DIFFERENTIAL CALCULUS AND ITS APPLICATIONS

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3.1 INTRODUCTION:

Calculus is a very versatile and valuable tool. It is a form of Mathematics which was developed from algebra and geometry. It is made up of two interconnected topics, differential calculus and integral calculus.

Differential calculus can be termed as the Mathematics of motion and change. Integral calculus covers the accumulation of quantities, such as areas under a curve. The two ideas work inversely together as defined by the Fundamental Theorem of Calculus.

Calculus is deeply integrated in every branch of the physical sciences, such as physics and biology. It is found in computer science, statistics, and engineering; in economics, business, and medicine. Modern developments such as architecture, aviation, and other technologies all make use of what calculus can offer and why calculus is so important and useful.

APPLICATIONS:

1. **Finding the Slope of a Curve**
2. **Calculating the Area of Any Shape**.
3. **Calculate Complicated X-intercepts**
4. **Visualizing Graphs**
5. **Finding the Average of a Function**
6. **Calculating Optimal Values**
   Differential calculus can be used to determine the stationary points of functions, in order to sketch their graphs. Calculating stationary points also lends itself to the solving of problems that require some variable to be maximised or minimised. These are referred to as optimisation problems.

   - Optimal Shape of an Irrigation Channel (Civil engineering)
   - Overcoming Friction and other Forces to move an Object (Mechanical, Aerospace, Civil engineering)
   - Beam Design (All Engineering)

3.1.1 DEFINTIONS:

Let $y = f(x)$ be a function continuous in the closed interval [a,b]. This means that if a < c < b,

$$\lim_{x \to c^-} f(x) = f(c) and \lim_{x \to a+0} f(x) = f(a), \lim_{x \to b-0} f(x) = f(b).$$

Let $y = f(x)$ be a function continuous in the closed interval [a , b]. This means that if a < c < b, the derivative of $f(x)$ at x = c exists i.e.,

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} exists.$$

Further, $\lim_{x \to a+o} \frac{f(x) - f(a)}{x - a}$ and $\lim_{x \to b-0} \frac{f(x) - f(b)}{x - b}$ exists.

Geometrically, if f(x) is a continuous function in the closed interval [a , b], the graph $y = f(x)$ is a continuous curve for the points x in [a , b]. If f(x) is derivable in the closed [a , b), there exists a unique tangent to the curve at every point in the curve [a , b].

Properties:

1. If f(x) is continuous in the closed interval [a , b], f(x) is bounded there in. Also it attains its glb and lub.
2. If f(x) is continuous is closed interval [a , b], it attains all values between f(a) and f(b).
3. If f(x) is continuous in the closed [a , b] and f(a), f(b) are of opposite signs, then there exists atleast one point c in the open interval (a , b) such that f(c) = 0.

3.1.2 ROLLES THEOREM:

Let f(x) be a function such that

i. it is continuous in the closed interval [a , b];
ii. it is differentiable in open interval (a , b) and
iii. f(a) = f(b)

Then there exists atleast one point c in open interval (a , b) such that $f^1(c) = 0$.

3.1.3 LARGANGE'S MEAN VALU THEOREM :

Let f(x) be a function such that

i. it is continuous in the closed interval [a , b] and

ii. it is differentiable in open interval (a , b)

Then there exists atleast one point c in open interval (a , b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3.2.1 TAYLOR'S THEOREM :

If f : [a , b] $\rightarrow$ R is such that

a) $f^{(n-1)}$ is continuous on [a , b]

b) $f^{(n-1)}$ is derivable on (a , b) (or) $f^{(n)}$ exists on (a , b) and p $\in z^{+}$ then there exist a point c $\in$ (a , b) such that

$$f(b) = f(a) + \frac{b-a}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \ldots\ldots\ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

Where $R_n = \dfrac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}$

Note:

i. Schlomilch – Roche's form of remainder

$$R_n = \frac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}$$

ii. Lagranges form of remainder

Putting p = n, we get

$$R_n = \frac{(b-a)^n f^{(n)}(c)}{n!}$$

iii. Cauchy's form of remainder:

Putting p = 1, we get

$$R_n = \frac{(b-a)\ (b-c)^{n-1} f^{(n)}(c)}{(n-1)!}.$$

3.2.2 MACLAURIN'S THEOREM :

If f : [0 , x] $\rightarrow$ R is such that

a) $f^{(n-1)}$ is continuous on [o , x]

b) $f^{(n-1)}$ is derivable on (0 , x) and p $\in z^+$ then there exists a real number θ∈ (0 , 1) such that

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots\ldots\ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(\theta x)$$

Note :

1. Schlomilch – Roche's form of remainder:

$$R_n = \frac{x^n(1-\theta)^{n-p} f^{(n)}(\theta x)}{(n-1)!\,p}$$

2. Lagrange's form of remainder:

Putting p = n, we get

$$R_n = \frac{x^n f^{(n)}(\theta x)}{n!}$$

3. Cauchy's form of remainder:

Putting p = 1, we get

$$R_n = \frac{x^n(1-\theta)^{n-1} f^{(n)}(\theta x)}{(n-1)!}.$$

PROBLEMS

1. **Obtain the Maclaurin's series expansion for *f*(x) =(1+x)^n (or)Expand (1+x)^n in power of x.**

Sol : Let f(x)=(1+x)^n. Then

$$f'(x) = n(1+x)^{n-1};\ f''(x) = n(n-1)(1+x)^{n-2};$$

$$f'''(x) = n(n-1)(n-2)(1+x)^{n-3} \text{ etc.}$$

Thus $f^k(x) = n(n-1)(n-2)\ldots.(n-k+1).(1+x)^{n-k}$

$\therefore f^k(0) = n(n-1)\ldots..(n-k+1).$

Hence $f(x)=\sum_{k=0}^{\infty} f^{k}(0)\frac{x^k}{k!}=\sum_{k=0}^{\infty}\frac{n(n-1).........(n-k+1)}{1.2.........k}.x^k$

i.e. $(1+x)^n=1+\frac{n}{1}x+\frac{n(n+1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+.....$

[This expansion is valid in -1<x<1.i.e., |x|<1].

2: Find the Maclaurin series for $\sin x$ for all x .

Sol: We arrange our computation in two columns as follows:

$$\begin{array}{ll} f(x)=\sin x & f(0)=0 \\ f'(x)=\cos x & f'(0)=1 \\ f''(x)=-\sin x & f''(0)=0 \\ f'''(x)=-\cos x & f'''(0)=-1 \\ f^{(4)}(x)=\sin x & f^{(4)}(0)=0 \end{array}$$

Since the derivatives repeat in a cycle of four, we can write the Maclaurin series as follows:

$$f(0)+\frac{f'(0)}{1!}x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\ldots=0+\frac{1}{1!}x+\frac{0}{2!}x^2+\frac{-1}{3!}x^3+\frac{0}{4!}x^4+\ldots$$

$$=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$$

$$\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+.......=\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!} forallx$$

3. Obtain the Maclaurin's series expansion of e^x, sinh x, cosh x

Sol : Let $f(x)=e^x$**.** Then

$f^1(x)=f^{11}(x)=f^{111}(x)=.......=e^x$

$\Rightarrow f(0)=f^1(0)=f^{11}(0)=f^{111}(0)=..........=e^0=1$

TheMaclaurin's series expansion of f(x) is given by

$$f(x) = f(0) + xf^1(0) + \frac{x^2}{2!} f^{11}(0) + \frac{x^3}{3!} f^{11}(0) + ... + \frac{x^n}{n!} f^n(0) + ... + ...$$

$$i.e., e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...$$

Similarly proceed for $\sinh x$, $\cosh x$.

4. Verify Taylor's theorem for *f*(x)=(1-x)$^{5/2}$ with Lagrange's form of remainder upto 2 terms in the interval[0,1].

Sol : Consider f(x) = (1-x) $^{5/2}$ in [0, 1]

(i) $f(x), f'(x)$ are continuous in [0,1]

(ii) $f''(x)$ is differentiable in(0,1)

Thus f(x) satisfies the conditions of Taylor's theorem.

We consider Taylor's theorem with Lagrange's form of remainder

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x) with 0 < \theta < 1 .....(1)$$

Here n=p=2,a=0 and x=1

$$f(x) = (1-x)^{5/2} \Rightarrow f(0) = 1$$

$$f'(x) = \frac{-5}{2}(1-x)^{3/2} \Rightarrow f'(0) = \frac{-5}{2}$$

$$f''(x) = \frac{15}{4}(1-x)^{1/2} \Rightarrow f''(\theta x) = \frac{15}{4}(1-\theta x)^{1/2} \Rightarrow f''(\theta) = \frac{15}{4}(1-\theta)^{1/2} \text{ and } f(1)=0$$

From(1), we have $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(\theta x)$

Substituting the above values, we get

$$0 = 1 + 1.\left(\frac{-5}{2}\right) + \frac{1^2}{2!}\frac{15}{4}(1-\theta)^{1/2}$$

$$\Rightarrow \theta = \frac{9}{25} = 0.36$$

$\therefore \theta$ lies between 0 and 1.

Thus the Taylor's theorem is verified.

5.Show that $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + 4\frac{x^3}{3!}$ **(or)Expand** $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$ **in powers of x.**

Sol : Let $f(x) = \frac{\sin^{-1}x}{\sqrt{1-x^2}}. Then f(0) = 0$

$$\Rightarrow \sqrt{(1-x^2)}\, f(x) = \sin^{-1}x \quad \ldots(1)$$

Differentiating (1) w. r. t 'x', we get

$$\sqrt{(1-x^2)}\; f'(x) + f(x).\frac{-2x}{2\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2) f'(x) - xf(x) = 1 \ldots(2)$$

Now $f'(0) = 1$

Differentiating (2)w.r.t'x' we get

$$(1-x^2) f''(x) + f'(x)(-2x) - xf'(x) - f(x) = 0$$

$$\Rightarrow (1-x^2) f''(x) - 3xf'(x) - f(x) = 0 \quad \ldots(3)$$

$$\Rightarrow f''(0) - f(0) = 0 \Rightarrow f''(0) = 0$$

Diff.(3)w.r.t.'x', we get

$$(1-x^2) f'''(x) - 2xf''(x) - 3f'(x) - 3xf''(x) - f'(x) = 0$$

$$\Rightarrow (1-x^2) f'''(x) - 5x - f''(x) - 4f'(x) = 0 \quad \ldots(4)$$

$$\Rightarrow f'''(0) - 4f'(0) = 0 \Rightarrow f'''(0) = 4$$

Similarly $f^4(0) = 0$

We have by Taylor's theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^4(0) + \ldots..$$

i.e., $\frac{\sin^{-1}x}{\sqrt{1-x^2}} = 0 + 1.x + \frac{x^2}{2!}(0) + \frac{x^3}{3!}.4 + ... = x + 4.\frac{x^3}{3!} + ...\infty$

6.Show that $\log\left(1+e^x\right) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + ...$ **and hence deduce that** $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + ...$**.**

Sol : Let f(x) =log (1+e^x). Then f(0) =log2

Differentiating successively w. r. t. x, we get

$$f'(x) = \frac{e^x}{1+e^x} \therefore f'(0) = \frac{1}{1+1} = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x)e^x - e^x.e^x}{(1+e^x)2} = \frac{e^x}{(1+e^x)2} \quad f''(0) = \frac{1}{(1+1)^2} = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - 2e^x(1+e^x)e^x}{(1+e^x)^4} = \frac{(1+e^x)\left[e^x + e^{2x} - 2e^{2x}\right]}{(1+e^x)^4} = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$\therefore f'''(0) = 0$$

$$f^{iv}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x}).3(1+e^x)^2.e^x}{(1+e^x)^6}$$

$$= \frac{(1+e^x)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2d})}{(1+e^x)^4}$$

$$\therefore f^{iv}(0) = \frac{(1+1)(1-2) - 3(1-1)}{(1+1)^4} = \frac{-2}{16} = \frac{-1}{8}$$

Substituting the values of $f\left(0\right), f'\left(0\right), f''(0)$ etc., in the Maclaurin's series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{iv}(0) + ...$$

We obtain

$$\log(1+e^x) = \log 2 + x.\frac{1}{2} + \frac{x^2}{2!}.\frac{1}{4} + \frac{x^3}{3!}(0) + \frac{x^4}{4!}\left(-\frac{1}{8}\right) + ...$$

$$=\log 2+\frac{x}{2}+\frac{x^2}{8}-\frac{x^4}{192}+.... \qquad ....(1)$$

Deduction:

Differentiating the result given by (1)w. r. t. x, we get

$$\frac{1}{1+e^x}.e^x=\frac{1}{2}+\frac{2x}{8}-\frac{4x^3}{192}+.... \text{or} \frac{e^x}{1+e^x}=\frac{1}{2}+\frac{x}{4}-\frac{x^3}{48}+.....$$

7.Calculate the approximate value of $\sqrt{10}$ correct to 4 decimal places using Taylor's theorem.

Sol :Let $f(x+h)=\sqrt{10}=\sqrt{9+1}=\sqrt{x+h}$

Here x=9 , h=1

Take f(x)= $\sqrt{x}$ **.**then

$$f'(x)=\frac{1}{2\sqrt{x}}, f''(x)=\frac{-1}{4x^{3/2}}, f'''(x)=\frac{3}{8x^{5/2}}$$

and $f(9)=3, f'(9)=\frac{1}{6}, f''(9)=\frac{-1}{4x27}=\frac{-1}{108}, f'''(9)=\frac{1}{8x81}=\frac{1}{648}$

Substituting these values in Taylor's series,

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(9)+.....$$

i.e., $\sqrt{10}=f(9)+1.f'(9)+\frac{1}{2!}f''(9)+...$

$$=3+\frac{1}{6}-\frac{1}{2}.\frac{1}{108}+\frac{1}{3!}.\frac{1}{648}+....$$

$$=3+\frac{1}{6}-\frac{1}{216}+\frac{1}{3888}+...$$

=3+0.1666-0.0046+0.0002

=3.1623

EXERCISE

1. Obtain the Taylor's series expansion of $\sin x$ in powers of $(x-\frac{\pi}{4})$.
2. Obtain the Taylor's series expansion of e^x about x = - 1.
3. Express $\tan^{-1} x$ in the powers of $(x-\frac{\pi}{4})$
4. Expand $e^{\sin x}$ by Maclaurin's series up to the term containing x^4.
5. Find the Maclaurin series for $\cos x$**.**
6. Expand $e^{x \sin x}$ in power of x**.**
7. Write Taylor's series for $f(x) = (1 - x)^{5/2}$ with Lagrange's from of remainder up to 3 terms in the interval[0,1].
8. Expand $\log e^x$ in powers of (x - 1) and hence evaluate log1.1 correct to 4 decimal places**.**
9. Obtain the Maclaurin's series expansion of log(1+x).
10. Show that $\sqrt{x} = 1+\frac{1}{2}(x-1)-\frac{1}{8}(x-1)^2+....... for 0 < x < 2$

ANSWERS

1. $\sin x = \frac{1}{\sqrt{2}}+(x-\frac{\pi}{4})\frac{1}{\sqrt{2}}-\frac{(x-\frac{\pi}{4})^2}{2!}\frac{1}{\sqrt{2}}-\frac{(x-\frac{\pi}{4})^3}{3!}\frac{1}{\sqrt{2}}+.....$
2. $\frac{1}{e}[1+(x+1)+\frac{(x+1)^2}{2!}+\frac{(x+1)^3}{3!}+......$
3. $\tan^{-1} x = 1+\frac{(x-\frac{\pi}{4})}{(1+\frac{\pi^2}{16})}-\frac{\pi}{4}\frac{(x-\frac{\pi}{4})^2}{(1+\frac{\pi^2}{16})^2}+.....$
4. $e^{\sin x} = 1+x+\frac{x^2}{2}-\frac{x^4}{8}+.........$
5. $\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+..........$
6. $e^{x\sin x} = 1+x^2+.......$
7. $f(x) = 1-\frac{5x}{2}+\frac{15x^2}{8}-\frac{5x^3}{16}(1-\theta x)^{-1/2}$
8. Log 1.1 = 0.0953
9. $\log(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+..........+(-1)^{n-1}\frac{x^n}{n}+.......$

3.3 CONTINUITY:

A function f(x, y) is said to be continuous at the point (a, b) if

- **i.** $\lim_{\substack{x \to a \\ y \to b}} f(x, y) exists$
- ii. the limit is same irrespective of the path along which the point (x, y) approaches (a, b).
- iii. the limit of f(x, y) as x $\rightarrow$ a and y $\rightarrow$ b is equal to the value of f(x, y)
  i.e., $\lim_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b)$

Examples:

1. Evalute $\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2 y}{x^2 + y^2 + 1}$

Sol : Consider $\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2 y}{x^2 + y^2 + 1} = \lim_{x \to 1} [\lim_{y \to 2} \frac{2x^2 y}{x^2 + y^2 + 1}]$

$$= \lim_{x \to 1} [\frac{4x^2}{x^2 + 5}]$$

$$= \frac{2}{3}$$

$$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2 y}{x^2 + y^2 + 1} = \lim_{y \to 2} [\lim_{x \to 1} \frac{2x^2 y}{x^2 + y^2 + 1}]$$

$$= \lim_{y \to 2} [\frac{2y}{y^2 + 2}]$$

$$= \frac{2}{3}$$

Hence $\lim_{\substack{x \to 1 \\ y \to 2}} \frac{2x^2 y}{x^2 + y^2 + 1} = \frac{2}{3}$.

2. Evalute $\lim_{\substack{x \to \infty \\ y \to 2}} \frac{xy + 1}{x^2 + 2y^2}$

Sol : Consider $\lim\limits_{\substack{x\to\infty \\ y\to 2}} \frac{xy+1}{x^2+2y^2}$ = $\lim\limits_{x\to\infty}[\lim\limits_{y\to 2} \frac{xy+1}{x^2+2y^2}]$

$= \lim\limits_{x\to\infty}[\frac{2x+1}{x^2+8}]$

=0

$\lim\limits_{\substack{x\to\infty \\ y\to 2}} \frac{xy+1}{x^2+2y^2}$ = $\lim\limits_{y\to 2}[\lim\limits_{x\to\infty} \frac{xy+1}{x^2+2y^2}]$

$= \lim\limits_{y\to 2}[0]$

=0

Hence $\lim\limits_{\substack{x\to\infty \\ y\to 2}} \frac{xy+1}{x^2+2y^2}$ **=0.**

3. Examine for continuity at the origin of the function defined by

$$f(x,y) = \frac{x^2}{\sqrt{x^2+y^2}} \quad for x \neq 0, y \neq 0$$

= 0 $\quad for x = 0, y = 0$ **Redefine the function to make it continuous.**

Sol : Notice that the value of f(x, y) for x =0 y=0 is not given in the problem let us discuss the continuity of the given function at(0,0).

$$\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} = \lim_{x\to 0}\left\{\lim_{y\to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right\} = \lim_{x\to 0}\left\{\frac{x^2}{x}\right\} = \lim_{x\to 0} x = 0$$

Also $\lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} f(x,y)\right\} = \lim\limits_{y\to 0}\left\{\lim\limits_{x\to 0} \frac{x^2}{\sqrt{x^2+y^2}}\right\}$

$$= \lim_{x\to 0}\left\{\frac{0}{\sqrt{0+y^2}}\right\} = \lim_{x\to 0}(0) = 0$$

$$\therefore \lim_{x\to 0}\left\{\lim_{x\to 0}(f(x,y)\right\} = \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}.$$

Also along the path y=mx

$$\lim_{x\to 0} f(x,y) = \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+m^2x^2}} = \lim_{y\to 0} \frac{x}{\sqrt{1+m^2}} = 0$$

Similarly along the path $y=mx^2$,

$$\lim_{x\to 0} f(x,y) = 0$$

Hence the function f(x, y) is continuous at the origin if f(x, y)=0for x=0,y=0.

Otherwise f(x, y) is not continuous at the origin.

If f(x,y) is not continuous at (0,0) then define f(x,y)=0 for x=0,y=0 so that f(x, y)is continuous at origin.

EXERCISE

1. If $f(x,y) = \frac{x-y}{2x+y}$. *Show that* $\lim_{x\to 0}\left\{\lim_{y\to 0} f(x,y)\right\} \neq \lim_{y\to 0}\left\{\lim_{x\to 0} f(x,y)\right\}$.
2. Discuss the continuity of the function

$$f(x,y) = \left\{\frac{2xy}{x^2+y^2}, (x,y) \neq (0,0) \text{ and } 0 \text{ for } (x,y) = (0,0)\right.$$

3. Discuss the continuity of

(i) $\lim_{\substack{x\to 1\\ y\to 1}} \frac{x(y-1)}{y(x-1)}$ (ii) $\lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{x^2+y^2}$

4. Investigate the continuity at (0, 0) of $f(x,y) = \frac{x^2-y^2}{x^2+y^2}, (x,y) \neq (0,0)$

= 0 (x, y) = (0, 0).

5. Show that the function $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}, (x,y) \neq (0,0)$

= 0, for (x, y) = (0,0).

ANSWERS

1. f(x, y) is continuous for given values of x and y but it is not continuous at (0, 0).

2. (i) does not exist (ii) does not exist.

3. Not continuous at (0, 0)

3.4 PARTIAL DIFFERENTIATION:

Let z = f(x, y) be a function of two variables x and y. Then $\lim_{\Delta x\to 0}\frac{f(x+\Delta x, y)-f(x,y)}{\Delta x}$, if it exists, is said to be partial derivative or partial differential coefficient of z or f(x, y), w.r.t. x. It is denoted by the symbol $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x.

Thus we see that for the partial derivative of z = f(x, y) w.r.t. y is kept constant

Similarly, the partial derivative of z = f(x, y) w.r.t. y keeping x as constant is defined $\lim_{\Delta y\to 0}\frac{f(x, y+\Delta y)-f(x,y)}{\Delta y}$, and is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.

In the same way, the partial derivatives of the function z = f $(x_1,x_2,\ldots..x_n)$ w.r.t. x_1 keeping other variables constant can be defined by

$\frac{\partial z}{\partial x_i}=\lim_{\Delta x_i\to 0}\frac{f\left(x_1,x_2,....,x_i+\Delta x_i.....,x_n\right)-f\left(x_1,x_2,....,x_i.....,x_n\right)}{\Delta x_i}$, i = 1, 2, ….n if the limit on the right hand exists.

HIGHERORDERPARTIALDERIVATIVES

In general the first order partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also functions of x and y and they can be differentiated repeatedly to get higher order partial derivatives.

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial x^2},\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial y^2},\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)=\frac{\partial^2 f}{\partial x\partial y},\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)=\frac{\partial^2 f}{\partial y\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial x^3},\ \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^3 f}{\partial y^3},\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial y^2}\right)=\frac{\partial^3 f}{\partial x\partial y^2},\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)=\frac{\partial^3 f}{\partial y\partial x^2}\text{ and so on.}$$

It is noted that $f_{xy}=f_{yx}$

PROBLEMS

1. If $U=\frac{1}{\sqrt{x^2+y^2+z^2}}, x^2+y^2+z^2\neq 0$ **then prove that** $\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}+\frac{\partial^2 U}{\partial z^2}=0.$

Sol : Given $U=(x^2+y^2+z^2)^{-1/2}$

$$\therefore \frac{\partial U}{\partial x}=-\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\, X\, \frac{\partial}{\partial x}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}(x^2 + y^2 + z^2)^{-3/2}(2x)$$

$= - (x^2+y^2+z^2)^{-3/2}(x)$

and $\frac{\partial^2 U}{\partial x^2} = -\left[(x^2 + y^2 + z^2)^{-3/2} + x\left(\frac{-3}{2}\right)(x^2 + y^2 + z^2)^{-5/2}.2x \right]$

$=-(x^2+y^2+z^2)^{-5/2}[x^2+y^2+z^2-3x^2]$

$= (x^2+y^2+z^2)^{-5/2}[2x^2-y^2-z^2]$ ....(1)

Similarly, we get

$$\frac{\partial^2 U}{\partial y^2} = (x^2 + y^2 + z^2)^{-5/2}(-x^2 + 2y^2 - z^2) \quad ..... (2)$$

and $\frac{\partial^2 U}{a z^2} = (x^2 + y^2 + z^2)^{-5/2}(-x^2 - y^2 + 2z^2)$ ..... (3)

(1) +(2)+(3) gives

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$$

2. If U=log(x^3+y^3+z^3-3xyz), prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 U = \frac{-9}{(x+y+z)^2}$ **.**

Sol : Given that U=log(x^3+y^3+z^3-3xyz)

$\therefore \frac{\partial U}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$ [Here y and z are constants]

$\frac{\partial U}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$ [Here x and z are constants]

and $\frac{\partial U}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$ [Here x and y are constants]

$$\therefore \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$\Rightarrow \frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z} = \frac{3}{x+y+z} \quad ....(1)$$

$$\text{Now}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 U = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial U}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \quad \text{[from (1)]}$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= -\frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} = -\frac{9}{(x+y+z)^2}$$

3. If $x^x y^y z^z = e$ show that at x=y=z, $\frac{\partial^2 z}{\partial x \partial y} = -(x \log ex)^{-1}$

Sol : Given that $x^x y^y z^z = e$

Taking logarithm on both sides, we get

$$x \log x + y \log y + z \log z = \log e$$

$$\Rightarrow z \log z = 1 - x \log x - y \log y$$

Differentiating partially w.r.t ‘x’ we get

$$\left(z.\frac{1}{z} + 1.\log z\right)\frac{\partial z}{\partial x} = -\left(x.\frac{1}{x} + 1.\log x\right)$$

$$\Rightarrow (1 + \log z)\frac{\partial z}{\partial x} = -(1 + \log x)$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{(1 + \log x)}{(1 + \log z)} \quad ..... (1)$$

$$\text{Similarly} \quad \frac{\partial z}{\partial y} = -\frac{(1 + \log y)}{1 + \log z} \quad .....(2)$$

When x=y=z, we have

$$\frac{\partial z}{\partial x} = -1 and \frac{\partial z}{\partial y} = -1$$

Now differentiating (2) partially w.r.t ‘x’ we get

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[-\frac{(1+\log y)}{(1+\log z)}\right]$$

$$= - (1+\log y)\left[-(1+\log z)^{-2}\frac{1}{z}\frac{\partial z}{\partial x}\right] = \frac{1+\log y}{z(1+\log z)^2}\frac{\partial z}{\partial x} \text{.... (3)}$$

When x=y=z from (3), we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1+\log x}{x(1+\log x)^2}(-1)\left(\because \frac{\partial z}{\partial x} = -1\right)$$

$$= -\frac{1}{x(1+\log x)} = -\frac{1}{x(\log e + \log x)} (\therefore \log e = 1)$$

$$= -\frac{1}{x \log ex} = -(x \log ex)^{-1}$$

4. If $u = \tan^{-1}\left[\frac{2xy}{x^2 - y^2}\right], prove that \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Sol : Given $u = \tan^{-1}\left[\frac{2xy}{x^2 - y^2}\right]$

$$\therefore \frac{\partial u}{\partial x} = \frac{1}{1+\frac{4x^2y^2}{(x^2-y^2)^2}}\frac{\partial}{\partial x}\left[\frac{2xy}{x^2 - y^2}\right]$$

$$= \frac{(x^2-y^2)^2}{(x^2-y^2)^2 + 4x^2y^2} \cdot \frac{(x^2-y^2)2y - 2xy.2x}{(x^2-y^2)^2}$$

$$= \frac{2y[x^2 - y^2 - 2x^2]}{(x^2+y^2)^2} = \frac{-2y(x^2+y^2)}{(x^2+y^2)^2} = \frac{-2y}{x^2+y^2}$$

and $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{2y(2x)}{(x^2+y^2)^2} = \dfrac{4xy}{(x^2+y^2)^2}$

Also $\dfrac{\partial u}{\partial y} = \dfrac{1}{1+\dfrac{4x^2y^2}{(x^2-y^2)^2}} \dfrac{\partial}{\partial y}\left(\dfrac{2xy}{x^2-y^2}\right)$ [Here x is constant]

$$= \frac{(x^2-y^2)^2}{(x^2+y^2)^2}.\frac{(x^2-y^2)2x - 2xy(-2y)}{(x^2-y^2)^2} = \frac{2x(x^2-y^2+2y^2)}{(x^2+y^2)^2} = \frac{2x}{x^2+y^2}$$

and $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{-2x.2y}{(x^2+y^2)^2} = \dfrac{-4xy}{(x^2+y^2)^2}$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

5. Verify $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ **for the function** $u = \tan^{-1}\dfrac{x}{y}$.

Sol : Let $u = \tan^{-1}\dfrac{x}{y}$.

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{1}{1+(x/y)^2}\left(\dfrac{1}{y}\right) = \dfrac{y}{y^2+x^2}$ [Here y is constant]

and $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{(y^2+x^2)1 - y(2y)}{(y^2+x^2)^2} = \dfrac{x^2-y^2}{(y^2+x^2)^2}$ ..... (1)

Now $\dfrac{\partial u}{\partial y} = \dfrac{1}{1+\left(\dfrac{x}{y}\right)^2}.\left(\dfrac{-x}{y^2}\right) = \dfrac{-x}{y^2+x^2}$ [Here x is constant]

and $\dfrac{\partial^2 u}{\partial x \partial y} = -\left[\dfrac{(y^2+x^2)1 - x(2x)}{(y^2+x^2)^2}\right] = -\dfrac{y^2-x^2}{(y^2+x^2)^2} = \dfrac{x^2-y^2}{(y^2+x^2)^2}$ ..... (2)

From (1) and (2), we have $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial x \partial y}$

6. IF z(x + y) =x²+y². show that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$.

Sol : Given $z = \frac{x^2 + y^2}{x + y}$

Differentiating 'z' partially w.r.t to 'x',

$$\frac{\partial z}{\partial y} = \frac{(x+y)(2x) - (x^2 + y^2)(1)}{(x+y)^2} = \frac{2x^2 + 2xy - x^2 - y^2}{(x+y)^2} = \frac{x^2 + 2xy - y^2}{(x+y)^2}$$

Differentiating 'z' partially w.r.t to 'y',

$$\frac{\partial z}{\partial y} = \frac{(x+y)(2y) - (x^2 + y^2)(1)}{(x+y)^2} = \frac{-x^2 + 2xy + y^2}{(x+y)^2}$$

Now $\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{x^2 + 2xy - y^2}{(x+y)^2} - \left[\frac{-x^2 + 2xy + y^2}{(x+y)^2}\right]$

$$\Rightarrow \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = \frac{2x^2 - 2y^2}{(x+y)^2} = \frac{2(x+y)(x-y)}{(x+y)^2} = \frac{2(x-y)}{(x+y)}$$

$$\therefore \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \qquad ....(1)$$

Now $1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 1 - \left[\frac{x^2 + 2xy - y^2}{(x+y)^2} + \frac{y^2 - x^2 + 2xy}{(x+y)^2}\right]$

$$= 1 - \frac{4xy}{(x+y)^2} = \frac{(x+y)^2 - 4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2}$$

$$\Rightarrow 4\left[1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right] = \frac{4(x+y)^2}{(x+y)^2} \qquad ....(2)$$

From (1) and (2), we have

$$\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2 = 4\left[1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right]$$

EXERCISE

1. If $u^3 + xv^2 - uy = 0, u^2 + xyv + v^2 = 0$ *find* $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$
2. If $u = x \log xy$ *where* $x^3 + y^3 + 3xy = 1$ *find* $\frac{du}{dx}$.
3. Find the first and second order partial derivatives of $ax^2 + 2hxy + by^2$ *and verify* $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.
4. If $z = f(x+ay) + \varphi(x-ay)$. Prove that $\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}$
5. If $f(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, then prove that $\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial y^2}$.
6. If $r^2 = x^2 + y^2 + z^2$ and $u = r^m$ then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$.
7. If $z = \log(e^x + e^y)$ show that $rt - s^2 = 0$, where $r = \frac{\partial^2 z}{\partial x^2}, t = \frac{\partial^2 z}{\partial y^2}, s = \frac{\partial^2 z}{\partial x \partial y}$.
8. If $u = f(r,s,t)$ where $r = \frac{x}{y}, s = \frac{y}{z}$ *and* $t = \frac{z}{x}$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$.
9. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\frac{\partial u}{\partial x} + (x^2 - yz)\frac{\partial u}{\partial y} + (z^2 - xy)\frac{\partial u}{\partial z} = 0$.
10. If $u = e^{xyz}$, show that $\frac{\partial^3 u}{\partial x \partial y \partial z} = (1 + 3xyz + x^2y^2z^2)e^{xyz}$.

ANSWERS

1. $\frac{\partial u}{\partial y} = \frac{2x^2v^2 + uxy + 2uv}{3u^2xy + 6u^2v - xy^2 - 2vy - 4xuv}$, $\frac{\partial u}{\partial x} = \frac{2xyv^2 - 2v^3 - xyv^2}{6u^2v + 3u^2xy - 2vy - xy^2 - 4xuv}$,

   $\frac{\partial v}{\partial x} = \frac{2uv^2 + vy^2 - 3u^2vy}{6u^2v + 3u^2xy - 2vy - xy^2 - 4xuv}$ and $\frac{\partial v}{\partial y} = \frac{xyv - 3xu^2v - 2u^2}{3u^2xy + 6u^2v - xy^2 - 2vy - 4xuv}$
2. $\frac{du}{dx} = 1 + \log(xt) - \frac{x(x^2+y)}{y(y^2+x)}$

3. 5 JACOBIAN:

Definition:

If u and v are functions of two independent variables x, y then the determinant

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$ is called the Jacobian named after German Mathematician Jacob Jacobi (1804 – 1851)

and is denoted by $\partial(\frac{u,v}{x,y})$ (or) $J(\frac{u,v}{x,y})$

Similarly the Jacobian of u, v, w w. r. to x, y, z is

$$J\left(\frac{u,v,w}{x,y,z}\right) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial w} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

Similarly, the Jacobian for four or more variables can be defined.

Properties:

1. If J is Jacobian of u, v w. r. to x, y and J' is the Jacobian of x, y w. r. to u,v then J J' = $\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$. This is called inverse property of Jacobians.
2. If u and v are functions of r and s and r and s are in turn functions of x, y then $J \equiv \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$. This is called the chain rule for the Jacobians.

Results:

1. If there is a change of variables from cartesian coordinates x, y in a plane to polar coordinates given by x = r sin θ , y = r cos θ then $J = \frac{\partial(x,y)}{\partial(r,\theta)} = r$.
2. If there is a change variables from Cartesian coordinates (x, y, z) to spherical polar coordinates given by x = r sin θ cos ∅, y = r sin θ sin ∅, z = r cos θ. Then

$$J \equiv \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta .$$

3. If there is a change variables from cartesian coordinates (x, y, z) to cylinderical coordinates given by $x = \rho\cos\varphi, y = \rho\sin\phi, z = z$ then the Jacobian of transformation

$$J \equiv \frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho .$$

1. If $u = \frac{yz}{x}, v = \frac{zx}{y}, w = \frac{xy}{z}\ show that\ \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$**.**

Sol : We have

$$\frac{\partial u}{\partial x} = \frac{-yz}{x^2}, \frac{\partial u}{\partial y} = \frac{z}{x}\ and\ \frac{\partial u}{\partial z} = \frac{y}{x}$$

$$\frac{\partial v}{\partial x} = \frac{z}{y}, \frac{\partial v}{\partial y} = \frac{-xz}{y^2}\ and\ \frac{\partial v}{\partial z} = \frac{x}{y}$$

and $\frac{\partial w}{\partial x} = \frac{y}{z}, \frac{\partial w}{\partial y} = \frac{x}{z}\ and\ \frac{\partial w}{\partial z} = \frac{-xy}{z^2}$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{xyz} \begin{vmatrix} \frac{-xyz}{x^2} & \frac{zy}{x} & \frac{yz}{x} \\ \frac{xz}{y} & \frac{-xyz}{y^2} & \frac{xz}{y} \\ \frac{xy}{z} & \frac{xy}{z} & \frac{-xyz}{z^2} \end{vmatrix}$$ [Multiplying C_1 by x , C_2 by y and C_3 by z]

$$= \frac{1}{xyz} . \frac{yz}{x} . \frac{xz}{y} . \frac{xy}{z} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$ [Taking common $\frac{yz}{x}, \frac{xz}{y}, \frac{xy}{z}$ from R_1, R_2, R_3 resp.]

$$=\frac{x^2y^2z^2}{(xyz)^2}[-1(1-1)-1(-1-1)+1(1+1)]$$

=-1(0)+2+2=4

Hence $\frac{\partial(u,v,w)}{\partial(x,y,z)}=4$

2. If $u=x^2-2y; v=x+y+z; w=x-2y+3z,\ find\ \frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol : We have $\frac{\partial u}{\partial x}=2x; \frac{\partial u}{\partial y}=-2; \frac{\partial u}{\partial z}=0$

$$\frac{\partial v}{\partial x}=1; \frac{\partial v}{\partial y}=1; \frac{\partial v}{\partial z}=1$$

$$and\ \frac{\partial w}{\partial x}=1; \frac{\partial w}{\partial y}=-2; \frac{\partial w}{\partial z}=3.$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}2x & -2 & 0\\ 1 & 1 & 1\\ 1 & -2 & 3\end{vmatrix}=2x(3+2)+2(3-1)=10x+4$$

3. If x + y + z = u, y + z = uv, z = uvw, then evaluate $\frac{\partial(x,y,z)}{\partial(u,v,w)}$.

(or)

If u = x + y + z, y + z = uv, z = uvw show that $\frac{\partial(x,y,z)}{\partial(u,v,w)}=u^2v$ **.**

Sol : Given u = x + y + z.....(1)

uv = y + z.....(2)

and uvw = z.....(3)

uv = y + z ⇒ y = uv –z = uv–uvw [using(3)]

u = x + y + z ⇒ x = u - (y + z) ⇒ x = u - uv [using (2)]

$$\therefore \frac{\partial x}{\partial u}=\frac{\partial}{\partial u}[u-uv]=1-v, \frac{\partial x}{\partial v}=-u\ and\ \frac{\partial x}{\partial w}=0$$

and $\dfrac{\partial y}{\partial u} = v - vw, \dfrac{\partial y}{\partial v} = u - uw, \dfrac{\partial y}{\partial w} = -uv$

and $\dfrac{\partial z}{\partial u} = vw, \dfrac{\partial z}{\partial v} = uw and \dfrac{\partial z}{\partial w} = uv$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

=(1-v)[(u-uw)(uv)+uv(uw)]+u[(v-vw)uv+uv(vw)]=u^2v

Thus $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = u^2 v$

4. If x = r sin θ cosϕ, y = r sin θ sinϕ, z = r cosθ , show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = r^2 \sin\theta and find \dfrac{\partial(r,\theta,\varphi)}{\partial(x,y,z)}$.

Sol : From the given spherical polar co-ordinates , we have

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi, \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi, \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi, \frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi, \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi,$$

$$\frac{\partial z}{\partial r} = \cos\theta, \frac{\partial z}{\partial\theta} = -\sin\theta, \frac{\partial z}{\partial\phi} = 0.$$

$$\therefore \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} \text{ [Expand by } R_3]$$

=cosθ [(r cosθ cosϕ)(r sinθ cosϕ)+(r cosθ sinϕ)(r sin θ sinϕ)]
+rsinθ [(sinθ cosϕ)(r sinθ cosϕ)+(sinθ sinϕ)(rsinθ sinϕ)]

$$= \cos\theta[r^2 \sin\theta\cos\theta(\cos^2\phi + \sin^2\phi)] + r\sin\theta[r\sin^2\theta(\cos^2\phi + \sin^2\phi)]$$

$= r^2 \sin\theta\cos^2\theta + r^2 \sin^3\theta$

$= r^2 \sin\theta(\cos^2\theta + \sin^2\theta) = r^2 \sin\theta$

Since $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)}.\dfrac{\partial(r,\theta,\varphi)}{\partial(x,y,z)} = 1, we have \dfrac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \dfrac{1}{r^2\sin\theta}$

5. If $u = x^2 - y^2, v = 2xy where x = r\cos\theta, y = r\sin\theta, show that \dfrac{\partial(u,v)}{\partial(r,\theta)} = 4r^3.$

Sol : Given $u=x^2-y^2$ and v = 2xy

Since $x = r cos\theta$ *and* $y = r sin\,\theta$ *, we have*

$u = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta \ldots(1)$

and $v = 2(r\cos\theta)(r\sin\theta) = r^2\sin 2\theta$ .....(2)

Differentiating (1) and (2) partially w.r.t 'r' and 'θ' we have

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta, \frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta and \frac{\partial v}{\partial r} = 2r\sin 2\theta, \frac{\partial v}{\partial \theta} = 2r^2\cos 2\theta$$

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} 2r\cos 2\theta & -2r^2\cos 2\theta \\ 2r\sin 2\theta & 2r^2\cos 2\theta \end{pmatrix} = 4r^3(\cos^2 2\theta + \sin^2 2\theta) = 4r^3$$

6. *If* $x = e^r\sec\theta, y = e^r\tan\theta prove that \dfrac{\partial(x,y)}{\partial(r,\theta)}.\dfrac{\partial(r,\theta)}{\partial(x,y)} = 1.$

Sol :*We have* $x = e^r\, sec\theta\ , y = e^r\, tan\theta$ ....(1)

$$\therefore \frac{\partial x}{\partial r} = e^r\sec\theta, \frac{\partial x}{\partial \theta} = e^r\sec\theta\tan\theta$$

$$\frac{\partial y}{\partial r} = e^r\tan\theta, \frac{\partial y}{\partial \theta} = e^r\sec^2\theta$$

$$\therefore \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} e^r\sec\theta & e^r\sec\theta\tan\theta \\ e^t\tan\theta & e^r\sec^2\theta \end{vmatrix}$$

$=e^{2r}(\sec^3\theta-\sec\theta\tan^2\theta)$

$=e^{2r}\sec\theta\,(\sec^2\theta - \tan^2\theta)$

$=e^{2r}\sec\theta$ .....(2)

Now from (1), we have

$$\frac{y}{x} = \frac{\tan\theta}{\sec\theta} = \sin\theta$$

And $x^2 - y^2 = e^{2r}(\sec^2\theta - \tan^2\theta) = e^{2r}$

Thus $r = \frac{1}{2}\log(x^2 - y^2), \theta = \sin^{-1}\left(\frac{y}{x}\right)$ ....(3)

These give

$$\frac{\partial r}{\partial x} = \frac{1}{2}.\frac{1}{x^2 - y^2}(2x) = \frac{x}{x^2 - y^2}, \frac{\partial r}{\partial y} = \frac{1}{2}.\frac{1}{x^2 - y^2}(-2y) = -\frac{y}{x^2 - y^2}$$

$$\frac{\partial\theta}{\partial x} = \frac{1}{\sqrt{1-(y^2/x^2)}}.\left(\frac{-y}{x^2}\right) = -\frac{y}{x\sqrt{x^2 - y^2}}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{\sqrt{1-(y^2/x^2)}}.\left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2 - y^2}}$$

$$\therefore \frac{\partial(r,\theta)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{x}{x^2 - y^2} & \frac{-y}{x^2 - y^2} \\ \frac{-y}{x\sqrt{x^2 - y^2}} & \frac{1}{\sqrt{x^2 - y^2}} \end{vmatrix}$$

$$= \frac{x}{(x^2 - y^2)^{3/2}} - \frac{y^2}{x(x^2 - y^2)^{3/2}}$$

$$\frac{1}{(x^2 - y^2)^{3/2}}\left[x - \frac{y^2}{x}\right] = \frac{1}{x\sqrt{x^2 - y^2}}$$ .....(4)

Equations (2) and (4) give the required Jacobeans.

Now substituting for x from (1) and for $\sqrt{x^2 - y^2}$ from (3) in (4) , we get

$$\frac{\partial(r,\theta)}{\partial(x,y)} = \frac{1}{e^r\sec\theta}.\frac{1}{e^r} = \frac{1}{e^{2r}\sec\theta}$$ .....(5)

Hence $\frac{\partial(x,y)}{\partial(r,\theta)}\cdot\frac{\partial(r,\theta)}{\partial(x,y)}=1$[From (2)and (5)

EXERCISE

1. $If x=\frac{u^2}{v}, y=\frac{v^2}{u}, find \frac{\partial(u,v)}{\partial(x,y)}.$

2. $If u=\frac{x+y}{1-xy} and \theta=\tan^{-1}x+\tan^{-1}y, find \frac{\partial(u,\theta)}{\partial(x,y)}$

3. If a = xu + v –y, b = u^2 + vy + w, c = zu – v + vw , show that
$\frac{\partial(a,b,c)}{\partial(u,v,w)}=x(yv+1-w)+z-2uv.$

4. If $x=u(1+v), y=v(1+u)$ then prove that $\frac{\partial(x,y)}{\partial(u,v)}=1+u+v.$

5. $If x=u(1-v), y=uv, prove that \frac{\partial(u,v)}{\partial(x,y)} x \frac{\partial(x,y)}{\partial(u,v)}=1.$

6. If $x=u(1-v), y=uv$ prove that $JJ'=1$

7. If u = xy + yz + zx, v = $x^2 + y^2 + z^2$ and w = x + y + z, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ and also show that w^2 – v -2u = 0.

ANSWERS

1. $\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{3}$ 2. 0

3.6 FUNCTIONAL DEPENDENCE:

Suppose u = f(x, y), v = ϕ(x, y) are two given differentiable functions of the two independent variables x and y. If these functions u and v are connected by a relation F(u, v) = 0, where F is differentiable. Then u and v are said to be functionally dependent on one another, that is one function, say, u is a function of the second function v if $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ *and* $\frac{\partial v}{\partial y}$ are not all zero simultaneously. To establish the functional dependence of two functions, we have a result.

Result:

1. Two functions u and v are functionally dependent if and only if their Jacobian

$$J\frac{(u,v)}{(x,y)} = \frac{\partial(u,v)}{\partial(x,y)} = 0.$$

2. If J ≠ 0, then u and v are functionally independent.

1. **Show that the functions u = xy + yz + zx, v = x^2 + y^2 + z^2 and w = x + y + z are functionally related. Find the relation between them.**

(or)

Prove that the functions u = x + y + z, v = xy + yz + zx, w = x^2 + y^2 + z^2 are functionally dependent and find the relation between them.

Sol : We have u = xy + yz + zx, v = x^2 + y^2 + z^2 and w = x + y + z**....(1)**

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} y+z & z+x & x+y \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} y+z & z+x & x+y \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} x+y+z & x+y+z & x+y+z \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} \text{(Applying } R_1 \to R_1 + R_2\text{)}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix}$$

=2(x + y + z) (0) (∴R_1 and R_3 are identiacal)

Hence u,v and we are functionally dependent .That is, the functional relationship exists between u,v and w.

Now w^2=$(x + y + z)^2$ = x^2 + y^2 + z^2 + 2(xy + yz + zx) = v + 2u, [by (1)]

∴w^2+2u+v is the functional relationship between u,v and w.

2. **Show that the functions u = x + y + z , v = $x^2 + y^2 + z^2$ - 2xy - 2yz - 2zx and w = $x^3 + y^3 + z^3$ - 3xyz are functionally related.**

Sol : Given u = x + y + z, v = $x^2 + y^2 + z^2$- 2xy - 2yz - zx and w = $x^3 + y^3 + z^3$ - 3xyz

$$\text{Now}\ \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2(x-y-z) & 2(y-x-z) & 2(z-y-x) \\ 3(x^2-yz) & 3(y^2-xz) & 3(z^2-xy) \end{vmatrix}$$

$$= 6\begin{vmatrix} 1 & 1 & 1 \\ x-y-z & y-x-z & z-y-x \\ x^2-yz & y^2-xz & z^2-xy \end{vmatrix}$$

$$= 6\begin{vmatrix} 0 & 0 & 0 \\ 2(x-y) & 2(y-z) & z-y-x \\ (x-y)(x+y+z) & (y-z)(x+y+z) & z^2-xy \end{vmatrix}$$ [Applying $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$]

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 12\begin{vmatrix} x-y & y-z \\ (x-y)(x+y+z) & (y-z)(x+y+z) \end{vmatrix}$$

$$=12(x-y)(y-z)\begin{vmatrix} 1 & 1 \\ x+y+z & x+y+z \end{vmatrix}$$

=12(x-y) (y-z) (0) [$\therefore C_1$ and C_2 are identical]

Hence the functional relationship exists between u,v and w.

3. **Prove that** $u = \frac{x^2-y^2}{x^2+y^2}, v = \frac{2xy}{x^2+y^2}$ **are functionally dependent and find the relation between them.**

Sol : We are given $u = \frac{x^2-y^2}{x^2+y^2}, v = \frac{2xy}{x^2+y^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{(x^2+y^2).2x-(x^2-y^2).2x}{(x^2+y^2)^2} = \frac{2x(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y}=\frac{(x^2+y^2)(-2x)-(x^2-y^2).2y}{(x^2+y^2)^2}=\frac{(-2y)(x^2+y^2+x^2-y^2)}{(x^2+y^2)^2}=\frac{-4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x}=2y\left[\frac{(x^2+y^{2)}.1-x.2x}{(x^2+y^2)^2}\right]=\frac{2y(y^2-x^2)}{(x^2+y^2)^2}$$

4. Prove that $u=\frac{x^2-y^2}{x^2-y^2},v=\frac{2xy}{x^2+y^2}$ **are functionally dependent and find the relation between them.**

Sol : We are given $u=\frac{x^2-y^2}{x^2+y^2},v=\frac{2xy}{x^2+y^2}$

$$\therefore\frac{\partial u}{\partial x}=\frac{(x^2+y^2).2x-(x^2-y^2).2x}{(x^2+y^2)^2}=\frac{2x(x^2+y^2-x^2+y^2)}{(x^2+y^2)^2}=\frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial u}{\partial y}=\frac{(x^2+y^2)(-2y)-(x^2-y^2).2y}{(x^2+y^2)^2}=\frac{(-2y)(x^2+y^2+x^2-y^2)}{(x^2+y^2)^2}=\frac{-4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x}=2y\left[\frac{(x^2+y^2).1-x.2x}{(x^2+y^2)^2}\right]=\frac{2y(y^2-x^2)}{(x^2+y^2)^2}\text{ and}$$

$$\frac{\partial v}{\partial x}=2x\left[\frac{(x^2+y^2).1-y.2y}{(x^2+y^2)^2}\right]=\frac{2x(x^2-y^2)}{(x^2+y^2)^2}$$

Thus $\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{4xy^2}{(x^2+y^2)^2} & \frac{-4x^2y}{(x^2+y^2)^2}\\ \frac{2y(y^2-x^2)}{(x^2+y^2)^2} & \frac{2x(x^2-y^2)}{(x^2+y^2)^2}\end{vmatrix}$

$$=\frac{8x^2y^2(x^2-y^2)}{(x^2+y^2)^4}+\frac{8x^2y^2(y^2-x^2)}{(x^2+y^2)^4}=\frac{8x^2y^2(x^2-y^2)-8x^2y^2(x^2-y^2)}{(x^2+y^2)^4}=0$$

∴u,v are functionally dependent

$$u^2+v^2=\frac{(x^2+y^2)^2}{(x^2+y^2)^2}+\frac{4x^2y^2}{(x^2+y^2)^2}=\frac{(x^2+y^2)^2}{(x^2+y^2)^2}=1$$

Hence $u^2+v^2=1$ is the functional relation between u and v.

5. Verify if u = 2x – y + 3z, v = 2x – y - z, w = 2x – y + z are functionally dependent and if so , find the relation between them.

Sol :We are given u = 2x – y + 3z, v = 2x – y - z, w = 2x – y + z

The functions u,v,w are functionally dependent if and only if $J\left(\frac{u,v,w}{x,y,z}\right)=0$

$$\text{Now } J\left(\frac{u,v,w}{x,y,z}\right)=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial a}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}=\begin{vmatrix}2 & -1 & 3\\ 2 & -1 & -1\\ 2 & -1 & 1\end{vmatrix}=2(-1)\begin{vmatrix}1 & 1 & 3\\ 1 & 1 & -1\\ 1 & 1 & 1\end{vmatrix}=(-2)(0)=0$$

∴u,v,w are functionally dependent

Now u + v - 2w = (2x – y + 3z) + (2x – y - z) - 2(2x – y + z)

= (4x - 2y + 2z) - (4x - 2y + 2z) = 0

Hence u + v- 2w = 0 is the functional relationship between u,v and w.

EXERCISE

1. Determine whether the following functions are functionally depedent or not. If they are functionally dependent, find a relation between them. (i). u = $e^x \sin y$, v= $e^x \cos y$
   (ii) u = $\frac{x}{y}$, v = $\frac{x+y}{x-y}$.
2. If u = $\frac{x+y}{1-xy}$, v = $\tan^{-1} x + \tan^{-1} y$. Find $\frac{\partial(u,v)}{\partial(x,y)}$. Hence prove that u and v are functionally dependent. Find the functional relation between them.
3. Show that the following functions are functionally dependent and hence find the functional relationship between them: (i). U =$\frac{x-y}{x+y}$, V = $\frac{xy}{(x+y)^2}$
   (ii) U = $\sin^{-1} x + \sin^{-1} y$, V = $x\sqrt{1-y^2} + y\sqrt{1-x^2}$
   (iii) U = $xe^y \sin z$, V = $xe^y \cos z$,W = $x^2 e^{2y}$.
4. Verify if u = 2x – y + 3z, v = 2x – y – z, w = 2x – y +z are functionally dependent and if so, find the relation between them.
5. Show that the following functions are functionally dependent and hence find the functional relationship between them.
   (i). $u=\sin^{-1}x+\sin^{-1}y, v=x\sqrt{1-y^2}+y\sqrt{1-x^2}$
   (ii) $u=xe^y\sin z, v=xe^y\cos z, w=x^2e^{2y}$
   (iii) $u=x^2e^{-y}\cosh z, v=x^2e^{-y}\sinh z, w=x^2+y^2+z^2-xy-yz-zx$
   (iv) $u=x+y+z, v=x^3+y^3+z^3-3xyz, w=x^2+y^2+z^2-xy-yz-zx$

(v) $u = \dfrac{x-y}{x+y}, v = \dfrac{xy}{(x+y)^2}$

ANSWERS

1. (i) U,V are functionally independent
   (i i) U and V are functionally dependent, V = $\frac{U+1}{U-1}$ is relationship.
2. V = $\tan^{-1} U$ is relationship.
3. (i). $u^2 + 4v = 1$. (ii) v = $\sin u$. (iii) $2w = u(3v - u^2)$
4. Functionally dependent with relationship u + v – 2w = 0.
5. (i) v = sin u (ii) $u^2 + v^2 = w$ (iii) $3(u^2 - v^2) = w$ (iv) uw = v . (v) $u^2 + 4v = 1$.

3.7 TAYLOR'S SERIES WITH TWO VARIABLES

Statement: Let f(x, y) be a function of two independent variables x and y. If h and k be small increments in x and y then

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) + \frac{1}{2!}\left[h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2} \right] + ... \qquad (1)$$

Note:

1. The above Taylor's expansion can also be written as

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right) + \frac{1}{2!}\left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} \right)^2 + ...$$

2. Put x = a, and y = b, Taylor's expansion is given as

$$f(a+h, b+k) = f(a, b) + \left(h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y} \right) +$$
$$\frac{1}{2!}\left[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2} \right]$$
$$+.....$$

3. Put a + h = x, b + k = y in above we have

$$f(x,y) = f(a,b) + \left((x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y} \right) +$$

$$\frac{1}{2!}\left[(x-a)^2 \frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x \partial y} + (y-b)^2 \frac{\partial^2 f(a,b)}{\partial y^2} \right] + ...$$

1. Expand $x^2 y + 3y - 2$ **in powers of (x - 1) and (y + 2) up to the terms of 3rd degree.**

Sol : We know that the Taylor's series expansion of *f*(x, y) in power of (x - a) and (y - b) is

f(x, y) = *f*(a, b) + (x - a) f_x(a, b) + (y - b) f_y(a, b)

$$+\frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] +$$

$$\frac{1}{3!}[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}(a,b)]$$

+ ………….

Here a = 1 and b = -2

Let $f(x,y) = x^2 y + 3y - 2$

Now $f(1,2) = (1)^2(-2) + 3(-2) - 2 = -10$

$f_x = \frac{\partial f}{\partial x} = 2xy \qquad \therefore f_x(1,2) = 2(1)(-2) = -4$

$f_y = \frac{\partial f}{\partial y} = x^2 + 3 \qquad \therefore f_y(1,2) = (1)^2 + 3 = 4$

$f_{xx} = \frac{\partial^2 f}{ax^2} = 2y \qquad \therefore f_{xx}(1-2) = 2(-2) = -4$

$f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 2x \qquad \therefore f_{xy}(1,-2) = 2(1) = 2$

$f_{yy} = \frac{\partial^2 f}{\partial y^2} = 0 \qquad \therefore f_{yy}(1,-2) = 0$

$f_{xxy} = \frac{\partial^3 f}{\partial x^2 \partial y} = \frac{\partial}{\partial x^2}(x^2 + 3) = 2 \qquad \therefore f_{xxy}(1,-2) = 2$

$$f_{xxx} = \frac{\partial^3 f}{\partial x^3} = 0 \qquad \therefore\ f_{xxx}(1,2) = 0$$

$$f_{xyy} = \frac{\partial^3 f}{\partial x \partial y^2} = 0 \qquad \therefore\ f_{xyy}(1,2) = 0$$

$$f_{yyy} = \frac{\partial^3 f}{\partial y^3} = 0 \qquad \therefore\ f_{yyy}(1,-2) = 0$$

Since all the third order partial derivatives are constant, therefore, all partial derivates of further higher order vanishes.

Substituting these values in (2), we get

$$f(x, y) = x^2 y + 3y - 2$$

$$= -10 + [-4(x-1) + 4(y-2)] + \frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y-2)(2) + (y+2)^2(0)]$$

$$+ \frac{1}{3!}[0 + 3(x-1)^2(y+2)(2) + 3(x-1)(y+2)^2(0) + 0]$$

$$= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + (x-1)^2(y+2)$$

2. Expand $x^3 + y^3 + xy^2$ in power of (x - 1) and (y - 2) using Taylor's series.

Sol : Here a = 1, b = 2

Let $f(x, y) = x^3 + y^3 + xy^2 . Then f(1,2) = 1 + 8 + 4 = 13.$

Now $f_x = 3x^2 + y^2 \qquad \therefore\ f_x(1,2) = 3(1)^2 + (2)^2 = 7$

$$f_y = 3y^2 + 2xy \qquad \therefore\ f_y(1,2) = 3(2)^2 + 2(1)(2) = 12 + 4 = 16$$

$$f_{xx} = 6x \qquad \therefore\ f_{xx}(1,2) = 6(6) = 6$$

$$f_{xy} = 2y \qquad \therefore\ f_{xy}(1,2) = 2(2) = 4$$

$$f_{yy} = 6y + 2x \qquad \therefore\ f_{yy}(1,2) = 6(2) + 2(1) = 14$$

$$f_{xxx} = 6 \qquad \therefore\ f_{xxx}(1,2) = 6$$

$$f_{xyy} = 2 \qquad \therefore\ f_{xyy}(1,2) = 2$$

$f_{xxy} = 2$ $\therefore f_{xxy}(1,2) = 2$

$f_{yyy} = 6$ $\therefore f_{yyy}(1,2) = 6$

The Taylor's series expansion of f(x, y) in powers of (x - 1) and (y - 2) is

$$f(x,y) = f(1,2) + [(x-1) f_x(1,2) + (y-2) f_y(1,2)] + \frac{1}{2!}[(x-1)^2 f_{xx}(1,2) +$$

$$2(x-1)(y-2) f_{xy}(1,2) + (y-2)^2 f_{yy}(1,2)] + \frac{1}{3!}[(x-1)^3 f_{xxx}(1,2) +$$

$$3(x-1)^2 (y-2) f_{xxy}(1,2) + 3(x-1)(y-2)^2 f_{xyy}(1,2) + (y-2)^3 f_{yyy}(1,2)] + \ldots$$

$$= 13 + [(x-1)7 + (y-2)16] + \frac{1}{2!}[(x-1)^2 6 + 2(x-1)(y-2)4 + (y-2)^2 14]$$

$$+ \frac{1}{3!}[(x-1)^3 6 + 3(x-1)^2 (y-2)2 + 3(x-1)(y-2)^2 2 + (y-2)^3 6] + \ldots$$

$= 13 + 7(x - 1) + 16(y - 2) + 3(x - 1)^2 + 4(x - 1)(y - 2) + 7(y - 2)^2 + (x - 1)^3 + (x - 1)^2 (y - 2) + (x-1)(y-2)^2 + (y-2)^3$

3. Expand e^x sin y in power of x and y.

Sol : Let f(x, y) = e^xsin y.then f(0,0)=0

Now $f_x(x,y) = e^x \sin y$; $f_x(0,0)=0$

$f_y(x,y) = e^x \cos y$; $f_y(0,0)=1$

$f_{xx}(x,y) = e^x \sin y$; $f_{xx}(0,0)=0$

$f_{xy}(x,y) = e^x \cos y$; $f_{xy}(0,0)=1$

$f_{yy}(x,y) = e^x \sin y$; $f_{yy}(0,0)=0$

$f_{xxx}(x,y) = e^x \sin y$; $f_{xxx}(0,0)=0$

$f_{xxy}(x,y) = e^x \cos y$; $f_{xxy}(0,0)=1$

$f_{xyy}(x,y) = e^x \sin y$; $f_{xyy}(0,0)=0$

And $f_{yyy}(x,y) = e^x \cos y$; $f_{yyy}(0,0)= -1$

∴By Taylor's theorem,

$$f(x,y) = f(0,0) + xf_x(0,0) + yf_y(0,0) + \frac{1}{2!}[x^2 f_{xx}(0,0) + y^2 f_{yy}(0,0)]$$

$$+\frac{1}{3!}\left[x^3 f_{xxx}(0,0) 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right] + \cdots$$

$$= 0 + 0 + y\frac{1}{2!}[0 + 2xy + 0] + \frac{1}{3!}[0 + 3x^2 + 0 - y^3] + ...$$

$$= y + xy + \frac{x^2 y}{2} - \frac{y^3}{6} + ...$$

4. **Expand e^{xy} in the neighborhood of (1,1).**

Sol : $f(x,y) = e^{xy} \Rightarrow f(1,1) = e$. then

$f_x(x,y) = ye^{xy} \quad \Rightarrow f_x(1,1) = e$

$f_y(x,y) = xe^{xy} \Rightarrow f_y(1,1) = e$

$f_{xx}(x,y) = y^2 e^{xy} \Rightarrow f_{xx}(1,1) = e$

$f_{xy}(x,y) = xye^{xy} + e^{xy} \Rightarrow f_{xy}(1,1) = 2e$

$f_{yy}(x,y) = x^2 e^{xy} \Rightarrow f_{yy}(1,1) = e$

From Taylor's series, we get

$$f(x,y) = f(1,1) + (x-1) f_x(1,1) + (y-1) f_y(1,1)$$

$$+\frac{1}{2!}[(x-1)^2 f_{xx}(1,1) + 2(x-1)(y-1) f_{xy}(1,1) + (y-1)^2 f_{yy}(1,1)] + ..$$

i.e., $e^{xy} = e + e(x-1) + e(y-1) + \frac{1}{2!}[e(x-1)^2 + 4e(x-1)(y-1)e(y-1)^2] + ...$

$$= e[1 + (x-1) + (y-1) + \frac{(x-1)^2}{2!} + 2(x-1)(y-1) + \frac{(y-1)^2}{2!} + ...]$$

5. **If** $f(x,y) = \tan^{-1} xy$ **compute f(0.9, -1.2) approx.**

Sol :Given $f(x,y) = \tan^{-1} xy$

Now Taylor's expansion of f(x +h ,y +k)gives

$$f(x+h,y+k) = f(x,y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \ldots \qquad \ldots.(1)$$

$$\therefore\ f(0.9,-1.2) = f(1-0.1,-1-0.2) = f(x+h,y+k)$$

$$= f(1,-1) + (-0.1)\frac{\partial f}{\partial x} + (-0.2)\frac{\partial f}{\partial y}$$

$$+\frac{1}{2!}\left[(-0.1^2\frac{\partial^2 f}{\partial x^2} + 2(-0.1)(-0.2)\frac{\partial^2 f}{\partial x \partial y} + (-0.2)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots.. \qquad \ldots..(2)$$

Here $f(x,y) = \tan^{-1}(xy)$ $\qquad \therefore\ f(1,-1) = \tan^{-1}(-1) = -\frac{\pi}{4}$

$\frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2}$ $\qquad \therefore\ \frac{\partial f}{\partial x}(1,-1) = \frac{-1}{1+1} = \frac{-1}{2}$

$\frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2}$ $\qquad \therefore\ \frac{\partial f}{\partial y}(1,-1) = \frac{1}{1+1} = \frac{1}{2}$

$\frac{\partial^2 f}{\partial x^2} = -\frac{2xy}{(1+x^2y^2)^2}$ $\qquad \therefore\ \frac{\partial^2 f}{\partial x^2}(1,-1) = \frac{-(2)(-1)}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{1+x^2y^2-2x^2y^2}{(1+x^2y^2)2}$

$= \frac{1-x^2y^2}{(1+x^2y^2)^2}$ $\qquad \therefore\ \frac{\partial^2 f}{\partial x \partial y}(1,-1) = \frac{1-1}{(1+1)^2} = 0$

$\frac{\partial^2 f}{\partial y^2} = \frac{-x(2x^2y)}{(1+x^2y^2)^2}$ $\qquad \therefore\ \frac{\partial^2 f}{\partial y^2}(1,-1) = \frac{2}{(1+1)^2} = \frac{2}{4} = \frac{1}{2}$

Substituting these values in(2), we get

$$f(0.9,-1.2) = -\frac{\pi}{4} + (-0.1)\left(\frac{-1}{2}\right) + (-0.2)\left(\frac{1}{2}\right)$$

$$+\frac{1}{2}\left[(-0.1)^2\frac{1}{2} + 2(-0.1\right])(-0.2)(0) + (-0.2)^2\left(\frac{1}{2}\right) + \ldots.$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + \frac{1}{2}(0.005 + 0.02)$$

$$= -\frac{\pi}{4} + 0.05 - 0.1 + 00.0125 = -0.823$$

EXERCISE

1. Expand $x^2y + 3y - 2$ in powers of (x - 1) and (y + 2) using Taylor's theorem.
2. Expand$e^x \cos y$ near $(1, \frac{\Pi}{4})$.

3. Expand the function f(x, y) = $e^x \log(1+y)$ in terms of x and y upto the terms of 3[rd] degree using Taylor's theorem.
4. Expand f(x, y) = $x^3 + y^3 + xy^2$ in powers of (x-1) and (y-2) using Taylor's theorem.
5. Expand f(x, y) = $x^2 + xy + y^2$.
6. Expand $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ in powers of (x - 1) and (y – 1) upto third degree terms. Hence compute f(1.1, 0.9) approximately.
7. Expand $(1+x+y^2)^{\frac{1}{2}}$ at (1, 0).

ANSWERS

1. $-10 - 4(x-1) + (y+2) - 2(x-1)^2 + (x-1)(y+2) + (x-1)^2(y-2)$.
2. $\frac{e}{\sqrt{2}}\left[1 + (x-1) - \left(y - \frac{\Pi}{4}\right) + \frac{(x-1)^2}{2!} - (x-1)\left(y - \frac{\Pi}{4}\right) - \frac{\left(y-\frac{\Pi}{4}\right)^2}{2!} + \ldots\right]$
3. $e^x \log(1+y) = y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^3 \ldots\ldots$

4.
$13 + 7(x-1) + 16(y-2) + 3(x-1)^2 + 4(x-1)(y-2) + 7(y-2)^2 + (x-1)^3 + (x-1)(y-2)^2 + (y-2)^3 + \ldots\ldots$

5. $7 + 4(x-1) + 5(y-2) + \frac{1}{2!}\left[2(x-1)^2 + 2(x-1)(y-2) + 2(y-2)^2\right] + \ldots$

6. f(1.1, 0.9) = 0.6857.

7. $\sqrt{2}\left[1 + \frac{x-1}{4} - \frac{(x-1)^2}{32} + \frac{y^2}{4} + \ldots\ldots\right]$

3.8 MAXIMA AND MINIMA OF FUNCTIONS OF TWO VARIABLES

Definition:

A function f(x, y) is said to have a maximum value at x = a, y = b if
f(a, b) > f (a + h, b + k) for small and independent values of h and k, positive or negative.

A function f(x, y) is said to have a maximum value at x = a, y = b if
f(a, b) < f (a + h, b + k) for small and independent values of h and k, positive or negative.

Stationary point : The point (a,b) is called a stationary point if $\frac{\partial f}{\partial x}(a,b) = 0, \frac{\partial f}{\partial y}(a,b) = 0$.

Statoniary value : f(a, b) is said to be stationary of f(x, y) if $\frac{\partial f}{\partial x}(a,b)$ = 0 &$\frac{\partial f}{\partial y}(a,b)$ =0, i.e., thefunction is stationary at (a, b).

Extreme Value: A maximum or minimum value of a function is called its extreme value.

Saddle Point : The point (a, b) at which f(x, y) has neither a maximum nor minimum, i.e., f(a, b) is not an extreme value is called a saddle point.

Note: It may be noted that every extreme value is a stationary value but the converse may not be true.

Working Procedure to find Maxima or Minima of f(x, y) :

Given f(x, y) a function of two variables.

Step 1: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each of them to zero and solve these as simultaneous equationin x and y. Let (a, b), (c, d)….. be the pair of values.

Step 2: Find r = l = $\frac{\partial^2 f}{\partial x^2}$, s = m = $\frac{\partial^2 f}{\partial x \partial y}$, t = n = $\frac{\partial^2 f}{\partial y^2}$ and δ = r t - s^2 = l n – $m^2$and evaluate these at each pair of values (a, b), (c, d) ……… obtain in step 1.

Step 3 : i) If δ> 0 and r> 0 at an extreme point (a, b), then we conclude that f(x, y) is maximum at (a, b) and maximum value is given by f(a, b).

ii) If δ > 0 and r < 0 at an extreme point (a, b), then we conclude that f(x, y) is minimumat (a, b) and minimum value is given by f(a, b).

iii) If δ = 0 then the case is doubtful and needs further investigation.

iv) If δ< 0, at an extreme point (a, b), then f(a, b) is neither maximum nor minimum, i.e., f(a, b) is not an extreme value. In this case (a, b) is called saddle point.

1. **Find the maximum and minimum values of $x^3 + y^3$- 3axy.**

Sol : Let z = $x^3 + y^3$- 3axy we have

$\frac{\partial z}{\partial x} = 3x^2 - 3ay$=0 ....(1)

$\frac{\partial z}{\partial x} = 3y^2 - 3ay$=0 .....(2)

From (1),-3ay =$-3x^2 \Rightarrow y = \frac{-3x^2}{-3a} = \frac{x^2}{a}$......(3)

Substituting this value of y in (2) , we have

$$3.\frac{x^4}{a^2} - 3ax = 0 \Rightarrow 3x^4 - 3a^3x = 0 \Rightarrow 3x(x^3 - a^3) = 0$$

∴ x = 0, x = a

Corresponding values of y are y = 0,y = a.

Now $l = \frac{\partial^2 z}{\partial x^2} = 6x; m = \frac{\partial^2 z}{\partial x \partial y} = -3a; n = \frac{\partial^2 z}{\partial y^2} = 6y.$

At the point (0,0), $ln - m^2 = 36xy - 9a^2 < 0$

At(a, a), $ln - m^2 = 36a^2 - 9a^2 = 27a^2 > 0$ and $l = 6a > 0$ if $a > 0$ and $l < 0$ if $a < 0$.

Thus if $a < 0$, $z = -a^3$ is the maximum value and if $a > 0$, $z = -a^3$ is the minimum value.

At(0,0), z does not have any extreme value.

2. Find the maximum and minimum, values of $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Sol : We have $\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x = 0$ ....(1)

and $\frac{\partial f}{\partial y} = 6xy - 6y = 0$ .....(2)

Solving (1) & (2) , we get x=0,1,2 and y = 0; ± 1

Now $\frac{\partial^2 f}{\partial x^2} = 6x - 6; \frac{\partial^2 f}{\partial x \partial y} = 6y; \frac{\partial^2 f}{\partial y^2} = 6x - 6$

At (0, 0), $ln - m^2 = (6x - 6)^2 - 36y^2 = 36 > 0$ and $l = 6x - 6 < 0$.

∴f(0,0) = 4 is the maximum value.

At(2, 0), $ln - m^2 = (6x - 6)^2 - 36y^2 = 36 > 0$ and $l = 6x - 6 > 0$

∴f(2,0) = 0 is the minimum value

At(1,±1), $ln - m^2 = (6x - 6)^2 - 36y^2 = -36 < 0$

∴f(1,±1) is not an extreme value.

3. Examine the function for extreme value $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ ($x > 0, y > 0$).

Sol : We have $\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y$ and $\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y.$

For f(x, y) to be maximum or minimum , $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\Rightarrow 4(x^3 - x + y) = 0$ and $4(y^3 + x - y) = 0$

Solving these two equations, we get

$x = \sqrt{2}, -\sqrt{2},$ 0 and corresponding values for y are $-\sqrt{2}, \sqrt{2}, 0.$

Now $l = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4, m = \frac{\partial^2 f}{\partial x \partial y} = 4, n = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$

$$At(\sqrt{2}, -\sqrt{2})(\ln - m^2) = 20.20 - 4^2 > 0$$
$$At(\sqrt{2}, -\sqrt{2}), l = 20 > 0$$

$\therefore$The function has a minimum at $(\sqrt{2}, -\sqrt{2}).$

Also at $(-\sqrt{2}, \sqrt{2}).$, ln - $m^2 > 0$ and l = 20 > 0

$\therefore$The function is minimum at $(-\sqrt{2}, \sqrt{2}).$

At (0,0),$\ln^2$ - $m^2 = 0$ and therefore we cannot say anything. It needs further investigation we can find points in the neighborhood of f(0,0) for which function assumes values greater than f(0,0) and values less than f(0,0).

$\therefore$f(0,0) is not an extreme value.

4. Find the maximum and minimum values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Sol : Let f(x, y) = $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

Than $\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 15x^2 - 15y^2 + 72, \frac{\partial f}{\partial y} = 6xy - 30y = 6y(x-5)$

Now $r = \frac{\partial^2 f}{\partial x^2} = 6x - 30 = 6(x-5)$

$$t = \frac{\partial^2 f}{\partial y^2} = 6x - 30 = 6(x-5) and s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(6xy - 30y) = 6y$$

The critical points of f are given by $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$

i.e,. $3x^2 + 3y^2 - 30x + 72 = 0$ and 6y (x - 5) = 0

i.e., $x^2 + y^2 - 10x + 24 = 0$ and (y = 0 or x = 5)

$\Rightarrow$(y = 0 and $x^2 + y^2 - 10x + 24 = 0$) or (x = 5 and $x^2 + y^2 - 10x + 24 = 0$)

$\Rightarrow$(y = 0, $x^2 - 10x + 24 = 0$) or (x = 5, $25 + y^2 - 50 + 24 = 0$)

$\Rightarrow$(y = 0 and x = 6, 4) or (x=5 and y = ± 1)

$\therefore$The critical points of f are A(4,0),B(6,0), C(5,1) and D(5,-1)

Now, $\delta = rt - s^2 = 36[(x - 5)^2 - y^2]$

At A (4,0),δ =36 $[(4 - 5)^2 - 0] = 36 > 0$

At B(6, 0),δ =36 $[(6 - 5)^2 - 0] = 36 > 0$

At C(5, 1), δ = 36(0 - 1) = -36 < 0

At D(5,-1),δ = 36 (0-1) = -36 < 0

Thus A and B are points of extremum for f , while C and D are saddle points.

But r = 6(x - 5) = 6(4 - 5) = -6 < 0 at A(4,0)

$\Rightarrow$ A is the point of maximum for f

and r = 6(x - 5) = 6(6 - 5) = 6 > 0 at B(6,0)

$\Rightarrow$ B is the point of minimum for f

Minimum value of f = $4^3 + 3(4)(0) - 15(4)^2 - 15(0) + 72(4) = 112$

Maximum value of f = $6^3 + 3(6)(0) - 15(6)^2 - 15(0) + 72(6) = 108$

5. Find the maximum and minimum values of $xy + \frac{a^3}{x} + \frac{a^3}{y}$.

Sol : Given function is f(x, y)= $xy + \frac{a^3}{x} + \frac{a^3}{y}$ ....(1)

$\therefore \frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}, \frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$ and

$\frac{\partial^2 f}{\partial x^2} = \frac{2a^3}{x^3}, \frac{\partial^2 f}{\partial y^2} = \frac{2a^3}{y^3}$ and $\frac{\partial^2 f}{\partial x \partial y} = 1.$

The condition for f(x,y) to have min (or) max. is $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$

$\Rightarrow y = \frac{a^3}{x^2}$ ....(2)

and $x = \frac{a^3}{y^2}$ ....(3)

Substituting (3) in (2), we get

$y = \frac{a^3 y^4}{a^6} = \frac{y^4}{a^3}$

$\Rightarrow y(y^3 - a^3) = 0$

$\Rightarrow y = 0$ or $y = a$

Now $y = 0 \Rightarrow x = \infty$

$\therefore$It is not possible.

Now $y = a \Rightarrow x = a$

$\therefore$The extremum point is (a, a)

f(x, y) will have max. (or) min at (a, a)

At(a,a), $l = \frac{\partial^2 f}{\partial x^2} = 2, m = l, n = 2$

Now $ln - m^2 = 4 – 13 > 0$, $l = 2 > 0$

$\therefore$f(x, y)has minimum at (a, a)

The minimum value is f(a, a) = $a^2 + \frac{a^3}{a} + \frac{a^4}{a} = 3a^2$

6. Find the Maximum and minimum value of $f = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

Sol : We have $f(x) = 3x^4 - 2x^3 - 6x^2 + 6x + 1$

$\therefore \frac{df}{dx} = 12x^3 - 6x^2 - 12x + 6 = 6(2x^3 - x^2 - 2x + 1)$

For maxima or minima, $\frac{df}{dx} = 0$

i.e., $2x^3 - x^2 - 2x + 1 = 0$ i.e., $(x - 1)(2x^2 + x - 1) = 0$

i.e., (x -1) (x + 1) (2x - 1) = 0

$\therefore x = 1, -1, \frac{1}{2}$

These are the possible extreme points.

Now $\frac{d^2 f}{dx^2} = 6(6x^2 - 2x - 2) = 12(x^2 - x - 1)$

When x = 1, $\frac{d^2 f}{dx^2} = -12 < 0$

∴f(x) is minimum at x=1

When x = -1, $\frac{d^2 f}{dx^2} = 12 > 0$

∴f(x) is maximum at x $= \frac{1}{2}$.

Hence Maximum values are given by $f(1)$ and $f\left(\frac{1}{2}\right)$

i.e., f(1)=3 – 2 – 6 + 6 + 1 = 2 and

$$f\left(\frac{1}{2}\right) = \frac{3}{6} - \frac{2}{8} - \frac{6}{4} + \frac{6}{2} + 1 = \frac{1}{16}(3 - 4 - 24 + 48 + 16) = \frac{39}{16}$$

Minimum value of f = f(-1) = 3 + 2 – 6 – 6 + 1 = -6

7. Examine for minimum and maximum values if $\sin x + \sin y + \sin(x + y)$

Sol : $Given\ u(x, y) = \sin x + \sin y + \sin(x + y)$....(1)

$\therefore \frac{\partial u}{\partial x} = \cos x + \cos(x + y)$ ....(2)

and $\frac{\partial u}{\partial y} = \cos y + \cos(x + y)$ ....(3)

Consider $\frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$.

From these we, get cos x = cos y ⇒ x = y

∴From (2), $\frac{\partial u}{\partial x} = cos\ x + cos\ 2x$

$\frac{\partial u}{\partial x} = 0 \Rightarrow cos\ x + cos\ 2x = 0 \Rightarrow 2\ cos\ \frac{3x}{2}\ cos\frac{x}{2} = 0$

$\Rightarrow \frac{3x}{2} = \pm\frac{\pi}{2} or \frac{x}{2} = \pm\frac{\pi}{2} \Rightarrow x = \pm\frac{\pi}{3} or x = \pm\pi$

$x = \pm\frac{\pi}{3}, y = \pm\frac{\pi}{3}\ i.e., \left(\pm\frac{\pi}{3}, \pm\frac{\pi}{3}\right)$

And $x = \pm\pi, y = \pm\pi$ i.e., $(\pm\pi, \pm\pi)$

Now $\frac{\partial^2 u}{\partial x^2} = -\sin x - \sin(x+y); \frac{\partial^2 u}{\partial x \partial y} = -\sin(x+y)$

And $\frac{\partial^2 u}{\partial x^2} = -\sin y - \sin(x+y)$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right), l = -\sqrt{3}, m = \frac{-\sqrt{3}}{2} and n = -\sqrt{3}$

∴ln-m^2= $\frac{9}{4}$ > 0, and l < 0⇒ u has maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$.

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, u=$\frac{3\sqrt{3}}{2}$.

∴ Maximum value of u=$\frac{3\sqrt{3}}{2}$.

We can prove that ln - m^2 is positive and l is positive at $\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$

U= has a minimum at $\left(\frac{-\pi}{3}, \frac{-\pi}{3}\right)$.

Minimum value of u=$\frac{-3\sqrt{3}}{2}$

At (±π, ± π), ln - m^2 = 0. There is a need for further investigation.

8. **Find the positive numbers whose sum is 100 and whose product is maximum**.

Sol :Let x, y, z be the required three numbers

Then x + y + z = k(=100) ....(1)

And f(x, y, z) = xyz .....(2)

Eliminating z form (2) with the help of(1),we get

f(x, y) = xy (k – x - y)

$$\therefore \frac{\partial f}{\partial x} = y[x(-1)+(k-x-y).1] = y(k-2x-y)$$

$$\frac{\partial f}{\partial y} = x[y(-1)+(k-x-y).1] = x(k-x-2y)$$

For f(x, y) to be maximum,

$\frac{\partial f}{\partial x} = 0 and \frac{\partial f}{\partial y} = 0$ gives

⇒2x + y = k and x + 2y = k

Solving these , we get x = y = $\frac{k}{3}$

Now $r = \frac{\partial^2 f}{\partial x^2} = -2y, s = \frac{\partial^2 f}{\partial x \partial y} = x(-1)+(k-x-2y).1 = k-2x-2y$

And $t = \frac{\partial^2 f}{\partial y^2} = -2x$

Now rt - s^2 = 4xy - (k - 2x - 2y)^2

At $x = y = \frac{k}{3}, rt - s^2 = \frac{4k^2}{9} - \left(k - \frac{2k}{3} - \frac{2k}{3}\right)^2 = \frac{4k^2}{9} - \frac{k^2}{9} = \frac{3k^2}{9} = \frac{k^2}{3} > 0$

Also at x = y = $\frac{k}{3}$, r = -2y = $\frac{-2k}{3} < 0$

Hence f(x, y) has a maximum at $\left(\frac{k}{3}, \frac{k}{3}\right)$.

From (1),z = k- (x + y) = k - $\frac{2k}{3} = \frac{k}{3}$

The required numbers are $\frac{k}{3}, \frac{k}{3}, \frac{k}{3}$ i.e., $\frac{100}{3}, \frac{100}{3}, \frac{100}{3}$ ($\therefore$ k=100).Thus the product is maximum when all the three numbers are equal.

9. Find the maximum and minimum values of the function f(x) = x^5 - $3x^4$ + 5.

Sol : Given f(x) = $x^5 - 3x^4 + 5$

$\therefore f'(x) = 5x^4 - 12x^3, f''(x) = 20x^3 - 36x^2 = 4x^2(5x-9)$

For f(x) to be maximum or minimum $f'(x) = 0$

i.e., $x^3(5x-12) = 0 \Rightarrow x = 0,.\frac{12}{5}$

When x=0, $f''(x) = 0$

$\therefore$f(x) has neither maximum nor minimum at x = 0

When $x = \frac{12}{5}, f''(x) = \frac{4(144)}{25}(12-9) > 0$

$\therefore$f(x) has minimum at$x = \frac{12}{5}$

Minimum value = $f\left(\frac{12}{5}\right) = \left(\frac{12}{5}\right)^5 - 3\left(\frac{12}{5}\right)^4 + 5$

$=(2.4)^5 - 3(2.4)^4 + 5 = -14.91$

10. A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction.

Sol : Let x ft, yft and z ft be the dimensions of the box and let S be the surface of the box. Then we have

S= $xy + 2yz + 2zx$ (since open at the top) ...(1)

Given that its volume, x. y.z=32 ...(2)

From (2), z = $\frac{32}{xy}$

Substituting the value of Z in (1) , we get

$$S = xy + 2y\left(\frac{32}{xy}\right) + 2\left(\frac{32}{xy}\right)x = xy + \frac{64}{x} + \frac{64}{y}$$

Now $\frac{\partial S}{\partial x} = y - \frac{64}{x^2} = 0 and \frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0$

Solving these , we get x = 4; y = 4.

Also $l = \frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}, m = \frac{\partial^2 S}{\partial x \partial y} = 1; n = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^2}$

At x=4 & y=4, ln-m^2=$\frac{128}{x^3} X \frac{128}{y^3} - 1 = 2X2 - 1 = 3 > 0$ $and\ l = \frac{128}{x^3}$ =2 > 0

Thus, S is minimum when x = 4,y = 4.

From (2) , we get z = 2

∴The dimension of the box for least material for its construction are 4, 4, 2.

11. Find the minimum value of$x^2 + y^2 + z^2$ given that xyz =a^3

Sol: Let f(x, y, z) = $x^2 + y^2 + z^2$ ...(1)

Given xyz =a^3 ....(2)

From (2), $z = \frac{a^3}{xy}$

Substituting in (1), we get

$$f = x^2 + y^2 + \frac{a^6}{x^2 y^2}$$

$$\therefore \frac{\partial f}{\partial x} = 2x - \frac{2a^6}{x^3 y^2}$$

and $\frac{\partial f}{\partial y} = 2x - \frac{2a^6}{x^2 y^3}$

Making $\frac{\partial f}{\partial x} = 0 and \frac{\partial f}{\partial y} = 0$ and solving them, we get x = y = a.

Now $l = \frac{\partial^2 f}{\partial x^2} = 2 + \frac{6a^6}{x^4 y^2}, m = \frac{\partial^2 f}{\partial x \partial y} = \frac{4a^6}{x^3 y^3}, n = \frac{\partial^2 f}{\partial y^2} = 2 + \frac{6a^6}{x^2 y^4}$

At (a, a), ln - m^2 = 64 – 16 = 48 > 0. Also l > 0

We have f is minimum at (a, a) and minimum value = $a^2 + a^2 + \frac{a^6}{a^2 X a^2} = 3a^2$

12. Find the stationary points of u(x, y) = $\sin x \sin y \sin(x+y)$ where $0 < x < \pi, 0 < y < \pi$ **and find the maximum u.**

Sol :u= $\sin x \sin y \sin(x+y)$, 0 < x < π and 0 < y < π

$$\therefore \frac{\partial u}{\partial x} = \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y)$$

$$= \sin y[\cos x \sin(x+y) + \sin x \ \cos x(x+y)] = \sin y . \sin(2x+y)$$

$$\frac{\partial u}{\partial x} = 0 \Rightarrow y = 0 or 2x + y = \pi$$

$$\frac{\partial u}{\partial y} = \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y)$$

$$= \sin x \ [\sin(x+y) \cos y + \cos(x+y) \sin y] = \sin x \ \sin(x+2y)$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow x = 0 (or) x + 2y = \pi$$

$$\therefore x = \pi/3, y = \pi/3$$

Now $l = \frac{\partial^2 u}{\partial x^2} = 2 \sin y \cos(2x+y) = 2.\frac{\sqrt{3}}{2}(-1) = -\sqrt{3}$

$$m = \frac{\partial^2 u}{\partial x \partial y} = \cos y \sin(2x+y) + \sin y \cos(2x+y) = \sin(2x+2y) = -\frac{\sqrt{3}}{2}$$

And $n = \frac{\partial^2 u}{\partial y^2} = 2 \sin x \cos(x-y) = 2.\frac{\sqrt{3}}{2}(-1) = -\sqrt{3}$

Also $\ln - m^2 = 3 - \frac{3}{4} > 0 and l < 0$

∴u will be maximum at $x = \frac{\pi}{3}, y = \frac{\pi}{3}$.

Hence the maximum value of $u = \frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2}.\frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$

Stationary points are $x = \frac{\pi}{3}, y = \frac{\pi}{3}$.

13. Find the points on the surface $z^2 = xy + 1$ that are nearest to the origin.

Sol : Let P(x,y,z) be any point on the surface

$\Phi(x,y,z)=z^2 - xy - 1=0$ ....(1)

Let $OP = P = \sqrt{x^2 + y^2 + z^2}$ .....(2)

We have to find the minimum values of (2) subject to the condition (1).

From (1) and (2), we have

$P^2 = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1.$ ...(3)

Differentiating (3) partially w.r.t 'x' and 'y' we get

$$2p\frac{\partial p}{\partial x} = 2x + y \quad ....(4)$$

and $2p\frac{\partial p}{\partial y} = 2y + x$ ....(5)

The critical points are given by $\frac{\partial p}{\partial x} = 0$ and $\frac{\partial p}{\partial y} = 0$

$\Rightarrow 2x + y = 0$ and $2y + x = 0 \Rightarrow x=0, y=0$

(1) $\Rightarrow z = \pm\sqrt{xy+1} = \pm 1$ ($\therefore$ x=0,y=0)

$\therefore$P(0,0,1) and Q (0,0,-1) are the critical points of p.

Differentiating (4) partially w.r.t 'x' and 'y' ,we get

$$2pr + 2\left(\frac{\partial p}{\partial x}\right)^2 = 2 \Rightarrow r = \frac{2}{2p} = 1 at P and Q \left(\therefore\ p = 1 and \frac{\partial p}{\partial x} = 0 at p and Q\right)$$

And $2ps + 2\frac{\partial p}{\partial x}.\frac{\partial p}{\partial y} = 1 \Rightarrow s = \frac{1}{2p} = \frac{1}{2}$ at P and Q ($\therefore$p=1, $\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$ at p and Q)

Diff (5) partially w.r.t 'y', we get

$$2pt + 2\left(\frac{\partial p}{\partial y}\right)^2 = 2 \Rightarrow t = \frac{2}{2p} = 1 at P and Q$$

∴At P andQ rt-s^2=1-$\frac{1}{4} = \frac{3}{4} > 0$

Hence P has minimum at P andQ.

∴Required points are (0,0,1) and (0,0,-1).

14. Find the shortest distance form origin to the surface xyz^2=2.

Sol : Let P(x, y, z) be any point on the surface xyz^2=2....(1)

Then OP=d=$\sqrt{x^2+y^2+z^2}$

Let f(x, y)= $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + \frac{2}{xy}$ [using (1)]

Now $f_x = 2x - \frac{2}{x^2 y}$; $f_y = 2y - \frac{2}{xy^2}$

Solving f_x = 0 and f_y = 0 weget

$\frac{x^3 y - 1}{x^2 y} = 0$ and $\frac{xy^3 - 1}{xy^2} = 0$

i.e., $x^3y = 1$ and $xy^3 = 1$

or $x^3 y = xy^3 \Rightarrow xy(x^2 - y^2) = 0 \Rightarrow x = \pm y (\because x \neq 0, y \neq 0)$

The stationary points are P_1(1,1) and P_2(1,-1).

Also $f_{xx} = 2 + \frac{4}{x^3 y}$, $f_{yy} = 2 + \frac{4}{xy^3}$ and $f_{xy} = \frac{2}{x^2 y^2}$

At (1,1), $l = f_{xx} = 6 > 0, n = f_{yy} = 6, m = f_{xy} = 2 and\ ln - m^2 = f_{xx}.f_{yy} - f_{xy}^2 = (6)(6) - (2)^2 = 32 > 0$

Now $z^2 = \frac{2}{1} = 2$ [using(1)]

∴z = ± $\sqrt{2}$. So minimum occurs at (1, 1,$\sqrt{2}$)

Hence the shortest distance from the origin is $\sqrt{1+1+2}$ or 2.

15. Find the minimum value of $x^2 + y^2 + z^2$ when ax + by + cz = p.

Sol: We have $ax+by+cz=p \Rightarrow z=\frac{1}{c}(p-ax-by)$

Let f(x, y)= $x^2 + y^2 + z^2$

$= x^2+y^2+\frac{1}{c^2}(p-ax-by)^2$ ....(1)

$\therefore \frac{\partial f}{\partial x}=2x-\frac{2a}{c^2}(p-ax-by)$

$\frac{\partial f}{\partial y}=2y-\frac{2b}{c^2}(p-ax-by)$

Now $r=\frac{\partial^2 f}{\partial x^2}=2+\frac{2a^2}{c^2}, s=\frac{\partial^2 f}{\partial x\partial y}=\frac{2ab}{c^2}$ and $t=\frac{\partial^2 f}{\partial y^2}=2+\frac{2b^2}{c^2}$

And $rt-s^2=\left(2+\frac{2a^2}{c^2}\right)\left(2+\frac{2b^2}{c^2}\right)-\left(\frac{2ab}{c^2}\right)^2$

$=4\left(1+\frac{a^2}{c^2}\right)\left(1+\frac{b^2}{c^2}\right)-\frac{4a^2b^2}{c^4}=4\left(1+\frac{a^2}{c^2}+\frac{b^2}{c^2}\right)>0$

Also $r=2\left(1+\frac{a^2}{c^2}\right)>0$

$\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$ implies

$x=\frac{a}{c^2}(p-ax-by)$ ....(2)

and $y=\frac{b}{c^2}(p-ax-by)$ ....(3)

(2) ÷ (3) gives , $\frac{x}{y}=\frac{a}{b}\Rightarrow y=\frac{bx}{a}$

Putting in (1), x= $\frac{a}{c^2}\left(p - ax - \frac{b^2 x}{a}\right)$

$$\Rightarrow x\left(1 + \frac{a^2 + b^2}{c^2}\right) = \frac{ap}{c^2} or x = \frac{ap}{a^2 + b^2 + c^2}$$

$$\therefore y = \frac{bx}{a} = \frac{bp}{a^2 + b^2 + c^2}$$

Substituting these values in ax + by + cz = p, we get

$$\frac{a^2 p}{a^2 + b^2 + c^2} + \frac{b^2 p}{a^2 + b^2 + c^2} + cz = p$$

i.e., $\frac{p}{a^2 + b^2 + c^2}(a^2 + b^2) + cz = p$

i.e., $cz = p - \frac{p(a^2 + b^2)}{a^2 + b^2 + c^2} = p\left(1 - \frac{a^2 + b^2}{a^2 + b^2 + c^2}\right)$

$$= \frac{pc^2}{a^2 + b^2 + c^2}$$

Or $z = \frac{cp}{a^2 + b^2 + c^2}$

Hence f is minimum at $\left(\frac{ap}{a^2 + b^2 + c^2}, \frac{bp}{a^2 + b^2 + c^2}, \frac{cp}{a^2 + b^2 + c^2}\right)$

∴Minimum value of $f = \frac{a^2 p^2 + b^2 p^2 + c^2 p^2}{(a^2 + b^2 + c^2)^2} = \frac{p^2}{a^2 + b^2 + c^2}$

16. Find the rectangular parallelepiped of maximum volume that can be inscribed in a sphere.

Sol : Let ‘a’ (constant) be the radius of the given sphere. Also let x, y, z be the length breadth and height of a rectangular parallelopied inscribed in the given sphere.

The equation of the sphere is $x^2 + y^2 + z^2 = a^2$ ....(1)

Volume of the rectangular parallelopiped is V = xyz

. ...(2)

From (1), z = $\sqrt{a^2 - x^2 - y^2}$ ....(3)

Substituting (3) in (2), we get

V= $xy\sqrt{a^2 - x^2 - y^2}$

$\therefore V^2 = x^2y^2(a^2 - x^2 - y^2) = x^2y^2a^2 - x^4y^2 - x^2y^4$ ....(4)

Let $f(x,y) = V^2 = x^2y^2a^2 - x^4y^2 - x^2y^4$

$$\therefore \frac{\partial f}{\partial x} = 2xy^2a^2 - 4x^3y^2 - 2xy^4 = 2xy^2(a^2 - 2x^2 - y^2)$$

$$\frac{\partial f}{\partial y} = x^2(2y)a^2 - 2x^4y - 4x^2y^3 = 2x^2y(a^2 - x^2 - 2y^2)$$

For V to be maximum,

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2xy^2(a^2 - 2x^2 - y^2) = 0$$

$\Rightarrow a^2 - 2x^2 - y^2 = 0 (\because x \neq 0, y \neq 0)$ ....(5)

$\frac{\partial f}{\partial y} = 0 \Rightarrow a^2 - x^2 - 2y^2 = 0$ ....(6)

(6) - (5) gives

$x^2 - y^2 = 0 \Rightarrow x = y$

$\therefore$From (5), x = y = $\frac{a}{\sqrt{3}}$

From (3), we get

$$z = \sqrt{a^2 + \frac{a^2}{3} - \frac{a^2}{3}} = \frac{a}{\sqrt{3}}$$

$$\therefore x = y = z = \frac{a}{\sqrt{3}}$$

$\therefore$The critical point is $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$.

Now $r = \frac{\partial^2 f}{\partial x^2} = 2a^2y^2 - 12x^2 - 2y^4; S = \frac{\partial^2 f}{\partial x \partial y} = 4a^2xy - 8x^3y - 8xy^3$

$t = \frac{\partial^2 f}{ay^2} = 2a^2x^2 - 2x^4 - 12x^2y^2$

At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right), r = 2a^2\left(\frac{a^2}{3}\right) - 12.\frac{a^2}{3}.\frac{a^2}{3} - 2.\frac{a^4}{9}$

$= \frac{-8a^4}{9} < 0 (\therefore a > 0)$

At $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right), s = 4a^2\left(\frac{a^2}{3}\right) - 8.\frac{a^3}{3\sqrt{3}}.\frac{a}{\sqrt{3}}. - 8.\frac{a}{\sqrt{3}}.\frac{a^3}{3\sqrt{3}}$

$= \frac{4a^4}{3} - \frac{8a^4}{9} - \frac{8a^4}{9} = \frac{-4a^4}{9}$

And at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right), t = 2a^2\left(\frac{a^2}{3}\right) - 2.\frac{a^4}{9} - 12.\frac{a^2}{3}.\frac{a^2}{3} = \frac{-8a^4}{9}$

Now $rt - s^2 = \frac{64a^8}{81} - \frac{16a^8}{81} = \frac{48a^8}{81} = \frac{16a^8}{27} > 0$

Since r < 0 and $(r t - s^2) > 0$,

$\therefore$f(x, y) i.e., V^2 and hence V is maximum at $\left(\frac{a}{\sqrt{3}}, \frac{a}{\sqrt{3}}\right)$

i.e., Volume is maximum for x = y = a = $\left(\frac{a}{\sqrt{3}}\right)$

$\therefore$Inscribed rectangular parallelopiped is a cube.

Maximum volume of the rectangular parallellopiped.

$= xyz = \left(\frac{a}{\sqrt{3}}\right) = \frac{a^3}{3\sqrt{3}}$ cu.units.

17. Divide 24 into three parts such that the continued product of the first square of the second and cube of the third is maximum.

Sol: Let 24 is divided into three parts x, y, z.

Then x + y + z = 24 .....(1)

Take $f(x,y,z) = x^3y^2$ such that x + y + z = 24 .....(2)

f(x, y) = $x^3y^2(24 - x - y)$ from (1)

$\Rightarrow f(x,y) = 24x^3y^2 - x^4y^2 - x^3y^3$

$$p = \frac{\partial f}{\partial x} = 72x^2y^2 - 4x^3y^2 - 3x^2y^3$$

$$r = \frac{\partial^2 f}{\partial x^2} = 144xy^2 - 12x^2y^2 - 6xy^3$$

$$q = \frac{\partial f}{\partial y} = 48x^3y - 2x^4y - 3x^3y^2$$

$$t = \frac{\partial^2 f}{\partial y^2} = 48x^3 - 2x^4 - 6x^3y$$

And $s = \dfrac{\partial^2 f}{\partial x \partial y} = 144x^2 - 8x^3 - 9x^2y^2$

For f(x,y) to be maximum , we have

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 72x^2y^2 - 4x^3y^2 - 3x^2y^3 = 0 \Rightarrow x^2y^2(72 - 4x - 3y) = 0$$

$\Rightarrow$72 - 4x - 3y = 0, x = 0,y = 0 ....(3)

And $\dfrac{\partial f}{\partial y} = 0 \Rightarrow 48x^3y - 2x^4y - 3x^3y^2 = 0$

$\Rightarrow x^3y(48 - 2x - 3y) = 0$

$\Rightarrow 48 - 2x - 3y = 0, x = 0, y = 0$ .....(4)

Solving (3) and (4), we get x=12 and y=8.

At(12, 8), we have

$r = 144{,}12(8)^2 - 12(12)^2(8)^2 - 6(12)(8)^3$

$=12(8)^2(144 - 144 - 48) = 12(8)^2(-48) < 0$

$t = 48(12)^3 - 2(12)^4 - 6(12)^3.8$

$=(12)^3(48-24-48) = 12^3(-24)$

$S = 144(12)^2.8 - 8\ (12)^2.8 - 9\ (12)^2(8)^2$

$=(12)^2.8(144 - 96-72)=(12)^2.8(-24)$

$\therefore rt - s^2 = [12.(8)^2(-48).(12)^3(-24)]-[(12)^2.8.(24)]^2$

$= (12)^4.8^2.24(48-24)$

$= (12)^4.8^2.24^2 > 0$

Since $rt - s^2 > 0$ and $r < 0$, f(x,y) is maximum at (12,8)

Putting x = 12, and y = 8 in(1), we get z = 4.

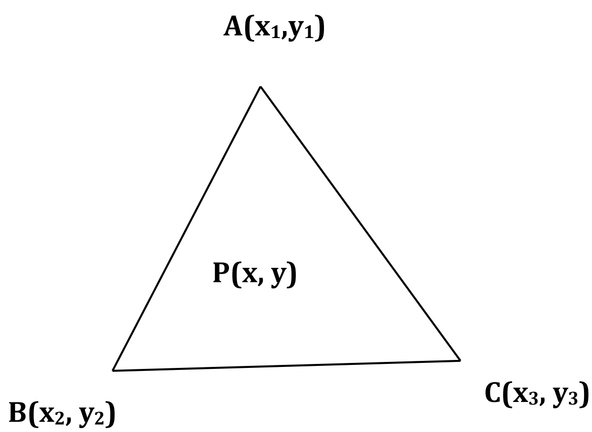
The values of x, y ,z are 12, 8, 4 respectively.

This is the division of 24 for maximum of f(x, y, z).

18. Find a point within a triangle such that the sum of the squares of its distances from the three vertices is a minimum.

Sol : Let $A(x_1, y_1)$,$B(x_2, y_2)$and $C(x_3, y_3)$ be the vertices of ΔABC

Also let P(x, y) be a point in the ΔABC.



Then we have $f(x, y) = AP^2 + BP^2 + CP^2$

$$= \sum_{i=1}^{3}[x - x_i)^2 + (y - y_i)^2]$$

Now $\frac{\partial f}{\partial x} = 2\sum_{i=1}^{3}(x - x_i) and \frac{\partial f}{\partial y} = 2\sum_{i=1}^{3}(y - y_1)$

For f(x, y) to be minimum , $\frac{\partial f}{\partial y} = 0 and \frac{\partial f}{\partial y} = 0$

$$\Rightarrow \sum_{i=1}^{3}(x - x_1) = 0 and \sum_{i=1}^{3}(y - y_1) = 0$$

$$\Rightarrow (x - x_1) + (x + x_2) + (x - x_3) = 0 and (y - y_1) + (y - y_2) + (y - y_3) = 0$$

$\Rightarrow 3x-(x_1+x_2+x_3)=0$ and $3y-(y_1+y_2+y_3)=0$

$\Rightarrow x=\dfrac{x_1+x_2+x_3}{3}$ and $y=\dfrac{y_1+y_2+y_3}{3}$

Now $l=\dfrac{\partial^2 f}{\partial x^2}=2(3)=6, m=\dfrac{\partial^2 f}{\partial x\partial y}=0$ and $n=\dfrac{\partial^2 f}{\partial y^2}=6$

At P(x, y), $ln - m^2 = 36$ = +ve and l is also +ve

Hence f is minimum for $x=\dfrac{x_1+x_2+x_3}{3}$ and $y=\dfrac{y_1+y_2+y_3}{3}$

$\therefore$The required points is $\left(\dfrac{x_1+x_2+x_3}{3},\dfrac{y_1+y_2+y_3}{3}\right)$ which is the centroid of Δ ABC.

3.8 EXERCISE

1. Investigate the maximum and minimum for $x^3y^2(1 - x - y)$.
2. Discuss the maximum and minimum of $x^2y + xy^2 - axy$.
3. Examine the extrema of $f(x,y)=x^2+xy+y^2+\dfrac{1}{x}+\dfrac{1}{y}$.
4. Find the maxima and minima of the function $f(x)=2(x^2-y^2)-x^4+y^4$.
5. Find the maximum and minimum values of $ax^3y^2 - x^4y^2 - x^3y^3$.
6. Show that the function $f(x, y) = x^3 + y^3 - 63(x + y) + 12xy$ is maximum at (-7,-7) and minimum at (3,3).
7. Find the extreme values of $u = x^2y^2 - 5x^2 - 8xy - 5y^2$.
8. Discuss the maximum and minimum of $x^2 + y^2 + 6x + 12$.
9. A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction.
10. Find the dimensions of the rectangular box open at the top of maximum capacity whose surface area 108 sq.inches.

ANSWERS

1. $Max\left(\dfrac{1}{2},\dfrac{1}{3}\right)=\dfrac{1}{432}$. 2. Min at $\left(\dfrac{a}{3},\dfrac{a}{3}\right)$ 3. Min at $\left(\left(\dfrac{1}{3}\right)^{\frac{1}{3}},\left(\dfrac{1}{3}\right)^{\frac{1}{3}}\right)$ 4. Max at $(-1,0),(1,0)$ and Min at $(0,-1),(0,1)$ 5. Max at $\left(\dfrac{a}{2},\dfrac{a}{3}\right)$ 7. Max at (0, 0) 8. Min at (-3, 0) = 3

9. $x=y=z=(2v)^{\frac{1}{3}}$ 10. x = 6, y = 6, z = 3.

3.9 LAGRANGE'S METHOD OF UNDETERMINEDMULTIPLIERS

Sometimes it is necessary to find the stationary values of a function of several variables which are not all independent but are connected by some given conditions. Such type of problems can be solved by using the method of Lagrange's undetermined mutlipers.

Working Rule:

Suppose it is required to find the extremum for the function f(x, y, z) subject to the condition ϕ(x, y, z) =0 ….(1)

Step 1. : Form Lagrangean function F(x, y, z) = f(x, y, z) + λϕ(x, y, z)….(2)

whereλ is called the Lagrange multiplier, which is determined by the following conditions.

Step 2: Obtain the equations

$$\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \quad \text{…(3)}$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \quad \text{….(4)}$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0 \quad \text{…(5)}$$

Step 3: Solving the equations (1), (3), (4) & (5).

The values of x, y, z so obtained will give the stationary point of f(x, y, z).

Examples:

1. **Find the minimum value of $x^2 + y^2 + z^2$, given that xyz = a^3.**

Sol : Let $u=x^2+y^2+z^2$ ....(1)

and $\phi \equiv xyz - a^3 = 0$ ...(2)

Consider the Lagrangean function

F(x, y, z) = u(x, y, z) + λ. ϕ(x, y, z)

i.e., $F(x, y, z) = (x^2 + y^2 + z^2) + \lambda (xyz - a^3)$ .....(3)

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 2x + \lambda yz = 0$ ....(4)

$$\frac{\partial F}{\partial y} = 0 \Rightarrow \frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 2y + \lambda xz = 0 \qquad ....(5)$$

and $\frac{\partial F}{\partial z} = 0 \Rightarrow \frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 2z + \lambda yx = 0$ ....(6)

From (4),(5) and(6) , we have $\frac{x}{yz} = \frac{y}{zx} = \frac{z}{xy} = -\frac{\lambda}{2}$ ....(7)

From the first two members , we have $\frac{x}{yz} = \frac{y}{zx} \Rightarrow x^2 = y^2$ . ...(8)

From the last two members , we have $\frac{y}{zx} = \frac{z}{xy} \Rightarrow y^2 = z^2$ ....(9)

From (8) and (9), we have $x^2 = y^2 = z^2 \Rightarrow x = y = z$ ....(10)

Solving (2) and (10), we get x = y = z = a

$\therefore$Minimum value of $u = a^2 + a^2 + a^2 = 3a^2$

2. **Find the minimum value of $f = x^2 + y^2 + z^2$ given $x + y + z = 3a$.**

Sol : Let $f = x^2 + y^2 + z^2$ ....(1)

and $\phi = x + y + z - 3a$ .....(2)

$$\therefore \frac{\partial f}{\partial x} = 2x, \frac{\partial \phi}{\partial x} = 1, \frac{\partial f}{\partial y} = 1 and \frac{\partial f}{\partial z} = 2z, \frac{\partial \phi}{\partial z} = 1$$

By Lagrange's method of multiplier , we have

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \varphi}{\partial x} = 0 \Rightarrow 2x + \lambda = 0 (or) x = -\frac{\lambda}{2}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \varphi}{\partial y} = 0 \Rightarrow 2y + \lambda = 0 (or) y = -\frac{\lambda}{2}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \varphi}{\partial z} = 0 \Rightarrow 2z + \lambda = 0 (or) z = -\frac{\lambda}{2}$$

Substituting these values of x, y, z in $\phi(x, y, z) = 0$, we get

$-\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} = 3a$ or $-\frac{3\lambda}{2} = 3a$ (or) $\lambda = -2a$

$\therefore x = a, y = a, z = a$

The possible extreme points is (a, a, a)

Thus , the minimum value of $f = a^2 + a^2 + a^2 = 3a^2$

3.Find the maximum and minimum values of x + y + z subject to constraint $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

Sol : This is a constrained extremum problem where the function f(x, y, z) = x + y + z subjected to the constraint $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$

So, consider the auxiliary function F(x, y, z) = x + y + z + $\lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$ ....(1)

Differentiating(1) partially w.r.t x, y, z and equating to zero , we get

$$\frac{\partial F}{\partial x}=1-\frac{\lambda}{x^2}=0 \quad ....(2)$$

$$\frac{\partial F}{\partial y}=1-\frac{\lambda}{y^2}=0. \quad ...(3)$$

$$\frac{\partial F}{\partial z}=1-\frac{\lambda}{z^2}=0 \quad ....(4)$$

Solving (2),(3) and (4) for x, y, z we get $x=\pm\sqrt{\lambda}, y=\pm\sqrt{\lambda}, z=\pm\sqrt{\lambda}$

Substituting these values of x, y, z in the given constraint , we have

$$\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\lambda}}=1 \Rightarrow 3=\sqrt{\lambda} \text{ or } \lambda=9$$

Using this λ we get x = ±3,y = ±3, z = ±3

Thus the maximum and minimum values are 9 and -9.

4. **Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

Sol :Let 2x, 2y, 2z be the length ,breadth and height of the rectangular parallelopiped that can be inscribed in the ellipsoid. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ …(1)

Then the centroid of the paralleopiped coincides with the centre O(0,0,0) of the ellipsoid and the corners of the parallelopiped lie on the surface of the ellipsoid(1).

If (x, y, z) is any corner of the parallelopied then it satisfies condition(1).

Let V be the volume of the parallelopiped i.e., V = 8xyz. We have to find the maximum value of V subject to the condition(1).

Consider the Lagrangean function.

$$F(x, y, z) = V + \lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right) \quad ....(2)$$

Where λis the multiplier to be determined such that

$$\frac{\partial F}{\partial x}=0,\frac{\partial F}{\partial y}=0,\frac{\partial F}{\partial z}=0$$

$$\Rightarrow 8yz+\frac{2x}{a^2}\lambda=0, \quad .....(3)$$

$$\Rightarrow 8yz+\frac{2x}{b^2}\lambda=0, \quad ....(4)$$

and $8yz+\frac{2x}{c^2}\lambda=0,$ ....(5)

Now (3),(4),(5) are combined as:

$$\frac{a^2yz}{x}=\frac{b^2zx}{y}=\frac{c^2xy}{z}=-\frac{\lambda}{4} \quad ...(6)$$

From first two fractions , we have

$$\frac{a^2y}{x}=\frac{b^2x}{y}\Rightarrow\frac{x^2}{a^2}=\frac{y^2}{b^2} \quad ....(7)$$

Similarly $\frac{y^2}{b^2}=\frac{z^2}{c^2}$ ....(8)

Substituting (7) and (8) in (1), we get

$$x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$$

Hence the possible extreme points is $P\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.

First we note that for fixed x and y, V=8xyz is an increasing functions of z.

If z = 0 , the parallelopiped reduces into a two dimensional lamina for which V=0.

As z increases , V also increases.

But for $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}$, the maximum value of z satisfying the condition (1) is z=$\frac{c}{\sqrt{3}}$.

Thus V is maximum at $P\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$.and its maximum value is V=8xyz=$\frac{8abc}{3\sqrt{3}}$.c.u.

5. Find a point on the plane 3x + 2y + z – 12 = 0 , which is nearest to the origin.

Sol : Let P(x, y, z) be a point on the given plane .then

$OP = \sqrt{x^2 + y^2 + z^2}$, where O is the origin.

Let u = $x^2 + y^2 + z^2$ ....(1)

Now we have to minimize (1) subject to the condition

$\Phi(x, y, z) = 3x + 2y + z - 12 = 0$ ...(2)

Consider the Lagrangean function

$F(x, y, z) = u(x, y, z) + \lambda\, \phi(x, y, z)$

i.e., $F(x, y, z) = x^2 + y^2 + z^2 + \lambda\,(3x + 2y + z - 12)$

For F to be minima,

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$$

$$\therefore \frac{\partial F}{\partial x} = 0 \Rightarrow 2x + 3\lambda = 0 \Rightarrow x = \frac{-3}{2}\lambda \quad ....(3)$$

$$\therefore \frac{\partial F}{\partial y} = 0 \Rightarrow 2y + 2\lambda = 0 \Rightarrow y = -\lambda \qquad ....(4)$$

$$\text{and} \therefore \frac{\partial F}{\partial z} = 0 \Rightarrow 2z + \lambda = 0 \Rightarrow z = \frac{-\lambda}{2} \qquad .....(5)$$

Substituting (3),(4) and (5)in(2), we get

$$3\left(\frac{-3}{2}\lambda\right) + 2(-\lambda) + \left(\frac{-\lambda}{2}\right) - 12 = 0 \Rightarrow \lambda = \frac{-12}{7}$$

Putting this value of λ in (3),(4) and(5), we get

$$x = \frac{18}{7}, y = \frac{12}{7}, z = \frac{6}{7}$$

$\therefore$ Hence $\left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$ is the point on the given plane which is nearest to the origin.

Note: Minimum value of $OP = \sqrt{\left(\frac{18}{7}\right)^2 + \left(\frac{12}{7}\right)^2 + \left(\frac{6}{7}\right)^2} = \sqrt{\frac{504}{7x7}} = \sqrt{\frac{72}{7}}$

6. Find the maximum value of $u = x^2y^3z^4$ if 2x + 3y + 4z = a.

Sol : Given $u = x^2y^3z^4$ .....(1)

Let $\phi(x, y, z) = 2x + 3y + 4z - a = 0$ ...(2)

Consider the Lagrangean function $F(x, y, z) = u(x, y, z) + \lambda\, \phi(x, y, z)$

i.e., $F(x, y, z) = x^2y^3z^4 + \lambda\,(2x + 3y + 4z - a)$ .....(3)

For maxima or minima $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$

$$\text{Now } \frac{\partial F}{\partial x} = 0 \Rightarrow 2xy^3z^4 + 2\lambda = 0 \Rightarrow xy^3z^4 = -\lambda \qquad ....(4)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 3x^2y^2z^4 + 3\lambda = 0 \Rightarrow x^2y^2z^4 = -\lambda \qquad ....(5)$$

$$\text{and } \frac{\partial F}{\partial y} = 0 \Rightarrow 4x^2y^3z^3 + 4\lambda = 0 \Rightarrow x^2y^3z^3 = -\lambda \qquad ....(6)$$

From (4) and (5), we have x = y .....(7)

From (5) and (6), we have y = z ....(8)

Hence combining (7) and (8) , we get x = y = z .....(9)

Solving (2) and (9) , we get $x = y = z = \frac{a}{9}$

$\therefore$Maximum value of $u = \left(\frac{a}{9}\right)^2\left(\frac{a}{9}\right)^3\left(\frac{a}{9}\right)^4 = \left(\frac{a}{9}\right)^9$

7. **Find the point on the plane x + 2y + 3z = 4 that is closest to the origin**.

Sol :Let P(x, y, z) be a point on the given plane .

Then OP$\sqrt{x^2 + y^2 + z^2}$

Let u=$x^2 + y^2 + z^2$ ....(1)

Now we have to minimize (1) subject to the condition

$\Phi(x, y, z) = x + 2y + 3z - 4 = 0$ .....(2)

Consider the Lagrangian function F(x,y,z)=u(x,y,z)+λ .ϕ(x,y,z)

i.e., F(x,y,z)=$x^2 + y^2 + z^2 + \lambda(x + 2y + 3z - 4)$

From F to be minima, $\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda = 0$ $\quad \therefore x = -\frac{\lambda}{2}$ ......(3)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + 2\lambda = 0$ $\quad \therefore y = -\lambda$ .....(4)

$\frac{\partial F}{\partial z} = 0 \Rightarrow 2z + 3\lambda = 0$ $\quad \therefore z = -\frac{3\lambda}{2}$ .....(5)

Substituting (3),(4) and (5) in(2) we get $-\frac{\lambda}{2} - 2\lambda - \frac{9\lambda}{2} - 4 = 0 \Rightarrow 14\lambda + 8 = 0$

$\therefore \lambda = -\frac{4}{7}$

Putting this value of λ in (3),(4) and (5) , we get $x = \frac{2}{7}, y = \frac{4}{7} and z = \frac{6}{7}$.

Hence $\left(\frac{2}{7},\frac{4}{7},\frac{6}{7}\right)$ is the point on the given plane which is nearest to the origin.

EXERCISE

1. Find the minimum of $u = x^2 + y^2 + z^2$ when $x + y + z = a$.
2. Find the minimum of $x^2 + y^2 + z^2$ when $ax + by + cz = p$.
3. Find the maximum and minimum of $x^2 + y^2 + z^2$ subject to $lx + my + nz = 0$.
4. Find the shortest and the longest distances from the point (1, 2, -1) to the sphere $x^2 + y^2 + z^2 = 24$.
5. Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $4x^2 + 4y^2 + 9z^2 = 36$.
6. Divide 24 into three points such that the continued product of the first, square of the second and cube of the third is maximum.
7. Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest and farthest from the point (3, 1, -1).
8. Given that $x + y + z = a$, find the maximum vlaue of $x^m y^n z^p$.
9. Find the minimum value of $u = x^4 + y^4 + z^4$ subject to $xyz = a^3$.
10. In a plane triangle ABC find the maximum value of cos A cos B cos C.

ANSWERS

1. $\frac{p^2}{a^2 + b^2 + c^2}$
2. $\left(\frac{5}{7},\frac{15}{14},\frac{-5}{14}\right)$
3. $\sqrt{6};3\sqrt{6}$
4. $16\sqrt{3}$ c.u.
5. 4, 8, 12
6. $\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right)$, $\left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right)$
7. Max is $\frac{a^{m+n+p}.m^m.n^n.p^p}{(m+n+p)^{m+n+p}}$.
8. $3a^4$
9. $f\left(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3}\right) = \frac{1}{2}.\frac{1}{2}.\frac{1}{2} = \frac{1}{8}$.