అమ్మ Unit - 1 నాన్న

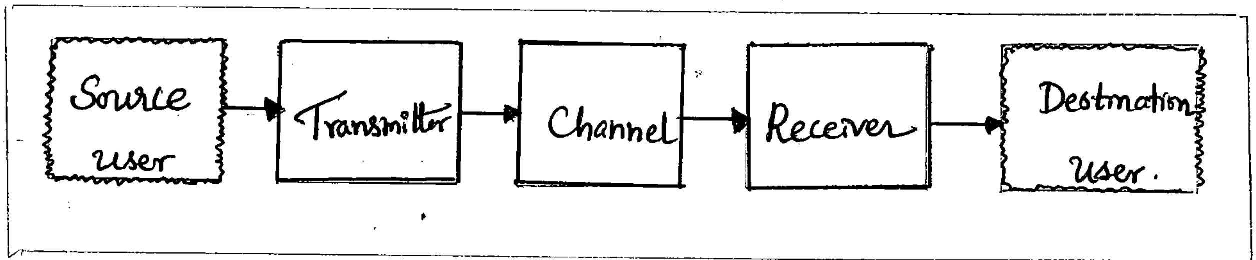
Base Band Data Transmission - I

Syllabus: Overview of Analog modulation, General block diagram of digital communication system, Analog to Digital conversion (ADC): Samplers, Quantizers: Uniform, non-uniform, differential, types of encoders, Pulse code modulation (PCM), Noise in PCM, Differential PCM (DPCM), Noise in DPCM, Delta modulation (DM), noise in DM, Adaptive DM (ADM), continuously variable slope DM (CVSDM), comparision between PCM, DPCM, DM and ADM.

Introduction:

Communication: Exchanging the information from source to destination (or) one user to another user.

A general communication system as.



Communication System.

* Source user : Speech, video etc.
* Transmitter : Conveys information
* Channel : Invariably distorts signals
* Receiver : Extracts information signal
* Destination/user : Utilizes information.

✓ The purpose of a communication system is to transport an information bearing signal from a source to a user destination via a communication channel.

* Basically a communication system is of analog & digital type.

✓ In <u>analog communication</u> system, the information bearing signal is continuously varying in both amplitude and time and it is used directly to modify some characteristics of a sinusoidal carrier wave such as Amplitude (AM), frequency (FM) & phase (PM).

✓ In a <u>digital communication system</u>, the information bearing signal is processed so that it can be represented by a sequence of discrete messages for <u>higher rate</u> and as <u>accurate</u> as possible.

General block diagram of Digital Communication System :

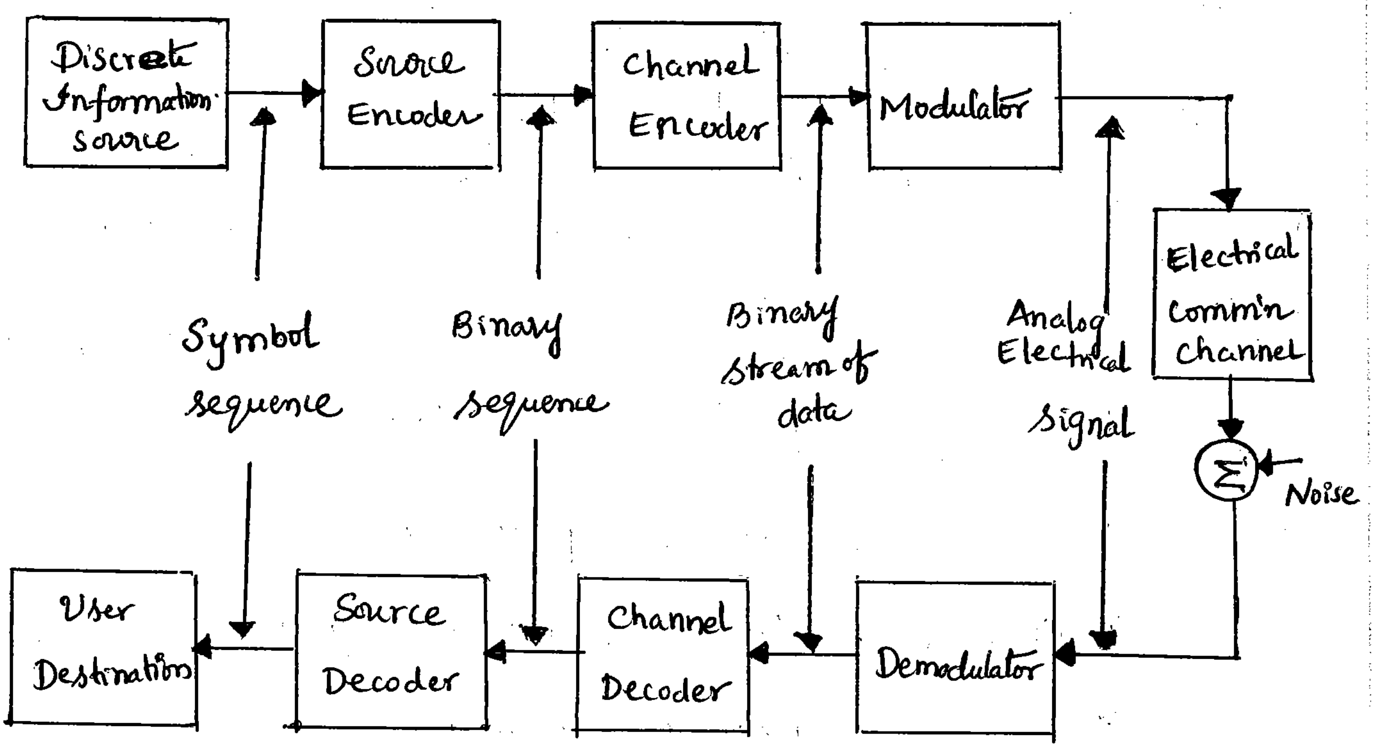


Fig: <u>Blockdiagram Digital Communication System.</u>

The basic building blocks of digital communication systems are

1. Information source
2. Source encoder and decoder.
3. Communication channel
4. Channel encoder and decoder
5. Modulator and Demodulator
6. User destination.

① Information Source :

~~Usually an information source may be analog or discrete.~~

✓ If the source is analog, then the analog information can be tran-formed into discrete information through the process of sampling and quantizing.

✓ If the source is discrete, it emits the discrete information symbols.

Ex: Analog information

A microphone activated by speech or a TV camera scanning a scene, emit one or more continuous amplitude signals.

Ex: Discrete information.

Teletype & the numerical output of a computer consisting of sequence of discrete symbols & letters.

✓ Discrete information sources are characterized by the following parameters.

a) Source alphabet (Symbols & letters)
b) Symbol rate
c) Source alphabet probabilities.
d) Probabilistic dependance of symbols in a Sequence.
e) Entropy (bits/sym) & Source information rate.

② Source encoder and decoder:

✓ The function of source encoder is to convert the symbol fed at the input into a binary sequence such as 0's & 1's, by assigning codewords to the symbols in the input sequence.

✓ The input to the source encoder (coder) is a string of symbols accuring at a rate of samples/sec.

✓ The output of source encoder is binary sequence.

* The function of source decoder converts the binary sequence of the channel decoder into a symbol sequence.

* The decoder for a system using fixed length codeword is simple but, the decoder for a system using variable length codewords will be very complex.

* Decoders for such systems must be able to scope with a no. of problems such as growing memory requirements and loss of synchronization due to bit errors.

③ Communication Channel:

✓ A communication channel provides the electrical connection between source and destination

✓ The channel may be a pair of wires & a teletype (telephone) link & free space etc. over which the information signal is radiated.

* Due to the physical limitations of communication channel is have only finite bandwidth and the information bearing signal often suffers amplitude and phase distortion, the signal power also decreases due to the attenuation of the channel.

* Further more the signal is corrupted by unwanted, unpredictable electrical signals referred to as noise.

- One of the way in which the effect of noise can be minimized is to increase the signal power.
- However signal power cannot be increased beyond certain levels because of non linear effect that become dominant as the signal amplitude is increased.

For the reason the SNR or (S/N) at the output of channel can be maintained to a proper value which is one of the important parameter.

- If the parameters of communication channel are known the Channel Capacity (C) which represents the max. rate at which nearly errorless data transmission is critically possible & given by

channel Capacity $$\boxed{C = B \cdot \log\left(1 + \frac{S}{N}\right) \text{ Bits/Sec}}$$

where B - Bandwidth
S/N - Signal to Noise ratio.

Ⓐ Channel encoder and decoder:

- Digital channel coding is practical method of realizing high transmission reliability & efficiency that otherwise may be achieved only by the use of signals of longer durations in the modulation, demodulation process.
- Error control is accomplished by the channel coding operation by adding extra bits to the output at the source coder, while these extra bits themselves convey no information make it possible for the receiver to detect and/or correct some of the errors in the information bearing bits.

There are two methods of channel coding

a) Block Coding method.

b) Convolutional Coding method.

* In block coding method the encoder takes a block of 'K' information bits from the source encoder and adds 'r' error & control bits.

* In convolution method information bearing message stream is encoded in a continuous fashion by continuously interleaving information bits and error control bits.

✓ The important parameter of channel encoder are
 a) Method of coding
 b) Rate or efficiency of the coder.
 c) Error control capabilities..
 d) Complexity of the encoder.

✓ The channel decoder recovers the information bearing bits from the coded binary stream error detection of possible corrections are also performed by the channel decoder.

✓ The complexity of the decoder and the time delay involved in the decoders are important design parameters.

⑤ Modulator and Demodulator:

✓ The modulator accepts a bit stream as it's input and converts to an electrical waveform suitable for transmission over the commin channel.

✓ Modulation can be effectively used to minimize the effect of channel noise, to match the characteristics, to provide the capability to multiplex many signals and to over come equipment limitations.

* The important parameters of the modulation are
 a) Type of waveform used.
 b) The duration of the waveform
 c) The power level.
 d) The bandwidth used.

✓ Demodulation is the reverse process of modulation by which the extraction of message from the modulated signal.

✓ The characteristics of modulator, demodulator & the channel establish an average bit error rate between channel encoder & decoder.

✓ The bit error rate and the corresponding symbol error rate will be higher than desired. A lower bit error rate can be accomplished by redesigning the modulator and the demodulator by using control coding ↳(MODEM)

⑥ User Destination:

The extracted symbols are received by the use destination with high data rate and accurately.

Advantages of Digital communication over Analog Communication:

① The digital comm'n provides greater immunity to noise when compared to analog comm'n systems.

② The digital comm'n provides detecting and correcting the errors which are occured during the Txion of the signal.

③ Digital signals which are inherently compatible with computers have a potential to be stored, retrived, processed and manipulated for signal enhancement & improved performance.

④ Using data encryption, only permitted receivers may be allowed to detect the transmitted data i.e. High degree of Security of information. This property is of its most importance in military applications.

⑤ Since the transmitted signal is digital in nature, therefore a large amount of noise interference may be tolerated.

(6) Since in digital communication, Channel Coding is used, therefore the errors may be detected and corrected in the receiver.

(7) In digital communication, the speech, video and other data may be merged and transmitted over a common channel using multiplexing.

(8) Digital communication is adaptive to other advanced branches of data processing such as digital signal processing (DSP), digital image processing (DIP) and data compression etc.

(9) Digital circuits are more reliable & greater dynamic range is possible.

(10) The digital communication systems are simpler and cheaper compared to analog communication systems because of the advances made in the IC technologies.

Disadvantages:

(1) More complex in Circuitary.

(2) Higher channel bandwidth, ie Due to analog to digital conversion the data rate becomes high.

∴ More transmission bandwidth is required for digital communication.

(3) Digital communication needs synchronization in case of synchronous modulation. However, the advantages of digital comm'n overweight the disadvantages.

Telephone channel	Coaxial Cable	Optical fiber
✓ Frequency range : (800 – 3400) Hz.	✓ Bandwidth: 39 GHz	✓ Largest bandwidth : 1000 THz.
✓ Transmission rate : 16.8 Kbps.	✓ Data rate : 274 Mbps.	✓ Free from interference
✓ For voice & data comm'n over long distance	✓ Free from external interference	T - Tera - 10^{12}
	✓ Repeater spacing (1 km - disadvantage).	

Analog to Digital Conversion (ADC) :

The basic operations are usually performed to convert analog to digital as

1. Samplers
2. Quantizers
3. Encoders.

Sampling :

The process of representing an analog signal by sequence of sampled segments is called sampling.

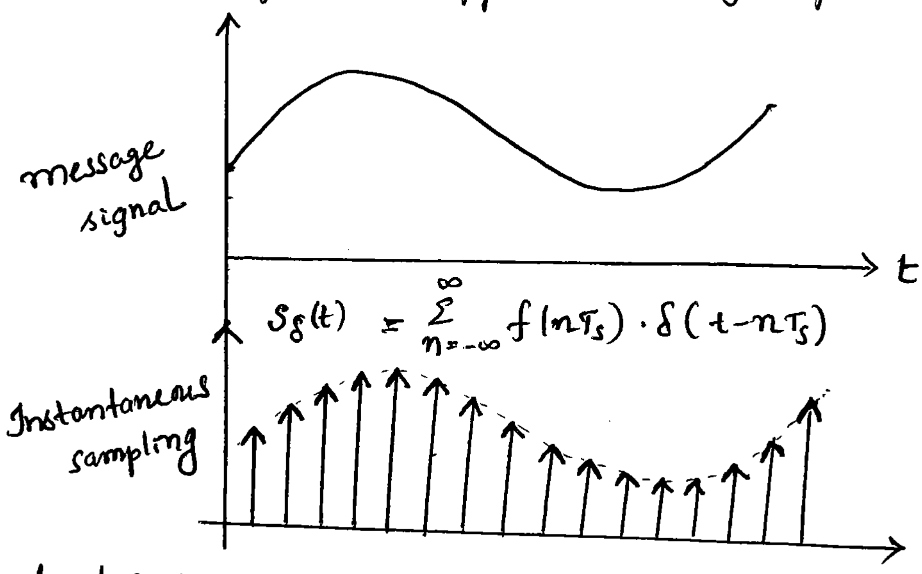
There are three types of sampling.

1. Instantaneous sampling / ideal sampling.
2. Natural sampling.
3. Flat-top sampling.

① Instantaneous Sampling :

Instantaneous sampling signal is a true impulse sequence represented by $\sum_{n=-\infty}^{\infty} \delta(t - nT_s)$.

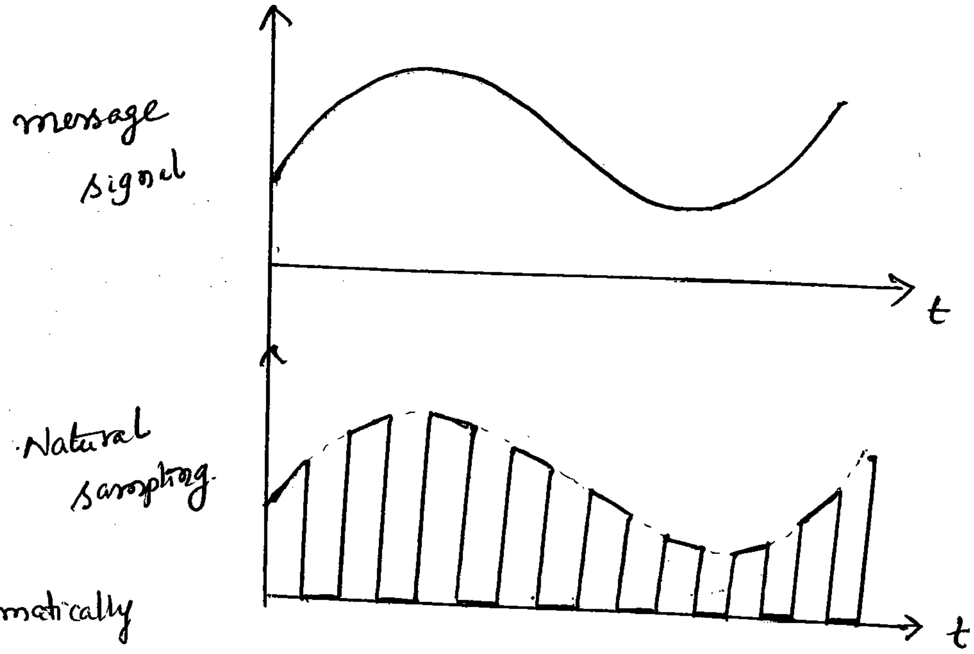
Each impulse is approximated by a pulse of infinite decimal.



Disadvantage : Strength of impulse is very small due to small width when this instantaneously sampled signal is transmitted through the channel. To reduce the effect of noise and the sampled signal the strength of each sample must be increased.

② Natural Sampling :

In natural sampling the samples are rectangular pulses of amplitude 'A' duration 'τ' with a time period of 'T_s'. These pulses are not flat but to follow the natural slope of input waveform.



Mathematically

The sampled signal $s(t) = c(t) \cdot g(t) \longrightarrow (1)$

where $g(t) \rightarrow$ message signal

$c(t) \rightarrow$ Complex fourier series as

$$c(t) = f_s T_A \cdot \sum_{n=-\infty}^{\infty} \text{Sinc}(n f_s T)\, e^{j2\pi f_s n t} \longrightarrow (2)$$

where $f_s = \frac{1}{T_s}$ - Sampling frequency.

From (1) & (2)

$$\therefore \quad s(t) = f_s \cdot T_A \cdot \sum_{n=-\infty}^{\infty} \text{Sinc}(n f_s T)\, e^{j2\pi f_s n t} \cdot g(t)$$

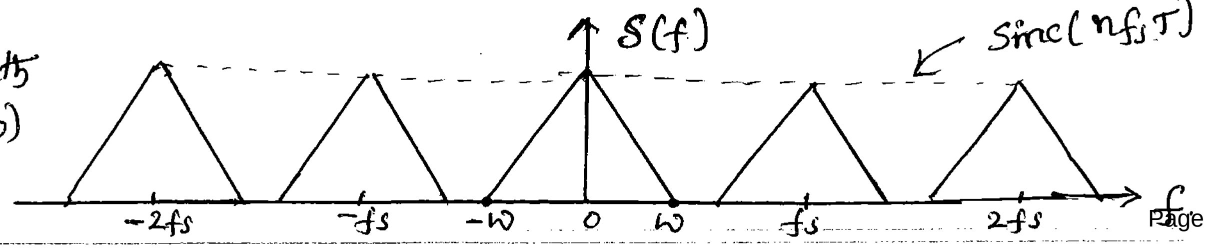
Apply fourier transform on both sides

$$S(f) = f_s \cdot T_A \cdot \sum_{n=-\infty}^{\infty} \text{Sin}(n f_s T)\, \delta(f - n f_s) * G(f).$$

Applying the frequency shifting property of the delta function.

$$\boxed{S(f) = f_s \cdot T_A \cdot \sum_{n=-\infty}^{\infty} \text{Sinc}(n f_s T)\, G(f - n f_s)}$$

Bandwidth
$= W - (-W)$
$= 2W.$



The spectrum is drawn assuming that g(t) is strictly band limited.

ie It contains no frequencies outside the band $-W$ to W and that the sampling rate is greater than $2W$ ($f_s > 2W$) nyquist rate.

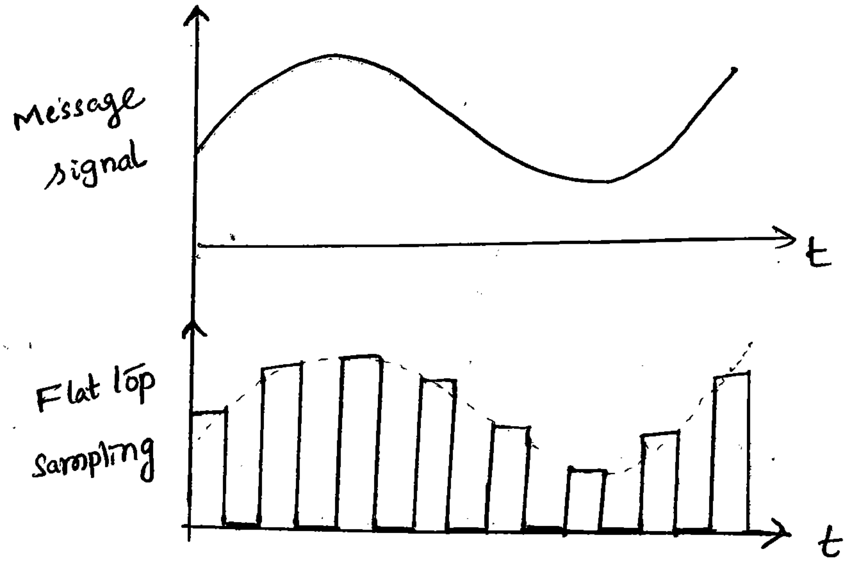
So there is no aliasing effect.

Advantage : This is usually used in time division multiplexing.

Disadvantage : The electronic circuitry needed to perform is complicated because the shape of the pulses has to be maintained.

③ Flat top sampling :

In flat top sampling the top of the pulses are flat & that each pulse is lengthened to the duration 'T' for the convience of Txion, to avoid the use of a excessive between pulses.



Disadvantages :

✓ Due to flat top sampling amplitude distortion & delay is introduced The distortion caused by lengthening the sampled pulse is called as apperature effect

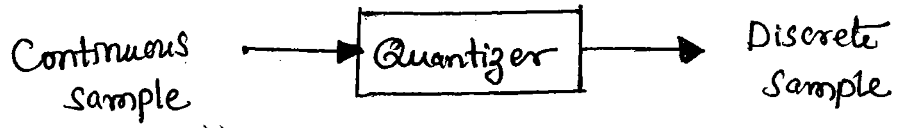
✓ The appertune effect can be corrected by connecting an equalizer to the low pass reconstructed filter, the equalizer will compensate the aperature effect by increasing the frequency.

✓ To ensure perfect reconstruction of the message signal at the Rxer. The sampling rate must be greater than twice the highest frequency component 'ω' of the message signal in accordance with the sampling theorem.

* In practice a low pass anti alliasing filter is used at the front end of sampler to exclude frequencies greater than ω before sampling.

Hence the application of sampling permitts the reduction of continuously varying message signal to a limited no. of discrete values per second.

Quantization :



✓ Representing the analog sample values by a finite set of levels is called Quantizing.

" The process of converting continuous amplitude, discrete time signal such as sampled version of analog signal into discrete amplitude discrete time signal is called <u>Quantization</u>. "

(OR)

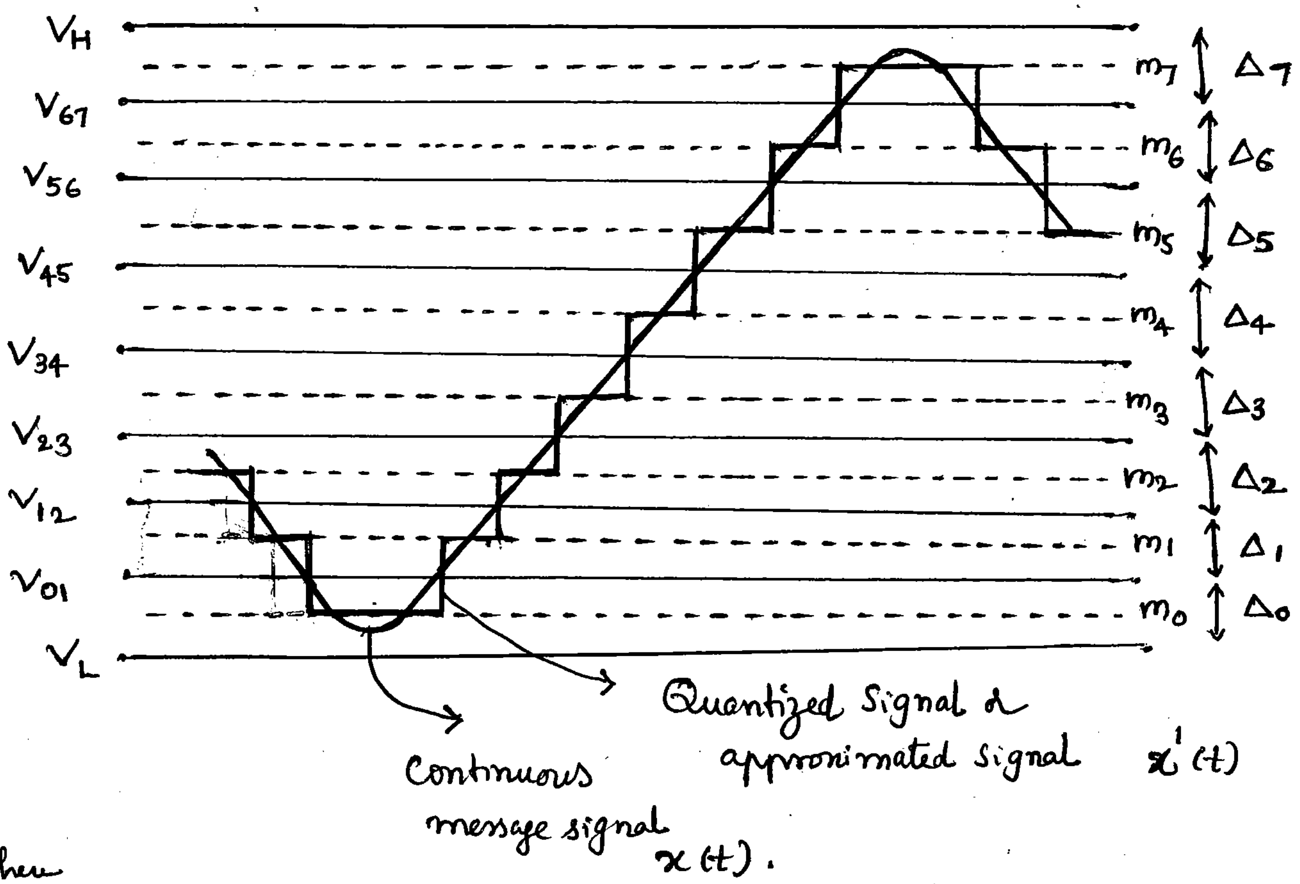
" The approximating the amplitude of each sample to the nearest value from a set of predetermined discrete amplitude levels called <u>Quantization</u> or <u>Representation levels</u>.

Need for quantization:

A continuous signal such as voice or picture signal has continuous range of amplitude when there signals are sampled an infinite no. of amplitude levels will result.

But human sense cannot detect finite differences in the amplitude of the signal. So, continuous amplitude signal may be approximated by a discrete amplitude signal on a min. error basis from a available set of values.

✓ Quantization is referred to breaking down a continuous amplitude range into a finite no. of amplitude values & steps.



where

$$\Delta_0 = \Delta_1 = \Delta_2 = \ldots\ldots\ldots = \Delta_7 = \Delta \rightarrow \text{Step Size}$$

$$\boxed{\text{Step Size } (\Delta) = \frac{V_H - V_L}{L}}$$ where L - No. of levels.

$m_0, m_1, m_2 \ldots\ldots m_7$ are quantization levels or representation levels.

Quatization Error :

The difference between the continuous message signal and the quantized signal is known as quantization error. It is denoted as 'e'

$$\boxed{e = x(t) - x'(t)} \; ; \; -\Delta/2 < e < \Delta/2$$

* At every instant of time $x'(t)$ doesn't change with time or it makes discrete quantized jump of step size 'Δ'.

✓ The quantization levels are seperated by an amount of step size 'Δ' & the other hand the seperation of the criterion levels V_L and V_H each from their nearest quantization levels is $\Delta/2$.

Thus the quantized signal approximated to the original signal. The approximation can be improved by reducing the size of step their by increasing the no. of quantization levels.

Types of Quantizers:

1. Uniform quantizer.
2. Non uniform quantizer.
3. Differential quantizer.

✓ In uniform quantization, step size remains same through out the input range.

ie The representation levels are uniformly placed.

✓ In non uniform quantization, step size varies according to the input signal values.

✓ In differential quantization which is used to reduce the Band-width by taking difference between two samples.

Uniform Quantizer:

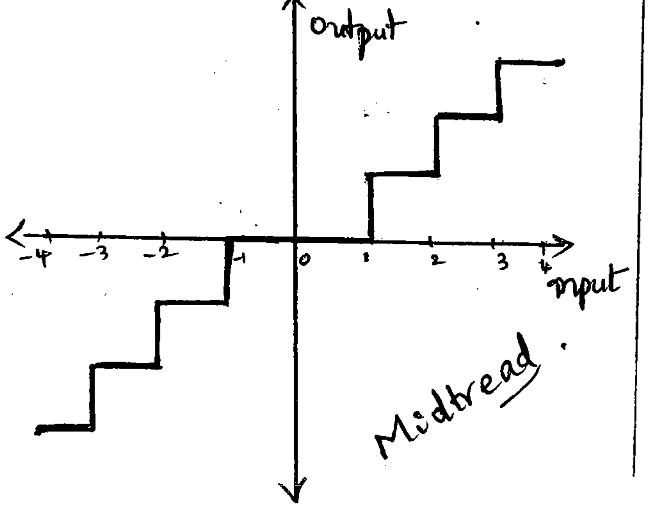
⇒ Quantizer Characteristics are two types which is in staircase fun.

a) Midtread quantization.
b) Midrisor quantization.

} Belogs to Uniform quantization

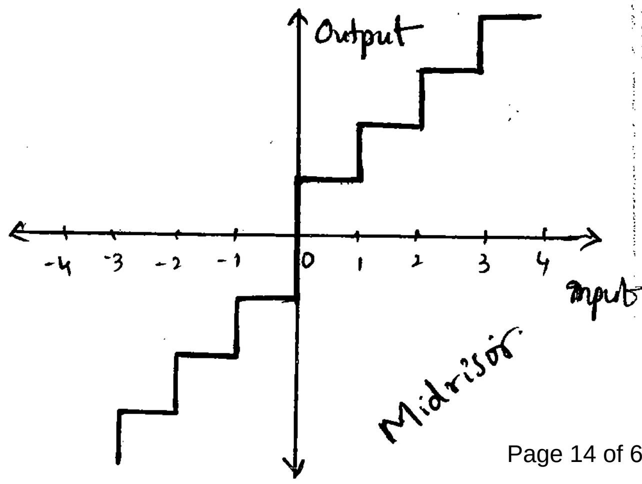
a) Midtread quantization:

Origin is lies middle of the staircase of thread.



b) Mid risor quantization:

Origin is lies middle of the risor of staircase.



**
Let the no-of quantization levels can be given as

L (or) $\boxed{Q = 2^N}$ where N - no-of bits/Sample

✓ Step size of each quantization level is denoted by 'Δ'

$$\Delta = \frac{V_H - V_L}{L} \text{ (or) } \frac{\text{input peak-to-peak voltage}}{\text{No-of quantization levels}}.$$

$$\therefore \boxed{\Delta = \frac{V_{P\text{-}P}}{2^N}}$$

✓ The bit duration is denoted by 'T_b' & it is given by

$\boxed{T_b = \frac{T_s}{N}}$ where T_s - Sampling period
N - No-of bits per sample

✓ The bit rate & max. bandwidth is calculated as

R_b = bit rate = Sampling rate × No-of bits/sample

$= \frac{1}{T_s} \times N$ ↳ sample/sec * bits/sample = bits/sec = bps.

$= \frac{N}{T_s} = \frac{N}{T_s} \Rightarrow \frac{1}{T_b}$

∴ Bitrate $\boxed{R_b = \frac{1}{T_b}}$ bps (or) $\boxed{(B.W)_{max} = \frac{1}{T_b} \text{ Hz}}$.

✓ The Quantization noise/error = Actual sample value − Quantized/approximate Value

$$e = x(t) - x'(t).$$

∴ The maximum Quantization error $\boxed{Q_{e_{max}} = \pm \Delta/2}$

where Δ - Step size

Note:

Let No-of bits $\boxed{N = 3}$.

The no-of Quantization levels $Q = 2^N$

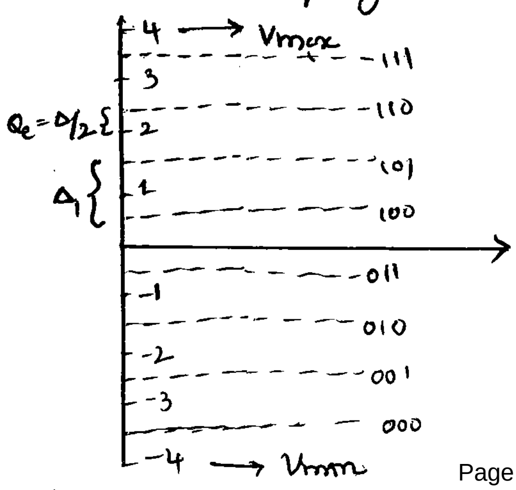
$Q = 2^3 = 8 \Rightarrow \boxed{Q = 8}$.

* The step size $\Delta = \frac{V_{PP}}{2^N} = \frac{V_{max} - V_{min}}{2^N}$

$= \frac{4-(-4)}{2^3} = \frac{8}{8} = 1 \Rightarrow \boxed{\Delta = 1}$

∴ The Max. Quantization error

$Q_{e_{max}} = \pm \Delta/2 \Rightarrow \boxed{Q_{e_{max}} = \pm 1/2}$



Signal to Quantization Noise Ratio (SQNR):

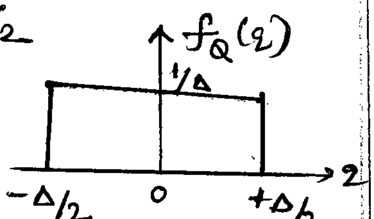
It is defined as the ratio of signal power to the quantization noise power ie $SQNR = (S/N_q)$.

Quantization Noise power (N_q):

The Quantization error is randomly varying from $-\frac{\Delta}{2}$ to $+\Delta/2$ & it is consider as uniform random variable.

Let 'q' be the random variable its probability density fun PSD as

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\Delta/2 < q < +\Delta/2 \\ 0, & \text{otherwise.} \end{cases}$$



The mean of the quantization error is being zero.

$\therefore$ The toal quantization noise power is equal to mean square value & Variance.

$$N_q = \sigma_{Q^2} = E[Q^2]$$

$$= \int_{-\infty}^{\infty} q^2 \cdot f_Q(q)\, dq \qquad \left[\because E[x^2] = \int_{x=-\infty}^{\infty} x^2 \cdot f_X(x)\,dx\right]$$

$$= \int_{-\Delta/2}^{+\Delta/2} q^2 \cdot \frac{1}{\Delta} \cdot dq$$

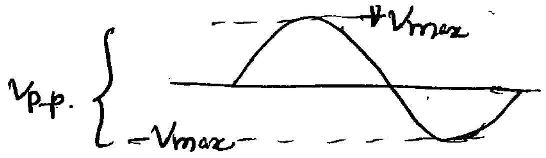
$$= \frac{1}{\Delta} \cdot \frac{q^3}{3}\Bigg|_{-\Delta/2}^{\Delta/2} = \frac{1}{\Delta}\left[\frac{\Delta^3}{24} + \frac{\Delta^3}{24}\right] = \frac{\Delta^3}{\Delta \times 12} = \frac{\Delta^2}{12}$$

$$\therefore \boxed{N_q = \frac{\Delta^2}{12}}$$

we know Step Size $\Delta = \frac{V_{pp}}{Q_{AL}} = \frac{V_{p\text{-}p}}{Q} = \frac{V_{max} - (-V_{max})}{Q}$

$Q = 2^N$ $\quad \therefore \boxed{\Delta = \frac{2V_{max}}{Q}}$

where N - no of bits/sample



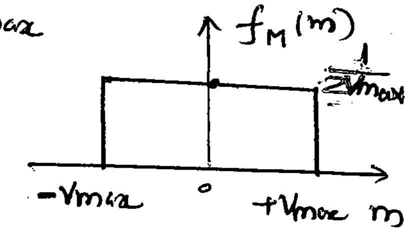
$\therefore$ Quantization noise power $N_q = \frac{\Delta^2}{12} = \frac{(2V_{max})^2}{12 \cdot Q^2} = \frac{4V_{max}^2}{12Q^2}$

$$\therefore \boxed{N_q = \frac{V_{max}^2}{3Q^2}}$$ where $Q = 2^N$.

Signal power (S) :

Let 'm' be the uniform random variable which represents input signal & probability density function can represented as

$$f_M(m) = \begin{cases} \dfrac{1}{2V_{max}}, & -V_{max} < m < +V_{max} \\ 0, & \text{Other wise} \end{cases}$$



∴ The total signal power is mean square value

$$S = \int_{-\infty}^{\infty} m^2 \cdot f_M(m) \cdot dm \qquad f_M(m) = \frac{1}{2V_{max}}.$$

$$= \int_{m=-V_{max}}^{V_{max}} m^2 \cdot \frac{1}{2V_{max}} \cdot dm$$

$$= \frac{1}{2V_{max}} \cdot \left[\frac{m^3}{3}\right]_{-V_{max}}^{+V_{max}}$$

$$= \frac{1}{2V_{max}} \cdot \left[\frac{V_{max}^3}{3} + \frac{V_{max}^3}{3}\right]$$

$$S = \frac{\cancel{2} \cdot V_{max}^3}{\cancel{2} \cdot 3 \cdot V_{max}} \Rightarrow \boxed{S = \frac{V_{max}^2}{3}}$$

∴ The signal to Quantization noise ratio $= S/N_q$

$$= \frac{V_{max}^2/3}{V_{max}^2/3Q^2}$$

We know $Q = 2^N$ $\qquad SQNR = Q^2$

$$= (2^N)^2 = 2^{2N}$$

∴ $\boxed{SQNR = Q^2 = 2^{2N}.}$ where N - no-of bits/sample.

$$(SQNR)_{\text{in dB}} = 10 \log 2^{2N}.$$

If $N = 1 \Rightarrow SQNR = 6\text{ dB}$
$N = 2 \Rightarrow SQNR = 12\text{ dB}$
$N = 3 \Rightarrow SQNR = 18\text{ dB}$

∴ In general $\boxed{(SQNR)_{dB} = 6N\text{ dB}}$

ie As no-of bits increases, Quantization error decreases, so SQNR increases thus bandwidth increases.

Average Signal to Quantization Noise ratio :

It is defined as the ratio of average signal power to the quantization noise power.

$\therefore$ Average (or) peak (or) DC signal power

$$S_i = \frac{V_{max}^2}{R} \quad (\text{let } R = 1\Omega)$$

$$\therefore \boxed{S_i = V_{max}^2}$$

We know

Quantization noise power $\boxed{N_q = \frac{V_{max}^2}{3Q^2}}$

$\therefore$ Average Signal to Quantization noise ratio.

$$(SQNR)_i = \frac{S_i}{N_q} = \frac{V_{max}^2}{V_{max}^2/3Q^2} = 3Q^2$$

$$\therefore \boxed{(SQNR)_i = 3Q^2 = 3 \cdot 2^{2N}}$$

In decibels.

$$(SQNR)_{i\,in\,dB} = 10\log\left(3 \cdot 2^{2N}\right)$$

$$= 10\log 3 + 10\log 2^{2N}$$

for uniform Quantization.

$$\boxed{\text{Average}\,(SQNR)_{dB} = 4 \cdot 8 + 6N} \text{ dB}$$

For RMS s/g power

$$S_r = \frac{\left(\frac{V_{max}}{\sqrt{2}}\right)^2}{R=1}$$

$$S_i = \frac{V_{max}^2}{2}$$

$$\text{rms } SQNR = \frac{3Q^2}{2}$$

$$= 1.5Q^2$$

RMS $(SQNR)_{in\,dB}$

$$= 10\log(1.5) + 20\log Q$$

$$= 1.8 + 6N$$

$$\therefore \boxed{RMS\,(SQNR)_{dB} = 1.8 + 6N}$$

Non-Uniform Quantization:

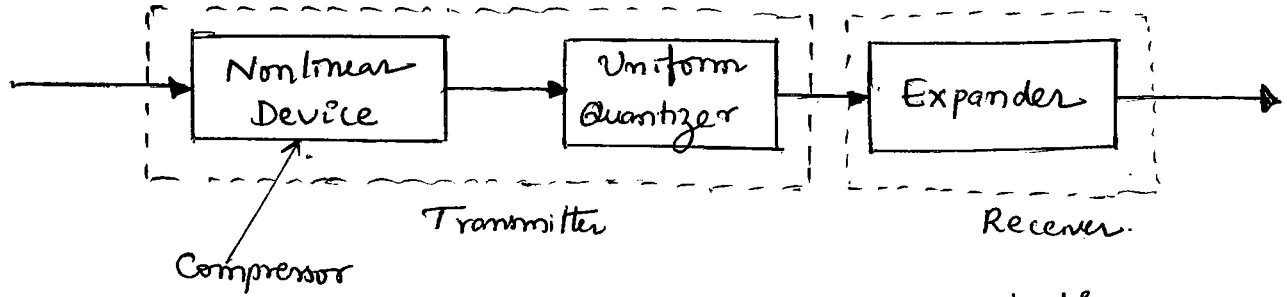


Fig: Model of non uniform quantization.

- ✓ To increase SQNR and to decrease bit rate (R_b) of the signal non uniform quantization is used.
- ✓ In non uniform quantization, the step size varies according to the input signal.

* Consider step size is uniform
  - ✓ Small amplitude signal will have high quantization error results in low SQNR.
  - ✓ High amplitude signal have low quantization error that results in high SQNR.

Companding is the process of maintaining a constant SNR for entire signal. ie Non uniform quantizing is basically a compressor followed by a uniform quantizer and expander.

$$\boxed{\text{Companding} = \text{Compressor} + \text{Expander}}$$

Example:

✓ Quantization of voice signal the signal power corresponding to the voice signal varies from talker to talker.

✓ Even for the same talker the quality of the signal ie the ratio of voltage levels covered by voice signals may range in the order of 1000 : 1.

However excursion of voice signal into the large amplitude ranges occurs in practice less frequently such variations are taken care of by making use non uniform quantizing.

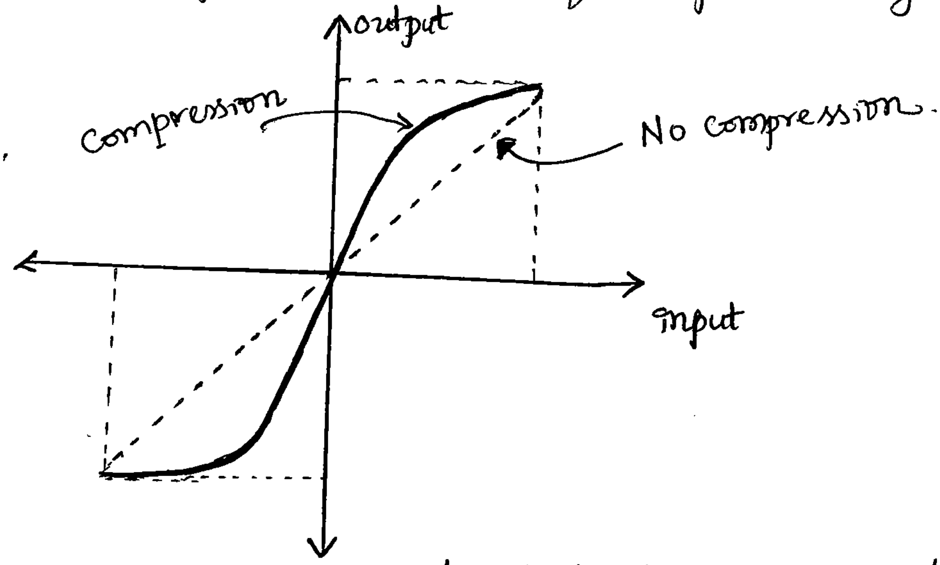
This achieved by a non uniform quantizer which operates in such a way that the step size automatically changes.

→ It is difficult to implement non-uniform quantization in practice because there is no prior information about the changes in the signal level.

Compressor: Compressor is used to amplify the weak signal (Low amplitudes) and attenuated strong signal (large amplitude) So, a signal transmitted through such a network will have extremities of its waveform compressed.

A Typically input output characteristics of compressor is given by

i/p - o/p Characteristics.

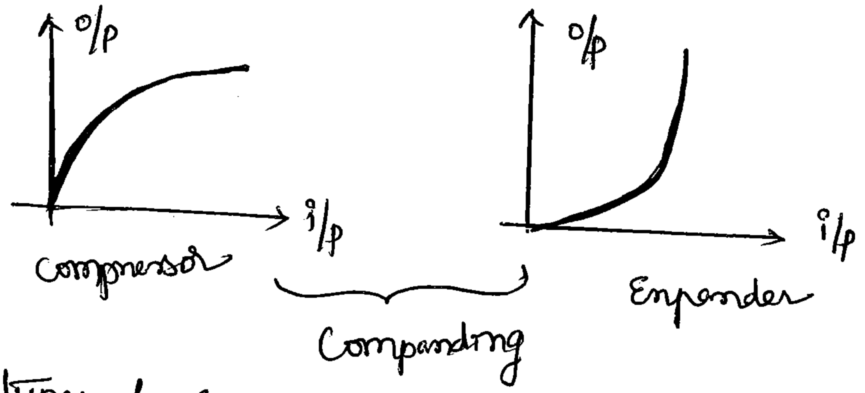


✓ The compression produces signal distortion. To undo the distortion, at the receiver we pass the recovered signal through an expander network.

* An Expander input output characteristics which the inverse of the characteristics of the compressor.

Thus The inverse distortions of compressor & expanders generates a signal output signal without distortions.

ie



* There are two types of Companding Law.

1. μ-law Companding.
2. A-law Companding.

① μ-Law Companding :

μ-law companding used in USA, Canada, Japan.

✓ Low input values → Linear characteristics.

✓ high input values → Logarithmic Characteristics.

It is mathematically expressed as

$$\frac{C(|V|)}{V_{max}} = \frac{1}{\ln(1+\mu)} \ln\left(1+\frac{\mu|v|}{V_{max}}\right) \; ; \; 0 \le \frac{|V|}{V_{max}} \le 1.$$

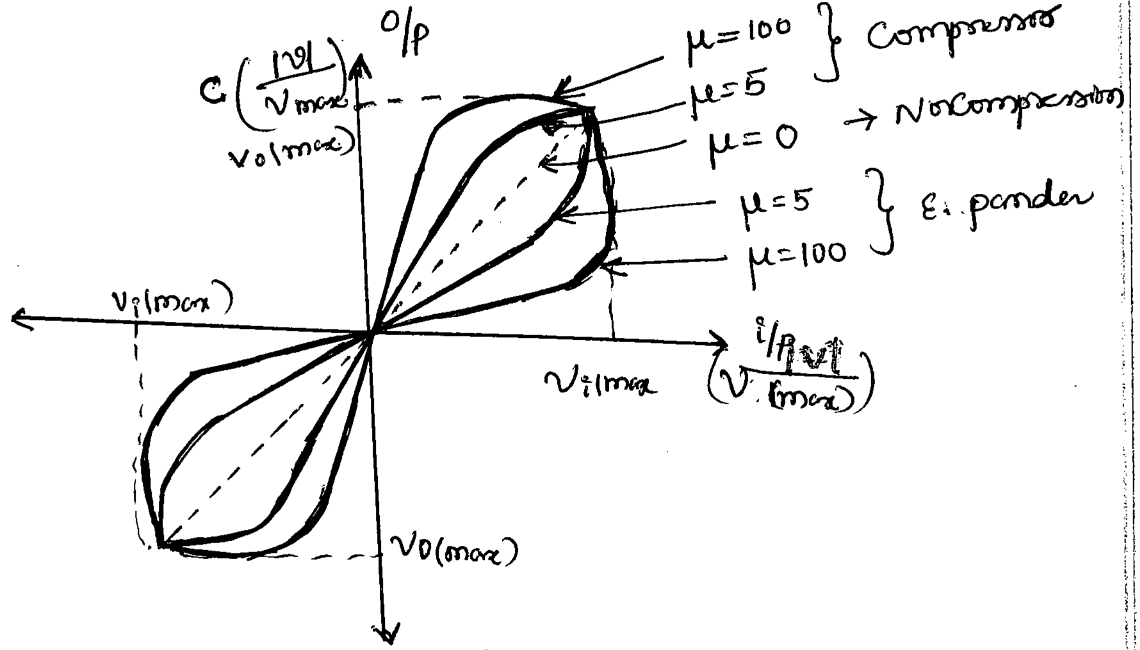
where v is the amplitude of input signal

V_{max} is max. amplitude of i/p signal.

$C(|v|)$ is compression of input signal

μ - Amount of compression.

Fig: μ-law companding



* $\mu = 0$ represents the uniform quantization
* Practically the value of μ is 255. $\boxed{\mu = 255}$

② A-law Companding:

A-law companding used in Europe and rest of the world.

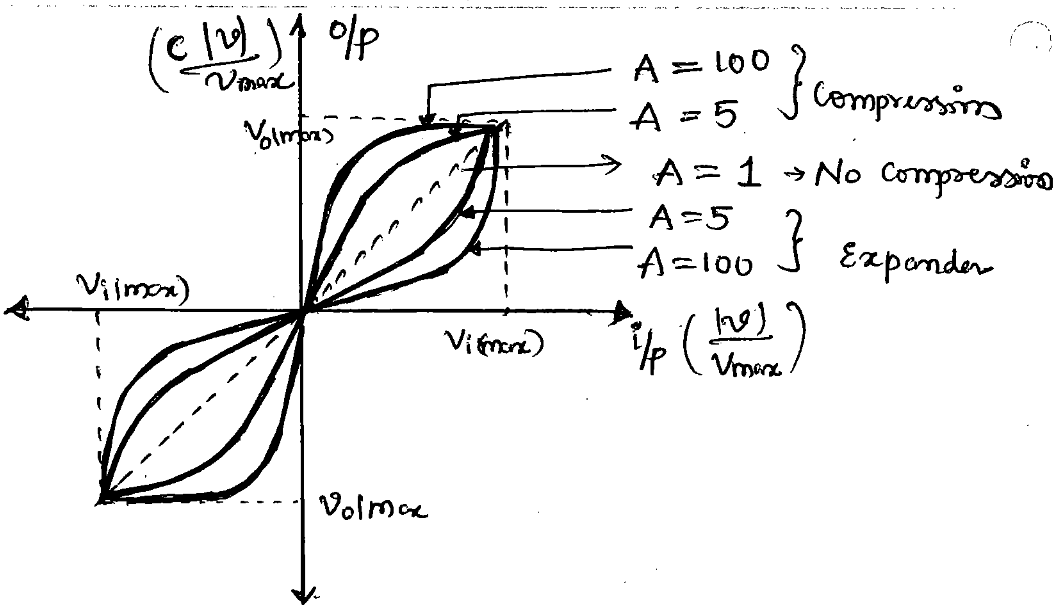
It is mathematically expressed as

$$\frac{C(|v|)}{V_{max}} = \begin{cases} \frac{A}{1+\ln(A)} \cdot \frac{|v|}{V_{max}} & ; \; 0 \le \frac{|v|}{V_{max}} \le \frac{1}{A} \\ \frac{1}{1+\ln(A)}\left[1+\ln\left(\frac{A|v|}{V_{max}}\right)\right] & ; \; \frac{1}{A} \le \frac{|v|}{V_{max}} \le 1. \end{cases}$$

A - Defines amount of compression.

$C(|v|)$ - Compression of input signal.

Fig: A-law Companding.



* A = 1 represents the uniform quantization
* Practically the value of A is 87.56 $\boxed{A = 87.56}$

<u>Output SQNR</u>

For Uniform Quantization $\frac{S_i}{N_q} = 3Q^2\left(\frac{m^2(t)}{V_{max}^2}\right)$

for Non uniform quantization $\frac{S_i}{N_q} = 3Q^2 \cdot \frac{1}{\{\ln(1+\mu)\}^2}$, ($\because$ From μ-law companding)

In general signal to Quantization noise ratio SQNR is given by

$$\boxed{\begin{aligned}\frac{S_i}{N_q} &= C \cdot Q^2, \\ \text{where} \quad C &= \begin{cases} 3 \cdot \frac{m^2(t)}{V_{max}^2}, & \text{for uniform quantization} \\ 3 \cdot \frac{1}{\{\ln(1+\mu)\}^2} & \text{for non-uniform quantization}\end{cases}\end{aligned}}$$

SQNR in dB

$$\begin{aligned}(SQNR)_{dB} &= 10 \log (S_i/N_q) \\ &= 10 \log CQ^2 \\ &= 10 \log C + 20 \log 2^N \quad (\because Q = 2^N) \\ &= 10 \log C + 6N\end{aligned}$$

$\therefore \boxed{(SQNR)_{dB} = \alpha + 6N}$, $\alpha = 10 \log C$.

Differential Quantization :

✓ The bit rate of the digitally encoded signal $\boxed{R_b = N f_s}$

where f_s - no. of samples/sec a sampling frequency

N - no. of bits / sample

* To decrease the bit rate of the signal, either N or f_s should be decreased, but f_s cannot be decreased below the nyquist sampling rate & hence 'N' is decreased.

* To decrease 'N' value, the peak-to peak voltage of the message signal is decreased by taking difference b/w the two samples.

∴ "The output of the difference between two samples is quantized and this type of quantization is called Differential Quantization".

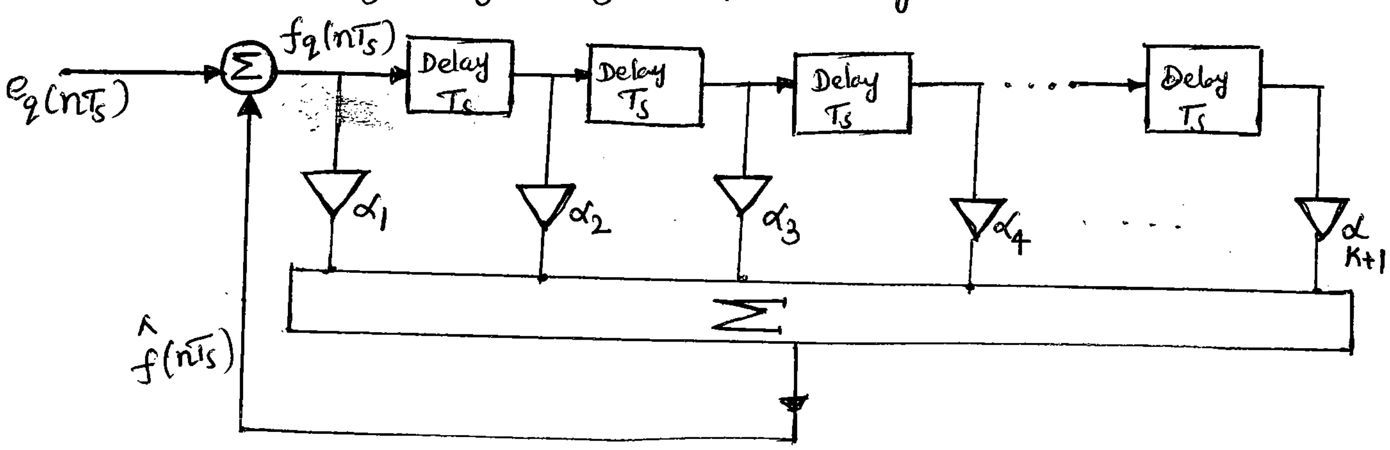
✓ The gap b/w two successive samples is 'T_s' seconds.

* To find the difference b/w the samples, predictor is employed along with the uniform quantization.

Predictor (or) Prediction filter :

✓ The main purpose of predictor is used to find the difference between the samples and also estimating the future values based on the previous values.

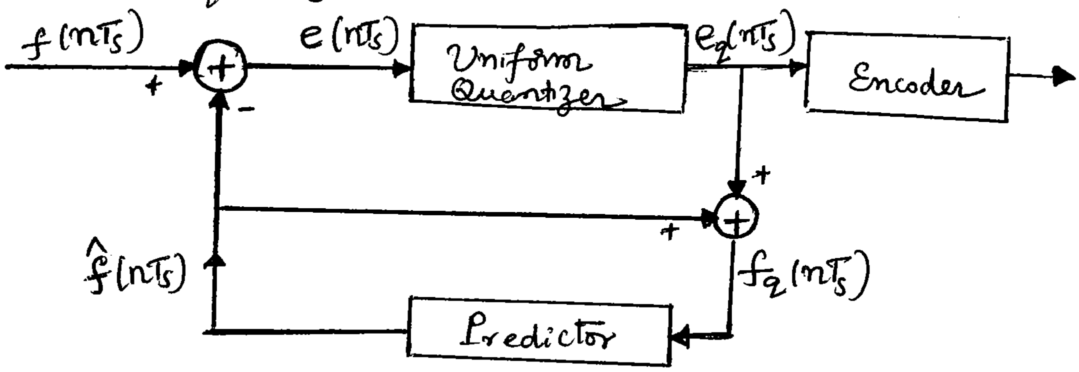
✓ The predictor filter is used in transmitter & Receiver of a DPCM s/m. & can be realized by using tapped delay line filters.



where $\alpha_1, \alpha_2, \alpha_3, \ldots \ldots \alpha_{K+1}$ are tape gains.

$$\hat{f}(nT_s) = \alpha_1 \cdot f_q(nT_s) + \alpha_2 f_q\{(n-1)T_s\} + \alpha_3 f_q\{(n-2)T_s\} + \cdots \cdots + \alpha_{K+1} f_q\{(n-K)T_s\}.$$

✓ A uniform quantizer along with the predictor will realize a differential quantizer as shown in below.



At Transmitter

* Consider a signal '$f(t)$' which is sampled at a nyquist rate and produce a sequence of correlated samples of 'T_s' seconds apart & denoted as $f(nT_s)$.

✓ The input to the quantizer is $e(nT_s)$ given by

$$\boxed{e(nT_s) = f(nT_s) - \hat{f}(nT_s)} \longrightarrow (1)$$

where $e(nT_s) \rightarrow$ prediction error & can be reduced by changing the prediction value.

$f(nT_s) \rightarrow$ Unquantized sample

$\hat{f}(nT_s) \rightarrow$ Predicted sample sequence consists of quantized signal

✓ The output of the quantizer $e_q(nT_s)$ is given by

$$\boxed{e_q(nT_s) = e(nT_s) \pm Q_e(nT_s)} \longrightarrow (2)$$

where $e_q(nT_s) \rightarrow$ Quantized output

$Q_e(nT_s) \rightarrow$ Quantization error

* The quantizer output is added to the prediction value $\hat{f}(nT_s)$ to produce the prediction input $f_q(nT_s)$

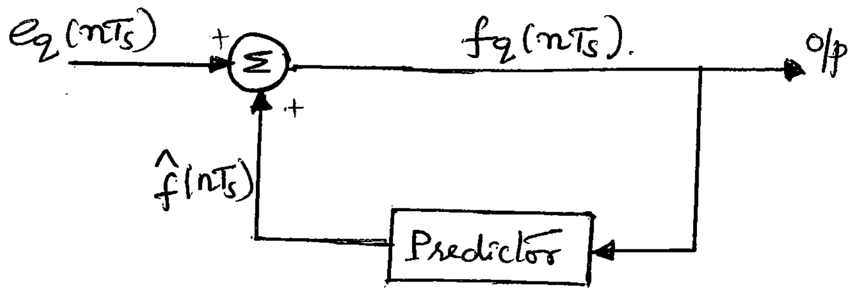
$$\boxed{f_q(nT_s) = e_q(nT_s) + \hat{f}(nT_s).} \longrightarrow (3)$$

From eqn ② & ③
$$f_q(nT_s) = e(nT_s) \pm Q_e(nT_s) + \hat{f}(nT_s) \longrightarrow ④$$

From eqn ① & ④
$$f_q(nT_s) = f(nT_s) - \cancel{\hat{f}(nT_s)} \pm Q_e(nT_s) + \cancel{\hat{f}(nT_s)}$$

$$\therefore \boxed{f_q(nT_s) = f(nT_s) \pm Q_e(nT_s)}$$

ie The input of the prediction filter differs from the input signal ($f(nT_s)$) by quantization error ($Q_e(nT_s)$).



At Receiver

✓ The input in both cases are same, ie: $e_q(nT_s)$

$\therefore$ The prediction output must be $\hat{f}(nT_s)$

$\therefore$ The receiver output is also same

$$f_q(nT_s) = e_q(nT_s) + \hat{f}(nT_s)$$

From eqn ②
$$f_q(nT_s) = e(nT_s) \pm Q_e(nT_s) + \hat{f}(nT_s)$$

from eqn ①
$$\therefore \boxed{f_q(nT_s) = f(nT_s) \pm Q_e(nT_s).}$$

This shows that we are able to receive the desired signal input plus the quantization noise associated with the difference between the sample $e(nT_s)$, which is generally smaller than $f(nT_s)$.

✓ The quantized version of the original input is reconstructed from the receiver output by using same prediction filter used in transmitter.

Encoding (or) Coding :

In combining the process of sampling & Quantizing, the continuous message & baseband signal becomes limited to a discrete set of values but not in the form best suited for transmission over common channel such as a telephone line (or) radio path & optical fiber etc.

" The process of assigning a binary value & binary code to each discrete set of samples is known as Coding."

✓ Set of symbols is called codeword.

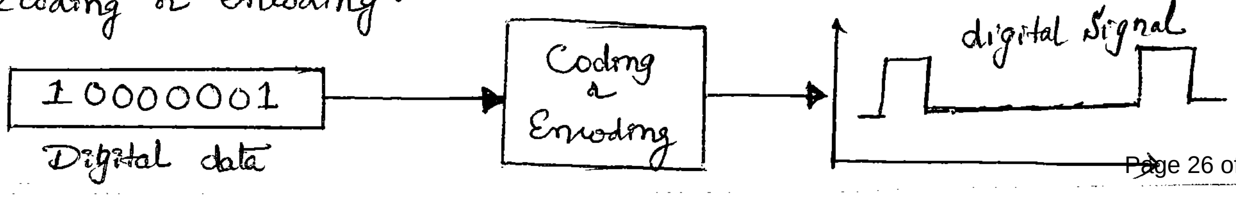
* In binary code, each symbol consists of two distinct values.
* In ternary code, each symbol consists of three distint values.

✓ Each codeword consists of n-bits, such a code represents a total of 2^n distinct numbers.

* Encoding is also called waveform coding & line coding or transmission coding & binary level representation.

Ordinary no. of representation levels	Level no. expressed as sum of power 2	Binary Numbers
1	2^0	001
2	2^1	010
3	2^1+2^0	011
4	2^2	100
5	2^2+2^0	101
6	2^2+2^1	110
7	$2^2+2^1+2^0$	111

* The first approach converts digital data into digital signal is known as line coding & encoding.



Electrical representation of binary value :

There are different types of coding techniques.

1. Unipolar NRZ Signalling.
2. Polar NRZ Signalling
3. Unipolar RZ signalling
4. Bipolar RZ Signalling
5. Split phase (or) Manchester Signalling
6. Differential Encoding.

where
NRZ - Non Return to Zero
RZ - Return to Zero.

1. Unipolar NRZ Signalling :

1 → 1
0 → 0

* Symbol 1 represented by presence of pulse and
* Symbol 0 represented by absence of pulse.
* It is also reffered as ON-OFF signalling.
* It is used for the Transmission of ASK signal (Amplitude Shift Keying)

2. Polar NRZ Signalling :

1 → +1
0 → −1

* Symbol 1 is represented by '+1' volts and
* Symbol 0 is represented by '−1' volts.
* It is used for PCM, DM, ADM, PSK, FSK Signal.

3. Unipolar RZ Signalling :

1 → +½ → +1 ; 1 → −½ → 0
0 → 0

* Symbol '1' is splitting into two half cycles, & 1st half cycles represents +1.
* Symbol '0' & 2nd half cycle of symbol '1' is represented by '0'.
* It is used for fiber optical communication.

4. Bipolar RZ Signalling :

1 → first, +1 ; Second, −1 & 0 → No pulse.

* Symbol '1' is represented by alternation of +1 and −1, no pulse for '0'
* It is also referred as AMI (Alternate Mark Inversion)

5. Split phase (or) Manchester Signalling :

1 → ½ → +1 ; 1 → −½ → −1 ; 0 → −½ → −1 ; 0 → +½ → +1

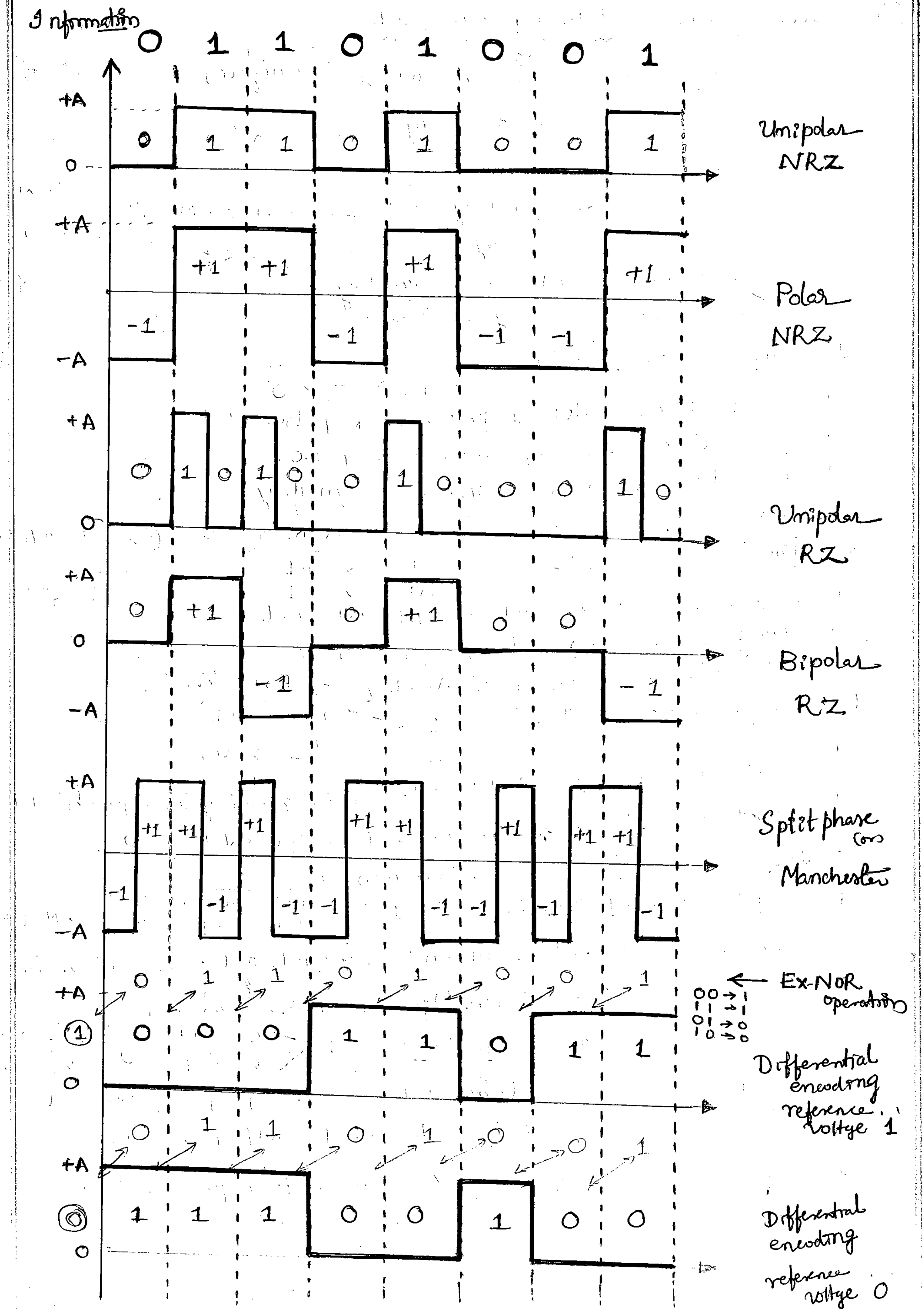
* Symbol '1' is represented by +1 and −1 half cycle pulse and Symbol '0' is represented by −1 and +1 half cycle pulse.

6. Differential encoding :

0 → complement of previous voltage
1 → previous voltage.

* Symbol '0' is represented by complement of previous voltage
* Symbol 1 is represented by previous voltage. → Exclusive NOR operation

00 − 1
11 − 1
01 − 0
10 − 0


Information
0 1 1 0 1 0 0 1
+A
0
Unipolar NRZ
+A
-A
+1 +1 +1 +1
-1 -1 -1 -1
Polar NRZ
Unipolar RZ
Bipolar RZ
Split phase (or) Manchester
Ex-NOR operation
00 → 1
11 → 1
01 → 0
10 → 0
Differential encoding reference voltage 1
Differential encoding reference voltage 0

Pulse Code Modulation (PCM) :

✓ Pulse digital modulation is a scheme that converts the analog signal to its corresponding digital form.

* The simplest form of pulse digital modulation (PDM) is known as Pulse Code modulation (PCM).

✓ In PCM as message signal is represented by sequence of coded pulses which is accomplished by representing the signal in discrete form in both time and amplitudes.

* PCM system consists of mainly three blocks.

(a) Transmitter (b) Transmission path (c) Receiver

(a)

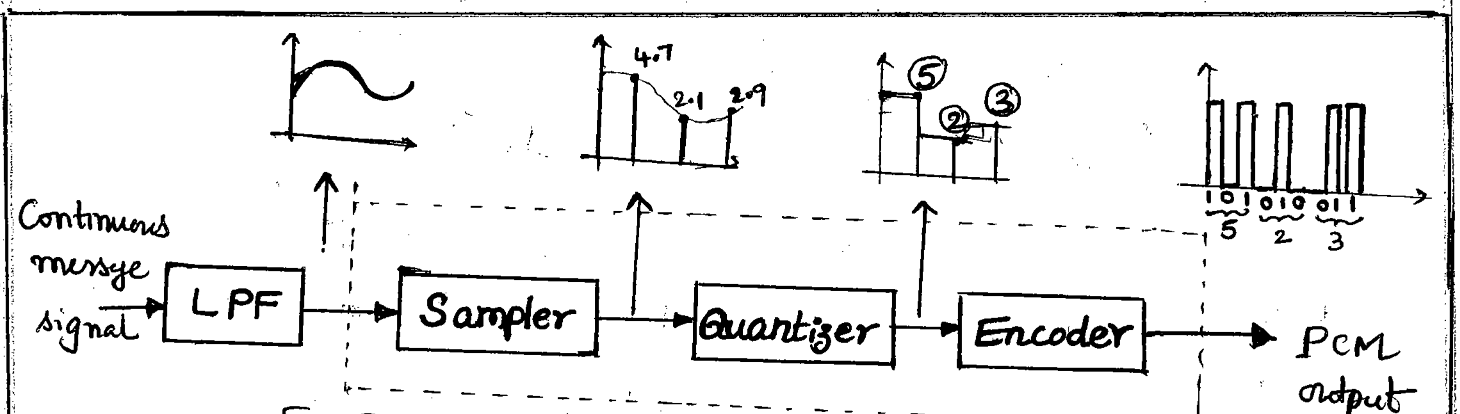


Fig (a) Transmitter

(b)

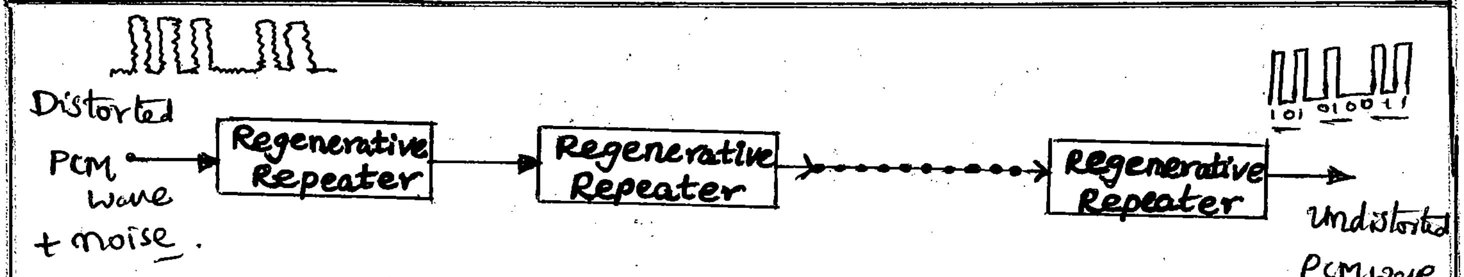


Fig (b) Transmission path → To reduce noise

(c)

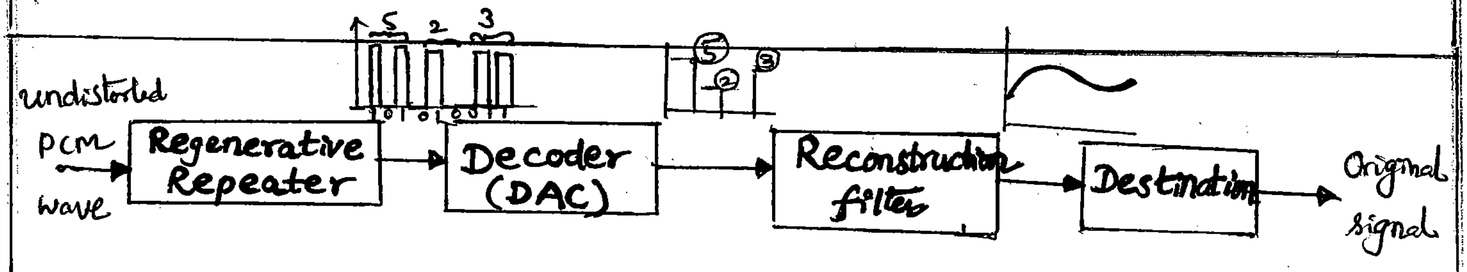


fig (c) Receiver

Principle of operation :

- ✓ The basic operation of PCM system involves the PAM signals are quantized & then coded, thus PCM is obtained and its amplitude and time are represented in discrete values.
- ✓ PCM is not modulation scheme, In the conventional scheme the term 'modulation' usually refers to variation of some characteristics of carrier wave in accordance with an information signal.

* Main use of discrete or digital representation of signals are
  - ✓ Ruggedness for Txion noise and Interference.
  - ✓ Security is obtained.

→ The essential operation in the <u>transmitter</u> of PCM system are Sampling, Quantizing and encoding & these three operations are usually performed in same ckt which is called ADC ie Analog to digital converter – as shown in fig ⓐ.

→ Regeneration of impaired signals occurs at intermediate points along the <u>transmission path</u> as in fig ⓑ.

→ At the <u>receiver</u>, the essential operations consists of one last state of regeneration followed by decoding then demodulation of the train of quantized samples ; the operations of decoding and reconstruction are usually performed in the same ckt as digital to Analog converter (DAC) – as in fig ⓒ.

ⓐ Transmitter :

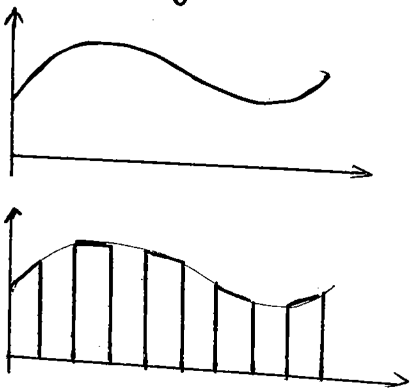
- ✓ The LPF prior to sampler is included to prevent aliasing effect (or) foldover effect.
- ✓ LPF is also called as prealliasing filter.

The basic operations performed in transmitter of PCM systems are
(i) Sampling (ii) Quantizing (iii) Encoding.

(i) **Sampling:** The process of representing an analog signal by a sequence of sampled segments is called sampling.

Sampling permits the reduction of continuously varying message signal to a limited no. of discrete values per second.

$$\boxed{f_s \geq 2f_m}$$

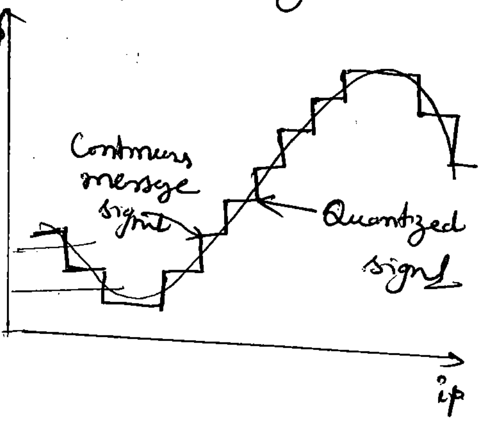


(ii) **Quantizing:**

The process of converting continuous amplitude discrete time signal into discrete amplitude discrete time signal.

& approximating the amplitude of each sample to the nearest value from a set of discrete amplitude levels called as Quantization.

PCM uses either uniform Quantization (or) non uniform quantization to convert analog to digital signal.



✓ The step size $\boxed{\Delta = \dfrac{V_{pp}}{Q} = \dfrac{V_{pp}}{2^N}}$

where

Q - no. of quantization levels $= 2^N$ where N - no. of bits / sample.

✓ The quantization error / noise can be calculated by the difference between continuous message signal and quantized signal.

ie

$$\boxed{e \text{ or } Q_e = x(t) - x'(t) \Rightarrow -\Delta/2 \leq e \leq +\Delta/2}$$

The maximum quantization error Q_e & $e = \Delta/2$.

(iii) **Encoding:** The process of assigning a binary value to a discrete set of samples known as coding & encoding for the best suited transmission over a communication channel such as telephone line & optical fiber etc.

✓ It's mainly converts digital data in digital signal representation.

(b) Transmission path : (Set of Regenerative Repeaters).

* The most important feature of PCM system lies in the ability to control the effect of distortion noise produced by transmitting a PCM wave through a channel.

✓ This capability is accomplished by reconstructing the PCM wave by means of a choice of regenerative repeaters located at sufficiently close space along the transmission path.

Regenerative Repeater :

The three basic functions performed by a regenerative repeater are (i) Equalization (ii) Timing and (iii) Decision making device.

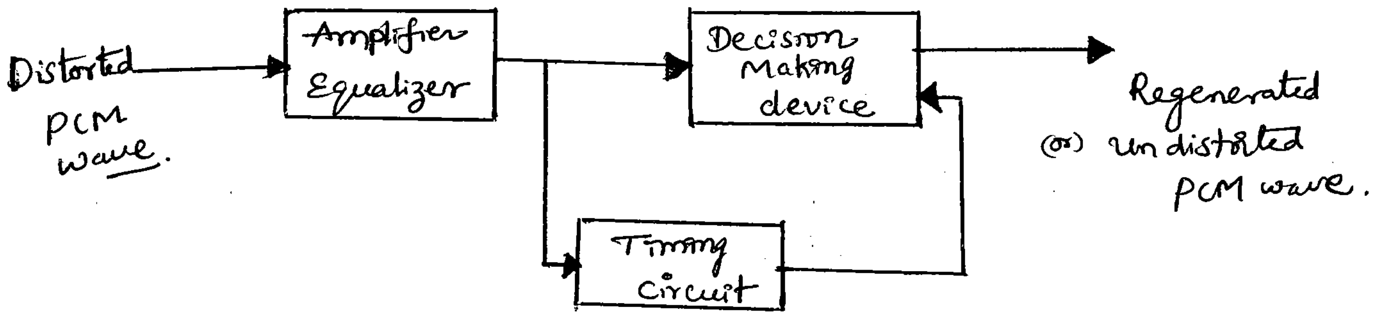


Fig: Regenerative Repeater.

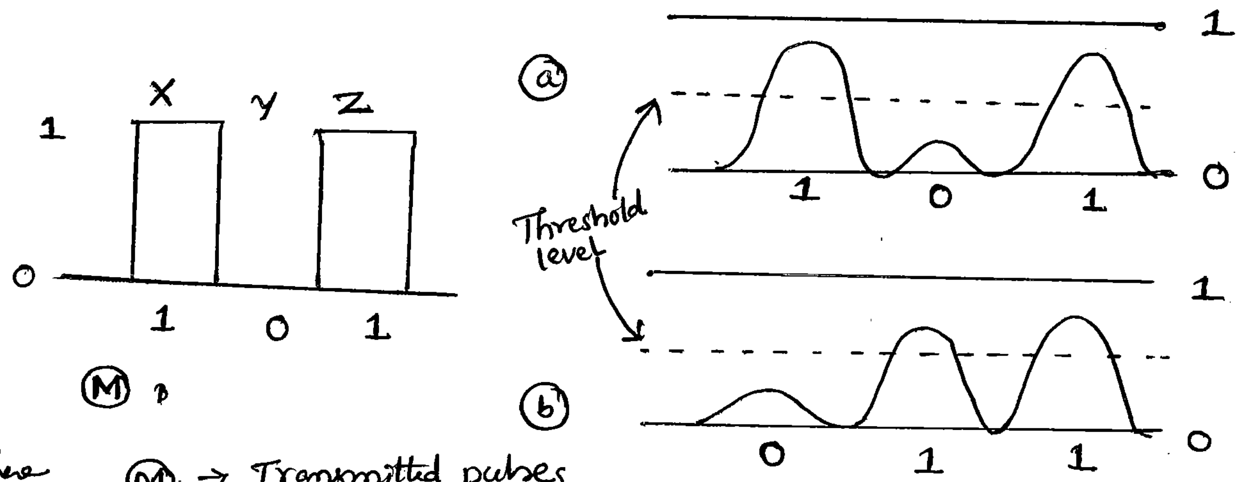
(i) Equalizer : It shapes the received pulses so as to compensate for the effect of amplitude and phase distortions produced by the channel.

(ii) Timing Circuit : It provides a periodic pulse train derived from received pulses, for sampling the equalized pulses at the instants of time where the signal to quantization noise ratio is maximum.

(iii) Decision making device : It enables the sampling time determined by the timing circuitry makes its decision based on whether or not the amplitude of quantized pulse plus noise exceeds a predetermined voltage level.

Example: Consider the case of PCM with unipolar signalling.

The major advantage of the PCM system is that the information does not lie in any property of pulse, but it lies in the presence (or) absence of the pulse.



Where
- (M) → Transmitted pulses
- (a) → Received pulse without error
- (b) → Received pulse with error.

(a) → when the voltage in a time slot crosses the threshold level (which is the centre of 0 and 1 level), it is received as a '1'.
→ when the voltage in a time slot does not cross the threshold level, it is received as a '0'.

* The received pulses are distorted due to noise, there is no error of decision.

(b) → In the time slot X, the voltage does not cross the threshold level & hence a '0' is received (an error),
→ In the time slot 'y', the voltage crosses the threshold level and hence a '1' is received (an error).
→ There is no error in the time slot 'z'.

* Thus 101 is received as 011, but the probability of the occurance of such error is extremely small.

Thus the difference between the '0' & '1' levels can be increased, resulting in a reduced effect of noise.

Hence practically error free transmission is possible.

The repeaters in PCM system used to generate pulses in time slots according to the presence of '0' or '1', thus eliminating effect of noise till that point.

© Receiver :

The basic operations in the PCM receiver are

(i) Regenerative repeater last stage.
(ii) Decoding. (DAC)
(iii) Reconstruction filter.

* At the receiver end the first operation is to regenerate the received pulse by cleaning and reshaping.
* The cleaned pulses are then regrouped into codewords and then mapped back into a quantized PAM signal.

" **Decoding** : The mapping back of codewords into a quantized PAM signal is called decoding.

✓ Decoding may be viewed as reverse process of encoding performed in the transmitter.

* The decoding process involves generating a pulse whose amplitude is a linear sum of all the pulses in the code word with each pulse weighted by its place value ($2^0, 2^1, \ldots$) in the code.

Reconstructing filter :

✓ It reconstructs the original message signal from the PAM signal appeared at the output of the decoder.

✓ Reconstructing filter as a Low pass filter whose cut off frequency equal to the message bandwidth 'w'.

However, the process of quantizing a signal at the transmitter is an "irreversible process". It introduces quantizing noise in the transmitter and results in a loss of information which cannot be recovered by any means at the receiver.

Bandwidth in PCM :

✓ The message bandwidth is 'f_m' when the quantized sample occur at rate of '$f_s \geq 2f_m$' sample/sec.

✓ If the PCM system uses binary channels to represents the Q-quantization levels then each code word consists of N digits.

where $\boxed{N = \log_2 Q}$ $(\because Q = 2^N)$

Channel symbol / bit rate is given by

$R_b = N \cdot f_s$ where N - no of bits / sample
f_s - Sampling frequency

$\therefore R_b \geq 2Nf_m$ $\therefore f_s \geq 2f_m$.

If a system has a bandwidth of 'B' Hz when we can transmit a min. of '2B' samples / sec.

$\therefore$ Transmission rate represented by R & R_b.

$$R_b = 2B \text{ bits/sec.}$$

$$B = R_b/2 \geq \frac{\cancel{2}N \cdot f_m}{\cancel{2}}$$

$\therefore \underline{\underline{B \geq Nf_m}}$ $\therefore$ $\boxed{\text{Min. B.W of PCM system, } B \geq Nf_m.}$

Advantages of PCM :

✓ Used for long distance communication ($\because$ Regenerative repeaters are used).
✓ High Immunity to noise ie noise can be removed.
✓ Storage is easy (because PCM signals are digital in nature

Disadvantages of PCM :

✓ Complexity is high ($\because$ PCM systems includes encoding, decoding, sampling, quantizing and repeaters).
✓ It requires more bandwidth compare to other systems.

Applications of PCM :

✓ PCM is used in telephone systems.
✓ Used for digital audio in computers.
✓ It is one of the formate for writing CD-ROMs, DVD's, Blue-ray Disc.

Noise in PCM System :

The performance of a PCM system is effected by major source of noise.

a) Channel noise & thermal noise & transmission noise

b) Quantization noise.

✓ Channel noise is introduced any where between the output of transmitter & input of the Receiver.

✓ Quantized noise is introduced in the transmitter and is carried along to the receiver output and it is a signal dependent in the sence that is disappear when the message signal is switched off.

* The output $\bar{X}(t)$ in a PCM system can be written as

$$\bar{X}(t) = X_0(t) + n_q(t) + n_0(t)$$

where $X_0(t) \rightarrow$ Signal component in the output

$n_q(t) \rightarrow$ Quantization noise

$n_0(t) \rightarrow$ Channel noise.

The overall SNR at the baseband output which is a measure of signal quality can be defined as

$$\boxed{(S/N)_0 = \frac{E\{[X_0(t)]^2\}}{E\{[n_q(t)]^2\} + E\{[n_0(t)]^2\}}}.$$

Quantization noise in PCM :

Let us assume that ideal impulse sample (sampling) is used in PCM system then the output of sample is

$$X_s(t) = X(t) \sum_{K=-\infty}^{\infty} \delta(t-KT_s)$$

T_s - Sampled interval.

The quantized signal is represented by $X_q(t)$ as

$$X_{sq}(t) = X_q(t) \cdot \sum_{K=-\infty}^{\infty} \delta(t-KT_s)$$

$$x_{Sq}(t) = x(t)\cdot\sum_{k=-\infty}^{\infty}\delta(t-kT_s) + [x_q(t)-x(t)]\sum_{k=-\infty}^{\infty}\delta(t-kT_s).$$

$$= x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT_s) + e_q(t)\sum_{k=-\infty}^{\infty}\delta(t-kT_s)$$

where $e_q(t) = |x_q(t) - x(t)|$

$$\therefore x_{Sq}(t) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT_s) + e_q(kT_s)\sum_{k=-\infty}^{\infty}\delta(t-kT_s)$$

where $e_q(kT_s)$ is the error due to the quantization process

✓ The power spectral density of $e_q(kT_s)$ is $G_{eq}(f)$ as

$$G_{eq}(f) = \frac{\text{Variance}}{\text{Timeperiod}} = \frac{E\{e_q^2(kT_s)\}}{T_s}$$

If uniform quantization operating on $x(t)$ having an uniform PDF

$$\sigma_q^2 = E\{e_q^2(kT_s)\} = \frac{\Delta^2}{12} \quad \text{where } \Delta\text{-Step size}$$

$$\therefore \boxed{G_{eq}(f) = \frac{\Delta^2}{12\cdot T_s}}$$

If we ignore the effects of channel noise temporarily, where noise components $n_q(t)$ has a PSD of $G_{nq}(f)$ as

$$\underline{G_{nq}(f) = G_{eq}(f)\cdot|H_R(f)|^2}$$

where $H_R(f)$ - transfer function of ideal LPF (reconstruction filter)

Assume sampling rate $f_s = 2f_m$ & $H_R(f)$ to be ideal LPF with bandwidth 'f_m'

$$\therefore G_{nq}(f) = \begin{cases} G_{eq}(f) & ; -f_m \le f \le f_m \\ 0 & \text{otherwise.} \end{cases}$$

Average quantization noise power is mean square value ie

$$E[n_q^2(t)] = \int_{-f_m}^{f_m} G_{nq}(f)\cdot df = \int_{-f_m}^{f_m} G_{eq}(f)\, df$$

$$= \int_{-f_m}^{f_m} \frac{\Delta^2}{12\cdot T_s}\cdot df$$

$$\therefore \quad E[n_q^2(t)] = \frac{\Delta^2}{12}\cdot\frac{1}{T_s}\cdot[f]_{-f_m}^{+f_m}$$

$$= \frac{\Delta^2}{12\cdot T_s}\cdot 2f_m$$

$$= \frac{\Delta^2}{12\cdot T_s^2} \qquad \left(\because f_s = 2f_m\right), \quad \frac{1}{T_s} = f_s$$

Quantization noise power

$$\therefore \boxed{E\{[n_q(t)]^2\} = \frac{\Delta^2}{12\cdot T_s^2}}$$

* The output signal component $x_0(t)$ is the response of the LPF to $x(t)\cdot\sum_{K=-\infty}^{\infty}\delta(t-KT_s)$.

$\therefore$ The output signal $x_0(t) = \frac{1}{T_s}\cdot x(t)$

Normalized signal output power is mean square value as

$$E\{[x_0(t)]^2\} = \frac{1}{T_s^2}\cdot\overline{x}^2(t).$$

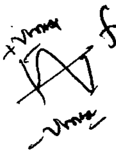
The PSD of the instantaneous value x is $f_X(x)$

$$\therefore \quad f_X(x) = \begin{cases} \frac{1}{Q\Delta} & ; \; -\frac{Q\Delta}{2} < x < \frac{Q\Delta}{2}, \\ 0 & ; \; \text{otherwise.} \end{cases}$$

$$\therefore \quad \text{Variance} \quad \overline{x}^2(t) = \int_{-\infty}^{\infty} x^2\cdot f_X(x)\, dx$$

$$= \int_{-Q\Delta/2}^{Q\Delta/2} x^2\cdot\frac{1}{Q\Delta}\cdot dx$$

$$\frac{1}{Q\Delta} = \frac{1}{Q\cdot\frac{V_{pp}}{Q}} = \frac{1}{V_{pp}}$$



$$f_X(x) = \frac{1}{2V_{max}}$$

$$= \frac{1}{Q\Delta}\cdot\left.\frac{x^3}{3}\right|_{-Q\Delta/2}^{+Q\Delta/2}$$

$$= \frac{1}{3Q\Delta}\left\{\left[\frac{Q\Delta}{2}\right]^3 + \left[\frac{Q\Delta}{2}\right]^3\right\}$$

$$= \frac{1}{3Q\Delta}\cdot\frac{Q^3\cdot\Delta^3}{4} \Rightarrow \boxed{\overline{x}^2(t) = \frac{Q^2\Delta^2}{12}}$$

$\therefore$ Signal output power $E\{x_0^2(t)\} = \frac{1}{T_s^2}\cdot\frac{Q^2\Delta^2}{12}$

$$\therefore \boxed{E\{[x_0(t)]^2\} = \frac{Q^2}{12}\cdot\frac{\Delta^2}{T_s^2}}$$

∴ The signal to quantization noise ratio (SQNR) of PCM S/m is

$$\left(S/N\right)_{QN} \text{ a } SQNR = \frac{E[x_0^2(t)]}{E[n_q^2(t)]}$$

$$= \frac{Q^2 \cdot \cancel{\frac{\Delta^2}{12 \cdot T_S^2}}}{\cancel{\frac{\Delta^2}{12 \cdot T_S^2}}}$$

$$\boxed{(S/N)_{QN} = Q^2}$$

$$= (2^N)^2 \quad (\because Q = 2^N)$$

For PCM system $\boxed{SQNR = 2^{2N}}$, $\boxed{(SQNR)\,dB = 6N}$

where N - no-of bits / sample.

Channel noise in PCM:

✓ Channel noise is measured in terms of average probability of symbol error it also called as 'error rate'.

ie The deviation of the recovered receiver output from the original message signal on an average is called average probability of symbol error.

* The main effect of channel noise introduces bit error into the received signal in binary PCM.

✓ The presence of a bit error causes symbol '1' to be mistaken for symbol '0' & viceversa. This deviation is called 'error rate'.

* In order to calculate the effect of bit errors induced by channel noise consider a PCM system using N-bit code word ie 2^N quantization levels.

✓ The most negative level is represented by

00000 000

✓ the next higher quantization level is represented by

00000 001

✓ The most possible positive level is represented by

$$1\,1\,1\,1\,1 \,\ldots\ldots\, 1\,1\,1 .$$

* Due to the channel noise bit error will be introduce.

Let us consider bit error is occured as 'Δ'

The error in the next (First) significant bit occurs an error is 2Δ, and the error i^{th} bit position causes an error of

$2^{i-1}.\Delta$ ie $\Delta, 2.\Delta, 2^2.\Delta, 2^3\Delta, \ldots, 2^{i-1}.\Delta$.

Let the error be represented by Q_Δ as

$$Q_\Delta = 2^{i-1}.\Delta$$

An error may occur any one of the N-bits in the codeword the variance of the error is.

$$E[\overline{Q_\Delta}]^2 = \frac{1}{N}\left[\Delta^2 + (2\Delta)^2 + (2^2\Delta)^2 + (2^3\Delta)^2 + \ldots (2^{N-1}\Delta)^2\right]$$

mean square value

$$= \frac{1}{N}.\Delta^2\frac{[(2^2)^N - 1]}{2^2-1}$$

$$= \frac{\Delta^2}{N}.\frac{2^{2N}-1}{4-1}$$

$$= \frac{\Delta^2}{N}.\frac{2^{2N}-1}{3}$$

$$= \frac{\Delta^2}{N}.\frac{2^{2N}}{3}$$

$$= \frac{\Delta^2}{N}.\frac{2^{2N}}{3} \quad [\because N \text{ is very large } 2^{2N} \gg 1]$$

Geometric progression

$r = \frac{a_2}{a_1} = \frac{(2\Delta)^2}{\Delta^2} = 4 = 2^2$

Sum of the N terms.

$S_n = a.\frac{r^{n}-1}{r-1}$

$a = \Delta^2$, $r = 2^2$ $\quad S_n = \frac{\Delta^2.2^{2N}-1}{2^2-1}$

(From geometric progression)

$$\therefore \boxed{E[Q_\Delta]^2 = \frac{\Delta^2}{N}.\frac{2^{2N}}{3}}$$

✓ The bit errors due to channel noise lead to incorrect values of $X_2(kT_s)$ as impulse sequence;

These errors appears as impulses of random amplitude & random time.

The mean seperation between bit error is $\frac{1}{P_e}$ bits.

Where P_e - Probability of bit error.

Since there are N-bits in codeword.

The mean seperation in code that are error is $\frac{1}{NP_e}$ bits.

$\therefore$ The mean time $T = T_s \times \frac{1}{NP_e}$ where T_s - Sampling period.

$\therefore$ The PSD of the noise power of channel noise

$$G_\Delta(f) \text{ \& } G_n(f) = \frac{E[Q_\Delta]^2}{T}$$

$$= \frac{\Delta^2}{\cancel{N}} \cdot \frac{2^{2N}}{3} \cdot \frac{\cancel{N} P_e}{T_s}$$

$$G_\Delta(f) = \frac{\Delta^2 \cdot 2^{2N} \cdot P_e}{3T_s}$$

$\therefore$ At the output of ideal LPF produces an average noise power $\therefore$

$$E\{n_o^2(t)\} = N_0 = \int_{-f_m}^{f_m} G_\Delta(f) \cdot df$$

$$= \int_{-f_m}^{f_m} \frac{\Delta^2 \cdot 2^{2N}}{3T_s} \cdot P_e \cdot df$$

$$= \frac{\Delta^2 \cdot 2^{2N}}{3T_s} \cdot P_e \cdot (f)_{-f_m}^{+f_m}$$

$$= \frac{\Delta^2 \cdot 2^{2N}}{3T_s} \cdot P_e \cdot (2f_m)$$

Channel noise power

$$\boxed{E\{[n_o(t)]^2\} = \frac{\Delta^2 \cdot 2^{2N} \cdot P_e}{3 \cdot T_s^2}} \quad (\because f_s = 2f_m, \ \frac{1}{T_s} = f_s.)$$

The overall SNR of the PCM system is given by

$$* \ (SNR)_{PCM} = \frac{E\{[x_o(t)]^2\}}{E[(n_q(t))^2] + E[(n_o(t))^2]}$$

✓ $E\{(x_o(t))^2\} = \frac{Q^2}{12} \cdot \frac{\Delta^2}{T_s^2}$

✓ $E\{[n_q(t)]^2\} = \frac{\Delta^2}{12 \cdot T_s^2}$

✓ $E\{[n_o(t)]^2\} = \frac{\Delta^2 \cdot 2^{2N}}{3T_s^2} \cdot P_e$.

$$(S/N)_o = (SNR)_o = \frac{\frac{Q^2}{12}\cdot\frac{\Delta^2}{T_s^2}}{\frac{\Delta^2}{12\cdot T_s^2} + \frac{\Delta^2}{T_s^2}\cdot\frac{2^{2N}\cdot Pe}{3}}$$

$$= \frac{\cancel{\frac{\Delta^2}{T_s^2}}\cdot\left(\frac{Q^2}{12}\right)}{\cancel{\frac{\Delta^2}{T_s^2}}\left[\frac{1}{12} + \frac{2^{2N}\cdot Pe}{3}\right]}$$

$$= \frac{Q^2/\cancel{12}}{\frac{1 + 4\cdot 2^{2N}\cdot Pe}{\cancel{12}}}$$

$$= \frac{Q^2}{1+4\cdot 2^{2N}\cdot Pe}$$

$$= \frac{Q^2}{[1+4\cdot 2^{2N}\cdot Pe]}$$

$$\boxed{(SNR)_{PCM} = \frac{2^{2N}}{1+4Pe\cdot 2^{2N}}} \qquad (\because Q = 2^N)$$

The above SNR shows the overall signal to quantization noise plus channel noise for the PCM system.

Case (i): For small noise case

$$4Pe\,2^{2N} << 1$$

& $4Pe\,2^{2N}$ can be neglected

$$\therefore (SNR)_o = 2^{2N} = 6N \text{ in dB}$$

$$\therefore \boxed{(SNR)_o = 2^{2N}} \qquad \boxed{(SNR)_{o\ \text{in dB}} = 6N}$$

Case (ii) For large noise case

$$4Pe\cdot 2^{2N} >> 1$$

'1' can be neglected.

$$\therefore (SNR)_o = \frac{\cancel{2^{2N}}}{4Pe\,\cancel{2^{2N}}} = \frac{1}{4Pe} \qquad \therefore \boxed{(SNR)_o = \frac{1}{4Pe}}$$

Where Pe - Probability of bit error, N - no of bits / sample.

Differential Pulse Code Modulation (DPCM):

- When a voice & Video signals are sampled. It is usually found that adjacent samples are highly correlated (ie close to the same value ie the signal does not change rapidly from one sample to the next sample).

* When these high correlated samples are encoded as in a standard PCM system, the resultant encoded signal contains "redundant Information"

- Due to redundant information (sample values), consequently the bandwidth of a PCM system is wasted when redundant sample values are retransmitted.

* To minimize the redundant data transmission and to reduce the bandwidth is to transmit samples corresponding to the difference in adjacent sample value, DPCM system is used.

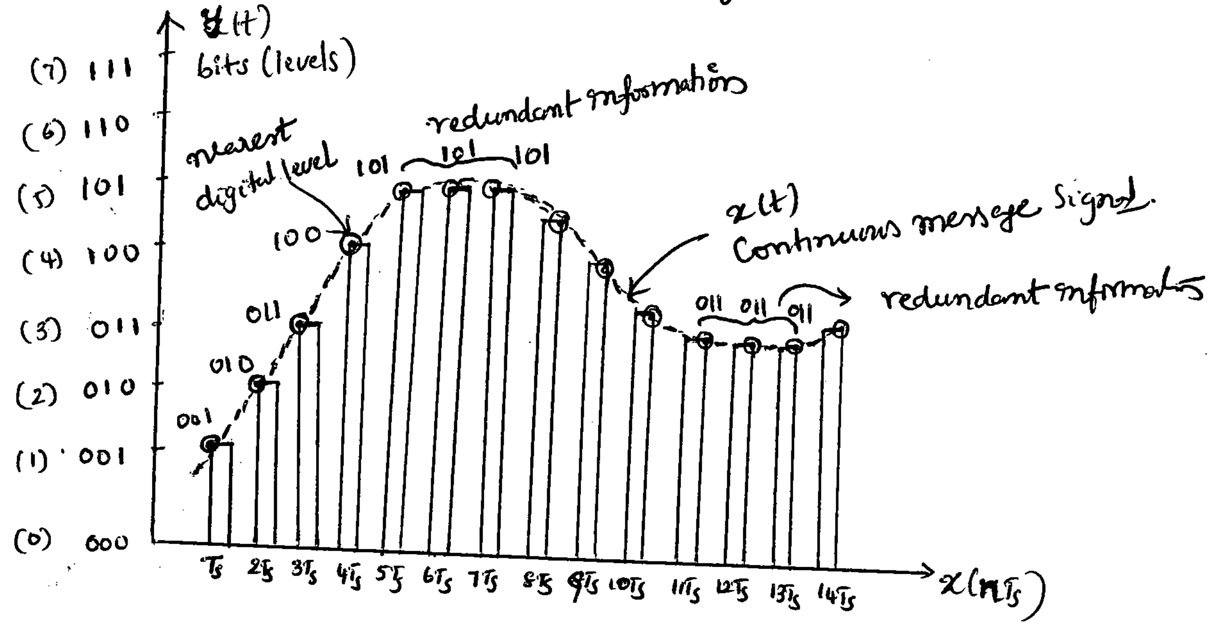


Fig: Redundant information in PCM

ie
- The signal is sampled by flat-top sampling at intervals $T_s, 2T_s, \dots nT_s$
- The samples are encoded by using 3 bit (7 levels) PCM.
- The samples are quantized to the nearest digital level.

ie The samples at $5T_s, 6T_s, 7T_s$ are encoded to same value of (101). This information carried only one sample, but these samples are carrying same information means that it is redundant.

* If redundancy is reduced, then overall bit rate will decrease and no. of bits required to transmit one sample will also be reduced and also Bandwidth will be reduced.

This type of digital pulse modulation scheme is called DPCM ie Differential pulse code Modulation.

✓ In DPCM system uses the difference between the original sample value and the predicted sample value.

ie The value of the present sample is predicted from the past samples.

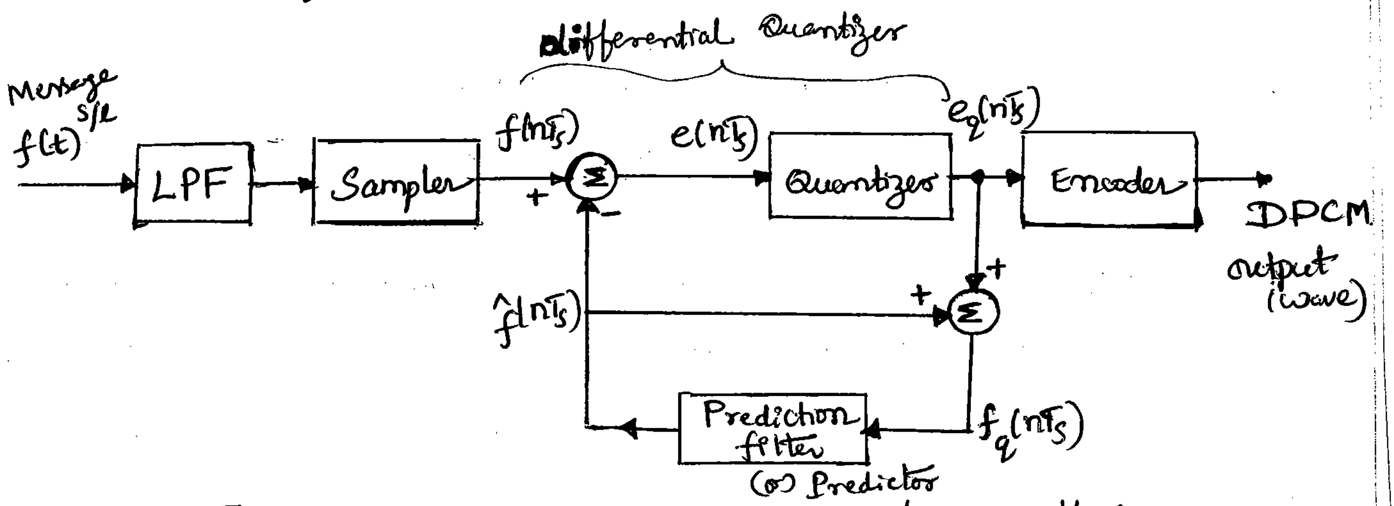


Fig: Block diagram of DPCM - transmitter.

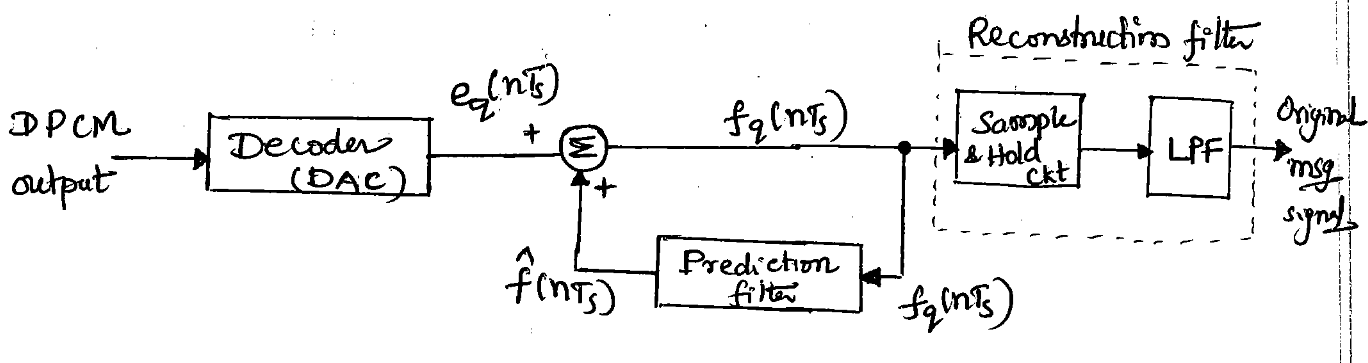
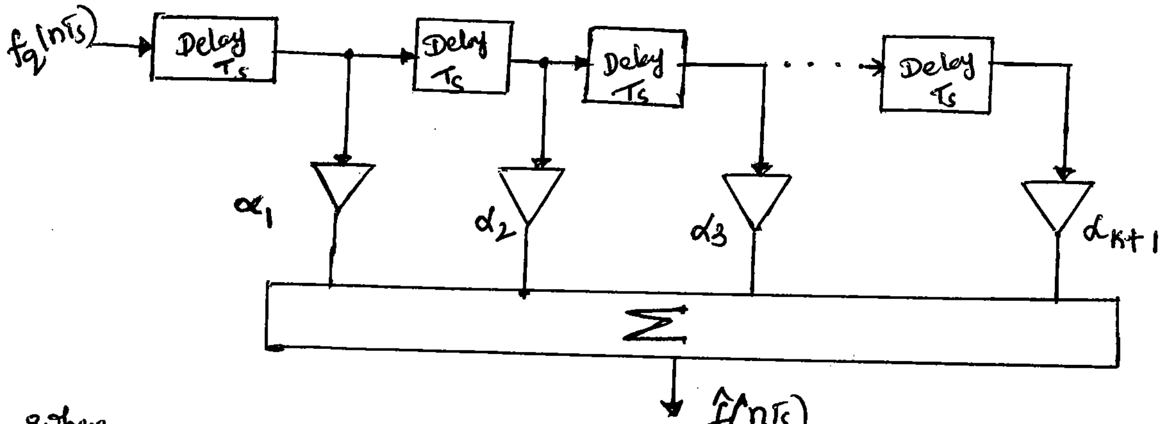


Fig. Block diagram of DPCM Receiver.

Predictor: (or) Prediction filter is a moderately sofisticated system it will need to incorporate the facility for storing past differences for carrying out some algorithms to predict next required increment.

* It is used for both transmitter and receiver of the DPCM system.

✓ Predictor can be realized by using tapped delay line filters.
ie The predicted value is modelled as a linear combination of past values of the quantized input.



where
$\alpha_1, \alpha_2, \alpha_3, \ldots\ldots \alpha_{K+1}$ are tape gains.

Fig: Tapped delay line filter used as a predictor.

$$\therefore \hat{f}(nT_s) = \alpha_1 f_q(nT_s) + \alpha_2 f_q[(n-1)T_s] + \alpha_3 f_q[(n-2)T_s] + \cdots + \cdots + \alpha_{K+1} f_q[(n-K)T_s].$$

Thus, Error is the difference between unquantized input signal $f(nT_s)$ and prediction of it $\hat{f}(nT_s)$.

✓ The predicted value is produced by using a prediction filter.

✓ The quantizer output signal gap $e_q(nT_s)$ and previous prediction is added and given as input to the prediction filter. This signal is called $f_q(nT_s)$.

* This makes the prediction more and more close to the actual sampled signal.

We can observe that the quantized error signal $e_q(nT_s)$ is very small and can be encoded by using small no of bits.

Thus the no-of bits per sample are reduced in DPCM.

The quantizer output can be written as

$$\boxed{e_q(nT_s) = e(nT_s) + Q_e(nT_s)} \longrightarrow (1)$$

where $Qe(nT_s)$ is the quantization error.

∴ The prediction filter input $f_q(nT_s)$ as

$$f_q(nT_s) = \hat{f}(nT_s) + e_q(nT_s) \longrightarrow (2)$$

From (1) & (2) $$f_q(nT_s) = \hat{f}(nT_s) + e(nT_s) \pm Qe(nT_s) \longrightarrow (3)$$

∴ The input of the Quantizer is written as

$$e(nT_s) = f(nT_s) - \hat{f}(nT_s) \longrightarrow (4)$$

From (3) & (4) $$f_q(nT_s) = \cancel{\hat{f}(nT_s)} + f(nT_s) - \cancel{\hat{f}(nT_s)} \pm Qe(nT_s)$$

$$\therefore \boxed{f_q(nT_s) = f(nT_s) \pm Qe(nT_s)}$$

Hence The quantized version of the signal $f_q(nT_s)$ is the sum/difference of original sample value & Quantization error.

∴ The quantization error can be positive or negative.

Thus $f_q(nT_s)$ doesnot depends on prediction filter characteristics.

Signal to Noise Ratio in DPCM :

- ✓ The power of message sequence $f(nT_s)$ is given by P_m
- ✓ The power of quantizing error sequence $q_e(nT_s)$ is given by 'P_q'

The output signal to Quantization noise ratio is

$$(SNR)_o = \frac{P_m}{P_q}$$

$$= \frac{P_m}{P_e} \times \frac{P_e}{P_q}$$

where P_e - prediction error power

$$(SQNR)_0 = G_p \cdot (SQNR)_{uniform}$$

where G_p - prediction gain → produced by the differential quantization.

$G_p = \frac{P_m}{P_e}$ is greater than unity.

$$(SQNR) = \left(\frac{V_{max}}{V_{dmax}}\right)^2 \times (SQNR)_{uniform}$$

where V_{max} - peak amplitude of original message signal

V_{dmax} - peak amplitude of difference signal.

Delta Modulation (DM): [Step Size is fixed]

- Delta modulation is also called Single bit modulation.
- * It is also a 1-bit PCM system (or) 1-bit DPCM system.
- PCM transmits all the bits which are used to code a sample So, bit rate and transmission channel bandwidth are large in PCM. To overcome this problem Delta modulation is used.
- * 1-bit Quantizer is equivalent to a two-level comparator.
- The block diagram is similar to DPCM but only change occured is in the place of a predictor we are using a single delay 'T_s'
- * The difference between the input and delayed value is quantized into only two levels namely '$+\Delta$' and '$-\Delta$'
- If the difference is positive, then approximated signal is increased by one step i.e 'Δ'

  If the difference is negative, then approximated signal reduced by one step 'Δ'.
- * When the step is reduced, '0' is transmitted and when the step is increased '1' is transmitted.

Hence For each sample, only one binary bit is transmitted. & step Size is fixed in delta modulation.

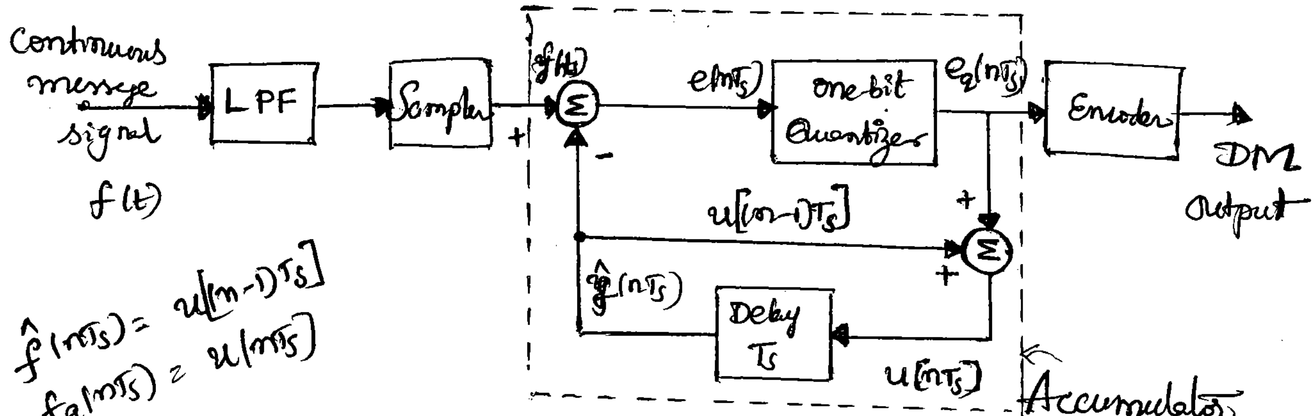


Fig : DM - Transmitter.

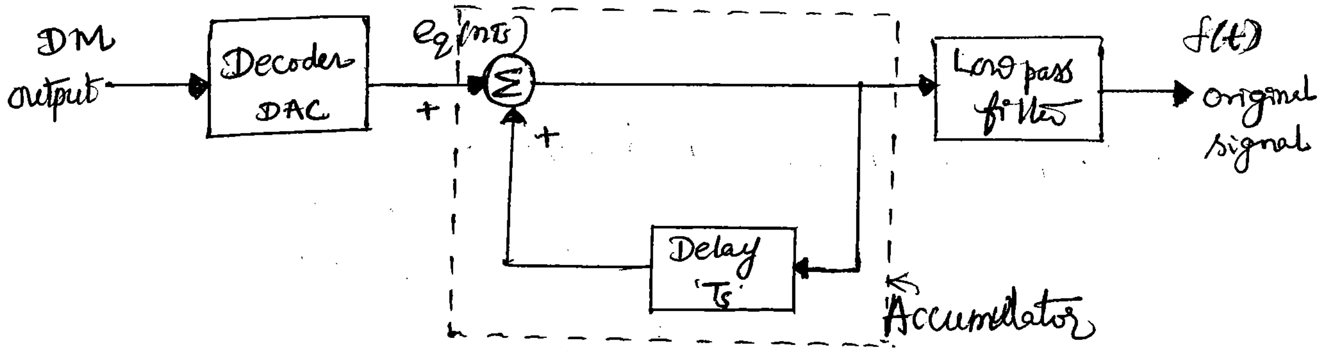


Fig: DM - Receiver

where $f(nT_s)$ - Sampled signal of $f(t)$

$\hat{f}(nT_s)$ - last sample approximation of the staircase waveform

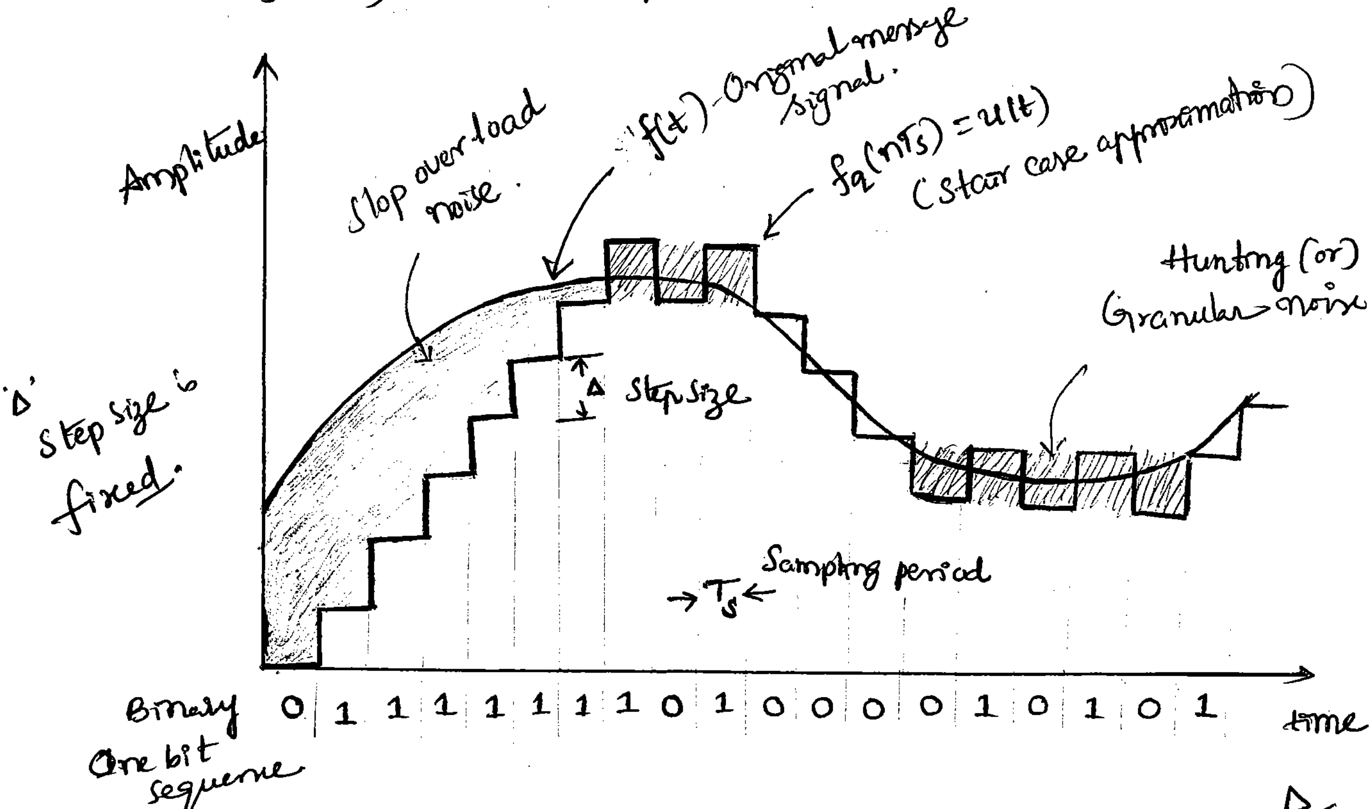


Fig : Delta modulation waveform

$$\text{slope} = \frac{\Delta}{T_s}$$

- In delta modulation, No. of bits/sample $\boxed{N=1}$

$\therefore$ No. of quantization levels $Q = 2^N = 2^1 \therefore \boxed{Q=2}$

Bit rate $R_b = N \times f_s = 1 \cdot f_s \Rightarrow \boxed{f_s = R_b}$

* When the step is reduced, '0' is transmitted, when the step is increased, '1' is transmitted, for each sample, only one bit is transmitted.

If $f(nT_s) \geq \hat{f}(nT_s)$ - $e_q(nT_s) = +\Delta \rightarrow$ '1'

If $f(nT_s) < \hat{f}(nT_s)$, $e_q(nT_s) = -\Delta$, $\rightarrow$ '0'

Hardware Implementation of Deltamodulation:

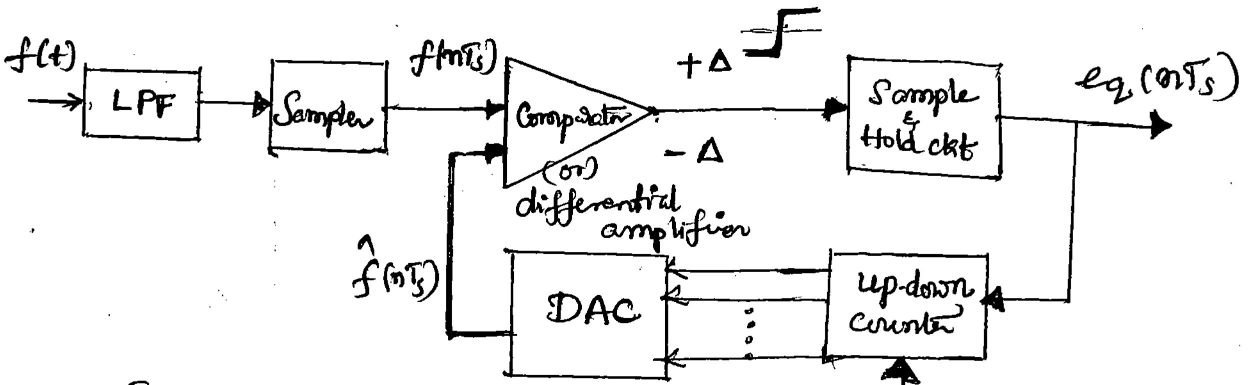


Fig: Transmitter

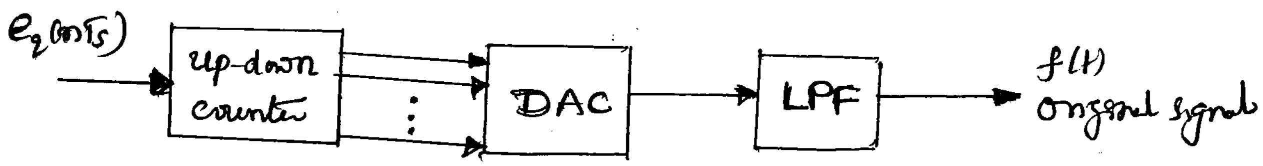


Fig: Receiver.

- **Comparator**: If $f(t) \geq \hat{f}(nT_s)$ then output $V(H)$ - logic-1

If $f(nT_s) < \hat{f}(nT_s)$ then output $V(L)$ - logic-0.

- **Sample and hold ckt**: It takes the sample of comparator output and maintain (Hold) the same peak value upto the completion of one clock cycle.

- **Up down Counter**: It increaments or decrements its count by '1' at each active edge of the clock waveform.

✓ DAC - Digital to analog Counter : The output of counter is 'n'-bits this n-bits applied to DAC, DAC Output ~~is~~ is say 5V.

ie If counter is creamented by 1 then output of DAC is incremented by 5V.
If Counter is decremented by 1 then output of DAC is decremented by 5V.

LPF : Low pass filter has the cut off frequency equal to the highest frequency of f(t).

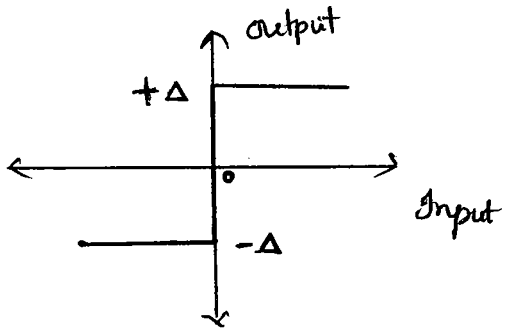
✓ It smoothens the stair case signal to reconstruct original message signal f(t).

ie At transmitter → Sampler. onebit quantizer & accumulator are interconnected

Initially set zero.

At receiver → Decoder, accumulator, LPF.

The input-output characteristics of two level quantizer as



Advantages of DM :

✓ The DM transmits only one bit for one sample, so the signalling rate and transmission channel Bandwidth is quite small.

✓ The transmitter & receiver implementation is very much simpler.

Disadvantages of DM :

The deltamodulation has two major drawbacks as under

(a) Slope over load distortion/noise

(b) Granular or idle or Hunting noise/distortion.

Let us consider the message signal m(t) & f(t) and the quantized approximate signal $\hat{f}_q(nT_s)$ & $\bar{m}(t)$ then the distortions as follows.

(a) Slope overload distortion :

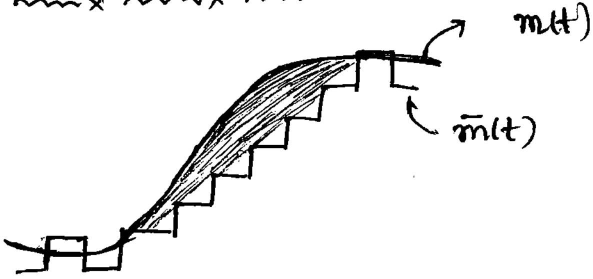


Fig (a) Slope over load (positive)

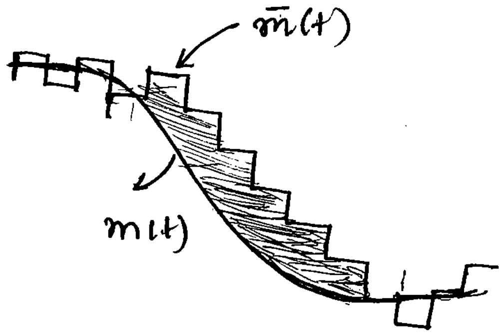


Fig (b) Slope over load (Negative)

✓ The slope of the deltamodulation is $\text{Slope} = \frac{\Delta}{T_s}$ & Δf_s.

where Δ - Step size

T_s - Sampling period.

Fig (a). The waveform $\bar{m}(t)$ is unable to follow $m(t)$ because the slope of $m(t)$ is greater than the slope of $\bar{m}(t)$ i.e slope is positive

Fig (b). The slope of $m(t)$ is more negative than the slope of $\bar{m}(t)$.

In both cases the recovered waveform will be distorted, The delta modulation is, then said to be have slope over load distortion.

To avoid slope over load error, condition that the slope of $\bar{m}(t)$ & $\hat{f}_q(nT_s)$ should be greater than or equal equals to maximum slope of message $m(t)$ & $f(t)$.

$\therefore$ The slope of $\bar{m}(t) = \frac{\Delta}{T_s} = \Delta \cdot f_s$.

The slope of $m(t) = \frac{d}{dt}(m(t))$

where $m(t) = A_m \sin \omega_m t$.

$$\therefore \frac{\Delta}{T_s} \geq \max. \frac{d}{dt}(m(t))$$

$$\geq \max. \frac{d}{dt}\left[A_m \sin 2\pi f_m t\right]$$

$$\geq \max. \left[A_m \cdot \cos 2\pi f_m t \cdot 2\pi f_m\right]$$

$$\Delta f_s = \frac{\Delta}{T_s} \geq 2\pi f_m \cdot (A_m)_{max} \qquad (\because \cos\ [-1, \underset{\uparrow max}{1}])$$

$$\therefore \boxed{(A_m)_{max} \ \& \ (V_m)_{max} \leq \frac{\Delta f_s}{2\pi f_m}}$$

The permissiable value of the output signal power is

$$P_{max} = \frac{\left(\frac{V_{max}^2}{2}\right)}{R} \ \& \ \frac{(A_{max})^2}{2} = \frac{\Delta^2 f_s^2}{4\pi^2 f_m^2 \times 2} \qquad \left(\because \left[\frac{V_{max}}{\sqrt{2}}\right]^2 \hookrightarrow \text{rms values}\right)$$

$$\therefore \boxed{P_{max} = \frac{\Delta^2 f_s^2}{8\pi^2 \cdot f_m^2}}$$

(b) Granular & Idle & hunting noise :

In both cases the variations in $m(t)$ are such that they are within the step size. Hence the waveform $\bar{m}(t)$ is like a square wave. This will be recovered as d.c whereas the original signal $m(t)$ is not dc. Thus, in this case also distortion is resulted and noise is known as granular & idel & hunting noise.

- To overcome these error Adaptive delta modulation is used.

Noise in Delta Modulation :

The output of the receiver differs from the input of the transmitter because of two noises. They are

(a) Quantization noise $[n_q(t)]$

(2) Channel noise $[n_c(t) \ \& \ n_o(t)]$

$\therefore$ The output consists at Receiver is $\bar{X}(t) = \underbrace{X_o(t)}_{\text{Signal}} + \underbrace{n_q(t)}_{\text{Quantization noise}} + \underbrace{n_c(t)}_{\text{Channel noise}}$

$\therefore$ The overall SNR of delta modulation is

$$(SNR)_{DM} = \frac{E\{(X_o(t))^2\}}{E\{[n_q(t)]^2\} + E\{[n_c(t)]^2\}} \quad (\text{or}) \quad \frac{S_o}{N_q + N_c}.$$

Signal power (S_o):

For calculating signal power we have a limitation that the slope of the signal waveform must be observed if slope overload error is to be avoided

Consider that there is no slope over load error, condition is

slope of $\hat{f}(nT_s) \geq$ max. slope of $f(t)$

let message signal $f(t) = V_m \sin 2\pi f_m t$ (∵ V_m & A_m)

& Staircase approximated signal is $\hat{f}(nT_s)$

$$\therefore \quad \frac{\Delta}{T_s} \geq \max \frac{d}{dt}(f(t))$$

$$\frac{\Delta}{T_s} \geq \max \frac{d}{dt}(V_m \sin 2\pi f_m t)$$

$$\frac{\Delta}{T_s} \geq (V_m)_{max} \cdot (\cos 2\pi f_m t \cdot)_{max} \cdot 2\pi f_m$$

$$\frac{\Delta}{T_s} = 2\pi f_m \cdot (V_m)_{max} \cdot \qquad (\because \cos \text{ fun } [-1\ 1])$$

$$\therefore (V_m)_{max} = \frac{\Delta / \cancel{T_s}}{2\pi f_m} = \frac{\Delta f_s}{2\pi f_m} = \left(\frac{\Delta}{2\pi}\right)\left(\frac{f_s}{f_m}\right)$$

∴ R.M.S power of signal component is

$$S_o = E\{x_o^2(t)\} = \left(\frac{(V_m)_{max}}{\sqrt{2}}\right)^2 \Big/ R = \frac{[V_m(max)]^2}{2} \qquad (R=1)$$

$$= \left(\frac{\Delta}{2\pi}\right)^2 \left(\frac{f_s}{f_m}\right)^2 \times \frac{1}{2}$$

$$= \frac{\Delta^2}{4\pi^2} \times \frac{f_s^2}{f_m^2} \times \frac{1}{2}$$

$$S_o = \frac{\Delta^2}{8\pi^2}\left(\frac{f_s}{f_m}\right)^2$$

$$\therefore \text{Signal power} \quad \boxed{S_o = E\{[X_o^2(t)]\} = \frac{\Delta^2}{8\pi^2}\cdot\left(\frac{f_s}{f_m}\right)^2}$$

Quantization Noise Power: (N_q).

To estimate quantization noise power in delta modulation consider error waveform

$\therefore$ Quantization error $\quad e_q(t) = f(t) - \hat{f}_q(t) \leq \Delta$

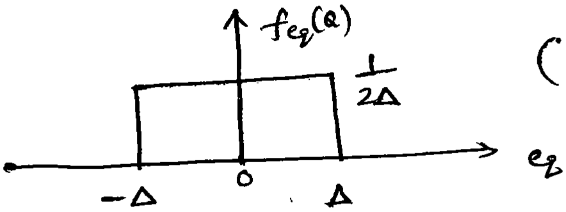
where $f(t)$ - original message signal

$\hat{f}_q(t)$ - Staircase approximated signal.

Consider in the absence of slope overload error.

let an uniform PDF of $e_q(t)$ as $\quad f_{e_q}(Q) = \begin{cases} \frac{1}{2\Delta}, & -\Delta \leq e_q \leq \Delta \\ 0, & \text{otherwise.} \end{cases}$

ie



$\left(\frac{1}{b-a} = \frac{1}{\Delta-(-\Delta)} = \frac{1}{2\Delta} \right)$

$\therefore$ The power of noise (error component) is mean square value of $e_q(t)$

$$\therefore E\{[e_q^2(Q)]\} = \int_{-\infty}^{\infty} q^2 \cdot f_{e_q}(Q)\, dq$$

$$= \int_{-\Delta}^{\Delta} q^2 \cdot \frac{1}{2\Delta} \cdot dq$$

$$= \frac{1}{2\Delta} \cdot \frac{q^3}{3} \Big|_{-\Delta}^{\Delta}$$

$$= \frac{1}{6\Delta} (\Delta^3 + \Delta^3)$$

$$= \frac{2\Delta^3}{6\Delta} = \frac{\Delta^2}{3}$$

$$\therefore \boxed{E\{[e_q^2(t)]\} = \Delta^2/3.}$$

The normalized total power of the waveform $e_q(t)$ is spread over the bandwidth 'f_s' & its is uniformly distributed over interval $(-f_s, f_s)$.

ie The PSD of $e_q(t)$ is $\quad G_{e_q}(f) = \begin{cases} \frac{\Delta^2}{3} \cdot \frac{1}{2f_s} & ; -f_s \leq f \leq f_s \\ 0 & ; \text{otherwise.} \end{cases}$

Now at the output of LPF, the Quantization noise power with LPF bandwidth is equal to the input signal bandwidth '$-f_m$ to f_m'

$\therefore$ Quantization noise power N_q as

$$N_q = E\{[n_q(t)]^2\} = \int_{-\infty}^{\infty} G_{eq}(f) \cdot df$$

$$= \int_{-f_m}^{f_m} \frac{\Delta^2}{6 f_s} \cdot df$$

$$= \frac{\Delta^2}{6 f_s} \cdot (f)_{-f_m}^{f_m}$$

$$= \frac{\Delta^2}{6 f_s} \cdot 2 f_m$$

$$= \frac{\Delta^2 \cancel{2} f_m}{3 \cancel{6} f_s}$$

$$\therefore \left(\frac{\Delta^2}{3}\right)\left(\frac{f_m}{f_s}\right)$$

$\therefore$ Quantization noise power $\boxed{N_q = E\{[n_q^2(t)]\} = \left(\frac{\Delta^2}{3}\right)\left(\frac{f_m}{f_s}\right)}$

Signal to Quantization noise in Delta modulation is

$$(SQNR)_{DM} = \frac{S_o}{N_q}$$

$$= \frac{\frac{\Delta^2}{8\pi^2} \cdot \left(\frac{f_s}{f_m}\right)^2}{\frac{\Delta^2}{3} \cdot \left(\frac{f_m}{f_s}\right)}$$

$$= \frac{3}{8\pi^2} \cdot \left(\frac{f_s}{f_m}\right)^2 \cdot \left(\frac{f_s}{f_m}\right)$$

$$= \frac{3}{8\pi^2} \cdot \left(\frac{f_s}{f_m}\right)^3$$

$$(S/N_q)_{DM} \text{ (or) } \boxed{(SQNR)_{DM} = \frac{3}{8\pi^2} \cdot \left(\frac{f_s}{f_m}\right)^3}$$

where f_s - Sampling Rate

f_m - modulating signal frequency (or) Cutoff freq of LPF.

Channel Noise power N_c (or) N_o :

The channel noise a thermal noise can be occured at the end of transmitter and beginning of receiver.

The mean time seperation between the impulses/samples is assumed to be equal to T_s/P_e ⇒ $\tau = \frac{T_s}{P_e} = \frac{1}{f_s \cdot P_e}$ → probability error.

The PSD of the impulse train can be assumed to be white noise with a magnitude of $4\Delta^2 \cdot P_e \cdot f_s$ $\left(\because \left(\frac{2\Delta}{\tau}\right)^2\right)$.

The PSD of the channel error noise at the input of the filter (LPF) is given by

$$G_{th}(f) = \frac{4\Delta^2 \cdot P_e \cdot f_s}{\omega^2} = \frac{4\Delta^2 \cdot P_e \cdot f_s}{(2\pi f)^2}$$

$$= \frac{\cancel{4}\Delta^2 \cdot P_e f_s}{\cancel{4}\pi^2 \cdot f^2}$$

$$\boxed{G_{th}(f) = \frac{\Delta^2}{\pi^2} \cdot P_e \cdot \left(\frac{f_s}{f^2}\right)}$$

If $f \to 0$ then $G_{th}(f) \to \infty$ & the integral of $G_{th}(f)$ over a range of frequencies including $f=0$ is infinite.

LPF will have a low frequency cut off 'f_1' > zero and a high frequency cut off 'f_x'. ($f_1 < f_x$).

∴ Average noise power $= 2\int_{f_1}^{f_x} G_{th}(f) \cdot df$

i.e. Channel noise power

$$N_c = E\{[n_o^2(t)]\} = 2\int_{f_1}^{f_x} \frac{\Delta^2}{\pi^2} \cdot P_e \cdot f_s \cdot \left(\frac{1}{f^2}\right) \cdot df$$

$$= \frac{2\Delta^2}{\pi^2} \cdot P_e \cdot f_s \cdot \left(\frac{f^{-1}}{-1}\right)_{f_1}^{f_x}$$

$$= \frac{2\Delta^2}{\pi^2} \cdot P_e f_s \cdot \left(\frac{1}{f_x} - \frac{1}{f_1}\right) \qquad \left(\because \int f^{-2} = \frac{f^{-1}}{-1}\right)$$

$$\therefore N_0 = \frac{2\Delta^2}{\pi^2} \cdot P_e \cdot f_s \cdot \left(\frac{1}{f_1} - \frac{1}{f_x}\right)$$

If $f_1 << f_x$ then $\frac{1}{f_1} >> \frac{1}{f_x}$ so, $\frac{1}{f_x}$ is neglected.

$\therefore$ Channel noise power $\boxed{N_0 = E\{[n_c(t)]^2\} = \frac{2\Delta^2}{\pi^2} \cdot P_e \cdot f_s \cdot \left(\frac{1}{f_1}\right)}$

$\therefore$ The over all SNR of the delta modulation as

$$(SNR)_{DM} = \frac{S_0}{N_q + N_0}$$

$$= \frac{\dfrac{\Delta^2 f_s^2}{8\pi^2 f_m^2}}{\dfrac{\Delta^2}{3} \cdot \dfrac{f_m}{f_s} + \dfrac{2\Delta^2}{\pi^2} \cdot P_e \cdot \dfrac{f_s}{f_1}}$$

$$= \frac{\cancel{\Delta^2} \cdot f_s^2 / 8\pi^2 f_m^2}{\dfrac{\cancel{\Delta^2} \cdot f_m}{3 f_s}\left[1 + \dfrac{2 \cancel{f_s} \cdot P_e}{\pi^2 f_1} \cdot \dfrac{3 \times f_s}{f_m}\right]}$$

$$= \frac{\dfrac{f_s^2 \times 3 f_s}{8\pi^2 f_m^2 \cdot f_m}}{1 + \dfrac{6 P_e}{\pi^2} \cdot \left(\dfrac{f_s^2}{f_1 f_m}\right)}$$

$$\boxed{\left(S/N\right)_{DM} \text{ or } (SNR)_{DM} = \frac{3}{8\pi^2}\left(\frac{f_s}{f_m}\right)^3 \cdot \frac{1}{\left[1 + \frac{6 P_e}{\pi^2} \cdot \left(\frac{f_s}{f_m}\right)^2\right]}} \quad (\because f_1 = f_m)$$

$(SQNR)_{DM} = \frac{3}{8\pi^2} \cdot \left(\frac{f_s}{f_m}\right)^3 \Rightarrow$ SQNR in dB $= 10 \log\left(\frac{3}{8\pi^2}\right) + 30 \log\left(f_s/f_m\right)$

If $f_s = 2f_m$ $(SQNR)_{dB} = -5$ dB $\downarrow +9$ dB

f_s doubling · If $f_s = 4f_m$ $(SQNR)_{dB} = +4$ dB

Hence. If $f_s = 8f_m$ $(SQNR)_{dB} = +13$ dB $\downarrow +9$ dB Increment

The max. output signal to noise ratio of a delta modulator is proportional to the sampling rate cubed.

$\therefore$ It indicates a 9 dB improvement with doubling of the sampling rate.

Adaptive Delta Modulation (ADM):

* To overcome the quantization errors due to slope overload distortion, and granular / idle noise.
* The step size (Δ) is made adaptive to variations in the input signal $f(t)$.
* Perticulary in the steep segment of the signal $f(t)$, the step size is increased also if the input is varying slowly, the step size is reduced.

Thus A delta modulation system which adjusts its step size is known as the adaptive delta modulation. (ADM) system.

Transmitter:

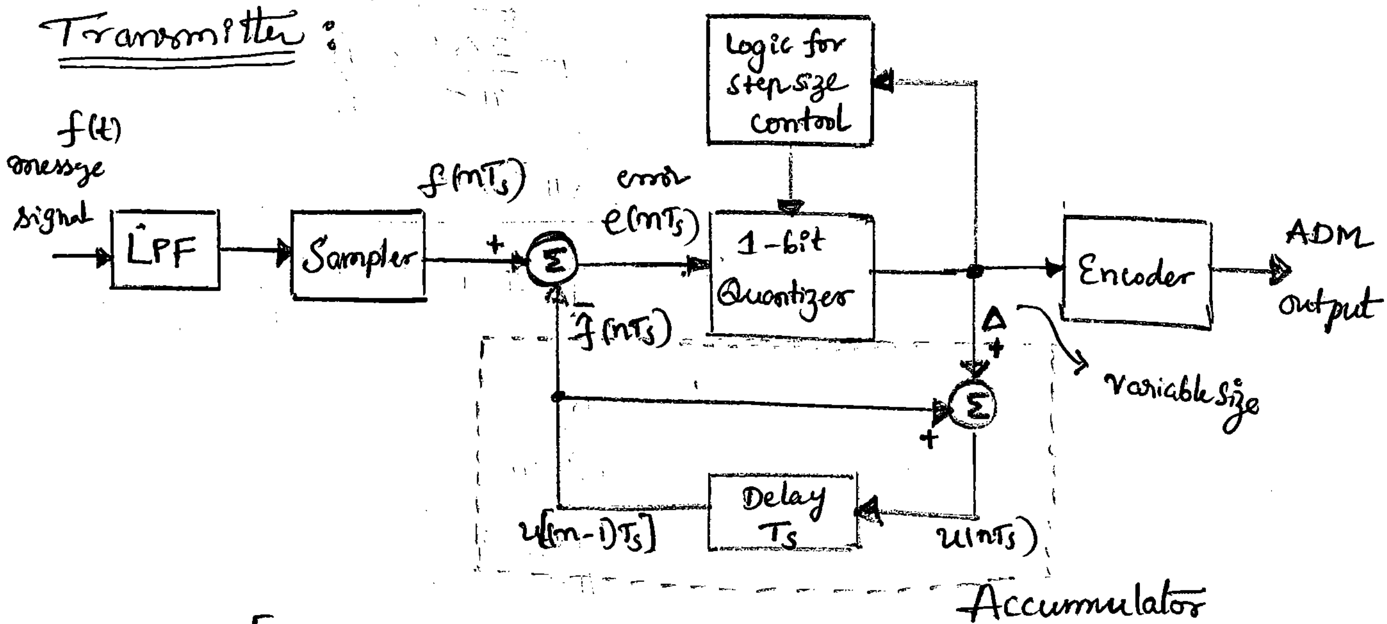


Fig: Adaptive Delta modulation - Transmitter.

Receiver:

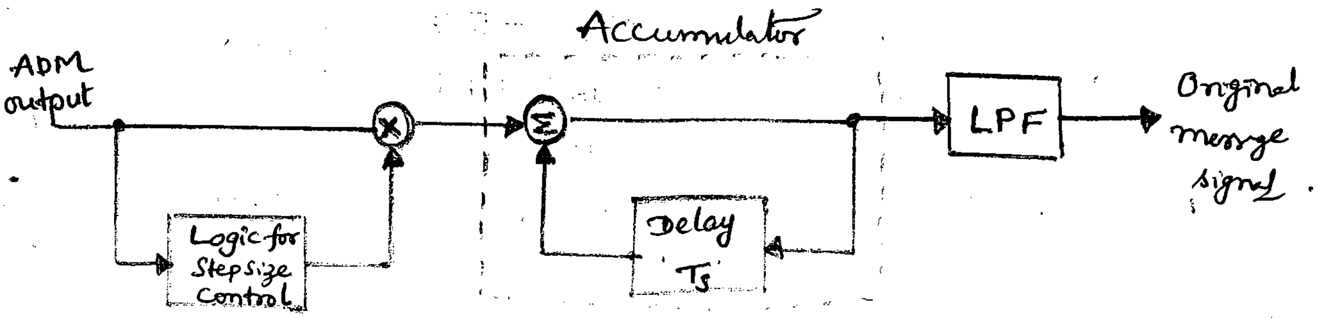


Fig: Adaptive Delta modulation - Receiver

* For both transmitter and receiver diagrams logic for step size control is added.

✓ The step size increases or decreases according to a specified rule depending on one bit quantizer output.

ie
* If one bit quantizer output is high (ie 1), then the step size may be doubled for next sample.
* If one bit quantizer output is low (ie 0), then the step size may be reduced by one step.

✓ The step size is constrained to lie between min. & max. values

ie $$\Delta_{min} \leq e_q(nT_s) \leq \Delta_{max}.$$

The upper limit 'Δ_{max}' controls the amount of slope overload distortion.

The lower limit 'Δ_{min}' controls the amount of granular/Idle noise.

Waveforms:

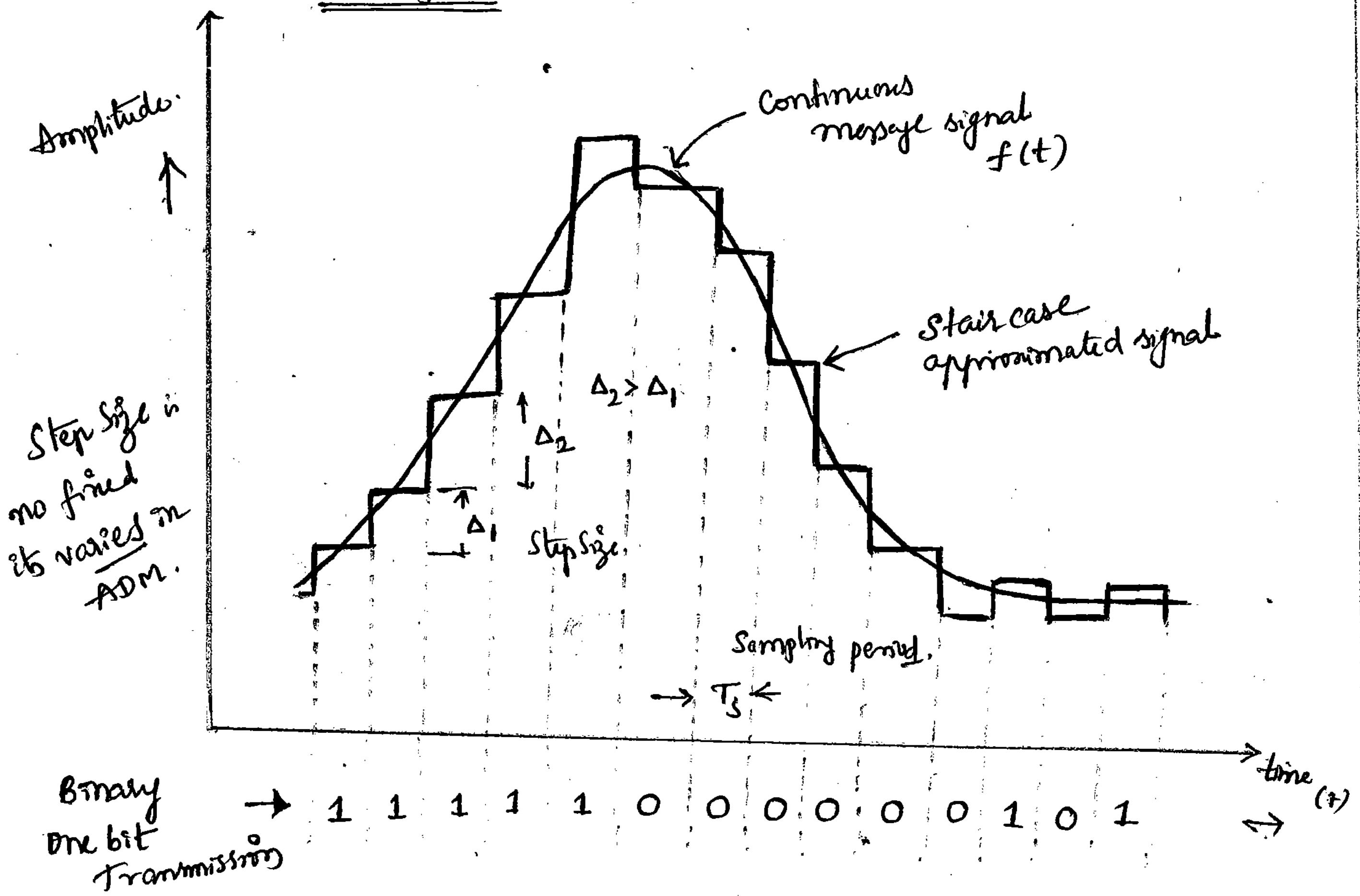
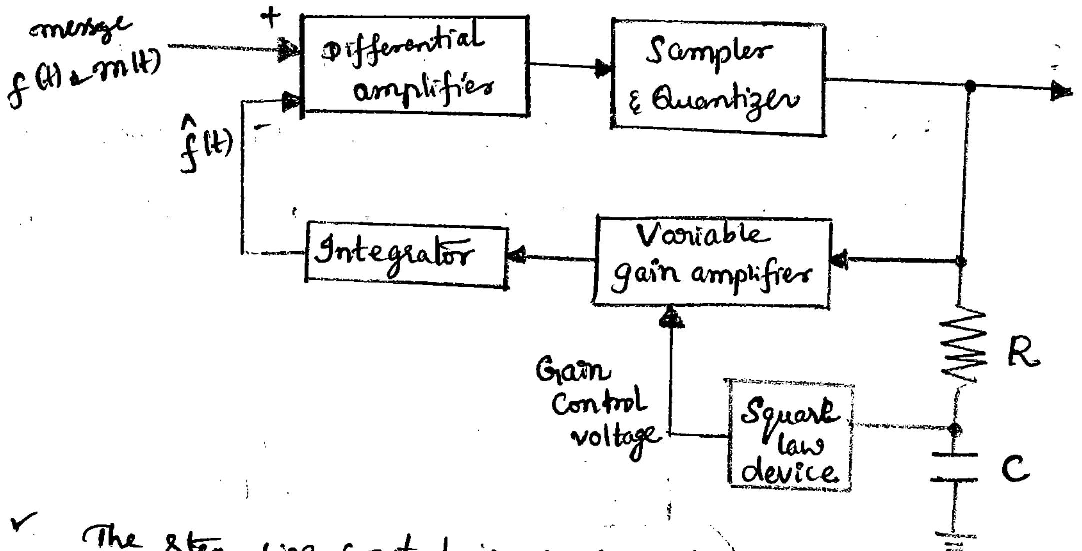


Fig: Waveform for Adaptive delta modulation.

✓ At the receiver, the first portion produces the step size from each incoming bit, the previous input and present input decides the step size. It is then applied to an accumulator which builds up stair case waveform

✓ The Low pass filter then smoothens out the staircase waveform to reconstruct the original signal.

Practical / Hardware Implementation of ADM : (or) CVSDM
Continuously Variable Slope Delta Modulation.



✓ The step size control is performed by a digital Integrator (accumulator) along with variable gain amplifier.

✓ The bit size is converted in to a voltage that is fed to a variable gain amplifier.

✓ The amplification is minimum when the input voltage corresponds to equal no. of 1's and 0's in the period.

✓ The amplifier controls the step size depending on the voltage feedback to it.

* The step size is varied by controlling the gain of the Integrator.
* The gain control ckt consists of a RC Integrator and a Square law device.
* When the modulator output is a sequence of alternate polarity pulses.
* these pulses when integrated by the RC filter yield an average output almost zero.

The gain control input and hence the gain & the step size are small.

Comparision of PCM, DPCM, DM and ADM Systems:

SL No	Parameter	PCM	DM	ADM	DPCM
1	Expansion	Pulse Code Modulation	Delta Modulation	Adaptive delta modulation	Differential pulse Code Modulation
2	No. of bits/sample (N)	4 (or) 8 (or) 16 (or) 32 bits	1-bit	1-bit	More than 1 but less than PCM.
3	Step size 'Δ'	Depends on no. of bits	Fixed.	Varied	Fixed.
4	Bandwidth	high (Since no. of bits are high).	Low	Low	Less than PCM.
5	No. of quantization level. $Q = 2^N$	High (∵ N is high) 2^N	$N = 1$ $Q = 2$	$N = 1$ $Q = 2$	Less than PCM 2^N.
6	Feedback in transmitter (or) Receiver	No feedback	Feedback Exists	feedback Exists	Feedback Exists.
7	Bit rate R_b - bits/sec	$N f_s$ (7-8)	f_s (N=1)	f_s (N=1)	less than PCM $N f_s$. (4-6)
8	SNR	Good	Poor	better	greater than PCM
9	Sampling rate 'f_s'	8 KHz	(64-128) KHz	(48-64) KHz	8 KHz
10	Complexity of implementation	Complex	Simple	Simple	Simple
11	Quantization error & distortion	Quantization error present	Slope overload distortion, idle noise.	Errors are absent! only quantization noise	Quantization noise present.
12	Applications	Telephony & TDM	Audio and Speech processing	Audio and Speech, telephony systems.	Space commn, telephony.

Problems :

① A ramp signal $m(t) = at$ is applied to a delta modulator which operate with a sampling period T_s and step size 'Δ' (or) δ.

a) Show that slope overload occurs if $\delta < aT_s$.

b) Sketch the modulator output for the following three values of step size $\delta = 0.75aT_s$, $\delta = aT_s$ & $\delta = 1.25aT_s$.

Sol. Given $m(t) = at$

Sampling period 'T_s'

Step size = 'Δ' or δ.

a) To avoid slope overload distortion, the condition is

$$\frac{\Delta}{T_s} \geq \max . \frac{d}{dt}(m(t))$$

Slope of staircase approximated s/g. ↙ ↳ slope of message signal

$$\frac{\delta}{T_s} \geq \max \left| \frac{d}{dt} . at \right|$$

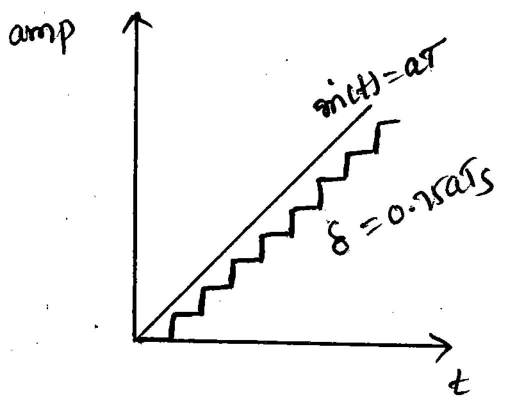
$$\frac{\delta}{T_s} \geq a$$

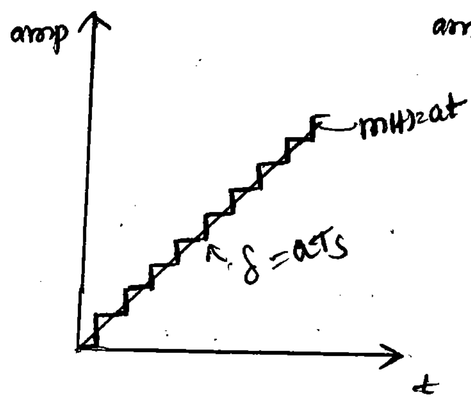
$$\therefore \underline{\delta \geq aT_s}. \quad \text{where } a \text{ - slope}$$

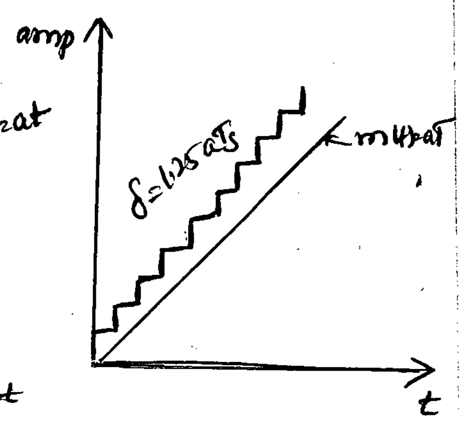
∴ ie

The slope overload error will occurs if $\underline{\delta < aT_s}$.

b) Step size $\underline{\delta = 0.75aT_s}$ $\quad$ $\underline{\delta = aT_s}$ $\quad$ $\underline{\delta = 1.25aT_s}$.







(2) If the step size is 4V in case of PCM and DM then what is the corresponding quantization noises.

Sol Step Size $\Delta = 4V$.

$\therefore$ PCM $\rightarrow$ Quantization noise $= \frac{\Delta^2}{12} = \frac{4^2}{12} = \frac{16}{12} = 1.33$.

DM $\rightarrow$ Quantization noise $= \frac{\Delta^2}{3} = \frac{4^2}{3} = \frac{16}{3} = 5.33$.

(3) Find out the output signal to quantization noise ratio of delta modulation If the sampling rate 8KHz and the modulating a base band frequency is 2KHz.

Sol. Deltamodulation.

$$f_s = 8KHz$$
$$f_m = 2KHz$$

$$\left(S/N_q\right)_{DM} = \frac{3}{8\pi^2}\left(\frac{f_s}{f_m}\right)^3.$$

$$= \frac{3}{8\pi^2} \times \left(\frac{\cancel{8}^{4} \times \cancel{10^3}}{\cancel{2} \times \cancel{10^3}}\right)^3$$

$$= \frac{3}{8\pi^2} \times 64$$

$$= 2.43$$

$\therefore (SQNR)_{DM}$ in dB $= 10\log(2.43) = 3.86$ dB.

(4) Find the sampling rate & step size if signal to quantization noise ratio is 40dB, signal frequency is 2KHz in deltamodulation. (Assume $A_m = 1$).

Sol Given $(SQNR)_{DM} = 40$ dB

$f_m = 2KHz$ $\qquad f_s = ?$

$A_m = 1$. $\qquad \Delta = ?$

$$(SQNR)_{DM} = 10\log\left(\frac{3}{8\pi^2}\cdot\left(\frac{f_s}{f_m}\right)^3\right).$$

$$40 = 10\log_{10}\left[\frac{3}{8\pi^2}\cdot\left(\frac{f_s}{f_m}\right)^3\right]$$

$$\frac{3}{8\pi^2}\left(\frac{f_s}{f_m}\right)^3 = 10^4$$

$$\left(\frac{f_s}{f_m}\right)^3 = \frac{8\pi^2 \times 10^4}{3}$$

$$\frac{f_s}{f_m} = \left[\frac{8\pi^2 \times 10^4}{3}\right]^{1/3} = 64$$

$$\therefore f_s = 64 \times f_m = 64 \times 2KHz = 128KHz$$

Sampling rate $\therefore \boxed{f_s = 128KHz.}$

$$(A_m)_{max} \leq \frac{\Delta f_s}{2\pi f_m}$$

$$\therefore \Delta = \frac{A_m \times 2\pi f_m}{f_s}$$

$$= \frac{1 \times 2\pi \times 2 \times 10^3}{128 \times 10^3}$$

$$\Delta = 0.098V$$

$\therefore$ Step size $\boxed{\Delta = 0.098V.}$

(5) A TV signal with a bandwidth of 4.2MHz is transmitted using PCM system. The no. of quantization levels are 512. Calculate

a) Codeword length
b) Final bitrate
c) Transmission bandwidth
d) SQNR. or (S/N_q).

Sol: Given Bandwidth $B.W = 4.2MHz$.

a) No. of Quantization levels $Q = 2^N = 512$ $(512 = 2^9)$

$$\therefore N = \log_2 Q = \log_2 2^9 = 9$$

$$\therefore \boxed{N = 9}$$

$\therefore$ Codeword length is no. of bits/sample $N = 9$ bits.

c) Transmission bandwidth $B.W_T \geq N f_m = N.W$ B.W of TV signal.

$$= 9 \times 4.2MHz$$

$$\therefore \boxed{B.W_T = 37.8MHz}$$

b) Final bit rate $R_b = N \times f_s$ → W → B.W of TV signal

$= N \times 2 f_m$

$= 2 \times 9 \times 4.2\,M.$

$= 75.6\,MHz$

$\therefore \boxed{R_b = 75.6\,MHz}$

d) $(SQNR)_{PCM} = Q^2 = 2^{2N} = 2^{2\times9} = 2^{18} = 262144$

SQNR of PCM in dB $= 10\log(262144) = 54.2\,dB.$

(or) SQNR in dB $= 1.8 + 6N$

$= 1.8 + 6\times9$

$\approx 55.8\,dB.$

$\boxed{(SQNR)_{dB} = 55\,dB}$

(6) In a single integration Deltamodulation system the voice signal is sampled at a rate of 64 KHz. The max. signal amplitude is 1v, the highest frequency component is 3 KHz.

a) Determine the min value of the step size σ (or) Δ (or) δ to avoid slope overload error.

b) Determine the granular noise power if the voice signal bandwidth is 3.5 KHz.

c) Assuming the voice signal is sinusoidal, determine the output signal power and SQNR. (or) (S/N) QN.

d) Determine the min. transmission bandwidth.

Sol. Given Deltamodulation. $f_m = 3\,KHz.$ $A_m = 1V.$

$f_s = 64\,KHz.$

a) To avoid slope overload distortion

$$\Delta f_s = \frac{\Delta}{T_s} \geq \frac{d}{dt}(m(t))$$

$$\Rightarrow (A_m)_m \leq \frac{\Delta f_s}{2\pi f_m}.$$

$$\Delta = \frac{A_m \times 2\pi \times f_m}{f_s} = \frac{1\times2\pi\times3\times10^3}{64\times10^3} = 0.29V$$

$\therefore$ Step Size $\boxed{\Delta = 0.29\,V.}$

b) Granular noise is quantization noise in delta modulation

$\therefore$ Granular noise $= \frac{\Delta^2}{3} \cdot \frac{f_m'}{f_s}$ → signal bandwidth

$= \frac{(0.29)^2}{3} \times \frac{3.5 \times 10^3}{64 \times 10^3} = 0.00153$

$\therefore$ $\boxed{\text{Granular noise} = 0.00153\,W}$

c) Output signal power $S_o = \frac{\Delta^2 f_s^2}{8\pi^2 f_m^2} = \frac{\Delta^2}{8\pi^2} \cdot \left(\frac{f_s}{f_m}\right)^2$

$\therefore S_o = \frac{(0.29)^2}{8\pi^2} \times \left(\frac{64 \times 10^3}{3 \times 10^3}\right)^2 = 0.485\,W.$

$\therefore$ $\boxed{S_o = 0.485\,W}$

SQNR or $(S/N)_{QN}$ in DM $= \frac{3}{8\pi^2} \left(\frac{f_s}{f_m}\right)^3$

$= \frac{3}{8\pi^2} \times \left(\frac{64 \times 10^3}{3 \times 10^3}\right)^3$

$= 368.8$

$(SQNR)_{DM}$ in dB $= 10 \log (368.8) = 25.6\,dB.$

$\therefore$ $\boxed{(SQNR)_{dB} = 25.6\,dB}$

d) Min. transmission bandwidth

$B.W = N f_m$

For delta modulation $N = 1$

$\therefore B.W = f_m = 3\,KHz$

$\therefore$ $\boxed{\text{Bandwidth} = 3\,KHz}$

Hence. a) Step size $\Delta = 0.29$

b) Granular noise $= 0.00153.$

c) Output signal power $S_o = 0.485W$; SQNR $= 25.6\,dB.$

d) Bandwidth $= 3\,KHz.$

Base Band Data Transmission - II

Syllabus: Fundamentals of time division multiplexing, T1 digital Carrier system, Synchronization and signalling of T_1, PCM hierarchy. different types of Binary encoding - M-ary encoding, correlative coding - Calculation of PSD, Inter Symbol Interference (ISI), Nyquist criteria, Eye diagram, probability of error in binary encoding & M-ary encoding, Equalizers, baseband signal receiver: Optimum filter - White noise, the matched filter, Correlation receiver.

Introduction:

* The term "base band" refers to the band of frequencies representing the original signal as delivered by the source.
* A digital data & message is an ordered sequence of symbols produced by a discrete information source.

Ex: A typical computer terminal is a source of digital data. When the terminal is operated it becomes a source of digital data Consisting of a sequence of binary digits 0 & 1 known as bits. Data rates & transfer rates within a computer may be 10^8 bits/sec & more.

✓ The purpose of a digital data communication system is to transfer a digital data (message) from source to destination.

* When a digital data is transmitted over a band limited channel, dispersion in the channel gives rise to an Interference called as "Intersymbol Interference" (ISI).

✓ This effect introduces deviations (errors) between the reconstructed data at the receiver and the original data transmitted.

Fundamentals of Time Division Multiplexing (TDM):

Multiplexing : Several message signals are transmitted through a single channel is called Multiplexing.

There are two types of multiplexing.

(a) FDM (Frequency division Multiplexing)

(b) TDM (Time division Multiplexing).

FDM : In FDM, total bandwidth can be divided into no. of sub channels and each sub channel is assigned to one user.

✓ It is a Analog Technique.

✓ Ex : TV signals and Radio signals.

TDM : In TDM, allots different time slots for different message signals & then transmitted using single channel.

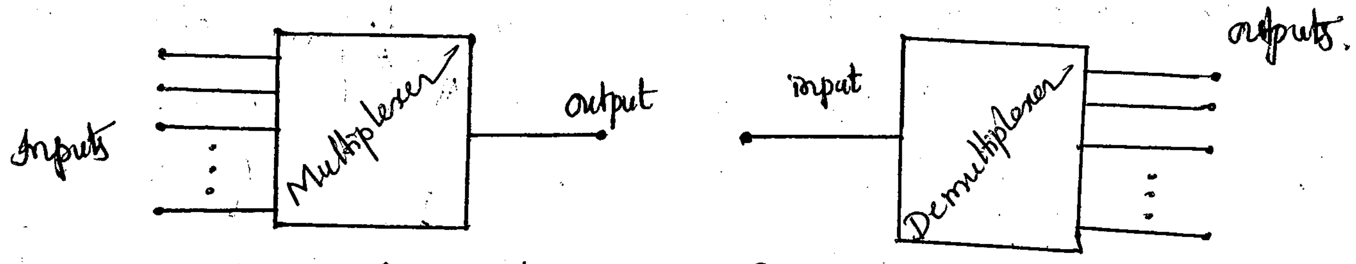
✓ It is a Digital technique.

✓ Ex : Telephone. (Voice transmission)

ie 300 Hz to 3.4 KHz.

Digital Multiplexer :

Multiplexer is a device which having more no. of inputs and single output (or) More no. of signals can be transmitted through a single channel.



Ex: Telephone (Voice transmission) ie 300 Hz − 3.4 KHz.

let N = 8 bits/sample

∴ No. of quantization levels $Q = 2^N = 2^8 = 256$ levels.

Sampling frequency $f_s \geq 2f_{max} = 2 \times 3.4K = 6.8\,KHz$.

But in standard $f_s = 8KHz = 8000$ samples/sec.

$\therefore$ Sampling time period $T_s = \frac{1}{f_s} = \frac{1}{8 \times 10^3} = 125 \times 10^{-6} = 125\,\mu sec$.

$\therefore$ Bit rate 'R_b' $= N \cdot f_s$

$= 8$ bits/sample $\times$ 8000 sample/sec

$$\boxed{R_b = 64\,Kbits/sec}$$

$\therefore$ Each bit time (bit duration) ie $T_b = \frac{T_s}{N} = \frac{125 \times 10^{-6}}{8} = 15.625\,\mu sec$

$\therefore$ Bit rate $R_b = \frac{1}{T_b} = \frac{1}{15.625 \times 10^{-6}} = 64 \times 10^3$ bits/sec

$$\therefore \boxed{R_b = 64K\,bits/sec.}$$

Block diagram of TDM:

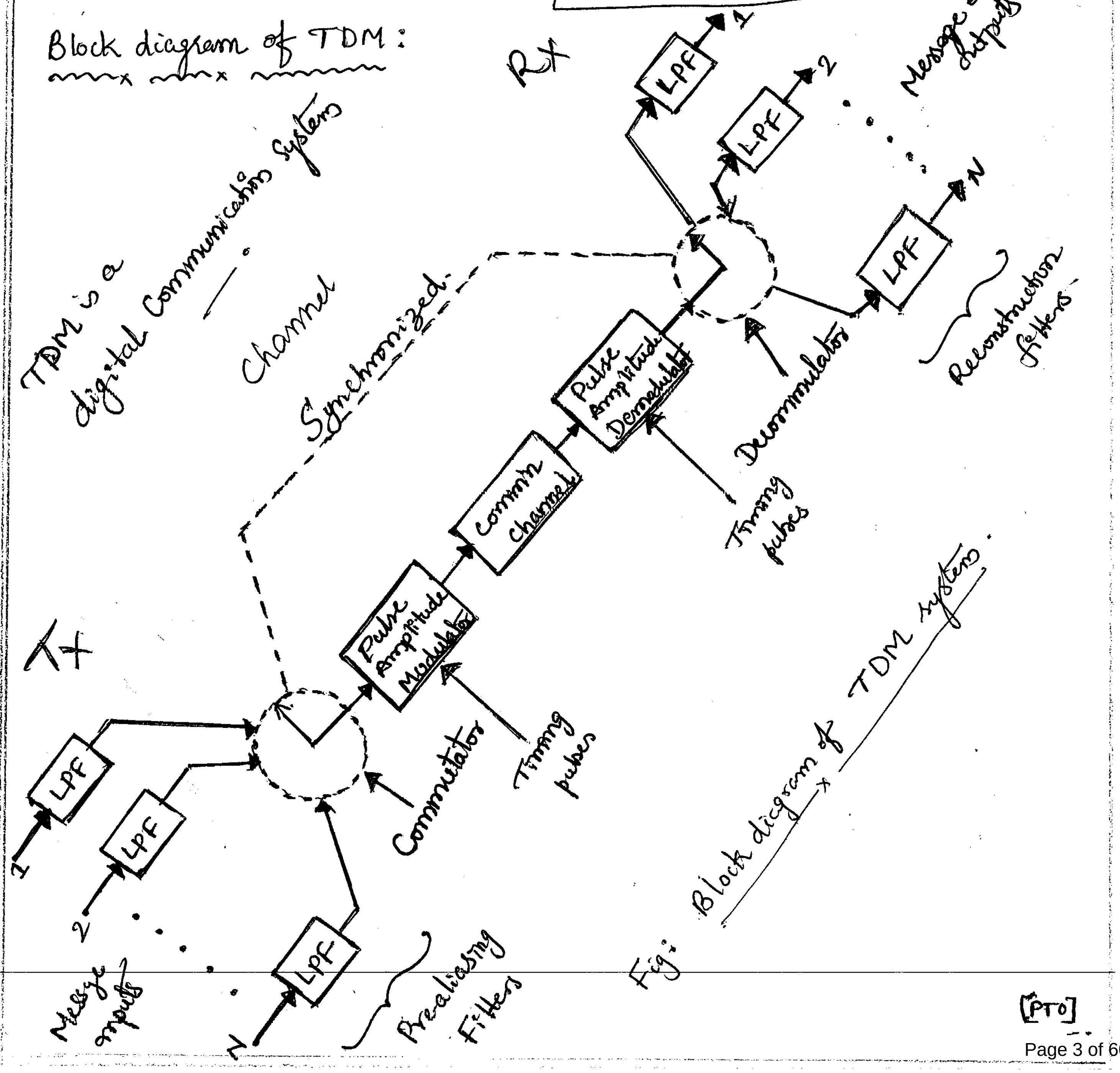
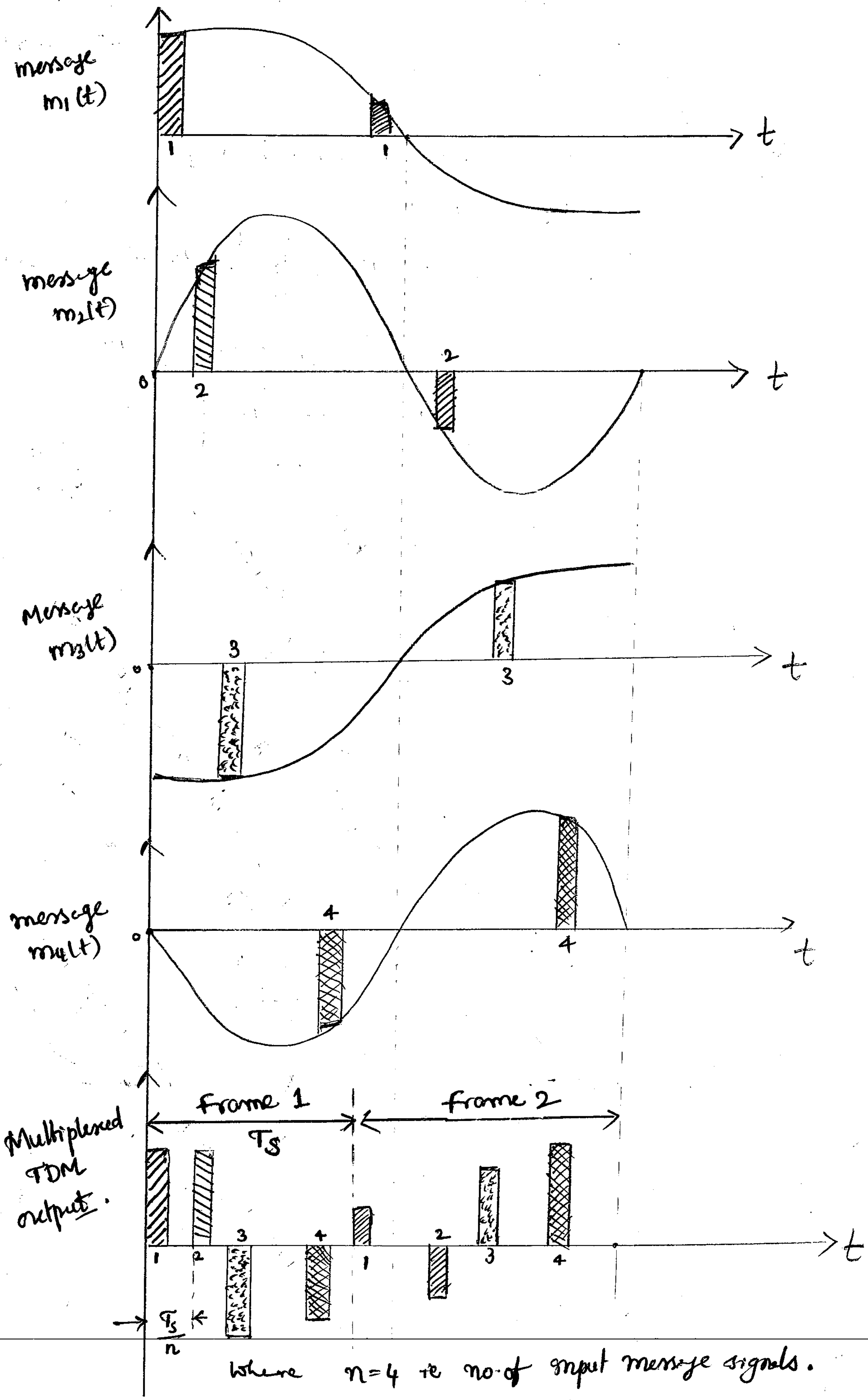


Fig: Block diagram of TDM system.

Waveforms:



where $n=4$ ie no. of input message signals.

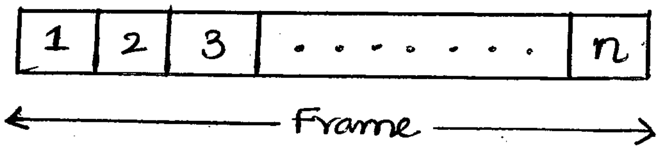
Operation :

- ✓ The input message signals are first converted to bandlimited signals by passing them through Low pass filters. These LPFs eliminate frequency components of the input signals which are not essential for their representation.
- ✓ The LPF outputs are then applied to a Commutator (or) rotating switch.
- ✓ All input signals are sequentially sampled at the transmitter by a commutator (or) Rotating sampling switch.
- ✓ The switch makes revolutions per second and extracts one sample from each input during each revolution.
- ✓ During one revolution, it collects sample data according to its time slots and form a frame.

Frame :

A frame consists of number of time slots one sample from each message signal is alloted in respected time slot.

Let a frame consists of n time slots.

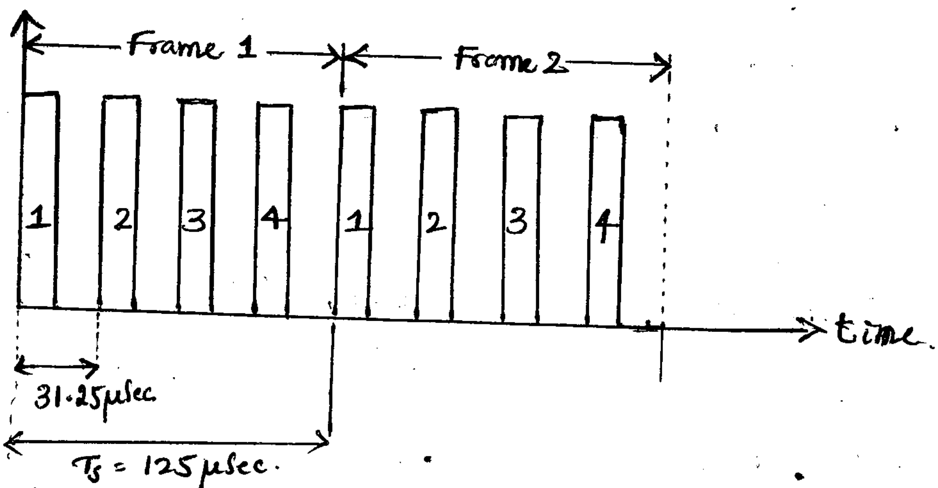


$f_s = 8KHz$
$= 8000$ samples/sec

$T_s = \frac{1}{f_s} = \frac{1}{8 \times 10^3}$

$T_s = 125\,\mu sec$

$T_{s_1} = \frac{125\mu}{4} = 31.25\,\mu sec.$



voice

- ✓ The output of switch is given to pulse modulator, the pulse modulator may be PAM (or) PCM.
- ✓ The sequence of samples may be transmitted by direct PAM (or)
- ✓ The sample values may quantized and transmitted by using PCM.

- The samples from adjascent input channels are seperated by $\frac{T_s}{n}$ where n is the no. of message input signals.

i.e a set of 'n' samples collected from n-input messages is called as frame.

- At the receiver, the samples from individual channels are seperated and distributed by another rotatary switch called as 'distributor' (or) decommutator.
- The samples from each channel are filtered to reproduce the original message signals.
- TDM system requires careful synchronization between commutator and decommutator.

Synchronization:

Synchronization is a critical consideration in TDM because each sample must be distributed to the correct output line at the appropriate time.

- There are two levels of synchronization in TDM.
  - (a) Frame Synchronization.
  - (b) Sample (or) Word Synchronization.
- Frame synchronization is necessary to seperate one frame to another.
- Sample (or) Word Synchronization is necessary to seperate the samples with each frame.

Applications:

- Time domain multiplexed PCM is used in a variety of application
- The most important use of TDM-PCM is in telephone systems, where voice and other signals are multiplexed and transmitted over a variety of transmission media including twin wires, wave guides and optical fibers.
- Applications of TDM in digital multiplexers for Telephony.

T1 digital Carrier System :

Synchronization & Signalling of T_1, PCM Hierarchy :

Let n - no. of message signals

N - no. of bits per frame

f_s - Sampling frequency. then

Bit rate $R_b = n \cdot N \cdot f_s$.

Bit duration $T_b = \frac{T_s}{n \cdot N} = \frac{1}{R_b}$.

In asynchronous TDM, No. of bits / frame $= n \cdot N$.

In Synchronous TDM, No. of bits / frame $= \underbrace{n \cdot N + 1}$ → Sync bit.

✓ Digital multiplexing patterns have been adopted for digital telecommunication.

* AT & T ie American Telephone & Telegraph company in north america & japan.

* CCIT in Europe ie CCIT → International telegraph and telephone Consultive Committee for the International Telecomm-unication unions.

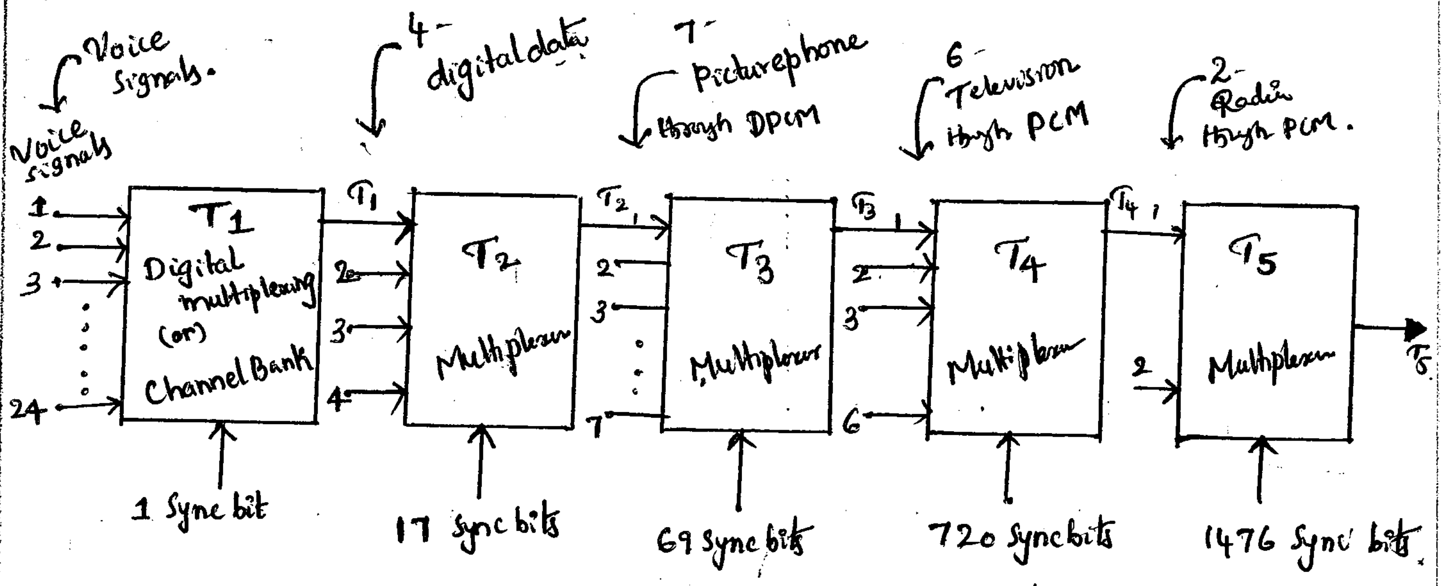


Fig: T - Digital Carrier Systems.

✓ The multiplexed PCM channels are transmitted various T carrier systems such as T_1 carrier system, T_2 carrier system, T_3 carrier system etc. as shown in above fig.

✓ A 24 channel TDM multiplexer is used as the basic system known as T_1 carrier system.

* 24 voice signals are sampled at a rate of 8 kHz and the resulting samples are quantized converted to 7 bit PCM codewords.

* At the end of each 7-bit codeword, an additional binary bit is added for synchronous purpose. ie $7+1=8$

* At the end of every frame contains 24 samples with 8 bit code words, another additional bit is inserted to give frame sychronization

ie $(8 \times 24)+1 = 193$ bits

Thus The overall frame size in the T_1 carrier system is <u>193 bits</u>

We know $f_s = 8\,KHz$. ∴ Bit rate $R_b = 193$ bits/frame × 8000 frames/sec

$$\boxed{R_b = 1.544\ M\ \text{bits/sec}}$$

* The main purpose of T_1 system is short distance coverage and high usage in metropolitan area.

✓ The maximum length of T_1 system is limited to 10–100 miles, with a repeaters spacing of 1 Mile.

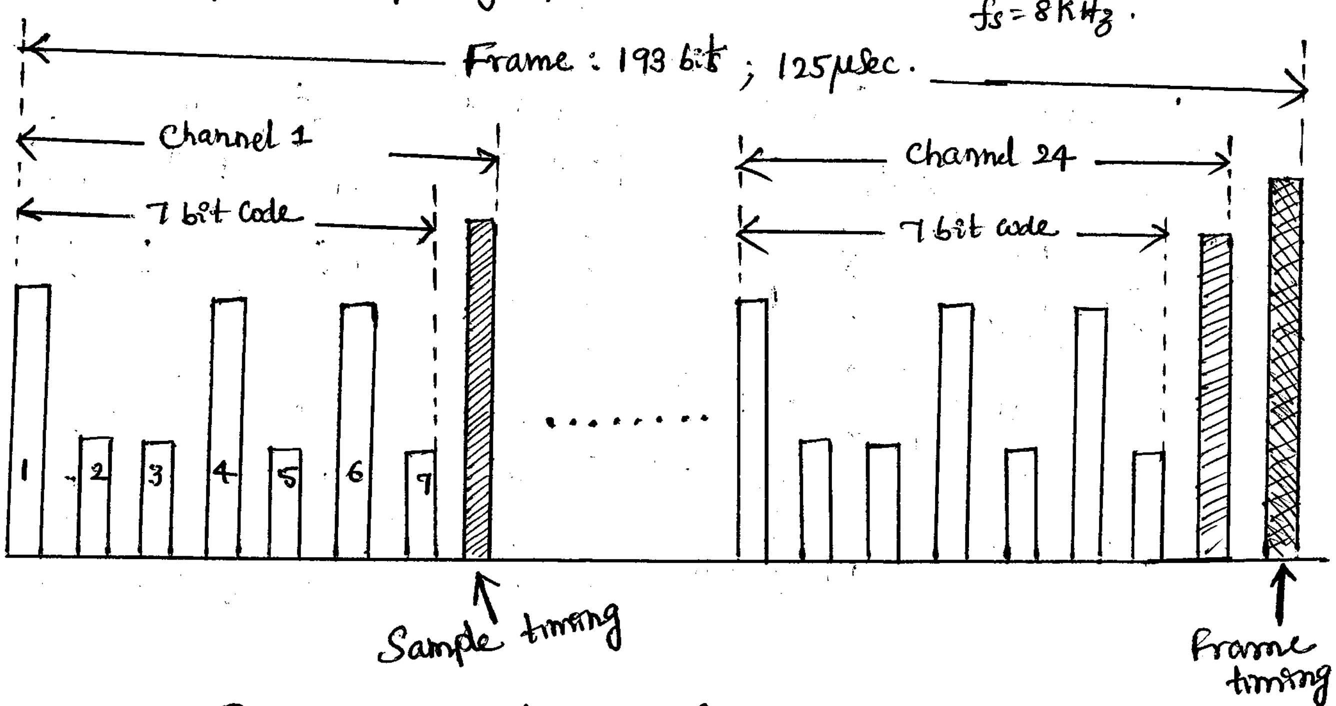


Fig: Frame formate for T1 carrier system.

✓ The first level T_1 signals include PCM voice and multiplexed digital data.

✓ The second level T_2 signals includes multiplexed T_1 signals along with telephone services ie picture phone

✓ The third level T_3 signals includes multiplexed T_2 signals along with TV (Television) signals. .... etc.

T_1 digital Multiplexer:

A 7 bit code word is added with synchronous bit '1' ie $7+1=8$ (Sample Synchronizat)

24 samples with 8-bit code word & another additional bit for frame synchronization

$\therefore$ No. of bit/frames $= (4 \times 24) + 1 = \underline{\underline{193}}$ bits/frame

$\therefore$ Bit rate of T_1 system $=$ 193 bits/frame $\times$ 8000 frames/sec $= 1.544 \times 10^6$ bits/sec

$$\boxed{R_{b_1} = 1.544\ \text{Mbps}}$$

T_2 digital Multiplexer:

It multiplexes 4 T_1 systems with 17 synchronization bits

$\therefore$ No. of bits/frame $= (4 \times 193)$ bits $+ 17$ bits $= \underline{\underline{789}}$ bits/frame

$\therefore$ The bit rate of T_2 system $=$ 789 bits/frame $\times$ 8000 frames/sec $= 6.312 \times 10^6$ bits/sec.

$$\boxed{R_{b_2} = 6.312\ \text{Mbps}}$$

T_3 digital Multiplexer:

It multiplexes 7 T_2 Systems with 69 synchronization bits.

$\therefore$ No. of bits/frames $= (7 \times 789) + 69 = \underline{\underline{5592}}$ bits/frame

$\therefore$ Bit rate of T_3 System $=$ 5592 bits/frame $\times$ 8000 frames/sec $= 44.736 \times 10^6$ bits/sec.

$$\boxed{R_{b_3} = 44.736\ \text{Mbps.}}$$

T_4 digital Multiplexer:

It multiplexes 6 T_3 systems with 720 synchronization bits

$\therefore$ No. of bits/frames $= (6 \times 5592) + 720 = \underline{\underline{34272}}$ bits/frame

$\therefore$ Bit rate of T_4 system $= 34272$ bits/frames $\times$ 8000 frames/sec

$= 274.176 \times 10^6$ bits/sec

$\therefore \boxed{R_{b_4} = 274.176 \text{ Mbps}}$

T_5 digital multiplexer :

It multiplexes $2T_4$ systems with 1476 synchronization bits.

$\therefore$ No of bits/frames $= (2 \times 34272 + 1476) = 70020$ bits/frames

$\therefore$ Bit rate of T_5 system $= 70020$ bits/frames $\times$ 8000 frames/sec

$= 560.16 \times 10^6$ bits/sec

$\therefore \boxed{R_{b_5} = 560.16 \text{ Mbps}}$

Hence

AT&T - T-carrier telephony system specifications..

System	Bit rate 'R_b'	Medium	Repeaters Spacing (Miles)	Maximum length (Miles)	System Error rate
T_1	1.544 Mbps	Twin wire	1	50	10^{-6}
T_2	6.312 Mbps	Coaxial Cable	2.5	500	10^{-7}
T_3	44.736 Mbps	Coaxial cable	Multiplexing only	-	-
T_4	274.176 Mbps	Coaxial cable	1	500	10^{-6}
T_5	560.16 Mbps.	Coaxial Cable.	1	500	$(0.4) \times 10^{-8}$.

where Mbps - Mega (10^6) bits per Second

Different types of Binary Encoding:

"The process of assigning a binary value & binary code to each discrete set of samples known as Encoding & coding."

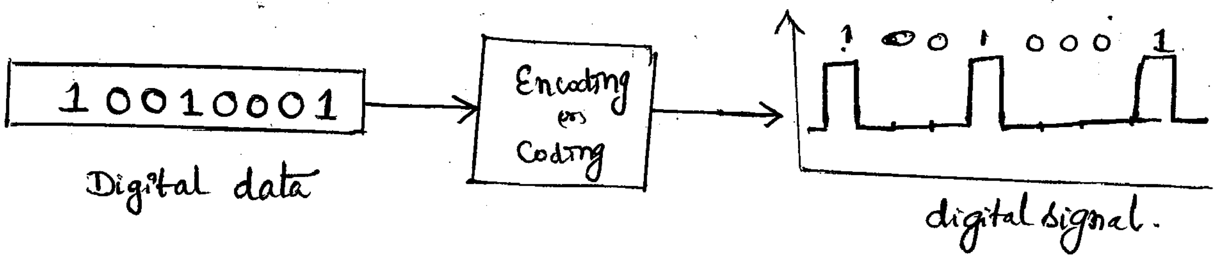
✓ Encoding is also called waveform coding & line coding & Txmn coding

✓ Set of symbols called code word.

✓ Each codeword consists of n-bits, such a code represents a total of 2^n distinct numbers.

* The first approach converts digital data into digital signal is known as line coding & simple encoding.

Ex:



There are different types of binary encoding Techniques.

1. Unipolar non return to zero (UNRZ) Signalling
2. Polar non return to zero (PNRZ) signalling.
3. Unipolar Return to zero (URZ) signalling.
4. Bipolar return to zero (BRZ) signalling.
5. Split phase (or) Manchester signalling.
6. Differential Encoding.

① Unipolar NRZ ie UNRZ Signalling

* Symbol 1 represents by presence of pulse
* Symbol 0 represents by absence of pulse.
* It is also reffered as ON-OFF signalling.

✓ It is used for Txon of ASK (Amplitude Shift Keying) signal.

ie $\boxed{UNRZ : \begin{array}{l} 1 \to 1 \\ 0 \to 0 \end{array}}$

② Polar NRZ Signalling:

✓ Symbol 1 is represented by +1 & Symbol 0 represents by -1

✓ It is used for PCM, DM, ADM, PSK, FSK signal.

ie $\boxed{PNRZ : \begin{array}{l} 1 \to +1 \\ 0 \to -1 \end{array}}$

③ Unipolar RZ (URZ) signalling:

✓ Symbol 1 is splitting into two half cycles, First half cycle represents +1, symbol 0 & Second half cycle of symbol 1 represented by 0

✓ It is used for fiber optical Communications.

$\boxed{URZ : \begin{array}{l} 1 \begin{cases} +1/2 \to +1 \\ -1/2 \to 0 \end{cases} \\ 0 \longrightarrow 0 \end{array}}$

④ Bipolar RZ ie (BRZ) signalling:

✓ Symbol 1 is represented by alternation of +1 and -1 & no pulse for symbol 0

✓ It is also reffured as "AMI (Alternate Mark Inversion)"

$\boxed{BRZ : \begin{array}{l} 1 \to \underset{\text{first}}{+1}, \underset{\text{second}}{-1} \\ 0 \to 0 \end{array}}$

⑤ Split phase (or) Manchester Signalling:

✓ symbol 1 is represented by +1 & -1 half cycle pulse and

✓ Symbol 0 is represented by -1 & +1 half cycle pulse.

$\boxed{1 \begin{cases} +1/2 \to +1 \\ -1/2 \to -1 \end{cases}, \quad 0 \begin{cases} -1/2 \to -1 \\ +1/2 \to +1 \end{cases}}$

⑥ Differential Encoding:

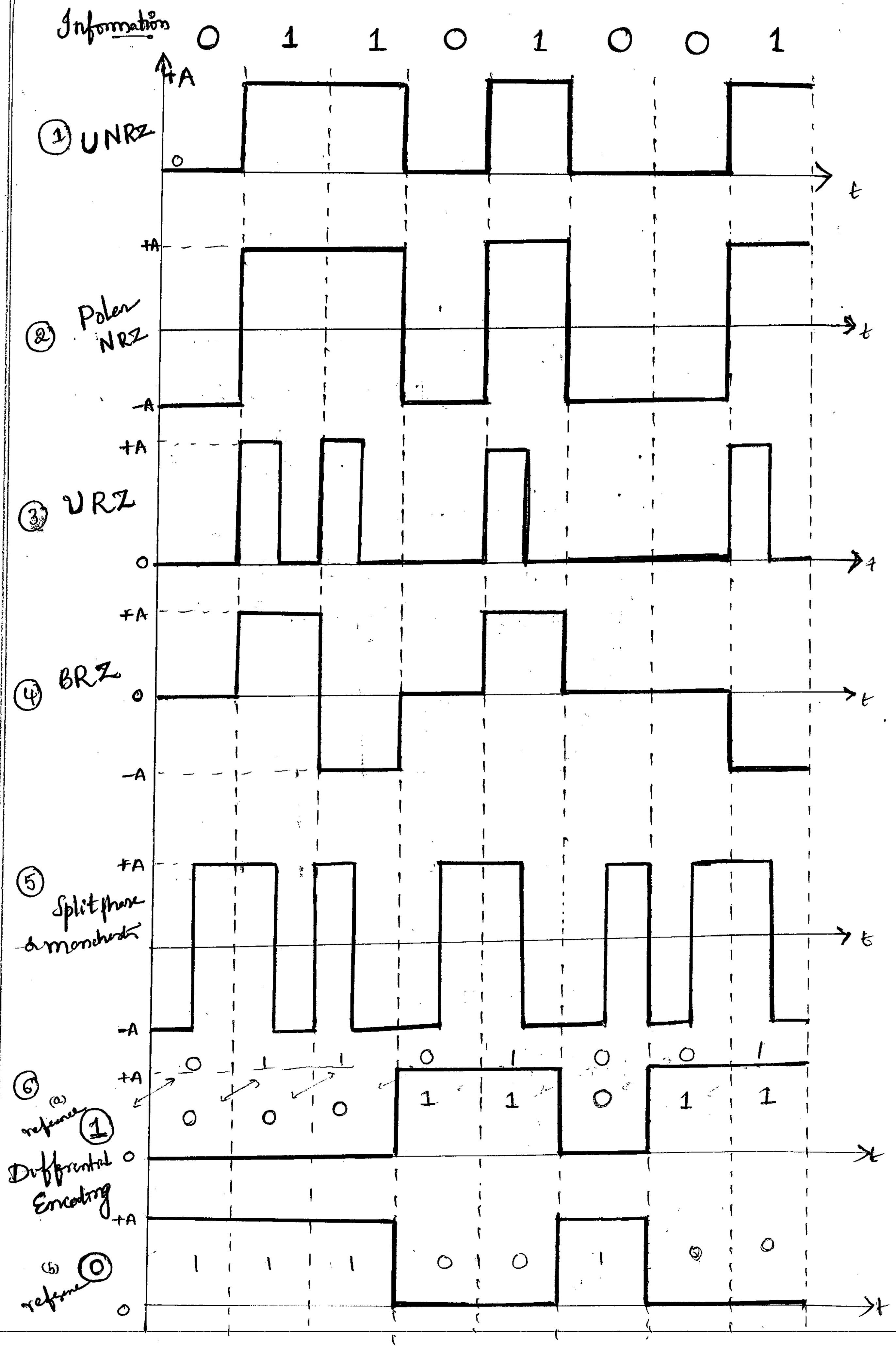
✓ Symbol 0 is represented by complement of previous voltge

✓ symbol 1 is represented previous Voltge.

$\boxed{\begin{array}{l} 1 \to \text{previous voltge} \\ 0 \to \text{Complement of previous voltge} \end{array}}$

Exclusive NOR operation

00 → 1
01 → 0
10 → 0
11 → 1


Information
0 1 1 0 1 0 0 1
+A
① UNRZ
0
t
+A
② Polar NRZ
-A
t
+A
③ URZ
0
t
+A
④ BRZ
0
-A
t
⑤ Split phase & manchester
+A
-A
t
⑥ (a) reference 1 Differential Encoding
+A
0
(b) reference 0
+A
0
t

M-ary Encoding (or) Signalling scheme:

M-ary signalling transmits 'm' symbols with 'm' voltage levels such that the no. of bits in each symbol is

$N = \log_2 M \Rightarrow$ If $M = 2$, $N = \log_2 2 = 1$ called Binary Encoding

If $M = 4$, $N = \log_2 4 = 2$ ie two bits in each symbol.

$\therefore$ The voltage levels in M-ary encoding

$$A_m = \begin{cases} 0, \pm 2A, \pm 4A, \dots \pm (m-1)A; & m\text{-odd} \\ \pm A, \pm 3A, \pm 5A, \dots \pm (m-1)A; & m\text{-even} \end{cases}$$

For Four symbols ie $m=4$. Assume A, B, C, D.

voltage levels $\therefore A_m = \pm A$ and $\pm 3A$.

let

$00 \rightarrow A = -A$

$01 \rightarrow B = +A$

$10 \rightarrow C = +3A$

$11 \rightarrow D = -3A$.

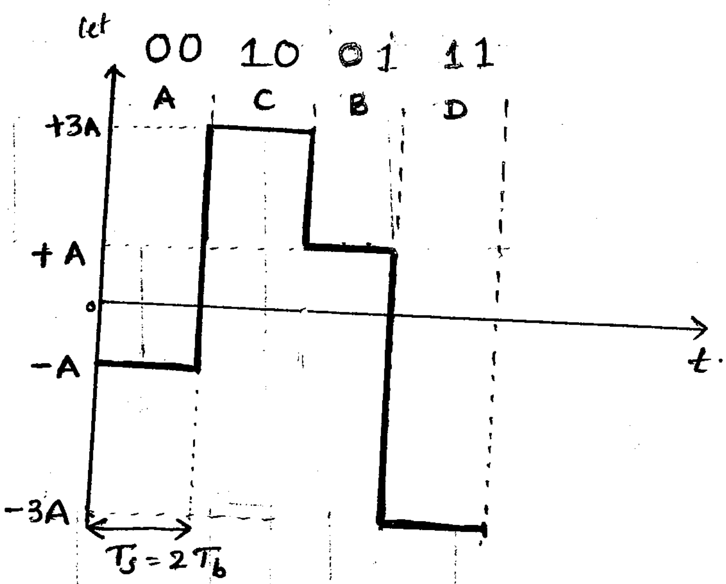


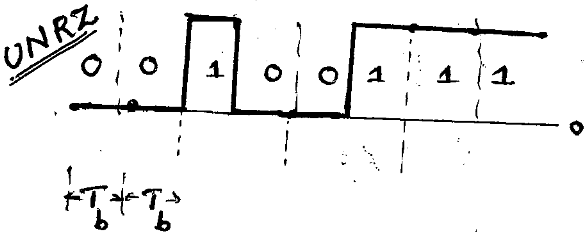
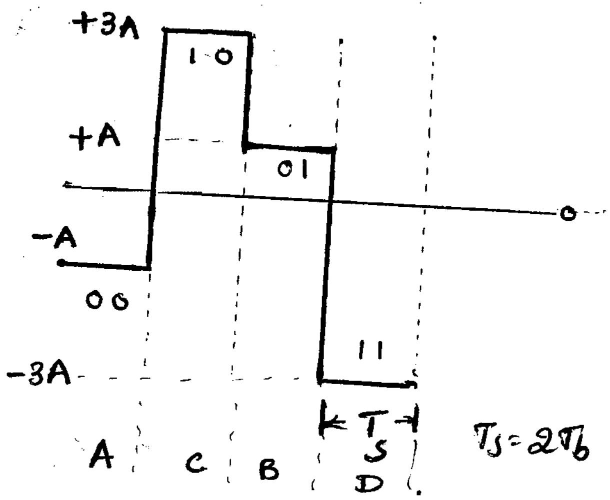
Fig: M-ary signalling.

- Band width is reduced by 2 ie $R_b/2$.
- Increase complexity at Receiver.
- Power level is high.
- The probability of error for m-ary coding is given by

$$P_e = \frac{2(m-1)}{m} Q(A/\sigma) \quad \text{where } Q(x) \rightarrow Q\text{-function.}$$

If $m = 4$ symbols $\therefore P_e = \frac{2(4-1)}{4} Q(A/\sigma) \Rightarrow \boxed{P_e = \frac{6}{4} Q(A/\sigma)}$

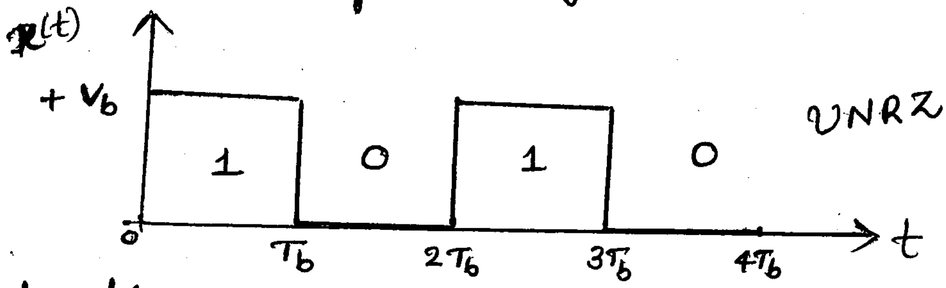
Comparision between Binary & M-ary Signalling Scheme :

Binary Scheme	M-ary scheme
(1) Representation let 00 10 01 11 data Data is 00100111 For Unipolar NRZ (non Return to Zero) Encoding method  r_s - Symbol rate of source	(1) Representation. let 00 → A → −A 01 → B → +A 10 → C → +3A 11 → D → −3A Sequence $\underline{00}\ \underline{10}\ \underline{01}\ \underline{11}$ A C B D  $T_s = 2T_b$
(2) Bit rate $R_b = N \times f_s$ bit/sec (or) $r_s \times \log_2 M \leftarrow N$ where N - no. of bits/symbol.	(2) Bit rate $R_b = r_s / N$ bits/sec.
(3) No. of threshold required = 1	(3) No. of threshol required = (m−1) levels where m - no. of symbols.
(4) Required transmission power is less	(4) Required transmission power is more.
(5) Hard Complexity is less	(5) Hardware Complexity is more
(6) Probability of error is less $P_e = Q(A/2\sigma)$ for UNRZ	(6) Probability of Error is more $P_e = \frac{2(m-1)}{m} \cdot Q(A/\sigma)$
(7) m-symbols can be represented only in two levels $m=2$ ie logic '0' & logic '1'	m-symbol can be represented with m-voltage levels.

Calculation of PSD :

Power Spectrum of Encoded Signal (UNRZ) — Uni polar Non Return to Zero

Consider UNRZ encoding signalling.



Fourier transform

$$X(\omega) = F\{x(t)\} = \int_{t=-\infty}^{\infty} x(t) \cdot e^{-j\omega t}\, dt$$

$$= \int_{0}^{T_b} (V_b)\, e^{-j\omega t}\, dt + \int_{T_b}^{2T_b} (0) \cdot e^{-j\omega t}\, dt$$

$$= V_b \cdot \left(\frac{e^{-j\omega t}}{-j\omega} \right)_0^{T_b} + 0$$

$$= V_b \left[\frac{e^{-j\omega T_b}}{-j\omega} - \frac{e^{0}}{-j\omega} \right]$$

$$= \frac{V_b}{-j\omega} \left[e^{-j\omega T_b} - 1 \right]$$

$$X(\omega) = j \cdot \frac{V_b}{\omega} \left(e^{-j\omega T_b} - 1 \right)$$

(or)

$$X(f) = \frac{j V_b}{2\pi f} \left[e^{-j2\pi f T_b} - 1 \right] \qquad (\because \omega = 2\pi f)$$

$$= \frac{j V_b}{2\pi f} \left[e^{-j\pi f T_b} \cdot e^{-j\pi f T_b} - e^{-j\pi f T_b} \cdot e^{+j\pi f T_b} \right]$$

$$= \frac{j V_b}{2\pi f} \left[e^{-j\pi f T_b} - e^{+j\pi f T_b} \right] e^{-j\pi f T_b}$$

$$= \frac{\cancel{j} V_b}{\pi f} \cdot \left[\frac{e^{+j\pi f T_b} - e^{-j\pi f T_b}}{2j \cdot \cancel{j}} \right] \cdot e^{-j\pi f T_b} \qquad (\because j^2 = -1)$$

$$= \frac{V_b}{\pi f} \cdot \sin(\pi f T_b) \cdot e^{-j\pi f T_b} \qquad \left(\because \frac{e^{x} - e^{-x}}{2j} = \sin x \right)$$

$$= \frac{V_b}{\cancel{\pi f}} \cdot \frac{\cancel{\pi f} T_b}{\pi f T_b} \cdot \sin(\pi f T_b) \cdot e^{-j\pi f T_b}$$

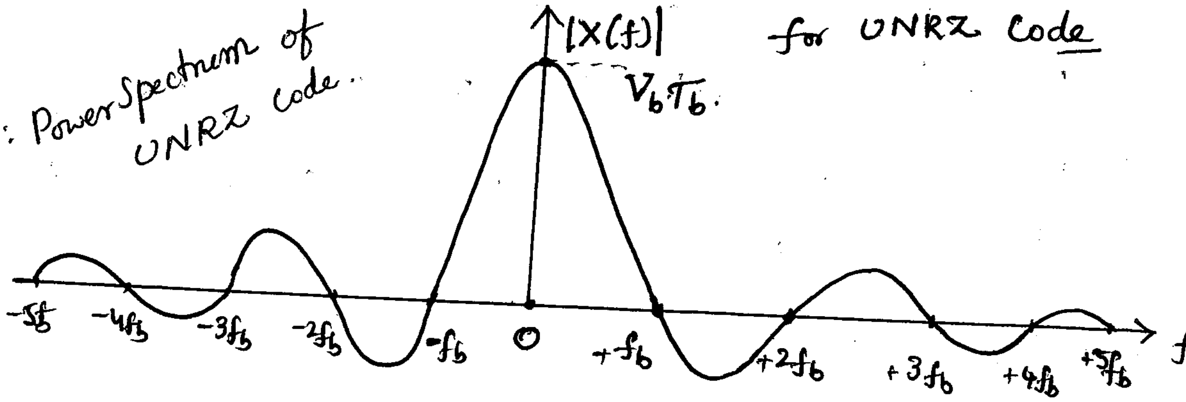
$$X(f) = V_b \cdot T_b \cdot \left[\frac{\sin(\pi f T_b)}{\pi f T_b} \right] \cdot e^{-j\pi f T_b}$$

$$X(f) = V_b \cdot T_b \cdot \text{Sinc}(fT_b) \cdot e^{-j\pi f T_b} \qquad \left(\because \frac{\sin \pi x}{\pi x} = \text{Sinc}(x)\right)$$

$$\therefore \boxed{X(f) = V_b \cdot T_b \cdot e^{-j\pi f T_b} \cdot \text{Sinc}(fT_b)}$$

The magnitude spectrum $\underline{|X(f)| = V_b \cdot T_b \cdot \text{Sinc}(fT_b)}$.

Fig: Power Spectrum of UNRZ code..



The PSD (power spectral density) of signal $x(t)$ for UNRZ is given by

$$\boxed{S_X(f) = \frac{1}{T_b} |X(f)|^2 \cdot \sum_{n=-\infty}^{\infty} R_A(n) \cdot e^{-j2\pi f n T_b}}$$

where $R_A(n) \rightarrow$ Auto correlation function.

Similarly for all encoding techniques the power spectrum as follows.

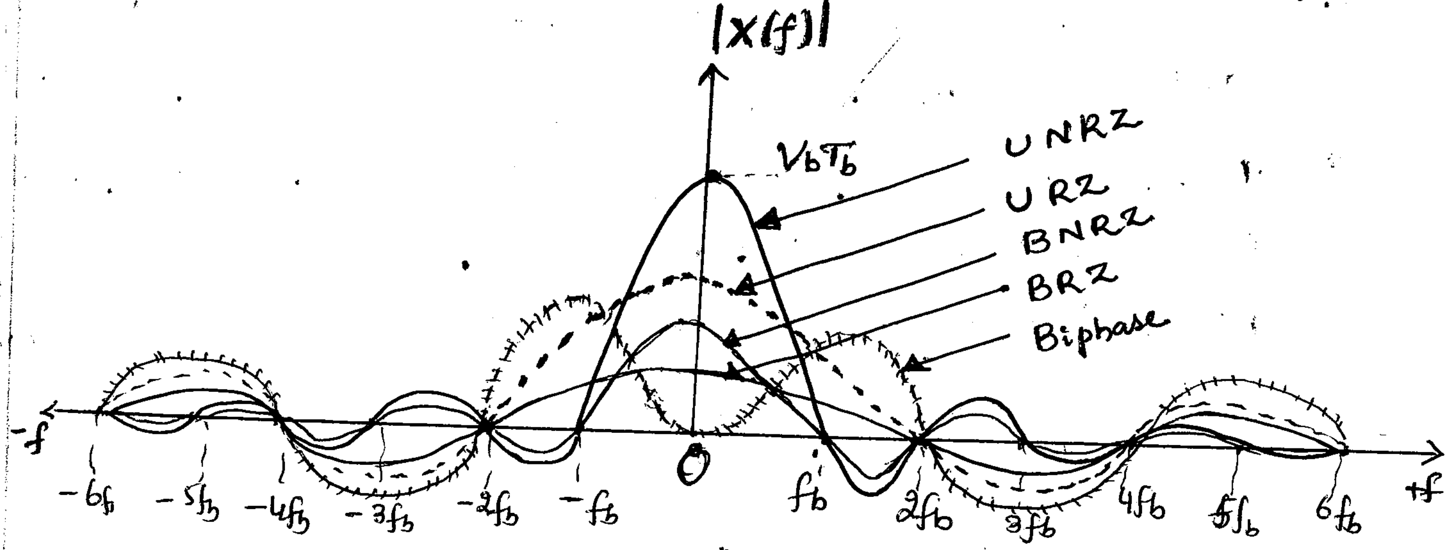


Fig: Power Spectrum of different line codes.

Power Spectral density

$$\text{PSD} \quad S_X(f) = \frac{1}{T} |X(f)|^2 \sum_{n=-\infty}^{\infty} R_A(n) \, e^{-j2\pi f n T}$$

$\hookrightarrow$ Autocorrelation function.

Note:

- Unipolar, most of signal power is centered around origin and there is waste of power due to dc component that is present.
- Polar form, most of signal power is centered around origin and they are simple to implement.
- Bipolar format does not have dc component & does not demand more bandwidth, but power requirements is double than other forms.
- Manchester format doesnot have dc component but provides proper clocking.

Desirable Characteristics of Encoded signal:

* ① The transmission of bandwidth of the encoded signal should be less.
* ② The average (or) dc value of encoded signal should be as low as possible.
* ③ The peak transmission power should be as low as possible.
* ④ The noise margin of the encoded signal should be as high as possibility ie noise immunity should be more.
* ⑤ The synchronization problem (or) timing error (or) clock recovery problem should be as low as possible.
  ie Transmitter and receiver should be operated same frequencies.

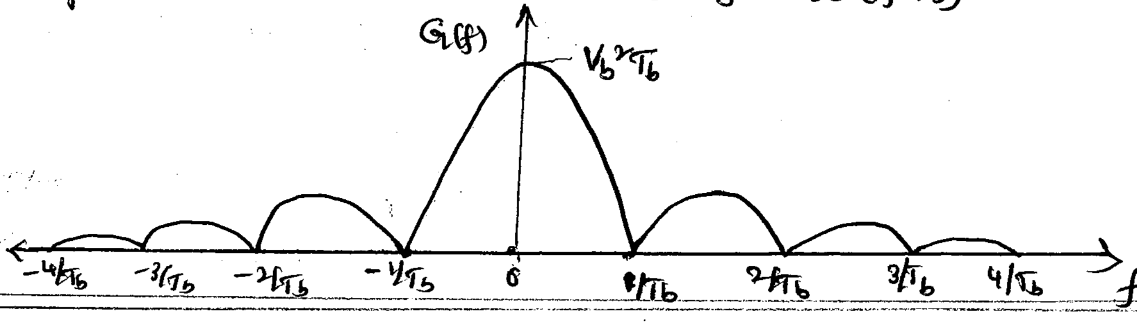
⇒ The PSD for UNRZ code.

$$S_x(f) = \frac{1}{T_b} |X(f)|^2$$
$$= \frac{1}{T_b} \cdot \left[V_b \cdot T_b \cdot \operatorname{Sinc}(f T_b)\right]^2$$

(or)

$$G(f) = V_b^2 \cdot T_b \cdot \operatorname{Sinc}^2(f T_b)$$

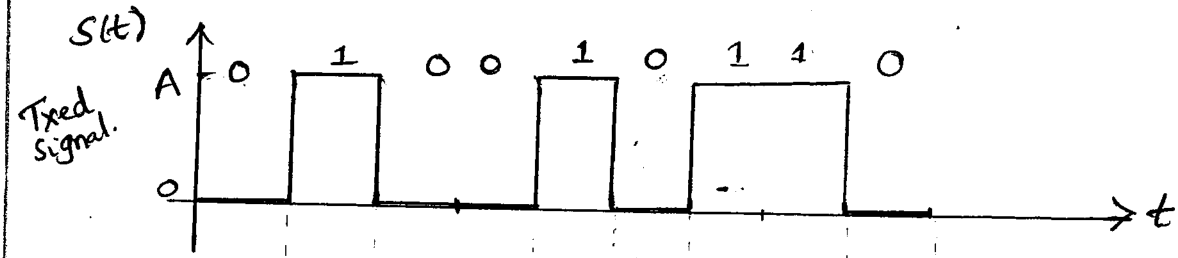
The PSD spectrum.



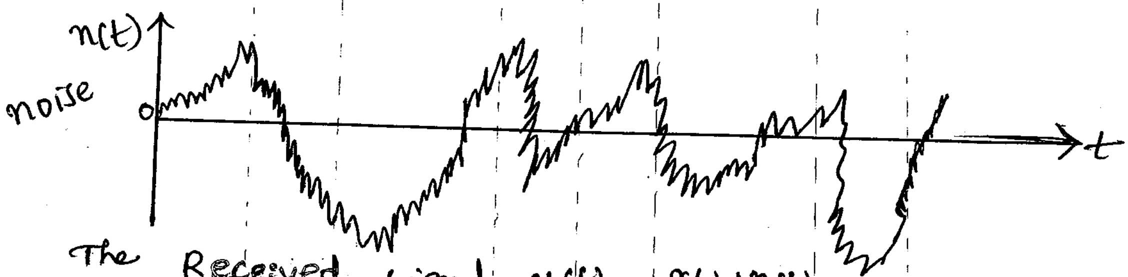
Probability of Error (Pe):

(a) <u>Binary Encoding</u>:

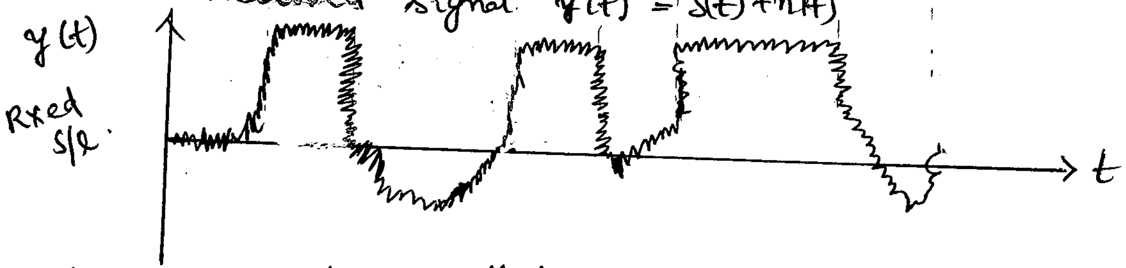
Let us consider the transmitted signal 0 1 0 0 1 0 1 1 0 using Unipolar Non Return to Zero code (UNRZ) ie.



let us consider $n(t)$ as zero mean Gaussian noise.



The Received signal $y(t) = s(t) + n(t)$



when '0' is transmitted, Received signal $y(t) = n(t)$

$y = n$

$E[y] = 0$ ($\because$ noise zero mean)

when '1' is transmitted, Received signal

$y(t) = s(t) + n(t)$

ie $y = s + n$

$n = y - s$ $\rightarrow A$

$\boxed{n = y - A}$

Let us consider the bandlimited white Gaussian function as

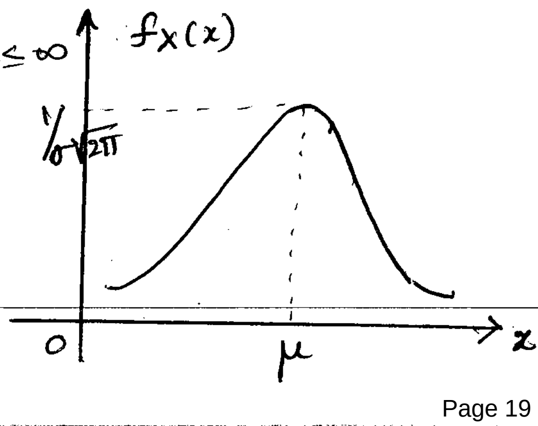
$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \; ; \; -\infty \le x \le \infty$$

where

μ - mean (or) Expectation of 'x'

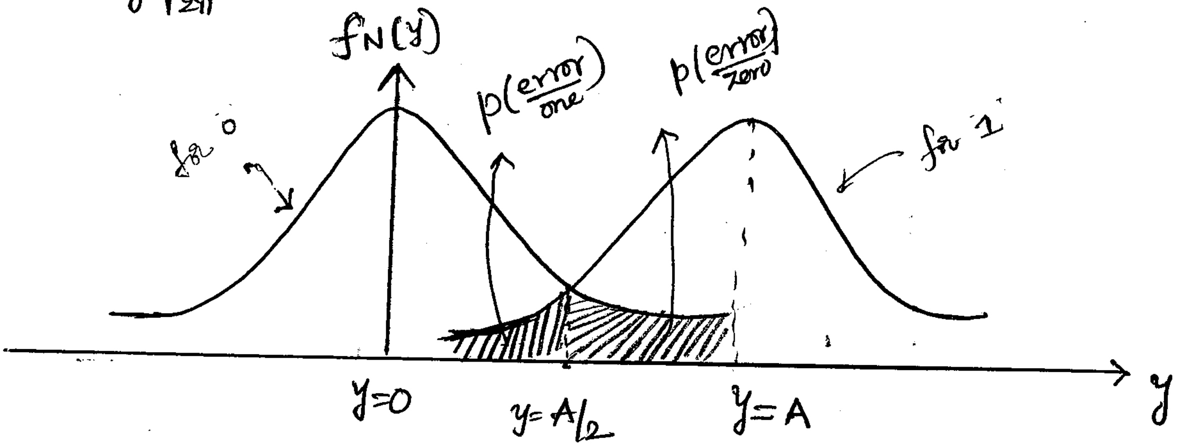
σ - Standard deviation

σ^2 - variance.



$$f_N(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-y^2/2\sigma^2} \quad ; \text{ noise, when '0' is transmitted.}$$

$$f_N(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{[(y-A)^2 - 0]}{2\sigma^2}} \quad ; \text{ Noise, when '1' is transmitted.}$$



Probability of '0' becomes '1', if $y > A/2$

$$\therefore \underline{\underline{p\left(\frac{\text{error}}{\text{zero}}\right)}} = P_{e_0} = \int_{u=A/2}^{\infty} f_N(u)\, du = \int_{u=A/2}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-u^2/2\sigma^2} du \longrightarrow ①$$

We know that

Q-function $\rightarrow$ $$\boxed{Q(x) = \frac{1}{\sqrt{2\pi}} \int_{u=x}^{\infty} e^{-u^2/2} . du} \longrightarrow ②$$

let $z = \frac{u}{\sigma} \Rightarrow dz = \frac{du}{\sigma} \Rightarrow du = \sigma dz$

Limits if $u = A/2$, $z = u/\sigma = A/2\sigma$

$u = \infty$, $z = \infty$

From eqn ①

$$P_{e0} = \int_{z=A/2\sigma}^{\infty} \frac{1}{\cancel{\sigma}\cdot\sqrt{2\pi}} \cdot e^{-(z^2\cancel{\sigma^2})/2\cancel{\sigma^2}} \cdot \cancel{\sigma} \cdot dz$$

$$P_{e0} = \frac{1}{\sqrt{2\pi}} \int_{z=A/2\sigma}^{\infty} e^{-z^2/2} . dz \longrightarrow ③$$

By comparing eqn ② & ③

$$\therefore p\left(\frac{\text{error}}{\text{zero}}\right) = P_{e0} = Q(A/2\sigma) \longrightarrow ④$$

$\therefore \boxed{P_{e0} = Q(A/2\sigma)}$. where A - Amplitude of logic data
σ - Standard deviation.

Similarly probability of 1 becomes 0 : if $y < A/2$

$$\underline{\underline{p\left(\frac{\text{error}}{\text{one}}\right) = P_{e_1}}} = \int_{u=-\infty}^{A/2} f_N(u)\,du = \int_{u=-\infty}^{A/2} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-A)^2/2\sigma^2}\,du$$

$$P_{e_1} = \int_{u=-\infty}^{\infty} f_X(u)\,du - \int_{u=A/2}^{\infty} f_X(u)\,du$$

$$= 1 - \int_{u=A/2}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-A)^2/2\sigma^2}\,du$$

Let $z = \frac{u-A}{\sigma}$, $dz = \frac{du}{\sigma}$ $\Rightarrow du = \sigma\, dz$

limits $u = A/2$, $\Rightarrow z = \frac{A/2 - A}{\sigma} = -A/2\sigma$

$u = \infty \Rightarrow z = \infty$

$$\therefore \quad P_{e_1} = 1 - \int_{z=-A/2\sigma}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \cdot e^{-z^2/2} \cdot \cancel{\sigma}\,dz$$

$$= 1 - \int_{z=-A/2\sigma}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\,dz.$$

$$= 1 - Q(-A/2\sigma)$$

$$\boxed{P_{e_1} = 1 - Q(A/2\sigma)} \longrightarrow (5) \quad (\because Q(x) \text{ is always even for } Q(-x) = Q(x)).$$

from (4) & (5) $P_{e_0} + P_{e_1} = Q(A/2\sigma) + 1 - Q(A/2\sigma) = 1$.

If the probability of error is equal in both cases ie

$$\boxed{P_{e_0} = P_{e_1} = Q(A/2\sigma)}. \quad \text{Equally probably}$$

$\Rightarrow$ From conditional probability theorem

Total probability error $P_e = p(\text{zero}) \cdot p\left(\frac{\text{error}}{\text{zero}}\right) + p(\text{one}) \cdot p\left(\frac{\text{error}}{\text{one}}\right)$.

if $p(\text{zero}) = p(\text{one}) = 1/2$

$$\therefore \quad P_e = \frac{1}{2}\left[Q(A/2\sigma) + Q(A/2\sigma)\right] = 1/2\left[2\,Q(A/2\sigma)\right]$$
$$= Q(A/2\sigma).$$

$\therefore$ $\boxed{\text{Probability of error } P_e = Q(A/2\sigma)}$ for binary system (UNRZ).

Probability of error using error function.

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{u=0}^{x} e^{-u^2} \cdot du \; ; \quad \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{u=x}^{\infty} e^{-u^2} \cdot du$$

↳ Complementary error function

& $\text{erf}(x) + \text{erfc}(x) = 1.$

$$P_{e_0} = P\left(\frac{\text{error}}{\text{zero}}\right) = \int_{u=A/2}^{\infty} f_X(u)\, du$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{u=A/2}^{\infty} e^{-u^2/2\sigma^2} \cdot du$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{z=A/2\sigma\sqrt{2}}^{\infty} e^{-z^2} \cancel{\sigma}\sqrt{2} \cdot dz$$

let $z = \frac{u}{\sigma\sqrt{2}}$

$dz = \frac{du}{\sigma\sqrt{2}} \Rightarrow du = \sigma\sqrt{2}\,dz$

If $u = A/2$, $z = A/2\sigma\sqrt{2}$

$u = \infty$, $z = \infty$.

$$= \frac{1}{\sqrt{\pi}} \int_{A/2\sigma\sqrt{2}}^{\infty} e^{-z^2} \cdot dz = \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} \int_{z=A/2\sigma\sqrt{2}}^{\infty} e^{-z^2}\, dz\right)$$

$$\therefore \boxed{P_{e_0} = \frac{1}{2}\,\text{erfc}\left(A/2\sigma\sqrt{2}\right)}$$

$$P_{e_1} = P\left(\frac{\text{error}}{\text{one}}\right) = \int_{u=-\infty}^{A/2} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-A)^2/2\sigma^2} \cdot du = \int_{u=-\infty} f_X(u)\, du - \int_{u=A/2} f_X(u)\, du$$

$$= 1 - \int_{u=A/2}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-A)^2/2\sigma^2} \cdot du.$$

$$= 1 - \int_{z=-A/\sigma 2\sqrt{2}}^{\infty} \frac{1}{\cancel{\sigma}\cancel{\sqrt{2}}\cdot\sqrt{\pi}} e^{-z^2}\, dz \cdot \cancel{\sigma}\cdot\cancel{\sqrt{2}}$$

$$= 1 - \left[\int_{z=-A/2\sigma\sqrt{2}}^{\infty} \frac{2}{\sqrt{\pi}} \cdot e^{-z^2}\, dz\right] \times \frac{1}{2}$$

let $z = \frac{u-A}{\sigma\sqrt{2}}$

$dz = \frac{du}{\sigma\sqrt{2}}$

$\Rightarrow du = \sigma\sqrt{2}\,dz$

If $u = A/2$, $z = -A/2\sigma\sqrt{2}$

$u = \infty$, $z = \infty$.

$$= 1 - \frac{1}{2}\,\text{erfc}\left(-A/2\sigma\sqrt{2}\right)$$

$$\boxed{P_{e_1} = 1 - \frac{1}{2}\,\text{erfc}\left(A/2\sigma\sqrt{2}\right)}$$

($\because$ Error function also always even $\therefore \text{erfc}(x) = \text{erfc}(-x)$)

$$\therefore P_{e_0} + P_{e_1} = \frac{1}{2}\,\text{erfc}\left(A/2\sigma\sqrt{2}\right) + 1 - \frac{1}{2}\,\text{erfc}\left(A/2\sigma\sqrt{2}\right) = 1.$$

$$P_e \Rightarrow \boxed{P_{e_0} = P_{e_1} = \frac{1}{2}\,\text{erfc}\left(A/\sigma 2\sqrt{2}\right)}$$

Probability of error

$$P_e = Q(A/2\sigma)$$

$$P_e = \frac{1}{2}\,\text{erfc}\left(\frac{A}{\sigma 2\sqrt{2}}\right)$$

for UNRZ code

Relation between $Q(x)$, $\text{erfc}(x)$ as $\boxed{\text{erfc}(x) = 2Q(x\sqrt{2})}$

(b) M-ary Encoding : Probability of error (P_e):

M-ary signalling transmits m symbols as voltage levels.

let m=4, assume A, B, C, D. Voltage levels $A_m = \pm A, \pm 3A$.

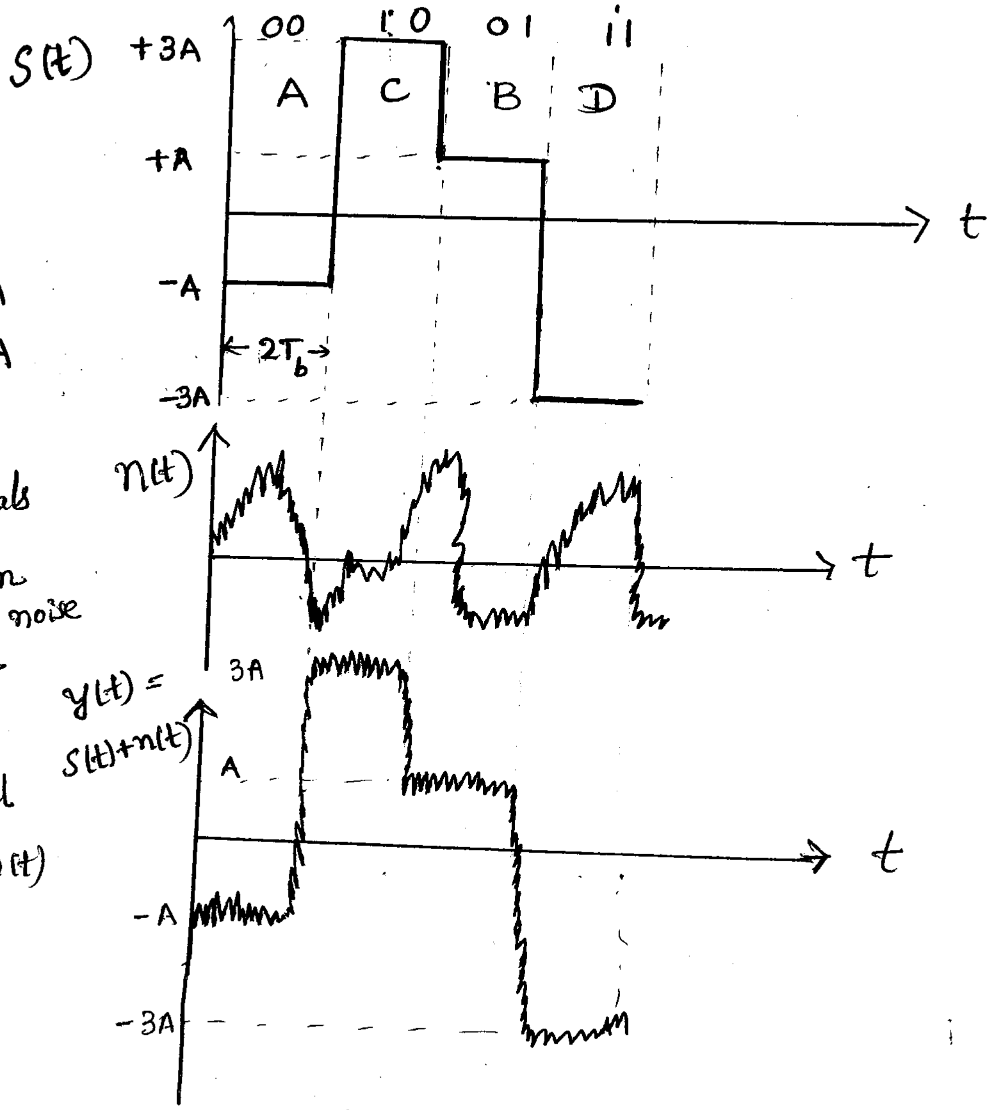
let

$00 \to A = -A$

$01 \to B = +A$

$10 \to C = +3A$

$11 \to D = -3A$



$S(t)$ = Txed signals

$n(t)$ = Zero mean gaussian noise & variance σ^2

$y(t)$ = Rxed signal

$y(t) = S(t) + n(t)$

$y = s + n$.

$y = -A + n$, when symbol 'A' is transmitted $\Rightarrow n = y + A$

$y = A + n$, when symbol B is transmitted $\Rightarrow n = y - A$

$y = 3A + n$, when symbol 'C' is transmitted $\Rightarrow n = y - 3A$.

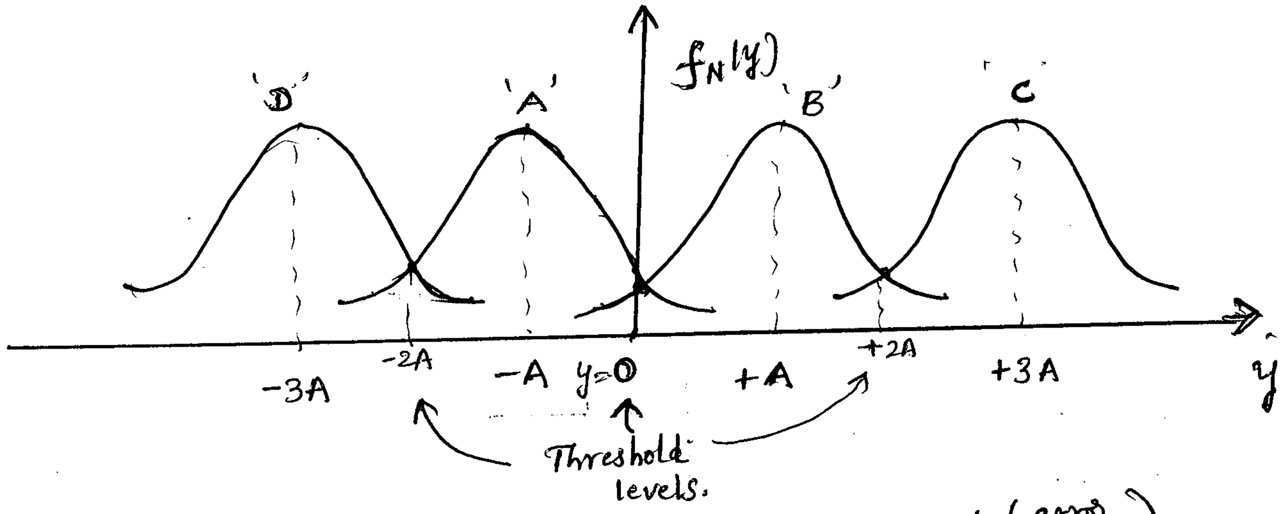
$y = -3A + n$, when symbol 'D' is transmitted $\Rightarrow n = y + 3A$.

* when 'A' is transmitted, $f_y(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y+A)^2/2\sigma^2}$; $-\infty \le y \le \infty$

when 'B' is transmitted, $f_y(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-A)^2/2\sigma^2}$; $-\infty \le y \le \infty$

when 'C' is transmitted, $f_y(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-3A)^2/2\sigma^2}$; $-\infty \le y \le -\infty$

when 'D' is transmitted, $f_y(n) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y+3A)^2/2\sigma^2}$; $-\infty \le y \le -\infty$



Total error probability $P_e = P(\text{'A' sent})\, P\left(\frac{\text{error}}{\text{'A' sent}}\right) + P(\text{'B' sent}) \cdot P\left(\frac{\text{error}}{\text{'B' sent}}\right) + P(\text{'C' sent}) \cdot P\left(\frac{\text{error}}{\text{'C' sent}}\right) + P(\text{'D' sent}) \cdot P\left(\frac{\text{error}}{\text{'D' sent}}\right)$.

If $P(\text{'A' sent}) = P(\text{'B' sent}) = P(\text{'C' sent}) = P(\text{'D' sent}) = 1/4$.

$\therefore$ Probability error $P_e = \frac{1}{4}\left\{ P\left(\frac{\text{error}}{\text{'A' sent}}\right) + P\left(\frac{\text{error}}{\text{'B' sent}}\right) + P\left(\frac{\text{error}}{\text{'C' sent}}\right) + P\left(\frac{\text{error}}{\text{'D' sent}}\right) \right\}$. $\longrightarrow$ (1)

Probability error when 'D' transmitted:

The receiver will wrongly decode the symbol when signal + noise voltage is exceeds '−2A' when 'D' is transmitted.

$$P\left(\frac{\text{error}}{\text{'D' sent}}\right) = P\{y \geq -2A\} = \int_{u=-2A}^{+\infty} f_Y(u) \cdot du$$

$$= \int_{u=-2A}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u+3A)^2/2\sigma^2} du$$

let $z = \frac{u+3A}{\sigma}$

$dz = \frac{du}{\sigma} \Rightarrow du = \sigma dz$

If $u = -2A$, $z = A/\sigma$

$u = \infty$, $z = \infty$

$$= \int_{z=A/\sigma}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \cdot e^{-z^2/2} \cdot \cancel{\sigma} \cdot dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z=A/\sigma}^{\infty} e^{-z^2/2} dz.$$

$\left(\because \text{Q-function } Q(x) = \frac{1}{\sqrt{2\pi}} \int_{u=x}^{\infty} e^{-u^2/2} du\right)$

$= Q(A/\sigma)$

$= Q(z)$

$$\therefore \boxed{P\left(\frac{\text{error}}{\text{'D' sent}}\right) = Q(A/\sigma)} \longrightarrow (2)$$

Probability of error when 'A' is transmitted :

The receiver will wrongly decode the symbol when signal + noise voltage is greater than '0' & less than '−2A', when symbol 'A' is transmitted.

$$P\left(\frac{\text{error}}{\text{'A' sent}}\right) = P\{y \geq 0\} + P\{y \leq -2A\}$$

$$= \int_{u=0}^{+\infty} f_y(u)\,du + \int_{u=-\infty}^{-2A} f_y(u)\,du$$

$$= \int_{u=0}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u+A)^2}{2\sigma^2}}\,du + \int_{u=-\infty}^{-2A} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u+A)^2/2\sigma^2}\,du.$$

let $z = \frac{u+A}{\sigma}$

$dz = \frac{du}{\sigma} \Rightarrow du = \sigma dz$

If $u = 0$, $z = A/\sigma$

$u = \infty$, $z = \infty$

If $u = -\infty$, $z = -\infty$

$u = -2A$, $z = -A/\sigma$

$$= \int_{z=A/\sigma}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2}\,dz\cdot\cancel{\sigma} + \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{z=-\infty}^{-A/\sigma} e^{-z^2/2}\,\cancel{\sigma}\,dz.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z=A/\sigma}^{\infty} e^{-z^2/2}\,dz + \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{-A/\sigma} e^{-z^2/2}\,dz$$

$$= Q(A/\sigma) + Q(-A/\sigma) \qquad (\because Q(-x) = Q(x))$$

$$\therefore \boxed{P\left(\frac{\text{error}}{\text{A' sent}}\right) = 2Q(A/\sigma)} \longrightarrow (3)$$

Probability of error when 'B' is transmitted :

The receiver will wrongly decode the symbol when signal + noise voltye exceeds '+2A' and less than '0', when Symbol 'B' is Txed.

$$P\left(\frac{\text{error}}{\text{B' sent}}\right) = P\{y \geq 2A\} + P\{y \leq 0\}$$

$$= \int_{u=2A}^{+\infty} f_y(u)\,du + \int_{u=-\infty}^{0} f_y(u)\,du$$

$$= \int_{u=2A}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(u-A)^2}{2\sigma^2}}\,du + \int_{u=-\infty}^{0} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-A)^2/2\sigma^2}\,du$$

let $z = \frac{u-A}{\sigma}$

$dz = \frac{du}{\sigma} \Rightarrow du = \sigma dz$

If $u = 2A$, $z = A/\sigma$

$u = \infty$, $z = \infty$

If $u = -\infty$, $z = -\infty$

$u = 0$, $z = -A/\sigma$

$$= \int_{z=A/\sigma}^{\infty} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2}\,\cancel{\sigma}\,dz + \int_{z=-\infty}^{-A/\sigma} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} e^{-z^2/2}\,\cancel{\sigma}\,dz.$$

$$= \frac{1}{\sqrt{2\pi}} \int_{z=A/\sigma}^{\infty} e^{-z^2/2}\,dz + \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{-A/\sigma} e^{-z^2/2}\,dz$$

$$= Q(A/\sigma) + Q(-A/\sigma)$$
$$= 2Q(A/\sigma) \qquad (\because Q(-x) = Q(x))$$

$$\therefore \boxed{P\left(\frac{\text{error}}{\text{B sent}}\right) = 2Q(A/\sigma)} \longrightarrow (4)$$

Probability error when 'C' is transmitted:

The receiver will wrongly decode the symbol when signal + noise voltage is less than +2A, when symbol 'C' is transmitted.

$$P\left(\frac{\text{error}}{\text{'C' sent}}\right) = P\{y \le +2A\}$$
$$= \int_{u=-\infty}^{+2A} f_y(u) = \int_{u=-\infty}^{+2A} \frac{1}{\sigma\sqrt{2\pi}} e^{-(u-3A)^2/2\sigma^2} du.$$

$$= \int_{z=-\infty}^{-A/\sigma} \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \cdot e^{-z^2/2} \cdot \cancel{\sigma} \cdot dz$$

let $z = \frac{u-3A}{\sigma}$

$dz = \frac{du}{\sigma} \Rightarrow du = \sigma dz$

if $u = -\infty$, $z = -\infty$

$u = +2A$, $z = -A/\sigma$.

$$= \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{-A/\sigma} e^{-z^2/2} dz$$

$$= Q(-A/\sigma) \qquad (\because Q(-x) = Q(x))$$
even function.
$$= Q(A/\sigma)$$

$$\therefore \boxed{P\left(\frac{\text{error}}{\text{'C' sent}}\right) = Q(A/\sigma).} \longrightarrow (5)$$

$\therefore$ From eqn (1), (2), (3), (4) & (5).

$$\therefore P_e = \frac{1}{4}\{Q(A/\sigma) + 2Q(A/\sigma) + 2Q(A/\sigma) + Q(A/\sigma)\}$$
$$= \frac{1}{4} \cdot 6 Q(A/\sigma)$$
$$P_e = 6/4 \cdot Q(A/\sigma) \qquad \boxed{\therefore P_e = 6/4 \cdot Q(A/\sigma)} \text{ for } \underline{\text{M-ary Scheme}}.$$

In general probability error

$$\therefore \boxed{P_e = \frac{2(m-1)}{m} Q[A/\sigma)} \quad \text{for 'm' symbols.}$$

Base Band Binary data Transmission System:

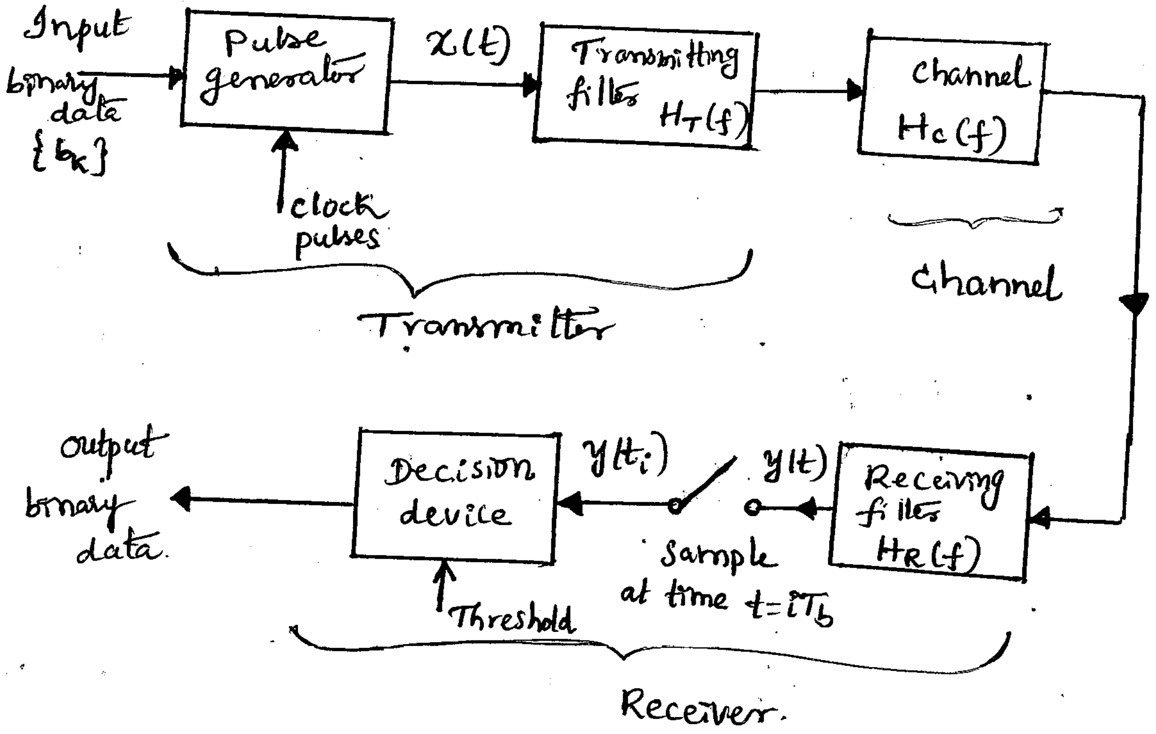


Fig: Baseband binary transmission system.

- Discrete pulse amplitude modulation is the most suitable technique for Txion of base band signal data.
- It is most efficient in terms of power and bandwidth
- Here, the amplitude of the transmitted pulse is varied in a discrete manner in accordance with the given digital data.

* The input to the system is a binary sequences (In the form of '0' and 1's) with a bit rate of r_b and bit duration of T_b

∴ The pulse generator output is a pulse wave form

$$x(t) = \sum_{K=-\infty}^{\infty} a_K \cdot P(t - K \cdot T_b) \longrightarrow (1)$$

where $P(t)$ is the basic pulse & normalize $p(0)=1$

a_K is represents the coefficient of the pulse & this varies based on the line code & Txion code. and a_K depends on the K^{th} Input bit

ie $p(t) = 1 \quad , t=0$
$= 0 \quad , t = \pm T_b, 2T_b, \ldots$

amplitude a_K depends on identity (1 or 0) of b_K

✓ Let us assume Bipolar NRZ scheme

$a_K = +A$; if the input bit b_K is symbol '1'
$= -A$; if the input bit b_K is symbol '0'.

✓ $x(t)$ passes through the transmitting filter having a transfer function. $H_T(f)$.

✓ The output of the Txing filter defines the Txed signal which is subsequently passed through the channel having a transfer function $H_c(f)$.

✓ At the receiver side the received signal is passed through a receiving filter of transfer function $H_R(f)$.

✓ The output of the receiving filter is sampled synchronously with the transmitter.
The sampling instants are determined by a clock pulse that is extracted from the receiving filter output.

✓ The sequence of samples obtained at the filter output are then fed to a decision device.

✓ The function of decision device is to reconstruct the original data.

* The decision device compares the amplitude of each sample to a threshold.

✓ If the threshold is exceeded, a decision is taken in favour of symbol '1'.

* If the amplitude of the sample is below the threshold, a decision is made in favour of symbol '0'.

✓ If the amplitude of a sample is exactly equal to the threshold, any one of the symbol '0' & '1' may be chosen without affecting the overall performance.

Inter Symbol Interference : (ISI)

" When a digital data is transmitted over a band limited Channel, dispersion in the channel gives rise to interference called ISI (Inter Symbol Interference)".

This effect introduces deviations (errors) between the reconstructed data at the receiver and the original data at the transmitter.

✓ It should be noted that noise is not only factor that distorts the signal on the channel. In the absence of noise also the channel cause a dispersion (or Spreading) of the pulse shaping.

ie

When the channel bandwidth is greater than bandwidth of pulse, Spreading of pulse is very less, but when channel bandwidth is close to signal bandwidth ie if we transmit digital data which demands more bandwidth, spreading will occur and cause signal pulses to overlap.

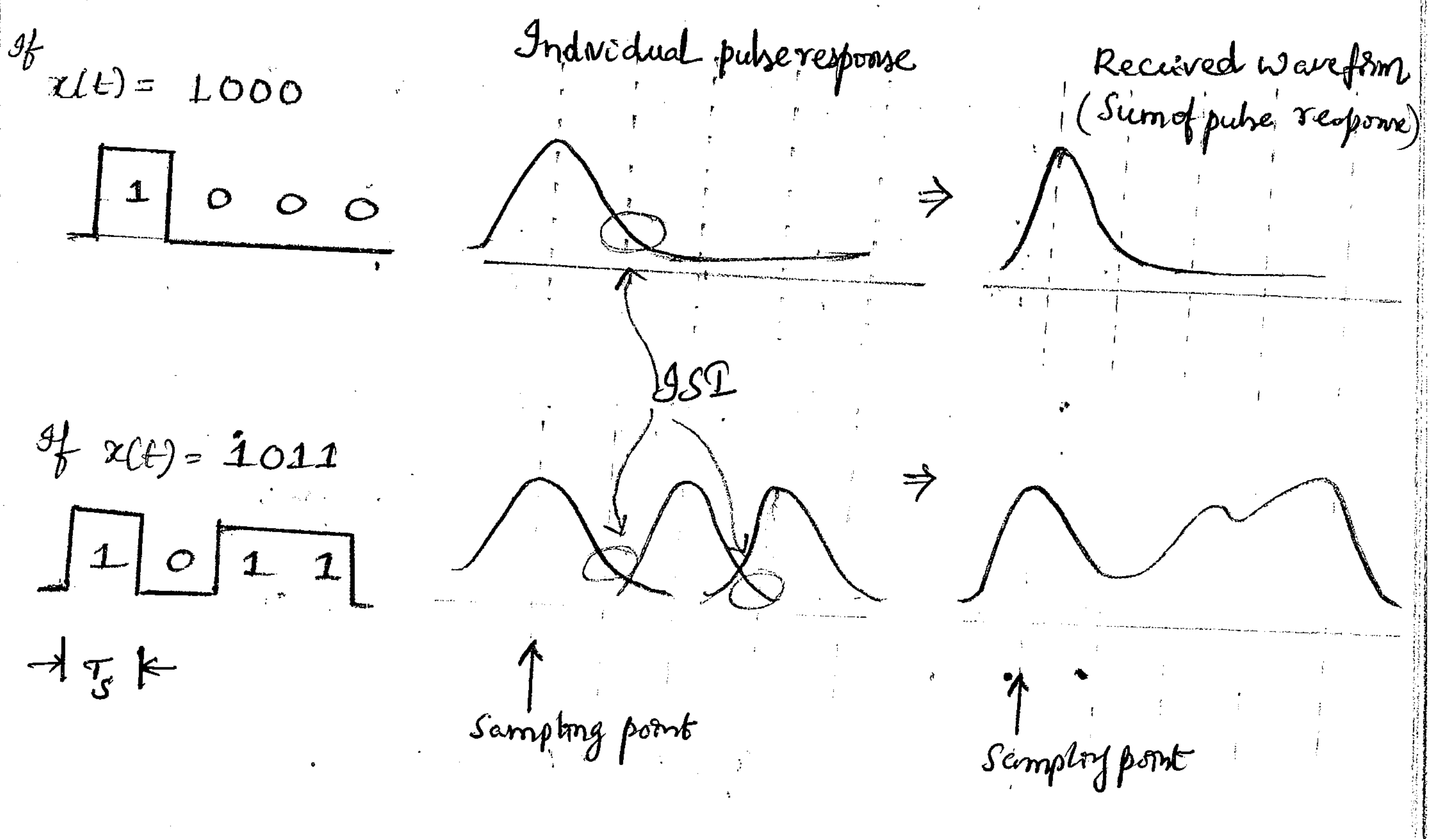


fig: Occurance of Intersymbol Interference.

The receiving filter output may be written as

Mathematically

$$y(t) = \mu \sum_{k=-\infty}^{\infty} a_k \cdot p(t - kT_b - t_d) \longrightarrow (2)$$

where μ - the scaling factor

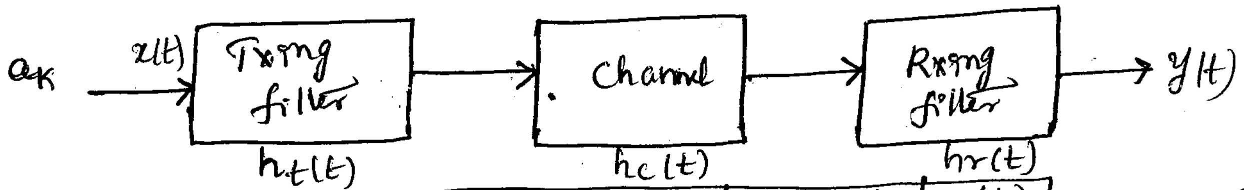
t_d - the time delay introduced by the channel

let us assume $p(t)$ is normalized in such a way that

$$p(0) = 1.$$

✓ The output $y(t)$ is obtained by double convolution involving the impulse response $h_t(t)$ of the Txing filter, the impulse response $h_c(t)$ of the channel & the impulse response $h_r(t)$ of the recvng filter

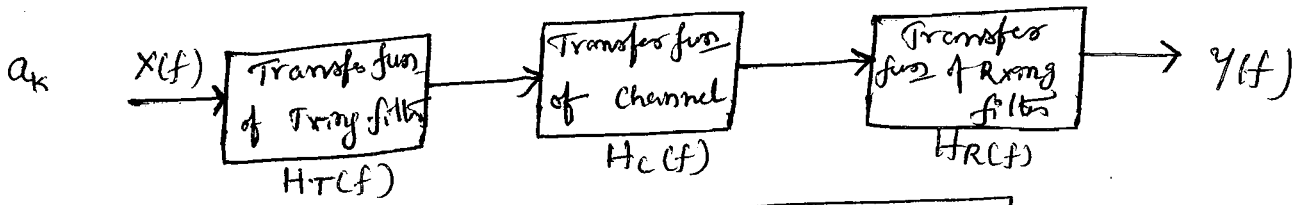
ie



$$\boxed{y(t) = x(t) * h_t(t) * h_c(t) * h_r(t)} \longrightarrow (3)$$

assume $t_d = 0$ (In time domain).

✓ In frequency domain the output of receiving filter is



$$\boxed{Y(f) = X(f) \cdot H_T(f) \cdot H_c(f) \cdot H_R(f)} \longrightarrow (4)$$

(In frequency domain).

✓ The output of Receiving filter is sampled at time $t_i = iT_b$ where i - is an Integer. & $t_d = 0$.

$$\therefore y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k \cdot p(iT_b - kT_b) + n(t) \longrightarrow (5)$$

$\hookrightarrow t = it_b,\ t_d = 0$ $\quad\hookrightarrow$ noise.

$$\therefore y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k \cdot p[(i-k)T_b] \qquad i = 0, \pm 1, \pm 2, \pm 3, \ldots$$

If $i = K$ then (In the absense of ISI)

$$y(t_i) = \mu a_i + n(t)$$

where μ - the scaling factor

$n(t)$ = noise added at the channel.

ie If $i \neq K$ then

$$y(t_i) = \underbrace{\mu a_i}_{\text{First}} + \underbrace{\mu \sum_{\substack{K=-\infty \\ i \neq K}}^{\infty} a_K \, p[(i-K)T_b] + n(t)}_{\text{Second}} \longrightarrow (6)$$

* The first term, μa_i is produced by the i^{th} Txed bit, but depends on the contribution of all other Txed bits.
* The second term represents the residual effect of all other txed bits on the decoding of the i^{th} bit. This residual effect is called Intersymbol Interference (ISI).

✓ The presence of channel introduced ISI & noise in the system however introduces errors in the decision device at the Rxing o/p.

* In order to reduce the errors caused by ISI to a min, it is necessary to design the Txing & Rxing filters suitably.

Nyquist Criterian :

For distortion less baseband data Txion :

✓ For Reconstructing the original binary data sequence $\{b_K\}$ and need to extract & then decode the corresponding sequence of samples, from the output $y(t)$.

We have to determine the transfer funs of the Txing & receiving filter and also shape of the Txed pulse is required.

✓ The extration of output involves sampling the output $y(t)$ at time $t = iT_b$.

For zero ISI, the condition should be

$$p[(i-K)T_b] = \begin{cases} 1 \; ; & i = K \\ 0 \; ; & i \neq K \end{cases} \longrightarrow (7)$$

ie The each pulse is zero at the sampling time of other pulses and this will results only when the pulse shape is a Sinc function

∴ The receiver output $y(t)$ is given as

$$\boxed{y(t) = \mu a_i} \quad \forall i \longrightarrow (8)$$

Hence above condition ensures perfect reception in the absence of noise.

Consider the sequence of samples $p\{nT_b\}$

where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

∴ Fourier transform of $p(nT_b)$

$$P_\delta(f) = F\{p(nT_b)\} = \frac{1}{T_b} \sum_{n=-\infty}^{\infty} P(f - nR_b) \longrightarrow (9)$$

The fourier transform of an infinite periodic sequence of delta function of period T_b.

$$\therefore P_\delta(f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[p(mT_b) \cdot \delta(t - mT_b)\right] e^{-j2\pi ft}\, dt \longrightarrow (10)$$

where $m = i-k$ an integer then $i=k,\ m=0$; $i \neq k,\ m \neq 0$.

According to the conditions of eqn (7), eqn (10) can be written as

ie $i=k$, $m=0$. $\quad \therefore P_\delta(f) = \int_{-\infty}^{\infty} p(0) \cdot \delta(t) \cdot e^{-j2\pi ft}\, dt. \longrightarrow (11)$

As per the property of delta functions. From eqn (9) & (11)

$$\underset{1}{p(0)} = R_b \sum_{n=-\infty}^{\infty} P(f - nR_b)$$

$$\therefore \boxed{\sum_{n=-\infty}^{\infty} P(f - nR_b) = \frac{1}{R_b} = T_b} \longrightarrow (12)$$

Thus,

The nyquist criterian for distortionless baseband Txion in the absence of noise, the frequency fun. $P(f)$ eliminates ISI for samples taken at Intervals T_b provided that is satisfies the eqn (12).

Ideal Nyquist Channel : (or) Ideal solution of ISI :

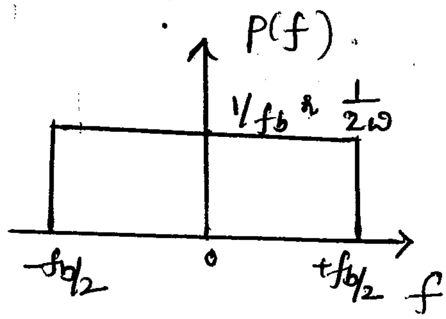
The simplest way of satisfying eqn (12) is to specify the frequency fun $P(f)$ to be in the form of a rectangular fun.

ie

$$P(f) = \begin{cases} \frac{1}{2W} & ; -W \le f < W \\ 0 & ; |f| > W. \end{cases} \longrightarrow (13)$$

The overall system bandwidth W is defined by

$$W = \frac{R_b}{2} = \frac{1}{2T_b} = f_b/_2 \longrightarrow (14)$$



Applying Inverse fourier transform

$$p(t) = F^{-1}\{P(f)\} = \int_{-\infty}^{\infty} P(f) \cdot e^{j2\pi f t} df.$$

$$= \int_{-f_b/2}^{f_b/2} \frac{1}{2W} \cdot e^{j2\pi f t} df \qquad \left(\because \frac{1}{2W} = \frac{1}{2 \cdot f_b/2} = \frac{1}{f_b} = T_b\right.$$

$$= T_b \cdot \left[\frac{e^{j2\pi f t}}{j2\pi t}\right]_{-f_b/2}^{f_b/2}$$

$$= T_b \left[\frac{e^{j2\pi f_b/_2 t} - e^{-j2\pi f_b/_2 t}}{j2\pi t}\right]$$

$$= \frac{T_b}{\pi t}\left[\frac{e^{j\pi f_b t} - e^{-j\pi f_b t}}{2j}\right]$$

$$= \frac{T_b}{\pi t} \sin(\pi f_b t) \qquad (f_b = 1/T_b)$$

$$p(t) = \frac{\sin(\pi f_b t)}{(\pi f_b t)} \qquad \left(\because \frac{\sin \pi x}{\pi x} = \operatorname{sinc} x\right)$$

$$p(t) = \operatorname{Sinc}(f_b t) \longrightarrow (15)$$

$$\therefore \boxed{p(t) = \operatorname{Sinc}(f_b t)}$$

The special value of the bit rate R_b & $f_b = 2W$ is called the nyquist rate & Sinc function produces zero ISI.

Fig (a): Ideal amplitude response

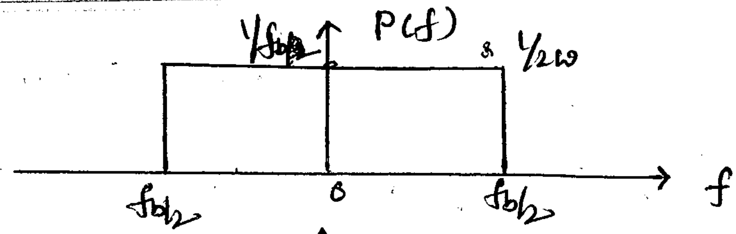
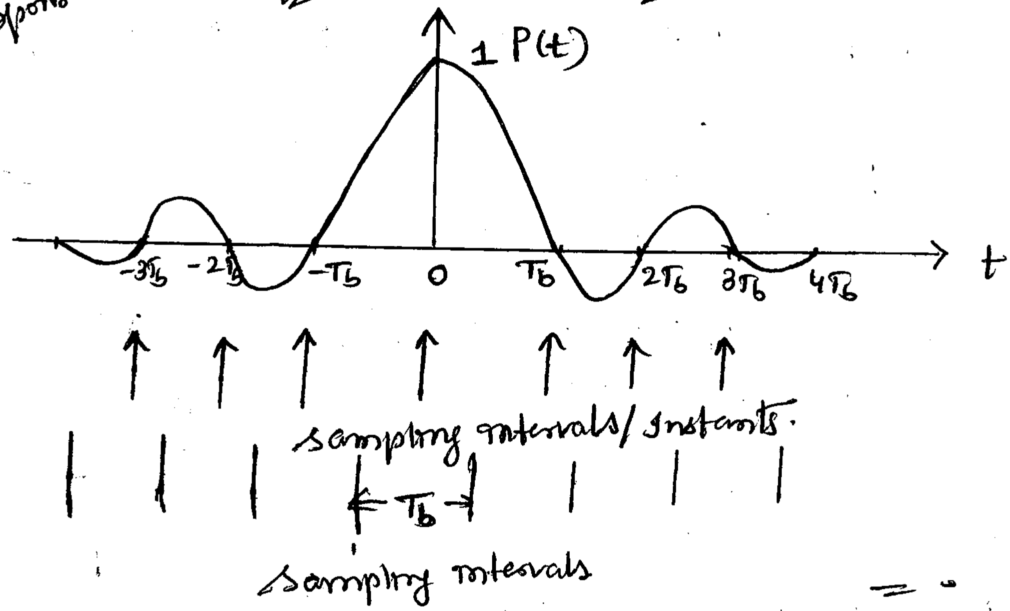


Fig (b): Ideal basic pulse shape.



Thus,

Nyquist criterian reduces ISI with min. bandwidth possible, there are two practical difficulties that make it an undesirable objective for system design.

① It requires that the amplitude characteristics of $P(f)$ be flat from $-f_b/2$ to $+f_b/2$ & zero elsewhere. This is physically undesirable because of the abrupt transition at the band edges $\pm f_b/2$.

② The function $p(t)$ decreases as $\frac{1}{|t|}$ for large $|t|$. resulting is a slow rate of decay, this is also caused by the discontinuity of $P(f)$ at $\pm f_b/2$.

To evaluate timing error, consider the sample of $y(t)$ are $t = \Delta t$, where Δt is the timing error.

$$y(\Delta t) = \mu \sum_{k=-\infty}^{\infty} a_k \cdot p(\Delta t - kT_b) \longrightarrow (16)$$

(∵ absence of noise)

$$= \mu \sum_{k=-\infty}^{\infty} a_k \cdot \frac{\sin \pi f_b (\Delta t - kT_b)}{\pi f_b (\Delta t - kT_b)}$$

Since the definition $f_b \cdot t = 1$
ie $T_b = 1$.

$$y(\Delta t) = \underset{\substack{\uparrow \\ \text{desired symbol}}}{\mu a_0 \cdot \operatorname{sinc}(f_b \Delta t)} + \underset{\substack{\uparrow \\ \text{ISI caused by} \\ \text{timing error } \Delta t}}{\mu \frac{\sin(\pi f_b \Delta t)}{\pi} \cdot \sum_{K \neq 0} \frac{(-1)^K \cdot a_K}{(f_b \cdot \Delta t - K)}} \longrightarrow (17)$$

$(\because \sin(x - \pi k) = (-1)^K \cdot \sin x.$

Practical Nyquist Channel : (or) Practical Solution of ISI : (or) Raised Cosine filter Characteristics :

- To overcome practical difficulties encountered with the ideal nyquist channel, here extending the bandwidth from minimum value $W = R_b/2$ to an adjustable value between W and $2W$.

We know $\sum_{n=-\infty}^{\infty} P(f - nR_b) = T_b = 1/f_b$.

The frequency band of Interest $[-f_b/2 ; f_b/2]$ as

$$P(f) + P(f - f_b) + P(f + f_b) = \frac{1}{f_b} \quad ; -f_b/2 \le f \le f_b/2$$

- The particular form of $P(f)$ that desirable features is provided by a Raised cosine spectrum.

This frequency characteristics consists of a flat portion and a roll-off portion that has a sinusoidal form.

$\therefore$ The raised consine function is defined as.

$$\boxed{P(f) = \begin{cases} T_b & ; |f| \le (f_b/2 - \beta) \\ T_b \cdot \cos^2\left[\frac{\pi}{4\beta}(|f| - f_b/2 + \beta)\right] & ; (f_b/2 - \beta) \le f \le (f_b/2 + \beta) \\ 0 & ; |f| > (f_b/2 + \beta) \end{cases}}$$

where β - roll-off factor " $0 \le \beta \le f_b/2$.

$$\boxed{p(t) = \frac{\cos(2\pi\beta t)}{1-(4\beta t)^2} \cdot \frac{\sin \pi f_b t}{\pi f_b t}}$$

at $t = nT_b$

$$\therefore p(nT_b) = \frac{\cos(2\pi\beta n T_b)}{1-(4\beta n T_b)^2} \cdot \frac{\sin n\pi}{n\pi}$$

If Roll-of factor $\beta = 0$ $\therefore$ $\boxed{p(0) = \frac{\sin n\pi}{n\pi}}$ $(\cos 0 = 1)$

If $\beta = f_b/4$ then

$$p(f) = \begin{cases} T_b & ; \ |f| \le f_b/4 \\ T_b \cdot \cos^2\left[\frac{\pi}{f_b}\left(|f| - f_b/4\right)\right] & ; \ f_b/4 \le |f| \le 3f_b/4 \\ 0 & ; \ |f| > 3f_b/4 \end{cases}$$

the roll off factor is a measure of the excess bandwidth of the filter

$$\therefore \boxed{p(nT_b) = \frac{\cos n\pi/2}{1-n^2} \cdot \frac{\sin n\pi}{n\pi}}$$

If $\beta = f_b/2$ then

$$p(f) = \begin{cases} T_b & ; \ |f| = 0 \\ T_b \cdot \cos^2\left(\frac{\pi}{2f_b}|f|\right) & ; \ 0 \le |f| \le f_b \\ 0 & ; \ |f| > f_b \end{cases}$$

$$\therefore \boxed{p(nT_b) = \frac{\cos n\pi}{1-4n^2} \cdot \frac{\sin n\pi}{n\pi}}$$

Thus The frequeny response $p(f)$ normalized by multiplying it by 'f_b' for three values of β namely $0, f_b/4, f_b/2$.

- The time response $p(t)$ is the inverse fourier transform of function $p(f)$.

* For non zero value of 'β', the function $p(f)$ cuts gradually and it is therefore easier to realize it in practice.

Raised Consine function in frequny domain.

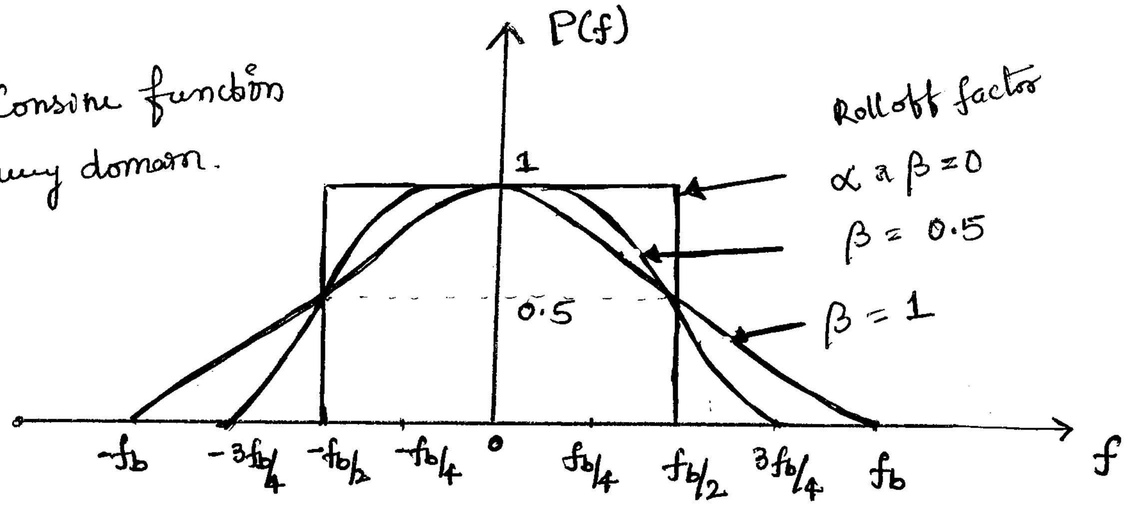
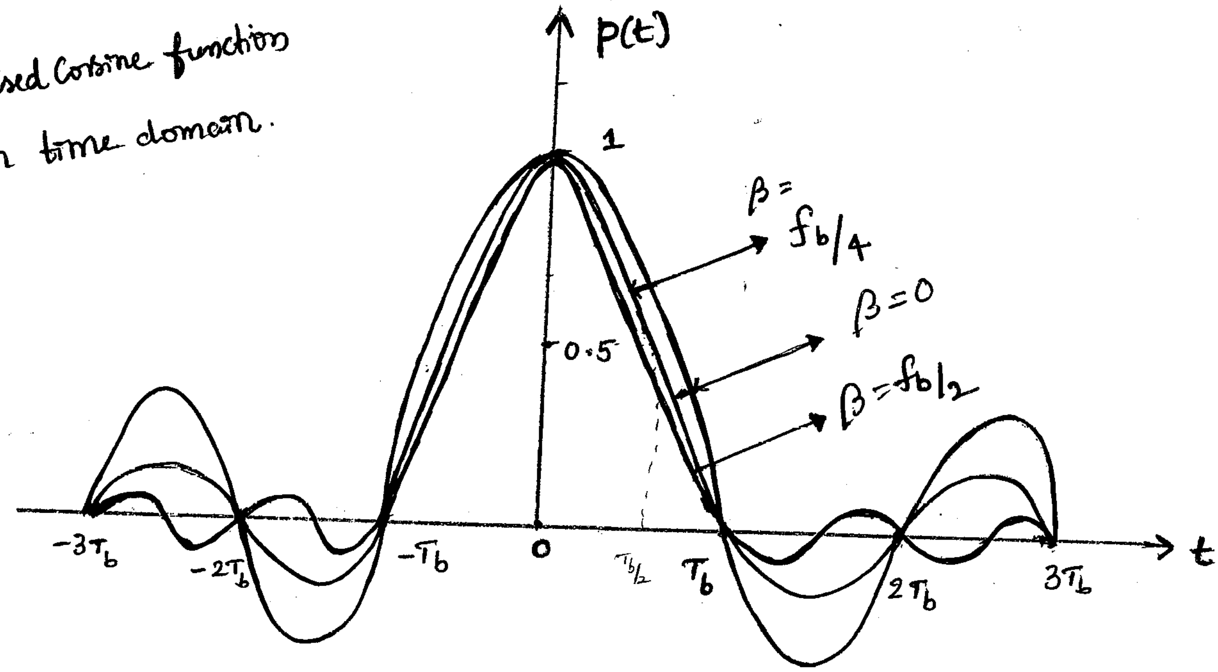


Fig: Frequeny response for different roll-off factors.

Raised Cosine function In time domain.



At $t = \pm T_b/2$, we have $p(t) = 0.5$ ie pulse width measured at half amplitude is exactly equal to the bit duration T_b

Further, there are zero crossing at $t = \pm 3T_b/2, \pm 5T_b/2 \ldots$ in addition to the usual zero crossing at the sampling rate (times) $t = \pm T_b, \pm 2T_b, \ldots$ etc.

These two properties of the time response are used for generating timing signals from the received signal for the purpose of Synchronization.

* The transmission bandwidth requirement of raised cosine solution is

$\boxed{\text{Bandwidth} = B_0(1+\beta)}$ where $B_0 = W = f_b/2 \rightarrow$ Nyquist bandwidth
β - rolloff factor. $\boxed{0 \le \beta \le f_b/2}$

For large values of β, the bandwidth is increased in frequny domain & faster decaying in the time domain.

Problem

Consider the T1 carrier system designed to accommodate 24 voice channels based on 8 bit PCM word. Calculate the bandwidth of the T1 system assuming.

(a) Ideal Low pass characteristics
(b) A raised cosine spectrum with α & $\beta = 1$ for base band pulse shaping

Sol.

The filtered voice signal in T1 system is usually sampled at 8 KHz - which is the standard sampling rate in digital telegraphy. Each frame of multiplexed signal occupies a period of 125 μsec. Each frame consists of 193 bits.

∴ The bit rate of the T1 system is

$$R_b = 193 \text{ bits/frame} \times 8000 \text{ frames/sec}$$

$$R_b = 1.544 \text{ M bits/sec}$$

$$\therefore \boxed{R_b = 1.544 \text{ M bits/sec}}$$

$$T_b = 1/R_b = 0.647 \mu\text{sec}$$

(i) For an ideal low pass characteristics of the channel, the nyquist bandwidth of the system is

$$B_0 \text{ \& } W = f_b/2 = \frac{1}{2T_b} = 772 \text{ KHz}.$$

$$\therefore \boxed{B_0 \text{ \& } W = 772 \text{ KHz}}$$

Which is the min. transmission bandwidth of T1 system for zero ISI

(ii) for a raised cosine spectrum with α & $\beta = 1$

The transmission bandwidth

$$B.W = B_0 (1+\beta)$$

$$= 2 B_0$$

$$= 2 \cdot W = 2 \cdot f_b/2 = f_b = 1/T_b = R_b$$

$$\therefore B.W = 1/T_b = R_b = 1.544 \text{ MHz}.$$

$$\therefore \boxed{\text{Bandwidth} = 1.544 \text{ MHz}.}$$

Eye pattern (or) Eye diagram :

" The intersymbol Interference (ISI) in data transmission can be studied with the help of a display on the oscilloscope called <u>eye diagram</u> ".

- ✓ The received distorted wave is applied to the vertical deflection plates of an oscilloscope and the saw-tooth wave at a rate equal to the txed symbol rate $1/T$ is applied to the horizontal deflection plates.

- ✓ The waveforms in successive symbol intervals are thus translated into one interval on the oscilloscope display.

Let us consider the distorted but noise free Bi-polar binary symbol s/m.

Binary data

+1 → 1
−1 → 0

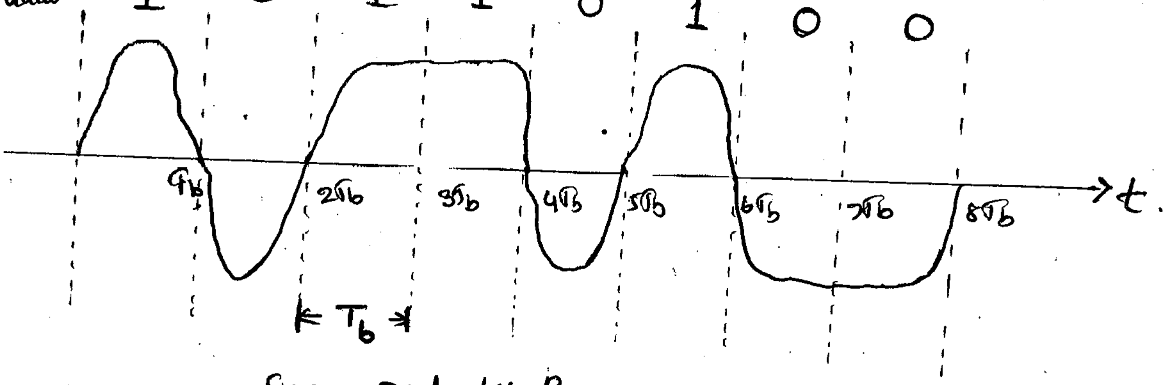
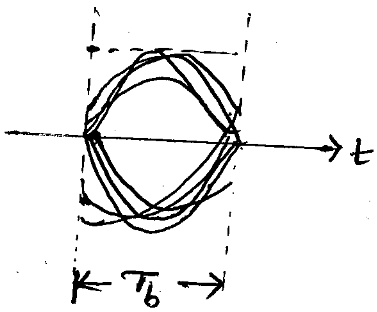


Fig : Distorted Binary wave.



← Fig: Eye pattern.

- ✓ When displayed on a Long persistence oscilloscope, we get superposition of successive symbol intervals to produce the eye-pattern.

" The pattern is So called because of its resemblance to the human eye. The interior region is called the "eye opening" ".

* A generalized binary eye pattern with labels identifying the significant features as follows.

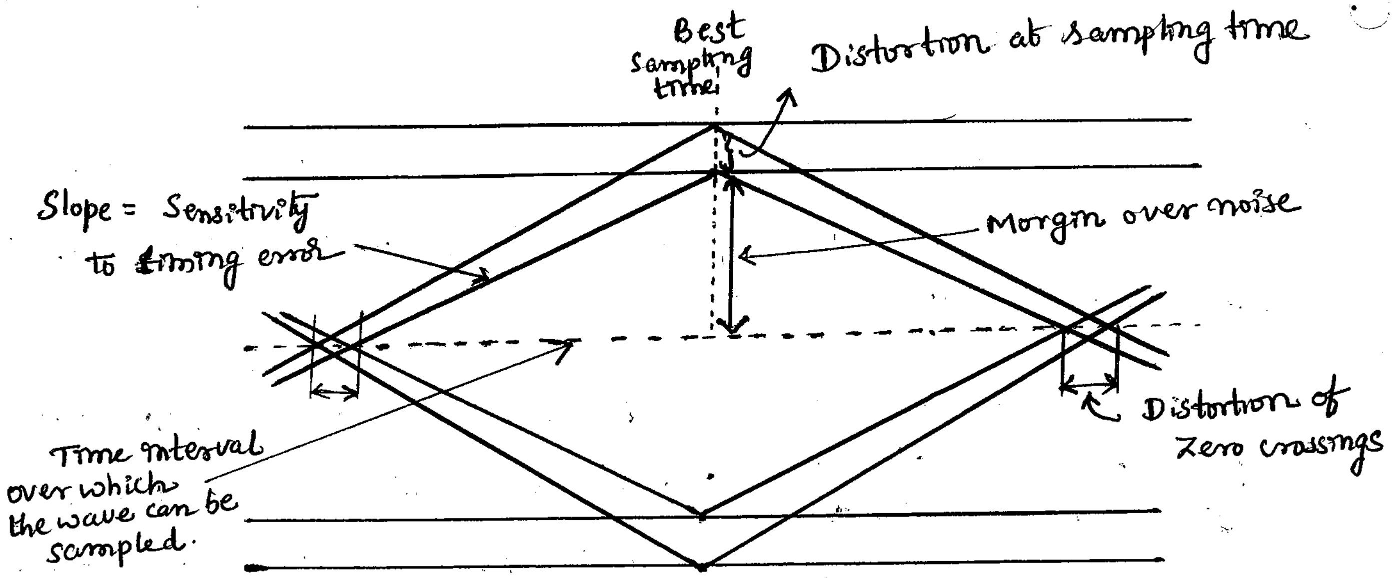


Fig: Interpretation of eye pattern.

* The eye pattern provide the following information about the performance of the system.

(1) The width of the eye opening defines the time interval over which the received wave can be sampled without error from ISI. The optimum sampling time corresponds to the maximum eye opening.

(2) The height of the eye opening at a specified sampling time is a measure of the margin over channel noise.

(3) The sensitivity of the system to timing error is determined by the rate of closure of the eye of the eye pattern in the vicinity of zero crossings as the sampling time is varied.

(4) Any non-linear transmission distortion would reveal itself in an asymmetric & squinted eye.

✓ When the effect of ISI is excessive, traces from the upper portion of the eye pattern cross traces from lower portion with the result that the eye is completed closed. In such a case, it is impossible to avoid error due to combined presence of Intersymbol Interference (ISI) and channel noise in the system.

Equalization:

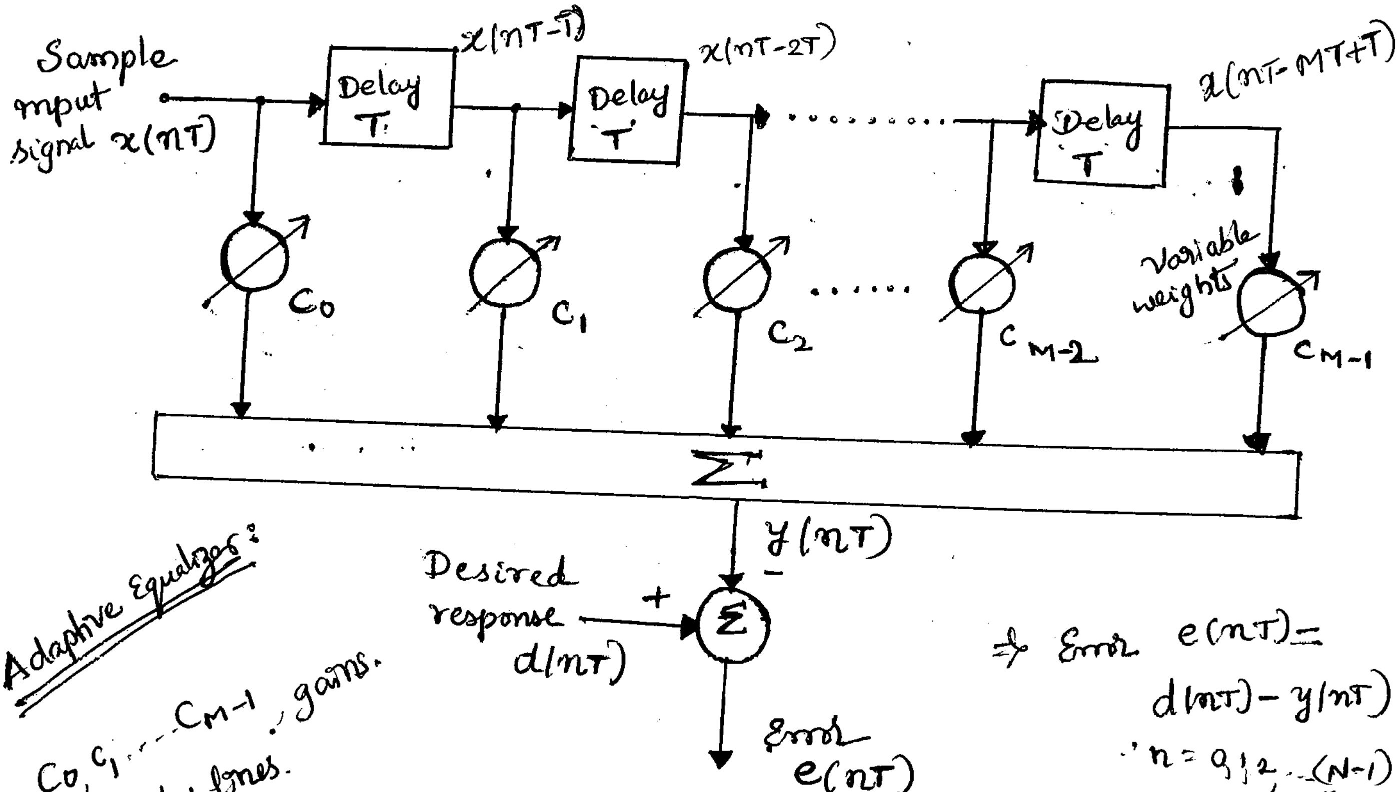
- " Due to an imperfect filter design and changes in the channel characteristics, Intersymbol Interference will be introduced in the received signal, The process of correcting channel induced distortion is called <u>equalization</u> ".

There are two types of equalizers.

1. Transversal equalizer
2. ~~Automatic~~ Equalizer
   (a) Preset Equalizer
   (b) Adaptive Equalizers

- The channel introduces signal distortion in the base band data, as a result any real channel needs equalization to approach an ideal frequency response for transmission of modulated digital signals.

* Equalization is very important in the high speed transmission of digital data over a voice grade telephone channel which is essentially linear and bandlimited having high signal-to-noise ratio.



Adaptive Equalizers:

$C_0, C_1, \ldots C_{M-1}$ gains.

tapped delay lines.

→ Error $e(nT) = d(nT) - y(nT)$

$n = 0, 1, 2, \ldots (N-1)$

length of the sequence

Fig: <u>Elements of Adaptive Equalizer/filter.</u>

✓ The structure is a tapped delay line filter that consists of a set of delay elements, a set of adjustable multipliers connected to the delay line taps and a summer for adding the multiplier outputs.

Let $x(nT)$ be the sequence is applied to the input of the tapped delay line filter.

∴ The output of the delay line filter will be

$$y(nT) = \sum_{i=0}^{M-1} c_i x(nT - iT) = \sum_{i=0}^{M-1} c_i\, x[nT - iT]$$

where c_i - the multiplying coefficients at the i^{th} tap.
M - the total number of taps.

* The tap spacing along the delay line is chosen equal to the symbol duration T of the Txed signal that is the reciprocal of the sampling rate.

∴ The Error Sequence $e(nT)$ is given by

$$\boxed{e(nT) = d(nT) - y(nT)}$$

wh $n = 0, 1, 2, \dots N-1$

wh N - the total length of the sequence.

where $d(nT)$ - the known Txed sequence as the desired Response.

✓ For optimization is one that minimises the total Error Energy defined by

$$\boxed{\mathcal{E} = \sum_{n=0}^{N-1} e^2(nT)}$$

The optimum values of the tap coefficients $c_0, c_1, c_2 \dots$ result when the total error energy is minimized.

The solution to this optimization problem may be achieved with the help of an algorithm that adjusts the tap weights of the filter in a recursive manner.

From the incoming data, sample by sample to automatically adjust the tap coefficients towards the optimum solution.

Correlative Coding :

Inter symbol Interference (ISI) is an undesirable phenomenon the produces a degradation in system performance.
However, by adding ISI to the transmitted signal in a controlled manner to achieve a bit rate higher than the bandwidth of the channel.

eg. Correlative Coding is transmit signal at a rate of $2R_b$ & $2W$ symbols per second in a channel having a bandwidth of R_b & W & B_0.
It is also known a partial response signalling (or) poly binary (or) Duo binary signalling ".

To avoid the problem associated with the ideal solution
$P(t) = \text{Sinc}(f_b \cdot t)$ ($\because$ Ideal nyquist channel).
The Example of correlating Coding is Duo Binary Signalling scheme.

Duo-Binary Signalling :

✓ It is the simplest type of correlative coding that employs a correlation span of one binary digit.

* Duo - stands to imply doubling of transmission capacity as compared to straight binary system.

✓ The modified duobinary signalling employes a correlation span of two binary digits.

Let us consider a binary input sequence $\{b_k\}$ consisting of un-correlated digits / symbols 1 and 0 each having duration T_b

ie
$$b_k = \begin{cases} +1 & ; \text{ if symbol } b_k \text{ is } 1 \\ -1 & ; \text{ if symbol } b_k \text{ is } 0 \end{cases} \longrightarrow (1)$$

✓ When the sequence is applied to a duo binary encoder, it is converted into a three level output ie $-2V$, $0V$, & $+2V$.

* The two level Sequence $\{b_k\}$ is first passed through a simple filter involving a single delay element, direct path & summer as follows.

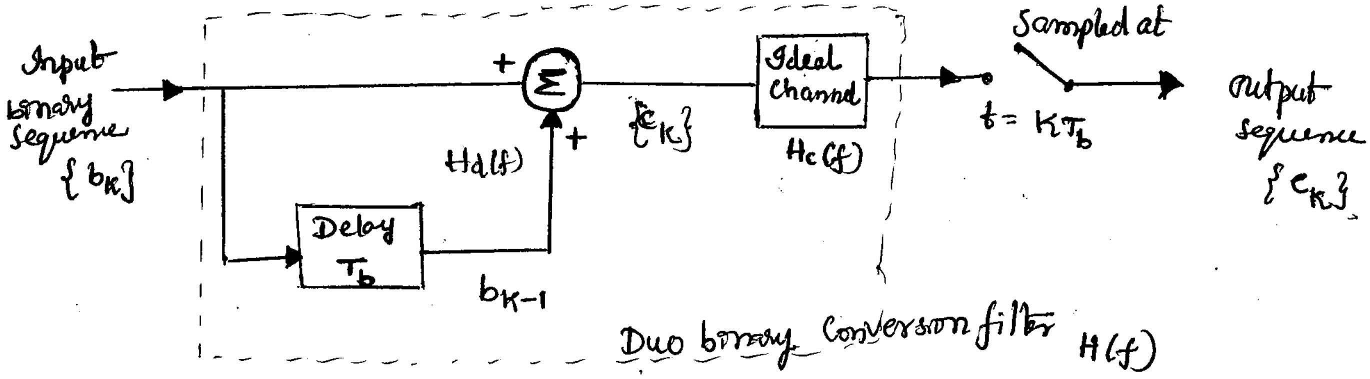


Fig: Duobinary Signalling Scheme.

Encoding:

✓ The summer output is denoded as C_K → Correlated output

ie
$$C_K = b_K + b_{K-1} \longrightarrow (2)$$

where
$$\left.\begin{aligned} C_K &= +2V \quad ; \; b_K = b_{K-1} = 1 \\ &= 0V \quad ; \; b_K \neq b_{K-1} \\ &= -2V \quad ; \; b_K = b_{K-1} = 0. \end{aligned}\right\} \longrightarrow (3)$$

" The correlation between the adjacent pulses may be viewed as introducing ISI into the transmitted signal in an artificial manner & this basis is called Correlative Coding."

✓ The transfer function of the Loop filter is

$$H_d(f) = 1 + \exp(-j2\pi f T_b) \longrightarrow (4)$$

✓ The overall transfer function of duobinary signalling filter is

$$\begin{aligned} H(f) &= H_d(f) \cdot H_c(f) \\ &= [1 + \exp(-j2\pi f T_b)] \cdot H_c(f) \\ &= \left[e^{-j\pi f T_b} \cdot e^{+j\pi f T_b} - e^{-j\pi f T_b} \cdot e^{-j\pi f T_b}\right] H_c(f). \\ &= \left[e^{+j\pi f T_b} - e^{-j\pi f T_b}\right] e^{-j\pi f T_b} \cdot H_c(f) \\ &= 2\cos(\pi f T_b) \cdot e^{-j\pi f T_b} \cdot H_c(f) \end{aligned}$$

$$\boxed{H(f) = 2H_c(f) \cdot \cos(\pi f T_b) \cdot e^{-j\pi f T_b}} \quad \left(\because \frac{e^{j\theta} + e^{-j\theta}}{2} = \cos\theta\right) \longrightarrow (5)$$

For an ideal nyquist channel of bandwidth

$$W = \frac{R_b}{2} = f_b/2 = \frac{1}{2T_b}.$$

$$H_c(f) = \begin{cases} \frac{1}{f_b} \text{ or } \frac{1}{2W} \text{ or } T_b & ; \ |f| \le f_b/2 \\ 0 & ; \ \text{otherwise} \end{cases} \longrightarrow (6)$$

From eqn (5) & (6)

The overall transfer function

$$H(f) = \begin{cases} 2T_b \cos(\pi f T_b)\, e^{-j\pi f T_b} & ; \ |f| \le f_b/2 \\ 0 & ; \ \text{otherwise.} \end{cases} \longrightarrow (7)$$

The magnitude $|H(f)| = \begin{cases} 2T_b \cos \pi f T_b & ; \ |f| \le f_b/2 \\ 0 & ; \ \text{otherwise} \end{cases}$

The phase angle $\underline{|H(f)}' = \begin{cases} -\pi f T_b & ; \ |f| \le f_b/2 \\ 0 & ; \ \text{otherwise.} \end{cases}$

<u>Representation:</u>

$f = +f_b/2 \Rightarrow \underline{|H(f)} = -\pi/2$

$f = -f_b/2 \Rightarrow \underline{|H(f)} = +\pi/2$

fig: frequency response of Ideal nyquist channel

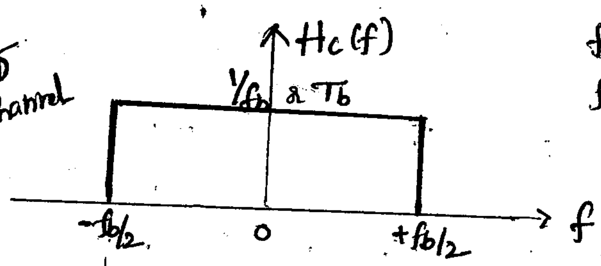


fig. magnitude response of conversion filter

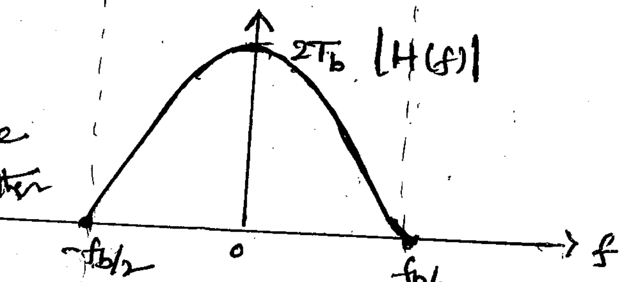
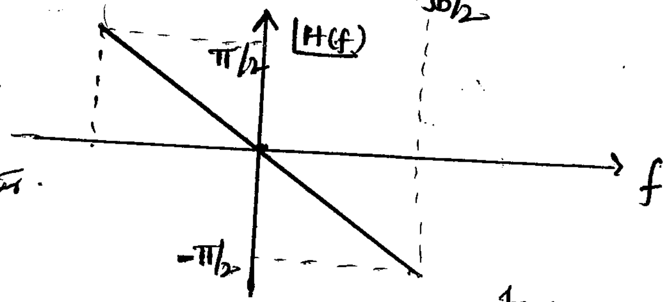


fig: phase response of conversion filter.



The impulse response can be obtained $^{\text{Inverse}}$ fourier transform of $H(f)$

$\therefore$ Transfer fun $H(f) = T_b\left[1 + e^{-j2\pi f T_b}\right]$ i.e. $H_c(f) \cdot H_d(f)$.

$$h(t) = F^{-1}\{H(f)\} = \int_{-\infty}^{\infty} H(f) \cdot e^{j2\pi f t}\, df$$

$$= T_b \int_{-f_b/2}^{f_b/2} \left[1 + e^{-j2\pi f T_b}\right] e^{j2\pi f t}\, df.$$

$$h(t) = T_b \int_{-f_b/2}^{+f_b/2} \left[e^{j2\pi f t} + e^{j2\pi f (t-T_b)}\right] df$$

$$= T_b \left\{ \frac{e^{j2\pi f t}}{j2\pi t}\Bigg|_{-f_b/2}^{f_b/2} + \frac{e^{j2\pi f (t-T_b)}}{j2\pi (t-T_b)}\Bigg|_{-f_b/2}^{f_b/2} \right\}$$

$$= T_b \left\{ \frac{e^{j\pi f_b t} - e^{-j\pi f_b t}}{\pi t \cdot 2j} + \frac{e^{j\pi f_b (t-T_b)} - e^{-j\pi f_b (t-T_b)}}{\pi (t-T_b) \cdot 2j} \right\}$$

$$= T_b \left\{ \frac{\sin \pi f_b t}{\pi t} + \frac{\sin \pi f_b (t-T_b)}{\pi (t-T_b)} \right\}$$

$\because \sin(\pi f_b t - \pi) = -\sin \pi f_b t$

$\because \sin(x-\pi) = -\sin(\pi - x) = -\sin x.$

$$= T_b \left\{ \frac{\sin(\pi t/T_b)}{\pi t} - \frac{\sin(\pi t/T_b)}{\pi (t-T_b)} \right\}$$

$(f_b = 1/T_b)$

$$= T_b \cdot \sin(\pi t f_b) \left[\frac{1}{\pi t} - \frac{1}{\pi (t-T_b)}\right]$$

$$= T_b \cdot \sin(\pi t f_b) \cdot \left[\frac{\cancel{\pi t} - \pi T_b - \cancel{\pi t}}{\pi^2 t (t-T_b)}\right]$$

$$h(t) = T_b \cdot \sin(\pi t f_b) \left[\frac{-\cancel{\pi} T_b}{-\pi^{\cancel{2}} t (T_b - t)}\right]$$

$$h(t) = \frac{T_b^2 \cdot \sin(\pi f_b t)}{\pi t (T_b - t)} \longrightarrow (8)$$

$\therefore$ Impulse response $\boxed{h(t) = \frac{T_b^2 \sin(\pi t f_b)}{\pi t (T_b - t)} \text{ \& } \frac{T_b^2 \sin(\pi t / T_b)}{\pi t (T_b - t)}}.$

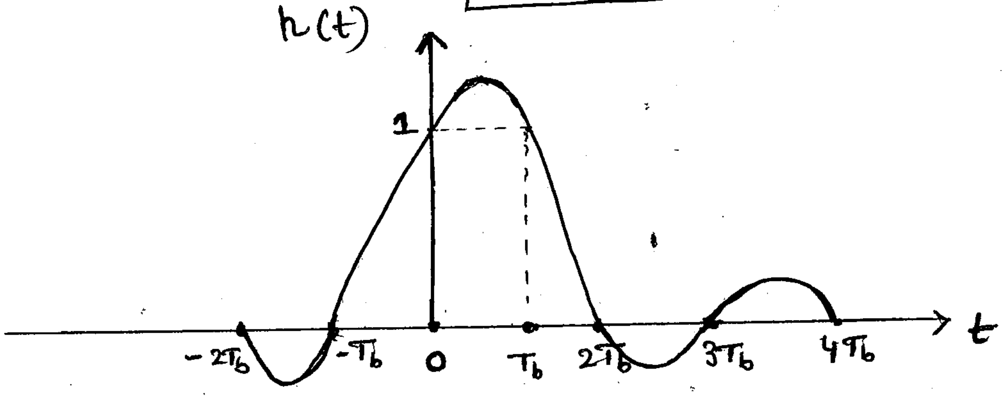


Fig: <u>Impulse response of duobinary conversion filter</u>

$\because$ L-H rule

$$h(t) = \frac{T_b^2 \cdot \cos \pi t f_b (\pi / T_b)}{\pi T_b - 2t\pi}$$

$$h(t) = \frac{T_b \cdot \cos(\pi t f_b)}{T_b - 2t}$$

$t = 0, \; h(t) = 1$

$t = T_b \quad h(t) = \frac{-T_b}{-T_b}$

$h(t) = 1.$

$t = \pm 2T_b, \; h(t) = 0.$

It shows that & explain why we also refer this type of correlative coding as partial signalling. The response to an input pulse is spread over more than on signalling interval.

Decoding :

The original two level sequence $\{b_K\}$ may be detected at receiver from the duo-binary sequence $\{C_K\}$ by substracting previous decoded binary digit from the presently received digit $\{C_K\}$.

The demodulation technique known as nonlinear decision feedback equalization is essentially an inverse of the operation of the digital filter at the transmitter.

ie $$\hat{b}_K = C_K - \hat{b}_{K-1} \longrightarrow (9)$$

where $\hat{b}_K$ - the estimate of original sequence b_K as conceived by the receiver at $t = KT_b$.

$\hat{b}_{K-1}$ is the estimate at $t = (K-1)T_b$

ie The detection procedure just described is essentially an inverse of the operation of the simply delay line filter at the transmitter.

Drawback : At $f = 0$ $|H(f)| = 2T_b$ ie DC value is high

So more transmitted power is required.

The major drawback of this detection procedure is that once errors are made they tend to propagate through the output known as Error propagation.

✓ A practical method of avoiding the error propagation phenomenon is to use pre coding before the due binary coding as shown in fig below

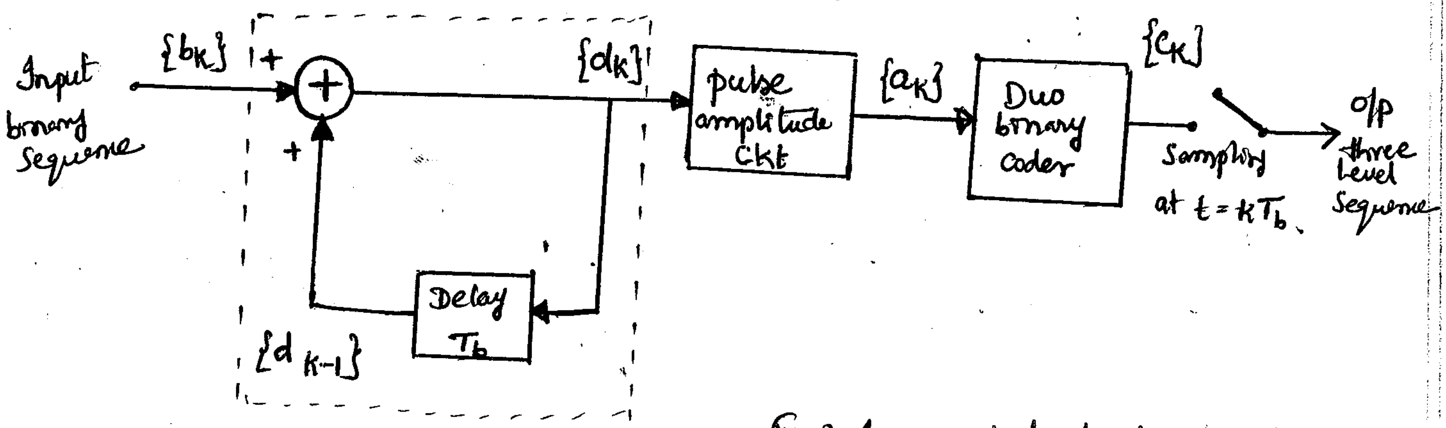


Fig: A precoded duobinary Scheme.

✓ The precoding operation performed on the binary data sequence $\{b_K\}$ converts into another binary sequence $\{d_K\}$, given by

$$d_K = b_K \oplus d_{K-1} \longrightarrow (10)$$

Non linear operation $\rightarrow$ $\boxed{d_K = b_K \oplus d_{K-1}}$

modulo-2 operation.. ie Exclusive OR operation

✓ Precoder prevents error propagation and makes it possible to recover the input message sequence.

ie The output of precoder is output '1' if exactly one input is a '1' otherwise the output remains '0' ie

Truth Table for precoder.

b_K	d_{K-1}	d_K
0	0	0
0	1	1
1	0	1
1	1	0

X-OR

✓ This d_K is applied to pulse amplitude ckt produces a corresponding two level sequence of short pulses $\{a_K\}$.

∴ The sequence c_K as

Linear operation $\rightarrow$ $\boxed{c_K = a_K + a_{K-1}}$ (we know that duobinary scheme o/p).

(or) $c_K = d_K + d_{K-1} \longrightarrow (11)$

The combination of eqn (10) & (11)

$$c_K = \begin{cases} \pm 2V & ; \text{ If data symbol } b_K = 0 \\ 0V & ; \text{ If data symbol } b_K = 1. \end{cases}$$

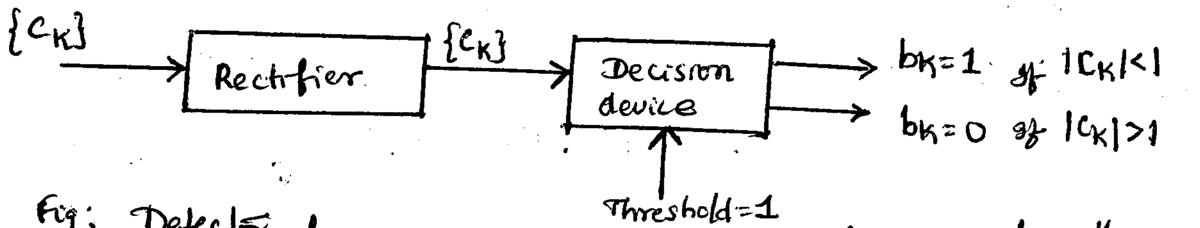


Fig: Detector for recovering original binary sequence from the precoded duobinary sequence.

✓ The detector consists of Rectifier, the output of which is compared in a decision device to a threshold of 1.

If $|c_K| < 1$ say symbol $b_K = 1$

If $|c_K| > 1$ say symbol $b_K = 0$

ie If $c_K = \pm 2V$, $b_K = 0$; $c_K = 0V$, $b_K = 1$.

~~If $|c_K| = 1$ the receiver simply makes a random gives in favour~~ of symbol 1 and 0.

Problem: For input binary data 1011101 obtain the output of duo-binary encoder and also the output of decoder.

1000101

Sol: The input binary data $\{b_K\} = \{1\ 0\ 1\ 1\ 1\ 0\ 1\}$

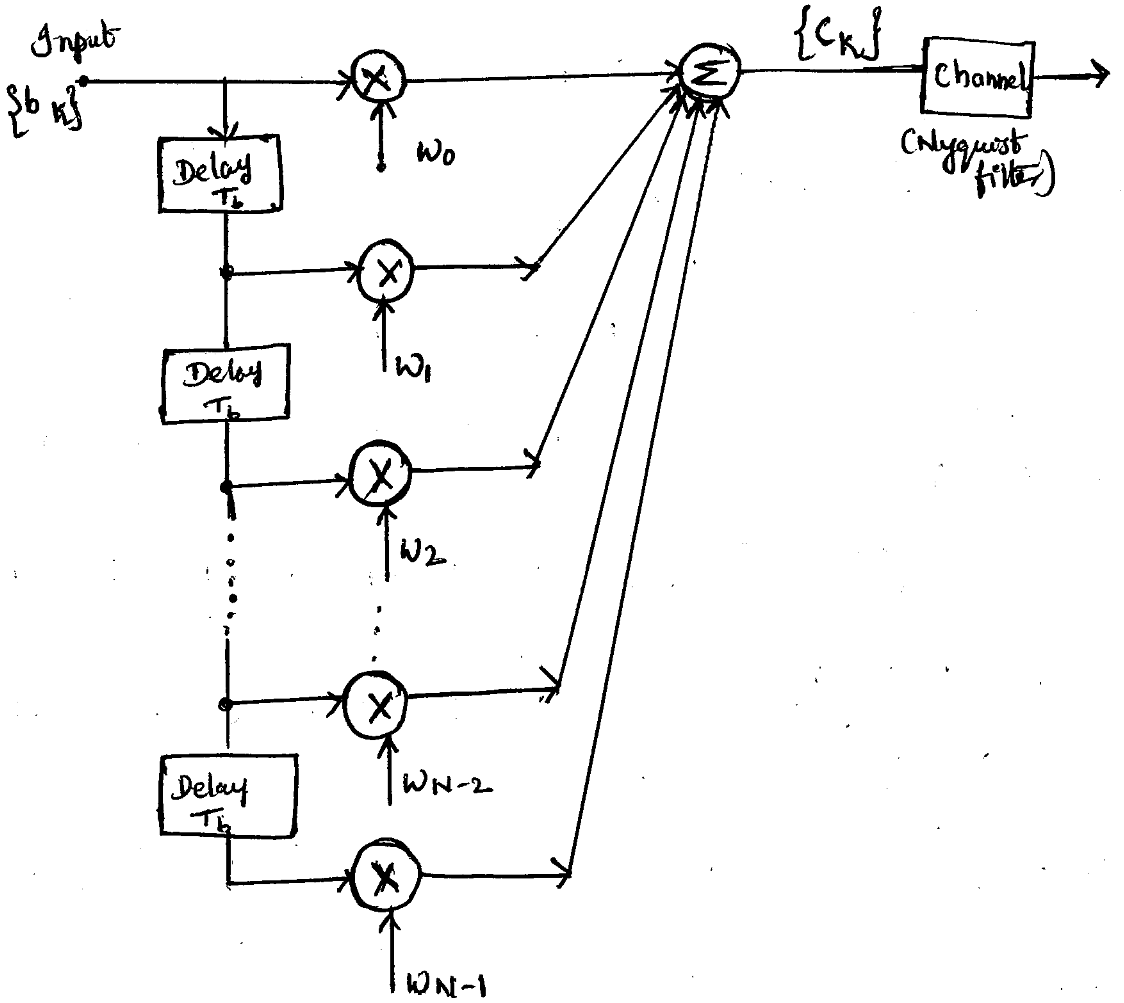
(a) With reference / extra bit '1' $\leftarrow d_{K-1}$

1.	Input binary Sequence $\{b_K\}$		1	0	1	1	1	0	1
2.	Precoded binary sequence $d_K = b_K \oplus d_{K-1}$ ↷ 1	1 (d_{K-1})	0	0	1	0	1	1	0
3.	Voltage representation of d_K ie $1 \to +1$, $0 \to -1$	+1	−1	−1	+1	−1	+1	+1	−1
4.	Voltage representation of Duobinary coder output $\{C_K\}$ ie $C_K = \pm 2V$, $b_K = 0$; $= 0$, $b_K = 1$		0	−2	0	0	0	+2	0
5.	Decoded binary sequence $\{\hat{b}_K\}$ $b_K = 0$, $C_K = \pm 2V$; $b_K = 1$, $C_K = 0$		1	0	1	1	1	0	1

(b) With reference / Extra bit 0 $\hookrightarrow d_{K-1}$

1.	Input binary Sequence $\{b_K\}$		1	0	1	1	1	0	1
2	Precoded binary sequence $d_K = b_K \oplus d_{K-1}$ ↷ 0	0 (d_{K-1})	1	1	0	1	0	0	1
3	Voltage representation of d_K ie $1 \to +1$, $0 \to -1$	−1	+1	+1	−1	+1	−1	−1	+1
4	Voltage representation of Duobinary Coder output $\{C_K\}$; $C_K = \pm 2V$; $b_K = 0$; $= 0$, $b_K = 1$		0	+2	0	0	0	−2	0
5	Decoded binary sequence $\{\hat{b}_K\}$ $b_K = 0$, $C_K = \pm 2V$; $b_K = 1$, $C_K = 0V$		1	0	1	1	1	0	1

Generalized form of Correlative Coding Scheme



✓ The duo-binary and modified duo binary scheme have correlation spans of binary digit and two binary digits respectively.

✓ In the duo binary scheme, the transfer fun H(f) & PSD of the Txed signal is non zero at the origin.
This can be reduced by using the technique called modified duo binary signalling. Here $C_K = a_K - a_{K-2}$.

* The generalization scheme involves the use of tapped delay line filters with tap weights $w_0, w_1, w_2 \cdots w_{N-1}$.

✓ The Correlative samples $\{C_K\}$ can be Obtained from a Superposition of 'N' successive input sample values $\{b_K\}$.

∴ The output of correlative coding $\boxed{C_K = \sum_{n=0}^{N-1} w_n \cdot b_{K-n}}$

If $N = 2 \Rightarrow C_K = \sum_{n=0}^{1} w_n b_{K-n} = w_0 b_K + w_1 b_{K-1}$

If $w_0 = w_1 = +1$ then $\boxed{C_K = b_K + b_{K-1}}$ ← Duobinary signalling scheme.

If $N=3$ $\quad C_K = \sum_{n=0}^{2} w_n b_{K-n} = w_0 b_K + w_1 b_{K-1} + w_2 b_{K-2}$

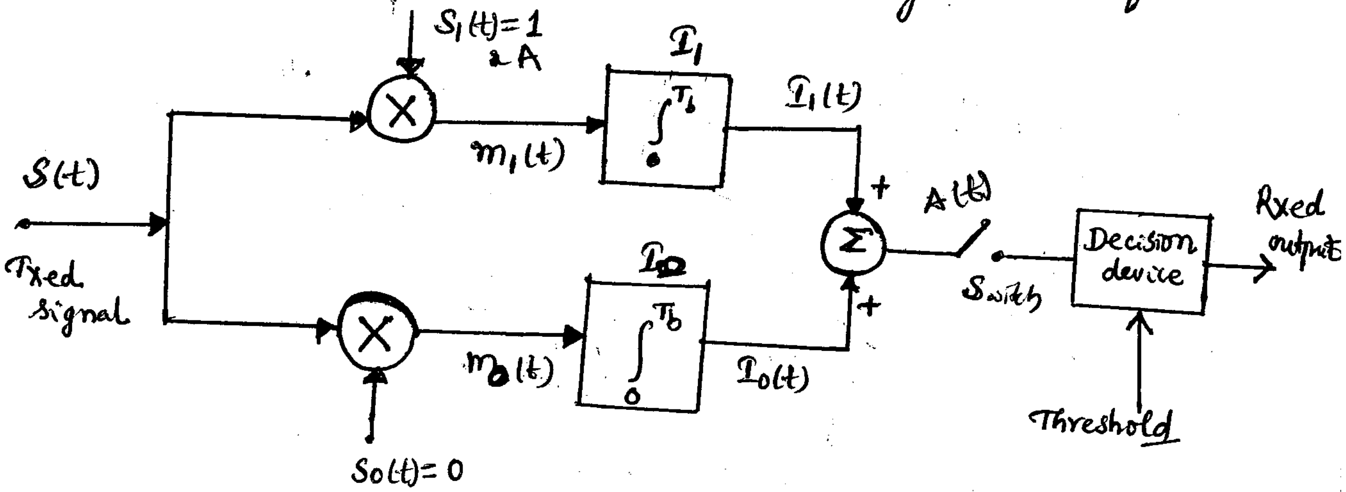
If $w_0 = +1$, $w_1 = 0$, $w_2 = -1$ then $C_K = b_K(1) + 0 + b_{K-2}(-1)$

$\therefore \boxed{C_K = b_K - b_{K-2}}$ ← Modified duo binary signalling scheme.

Correlation Receiver:

✓ The correlation detection/receiver means synchronous detection.

It consists of multiplier and LPF (or) Integrator as follows.



✓ Let us consider UNRZ Encoding scheme, the txed signal as

$$S(t) = \begin{cases} A + n(t) & ; \text{logic '1'} \\ n(t) & ; \text{logic '0'} \end{cases}$$

Ily for BNRZ scheme $\quad S(t) = \begin{cases} A + n(t) & ; \text{logic 1} \\ -A + n(t) & ; \text{logic '0'} \end{cases}$

Assume noise $n(t)$ has zero mean so $\int n(t) = 0$.

For UNRZ:

✓ If logic '1' is received then $S(t) = A + n(t)$

The output of multiplier is $m_1(t) = S(t) \times S_1(t)$

$$= [A + n(t)]\, A$$

$$= A^2 + A \cdot n(t).$$

The output of Integrator is $I_1(t) = \int_0^{T_b} m_1(t)\, dt$

$$= \int_0^{T_b} [A^2 + A \cdot n(t)]\, dt$$

$$I_1(t) = \int_0^{T_b} A^2 dt + \int_0^{T_b} A \cdot n(t)\, dt$$

$$= A^2 \cdot T_b + 0 \qquad (\because \int n(t) = 0) \text{ zero mean}$$

$$\therefore \boxed{I_1(t) = A^2 \cdot T_b}$$

✓ If logic '0' is received then $S(t) = n(t)$

The output of multiplier $m_0(t) = S(t) \times \phi_0(t)$

$$= n(t) \cdot 0$$

$$= 0 \quad \therefore m_0(t) = 0$$

The output of Integrator is $I_0(t) = \int_0^{T_b} m_0(t)\, dt = 0$

$$\boxed{I_0(t) = 0}$$

✓ The adder / summer output $A(t)$ is given by

$$\therefore A(t) = I_1(t) + I_0(t)$$

$$= A^2 \cdot T_b + 0$$

$$= A^2 \cdot T_b \quad \therefore A(t) = A^2 \cdot T_b.$$

✓ The threshold value V_{th} can be taken as

$$V_{th} = \frac{A^2 T_b + 0}{2} = \frac{A^2 T_b}{2}$$

ie

If the adder output is larger than the predetermined threshold V_{th} & λ we chose logic '1'

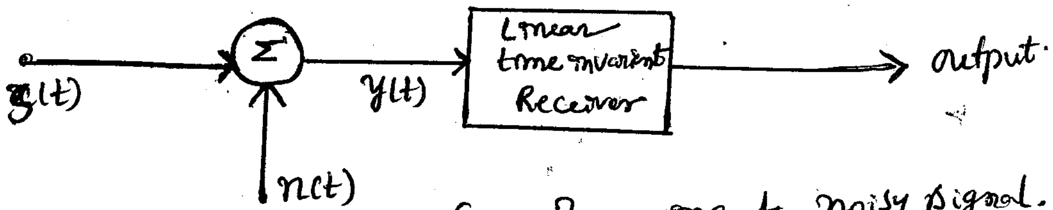
otherwise the received output, we choose logic '0'.

Note: An Intimate relationship between the two types of criterion

It should be noted that of the two filters (Correlator and matched filter are equivalent only at time $t = T$.

Optimum Receiver :

- Consider a received signal $y(t)$ consisting of a signal $s(t)$ of known form and additive white Gaussian Noise (AWGN) $n(t)$.
- The noise is white in the sense that it has a constant power spectral density $\eta/2$ & $N_0/2$ & Gaussian in the sense that a sample drawn at random from such a process has a gaussian probability distribution of its amplitude & has zero mean.



∴ fig: Processing of noisy signal.

* The purpose of detection is to establish the presence & absense of the signal, we should estimate which one of the two hypotheses noise alone (or) noise plus signal is true.

This can be done by processing the noisy received signal $y(t)$ with the help of a linear time invarient receiver in such a way that the receiver output at some arbitrary time $t = T$ is considerably greater when $s(t)$ is present than that when $s(t)$ is absent.

- There are two approaches to realize a receiver for detection of signal in the presence of noise.

1. Matched filter based on maximisation of SNR at the Receiver. This involves the use of a filter matched to the signal component of the received signal.
2. Correlation Receiver based on probabilistic criteria that ~~the~~ is directly related to performance ratings of a particular digital comm'n system of interest. This involves a correlation of the received signal with a stored replica of the transmitted signal.

Matched Filter :

* A matched filter is an optimum filter in the sense that it <u>maximizes the output signal to noise ratio (SNR)</u>.

✓ In digital communication matched filters are very useful.

* The main purpose of detection in digital communication is to recognize a pulse signal in presense of noise rather than improving the pulse shape.

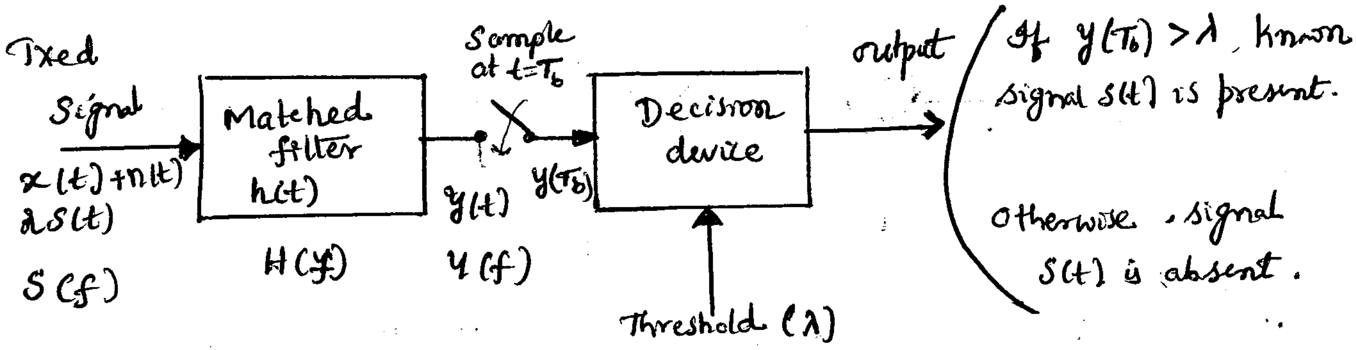


fig: Matched filter receiver.

✓ Let us consider the matched filter with impulse response $h(t)$ and frequency transfer function $H(f)$.

For calculating the SNR value, we have to calculate both signal power and noise power.

The input txed signal $s(t) = \underset{\substack{\uparrow \\ \text{signal}}}{x(t)} + \underset{\substack{\uparrow \\ \text{noise}}}{n(t)}$ ← AWGN with zero mean.

Consider only input signal $s(t) = x(t)$.

∴ The output of the matched filter $y(t) = x(t) * h(t)$

by taking fourier transform.

$$Y(f) = X(f) \cdot H(f) = \int_{t=-\infty}^{\infty} [x(t) * h(t)]\, e^{-j2\pi ft}\, dt$$

The inverse fourier transform as.

$$y(t) = F^{-1}\{Y(f)\} = \int_{f=-\infty}^{\infty} [X(f) \cdot H(f)] \cdot e^{+j2\pi ft}\, df$$

- The matched filter integrates incoming signal (ie signal + noise) for entire bit duration & then compare with threshold value at the end of bit based on the condition, Decision device decides either '1' or '0'. ie

Logic 1 $\rightarrow$ $y(T_b) \geq \lambda$ Threshold value.

Logic 0 $\rightarrow$ $y(T_b) < \lambda$

$\therefore$ at $t = T_b$

$$y(T_b) = \int_{-\infty}^{\infty} X(f) \cdot H(f) \cdot e^{j2\pi f T_b} \, df$$

The output signal power of matched filter as

$$\boxed{S_o = |y(T_b)|^2 = \left| \int_{-\infty}^{\infty} X(f) \cdot H(f) \cdot e^{j2\pi f T_b} \, df \right|^2} \rightarrow (1)$$

- Let us consider, the noise generated in the channel is gaussian noise (AWGN) with noise PSD $N_o/2$ & $\eta/2$.

ie Noise PSD at input of matched filter $S_{in}(f) = \frac{N_o}{2}$

The output noise PSD of matched filter can be written as

$$S_o(f) = S_{in}(f) \cdot |H(f)|^2$$

$$= \frac{N_o}{2} \cdot |H(f)|^2$$

$\therefore$ the output noise power of matched filter as

$$N_o = R_{xx}(0) = \int_{f=-\infty}^{\infty} S_{xx}(f) \, df$$

$$\therefore N_o = \int_{f=-\infty}^{\infty} \frac{N_o}{2} \cdot |H(f)|^2 \, df$$

$$\therefore \boxed{N_o = \frac{N_o}{2} \int_{f=-\infty}^{\infty} |H(f)|^2 \, df} \rightarrow (2)$$

The Signal to noise ratio (SNR) at output of matched filter is given by

$$SNR_o = \frac{S_o}{N_o} = \frac{\left| \int_{-\infty}^{\infty} X(f) \cdot H(f) \, e^{+j2\pi f T_b} \, df \right|^2}{\frac{N_o}{2} \cdot \int_{-\infty}^{\infty} |H(f)|^2 \, df}$$

Using Schwarz inequality ie

$$|f_1(t) \cdot f_2(t)|^2 \leq |f_1(t)|^2 \cdot |f_2(t)|^2 \quad \text{if } f_1(t) = K \cdot f_2^*(t).$$

ie
The Squared magnitude of the total area under the product of two such functions is less than & equal to the product of the total area under the squared magnitude of each of the two functions.

$$\Rightarrow \quad \left|\int_{-\infty}^{\infty} f_1(t) \cdot f_2(t)\, dt\right|^2 \leq \int_{-\infty}^{\infty} |f_1(t)|^2 dt \cdot \int_{-\infty}^{\infty} |f_2(t)|^2 dt.$$

∴ The output signal to noise ratio of matched filter

$$SNR = \frac{S_0}{N_0} \leq \frac{\cancel{\int_{-\infty}^{\infty} |H(f)|^2 df} \cdot \int_{-\infty}^{\infty} \left|X(f) \cdot e^{+j2\pi f T_b}\right|^2 df}{\frac{N_0}{2} \cdot \cancel{\int_{-\infty}^{\infty} |H(f)|^2 df}}$$

$$\leq \frac{2}{N_0} \cdot \int_{-\infty}^{\infty} |X(f)|^2 df \qquad \left[\because |e^{j\theta}| = 1\right]$$

$$\leq \frac{2}{N_0} \cdot E \qquad \text{where } E \text{ - Energy of signal} = \int_{-\infty}^{\infty} |X(f)|^2 df$$

$$\therefore \boxed{SNR = \frac{2E}{N_0}}$$

Thus the maximum output SNR depends on the input signal energy, and the power spectral density of the noise, Not on the particular shape of the waveform that is used.

* The mathematic operation of matched filter is convolution, a signal is convolved with the impulse response of the filter.

→ The mathematic operation of correlator is Correlation, a signal is correlated with a replica of itself.

The term matched filter is often used synonymously with "correlator."

→ The condition for Schwarz inequality is $f_1(x) = K \cdot f_2^*(x)$.

$$\therefore \quad H(f) = K \cdot \left[X(f) \cdot e^{j2\pi f T_b}\right]^*$$

$$H(f) = K \cdot X^*(f) \cdot e^{-j2\pi f T_b}$$

by taking inverse fourier transform on both sides

$$h(t) = F^{-1}\{H(f)\} = \int_{f=-\infty}^{\infty} K \cdot X^*(f) \cdot e^{-j2\pi f T_b} \cdot e^{j2\pi f t}\, df$$

$$= K \cdot \int_{-\infty}^{\infty} X^*(f) \cdot e^{-j2\pi f (T_b - t)}\, df$$

$$= K \cdot \int_{-\infty}^{\infty} \left[X(f) \cdot e^{j2\pi f (T_b - t)}\right]^* \cdot df$$

Replacing $(-f = f)$

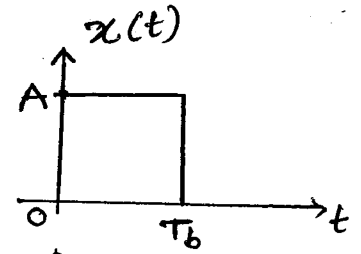
$\therefore$ The impulse response of matched filter is

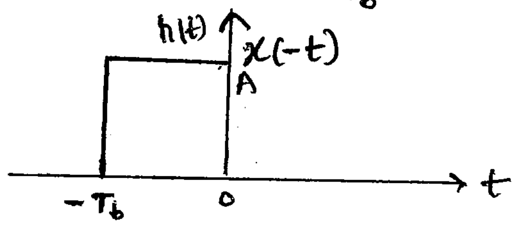
$$h(t) = K \cdot x^*(T_b - t)$$

If $x(t)$ is real function $\therefore \boxed{h(t) = K \cdot x(T_b - t)}$

~ The output of matched filter

let





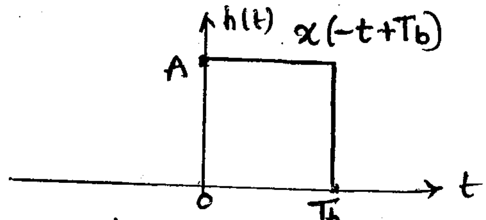
$$y(t) = x(t) * h(t)$$

$$= \int_{\tau=-\infty}^{\infty} h(\tau) \cdot x(t - \tau)\, d\tau$$

at $t = 0$, $y(0) = \int_{\tau=-\infty}^{\infty} h(\tau) \cdot x(-\tau)\, d\tau = 0$.

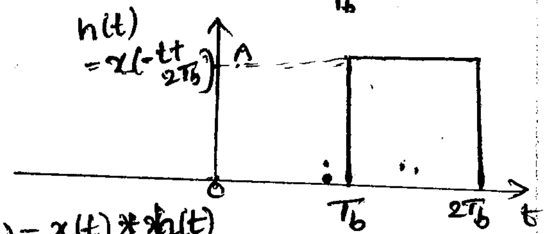
at $t = T_b$

$$y(T_b) = \int_{\tau=-\infty}^{\infty} h(\tau)\, x(T_b - \tau)\, d\tau.$$



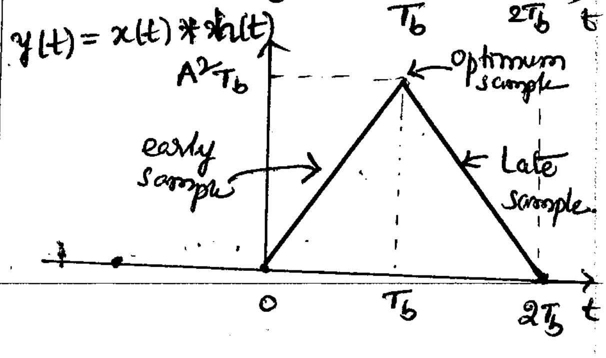
$$\therefore \boxed{y(T_b) = \int_{\tau=-\infty}^{\infty} h(\tau) \cdot x(T_b - \tau)\, d\tau}$$

$$y(T_b) = A^2 \cdot T_b$$



at $t = 2T_b$ $\quad y(2T_b) = \int_{-\infty}^{\infty} h(\tau) \cdot x(2T_b - \tau)\, d\tau = 0$

Hence the impulse response of a filter that produces the maximum output signal to noise ratio is mirror image of the signal $x(t)$, delayed by the symbol time duration T_b.



—*—

Properties of Matched filter :

Property ①: The spectrum of output signal of a matched filter with the matched signal as the input is proportional to the energy density of the input signal.

Proof : The output of matched filter $Y(f) = X(f) \cdot H(f)$

Consider In frequency domain, the matched filter is characterized by the transfer function

$H(f) = X^*(f) \cdot e^{-j2\pi f T_b}$ which except for a delay factor, is the complex conjugate of the spectrum of $x(t)$.

$$\therefore \quad Y(f) = X(f) \cdot H(f)$$

$$= X(f) \cdot X^*(f)\, e^{-j2\pi f T} \text{ delay.}$$

$$= |X(f)|^2 \cdot e^{-j2\pi f T}$$

$$Y(f) = \Psi_s(f) \cdot e^{-j2\pi f T} \Rightarrow \boxed{Y(f) \propto \Psi_s(f)}$$

where $\Psi_s(f) = |X(f)|^2$ is the Energy Spectral density of the input signal $x(t)$.

Thus apart from a time delay factor, the spectrum of the input signal is proportional to the Energy spectral density of the input signal.

Property ②: The output signal of a matched filter is proportional to a shifted version of the autocorrelation function of the input signal to which the filter is matched.

Proof : From the property ①, recognizing that the autocorrelation function and Energy spectral density of a signal form a fourier transform pair.

By taking inverse fourier transform on both sides of $Y(f)$.

$$\therefore y(t) = F^{-1}\{Y(f)\} = F^{-1}\{\Psi_s(f) \cdot e^{-j2\pi f T}\}$$

using fourier transform pair $R_s(\tau) \leftrightarrow \Psi_s(f)$

$R_s(\tau)$ being the auto correlation function of the input signal and the time shifting property of fourier transforms

we get $\boxed{y(t) = R_s(t-T)}$

Hence proved.

Property (3): The output signal to noise ratio of a matched filter depends only on the ratio of the signal energy to the power spectral density of the white noise at the filter output.

Proof: The signal to noise ratio of the matched filter at output is

$$(SNR)_0 = \frac{S_0}{N_0} = \frac{2}{N_0} \cdot \int_{-\infty}^{\infty} |X(f)|^2 df$$

where $X(f)$ is the fourier transform of the signal $x(t)$ to which the filter is matched.

Using the Raleigh's Energy theorem we may write the signal Energy $E = \int_{-\infty}^{\infty} x^2(t) dt = \int_{-\infty}^{\infty} |X(f)|^2 df$

∴ PSD of white noise is $N_0/2$

∴ The output signal to noise ratio of the matched filter as

$$\boxed{(SNR)_0 = \frac{2E}{N_0} = \frac{(E)}{(N_0/2)}}$$

Hence the SNR at the output of matched filter depends only on the ratio of the signal Energy to PSD of white noise hence proved.

*

Unit-3

Source Encoder & Decoder

Syllabus: Information, entropy, source coding and decoding: Fixed length, Shanon fano, huffmann, joint & conditional entropy, redundancy, mutual & self informations, Binary Symmetric Channel, Binary Erase channel, Shanon Hartley theorem (or) channel capacity theorem.

Information:

The basic principle in determining the information content of a message is " more the uncertainity (Surprise / unexpected) of the event. Most the information content is occurance of the event.

Example: Let us consider the message as

* ① " Sun rises in the East " .

The probability of the above event is the day-to-day phenomenon. no body shows interest in it, so the uncertainity of the above event is less and the information content is less.

* If some body say that " Sun is going to rise in the West ". then uncertainity of the above event is more & the information content is more.

② Dog bites a man (E_1)
A man bites a dog (E_2).

If event 'E_1' is most occurance so no information is there where as in event 'E_2', it is not most occurance hence it has more information probability of occurance is less.

∴ Information is inversly proportional to the probability of occurance of the event ie (or) message

$$\boxed{\begin{aligned} I &\propto \frac{1}{P(E)} \\ I &= \log_x(1/P) = -\log_x(P) \end{aligned}}$$

" $P(E) = P$ "

where '$\log_x$' is proportionality constant.

Units of Information:

$$I = \log_x(1/p) \text{ (or) } -\log_x(p)$$

a) If the base $x=2$ then $I = \log_2(1/p) = -\log_2 p$ Units : **bits**

b) If the base $x=10$ then $I = \log_{10}(1/p) = -\log_{10} p$ Units : Decits (or) Hartley.

c) If the base $x=e$ then $I = \log_e(1/p) = -\log_e p$ Units : nats.

* The source is generating 'x' and 'y' are two independent messages with probability $p(x)$ and $p(y)$, then the total information carried by x & y are

$$I(x,y) = \log_2 \frac{1}{p(x,y)} \text{ bits}$$

$$p(x,y) = p(x) \cdot p(y) \quad (\because \text{Independent events})$$

$$I(x,y) = \log_2\left[\frac{1}{p(x)\cdot p(y)}\right]$$

$$= \log_2(1/p(x)) + \log_2(1/p(y))$$

$$= I(x) + I(y)$$

$$\therefore \boxed{I(x,y) = I(x) + I(y)}$$

where $I(x) = \log_2 1/p(x)$, $I(y) = \log_2(1/p(y))$.

Entropy:

"The average amount of information per individual messages in a particular interval is called Entropy".

✓ The entropy (or) Average information transmitted under the following assumptions.

a) The source is stationary ie the probability of occurance of each message remain constant with time.

b) Each message Emitted from the source is independent of previous messages.

✓ Entropy is denoted as 'H' and the Units for entropy

$H \rightarrow$ bits/message.

Symbol → 1010 → Codeword
↘ length = 4

Let us consider 'M' different messages $m_1, m_2, m_3 \cdots\cdots$ with their respective probabilities of occurance is $P_1, P_2, P_3, \cdots\cdots$

For along time interval 'L' messages have been generated then

The no. of messages $m_1 = P_1 \times L$ $(L >> M)$

The amount of information in message $m_1 = \log(1/P_1)$ & $-\log(P_1)$

∴ The total amount of information in all m_1 messages

$$P_1 = L \log\left(\frac{1}{P_1}\right)$$

∴ The amount of information in all 'L' messages will be

$$I_t = P_1 \cdot L \cdot \log\left(\frac{1}{P_1}\right) + P_2 \cdot L \log(1/P_2) + \cdots\cdots$$

The average amount of information per message & Entropy is given by

$$H = \frac{I_t}{L} = P_1 \log(1/P_1) + P_2 \log(1/P_2) + \cdots\cdots$$

$$\therefore \boxed{\text{Entropy, } H = \sum_{K=1}^{M} P_K \log\left(\frac{1}{P_K}\right) \text{ (or) } -\sum_{K=1}^{M} P_K \cdot \log(1/P_K)}$$

Properties of Entropy :

① The entropy function is symmetrical

$$H(P_K, P_{K-1}) = H(P_{K-1}, P_K).$$

② The entropy 'H' is continuous in the interval $0 \le P_K \le 1$.

③ Dividing the entropy into different subsets does not effect the value of entropy.

ie $H(M) = H(m_1, m_2, m_3, \ldots\ldots) + H(m_{n+1}, m_{n+2}, m_{n+3}, \ldots\ldots)$

$$\therefore H(M) = H(m_1, m_2, m_3, \cdots\cdots m_{n+1}, m_{n+2}, m_{n+3}, \cdots\cdots).$$

④ $0 \le H \le \log(M)$.

⑤ $H = 0$ if all the probabilities are '0' except one which must be unit.

⑥ $H = \log M$, if all the probabilities are equally likely.

⑦ Entropy is maximum when uncertainity is more

Proof :

4 * Let us Examine entropy under different cases for 'M=2'

Case (i) : $P_1 = 0.01$, $P_2 = 0.99$

Case (ii) : $P_1 = 0.4$, $P_2 = 0.6$

Case (iii) : $P_1 = 0.5$, $P_2 = 0.5$.

case (i) : $H = P_1 \log(1/P_1) + P_2 \log(1/P_2)$

$= 0.01 \log_2 (1/0.01) + 0.99 \log_2 \left(\frac{1}{0.99}\right)$ $\left[\because \frac{\log_{10}(1/0.01)}{\log_{10} 2} \times 0.01 = 0.0664 \right]$

$= 0.0664 + 0.0143$

$\boxed{H = 0.08 \text{ bits/msg}}$

case (ii) : $H = 0.4 \log_2 (1/0.4) + 0.6 \log_2 (1/0.6)$

$= 0.528 + 0.442$

$\boxed{H = 0.97 \text{ bits/msg}}$

case (iii) : $H = 0.5 \log_2 (1/0.5) + 0.5 \log_2 (1/0.5)$

$= 0.5 + 0.5$

$= 1$

$\therefore \boxed{H = 1 \text{ bits/msg.}}$

Hence

Case (i) : The probabilities of messages m_1, m_2 are 0.01 , 0.99 , So uncertainity is less.

Case (ii) : The uncertainity is more compared to case (i), Since $P_1 = 0.4$, $P_2 = 0.6$.

Case (iii) : The uncertainity is maximum because messages are equally likely.

5 * If there is only single possible message is there ie

$m_1 = 1$ & $P_1 = 1$ then

$H = P_1 \log(1/P_1) \Rightarrow H = 1 \cdot \log(1/1) = 0$ $\therefore \boxed{H = 0}$

* Let there be only one message out of 'M' messages having probability '1' and all others are '0' in that case.

$$H = \sum_{K=1}^{M} P_K \log\left(\frac{1}{P_K}\right)$$

$$H = P_1 \log\left(\frac{1}{P_1}\right) + P_2 \log\left(\frac{1}{P_2}\right) + \cdots + P_M \log\left(\frac{1}{P_M}\right)$$

$$= 1 \log_2(1/1) + 0 + \cdots + 0$$

$$= 0$$

$$\therefore \boxed{H=0}$$

* For a binary system is $M=2$ then the entropy is 1 $\therefore \boxed{H=1}$

Proof:

$$H = \sum_{k=1}^{2} P_K \log(1/P_K)$$

$$H = P_1 \log(1/P_1) + P_2 \log(1/P_2)$$

let $P_1 = P$ and $P_2 = 1-P$ [equally likely]

$$\therefore H = P \log(1/p) + (1-p)\log(1/1-p)$$

Differentiating above eqn w.r.t. 'p' & equating it to zero.

$$\therefore \frac{dH}{dp} = 0$$

$$\Rightarrow \frac{dH}{dp} = \frac{d}{dp}\left\{-\left[p\log p + (1-p)\log(1-p)\right]\right\}$$

$$0 = -\left\{ p \cdot \frac{1}{p} + \log p\,(1) + (1-p)\cdot\frac{1}{(1-p)}(-1) + \log(1-p)\cdot(-1)\right\}$$

$$= -\left[\cancel{1} + \log p - \cancel{1} - \log(1-p)\right]$$

$$= -\left[\log p - \log(1-p)\right]$$

$$0 = -\log p + \log(1-p)$$

$$\log p = \log(1-p)$$

$$\therefore p = 1-p \rightarrow \text{Equally likely.}$$

The maximum value of entropy at $P = 1/2$, $1-P = 1/2$

$$H = -\left[\frac{1}{2}\log\frac{1}{2} + \left(1-\frac{1}{2}\right)\log\left(1-\frac{1}{2}\right)\right]$$

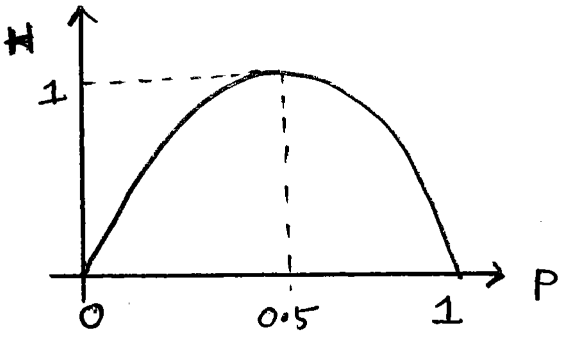
$$= -\left[\frac{1}{2}\log_2\frac{1}{2} + \frac{1}{2}\log\frac{1}{2}\right]$$

$$= -\left[1\log_2\frac{1}{2}\right]$$

$$= -\log_2 2^{-1}$$

$$H = 1 \qquad \therefore \boxed{H=1} \text{ bits/msg (or) bits/symbol.}$$

ie The entropy is maximum when both the messages are equally likely ie when $p=\frac{1}{2}$.



⑥ * Similarly in the case of M messages, the entropy is maximum when all the messages are equally likely.

ie $P_1 = P_2 = P_3 = \cdots = P_M = \frac{1}{M}$

$$\therefore H = \sum_{K=1}^{M} P_K \log(1/P_K)$$

$$= P_1 \log(1/P_1) + P_2 \log\left(\frac{1}{P_2}\right) + \cdots + P_M \log(1/P_M)$$

$$= \frac{1}{M}\log(M) + \frac{1}{M}\log(M) + \cdots + \frac{1}{M}\log(M)$$

$$= \cancel{M} \times \frac{1}{\cancel{M}} \log_2 M$$

$$\therefore \boxed{H_{max} = \log_2 M} \text{ bits/msg}.$$

ie The Entropy ranges from 0 to $\log_2 M$ $\therefore$ $\boxed{0 \le H \le \log_2 M}$

Information rate (R): The average no. of bits per sec

ie If a message source generate messages at a rate of 'r' messages per sec, the rate of information & Information rate is given by

$$\boxed{R = r \times H} = \text{msg/sec} \times \text{bits/msg}$$
$$= \text{bits/sec}$$

$\therefore$ The unit for information rate $\boxed{R = \text{bits/sec}}$

where R - Information rate
H - Entropy
r - no. of messages per sec.

Let us consider two sources of equal entropy, generating 'r_1 & r_2' messages per sec respectively

a) The first source will transmit the information at a rate

$$R_1 = r_1 \times H$$

b) The Second source will transmit the information at a rate

$$R_2 = r_2 \times H.$$

c) If $r_2 > r_1$ then $R_2 > R_1$ ie more information transmitted from the second source than the first source in a given second.

Hence the source is not described by its entropy alone but also by its rate of information.

Problem: An event has 6 possible outcomes with probabilities $P_1 = 1/2$, $P_2 = \frac{1}{4}$, $P_3 = 1/8$, $P_4 = \frac{1}{16}$, $P_5 = \frac{1}{32}$, $P_6 = \frac{1}{32}$. Find the entropy (H) and the rate of information (R) if there are 16 outcomes per sec.

Sol

P_1	P_2	P_3	P_4	P_5	P_6
$1/2$	$1/4$	$1/8$	$1/16$	$1/32$	$1/32$

$r = 16$ msg/se

Entropy $H = \sum_{K=1}^{M} P_K \log(1/P_K) = \sum_{K=1}^{6} P_K \log(1/P_K)$

$$= \frac{1}{2}\log 2 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16 + \frac{1}{32}\log 32 + \frac{1}{32}\log 32$$

$$= \frac{1}{2} + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{32}\times 5 + \frac{1}{32}\times 5$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{3}{8} + \frac{4}{16} + \frac{5}{32} + \frac{5}{32}$$

$$= \frac{16+16+12+8+5+5}{32}$$

$$= \frac{62}{32}$$

$$= 1.9375$$

$\therefore \boxed{H = 1.9375 \text{ bits/msg}}$

Information rate $R = r \times H = 16 \times 1.9375 = 31$

$\therefore \boxed{R = 31 \text{ bits/sec.}}$

(2) A continuous signal is band limited to 5 KHz, The signal is sampled and quantized into 8 levels of a PCM system, each quantization levels (or) the probability of occurance of each level are 0.25, 0.2, 0.2, 0.1, 0.1, 0.05, 0.05, 0.05. Calculate Entropy and information rate.

Sol

Given $f_m = 5 KHz$, PCM system

8 - quantization levels

ie $\{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\} = \{0.25, 0.2, 0.2, 0.1, 0.1, 0.05, 0.5, 0.05\}$

$$\therefore \text{Entropy } H = \sum_{K=1}^{M} P_K \log_2 (1/P_K) = -\sum_{K=1}^{8} P_K \log P_K.$$

$$H = -\Big[0.25 \log_2 (0.25) + 0.2 \log_2 (0.2) + 0.2 \log_2 (0.2) + 0.1 \log_2 (0.1) + 0.1 \log_2 (0.1) + 0.05 \log_2 (0.05) + (0.05) \log (0.05) + (0.05) \log (0.05)\Big]_{\log 2}$$

$$= -[-0.5 - 0.464 - 0.464 - 0.332 - 0.332 - 0.216 - 0.216 - 0.216]$$

$$= -[-2.74]$$

$$= 2.74 \qquad \therefore \text{Entropy } \boxed{H = 2.74 \text{ bits/msg.}}$$ Quantization level

Given

$f_m = 5 KHz$

$f_s = 2 f_m = 2 \times 5 KHz = 10 KHz$

$\therefore r = 10{,}000$ Quantization levels / sec.

Information rate $R = r \times H$

$= 10{,}000 \times 2.74$ × Quantization/sec × bits/Quantization

$= 27.4 \times 10^3$ bits/sec

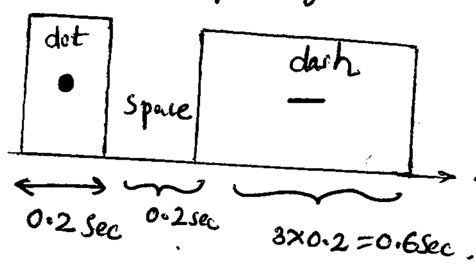
Information rate $\boxed{R = 27.4K \text{ bits/sec}}$

3 A given telegraph source having two symbols '•' and '—'. Dot duration is 0.2 sec and dash duration is 3 times the dot duration. The probability of dot occuring is twice that of dash occurance and the time difference between Dot and Dash is 0.2 sec. Find entropy (H) and Information rate (R) of telegraph source.

Sol Given

Dot duration $t_{dot} = 0.2\,sec$

Telegraph system



Dash duration $t_{dash} = 3 \times t_{dot} = 0.6\,sec$

$t_{space} = 0.2\,sec.$

$$P_{(dot)} = 2 \cdot P_{(dash)}$$

Total probability $= 1$

$$P_{dot} + P_{dash} = 1$$

$$2P_{dash} + P_{dash} = 1$$

$$3P_{dash} = 1$$

$$\boxed{P_{dash} = 1/3} \qquad \boxed{P_{dot} = 2/3}$$

∴ Entropy $H = \sum_{K=1}^{2} P_K \cdot \log(1/P_K)$ & $- \sum_{K=1}^{2} P_K \log(P_K)$.

$$= -\left[P_{dash} \cdot \log(P_{dash}) + P_{dot} \cdot \log(P_{dot})\right].$$

$$= -\left[\tfrac{1}{3}\log_2(1/3) + 2/3 \log_2(2/3)\right]$$

$$= -\left[\frac{-0.1590 - 0.11739}{\log 2}\right]$$

$$= \frac{0.27639}{0.30102} = 0.918 \quad \therefore \boxed{H = 0.918\ bits/msg}$$

Avg time per symbol

Time duration $t_s = P_{dot} \cdot t_{dot} + P_{dash} \cdot t_{dash} + P_{space} \cdot t_{space}$

$$= 2/3 \cdot (0.2) + \tfrac{1}{3} \times 0.6 + 1 \times 0.2 \quad (\because \text{let } P_{space} = 1)$$

$$t_s = 0.533\,sec$$

$$\boxed{r = \frac{1}{t_s} = 1.875\ msg/sec}$$

∴ Information rate

$R = r \times H = 1.875 \times 0.918 \Rightarrow \boxed{R = 1.72125\ bits/sec}$

④ A high resolution black & white TV picture consists of 2×10^6 pixels/frame and 16 different bright levels, pixel is repeated at a rate of 32 frames/sec. All pixel elements are assumed to be independent & equiprobability. Calculate Entropy (H) and Information rate (R)?

Sol Given Different bright levels $M = 16$ (Independent & equally likely)

Entropy $H = \log_2 M = \log_2 2^4 = 4$

$\therefore \boxed{H = 4 \text{ bits/pixel}}$

Information rate $R = r \times H$

where $r = 2\times10^6$ pixels/frame $\times$ 32 $\times$ frames/sec

$r = 64\times10^6$ pixels/sec

$\therefore$ Information rate $R = r \times H$

$= 64\times10^6$ pixels/sec $\times$ 4 bits/pixel

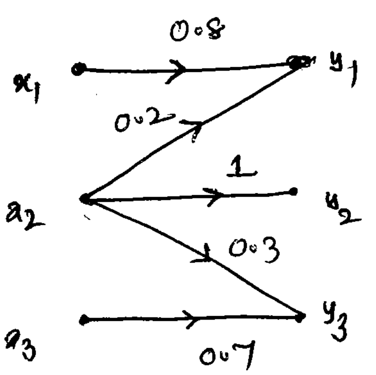
$= 256$ Mbits/sec

$\therefore \boxed{R = 256 \text{ M bits/sec}}$

Problems: →

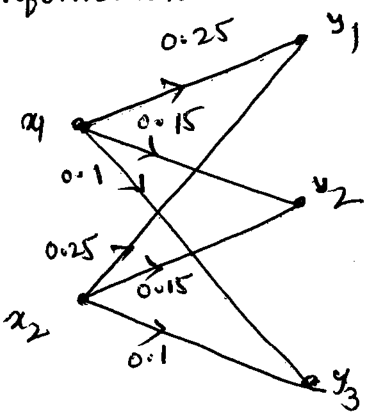
● Find All entropies and Mutual Information.

ⓐ



P(X/Y)

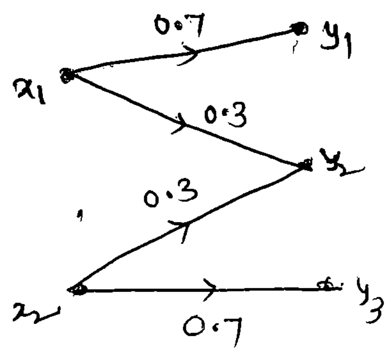
ⓑ



P(XY)

$p(y_1) = 0.2$, $p(y_2) = 0.5$, $p(y_3) = 0.3$.

ⓒ



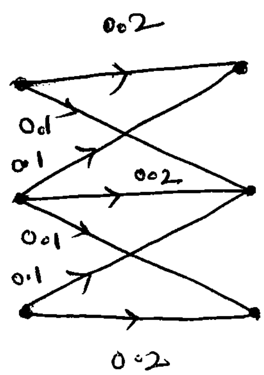
P(Y/X)

$p(x_1) = 0.5$

$p(x_2) = 0.5$

Channel capacity C = ?

ⓓ

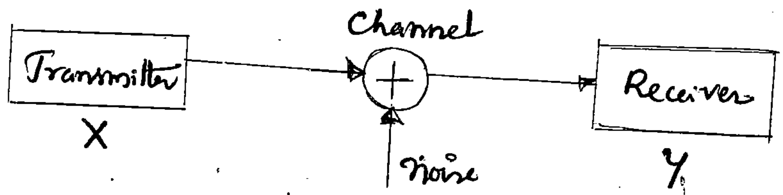


P(XY)

Joint Entropy and Conditional Entropy :

To study the behaviour of a communication system, we must study the behaviour of transmitter and the receiver, This gives rises to the concept of a two dimentional probability scheme.

* Let us consider two finite discrete sample space S_1 & S_2 let their product space $S = S_1 S_2$.



Let us consider transmitter 'x' generating 'm' symbols $\{x_1, x_2, x_3 \dots\dots x_m\}$ with probabilities $\{p(x_1), p(x_2), p(x_3) \dots\dots p(x_m)\}$

Let us consider the receiver 'y' receiving 'n' symbols $\{y_1, y_2, y_3 \dots y_n\}$ with probabilities $\{P(y_1), P(y_2), P(y_3), \dots\dots p(y_n)\}$.

$$[XY] = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & \cdots & x_1y_n \\ x_2y_1 & x_2y_2 & x_2y_3 & \cdots & x_2y_n \\ \vdots & \vdots & \vdots & & \vdots \\ x_my_1 & x_my_2 & x_my_3 & \cdots & x_my_n \end{bmatrix}$$

Thus we have 3 sets of complete probability schemes.

$$P(X) = [p(x_i)]$$

$$P(Y) = [P(y_j)]$$

$$P(XY) = [P(x_i y_j)]$$

Three associated entropies are

$$H(X) = -\sum_{i=1}^{m} P(x_i) \cdot \log [p(x_i)], \quad \text{where } p(x_i) = \sum_{j=1}^{n} P(x_i y_j)$$

$$H(Y) = -\sum_{j=1}^{n} P(y_j) \log [P(y_j)], \quad \text{where } P(y_j) = \sum_{i=1}^{m} p(x_i y_j)$$

$\therefore$ The joint entropy

$$\boxed{H(XY) := -\sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \log [p(x_i y_j)]}$$

$H(X)$ and $H(Y)$ are marginal entropies of X & Y.

$H(XY)$ is the joint entropy of X and Y.

- The conditional probability, from Baye's theorem

$$P(X/Y) = \frac{P(XY)}{P(Y)} \Rightarrow P(XY) = P(X/Y) \cdot P(Y)$$

(or)

$$P(Y/X) = \frac{P(XY)}{P(X)} \Rightarrow P(XY) = P(Y/X) \cdot P(X).$$

We know that 'y_j' may occur in conjuction with $x_1, x_2, \ldots x_m$.

thus $[X/y_j] = [x_1/y_j \;\; x_2/y_j \;\; x_3/y_j \; \ldots \; x_m/y_j]$

Probability $P(X/y_j) = [\,p(x_1/y_j) \;\; p(x_2/y_j) \;\; p(x_3/y_j) \; \ldots \; p(x_m/y_j)\,]$

& $p(x_1 y_j) + p(x_2 y_j) + p(x_3 y_j) + \ldots p(x_m, y_j) = P(y_j)$

Thus $\sum_{i=1}^{m} p(x_i/y_j) = 1$.

∴ The entropy may be associated with

$$H(X/y_j) = -\sum_{i=1}^{m}\left[\frac{p(x_i\, y_j)}{p(y_j)}\right] \cdot \log\left[\frac{p(x_i, y_j)}{p(y_j)}\right]$$

$$= -\sum_{i=1}^{m} p(x_i/y_j) \log p(x_i/y_j)$$

Consider all the messages of source & 'y_j' of receiver &

The measure of an average conditional entropy of the system.

$$H(X/Y) = \overline{H(X/y_j)}$$

$$= \sum_{j=1}^{n} p(y_j) \cdot H(X/y_j)$$

$$= -\sum_{j=1}^{n} p(y_j)\left[\sum_{i=1}^{m} p(x_i/y_j) \log p(x_i/y_j)\right]$$

Conditional Entropy

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} p(y_j) \cdot p(x_i/y_j) \cdot \log p(x_i/y_j)$$

$$\therefore \boxed{H(X/Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) \log p(x_i/y_j)} \quad (\because \text{Baye's theorem})$$

Similarly

$$\boxed{H(Y/X) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) \log P(y_j/x_i)}$$

$\therefore$ There are five entropies associated with two dimentional probability scheme. They are : $H(X)$, $H(Y)$, $H(XY)$, $H(X/Y)$ & $H(Y/X)$.

let

X - represents a transmitter

Y - represents a Receiver.

H(X) : Average information per character at the transmitter (or) Entropy of the transmitter.

H(Y) : Averge information per character at the receiver (or) Entropy of the receiver.

H(XY) : Entropy of a communication system.

H(X/Y) : Measure of information about the transmitter, where it is known that 'Y' is Received.

H(Y/X) : Measure of information about the receiver, where it is known that 'X' is transmitted.

Hence :

$$H(X) = -\sum_{i=1}^{m} p(x_i)\log p(x_i)$$

$$H(Y) = -\sum_{j=1}^{n} p(y_j)\log p(y_j)$$

$$H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(x_i y_j)$$

$$H(X/Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(x_i/y_j)$$

$$H(Y/X) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(y_j/x_i)$$

Relation b/w Different Entropies :

Case (i) : $H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\cdot\log[p(x_i y_j)]$

Using Baye's theorem

$$p(x_i/y_j) = \frac{p(x_i y_j)}{p(y_j)} \quad \text{(or)} \quad p(y_j/x_i) = \frac{p(x_i y_j)}{p(x)}$$

$$\therefore \quad H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\cdot\log\left\{p(x_i/y_j)\cdot p(y_j)\right\}$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\left\{\log p(x_i/y_j) + \log p(y_j)\right\}$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n}\left[p(x_i y_j)\log p(x_i/y_j) + p(x_i y_j)\log p(y_j)\right]$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(x_i/y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(y_j)$$

$$H(XY) = H(X/Y) - \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(y_j).$$

$$\left\{\begin{array}{l} \because \; p(y_j) = \sum_{i=1}^{m} p(x_i y_j) \\ p(x_i) = \sum_{j=1}^{n} p(x_i y_j). \end{array}\right\}$$

$$\therefore \quad H(XY) = H(X/Y) - \sum_{j=1}^{n} p(y_j)\log p(y_j)$$

$$H(XY) = H(X/Y) + H(Y).$$

$$\therefore \quad \boxed{H(X/Y) = H(XY) - H(Y).}$$

Similarly $\quad H(XY) = H(Y/X) + H(X)$

$$\therefore \quad \boxed{H(Y/X) = H(XY) - H(X)}$$

Case (ii):

If X and Y are independent events then

$$p(x_i y_j) = p(x_i)\cdot p(y_j).$$

$$\therefore \quad H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log\left[p(x_i y_j)\right]$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\left[\log p(x_i) + \log p(y_j)\right]$$

$$= -\sum_{i=1}^{m}\left[\sum_{j=1}^{n} p(x_i y_j)\right]\log p(x_i) - \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\cdot\log p(y_j)$$

$$= -\sum_{i=1}^{m} p(x_i)\log\left[p(x_i)\right] - \sum_{j=1}^{n} p(y_j)\log\left[p(y_j)\right]$$

$$= H(X) + H(Y)$$

$$\therefore \quad \boxed{H(XY) = H(X) + H(Y)}$$

Case iii: $H(X) \geq H(X/Y)$ and $H(Y) \geq H(Y/X)$.

The relation is proved by using the inequality as $\ln\left(\frac{1}{x}\right) \geq 1-x$ by writing the entropy in 'nats'.

$$\therefore H(X) - H(X/Y) \geq -\sum_{i=1}^{m} p(x_i)\log p(x_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\cdot \log[p(x_i/y_j)]$$

$$\geq -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(x_i) + \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(x_i/y_j)$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\left[\log p(x_i/y_j) - \log p(x_i)\right]$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log\left[\frac{p(x_i/y_j)}{p(x_i)}\right]$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\left(1 - \frac{p(x_i)}{p(x_i/y_j)}\right) \qquad \left(\because \log\frac{1}{x} = 1-x\right)$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p(x_i y_j)\cdot p(x_i)}{p(x_i/y_j)}$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i, y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p(\cancel{x_i y_j})\cdot p(x_i)\cdot p(y_j)}{p(\cancel{x_i y_j})}.$$

$$\geq \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)} - \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i)\cdot p(y_j) \qquad \text{(Bayes theorem)}$$

$$\geq \sum_{i=1}^{m} p(x_i) - \sum_{i=1}^{m} p(x_i) \qquad \left[\because \sum_{j=1}^{n} p(x_i y_j) = p(x_i),\ \sum_{j=1}^{n} p(y_j) = 1.\right.$$

$$\geq 0$$

$$H(X) - H(X/Y) \geq 0$$

$$\Rightarrow \boxed{H(X) \geq H(X/Y).}$$

Similarly $\boxed{H(Y) \geq H(Y/X)}$

$\Rightarrow$ ① $H(X/Y) = H(XY) - H(Y)$, $H(Y/X) = H(XY) - H(X)$

② $H(XY) = H(X) + H(Y)$, X & Y are independent events

③ $H(X) \geq H(X/Y)$, $H(Y) \geq H(Y/X)$

① Consider that the two sources S_1 & S_2 emit messages x_1, x_2, x_3 and y_1, y_2, y_3, with the joint probability $P(XY)$ as shown in below. Calculate $H(X)$, $H(Y)$, $H(XY)$, $X(X/Y)$, $H(Y/X)$.

$$P(XY) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 3/40 & 1/40 & 1/40 \\ x_2 & 1/20 & 3/20 & 1/20 \\ x_3 & 1/8 & 1/8 & 3/8 \end{array}$$

Sol: Given joint probability $P(XY) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 3/40 & 1/40 & 1/40 \\ x_2 & 1/20 & 3/20 & 1/20 \\ x_3 & 1/8 & 1/8 & 3/8 \end{array}$

$$P(x_i) = \sum_{j=1}^{n} p(x_i y_j)$$

let $i=1$ $\quad p(x_1) = \sum_{j=1}^{3} p(x_i y_j) = p(x_1 y_1) + p(x_1 y_2) + p(x_1 y_3)$

$= 3/40 + 1/40 + 1/40 = 5/40 = 1/8$. $\quad \therefore \boxed{p(x_1) = 1/8}$

$i=2$ $\quad p(x_2) = \sum_{j=1}^{3} p(x_i y_j) = p(x_2 y_1) + p(x_2 y_2) + p(x_2 y_3)$

$= 1/20 + 3/20 + 1/20 = 5/20 = 1/4$ $\quad \therefore \boxed{p(x_2) = 1/4}$

$i=3$ $\quad p(x_3) = \sum_{j=1}^{3} p(x_i y_j) = p(x_3 y_1) + p(x_3 y_2) + p(x_3 y_3)$

$= 1/8 + 1/8 + 3/8 = 5/8$ $\quad \therefore \boxed{p(x_3) = 5/8}$

$$P(y_j) = \sum_{i=1}^{m} p(x_i y_j)$$

let $j=1$ $\quad p(y_1) = \sum_{i=1}^{3} p(x_i y_j) = p(x_1 y_1) + p(x_2 y_1) + p(x_3 y_1)$

$= 3/40 + \frac{1}{20} + \frac{1}{8} = \frac{3+2+5}{40} = \frac{1}{4}$ $\quad \therefore \boxed{p(y_1) = \frac{1}{4}}$

$j=2$ $\quad p(y_2) = \sum_{i=1}^{3} p(x_i y_j) = p(x_1 y_2) + p(x_2 y_2) + p(x_3 y_2)$

$= 1/40 + 3/20 + \frac{1}{20} = \frac{1+6+5}{40} = 3/10$ $\quad \therefore \boxed{p(y_2) = 3/10}$

$j=3$ $\quad p(y_3) = \sum_{j=1}^{3} p(x_i y_j) = p(x_1 y_3) + p(x_2 y_3) + p(x_3 y_3)$

$= \frac{1}{40} + \frac{1}{20} + \frac{3}{8} = \frac{1+2+15}{40} = \frac{9}{20}$ $\quad \therefore \boxed{p(y_3) = 9/20}$

$\therefore$ Entropy $\quad H(X) = -\sum_{i=1}^{m} p(x_i) \cdot \log_2 p(x_i)$

$= -\left[p(x_1) \log p(x_1) + p(x_2) \log p(x_2) + p(x_3) \log p(x_3) \right]$

$= -\left[\frac{1}{8} \log 1/8 + 1/4 \log 1/4 + 5/8 \log 5/8 \right] / \log 2$

$= 1.2988$

$\therefore \boxed{H(X) = 1.2988 \text{ bits/msg}}$

Entropy $H(Y) = -\sum_{j=1}^{n} P(y_j) \log_2 P(y_j)$

$= -\left[\frac{1}{4}\log 1/4 + 3/10 \log 3/10 + 9/20 \log 9/20\right] / \log 2$

$= 1.5394$

$\therefore \boxed{H(Y) = 1.5394 \text{ bits/msg.}}$

joint Entropy

$H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) \log p(x_i y_j)$

$= -\left[3/40 \log 3/40 + 1/40 \log 1/40 + \frac{1}{40}\log 1/40 + \frac{1}{20}\log 1/20 + 3/20 \log 3/20 + \frac{1}{20}\log 1/20 + \frac{1}{8}\log 1/8 + \frac{1}{8}\log 1/8 + 3/8 \log 3/8\right]$

$= 2.67$ $\therefore \boxed{H(XY) = 2.67 \text{ bits/msg.}}$

$\underline{H(X/Y)}$:

Using Baye's theorem $p(x/y) = \frac{p(xy)}{p(y)}$

ie First column of $p(XY)$ is divided by $p(y_1)$, 1/4

Second column of $p(XY)$ is divided by $p(y_2)$ 4 3/10

Third column of $p(XY)$ is divided by $p(y_3)$. 9/20

$$P(X/Y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 3/10 & 1/12 & 1/18 \\ x_2 & 1/5 & 1/2 & 1/9 \\ x_3 & 1/2 & 5/12 & 5/6 \end{array}$$

$3/40 \times 4 = 3/10$
$3/20 \times 4 = 1/5$
$1/8 \times 4 = 1/2$
$\frac{1}{40} \times \frac{10}{3} = \frac{1}{12}$ $\frac{1}{40} \times \frac{20}{9} = \frac{1}{18}$
$3/20 \times \frac{10}{3} = \frac{1}{2}$ $\frac{1}{20} \times \frac{20}{9} = \frac{1}{9}$
$1/8 \times \frac{10}{3} = 5/12$ $\frac{3}{8} \times \frac{20}{9} = 5/6$

$H(X/Y) = -\sum_{i=1}^{3}\sum_{j=1}^{3} p(x_i y_j) \log[p(x_i/y_j)]$

$= -\left[3/40 \log 3/10 + 1/40 \log 1/12 + 1/40 \log 1/18 + 1/20 \log 1/5 + 3/20 \log 1/2 + \frac{1}{20}\log 1/9 + 1/8 \log 1/2 + 1/8 \log 5/12 + 3/8 \log 5/6\right]$

$= 1.12$ $\therefore \boxed{H(X/Y) = 1.12 \text{ bits/msg.}}$

$\underline{H(Y/X)}$:

Using Baye's theorem $P(Y/X) = \frac{P(XY)}{P(X)}$

ie First row of $p(XY)$ is divided by $p(x_1) \rightarrow 1/8$

Second row of $p(XY)$ is divided by $p(x_2) \Rightarrow 1/4$

Third row of $p(XY)$ is divided by $p(x_3) \rightarrow 5/8$

$$P(Y/X) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 3/5 & 1/5 & 1/5 \\ x_2 & 1/5 & 3/5 & 1/5 \\ x_3 & 1/5 & 1/5 & 3/5 \end{array}$$

$3/40 \times 8 = 3/5$
$1/40 \times 8 = 1/5, \frac{1}{40} \times 8 = 1/5$
$1/20 \times 4 = 1/5$
$3/20 \times 4 = 3/5$
$1/20 \times 4 = 1/5$
$\frac{1}{8} \times \frac{8}{5} = 1/5$
$\frac{3}{8} \times \frac{8}{5} = 3/5$

Conditional Entropy

$$H(Y/X) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) \log p(y_j/x_i)$$

$$= -\left[\tfrac{3}{40}\log 3/5 + \tfrac{1}{40}\log 1/5 + \tfrac{1}{40}\log 1/5 + \tfrac{1}{20}\log 1/5 + \tfrac{3}{20}\log 3/5 + \tfrac{1}{20}\log 1/5 + \tfrac{1}{8}\log 1/5 + \tfrac{1}{8}\log 1/5 + \tfrac{3}{8}\log 3/5 \right]$$

$$= 1.369$$

$$\therefore \boxed{H(Y/X) = 1.369 \text{ bits/msg.}}$$

(OR)

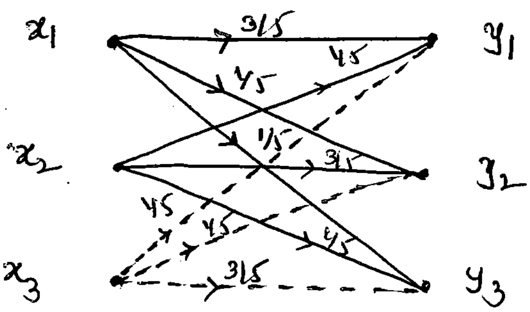
$H(X) = 1.2988$
$H(Y) = 1.5394$
$H(XY) = 2.67$

$$\therefore H(X/Y) = H(XY) - H(Y)$$
$$= 2.67 - 1.5394$$
$$= 1.13$$
$$\therefore \boxed{H(X/Y) = 1.13 \text{ bits/msg}}$$

III

$$H(Y/X) = H(XY) - H(X)$$
$$= 2.67 - 1.2988$$
$$= 1.37$$
$$\therefore \boxed{H(Y/X) = 1.37 \text{ bits/msg}}$$

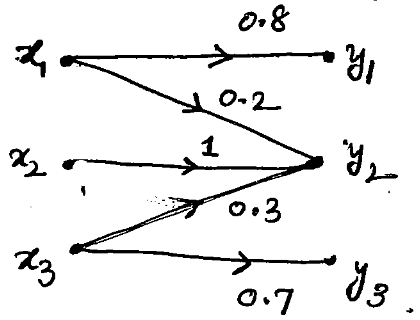
The matrix $p(y/x)$ is also known as channel matrix (or) noise matrix & graphically represented as



Note: Sum of the probabilities $= 1$

$$\sum_{i=1}^{m} P(x_i) = 1 \quad ; \quad \sum_{j=1}^{n} p(y_j) = 1$$

② A discrete source transmitter messages x_1, x_2, x_3 with the probabilities 0.3, 0.4, 0.3 respectively. The source is connected to the channel given in fig. Calculate all the Entropies.



Sol. Given

$$p(x_1) = 0.3$$
$$p(x_2) = 0.4$$
$$p(x_3) = 0.3$$

0.2
0.5
0.3

$$P(Y/X) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.8 & 0.2 & 0 \\ x_2 & 0 & 1 & 0 \\ x_3 & 0 & 0.3 & 0.7 \end{array}$$

$\therefore$ $P(X) = [0.3 \quad 0.4 \quad 0.3]$

Using Baye's theorem $P(Y/X) = \dfrac{P(XY)}{P(X)} \Rightarrow P(XY) = P(Y/X) \cdot P(X)$

$$\therefore \quad P(XY) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.24 & 0.06 & 0 \\ x_2 & 0 & 0.4 & 0 \\ x_3 & 0 & 0.09 & 0.21 \end{array}$$

$\therefore \; \sum P(XY) = 1$.

Entropy

$$H(X) = -\sum_{i=1}^{m} p(x_i) \log p(x_i)$$
$$= -\left[p(x_1)\log p(x_1) + p(x_2)\log p(x_2) + p(x_3)\log p(x_3) \right]$$
$$= -\left[0.3\log 0.3 + 0.4\log 0.4 + 0.3\log 0.3 \right]$$
$$= 1.57$$

$$\therefore \boxed{H(X) = 1.57 \text{ bits/msg}}$$

Entropy

$$H(Y) = -\sum_{j=1}^{n} p(y_j) \log p(y_j)$$

$$\Rightarrow \quad P(y_j) = \sum_{i=1}^{3} P(x_i y_j)$$

let $j = 1 \Rightarrow P(y_1) = p(x_1 y_1) + p(x_2 y_1) + p(x_3 y_1)$

$$= 0.24 + 0 + 0 = 0.24 \qquad \therefore \boxed{P(y_1) = 0.24}$$

$j = 2 \Rightarrow p(y_2) = p(x_1 y_2) + p(x_2 y_2) + p(x_3 y_2)$

$$= 0.06 + 0.4 + 0.09 = 0.55 \qquad \therefore \boxed{P(y_2) = 0.55}$$

$j = 3 \Rightarrow p(y_3) = P(x_1 y_3) + P(x_2 y_3) + P(x_3 y_3)$

$$= 0 + 0 + 0.21 = 0.21 \qquad \therefore \boxed{P(y_3) = 0.21}$$

$$\therefore \quad P(Y) = [0.24 \quad 0.55 \quad 0.21]$$

Entropy $H(Y) = -\left[p(y_1)\log p(y_1) + p(y_2)\log p(y_2) + p(y_3)\log p(y_3)\right]$

$= -\left[0.24\log 0.24 + 0.55\log 0.55 + 0.21\log 0.21\right] / \log 2$

$\doteq 1.44$ $\therefore$ $\boxed{H(Y) = 1.44 \text{ bits/msg}}$

Joint Entropy

$$H(XY) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(x_i y_j)$$

$$= -\left[0.24\log(0.24) + 0.06\log(0.24) + 0.4\log(0.4) + 0.09\log(0.09) + 0.21\log(0.21)\right]$$

$= 2.051$

$\therefore$ $\boxed{H(XY) = 2.051 \text{ bits/msg}}$

We know

$H(X/Y) = H(XY) - H(Y)$

$= 2.051 - 1.44 = 0.611$ $\therefore$ $\boxed{H(X/Y) = 0.611 \text{ bits/msg}}$

$H(Y/X) = H(XY) - H(X)$

$= 2.051 - 1.57 = 0.481$ $\therefore$ $\boxed{H(Y/X) = 0.481 \text{ bits/msg}}$

(OR)

$$H(Y/X) = -\sum_{i=1}^{3}\sum_{j=1}^{3} p(x_i y_j)\log p(y_j/x_i)$$

$$= -\left[0.24\log 0.8 + 0.06\log 0.2 + 0.4\log 1 + 0.09\log 0.3 + 0.21\log 0.7\right]$$

$= -[-0.077 - 0.139 - 0 - 0.156 - 0.108]$

$= 0.48$ $\therefore$ $\boxed{H(Y/X) = 0.48 \text{ bits/msg}}$

$$H(X/Y) = -\sum_{i=1}^{3}\sum_{j=1}^{3} p(x_i y_j)\log p(x_i/y_j)$$

$$P(X/Y) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 1 & 0.109 & 0 \\ x_2 & 0 & 0.727 & 0 \\ x_3 & 0 & 0.163 & 1 \end{array}$$

$$= -\left[0.24\log(1) + 0.06\log(0.1) + 0.4\log 0.7 + 0.09\log 0.163 + 0.21\log(1)\right]$$

$= -[0 - 0.199 - 0.205 - 0.235 + 0]$

$= 0.561$ $\therefore$ $\boxed{H(X/Y) = 0.612 \text{ bits/msg}}$

Hence

$H(X) = 1.57$ bits/msg

$H(Y) = 1.44$ bits/msg

$H(XY) = 2.05$ bits/msg

$H(X/Y) = 0.611$ bits/msg

$H(Y/X) = 0.48$ bits/msg.

(3) A transmitter has an alphabet of four letters $[x_1\ x_2\ x_3\ x_4]$ and the receiver has an alphabet of three letters $[y_1\ y_2\ y_3]$. The joint probability matrix is

$$P(XY) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.3 & 0.05 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0.15 & 0.05 \\ 0 & 0.05 & 0.15 \end{bmatrix} \end{array}$$

calculate all the Entropies.

Sol Given joint probability

$$P(XY) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.3 & 0.05 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0.15 & 0.05 \\ 0 & 0.05 & 0.15 \end{bmatrix} \end{array}$$

$$\therefore P(X) = [\,0.35 \quad 0.25 \quad 0.2 \quad 0.2\,]$$

$$P(Y) = [\,0.3 \quad 0.5 \quad 0.2\,]$$

$$\therefore H(X) = -\sum_{i=1}^{4} p(x_i) \log p(x_i)$$

$$= -[0.35 \log 0.35 + 0.25 \log 0.25 + 0.2 \log 0.2 + 0.2 \log 0.2]$$

$$= 1.96 \qquad \therefore \boxed{H(X) = 1.96 \text{ bits/msg}}$$

$$H(Y) = \sum_{j=1}^{3} p(y_j) \log p(y_j)$$

$$= -[0.3 \log 0.3 + 0.5 \log 0.5 + 0.2 \log 0.2]$$

$$= 1.49 \qquad \therefore \boxed{H(Y) = 1.49 \text{ bits/msg}}$$

$$H(XY) = -\sum_{i=1}^{4} \sum_{j=1}^{3} p(x_i y_j) \log p(x_i y_j)$$

$$= -[0.3 \log 0.3 + 0.05 \log 0.05 + 0.25 \log 0.25 + 0.15 \log 0.15 + 0.05 \log 0.05 + 0.05 \log(0.05) + 0.15 \log(0.15)]$$

$$= 2.49$$

$$\therefore \boxed{H(XY) = 2.49 \text{ bits/msg.}}$$

$$\therefore H(X/Y) = H(XY) - H(Y)$$

$$= 2.49 - 1.49 = 1.00 \qquad \therefore \boxed{H(X/Y) = 1 \text{ bit/msg}}$$

$$H(Y/X) = H(XY) - H(X)$$

$$= 2.49 - 1.96$$

$$= 0.53 \qquad \therefore \boxed{H(Y/X) = 0.53 \text{ bit/msg.}}$$

Mutual Information :

✓ The transfer of information from a transmitter through a channel to a receiver.

✓ The information provided about the event 'x_i' by the reception of the event 'y_j' is known as mutual information & it is denoted as $I(x_i y_j) \Rightarrow I(X Y)$.

* Before 'y_j' is received the uncertainity is $-\log p(x_i)$. is called <u>priori probability</u>.

* After 'y_j' is received the uncertainity becomes $-\log p(x_i/y_j)$ is called <u>posteriori probability</u>.

ie " The information gained about 'x_i' by the reception of 'y_j' is the net reduction in its uncertainity and is known as <u>mutual Information</u> ".

$$\therefore \quad I(x_i y_j) = \text{initial uncertainity} - \text{Final uncertainity}$$

$$= -\log p(x_i) - [-\log p(x_i/y_j)]$$

$$= \log p(x_i/y_j) - \log p(x_i)$$

$$\boxed{I(x_i y_j) = \log\left[\frac{p(x_i/y_j)}{p(x_i)}\right]}$$

$$I(x_i y_j) = \log\left[\frac{p(x_i y_j)}{p(x_i)\, p(y_j)}\right] \longrightarrow (1)$$

$\because$ Baye's theorem $p(x/y) = \dfrac{p(xy)}{p(y)}$.

lly

$$I(y_j x_i) = \log\left[\frac{p(x_i y_j)}{p(x_i)\, p(y_j)}\right] \longrightarrow (2)$$

From eqn (1) & (2) the mutual information is symmetrical in 'x_i', and 'y_j'.

$$\therefore \quad I(x_i y_j) = \underline{I(y_j x_i)}.$$

Self Information:

Self Information may be treated as a special case of mutual information when $y_j = x_i$ thus.

$$I(x_i x_i) = \log \frac{p(x_i/x_i)}{p(x_i)} = \log \frac{1}{p(x_i)} = I(x_i)$$

$\therefore$ Self Information $I(x_i x_i) = I(x_i)$.

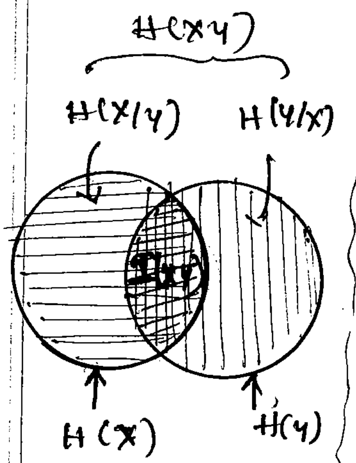
$\because$ The average of mutual Information ie, the entropy corresponding to mutual information is given by

$$I(XY) = \overline{I(x_i y_j)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \cdot I(x_i y_j)$$

$$\boxed{I(XY) = \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \cdot \log \left[\frac{p(x_i/y_j)}{p(x_i)} \right]}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \left[\log p(x_i/y_j) - \log p(x_i) \right]$$



$$= \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \log p(x_i/y_j) - \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \log p(x_i)$$

$$= -\sum_{i=1}^{m} \Big[\underbrace{\sum_{j=1}^{n} p(x_i y_j)}_{p(x_i)} \Big] \log p(x_i) - \left[- \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \log p(x_i/y_j) \right]$$

$$= -\sum_{i=1}^{m} p(x_i) \log p(x_i) - \left[H(X/Y) \right]$$

$$I(XY) = H(X) - H(X/Y)$$

$\therefore$ $\boxed{I(XY) = H(X) - H(X/Y)}$ → (3)

lly $I(YX) = H(Y) - H(Y/X)$

$\therefore$ $\boxed{I(YX) = H(Y) - H(Y/X)}$ → (4)

We know

$$H(XY) = H(X) + H(Y/X)$$

& $H(XY) = H(Y) + H(X/Y)$ $\quad \therefore H(X/Y) = H(XY) - H(Y)$

$\therefore$ $I(XY) = H(X) - H(XY) + H(Y)$

$\therefore$ Mutual Information $\boxed{I(XY) = H(X) + H(Y) - H(XY)}$ → (5)

Hence $I(XY)$ does not depends on the individual symbols x_i & y_j and it is a property of the entire communication system.

eqn ③, ④ & ⑤ states that the entropy corresponding to mutual information.

$$I(XY) = H(X) - H(X/Y)$$

where $H(X/Y)$ - the average information loss in the channel. This is also known as 'Equivocation'

∴ Mutual Information = Source information − loss of Information.

Illy

$$I(YX) = H(Y) - H(Y/X)$$

Mutual Information = Receiver Entropy − loss of Information

∴ For noise free channel

ie $\boxed{H(X/Y) = H(Y/X) = 0.}$

$$I(XY) = H(X) = H(Y) \longrightarrow ⑥$$

Eqn ⑥ states that the mutual information is equal to the source entropy and also equal to receiver entropy. Thus the information transmitted at source is completely received by the receiver through a channel without any loss of information.

Properties of Mutual Information:

① The mutual information of a channel is symmetry

ie $\boxed{I(XY) = I(YX)}$

② Mutual information is always non negative

ie $I(XY) \geq 0$.

Proof: We know that

$$I(XY) = H(X) - H(X/Y)$$

(or)

$$I(XY) = \sum_{i=1}^{m} \sum_{j=1}^{n} p(x_i y_j) \cdot \log \left[\frac{p(x_i/y_j)}{p(x_i} \right]$$

By using log inequality ie $\log x \geq (1 - \frac{1}{x})$.

$$I(XY) \geq \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\left[1 - \frac{p(x_i)}{p(x_i/y_j)}\right]$$

$$\geq \sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) - \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{p(x_i y_j)\cdot p(x_i)}{p(x_i/y_j)}.$$

Apply Baye's theorem

$$p(XY) = p(X)\cdot p(Y/X)$$

$$p(XY) = p(Y)\cdot p(X/Y) \Rightarrow p(XY)/p(X/Y) = p(Y)$$

$$\therefore I(XY) \geq \underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)}_{1} - \sum_{i=1}^{m}\sum_{j=1}^{n} p(y_j)\cdot p(x_i)$$

$$\geq 1 - \underbrace{\sum_{i=1}^{m} p(x_i)}_{1}\cdot \underbrace{\sum_{j=1}^{n} p(y_j)}_{1}.$$

$$\geq 1-1$$

$$\geq 0 \qquad \therefore \boxed{I(XY) \geq 0.}$$

③ Mutual information is related to entropies.

ie

$$I(XY) = H(X) - H(X/Y)$$

$$I(XY) = H(Y) - H(Y/X)$$

$$I(XY) = H(X) + H(Y) - H(XY)$$

④ Maximum value of mutual information is called channel capacity

ie $\boxed{C = \text{Max.}\,[I(XY)]}$

⑤ The channel efficiency (or) Transmission efficiency is defined as The ratio of actual trans information $I(XY)$ to the maximum transformation (C). ie

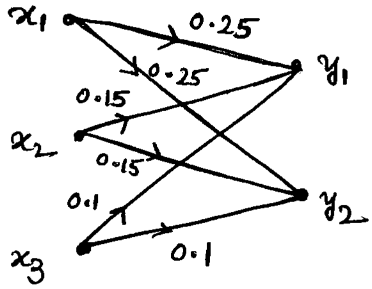
$$\eta = \frac{I(XY)}{\text{Max}\,(I(XY))} \Rightarrow \boxed{\eta = \frac{I(XY)}{C}}$$

⑥ Redundancy : The redundancy of the channel is defined as

$$\boxed{\delta = 1-\eta} \Rightarrow \delta = 1 - \frac{I(XY)}{C} \Rightarrow \boxed{\delta = \frac{C - I(XY)}{C}}$$

④ Find the mutual information for the channel shown in below.

Sol The joint probability matrix for the channel.



$$p(XY) = \begin{array}{c|cc} & y_1 & y_2 \\ x_1 & 0.25 & 0.25 \\ x_2 & 0.15 & 0.15 \\ x_3 & 0.1 & 0.1 \end{array}$$

$\therefore \; p(x_1) = 0.25 + 0.25 = 0.5$

$p(x_2) = 0.15 + 0.15 = 0.3$

$p(x_3) = 0.1 + 0.1 = 0.2$

$p(y_1) = 0.25 + 0.15 + 0.1 = 0.5$

$p(y_2) = 0.25 + 0.15 + 0.1 = 0.5$

$\therefore p(X) = [\,0.5 \;\; 0.3 \;\; 0.2\,]$ $\qquad p(Y) = [\,0.5 \;\; 0.5\,]$

$$H(X) = -\sum_{i=1}^{m} p(x_i).\log p(x_i)$$

$$= -[0.5 \log 0.5 + 0.3 \log 0.3 + 0.2 \log 0.2]$$

$$= 1.485 \text{ bits/msg} \qquad \therefore \boxed{H(X) = 1.485 \text{ bits/msg}}$$

$$H(Y) = -\sum_{j=1}^{2} p(y_j) \log p(y_j)$$

$$= -[0.5 \log(0.5) + 0.5 \log(0.5)]$$

$$= 1 \text{ bit/msg.} \qquad \therefore \boxed{H(Y) = 1 \text{ bit/msg}}$$

$$H(XY) = -\sum_{i=1}^{3} \sum_{j=1}^{2} p(x_i y_j) \log p(x_i y_j)$$

$$= -[0.25 \log(0.25) + 0.25 \log(0.25) + 0.15 \log(0.15) + 0.15 \log(0.15) + 0.1 \log(0.1) + 0.1 \log(0.1)].$$

$$= 2.485 \text{ bits/msg} \qquad \therefore \boxed{H(XY) = 2.485 \text{ bits/msg}}$$

$$H(X/Y) = H(XY) - H(Y)$$

$$= 2.485 - 1 = 1.485 \qquad \therefore \boxed{H(X/Y) = 1.485 \text{ bits/msg}}$$

$$H(Y/X) = H(XY) - H(X)$$

$$= 2.485 - 1.485 = 1 \qquad \therefore \boxed{H(Y/X) = 1 \text{ bit/msg}}$$

$\therefore$ Mutual information $I(XY) = H(X) + H(Y) - H(XY).$

$$= 1.485 + 1 - 2.485$$

$$= 2.485 - 2.485 = 0 \qquad \therefore \boxed{I(XY) = 0}$$

Hence

The channel shown above is with an independent input & output so mutual information becomes zero.

Note: A channel is said to be with an independent input and output when the joint probability matrix satisfies at least one of the following conditions
a) Each row consists of the same element.
b) Each column consists of the same element.

noisy channel

Discrete Memoryless channel:

* A discrete channel to be a system with input 'x' and output 'y'. The channel accepts input signal 'x' selected from an alphabet 'A'. and in response it delivers and output response 'y' from an alphabet 'B'.
* The probability transition matrix $P(Y/X)$ that express the probability of observing the output symbol 'y' given that the symbol 'x'.

✓ The channel is said to be discrete, both of the alphabets A & B have a finite length.

✓ It is said to be memoryless if the output depends only on the presence input and it is independent of previous inputs and outputs.

Channel Capacity:

Shannon has introduced a significant concept of Channels capacity & defined as the maximum of mutual information

(OR) The maximum rate at which the channel supplies the information to the receiver.

ie $$\boxed{C = \text{Max.}\, I(XY)} \text{ bits/message.}$$

where $$I(XY) = H(X) - H(X/Y)$$

(or) $$I(XY) = H(Y) - H(Y/X)$$

(or) $$I(XY) = H(X) + H(Y) - H(XY).$$

✓ Transmission efficiency & Channel efficiency $\boxed{\eta = \dfrac{I(XY)}{C}}$

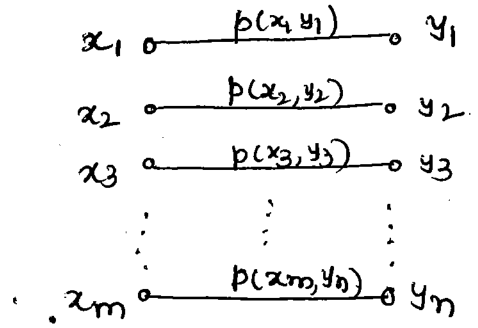
✓ The redundancy of the channel $\boxed{\delta = 1 - \eta}$

Classification of Channels :

① Noisy free Channel :

Noise free channel is also called noiseless binary channel. Noise free channel means there is one to one correspondance between input and output. ie. Each input symbol is received as one and only one output symbol.

ie



$$P(XY) = \begin{bmatrix} p(x_1 y_1) & 0 & 0 & \cdots & 0 \\ 0 & p(x_2 y_2) & 0 & \cdots & 0 \\ 0 & 0 & p(x_3 y_3) & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots 0 & \cdots 0 & \cdots & p(x_m y_n) \end{bmatrix}$$

For a noise free channel $H(X/Y) = H(Y/X) = 0$.

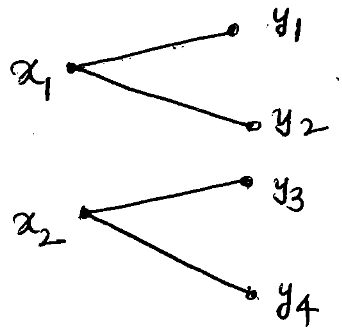
channel capacity
$$C = Max[I(XY)]$$
$$= Max[H(X) - \underbrace{H(X/Y)}_{0}]$$
$$= Max[H(X)]$$
$$C = \log_2 M$$

$$\therefore \boxed{C = \log_2 M} \text{ bits/msg.}$$

whe M - No of messages.

② Noisy channel with non overlapping output :

This channel has two possible output corresponds to single input. The channel appears to be noisy but really is not.

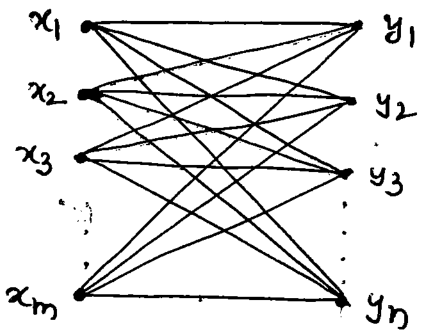


If x_1 is transmitted it may receved as y_1 and y_2.

Even though the output of channel is a random sequence of the input, the input can be determined from the output.

③ Noisy channel with overlapping :

When the channel has noise it becomes difficult to reconstruct the transmitted signal faithfully.



$$P(XY) = \begin{bmatrix} p(x_1y_1) & p(x_1,y_2) & \cdots & p(x_1y_n) \\ p(x_2y_1) & p(x_2y_2) & \cdots & p(x_2y_n) \\ \vdots & \vdots & & \vdots \\ p(x_my_1) & p(x_my_2) & \cdots & p(x_my_n) \end{bmatrix}$$

If the input and output symbol probabilities are statistically independent with each other.

$$p(x_i y_j) = p(x_i)\cdot p(y_j)$$

From Baye's theorem

$$p(x_i y_j) = p(x_i/y_j)\cdot p(y_j)$$

$$p(x_i)\, \cancel{p(y_j)} = p(x_i/y_j)\cdot \cancel{p(y_j)}$$

$$\therefore\ p(x_i/y_j) = p(x_i)$$

lly

$$p(y_j/x_i) = P(y_j).$$

$$\therefore\ H(Y/X) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\log p(y_j/x_i)$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j)\cdot \log p(y_j).$$

$$= -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i)\cdot p(y_j)\cdot \log P(y_j)$$

$$= -\sum_{j=1}^{n}\Big[\underbrace{\sum_{i=1}^{m} p(x_i)}_{\text{is } 1}\Big] p(y_j)\cdot \log p(y_j).$$

$$= -\sum_{j=1}^{n} p(y_j)\log p(y_j).$$

$$\boxed{H(Y/X) = H(Y)}$$

$\therefore$ $\boxed{H(Y/X) = H(Y)}$ lly

Similarly $\boxed{H(X/Y) = H(X)}$

$\therefore$ Mutual Information $I(XY) = H(X) - H(X/Y) = H(X) - H(X) = 0.$

Channel Capacity $C = Max[I(XY)] = 0$ $\therefore$ $\boxed{C = 0}$

ie No information is transmitted from the channel to the receiver.

④ Symmetric channel :

A symmetric channel is defined as the rows and columns of the channel matrix $P(Y/X)$ are identical except permitations (not in order).

(OR)

It is one for which ⓐ $H(Y/x_i)$ is independent of 'i'

ie the entropy corresponding to each row of $P(Y/X)$ is the same.

ⓑ $\sum_{i=1}^{m} P(y_j/x_i)$ is independent of 'j' ie. The sum of all the collums of $P(Y/X)$ is the same.

<u>Example</u> : a) $P(Y/X) = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$

This is a symmetric channel as rows and columns are identical except for permutations. (variation of the order of the series).

b) $P(Y/X) = \begin{bmatrix} 1/2 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/2 \\ 1/2 & 1/4 & 1/4 \end{bmatrix}$ ⇒ This is not a symmetric channel as the rows are identical except for permutations but not the columns.

c) $P(Y/X) = \begin{bmatrix} 0.4 & 0.6 \\ 0.3 & 0.7 \\ 0.6 & 0.4 \\ 0.7 & 0.3 \end{bmatrix}$ ⇒ This is not a symmetric channel as the columns are idetitical except for permutations but not the rows.

d) $P(Y/X) = \begin{bmatrix} 1/3 & 1/6 & 1/3 & 1/6 \\ 1/6 & 1/3 & 1/6 & 1/3 \end{bmatrix}$ ⇒ This is a symmetric channel as the row and columns are identical except for permutations.

e) $P(Y/X) = \begin{bmatrix} 1/3 & 1/6 & 1/3 & 1/6 \\ 1/3 & 1/3 & 1/6 & 1/6 \end{bmatrix}$ ⇒ This is not a symmetric channel as the rows and columns are not identical.

Channel Capacity for Symmetric channel :

Mutual information $I(XY) = H(Y) - H(Y/X)$

where $H(Y/X) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) \log p(y_j/x_i)$

$= -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i) \cdot p(y_j/x_i) \log p(y_j/x_i)$

(∵ Baye's theorem $p(XY) = p(Y/X) \cdot p(X)$)

$= -\left[\sum_{i=1}^{m} p(x_i)\right] \sum_{j=1}^{n} p(y_j/x_i) \log [p(y_j/x_i)]$

$\quad\; \left[\sum_{i=1}^{m} p(x_i)\right] = 1.$

$= -\sum_{j=1}^{n} p(y_j/x_i) \log p(y_j/x_i)$

$H(Y/X) = h$

Consider where $h = -\sum_{j=1}^{n} p(y_j/x_i) \log p(y_j/x_i).$

∴ $I(XY) = H(Y) - H(Y/X)$

$= H(Y) - h$

∴ The channel capacity $C = \text{Max}\, [I(XY)]$

$= \text{Max}\,[H(Y) - h]$

$= \text{Max}\,[H(Y)] - h$

$C = \log_2 M - h$ (∵ $\text{Max}(H(Y)) = \log_2 M$)

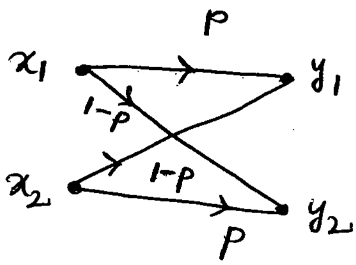
∴ $\boxed{C = \log_2 M - h}$

where $h = -\sum_{j=1}^{n} p(y_j/x_i) \log p(y_j/x_i).$

Binary Symmetric Channel (BSC) :

The most important case of symmetrical channel is Binary symmetric Channel (BSC).

Here $m = 2,\; n = 2$ (binary)



∴ The channel matrix $P(Y/X) = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$

The noise characteristics of the channel is

$p(y_1/x_1) = p$, $p(y_2/x_1) = 1-p$, $p(y_1/x_2) = 1-p$, $p(y_2/x_2) = p$.

∴ Channel capacity of binary symmetric channel

$$C = Max\,[I(X;Y)]$$

The channel capacity for symmetric channel is

$$C = \log_2 M - h$$

binary $M=2$

$$C = \log_2 2 - \left[- \sum_{j=1}^{n} p(y_j/x_i) \log p(y_j/x_i)\right]$$

$$= 1 + \sum_{j=1}^{2} p(y_j/x_i) \log p(y_j/x_i)$$

$$= 1 + p(y_1/x_i) \log p(y_1/x_i) + p(y_2/x_i) \log p(y_2/x_i)$$

$x_i = 1$ & 2

for $x=1$

$$= 1 + p\log p + (1-p)\log(1-p)$$

$$= 1 - [-\{p\log p + (1-p)\log(1-p)\}]$$

$$C = 1 - H(P)$$

∴ Channel capacity $\boxed{C = 1 - H(p)}$ where $H(p) = -[p\log p + (1-p)\log(1-p)]$

5 Find the channel capacity for the Binary Symmetric Channel for the given data: (i) $p = 0.9$ (ii) $p = 0.6$.

Sol Binary symmetric channel $C = 1 - H(p) = 1 + [-H(p)]$

(i) $H(p) = -[p\log p\,(1-p)\log(1-p)]$

$p = 0.9$
$1-p = 0.1$

$$= -[0.9\log(0.9) + 0.1\log(0.1)]$$

$$= -0.468$$

∴ $\boxed{C = 0.532 \text{ bits/msg}}$

(ii) $p = 0.6$
$1-p = 0.4$

$$H(p) = -[0.6\log(0.6) + (0.4)\log(0.4)]$$

$$= -0.97$$

$$C = 1 + [-H(p)]$$

$$= 1 - 0.97$$

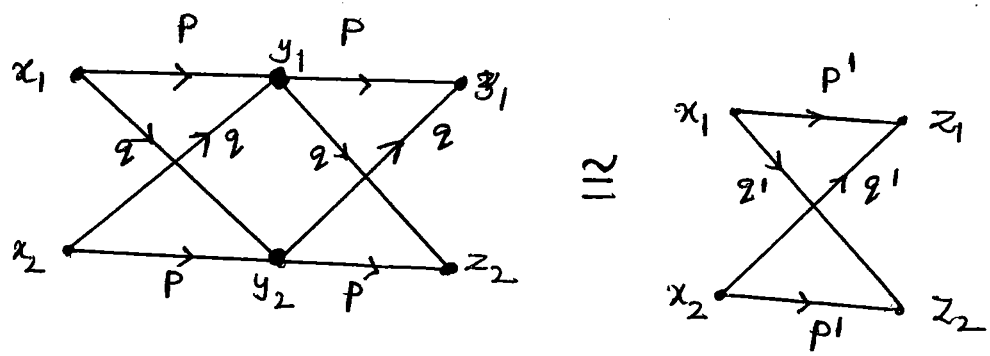
$$C = 0.03$$

∴ $\boxed{C = 0.03 \text{ bits/msg}}$

⑤ Cascaded Channel :

Some times channels are to be cascaded for some applications. Let us consider the code of two cascaded identical binary symmetric channel.

$q = 1 - p$



✓ The message from 'x_1' reaches to 'z_1' in two ways

$$x_1 \text{ to } z_1 \Rightarrow \quad x_1 - y_1 - z_1$$
$$x_1 - y_2 - z_1$$

∴ The respective path probabilities are $p \cdot p$ and $q \cdot q$.

$$\therefore \quad p' = p \cdot p + q \cdot q = p^2 + q^2$$
$$= (p+q)^2 - 2pq$$
$$= [p + (1-p)]^2 - 2pq$$
$$= 1 - 2pq \qquad \therefore \boxed{p' = 1 - 2pq}$$

✓ The message from 'x_1' reaches to 'z_2' in two ways

$$x_1 \text{ to } z_2 \Rightarrow \quad x_1 - y_2 - z_2$$
$$x_1 - y_1 - z_2$$

The respective path probabilities are qp and $p \cdot q$.

$$\therefore \quad q' = q \cdot p + p \cdot q = 2pq \qquad \therefore \boxed{q' = 2pq}$$

$$\Rightarrow \quad p' + q' = 1 - 2pq + 2pq = 1 \qquad \therefore p' + q' = 1.$$

The channel matrix of the cascaded channel is

$$P(Z/X) = \begin{bmatrix} p' & q' \\ q' & p' \end{bmatrix} = \begin{bmatrix} 1-2pq & 2pq \\ 2pq & 1-2pq \end{bmatrix}$$

Thus the cascaded channel is equavallent to a single binary symmetric channel with error probability equal to '$\underline{2pq}$'

∴ The channel capacity of a binary symmetric channel is

$$C = 1 - H(p) \quad \text{where} \quad H(p) = -[p \log p + (1-p) \log(1-p)]$$

∴ The channel capacity of a cascaded channel is

$$C = 1 - H(q') \quad , q' = 2pq \qquad \therefore \boxed{C = 1 - H(2pq)}.$$

⑥ Binary Erasure Channel (BEC):

Binary erasure channel has two inputs (0 & 1) and three outputs (0, n, 1). BEC is also very important.

Here '0' and '1' are transmitted & they are received 0, n, 1.

The symbol 'n' indicates that, due to noise, no deterministic decision can be made as to whether the received symbol is a '0' or a '1'.

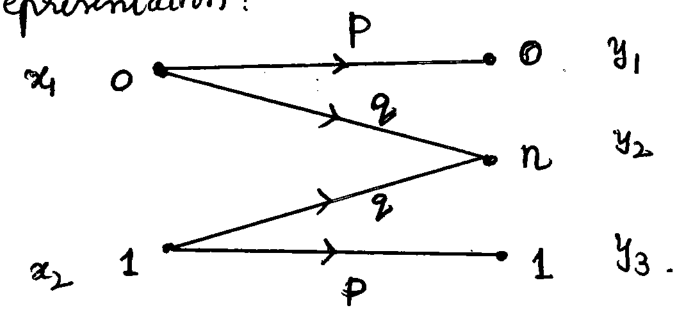
ie The symbol 'n' indicates that the output is erased.

Hence the name Binary Erasure Channel.

✓ If the output is 'n' then the receiver request the transmitter for retransmission till the decision is taken either the symbol '0' or '1'. This can be consider equivallent to error detection and requesting for retransmission.

ie **ARQ** (Automatic Repeat Request) method of error correction.

Graphical representation:



The channel matrix for binary Erasure channel is

$$p(Y|X) = \begin{array}{c} \\ x_1 \\ x_2 \\ \\ \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} p & q & 0 \\ 0 & q & p \end{bmatrix} \\ \begin{array}{ccc} 0 & n & 1 \end{array} \end{array} \qquad q = 1-p$$

Let $p(x_1) = \alpha$ and $p(x_2) = 1-\alpha$

$\therefore$ Entropy $H(X) = -\sum_{i=1}^{m} p(x_i) \log p(x_i)$

$$= -\left[p(x_1)\log p(x_1) + p(x_2)\log p(x_2) \right]$$

$$\therefore H(X) = -\left[\alpha \log \alpha + (1-\alpha)\log(1-\alpha) \right]$$

The joint probability matrix $p(XY)$ can be obtained by multiplying the rows of $p(Y/X)$ by 'α' and $(1-\alpha)$ respectively.

Since $p(x_1) = \alpha$, $p(x_2) = 1-\alpha$.

[$\because$ Baye's theorem $p(XY) = p(Y/X) \cdot p(X)$]

$$p(XY) = \begin{bmatrix} \alpha p & \alpha q & 0 \\ 0 & (1-\alpha)q & (1-\alpha)p \end{bmatrix}$$

The sum of the columns gives

$$p(y_1) = \alpha p$$
$$p(y_2) = \alpha q + q - \alpha q = q$$
$$p(y_3) = (1-\alpha)p.$$

The conditional probability matrix $p(X/Y)$ can be obtained by dividing the columns of $p(XY)$ by $p(y_1)$, $p(y_2)$, $p(y_3)$ respectively.

$$\therefore \; P(X/Y) = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1-\alpha & 1 \end{bmatrix} \qquad \left[\because P(X/Y) = \frac{P(XY)}{P(Y)}\right]$$

$$\because H(X/Y) = -\sum_{i=1}^{m}\sum_{j=1}^{n} p(x_i y_j) \log p(x_i/y_j)$$

$m=2$
$n=3$

$$= -\left[\alpha p \log(1) + \alpha q \log \alpha + 0 + 0 + (1-\alpha) q \log(1-\alpha) + (1-\alpha) p \log(1)\right]$$

$$= -\left[\alpha q \log \alpha + (1-\alpha) q \log(1-\alpha)\right]$$

$$= -q\left[\alpha \log \alpha + (1-\alpha)\log(1-\alpha)\right]$$

$$\therefore H(X/Y) = q \cdot H(X). \quad \text{(or)} \quad \boxed{H(X/Y) = (1-p)H(X)}$$

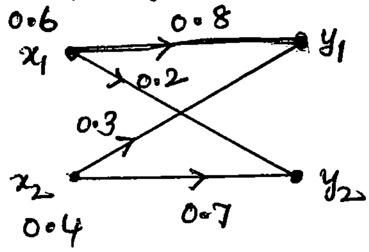
The channel capacity of Binary Erasure channel is

$$C = \text{Max}\left[I(XY)\right]$$
$$= \text{Max}\left[H(X) - H(X/Y)\right]$$
$$= \text{Max}\left[H(X) - (1-p)H(X)\right]$$
$$= \text{Max}\left[\cancel{H(X)} - \cancel{H(X)} + p \cdot H(X)\right] \qquad \boxed{I(XY) = p \cdot H(X)}$$
$$= \text{Max}\left[p \cdot H(x)\right]$$
$$= p.\, \text{Max}\left[H(x)\right]$$
$$= p.\, \log_2 M \qquad (\because \text{Max}[H(x)] = \log_2 M)$$
$$= p.\, \log_2 2 \qquad (\because \text{Binary } M=2)$$

$\underline{C = p}$

$\therefore$ Channel capacity for BEC, $\boxed{C = p.}$

⑥ Find the mutual information and channel capacity of the channel shown in fig below.



Sd Given

$$p(Y/X) = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

$p(x_1) = 0.6$, $p(x_2) = 0.4$

$P(XY) = p(Y/X) \cdot P(X)$

$$\therefore p(XY) = \begin{bmatrix} 0.48 & 0.12 \\ 0.12 & 0.28 \end{bmatrix}$$

$\Rightarrow$ $p(y_1) = 0.48 + 0.12 = 0.6$ $\quad \therefore p(y_1) = 0.6$

$p(y_2) = 0.12 + 0.028 = 0.4$ $\quad p(y_2) = 0.4$

$$P(X/Y) = \frac{P(XY)}{P(Y)} \Rightarrow p(X/Y) = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$$

$$\begin{bmatrix} 0.48/0.6 = 0.8 \\ 0.12/0.6 = 0.2 \\ 0.12/0.4 = 0.3 \\ 0.28/0.4 = 0.7 \end{bmatrix}$$

$$\therefore H(X) = -\sum_{i=1}^{2} p(x_i) \log p(x_i)$$

$= -[0.6 \log 0.6 + 0.4 \log 0.4]$

$= 0.971$ bits/msg $\quad \therefore \boxed{H(X) = 0.970 \text{ bits/msg}}$

$$H(Y) = -\sum_{i=1}^{2} p(y_j) \log p(y_j)$$

$= -[0.6 \log 0.6 + 0.4 \log 0.4]$ $\quad \therefore \boxed{H(Y) = 0.970 \text{ bits/msg}}$

$= -0.970$ bits/msg

$$H(XY) = -\sum_{i=1}^{2}\sum_{j=1}^{2} p(x_i y_j) \log p(x_i y_j).$$

$= -[0.48 \log(0.48) + 0.12 \log(0.12) + 0.12 \log(0.12) + 0.28 \log(0.28)]$

$= -[-0.508 - 0.367 - 0.367 - 0.514]$

$= 1.756$ bits/msg $\quad \therefore \boxed{H(XY) = 1.756 \text{ bits/msg}}$

$H(X/Y) = H(XY) - H(Y) = 1.756 - 0.97 = 0.786$ $\quad \therefore \boxed{H(X/Y) = 0.786 \text{ bits/msg}}$

$H(Y/X) = H(XY) - H(X) = 1.756 - 0.97 = 0.786$ $\quad \therefore \boxed{H(Y/X) = 0.786 \text{ bits/msg}}$

Mutual Information

$I(XY) = H(X) + H(Y) - H(XY)$

$= 0.97 + 0.97 - 1.756 = 0.184$ $\quad \therefore \boxed{I(XY) = 0.184 \text{ bits/msg.}}$

Channel capacity $\quad C = 1 - H(p)$

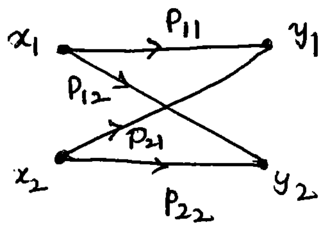
where $H(p) = -\{p \log p + (1-p) \log(1-p)\}$

for symmetric structures in channel ie $\begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}$

If Not perform as follows.

Note: A binary channel with non-symmetric structures

$$D = P(Y/X) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$



To find the channel capacity of a binary channel, the auxillary variables Q_1 & Q_2 are defined by

$$[D][Q] = -[H]$$

$$\Rightarrow \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} P_{11}\log P_{11} + P_{12}\log P_{12} \\ P_{21}\log P_{21} + P_{22}\log P_{22} \end{bmatrix}$$

$$\Rightarrow P_{11}Q_1 + P_{12}Q_2 = P_{11}\log P_{11} + P_{12}\log P_{12} \longrightarrow (1)$$

$$P_{21}Q_1 + P_{22}Q_2 = P_{21}\log 21 + P_{22}\log P_{22} \longrightarrow (2)$$

By solving eqn (1) & (2), we get Q_1 and Q_2.

$\therefore$ The channel capacity $\boxed{C = \log(2^{Q_1} + 2^{Q_2})}$

From problem.

$$\begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} 0.8\log 0.8 + 0.2\log 0.2 \\ 0.3\log 0.3 + 0.7\log 0.7 \end{bmatrix}$$

$$\Rightarrow 0.8Q_1 + 0.2Q_2 = -0.721$$

$$0.3Q_1 + 0.7Q_2 = -0.881$$

$\rightarrow$ $a_1 = 0.8$ $b_1 = 0.2$, $c_1 = -0.721$
$a_2 = 0.3$ $b_2 = 0.7$ $c_2 = -0.881$

By solving above eqns $Q_1 = -0.656$
$Q_2 = -0.976$

$\therefore$ The channel capacity

$$c = \log(2^{Q_1} + 2^{Q_2})$$

$$= \log\left[2^{-0.656} + 2^{-0.976}\right]$$

$$= \log(0.633 + 0.513)$$

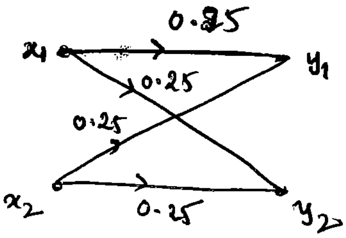
$$= \log_2 1.146 = \log(1.146)/\log 2 = 0.196 \text{ bits/msg}$$

$$= 0.2$$

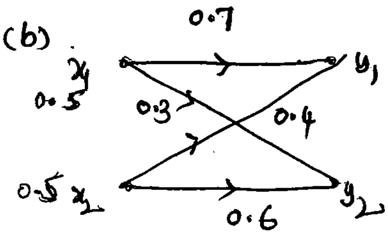
$\therefore$ $\boxed{C = 0.2 \text{ bits/message}}$

Problems.

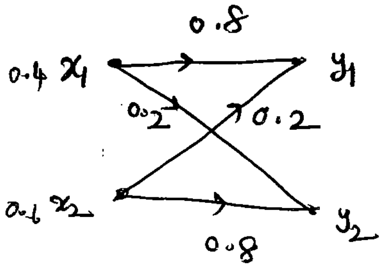
(a)



(b)



(c)



Source Coding :

Coding offers the most significant application of the information theory, the main purpose of the coding is to improve the efficiency of the communication system.

* A conversion of the output of discrete memoryless source (DMS) ie Voice, image & data into a sequence of binary symbols (ie binary code word) is called 'Source coding' 1's and 0's.

✓ The device the performs this conversion is called 'Source encoder'

* Source coding is mainly used to minimize the average bit rate required for representation of the source by reducing the redundancy of the information source.

Code length & Coding Efficiency :

✓ Let X be the discrete memoryless source with an alphabet $(x_1, x_2, \cdots x_m)$ with corresponding probabilities $[p(x_1), p(x_2), \cdots p(x_m)]$.

✓ Let binary code word assigned to symbol/alphabet 'x_i' by the encoder have length 'n_i' measured in bits.

"The length of the codeword is the no. of binary digits in the code word"

∴ Code length : The average length of the message (or) average length per code word is given by

$$\boxed{L = \sum_{i=1}^{m} p(x_i) \cdot n_i} \quad \text{Letters/msg.}$$

ie 'L' should be minimum to have efficient transmission.

Coding Efficiency :
$$\eta = \frac{L_{min}}{L}$$

where L_{min} - minimum possible value of 'L' when 'η' approaches unity, the code is said to be efficient.

Let $H(X)$ be the entropy of the source in bits/msg.

Let $\log_2 M$ be the maximum average information associated with each letter in bits/letter.

The minimum average no. of letters per message

$$\boxed{L_{min} = \frac{H(X)}{\log_2 M}} \qquad \text{Units} = \frac{\text{bit/msg}}{\text{bits/letter}} = \text{Letters/message}$$

∴ Hence

The coding efficiency $\eta = \frac{L_{min}}{L} = \frac{H(X)}{\log M} \Big/ L$

$$\therefore \boxed{\eta = \frac{H(X)}{L \cdot \log_2 M}}$$

and the redundancy is given by $\boxed{\delta = 1 - \eta}$

Example:

Let $M = [m_1 \; m_2 \; m_3 \; m_4]$

$P(M) = [\frac{1}{2} \; \frac{1}{4} \; \frac{1}{8} \; \frac{1}{8}]$

(a) Without coding & considering a one-one correspondance ie a noiseless channel, The efficiency is

$\eta = \frac{I(XY)}{C}$ → mutual information, ↳ channel capacity.

$I(XY) = H(X) - H(X/Y)$ → for noiseless channel $H(X/Y) = 0$.

∴ $I(XY) = H(X)$ & $C = \text{Max}[I(XY)] = \text{Max}[H(X)] = \log_2 M$

∴ $C = \log M$

$$\therefore \eta = \frac{H(X)}{\log_2 M}$$

$$H(X) = -\sum_{i=1}^{4} p(x_i) \log p(x_i)$$

$$= -\left[\frac{1}{2}\log\frac{1}{2} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{8}\log\frac{1}{8}\right]$$

$$= 7/4$$

$$\log_2 M = \log_2 4 = \log_2 2^2 = 2\log_2 2 = 2$$

∴ Efficiency $\eta = \frac{7/4}{2} = \frac{7}{8} = 0.875$ ∴ $\boxed{\% \eta = 87.5\%}$

By using Coding

Fixed length coding:

A fixed length code is one whose codeword length is fixed. Ex: 8421 code.

b) Let u use binary code for coding

let the code letters be '0' and 1. So, $M = 2$

using fixed length coding.

Probabity	Message	Code	Length of Code
$p(m_1) = 1/2$	m_1	$c_1 = 00$	$n_1 = 2$
$p(m_2) = 1/4$	m_2	$c_2 = 01$	$n_2 = 2$
$p(m_3) = 1/8$	m_3	$c_3 = 10$	$n_3 = 2$
$p(m_4) = 1/8$	m_4	$c_4 = 11$	$n_4 = 2$

$\therefore$ Entropy $H(x) = -\sum_{i=1}^{m} p(x_i) \log p(x_i) = 7/4$ bits/msg.

length $L = \sum_{i=1}^{4} n_i \times p(x_i)$

$= 2 \times \frac{1}{2} + 2 \times \frac{1}{4} + 2 \times \frac{1}{8} + 2 \times \frac{1}{8}$

$L = 2$ letters/msg $\qquad \therefore \log_2 M = \log_2 2 = 1$. bits/letter

$\therefore \frac{H(x)}{\log M} = \frac{7/4}{1} = \frac{7}{4}$ letters/msg.

$\therefore$ Efficiency $\eta = \frac{H(x)}{L \cdot \log M} = \frac{7/4}{2} = \frac{7}{8} = 87.5\%$

$\therefore \boxed{\%\eta = 87.5\%}$

By using fixed length coding procedure is not improving the efficiency.

Variable length Coding:

A variable length code is one whose codeword is not fixed. ie Length of the code word is variable.

* There are two functional requirements for the source encoder to be Efficient.

a) Codeword should be binary form

b) Code is uniquely decodable.

Uniquely decodable: A distinct code uniquely decodable if the original source sequence can be reconstructed perfectly from the encoded binary sequence.

c) Let us consider in this code

message	Code	Length of the Code	Probability
m_1	$c_1 = 0$	$n_1 = 1$	$p(m_1) = 1/2$
m_2	$c_2 = 10$	$n_2 = 2$	$p(m_2) = 1/4$
m_3	$c_3 = 110$	$n_3 = 3$	$p(m_3) = 1/8$
m_4	$c_4 = 111$	$n_4 = 3$	$p(m_4) = 1/8$

$$\therefore \quad H(X) = -\sum_{i=1}^{4} p(x_i) \log p(x_i) = 7/4 \text{ bits/msg.}$$

$$\log_2 M = \log_2 2 = 1 \qquad (\because M = 2 \text{ uses '0' and '1' letters})$$

$$\therefore \frac{H(X)}{\log M} = \frac{7}{4} = 7/4 \text{ letters/msg.}$$

$$\text{Length} \quad L = \sum_{i=1}^{4} n_i \cdot p(x_i)$$

$$= 1\times\frac{1}{2} + 2\times\frac{1}{4} + 3\times\frac{1}{8} + 3\times\frac{1}{8}$$

$$= 1 + \frac{3}{4}$$

$$L = 7/4 \text{ letters/msg.}$$

$$\therefore \text{Efficiency} \quad \eta = \frac{H(X)}{L\cdot\log_2 M} = \frac{7/4}{7/4} = 1$$

$$\therefore \boxed{\%\eta = 100\%}$$

Hence

Variable length coding is better than fixed length coding.

Note: Encode a message with high probability in a short cord word. Only then the average length of the code word 'L' will decrease resulting in an increase in efficiency.

ie

$$\longrightarrow \quad \begin{aligned} 1/2 &\to 0 \\ 1/4 &\to 10 \\ 1/8 &\to 110 \\ 1/8 &\to 111 \end{aligned}$$

Variable length coding is two types.

(a) Shannon Fano Coding

(b) Huffman coding.

Shannon-Fano Coding : ***

{Top-Bottom Approach}

- ✓ Shanon fano coding generates an efficient code in which "Word length increases as symbol probability decreases".
- ✓ Shannon fano coding is variable length code.
- ✓ It is a top-bottom approach.
- ✓ The transmission of an encoded message is reasonably efficient ie '1' and '0' appear independently, with almost equal probabilities.

Algorithm / Procedure :

1. List the source symbols / messages in order of decreasing probabilities.
2. The message set then is partitioned into two equi-probable subsets $[X_1]$ and $[X_2]$.
3. A '0' is assigned to each message contained in one subset ie upper set. & A '1' is assigned to each message contained in the other subset ie Lower set.
4. The same procedure is repeated for the subsets of $\overset{0}{[X_1]}$ and $\overset{1}{[X_2]}$. ie $[X_1]$ will be partitioned into two subsets $\overset{00}{[X_{11}]}$ and $\overset{01}{[X_{12}]}$; $[X_2]$ will be partitioned into two subsets $\overset{10}{[X_{21}]}$ and $\overset{11}{[X_{22}]}$.
5. The procedure is continued, each time partitioning the sets with as nearly equal probabilities as possible untill each subset contains only one message / further partitioning is not possible.
6. Calculate Entropy and length of the code to obtain Coding efficiency.

$$H(x) = -\sum_{i=1}^{m} p(x_i) \cdot \log p(x_i)$$

$$L = \sum_{i=1}^{m} p(x_i) \cdot n_i \qquad \therefore \eta = \frac{H(X)}{L \cdot \log M}$$

where $M = 2$ ('0' & '1')

(1) A source is transmitting messages A, B, C, D, E, F with corresponding probabilities $1/2, 1/4, 1/8, 1/16, 1/32, 1/32$. Find the coding efficiency using Shannon fano encoding method.

Sol Shannon Fano Coding.

	Message (X)	Probability P(X)	Encoded Message I ↓	II ↓	III ↓	IV ↓	V ↓	length of the code (n_i)
1.	A	$1/2$	0					1
2.	B	$1/4$	1	0				2
3.	C	$1/8$	1	1	0			3
4.	D	$1/16$	1	1	1	0		4
5.	E	$1/32$	1	1	1	1	0	5
6.	F	$1/32$	1	1	1	1	1	5

✓ The encoding alphabets 0 and 1 So $M=2$, $\log_2 M = \log_2 2 = 1$.

✓ The entropy $H(X) = -\sum_{i=1}^{6} p(x_i) \log p(x_i)$

$$= -\left[\frac{1}{2}\log 1/2 + \frac{1}{4}\log 1/4 + \frac{1}{8}\log 1/8 + \frac{1}{16}\log 1/16 + 1/32 \log 1/32 + \frac{1}{32}\log 1/32\right]$$

$$= \frac{62}{32} = \frac{31}{16} = 1.9375$$

$\therefore \boxed{H(X) = 1.9375 \text{ bits/msg.}}$

✓ Coding length $L = \sum_{i=1}^{6} p(x_i) \cdot n_i$

$$= \frac{1}{2}\times 1 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{32}\times 5 + \frac{1}{32}\times 5$$

$$= \frac{31}{16} = 1.9375$$

$\therefore \boxed{L = 1.9375 \text{ letters/msg}}$

$$\frac{H(X)}{\log_2 M} = \frac{1.9375}{1} \quad \frac{\text{bits/msg}}{\text{bits/letter}} = \text{letters/msg}$$

Coding efficiency $\% \eta = \frac{H(X)}{L \cdot \log_2 M} = \frac{1.9375}{1.9375 \times 1} \times 100 = 100\%$

$\therefore \boxed{\eta = 100\%}$

Redundancy $(1-\eta) = \delta = 0$

$\therefore \boxed{\delta = 0}$

2) Apply the Shanon-Fano coding procedure for the following messge ensemple:

$$[X] = [\, x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5 \;\; x_6 \;\; x_7 \;\; x_8 \,]$$

$$[P] = [\, 1/4 \;\; 1/8 \;\; 1/16 \;\; 1/16 \;\; 1/16 \;\; 1/4 \;\; 1/16 \;\; 1/8 \,]$$

Sol

Shannon-Fano Coding.

SL No	Messge X	Probability	Encoded Messge I	II	III	IV		Code	Length of the code (n_i)
1	x_1	$\frac{1}{4}=0.25$	0	0			→	00	2
2	x_6	$1/4=0.25$	0	1			→	01	2
3	x_2	$1/8=0.125$	1	0	0		→	100	3
4	x_8	$1/8=0.125$	1	0	1		→	101	3
5	x_3	$1/16=0.0625$	1	1	0	0	→	1100	4
6	x_4	$1/16=0.0625$	1	1	0	1	→	1101	4
7	x_5	$1/16=0.0625$	1	1	1	0	→	1110	4
8	x_7	$1/18=0.0625$	1	1	1	1	→	1111	4

✓ The Encoding alphabets/Letters '0' & '1' So, $M=2$, $\log_2^M = \log_2^2 = 1$. bit/letter

✓ The Entropy $H(X) = -\sum_{i=1}^{8} p(x_i)\log p(x_i)$

$$= -\left[\frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{8}\log\frac{1}{8} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16} + \frac{1}{16}\log\frac{1}{16}\right]$$

$$= \frac{2}{4} + \frac{2}{4} + \frac{3}{8} + \frac{3}{8} + \frac{4}{16} + \frac{4}{16} + \frac{4}{16} + \frac{4}{16}$$

$$= 22/8 = 2.75 \qquad \therefore \boxed{H(X) = 2.75 \text{ bits/msg.}}$$

✓ Code length $L = \sum_{i=1}^{8} p(x_i)\cdot n_i$.

$$= \frac{1}{4}\times 2 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{16}\times 4 + \frac{1}{16}\times 4 + \frac{1}{16}\times 4$$

$$= 22/8 = 2.75 \qquad \therefore \boxed{L = 2.75 \text{ letters/msg}}$$

$$\frac{H(X)}{\log M} = \frac{2.75}{1} = 2.75 \text{ letters/msg}$$

$\therefore$ Coding efficiency $\eta = \dfrac{H(X)}{L\cdot\log_2 M} = \dfrac{2.75}{2.75\times 1}\times 100 = 100\%$ $\boxed{\therefore \eta = 100\%}$

Redundancy $\delta = 1-\eta = \delta = 0 \Rightarrow \boxed{\delta = 0}$

③ Apply the Shannon-Fano Coding procedure for the following message ensemble.

(a) Take $\boxed{M=2}$ (b) Take $\boxed{M=3}$

$$[X] = [x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6 \ x_7]$$

$$[P] = [0.4 \ 0.2 \ 0.12 \ 0.08 \ 0.08 \ 0.08 \ 0.04]$$

Sol (a) $\boxed{M=2}$ The encoding alphabets / letters are '0' and '1' $\log_2 M = \log_2 2 = 1$ bit/letter

Shannon Fano Coding.

SL No	Message (x_i)	Probability $P(x_i)$	Encoded Message I	II	III	IV	Code	Length of the code (n_i)
1	x_1	0.4	0				→ 0	1
2	x_2	0.2	1	0	0		→ 100	3
3	x_3	0.12	1	0	1		→ 101	3
4	x_4	0.08	1	1	0	0	→ 1100	4
5	x_5	0.08	1	1	0	1	→ 1101	4
6	x_6	0.08	1	1	1	0	→ 1110	4
7	x_7	0.04	1	1	1	1	→ 1111	4

Coding length $L = \sum_{i=1}^{7} p(x_i) \cdot n_i$

$= 0.4\times1 + 0.2\times3 + 0.12\times3 + 0.08\times4 + 0.08\times4 + 0.08\times4 + 0.04\times4$

$= 2.48$ letters/msg. $\therefore \boxed{L = 2.48 \text{ letters/msg}}$

Entropy $H(x) = -\sum_{i=1}^{7} p(x_i) \log p(x_i)$

$= -[0.4\log 0.4 + 0.2\log 0.2 + 0.12\log 0.12 + 0.08\log 0.08 + 0.08\log 0.08 + 0.08\log 0.08 + 0.04\log 0.04]$

$= 2.42$ bits/msg $\therefore \boxed{H(x) = 2.42 \text{ bits/msg}}$

$\therefore \dfrac{H(X)}{\log_2 M} = \dfrac{2.42}{1} = 2.42$ letters/msg.

$\therefore$ Coding efficiency $\therefore \eta = \dfrac{H(X)}{L \cdot \log_2 M} = \dfrac{2.42}{2.48} \times 100 = 97.6\%$

$\boxed{\% \eta = 97.6\%}$

Redundancy, $\delta = 1 - \eta = \boxed{2.4\%}$

(b) M=3 ie let the encoding alphabet be 0, 1, 2.

Sl No	Message (X_i)	Probability $P(X_i)$	Encoded Message / Code I	II	III	Code	Length of the code (n_i)
1	x_1	0.4	0			→ 0	1
2	x_2	0.2	1	0		→ 10	2
3	x_3	0.12	1	1		→ 11	2
4	x_4	0.08	2	0		→ 20	2
5	x_5	0.08	2	1		→ 21	2
6	x_6	0.08	2	2	0	→ 220	3
7	x_7	0.04	2	2	1	→ 221	3

Coding Length $L = \sum_{i=1}^{7} p(x_i)\cdot n_i$

$= 0.4\times1 + 0.2\times2 + 0.12\times2 + 0.08\times2 + 0.08\times2 + 0.08\times3 + 0.04\times3$

$= 1.72$ letters/message. $\therefore \boxed{L = 1.72 \text{ letters/msg}}$

Entropy $H_{bits} = -\sum_{i=1}^{7} p(x_i)\log p(x_i)$

$\therefore \boxed{H(x) = 2.42 \text{ bits/msg.}}$

$\therefore$ Coding Efficiency $\therefore \eta = \dfrac{H(x)}{L\cdot\log_2 M} = \dfrac{2.42}{1.72\times\log_2 3} = 88.7\%$

$\therefore \boxed{\%\eta = 88.7\%}$

Redundancy $\delta = 1-\eta = 0.113$

$\therefore \boxed{\delta = 11.3\%}$

Problems:

① Find the coding efficiency for the following data using coding techniques

(i) $[x] = [x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7\ x_8]$

$[P] = [0.1\ 0.25\ 0.15\ 0.05\ 0.15\ 0.1\ 0.05\ 0.15]$

(ii) $[X] = [x_1\ x_2\ x_3\ x_4\ x_5]$

$[P] = [0.4\ 0.15\ 0.15\ 0.15\ 0.15]$

(a) Fixed length coding.

(b) Shannon-Fano coding procedure for M=2 and M=3

(c) Huffman coding procedure for M=2 and M=3.

Solutions for Page: 30

② $[P] = \left[\frac{1}{2}\ \frac{1}{8}\ \frac{1}{8}\ \frac{1}{16}\ \frac{1}{16}\ \frac{1}{16}\ \frac{1}{32}\ \frac{1}{32}\right]$ for M=2, a) Shannon Fano coding b) Huffman coding.

8-messages.

③ $[P] = [0.27\ 0.20\ 0.17\ 0.16\ 0.06\ 0.06\ 0.04\ 0.04]$ a) Shanon Fano coding b) Huffman coding.

8 messages

*** Huffman Coding: (Bottom-top approach)

✓ An another type of source code is "Huffman code" which leads to the lowest possible value of 'L' for a given Messages 'M', resulting in a maximum efficiency & minimum redundancy. Hence it is also known as "minimum redundancy code" (or) optimum code.

Algorithm / Procedure:

1. Arrange the probabilities in the order of descending.
2. Combine the last two probabilities into one probability by adding their probabilities.
3. Rearrange the probabilities in the descending order.
4. Combine the last two probabilities into one probability.
5. Repeat the procedure until no of messages is reduced to two elements.
6. Assign '0' (up) and '1' (down) to this last two messages as their first digit in the code sequence.
7. Go back to assign the numbers '0' and '1' to second digit for the two messages that were combine in previous steps.
8. Keep this way until first column is reached.
9. The first column coding gives the final Optimal Code.
10. Calculate entropy $H(X)$, lengths of the code (L) to obtain Coding efficiency $\boxed{\eta = \frac{H(X)}{L \cdot \log_2 M}}$ and Redundancy $\boxed{\delta = 1-\eta}$

(1) A source is trasmitting six messages with probabilities 0.3, 0.25, 0.15, 0.12, 0.10, 0.08 respectively. Find the efficiency of the code using Huffman encoding technique.

Sol Given 0.3, 0.25, 0.15, 0.12, 0.10, 0.08.

Huffman encoding :

Length n_i	Probability		First reduction	Second reduction	Third reduction	Fourth reduction
2 ←	0.3	00	0.3 00	0.3 00	0.43 1	0.57 0
2 ←	0.25	10	0.25 10	0.27 01	0.3 00	0.43 1
3 ←	0.15	010	0.18 11	0.25 10	0.27 01	
3 ←	0.12	011	0.15 010	0.18 11		
3 ←	0.10	110	0.12 011			
3 ←	0.08	111				

Length of the code $L = \sum_{i=1}^{6} p(x_i) \cdot n_i$

$= 0.3 \times 2 + 0.25 \times 2 + 0.15 \times 3 + 0.12 \times 3 + 0.10 \times 3 + 0.08 \times 3$

$L = 2.45$ letters/msg. $\therefore \boxed{L = 2.45 \text{ Letters/msg.}}$

Entropy $H(x) = -\sum_{i=1}^{6} p(x_i) \log p(x_i)$

$= -[0.3 \log 0.3 + 0.25 \log 0.25 + 0.15 \log 0.15 + 0.12 \log 0.12 + 0.10 \log 0.10 + 0.08 \log 0.08]$

$H(x) = 2.422$ bits/msg.

$\therefore \boxed{H(x) = 2.422 \text{ bits/msg.}}$

In encoding alphabets are '0' & '1' so $M = 2$, $\log_2 M = \log_2 2 = 1$ bits/letter.

$\therefore \frac{H(X)}{\log_2 M} = \frac{2.422}{1}$ = letters/msg.

$\therefore$ Coding Efficiency $\eta = \frac{H(X)}{L \cdot \log_2 M} = \frac{2.422}{2.45} = 0.9885$ $\boxed{\% \eta = 98.85\%}$

Redundancy $\delta = 1 - \eta = 1 - 0.9885 \Rightarrow \boxed{\% \delta = 1.15\%}$

(2) Apply the Huffman coding procedure for the following messages

(a) Take $\boxed{M=2}$ (b) Take $\boxed{M=3}$

$$[X] = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8]$$

$$[P] = \left[\frac{1}{4} \quad \frac{1}{8} \quad \frac{1}{16} \quad \frac{1}{16} \quad \frac{1}{16} \quad \frac{1}{4} \quad \frac{1}{16} \quad \frac{1}{8}\right]$$

Sol

Huffman encoding:

(a) $\boxed{M=2}$, the encoding alphabets / letters are '0' and '1'. $\therefore \log_2^M = \log_2^2 = 1$ bit/letter

Code length (n_i)	Probability	First reduction	Second Reduction	Third reduction	Fourth reduction	Fifth Reduction
2 ← x_1	$\frac{1}{4}=0.25$ (10)	0.25 10	0.25 10	0.25 01	0.25 00	0.5 1 → 0.5 0
2 ← x_6	$\frac{1}{4}=0.25$ (11)	0.25 11	0.25 11	0.25 10	0.25 01	0.25 00 → 0.5 1
3 ← x_2	$\frac{1}{8}=0.125$ (010)	0.125 001	0.125 000	0.25 11	0.25 10	0.25 01
3 ← x_8	$\frac{1}{8}=0.125$ (011)	0.125 010	0.125 001	0.125 000	0.25 11	
4 ← x_3	$\frac{1}{16}=0.0625$ (0000)	0.125 011	0.125 010	0.125 001		
4 ← x_4	$\frac{1}{16}=0.0625$ (0001)	0.0625 0000	0.125 011			
4 ← x_5	$\frac{1}{16}=0.0625$ (0010)	0.0625 0001				
4 ← x_7	$\frac{1}{16}=0.0625$ (0011)					

Length of the Code $L = \sum_{i=1}^{8} P(x_i) \cdot n_i$

$$= \frac{1}{4}\times 2 + \frac{1}{4}\times 2 + \frac{1}{8}\times 3 + \frac{1}{8}\times 3 + \frac{1}{16}\times 4 + \frac{1}{16}\times 4 + \frac{1}{16}\times 4 + \frac{1}{16}\times 4$$

$$= 22/8 = 11/4 = 2.75$$

$\therefore \boxed{L = 2.75 \text{ letters/message}}$

Entropy $H(X) = -\sum_{i=1}^{8} P(x_i) \log p(x_i)$

$$= +\left[\frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16 + \frac{1}{16}\log 16 + \frac{1}{16}\log 16 + \frac{1}{16}\log 16\right].$$

$$= 22/8 = 11/4 = 2.75$$

$\therefore \boxed{H(X) = 2.75 \text{ bits/message}}$

$\therefore \dfrac{H(X)}{\log_2^M} = \dfrac{2.75}{1} = 2.75$ letters/msg.

$\therefore$ Coding efficiency $\eta = \dfrac{H(X)}{L \cdot \log_2^M} = \dfrac{2.75}{2.75 \times 1} = 1$ $\boxed{\% \eta = 100\%}$

Redundancy $\delta = 1 - \eta = 1 - 1 = 0$ $\therefore \boxed{\delta = 0}$

(b) $\underline{M = 3}$

The encoding alphabets/letters are 0, 1, 2. So, $M = 3$.

$$\therefore \log_2 M = \log_2 3 = \frac{\log 3}{\log 2} = 1.585 \text{ letters/msg.}$$

Huffman encoding:

Length (n_i)	Probability $P(x_i)$	First Reduction I	Second Reduction II	Third reduction III
2	$\frac{1}{4} = 0.25$ (00)	0.25 00	0.3125 1	0.6875 0
2	$\frac{1}{4} = 0.25$ (01)	0.25 01	0.25 00	0.3125 1
2	$\frac{1}{8} = 0.125$ (10)	0.1875 02	0.25 01	
2	$\frac{1}{8} = 0.125$ (11)	0.125 10	0.1875 02	
2	$\frac{1}{16} = 0.0625$ (12)	0.125 11		
3	$\frac{1}{16} = 0.0625$ (020)	0.0625 12		
3	$\frac{1}{16} = 0.0625$ (021)			
3	$\frac{1}{16} = 0.0625$ (022)			

Length of the code $L = \sum_{i=1}^{8} P(x_i) \cdot n_i$

$$L = \frac{1}{4}\times 2 + \frac{1}{4}\times 2 + \frac{1}{8}\times 2 + \frac{1}{8}\times 2 + \frac{1}{16}\times 2 + \frac{1}{16}\times 3 + \frac{1}{16}\times 3 + \frac{1}{16}\times 3$$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{3}{16} + \frac{3}{16} + \frac{3}{16}$$

$$L = 2.1875 \text{ letters/msg.}$$

$$\therefore \boxed{L = 2.1875 \text{ letters/msg.}}$$

Entropy $H(x) = \sum_{i=1}^{8} P(x_i) \cdot \log \left[\frac{1}{P(x_i)}\right]$

$$= \frac{1}{4}\log 4 + \frac{1}{4}\log 4 + \frac{1}{8}\log 8 + \frac{1}{8}\log 8 + \frac{1}{16}\log 16 + \frac{1}{16}\log 16 + \frac{1}{16}\log 16 + \frac{1}{16}\log 16]$$

$$= 2.75 \text{ bits/msg}$$

$$\therefore \boxed{H(x) = 2.75 \text{ bits/msg.}}$$

$$\therefore \frac{H(x)}{\log_2 M} = \frac{2.75}{1.585} = 1.735 \text{ letters/msg.}$$

$$\therefore \text{Coding efficiency } \frac{H(x)}{\log_2 M} \times \frac{1}{L} = \frac{1.735}{2.1875} = 0.793 \quad \therefore \boxed{\%\eta = 79.3\%}$$

$$\text{Redundancy } \delta = 1 - \eta = 1 - 0.793 = 0.207 \quad \boxed{\therefore \delta = 20.7\%}$$

Continuous Channel :

* A no. of communication systems use continuous sources and thus use the Channel continuously.
* AM, FM, PM are examples of systems using continuous channel.

In a similar way the different entropies in continuous distributions

If $p(x)$ probability density function

$$\therefore \; h(x) = E[-\log p(x)] = -\int_{-\infty}^{\infty} p(x) \cdot \log p(x)\, dx.$$

$$h(y) = -\int_{-\infty}^{\infty} p(y) \cdot \log p(y)\, dy.$$

$$h(xy) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \cdot \log p(x,y)\, dx\, dy.$$

$$h(x/y) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(x/y)\, dx\, dy$$

$$h(y/x) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y) \log p(y/x)\, dx\, dy.$$

$$h(x/y) = h(xy) - h(y), \qquad i(x,y) = h(x) - h(x/y)$$

$$h(y/x) = h(xy) - h(x) \qquad i(x,y) = h(y) - h(y/x)$$

$$\therefore \; i(x,y) = h(x) + h(y) - h(xy).$$

where $p(x), p(y)$ are marginal densities.

$p(x,y)$ are joint density

$p(x/y), p(y/x)$ are conditional densities.

$$\Rightarrow \int_{-\infty}^{\infty} p(x)\, dx = 1, \quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(x,y)\, dx\, dy = 1.$$

Gaussian PDF : A gaussian random variable is a continuous whose probability density function (PDF) is given by.

$$\boxed{p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}} \qquad -\infty \leq x \leq \infty.$$

where μ and σ^2 are the mean & variance respectively.

*** Shannon Hartley Theorem : (OR) Channel Capacity of Gaussian Channel : (OR) Channel Capacity theorem.

Statement : The channel capacity of a white-band limited Gaussian channel is

$$\boxed{C = B \log\left[1+\frac{S}{N}\right]} \text{ bits/sec.}$$

where B - the channel bandwidth

S - the average signal power

& N - the average noise power.

It is also called Shannon's Third theorem.

Proof :

Let us consider the channel is assumed to be gaussian Channel ie. It generates gaussian noise with zero mean and Variance $\sigma^2 = N$. (AWGN) Additive White Gaussian Noise. whose probability density function (PDF) is

$$\boxed{P(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}} \; ; \; -\infty \le x \le \infty \longrightarrow (1)$$

The entropy of Gaussian noise can be written as

$$h[x] = -\int_{-\infty}^{\infty} p(x) \cdot \log_2 p(x) \, dx \longrightarrow (2)$$

Consider

$$-\log_2 p(x) = \log_2\left(\frac{1}{p(x)}\right)$$

$$= \log_2\left[\frac{1}{\frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-x^2/2\sigma^2}}\right] \quad (\because \text{From eqn } (1))$$

$$= \log_2\left[\sqrt{2\pi\sigma^2} \cdot e^{+x^2/2\sigma^2}\right]$$

$$= \log_2\sqrt{2\pi\sigma^2} + \log_2 e^{x^2/2\sigma^2}$$

$$= \log(2\pi\sigma^2)^{1/2} + \frac{x^2}{2\sigma^2} \cdot \log_2 e$$

$$-\log_2 p(x) = \frac{1}{2}\log(2\pi\sigma^2) + \frac{x^2}{2\sigma^2} \cdot \log_2 e \longrightarrow (3)$$

From eqn (2) & (3)

$$h(x) = \int_{-\infty}^{\infty} p(x) \cdot \left[\frac{1}{2}\log(2\pi\sigma^2) + \frac{x^2}{2\sigma^2} \cdot \log_2 e\right] dx$$

$$h(x) = \frac{1}{2}\log_2(2\pi\sigma^2)\int_{-\infty}^{\infty} p(x)\,dx + \frac{1}{2\sigma^2}\log_2 e \int_{-\infty}^{\infty} x^2\, p(x)\,dx$$

$$= \frac{1}{2}\log_2(2\pi\sigma^2)\,(1) + \frac{\log_2 e}{2\cancel{\sigma^2}}\cdot\cancel{\sigma^2} \qquad \left[\because \int_{-\infty}^{\infty} p(x)\,dx = 1, \quad \int_{-\infty}^{\infty} x^2\, p(x)\,dx = \sigma^2 \text{ variance}\right]$$

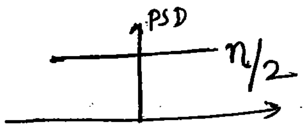
$$= \frac{1}{2}\log_2(2\pi\sigma^2) + \frac{1}{2}\log_2 e$$

$$\boxed{h(x) = \frac{1}{2}\log_2(2\pi e\sigma^2)} \text{ bits/symbol}$$

(or) $h(x) = \log_2\sqrt{2\pi e\sigma^2}$ bits/symbol. $\longrightarrow$ (4)

Consider white gaussian noise with mean square value N

ie $\sigma^2 = N$ = noise power



$\therefore$ Entropy $h(x) = \frac{1}{2}\log_2(2\pi e N)$

$$h(x) = \log_2\sqrt{2\pi e N} \text{ bits/symbol.}$$

$\therefore$ The Rate of information $\boxed{R(x) = 2B\cdot h(x)}$ bits/sec

$2B$ → Sampling theorem; B → Channel bandwidth

$$\therefore R(x) = 2B\log_2\sqrt{2\pi e N} \text{ bits/sec.}$$

$\therefore$ The rate of information received by the receiver can be written as

channel capacity $C \geq \max[\underbrace{R(Y)}_{S+N} - \underbrace{R(N)}_{N}]$ bits/sec. $[\because C \geq R]$

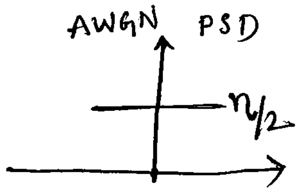
$$\geq \max\left[2B\log_2\sqrt{2\pi e(S+N)} - 2B\log_2\sqrt{2\pi e N}\right]$$

$$\geq \max\left[\cancel{2}B\cdot\frac{1}{\cancel{2}}\log_2(2\pi e(S+N)) - \cancel{2}B\cdot\frac{1}{\cancel{2}}\log_2(2\pi e N)\right]$$

$$\geq \max\left[B\log_2[2\pi e(S+N)] - B\log_2(2\pi e N)\right]$$

$$\geq \max\left\{B\log\left[\frac{\cancel{2\pi e}(S+N)}{\cancel{2\pi e}N}\right]\right\}$$

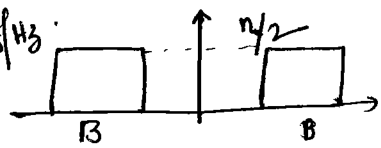
$$\geq \max\left\{B\log\left[\frac{S}{N}+1\right]\right\}$$



$\therefore$ $\boxed{C = B\log[1+S/N]}$ bits/sec Hence proved.

$\therefore$ $\boxed{C = B\log\left[1+\frac{S}{\eta B}\right]}$ bits/sec

where η - noise pdf watts/Hz



S - signal power

B - Channel Bandwidth

$$N = 2\times\frac{\eta}{2}\times B = \eta B$$

Problem:

1 Calculate the bandwidth of the picture (video) signal in a television. The following are the available data.

(i) No. of distinguishable brightness levels = 10.
(ii) The no. of elements per picture frame = 3,00,000
(iii) Picture frames transmitted per second = 30
& (iv) S/N required = 30 dB.

Sol
∴ No. of distinguishable brightness levels = 10

∴ The information per picture element = $\log_2 10$ = 3.32 bits/element

The information per picture frame = $3,00,000 \times 3.32 = 9,96,000$ bit/picture frame.

Since no. of elements per picture frame = 3,00,000.

∴ The no. of picture transmitted per second = 30

∴ The 30 picture frames are transmitted per second ie

the information rate $R = 9,96,000 \times 30$

$R = 29.9 \times 10^6$ bits/sec

∴ To transmit information at this rate, the channel capacity 'c' is

$$C \geq R$$

where

$$C = B \log\left(1 + \frac{S}{N}\right)$$

Given

$$S/N \text{ in dB} = 30\,\text{dB}$$

$$10\log_{10}(S/N) = 30$$

$$S/N = 10^3 = 1000$$

∴ Hence

$$C = B \cdot \log(1 + 1000)$$

$$R \leq C$$

$$29.9 \times 10^6 \leq B \cdot \log(1001)$$

$$\therefore \quad B \geq \frac{29.9 \times 10^6}{\log(1001)} = 3.02 \times 10^6 \text{ Hz}.$$

$$\therefore \boxed{\text{Bandwidth } B \cong 3\text{MHz}.}$$

Thus the minimum bandwidth required to transmit the picture (video) signal of the given television is approximately 3MHz.

Apr/May 2012 Q. Imp.

Find the efficiency of transmission using Shannon-Fano Coding and Huffman coding for the following messages sequence

$[X] = [A \quad B \quad C \quad D \quad E \quad F \quad G \quad H]$ with probabilities.

$[P] = [0.50 \quad 0.15 \quad 0.15 \quad 0.08 \quad 0.08 \quad 0.02 \quad 0.01 \quad 0.01]$.

Sol Given $[X] = [A \quad B \quad C \quad D \quad E \quad F \quad G \quad H]$

$[P] = [0.50 \quad 0.15 \quad 0.15 \quad 0.08 \quad 0.08 \quad 0.02 \quad 0.01 \quad 0.01]$

(a) Shannon fano coding:

Sl No	Message	Probability	I	II	III	IV	V	VI	Code	Length n_i
1	A	0.50	0						0	1
2	B	0.15	1	0	0				100	3
3	C	0.15	1	0	1				101	3
4	D	0.08	1	1	0				110	3
5	E	0.08	1	1	1	0			1110	4
6	F	0.02	1	1	1	1	0		11110	5
7	G	0.01	1	1	1	1	1	0	111110	6
8	H	0.01	1	1	1	1	1	1	111111	6

The alphabet in encoded message/code are 1 & 0 So $M=2$

$\therefore \log_2^M = \log_2^2 = 1$ bit/letter.

The length $L = \sum_{i=1}^{8} p(x_i) \cdot n_i$

$= 0.50\times1 + 0.15\times3 + 0.15\times3 + 0.08\times3 + 0.08\times4 + 0.02\times5 + 0.01\times6 + 0.01\times6$

$L = 2.18$ letters/msg. $\therefore \boxed{L = 2.18 \text{ letters/msg}}$

The Entropy $H(X) = -\sum_{i=1}^{8} p(x_i) \cdot \log p(x_i)$.

$= 2.15$ bits/msg $\therefore \boxed{H(X) = 2.15 \text{ bits/msg}}$

$\therefore$ Efficiency $\eta = \frac{H(X)}{L \cdot \log_2^M} = \frac{2.15}{1\times2.18}\times100 = 98.6\%$ $\therefore \boxed{\eta = 98.6\%}$

$\therefore$ Redundancy $\delta = 1-\eta = 1-0.986 = 0.014$ $\therefore \boxed{\%\delta = 1.4\%}$

Huffman coding:

length n_i	Msg	Probability	Reductions					
1	A	0.5 1	0.5 1	0.5 1	0.5 1	0.5 1	0.5 1	0.5 0
3	B	0.15 000	0.15 000	0.15 000	0.15 000	0.2 01	0.3 00	0.5 1
3	C	0.15 001	0.15 001	0.15 001	0.15 001	0.15 000	0.2 01	
3	D	0.08 011	0.08 011	0.08 011	0.12 010	0.15 001		
4	E	0.08 0100	0.08 0100	0.08 0100	0.08 011			
5	F	0.02 01011	0.02 01010	0.04 0101				
6	G	0.01 010100	0.02 01011					
6	H	0.01 010100						

$\therefore$ The encoded alphabets are '0' & '1', so $M=2$ $\therefore \log_2 M = 1$ bit/letter

The length $L = \sum_{i=1}^{\infty} p(x_i) \cdot n_i$

$$= 0.5\times1 + 0.15\times3 + 0.15\times3 + 0.08\times3 + 0.08\times4 + 0.02\times5 + 0.01\times6 + 0.01\times6.$$

$L = 2.18$ letters/msg. $\quad \therefore \boxed{L = 2.18 \text{ letters/msg.}}$

Entropy $H(X) = -\sum_{i=1}^{8} p(x_i)\log p(x_i)$

$= 2.15$ bits/msg. $\quad \therefore \boxed{H(X) = 2.15 \text{ bits/msg}}$

$$\therefore \frac{H(X)}{\log_2 M} = \frac{2.15}{1} = 2.15 \text{ letters/msg.}$$

$$\therefore \text{Efficiency } \eta = \frac{H(X)}{L\cdot\log_2 M} = \frac{2.15}{2.18}\times100 = 98.6\%$$

$$\therefore \boxed{\eta = 98.6\%}$$

Redundancy $\delta = 1-\eta = 100 - 98.6\% = 1.4\%$ $\quad \boxed{\therefore \delta = 1.4\%}$

Hence: In both coding techniques i.e. Shannon fano coding & Huffman coding the coding efficiency is same, $\eta = 98.6\%$

Sollutions for Problems:

1

$[X] = [x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8]$

$[P] = [0.1 \quad 0.25 \quad 0.15 \quad 0.05 \quad 0.15 \quad 0.1 \quad 0.05 \quad 0.15]$

Fixed length coding

Mss	Probability	Coding	length n_i
x_2	0.25	000	3
x_3	0.15	001	3
x_5	0.15	010	3
x_8	0.15	011	3
x_1	0.1	100	3
x_6	0.1	101	3
x_4	0.05	110	3
x_7	0.05	111	3

$L = \sum_{i=1}^{8} p(x_i) \cdot n_i$

$= 0.25\times 3 + 0.15\times 3 + 0.15\times 3 + 0.15\times 3 + 0.1\times 3 + 0.1\times 3 + 0.05\times 3 + 0.05\times 3$

$L = 3 \quad \therefore \boxed{L = 3 \text{ letters/msg}}$

$H(X) = -\sum_{i=1}^{8} p(x_i) \log p(x_i)$

$\Rightarrow \boxed{H(X) = 2.83 \text{ bits/msg}}$

Efficiency $\eta = \dfrac{H(X)}{L \log_2 M} = 94.33\%$

$M = 2$
i.e. 0, & 1

$\therefore \boxed{\eta = 94.33\%}$

Redundancy $1-\eta \Rightarrow \boxed{\delta = 5.67\%}$

Shannon Fano Coding:

Mss	Probability	Encoded Msg/code	Lengths n_i
x_2	0.25	0 0	2
x_3	0.15	0 1 0	3
x_5	0.15	0 1 1	3
x_8	0.15	1 0 0	3
x_1	0.1	1 0 1	3
x_6	0.1	1 1 0	3
x_4	0.05	1 1 1 0	4
x_7	0.05	1 1 1 1	4

Lengths $L = -\sum_{i=1}^{8} p(x_i) \cdot n_i$

$\therefore \boxed{L = 2.85 \text{ letters/msg}}$

Entropy $H(X) = -\sum_{i=1}^{m} p(x_i) \log p(x_i)$

$\therefore \boxed{H(X) = 2.83 \text{ bits/msg}}$

$\therefore$ Efficiency $\eta = \dfrac{H(X)}{L \cdot \log_2 M}$

$\boxed{\eta = 99.3\%}$

Redundancy $1-\eta \Rightarrow \boxed{\delta = 0.7\%}$

Huffman Coding:

n_i								
2	x_2	0.25 (10)	0.25 10	0.25 10	0.25 01	0.3 00	0.45 1	0.55 0
3	x_3	0.15 (000)	0.15 000	0.2 11	0.25 10	0.25 01	0.3 00	0.45 1
3	x_5	0.15 (001)	0.15 001	0.15 000	0.2 11	0.25 10	0.25 01	
3	x_8	0.15 (010)	0.15 010	0.15 001	0.15 000	0.2 11		
3	x_1	0.1 (110)	0.1 011	0.15 010	0.15 001			
3	x_6	0.1 (111)	0.1 110	0.1 011				
4	x_4	0.05 (0110)	0.1 111					
4	x_7	0.05 (0111)						

$$L = + \sum_{i=1}^{m} p(x_i) \cdot n_i$$

$\therefore \boxed{L = 2.85 \text{ letters/msg}}$

Entropy $\boxed{H = 2.83 \text{ bits/msg}}$

Efficiency $\boxed{\eta = 99.3\%}$

Redundancy $\boxed{\delta = 0.7\%}$

(2) Given : $[P] = \left[\frac{1}{2} \ \frac{1}{8} \ \frac{1}{8} \ \frac{1}{16} \ \frac{1}{16} \ \frac{1}{16} \ \frac{1}{32} \ \frac{1}{32}\right]$

Shannon Fano Coding:

1.	$\frac{1}{2}$	0					(1)
2.	1/8	1	0	0			(3)
3.	1/8	1	0	1			(3)
4.	1/16	1	1	0	0		(4)
5.	1/16	1	1	0	1		(4)
6.	1/16	1	1	1	0		(4)
7.	1/32	1	1	1	1	0	(5)
8.	1/32	1	1	1	1	1	(5)

Length $L = 2.3125$ letters/msg

Entropy $H = 2.3125$ bits/msg

Efficiency $\eta = 100\%$

Redundancy $\delta = 0\% \quad \therefore \delta = 0\%$

Huffman coding

n_i							
1	$1/2 = 0.5$ (1)	0.5 1	0.5 1	0.5 1	0.5 1	0.5 1	0.5 0
3	$1/8 = 0.125$ (010)	0.125 010	0.125 001	0.125 000	0.250 01	0.250 00	0.5 1
3	$1/8 = 0.125$ (011)	0.125 011	0.125 010	0.125 001	0.125 000	0.250 01	
4	$1/16 = 0.0625$ (0001)	0.0625 0000	0.125 011	0.125 010	0.125 001		
4	$1/16 = 0.0625$ (0010)	0.0625 0001	0.0625 0000	0.125 011			
4	$1/16 = 0.0625$ (0011)	0.0625 0010	0.0625 0001				
5	$1/32 = 0.03125$ (00000)	0.0625 0011					
5	$1/32 = 0.03125$ (00001)						

Length = 2.3125 letters/msg

Entropy = 2.3125 bits/msg

Efficiency = 100%

Redundancy = 0%

③ $p = [0.4 \quad 0.15 \quad 0.15 \quad 0.15 \quad 0.15]$

Shannon Fano coding

0.4	0			→	0	(1)
0.15	1	0	0	→	100	(3)
0.15	1	0	1	→	101	(3)
0.15	1	1	0	→	110	(3)
0.15	1	1	1	→	111	(3)

Length (L)

$$L = \sum_{i=1}^{5} p(x_i)\cdot(n_i)$$

$$\boxed{L = 2.2 \text{ letters/msg}}$$

Huffman coding

(1)	0.4 1	0.4 1	0.4 1	0.6 0
(3)	0.15 000	0.3 01	0.3 00	0.4 1
(3)	0.15 001	0.15 000	0.3 01	
(3)	0.15 010	0.15 001		
(3)	0.15 011			

Entropy H

$$H = -\sum_{i=1}^{5} p(x_i)\log p(x_i)$$

$$\boxed{H = 2.17 \text{ bits/msg}}$$

∴ Efficiency $\eta = \frac{H(X)}{L \log_2 M} = \frac{2.17}{2.2}$

∴ $\boxed{\eta = 98.64\%}$

Redundancy $\boxed{\delta = 1.4\%}$

④ Shannon fano coding

0.27	0	0			→	00	(2)
0.2	0	1			→	01	(2)
0.17	1	0	0		→	100	(3)
0.16	1	0	1		→	101	(3)
0.06	1	1	0	0	→	1100	(4)
0.06	1	1	0	1	→	1101	(4)
0.04	1	1	1	0	→	1110	(4)
0.04	1	1	1	1	→	1111	(4)

Length L

$$\boxed{L = 2.73 \text{ letters/msg}}$$

Huffman coding

(2)	0.27 01	0.27 01	0.27 01	0.27 01	0.33 00	0.4 1	0.6 0
(2)	0.2 11	0.2 11	0.2 11	0.2 10	0.27 01	0.33 00	0.4 1
(3)	0.17 000	0.17 000	0.17 000	0.2 11	0.2 10	0.27 01	
(3)	0.16 001	0.16 001	0.16 001	0.17 000	0.2 11		
(4)	0.06 1000	0.08 101	0.12 100	0.16 001			
(4)	0.06 1001	0.06 1000	0.08 101				
(4)	0.04 1010	0.06 1001					
(4)	0.04 1011						

Entropy H

$$\boxed{H = 2.69 \text{ bits/msg}}$$

∴ Efficiency $\boxed{\eta = 98.5\%}$

Redundancy $1-\eta \Rightarrow \boxed{\delta = 1.5\%}$

*

Unit-4

Channel Encoder & Decoder

Syllabus: Introduction, types of error control coding: Automatic Repeat Request, Forward Error Control - Linear block codes, error detection & Correction capabilities of linear block codes, Binary Cyclic codes, error detection & Correction capabilities in BCC, convolutional codes: Time domain & frequency domain approach, state diagram, trellis diagram, convolutional decoder: Viterbi & sequential decoder.

Introduction:

✓ The purpose of source coding (Encoding & decoding) is to convert the discrete message into bits ('0' and '1').

✓ The purpose of channel coding is to detect and correct the errors.

* Channel coding is a combination of channel encoding and channel decoding.

✓ The signal passes through some noisy channel, because of noise errors are generated in the received data. This errors can be detected and corrected using coding techniques.

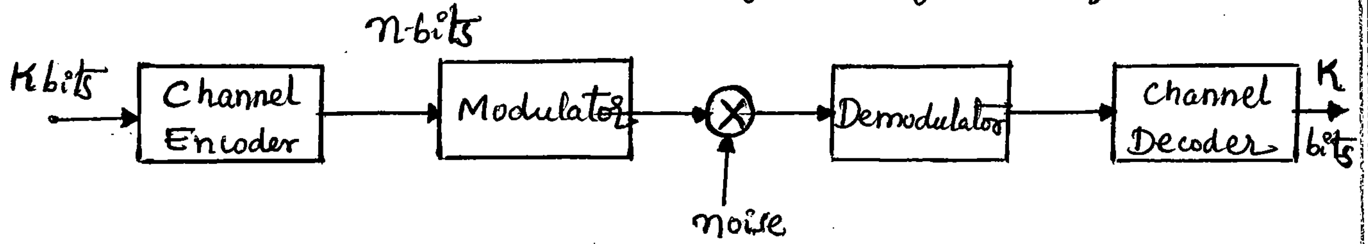


Fig: Channel Coding.

* **Channel Encoder:** The channel encoder add extra bits to the message sequence, The adding of extra bits is called parity bits (or) check bits (or) Redundant bits.

(PTO)

* Channel decoder :

The channel decoder identifies this redundant bits and use that to detect and correct the errors in the message bit.

✓ The channel encoder converts 'K' bits into 'n' bits by adding extra bits, such a code is called (n, K) block codes. Redundant bits is equal to $(n-K)$.

Parameters :

① **Code Word :** The encoded block of n bits is called a code word. It contains message bits 'K' and parity bits $(n-K)$.

② **Block length :** The no. of bits 'n' after coding is called block length of the code.

③ **Code rate :** The rate of the message bits (K) and the encoded bits (n) is called code rate.

$$\boxed{r_c = \frac{K}{n}}$$

④ **Channel data rate :** If the bit rate at the input of the encoder is 'R_s' then the channel data rate at the output of the encoder will be

$$\boxed{R_c = \frac{n}{K} \cdot R_s}$$

$$\rightarrow \boxed{R_c = \frac{R_s}{r_c}}$$

where

$R_s \rightarrow$ Source data bit rate.

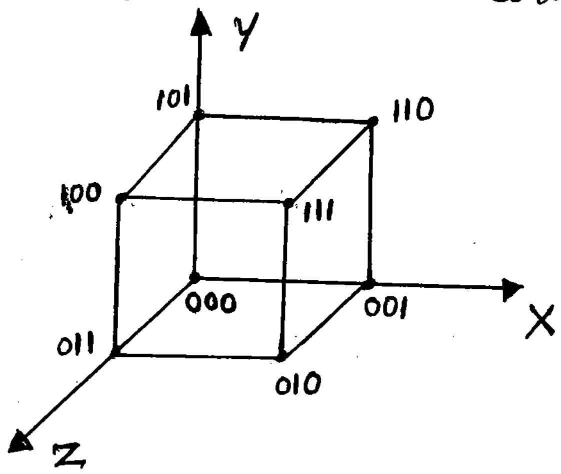
$r_c \rightarrow$ Code rate.

$R_c \rightarrow$ Channel data rate.

⑤ Code vector (or) Code book :

As n-bit code word can be visualize in an n-dimentional space as a vector whose elements & co-ordinates at the bits in the codeword.

ie Collection of all code words called as code book.



$K=3 \Rightarrow 2^3 = 8$ codewords.

$K=4 \Rightarrow 2^4 = 16$ codewords.

⑥ Hamming distance :

The hamming distance between two code words is equal to the number of elements (or) bits in which they differ. ie The difference between two code words.

The hamming distance is denoted as 'd'

Ex: $X = 1\ 0\ 0$
$Y = 1\ 1\ 1$ $\therefore d = 2.$ $\therefore \underline{d(x,y) = 2}.$

⑦ Weight of the Code :

The number of non zero's (1's) elements in the transmitted code word is called weight of the code (or) vector rate.

It is denoted as $W(X)$, where X is Code word.

Ex: $X = 01110101$.

Weight of the Codeword is $\underline{W(X) = 5}$

⑧ Minimum distance (d_{min}):

It is the smallest hamming distance between the Code Words except zero length. the following table list shows some of the requirements of error control capability of the code.

S.No	Error detection & Correction	Distance Requirement
1	Detect upto S-errors per Codeword.	$d_{min} \geq s+1$.
2	Correct upto 't' error per code word.	$d_{min} \geq 2t+1$
3 *	Correct upto 't' errors and detect upto 's' errors for code	$d_{min} \geq t+s+1$ $(s>t)$

For (n,K) block code ~~for~~ the minimum distance is given by

$$\boxed{d_{min} \leq n-K+1}$$

The min-hamming distance should be two (2) for detection of errors and ~~it~~ it should be more than two for correction of errors.

1 → Error can not be detected
2 → Single error detection
3 → Single error Correction

Ex: If $d_{min} = 3$ then

No of detections are '2'

ie $d_{min} \geq s+1$

$3 \geq s+1$

$2 \geq s$

$\therefore \boxed{s \leq 2}$

No of Corrections are '1'

ie $d_{min} \geq 2t+1$

$3 \geq 2t+1$

$2 \geq 2t$

$1 \geq t$

$\therefore \boxed{t \leq 1}$

4 → Single error Correction + double error detection
5 → Double error ~~detection~~ correction
6 → Double error Correction + triple error detection.

* $\frac{n-1}{2}$ error canbe corrected if n-odd.

$\frac{n-2}{2}$ errors canbe corrected & n/2 errors ~~can be~~ detected if n even.

Error Control Coding :

* The probability of error for a particular signalling scheme is a function of signal to noise ratio at the receiver input and the information rate.

* In practical systems, the maximum signal power and the bandwidth of the channel are restricted to some fixed values. Also, the noise power spectral density '$\frac{\eta}{2}$' is fixed for a particular operating environment.

✓ With all these constraints, it is often not possible to arrive at a signalling scheme which will yield an acceptable probability of error for a given application.

★ The only practical alternative for reducing the probability of error is the use of 'error-control coding'.

Types of Error Control Coding:

There are two methods of error control coding.

① . ARQ [Automatic Retransmission Query (or) Automatic Repeat Request) (or) (Error detection with retransmission)

② FEC. [Forward Error Correction]

① ARQ [Automatic Repeat Request]:

✓ In this method the receiver checks the input sequence if there is any error occurs then the receiver discard that part of the sequence and request the transmitter for retransmission. So, it is called automatic repeat request.

ie. When an error is detected, the receiver send a request to the transmitter for repeat transmission, after which the transmitter retransmits.

- ✓ Consider a source which emits 'm' messages bits and receiver receives those m-messages and finds one bit is corrupted. Hence the receiver send a request to source for retransmission.
- ✓ The source again sends the m-messages in this method, the time taken for retransmission is more.
- ✓ The ARQ method needs duplex arrangement as apart from the conventional transmitter to receiver signal, the request signal is to travel from receiver to transmitter.

It cannot be used in real-time systems.

Advantage:

① ARQ can be used to transmit messages more accurately.

Disadvantage:

① The probability of error is low its process is also slow.

② Even though P_e is low the retransmission of entire message takes place which leads a delay / time waste in transmission.

③ Only detection possible and can't be correct the errors.

② FEC [Forward Error Correction]:

- ✓ In this method the errors are detected and corrected by proper coding techniques.
- ✓ In FEC method, an automatic mechanism of error correction in the form of error correction code is employed & hence retransmission of data is not necessary.
- ✓ The FEC method needs simplex arrangement (Simple hardware as compared to duplex arrangement) as the signal has to travel only from the transmitter to the receiver.
- ✓ The FEC method is more popular & used in real time systems than ARQ method.

- In FEC, the channel encoder systematically adds digits/bits to the transmitted message bits. Although these additional bits convey no new information, they make it possible for the channel decoder to detect & correct the errors in the information bearing digits/bits.
- The overall probability of error is reduced due to error detection and error correction.

Error Control Coding Techniques (or) Structured Sequences:

Structured Sequences are divided into 3 sub categories.

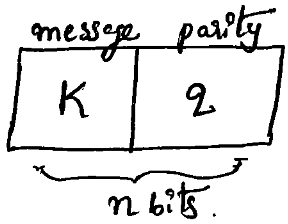
(a) Block Codes
(b) Cyclic Codes
(c) Convolutional Codes
(d). Turbo codes

Block Codes:

Block code is also known as arithmetic code (or) group code. There are two types of block codes.

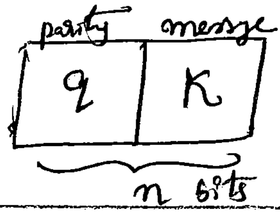
(i) Systematic block code.
(ii) Non-Systematic block code.

Systematic block code: In resultant 'n' bits, the first bits are message bits and remaining 'n-K' bits are check bits called as systematic block code.



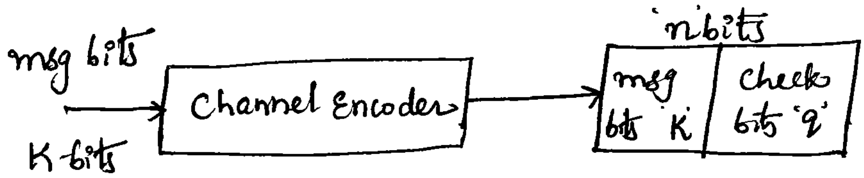
$q = n - K$

Non Systematic block code: In resultant 'n' bits the first bits are check bits / parity bits and remaining bits are message bits, called as non systematic block code.



$q = n - K$

- Each block of 'K' messages or message bits is encoded into a block of n-bits. $(n > K)$.
- The check bits are derived from the message bits and that check bits are added to the message bits.



$n = K + q$

Parity Check Codes:

The simplest possible block code is parity check codes. In this method a no. of check bits added to the message bit is one.

There are two types of parity check codes.

(a) Even parity check code.

(b) Odd parity check code.

Even parity checkcode:

When the check bit may be '1' or '0' such that the total no. of 1's in the encoded of 'n' bits code word is even then it is called Even parity check code.

Ex:

	msg (K)	Checkbit (q)		Resultant bit (n)
X =	010011	1	→	0100111
Y =	101110	0	→	1011100

Odd parity check code:

When the check bit may be '1' & '0' such that ~~the~~ total no. of 1's in the encoded of 'n' bits code word is odd then it is called odd parity check code.

Ex:

	msg (K)		Checkbit (q)		Resultant bit (n)
X =	010011	→	0	→	0100110
Y =	101110	→	1	→	1011101

Linear Block Codes :

A code is said to be linear block code if the sum (XOR) of any two code words in the codebook produced another codeword which is in the code book → set of code words.

- ✓ The basic notation for describing linear block codes in a matrix form.
- ✓ The error control capabilities of a linear block codes are determined by its minimum distance.

* The encoding operation in a linear block encoding scheme consists of two basic steps.

(i) The information sequence is segmented / divided into message blocks, each block consists of 'K' bits.

(ii) The channel encoder transform each block of K bits into large block of n-bits.

Consider the message block as a row vector $[M_1, M_2, \ldots M_K]$ each message bit can be '1' or '0'.

- ✓ There are 2^K different message blocks (or) 2^K different code words.
- ✓ Each message block is transformed to a code word 'C' of length n bits. $C = [C_1, C_2, C_3 \ldots C_n]$
- ✓ The set of 2^K code words is called Codebook & Code Vector.
- ✓ The check bits are generated from the message bits.
- ✓ The output of the channel encoder is 'n' bits which can be generated by using matrix form.

Code words ← $\boxed{C = MG}$ → message bits, → Generator matrix

$$[C]_{1\times n} = [M]_{1\times K} \cdot [G]_{K\times n}.$$

$$[C_1 \; C_2 \; C_3 \ldots C_n] = [M_1 \; M_2 \; M_3 \ldots M_K] \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & \ldots & P_{11} & P_{12} & \ldots & P_{1q} \\ 0 & 1 & 0 & \ldots & P_{21} & P_{22} & \ldots & P_{2q} \\ 0 & 0 & 1 & \ldots & P_{31} & P_{32} & \ldots & P_{3q} \\ \vdots & \vdots & \vdots & & & & & \\ 0 & 0 & 0 & \ldots 1 & P_{K1} & P_{K2} & \ldots & P_{Kq} \end{array}\right]$$

where $[G]_{K\times n} = [I_{K\times K} : P_{K\times q}]_{K\times n}.$

where $[G]_{K\times n}$ is called generator matrix

$[I]_{K\times K}$ is called Identity matrix.

$[P]_{K\times q}$ is called parity matrix. ← Impertent role.

An importance steps in the design of (n,K) block codes is the selection of a 'p' matrix so that the code generated by 'G' has some properties such as

* Easy of Implementation.
* Ability of correct errors.
* High rate efficiency.

(1) The generator matrix for a (6,3) block code is given below. Find all the Code words.

$$G = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\underbrace{\quad}_{I_{3\times3}} \quad \underbrace{\quad}_{P_{3\times3}}$$

Sol

The given block code $(n,K) = (6,3)$

$n = 6$
$K = 3$

∴ Length of the message block $K = 3$.
Length of the encoded codewords $n = 6$.
Check bits $q = n - k = 6 - 3 = 3$

Systematic system block code.

∴ There are $2^K = 2^3 = 8$ different message block, each message block consists of 3 bits.

M_1	M_2	M_3
0	0	0
0	0	1
0	1	0
0	1	1
1	0	0
1	0	1
1	1	0
1	1	1

8 Each 3 bits

$$C = M.G$$

Consider $M_1 = [0\,0\,0]$

$$\therefore \underset{1\times6}{C} = \underset{1\times3}{[0\,0\,0]} \begin{bmatrix} 1&0&0&0&1&1 \\ 0&1&0&1&0&1 \\ 0&0&1&1&1&0 \end{bmatrix}_{3\times6}$$

$$\boxed{C_1 = [0\;0\;0\;\;0\;0\;0]}$$

ie Each bit in the row matrix 'M' should be multiplied by each bits in the column of 'G' matrix.

Then all the values are added by XOR operation

XOR operation :

0	0	0
0	1	1
1	0	1
1	1	0

$$M_2 = [0\,0\,1], \quad C_2 = [0\,0\,1]\begin{bmatrix} 1\,0\,0 & 0\,1\,1 \\ 0\,1\,0 & 1\,0\,1 \\ 0\,0\,1 & 1\,1\,0 \end{bmatrix}$$

$$= [0+0+0 \quad 0+0+0 \quad 0+0+1 \quad 0+0+1 \quad 0+0+1 \quad 0+0+0]$$

$$\boxed{C_2 = [0\ 0\ 1\ 1\ 1\ 0]}$$

$$M_3 = [0\,1\,0], \quad C_3 = [0\,1\,0]\begin{bmatrix} 1\,0\,0 & 0\,1\,1 \\ 0\,1\,0 & 1\,0\,1 \\ 0\,0\,1 & 1\,1\,0 \end{bmatrix}$$

$$= [0+0+0 \quad 0+1+0 \quad 0+0+0 \quad 0+1+0 \quad 0+0+0 \quad 0+1+0]$$

$$\boxed{C_3 = [0\ 1\ 0\ 1\ 0\ 1]}$$

$$M_4 = [0\,1\,1], \quad C_4 = [0\,1\,1]\begin{bmatrix} 1\,0\,0 & 0\,1\,1 \\ 0\,1\,0 & 1\,0\,1 \\ 0\,0\,1 & 1\,1\,0 \end{bmatrix}$$

$$= [0+0+0 \quad 0+1+0 \quad 0+0+1 \quad 0+1+1 \quad 0+0+1 \quad 0+1+0]$$

$$\boxed{C_4 = [0\ 1\ 1\ 0\ 1\ 1]}$$

$$M_5 = [1\,0\,0] \quad C_5 = [1\,0\,0]\begin{bmatrix} 1\,0\,0 & 0\,1\,1 \\ 0\,1\,0 & 1\,0\,1 \\ 0\,0\,1 & 1\,1\,0 \end{bmatrix}$$

$$= [1+0+0 \quad 0+0+0 \quad 0+0+0 \quad 0+0+0 \quad 1+0+0 \quad 1+0+0]$$

$$\boxed{C_5 = [1\ 0\ 0\ 0\ 1\ 1]}$$

$$M_6 = [1\,0\,1], \quad C_6 = [1\,0\,1]\begin{bmatrix} 1\,0\,0 & 0\,1\,1 \\ 0\,1\,0 & 1\,0\,1 \\ 0\,0\,1 & 1\,1\,0 \end{bmatrix}$$

$$= [1+0+0 \quad 0+0+0 \quad 0+0+1 \quad 0+0+1 \quad 1+0+1 \quad 1+0+0]$$

$$\boxed{C_6 = [1\ 0\ 1\ 1\ 0\ 1]}$$

$$M_7 = [1\,1\,0], \quad C_7 = [1\,1\,0]\begin{bmatrix} 1\,0\,0 & 0\,1\,1 \\ 0\,1\,0 & 1\,0\,1 \\ 0\,0\,1 & 1\,1\,0 \end{bmatrix}$$

$$= [1+0+0 \quad 0+1+0 \quad 0+0+0 \quad 0+1+0 \quad 1+0+0 \quad 1+1+0]$$

$$\boxed{C_7 = [1\ 1\ 0\ 1\ 1\ 0]}$$

$$M_8 = [1\,1\,1] \quad C_8 = [1\,1\,1]\begin{bmatrix} 1\,0\,0 & 0\,1\,1 \\ 0\,1\,0 & 1\,0\,1 \\ 0\,0\,1 & 1\,1\,0 \end{bmatrix}$$

$$= [1+0+0 \quad 0+1+0 \quad 0+0+1 \quad 0+1+1 \quad 1+0+1 \quad 1+1+0]$$

$$\boxed{C_8 = [1\ 1\ 1\ 0\ 0\ 0]}$$

SL NO	Message	Code word
1	0 0 0	0 0 0 0 0 0
2	0 0 1	0 0 1 1 1 0
3	0 1 0	0 1 0 1 0 1
4	0 1 1	0 1 1 0 1 1
5	1 0 0	1 0 0 0 1 1
6	1 0 1	1 0 1 1 0 1
7	1 1 0	1 1 0 1 1 0
8	1 1 1	1 1 1 0 0 0

Alternative Method :

Consider $C = MP$.

$$C = [M_1 \ M_2 \ M_3] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$C = [M_2 \oplus M_3 \quad M_1 \oplus M_3 \quad M_1 \oplus M_2]$$

$\therefore$ $c_1 \ c_2 \ c_3 \rightarrow M_1 \ M_2 M_3$

$c_4 = M_2 \oplus M_3 \longrightarrow (1)$

$c_5 = M_1 \oplus M_3 \longrightarrow (2)$

$c_6 = M_1 \oplus M_2 \longrightarrow (3)$

$\therefore$ The codeword can be written as $\begin{bmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 \\ M_1 & M_2 & M_3 & ? & ? & ? \end{bmatrix}$

S No	Messages $M_1 M_2 M_3$	Check bits c_4 (1) c_5 (2) c_6 (3)	Complete Code Word $M_1 M_2 M_3 \ c_4 c_5 c_6$
1	0 0 0	0 0 0	0 0 0 0 0 0
2	0 0 1	1 1 0	0 0 1 1 1 0
3	0 1 0	1 0 1	0 1 0 1 0 1
4	0 1 1	0 1 1	0 1 1 0 1 1
5	1 0 0	0 1 1	1 0 0 0 1 1
6	1 0 1	1 0 1	1 0 1 1 0 1
7	1 1 0	1 1 0	1 1 0 1 1 0
8	1 1 1	0 0 0	1 1 1 0 0 0

(2) The generator Matrix for a (6,3) block code is given below. Find all the code words.

$$G = \begin{bmatrix} 1\,0\,0 & 1\,1\,0 \\ 0\,1\,0 & 0\,1\,1 \\ 0\,0\,1 & 1\,1\,1 \end{bmatrix}$$

Sol.

The given block code $(n, K) = (6,3)$.

$$G = \begin{bmatrix} 1\,0\,0 & 1\,1\,0 \\ 0\,1\,0 & 0\,1\,1 \\ 0\,0\,1 & 1\,1\,1 \end{bmatrix}$$
$$\quad\; I_{3\times3} \quad P_{3\times3}.$$

$n = 6$
$K = 3$

Length of the message block $K = 3$

length of the encoded code words $n = 6$.

Check bits $q = n - K = 6 - 3 = 3 = 3$.

∴ There are $2^K = 2^3 = 8$ different message block, each message block consists of 3 bits.

$$C = M \cdot [G]$$

and

$$C = [M][P]$$
$$= [M_1\ M_2\ M_3] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$C = [M_1 \oplus M_3 \quad M_1 \oplus M_2 \oplus M_3 \quad M_2 \oplus M_3]$$

∴ $c_1\ c_2\ c_3 = M_1\ M_2\ M_3$

$c_4 = M_1 \oplus M_3$

$c_5 = M_1 \oplus M_2 + M_3$

$c_6 = M_2 \oplus M_3$.

SL NO	Messages $M_1\ M_2\ M_3$	Check bits $C_4\ C_5\ C_6$	Comple code word $M_1\ M_2\ M_3\ C_4\ C_5\ C_6$
1	0 0 0	0 0 0	0 0 0 0 0 0
2	0 0 1	1 1 1	0 0 1 1 1 1
3	0 1 0	0 1 1	0 1 0 0 1 1
4	0 1 1	1 0 0	0 1 1 1 0 0
5	1 0 0	1 1 0	1 0 0 1 1 0
6	1 0 1	0 0 1	1 0 1 0 0 1
7	1 1 0	1 0 1	1 1 0 1 0 1
8	1 1 1	0 1 0	1 1 1 0 1 0

Parity Check Matrix (H):

✓ The parity check matrix can be used to verify wether the code word is generated by the matrix 'G' or not.

✓ The code word in the (n,K) block code generated by 'G' if and only if $CH^T = 0$

$$\Rightarrow \quad MGH^T = 0$$

where M - message matrix

G - Generator matrix, $G = \left[I_{K\times K} : P_{K\times q} \right]_{K\times n}$

H - parity check matrix $\boxed{H = \left[P^T_{q\times K} \,\vdots\, I_{q\times q} \right]_{q\times n}}$

$$H^T = \begin{bmatrix} (P^T)^T \\ \hline I \end{bmatrix} = \begin{bmatrix} P_{K\times q} \\ \hline I_{q\times q} \end{bmatrix}_{n\times q}.$$

$$P = \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1q} \\ P_{21} & P_{22} & \cdots & P_{2q} \\ \vdots & & & \\ P_{K1} & P_{K2} & \cdots & P_{Kq} \end{bmatrix}, \quad P^T = \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{K1} \\ P_{12} & P_{22} & \cdots & P_{K2} \\ \vdots & \vdots & & \vdots \\ P_{1q} & P_{2q} & \cdots & P_{Kq} \end{bmatrix}$$

Parity check matrix

$$H = \left[\begin{array}{cccc|ccccc} P_{11} & P_{21} & \cdots & P_{K1} & 1 & 0 & 0 & \cdots & 0 \\ P & P_{22} & \cdots & P_{K2} & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots \\ P_{1q} & P_{2q} & \cdots & P_{Kq} & 0 & 0 & 0 & \cdots & 1 \end{array}\right]_{q\times n}$$

Error detection & Correction Capabilities of Linear block code:

Hamming codes are (n,K) linear block codes.

These codes satisfies the following conditions.

1. The number of check bits $q \geq 3$
2. Block length $n = 2^q - 1$
3. The number of message bits $K = n - q$

(4) The minimum distance $d_{min} \geq 2$.

If $d_{min} = 2$ then single error can be detected.

If $d_{min} = 3$ then two errors can be detected and one error can be corrected.

(5) The code rate is given as

$$r_c = \frac{K}{n} = \frac{n-q}{n} = 1 - q/n$$

$$\boxed{r_c = 1 - \frac{q}{2^q - 1}}$$

(6) Error detection & Correction capabilities of hamming codes are.

* $d_{min} \geq s+1$ → Detect upto 's' error per codeword
* $d_{min} \geq 2t+1$ → Correct upto 't' errors per codeword.
* $d_{min} \geq s+t+1$ → Detect upto 's' errors and correct upto 't' errors for codeword.
  $(s>t)$.

(3) The parity check matrix of a particular (7,4) linear block code is given by

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(a) Find the generator matrix G.

(b) List all the code words.

(c) Minimum distance between the codewords.

(d) How many errors can be detected and how many error corrected.

(e) Draw the encoded diagram for given hamming code.

Sol

Linear block code $(n, K) = (7, 4)$ i.e. $n = 7$, $K = 4$.

The length of the message block $k = 4$

The length of the encoded code word $n = 7$.

Check bits $q = n - K = 7 - 4 = 3$

$\therefore$ There are $2^K = 2^4 = 16$ different message block, each message block consists of 4 bits.

Given parity check matrix $H = \left[\begin{array}{cccc:ccc} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]_{3\times 7}$

$$H = \left[P^T_{q\times K} \vdots I_{q\times q} \right]_{q\times n}$$

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{K\times q \text{ i.e. } 4\times 3} \qquad \left[\because P = (P^T)^T\right]$$

(a) <u>Generator matrix</u>: $G = \left[I_{K\times K} \; P_{K\times q} \right]_{K\times n}$

$$\therefore G = \left[\begin{array}{cccc:ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]_{4\times 7}.$$

(b) <u>All code words</u>:

$$C = MG$$

$$G = [I \vdots P]$$

$$C = [M][P]$$

$$\therefore \underset{1\times 3}{C} = \underset{1\times 4}{[M_1 \; M_2 \; M_3 \; M_4]} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}_{4\times 3}$$

$$C = [M_1 \oplus M_2 \oplus M_3 \quad M_1 \oplus M_2 \oplus M_4 \quad M_1 \oplus M_3 \oplus M_4]$$

$$\therefore [c_1, c_2, c_3, c_4] = [M_1 \; M_2 \; M_3 \; M_4]$$

$$c_5 = M_1 \oplus M_2 \oplus M_3 \longrightarrow (1)$$

$$c_6 = M_1 \oplus M_2 \oplus M_4 \longrightarrow (2)$$

$$c_7 = M_1 \oplus M_3 \oplus M_4 \longrightarrow (3)$$

$C_5 = M_1 \oplus M_2 \oplus M_3$, $C_6 = M_1 \oplus M_3 \oplus M_4$, $C_6 = M_1 \oplus M_3 \oplus M_4$

SL No	Message $M_1\ M_2\ M_3\ M_4$	Check bits $C_5\ C_6\ C_7$	Complete Codeword $M_1\ M_2\ M_3\ M_4\ C_5\ C_6\ C_7$	Vector wt. Weight
1	0 0 0 0	0 0 0	0 0 0 0 0 0 0	0
2	0 0 0 1	0 1 1	0 0 0 1 0 1 1	3
3	0 0 1 0	1 0 1	0 0 1 0 1 0 1	3
4	0 0 1 1	1 1 0	0 0 1 1 1 1 0	4
5	0 1 0 0	1 1 0	0 1 0 0 1 1 0	3
6	0 1 0 1	1 0 1	0 1 0 1 1 0 1	4
7	0 1 1 0	0 1 1	0 1 1 0 0 1 1	4
8	0 1 1 1	0 0 0	0 1 1 1 0 0 0	3
9	1 0 0 0	1 1 1	1 0 0 0 1 1 1	4
10	1 0 0 1	1 0 0	1 0 0 1 1 0 0	3
11	1 0 1 0	0 1 0	1 0 1 0 0 1 0	3
12	1 0 1 1	0 0 1	1 0 1 1 0 0 1	4
13	1 1 0 0	0 0 1	1 1 0 0 0 0 1	3
14	1 1 0 1	0 1 0	1 1 0 1 0 1 0	4
15	1 1 1 0	1 0 0	1 1 1 0 1 0 0	4
16	1 1 1 1	1 1 1	1 1 1 1 1 1 1	7

$d_{min} = 3$ Except 0.

(c)

The minimum distance of the (n,k) block code is $\leq n-K+1$

$\therefore$ The minimum hamming distance $d_{min} = 3$

$\therefore$ $d_{min} \leq n-K+1$

$3 \leq (7-4+1)$

$3 \leq 4$ (True)

$3 < 4$

$\therefore$ $\boxed{d_{min} = 3}$

(d) No. of detections of error 's' $\Rightarrow$ $d_{min} \geq s+1$

$3 \geq s+1$

$2 \geq s$ $\therefore$ $s \leq 2$

$\therefore$ $\boxed{\text{No. of error detections } s = 2}$

No. of error Corrections 't' $\Rightarrow$ $d_{min} \geq 2t+1$

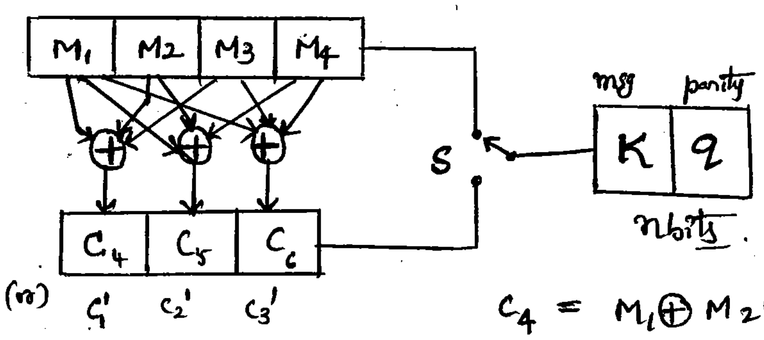
$3 \geq 2t+1$

$1 \geq t$ $\therefore$ $t \leq 1$

$\therefore$ $\boxed{\text{No. of error corrections } t = 1}$

(e) Encoder diagram :

The encoder of (7,4) hamming code is implemented for the given generator matrix.



S- switch.

$$C_4 = M_1 \oplus M_2 \oplus M_3$$
$$C_5 = M_1 \oplus M_2 \oplus M_4$$
$$C_6 = M_1 \oplus M_3 \oplus M_4.$$

- The check bits c_1', c_2', c_3' are obtained from the message bits by modulo-2 & XOR operation.
- The switch 'S' is connected to the message register first then transmit all the message bits.
- Then switch 'S' is then connected to the check bit register then all check bits are transmitted.

Thus it form a 7 bit block codes

Syndrome Decoding :

* The generator matrix (G) is used in the encoding operation & the parity check matrix (H) is used in the decoding operation.
- The encoder has to store the G-matrix & perform binary arithmetic operations to generate the check bits.
- The complexity of the encoder increases as the no-of check bits are increases.

* Let transmitted code vector be 'X', the received code vector be 'Y'
If $X = Y$ then there are no transmission errors.
If $X \neq Y$ then there are errors created during transmission.

✓ The decoder detects & Corrects the errors in 'y' using the stored information about the codeword.

✓ A direct way of performing error detection would be comparing 'y' with every codeword in the code book.

Let 'c' be a codeword that was transmitted over a noisy channel & it is received as 'y'.

The syndrome is represented by 's' & it can be written as

Detection. $\boxed{S = YH^T}$ where $[H]_{q \times n}$ - parity check matrix.

$$[S]_{1\times q} = [Y]_{1\times n} [H^T]_{n\times q}$$

Let us consider error vector 'E' & this vector represent the position of transmission errors in 'y'.

So $\quad Y = X \oplus E \quad \Rightarrow \quad \boxed{X = Y \oplus E}$ correction.

Substitute 'y' in 'S'

ie

$$S = [X \oplus E] \cdot H^T$$

$$= XH^T \oplus EH^T \qquad (\because XH^T = 0).$$

$$S = EH^T$$

$$\therefore \boxed{S = EH^T}$$ For Single bit error.

This relation shows that ~~the~~ syndrome depends upon error pattern only. It does not depends upon a particular message.

Syndrome vector 'S' of size '$1 \times q$'.

q bits of syndrome can only represents '$2^q - 1$' syndrome vectors.

Each syndrome vector corresponds to a particular error pattern.

Hence in syndrome decoding we can detect and correct only one error.

$\therefore$ The no. of errors can be corrected $\Leftrightarrow$ $1 - r_c \geq \left[\frac{1}{n}\log_2 \sum_{i=1}^{t} {}^{n}c_i\right]$

This eqn also called Hamming bound.

④ The parity check matrix of a particular (6,3) block code is given as

$$H = \begin{bmatrix} 1\,0\,1 & 1\,0\,0 \\ 1\,1\,0 & 0\,1\,0 \\ 0\,1\,1 & 0\,0\,1 \end{bmatrix}$$

(a) Determine the generator matrix G

(b) Find all the code words.

(c). Find the min distance between the codeword.

(d) How many errors can be detected & Corrected.

(e) Draw the encoder diagram.

(f) Calculate the syndrome vector for single bit error.

(g) Suppose that the received codeword is '110110', decode this code word.

(h) Draw the syndrome diagram.

Sol.: Given block code $(n,k) = (6,3)$ $\quad n = 6$, $k = 3$.

Length of the message blocks $k = 3$

Length of the Encoded code word $n = 6$.

Check bits $q = n - k = 6 - 3 = 3$.

$\therefore$ $k = 3$. There are $2^k = 2^3 = 8$ different message block, each message block consists of 3 bits.

$$H = \begin{bmatrix} 1\,0\,1 & 1\,0\,0 \\ 1\,1\,0 & 0\,1\,0 \\ 0\,1\,1 & 0\,0\,1 \end{bmatrix}_{3\times 6}$$

$$H = [\underset{3\times3}{P^T} \vdots \underset{3\times3}{I}]_{3\times6}$$

where $P^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}_{3\times3}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

(a) <u>Generator matrix</u> $G = [I_{k\times k} \vdots P_{k\times q}]_{k\times n}$

$$G = \begin{bmatrix} 1\,0\,0 & 1\,1\,0 \\ 0\,1\,0 & 0\,1\,1 \\ 0\,0\,1 & 1\,0\,1 \end{bmatrix}_{3\times6}$$

(b) <u>Code words</u>. $C = MG$

(or) $C = [M][P] = [M_1\ M_2\ M_3] \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$\therefore\ c_1\ c_2\ c_3 \rightarrow M_1\ M_2\ M_3$

$$C_4 = M_1 \oplus M_3$$
$$C_5 = M_1 \oplus M_2$$
$$C_6 = M_2 \oplus M_3.$$

Sl NO	Messages $M_1\ M_2\ M_3$	Check bits $C_4\ C_5\ C_6$	Complete Codeword $M_1\ M_2\ M_3\ C_4\ C_5\ C_6$	Vector rate $W(X)$
1	0 0 0	0 0 0	0 0 0 0 0 0	0
2	0 0 1	1 0 1	0 0 1 1 0 1	3
3	0 1 0	0 1 1	0 1 0 0 1 1	3
4	0 1 1	1 1 0	0 1 1 1 1 0	4
5	1 0 0	1 1 0	1 0 0 1 1 0	3
6	1 0 1	0 1 1	1 0 1 0 1 1	4
7	1 1 0	1 0 1	1 1 0 1 0 1	4
8	1 1 1	0 0 0	1 1 1 0 0 0	3

(c) The minimum distance $d_{min} = 3$ except 0. $\therefore\ \boxed{d_{min} = 3.}$

(d) No. of detection of errors.

$$d_{min} \geq s+1$$
$$3 \geq s+1$$
$$s \leq 2$$

$\therefore$ No. of error detections $\boxed{s=2}$

No. of error corrections.

$$d_{min} \geq 2t+1$$
$$3 \geq 2t+1$$
$$t \leq 1$$

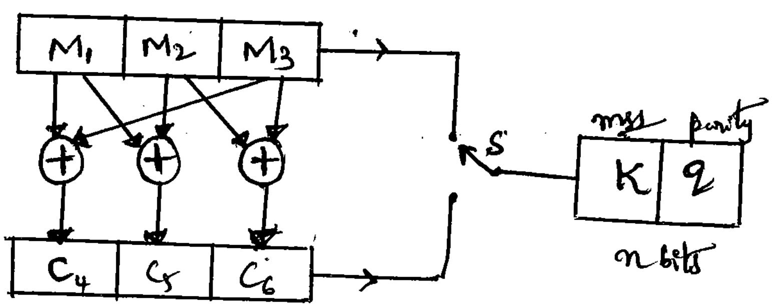
$\therefore\ \boxed{t=1}$. No. of error corrections.

For (n,K) block code the minimum distance is given by

$$d_{min} \leq n-K+1$$
$$3 \leq 6-3+1 \qquad n=6,\ K=3$$
$$3 \leq 4 \text{ (True)}.$$

(e) The <u>encoder diagram</u> of (6,3) block code is implemented for the given generator matrix.

$C_4 = M_1 \oplus M_3$
$C_5 = M_1 \oplus M_2$
$C_6 = M_2 \oplus M_3.$



(f) Syndrome vector $S = EH^T$ for single bit error

$$H^T = \begin{bmatrix} (P^T)^T \\ I \end{bmatrix} = \begin{bmatrix} P \\ I \end{bmatrix}_{n\times q}$$

$$\therefore H^T = \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 1&0&1 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}_{6\times 3} \quad \text{where} \quad P = \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 1&0&1 \end{bmatrix}; \quad I = \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}$$

Error Vector E as

$$E = \begin{bmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&0&0 \\ 0&0&0&0&1&0 \\ 0&0&0&0&0&1 \end{bmatrix}_{6\times 6}.$$

Syndrome vector $S = EH^T$

for $E = 100000$, $S = [1\,0\,0\,0\,0\,0] \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 1&0&1 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} = [1\ 1\ 0]$

lly.

Sl No	Bit in error	Bit in Error vector	Syndrome Vector S
1	1st	1 0 0 0 0 0	1 1 0
2	2nd	0 1 0 0 0 0	0 1 1
3	3rd	0 0 1 0 0 0	1 0 1
4	4th	0 0 0 1 0 0	1 0 0
5	5th	0 0 0 0 1 0	0 1 0
6	6th	0 0 0 0 0 1	0 0 1

(g) For the detection of the error by using syndrome decoding is given as $S = YH^T$

Given received bits (codeword) $Y = [1\ 1\ 0\ 1\ 1\ 0]$

$$S_{1\times 3} = [1\,1\,0\,1\,1\,0]_{1\times 6} \begin{bmatrix} 1&1&0 \\ 0&1&1 \\ 1&0&1 \\ 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}_{6\times 3}$$

$$S = [1\oplus 0\oplus 0\oplus 1\oplus 0\oplus 0 \quad 1\oplus 1\oplus 0\oplus 0\oplus 1\oplus 0 \quad 0\oplus 1\oplus 0\oplus 0\oplus 0\oplus 0]$$

$$S = [0\ 1\ 1]$$ $\therefore$ 2nd bit Error

Error correction is possible by using

$$X = Y \oplus E$$

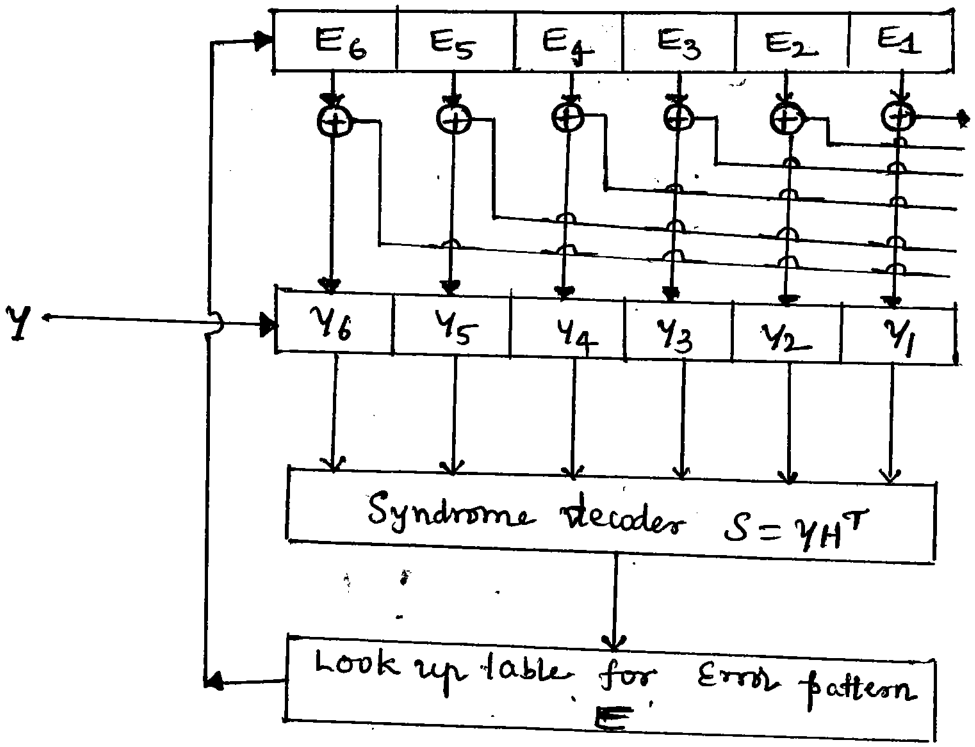
$$= [1\,1\,0\,1\,1\,0] \oplus [0\,1\,0\,0\,0\,0]$$

$$= [1\,0\,0\,1\,1\,0]$$

$$\therefore X = [\underbrace{1\,0\,0}_{\text{message bits}}\ \underbrace{1\,1\,0}_{\text{parity bits}}]$$

(h) <u>Syndrome diagram</u> :

Error bits → n bits
Rxed bits → n bits.



$X = Y \oplus E$

For the detection of error
$S = YH^T$

For the correction of error
$X = Y \oplus E$.

(5) The parity check matrix of a particular (7,4) linear block codes is given by :

$$H = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(a) Calculate syndrome vector for single bit error

(b) Suppose that the Rxed code word is 0110110, decode this code word.

(c) Draw the syndrome diagram.

Sol Given linear block code $(n,k) = (7,4)$

Length of the message block $k = 4$

Length of the Encoded code word $n = 7$.

check bits $q = n - k = 7 - 4 = 3$

$k = 4, \; n = 7, \; q = 3.$

$$H = \begin{bmatrix} 1110 & 100 \\ 1101 & 010 \\ 1011 & 001 \end{bmatrix} \qquad H = [\, P^T \,\vdots\, I \,]$$

$$P^T = \begin{bmatrix} 1110 \\ 1101 \\ 1011 \end{bmatrix}_{3\times4} \Rightarrow P = \begin{bmatrix} 111 \\ 110 \\ 101 \\ 011 \end{bmatrix}_{4\times3}$$

$$H^T = \begin{bmatrix} 111 \\ 110 \\ 101 \\ 011 \\ 100 \\ 010 \\ 001 \end{bmatrix}_{7\times3} \Rightarrow H^T = \left[\frac{(P^T)^T}{I}\right] = \left[\frac{P}{I}\right]$$

$$I = \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}$$

Error matrix

(a)

$$E = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}_{7\times7}$$

Syndrome Vector $S = E \cdot H^T$

For $E = [0\,1\,0\,0\,0\,0\,0] \begin{bmatrix} 111 \\ 110 \\ 101 \\ 011 \\ 100 \\ 010 \\ 001 \end{bmatrix} = \begin{matrix} 0\oplus1\oplus0\oplus0\oplus0\oplus0\oplus0 \\ 0\oplus1\oplus0\oplus0\oplus0\oplus0\oplus0 \\ 0\oplus0\oplus0\oplus0\oplus0\oplus0\oplus0 \end{matrix}$

$= [1\;1\;0]$

$\underline{\text{lly}}$

S.No	Bit in error	Bits in Error Vector	Syndrome vector S
1	1^{st}	1 0 0 0 0 0 0	1 1 1
2	2^{nd}	0 1 0 0 0 0 0	1 1 0
3	3^{rd}	0 0 1 0 0 0 0	1 0 1
4	4^{th}	0 0 0 1 0 0 0	0 1 1
5	5^{th}	0 0 0 0 1 0 0	1 0 0
6	6^{th}	0 0 0 0 0 1 0	0 1 0
7	7^{th}	0 0 0 0 0 0 1	0 0 1

$\cong H^T$

(b) For the detection of the error by using syndrome decoding

$$S = YH^T$$

Given $Y = [0\,1\,1\,0\,1\,1\,0]$ (MSB … LSB)

$$\therefore \underset{1\times 3}{S} = \underset{1\times 7}{\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{7\times 3}$$

$$S = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}_{1\times 3}$$

$\therefore$ $\boxed{S = [1\,0\,1] \text{ ie } 3^{rd} \text{ bit Error}}$

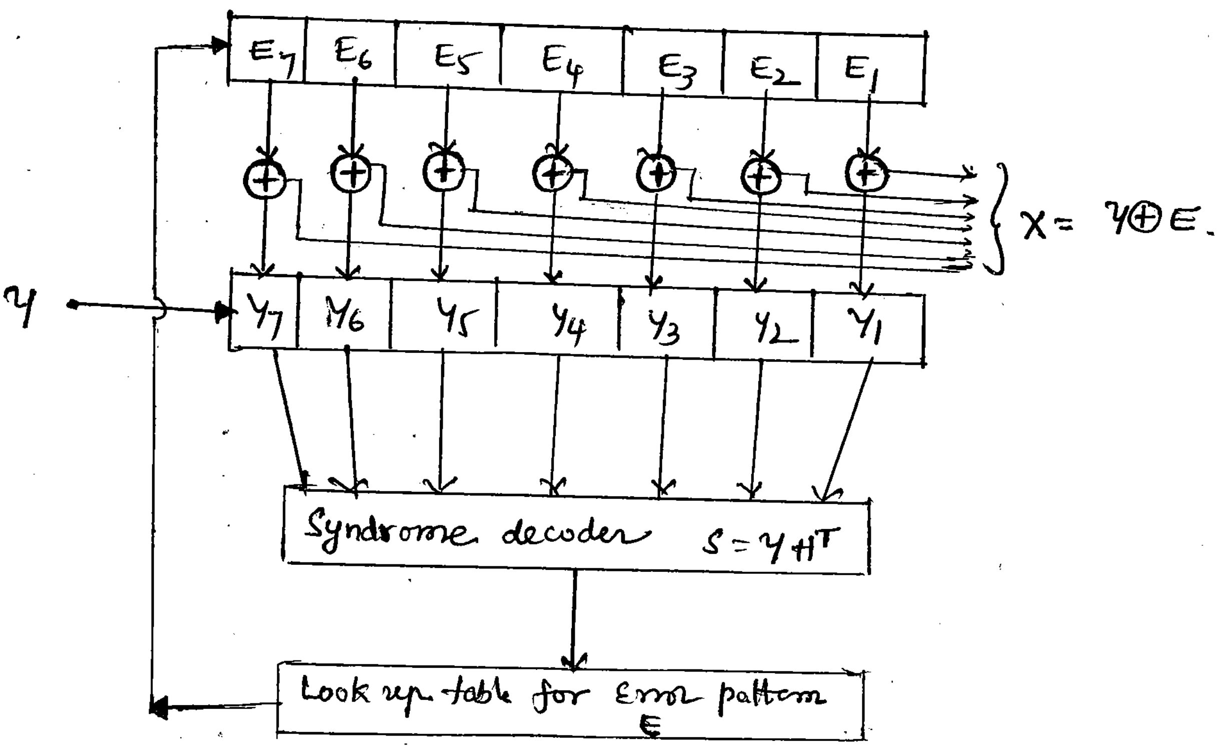
Error Correction can be possible by using

$$X = Y \oplus E$$
$$= [0\,1\,1\,0\,1\,1\,0] \oplus [0\,0\,1\,0\,0\,0\,0]$$
$$X = 0\,1\,0\,0\,1\,1\,0$$

$$\therefore \boxed{X = [\underbrace{0\,1\,0\,0}_{\text{message bits}}\ \underbrace{1\,1\,0}_{\text{parity bits}}]}$$

(c) <u>Syndrome diagram:</u>

Error bits → n bits
Received bits → n bits.



2011

* (6) Consider (7,4) code whose generator matrix is

$$G = \begin{bmatrix} 1111000 \\ 1010100 \\ 0110010 \\ 1100001 \end{bmatrix}$$

(a) Find all the code words of the code.

(b) Find parity check matrix H.

(c) Calculate the syndrome for the received vector '1101101'.

Sol. Given Linear block code $(n, k) = (7,4)$

$n = 7$
$k = 4$
$q = n - k = 3$.

Given generator Matrix

$$G = \begin{bmatrix} 111 & \vdots & 1000 \\ 101 & \vdots & 0100 \\ 011 & \vdots & 0010 \\ 110 & \vdots & 0001 \end{bmatrix}_{4\times 7}$$

Code word $C = MG$

where $G = [I \vdots P]_{k\times n}$ (or) $G = [\underset{k\times q}{P} \vdots \underset{k\times k}{I}]_{k\times n}$

$k = 4$

$2^k = 16$ different message blocks.

$$I = \begin{bmatrix} 1000 \\ 0100 \\ 0010 \\ 0001 \end{bmatrix}_{4\times 4}, \quad P = \begin{bmatrix} 111 \\ 101 \\ 011 \\ 110 \end{bmatrix}_{4\times 3}$$

(a) $C = M[P]$

$$= [M_1\ M_2\ M_3\ M_4] \begin{bmatrix} 111 \\ 101 \\ 011 \\ 110 \end{bmatrix}$$

$\Rightarrow C_1\ C_2\ C_3\ C_4 \rightarrow M_1 M_2 M_3 M_4 \Rightarrow$

$C_4 = M_1 + M_2 + M_4$
$C_5 = M_1 + M_3 + M_4$
$C_6 = M_1 + M_2 + M_3$.

Sl No	Message bits $M_1\ M_2\ M_3\ M_4$	Check bits $C_5\ C_6\ C_7$	Complete code word $M_1\ M_2\ M_3\ M_4\ C_5\ C_6\ C_7$	Weight
1	0 0 0 0	0 0 0	0 0 0 0 0 0 0	0
2	0 0 0 1	1 1 0	0 0 0 1 1 1 0	3
3	0 0 1 0	0 1 1	0 0 1 0 0 1 1	3
4	0 0 1 1	1 0 1	0 0 1 1 1 0 1	4
5	0 1 0 0	1 0 1	0 1 0 0 1 0 1	3
6	0 1 0 1	0 1 1	0 1 0 1 0 1 1	4
7	0 1 1 0	1 1 0	0 1 1 0 1 1 0	4
8	0 1 1 1	0 0 0	0 1 1 1 0 0 0	3
9	1 0 0 0	1 1 1	1 0 0 0 1 1 1	4
10	1 0 0 1	0 0 1	1 0 0 1 0 0 1	3
11	1 0 1 0	1 0 0	1 0 1 0 1 0 0	3
12	1 0 1 1	0 1 0	1 0 1 1 0 1 0	4
13	1 1 0 0	0 1 0	1 1 0 0 0 1 0	3
14	1 1 0 1	1 0 0	1 1 0 1 1 0 0	4
15	1 1 1 0	0 0 1	1 1 1 0 0 0 1	4
16	1 1 1 1	1 1 1	1 1 1 1 1 1 1	7

The minimum distance $\boxed{d_{min} = 3}$ Except '0'.

$$d_{min} \geq s+1$$
$$3 \geq s+1$$
$$s \leq 2$$

$\therefore$ No of detection of errors $\boxed{s=2}$

$$d_{min} \geq 2t+1$$
$$3 \geq 2t+1$$
$$t \leq 2$$

$\therefore$ No of ~~Coee~~ correction of errors $\boxed{t=1}$

(b) Parity check matrix 'H'.

$$H = [\underbrace{P^T}_{q\times k} : \underbrace{I}_{q\times q}]_{q\times n}$$

$$P = \begin{bmatrix} 111 \\ 101 \\ 011 \\ 110 \end{bmatrix} \Rightarrow P^T = \begin{bmatrix} 1101 \\ 1011 \\ 1110 \end{bmatrix}_{3\times 4}, \quad I = \begin{bmatrix} 100 \\ 010 \\ 001 \end{bmatrix}_{3\times 3}$$

$\therefore$

$$H = \begin{bmatrix} 1101 & 100 \\ 1011 & 010 \\ 1110 & 001 \end{bmatrix} \Rightarrow H^T = \begin{bmatrix} P \\ I \end{bmatrix} = \begin{bmatrix} 111 \\ 101 \\ 011 \\ 110 \\ 100 \\ 010 \\ 001 \end{bmatrix}_{7\times 3}$$

Syndrome vector

$$S = EH^T$$

where Error Vector

$$E = \begin{bmatrix} 1&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0 \\ 0&0&1&0&0&0&0 \\ 0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&1 \end{bmatrix}$$

$$S = [E]_{1\times 7}[H^T]_{7\times 3}$$
$$S = 1\times 3.$$

SNo	Bit in Error	Bit in Error Vector E	Syndrome Vector $S = \in H^T$
1	1st	1 0 0 0 0 0 0	1 1 1
2	2nd	0 1 0 0 0 0 0	1 0 1
3	3rd	0 0 1 0 0 0 0	0 1 1
4	4th	0 0 0 1 0 0 0	1 1 0
5	5th	0 0 0 0 1 0 0	1 0 0
6	6th	0 0 0 0 0 1 0	0 1 0
7	7th	0 0 0 0 0 0 1	0 0 1

For the detection of error by using syndrome decoding is

$$S = yH^T.$$

Given . Received vector $y = [1101101]$

$$S = [1\ 1\ 0\ 1\ 1\ 0\ 1]\begin{bmatrix} 111 \\ 101 \\ 011 \\ 110 \\ 100 \\ 010 \\ 001 \end{bmatrix} \Rightarrow S = [0\ 0\ 1]$$

$\therefore$ $\boxed{S = [0\ 0\ 1]}$ ie 7th bit error

Error Correction is possible by using. $X = Y \oplus E$

$$X = [1101101] \oplus [0000001]$$

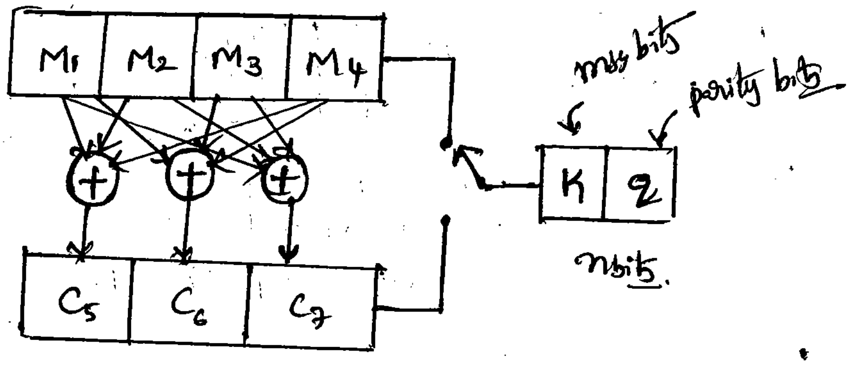
$$X = [\underbrace{1101}_{\text{msg bits}}\ \underbrace{100}_{\text{parity bits}}]$$

Encoder diagram:

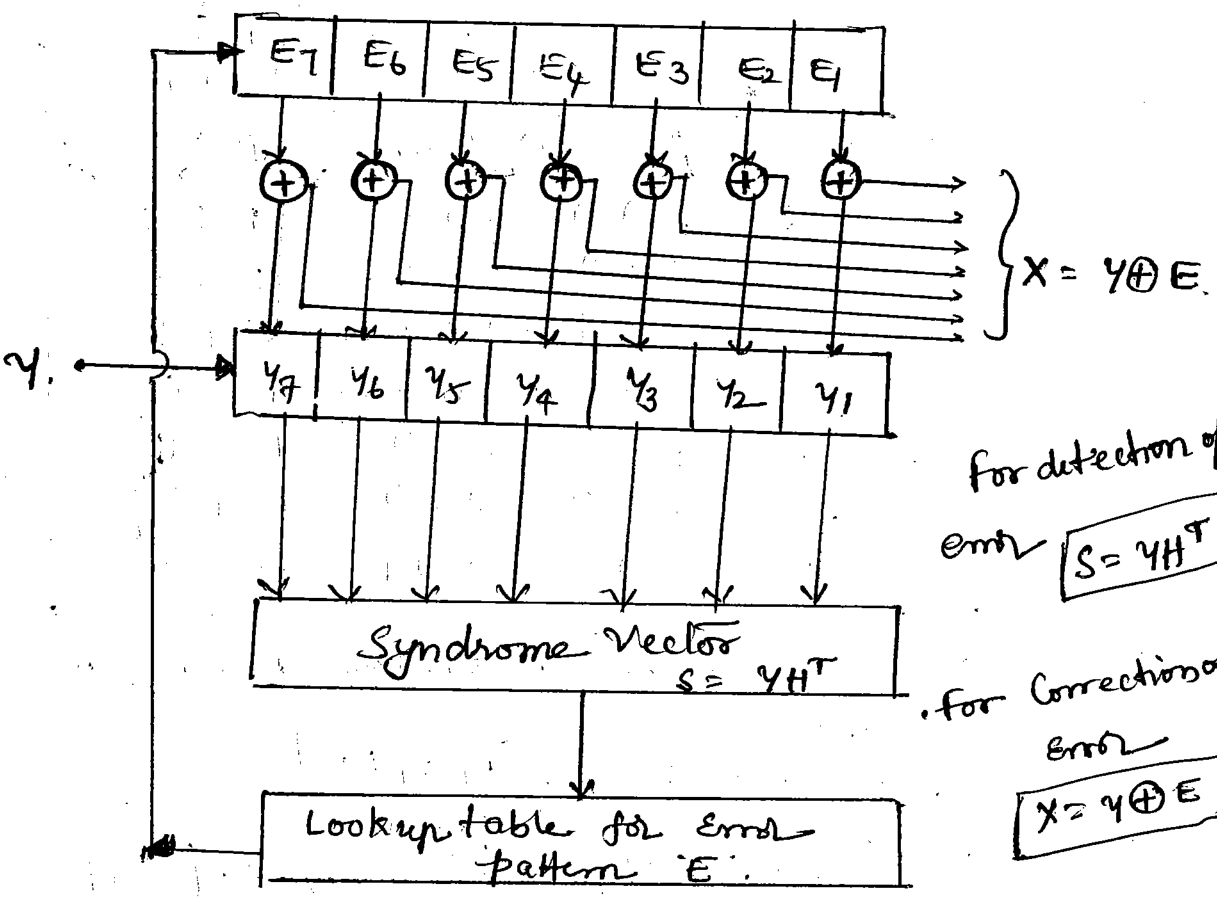
$C_5 = M_1 \oplus M_2 \oplus M_4$

$C_6 = M_1 \oplus M_3 \oplus M_4$

$C_7 = M_1 \oplus M_2 \oplus M_3$



Syndrome diagram: Error bits - n bits.
Received bits - n bits.



For detection of error $\boxed{S = YH^T}$

For Correction of Error $\boxed{X = Y \oplus E}$

Cyclic Codes: (Polynomial form)

In linear block codes the design of hardware requirements are complex, so the alternative method to generate code vector is "cyclic codes".

* A linear code is called cyclic code, if every cyclic shift of the code vector is produced another code vector.

Ex: Consider an n-bit code vector

$$[X] = [\underset{MSB}{x_{n-1}}\ x_{n-2} \cdots\cdots x_1\ \underset{LSB}{x_0}].$$

Shift the above code vector cyclicly produced another code vector.

$$[X'] = [\overset{MSB}{x_{n-2}}\ x_{n-3} \cdots\cdots x_1\ x_0\ \overset{LSB}{x_{n-1}}]$$

Actually x_{n-1} is MSB position but after cyclic shift it becomes LSB position.

A binary code is said to be a cyclic codes if exhibits two fundamental properties.

1. Linear property: The sum of two code words is also a code word.
2. Cyclic property: Any cyclic shift of a codeword is also a codeword.

Ex:

1 0 1 1
1 1 0 1
1 1 1 0
0 1 1 1
---
1 0 1 1

obtained by a cyclic shift of n-typle (tuple) $n = 4$

Cyclic codes are important for two reasons.

1. Encoding & syndrome calculations can be easily implemented by using simple shift registers with feedback connection.
2. The matematical structure of these codes is such that it is possible to design codes having useful error-correcting properties.

Analysis : The code vector is represented in mathematical form of polynomials.

$$X(p) = \left[\overset{MSB}{x_{n-1}p^{n-1}} + x_{n-2}p^{n-2} + \cdots\cdots + x_1 p^1 + \overset{LSB}{x_0} \right] \rightarrow (1)$$

where p- arbitrary value

power of p represents the position of the codeword bit.

Multiply eqn (1) with 'p' we get

$$p.X(p) = x_{n-1}p^{n} + x_{n-2}p^{n-1} + \cdots\cdots + x_1 p^2 + x_0.p^1 \longrightarrow (2)$$

Cyclic shift of eqn (2) produced another code vector.

$$X'(p) = x_{n-2}p^{n-1} + x_{n-3}p^{n-2} + \cdots + x_1 p^2 + x_0 p^1 + x_{n-1}p^0 \longrightarrow (3)$$

Add eqn (2) & (3) by using modulo 2 operation ie XOR operation.

$$pX(p) \oplus X'(p) = x_{n-1}p^n \oplus x_{n-1} + x_{n-2}p^{n-1} \oplus x_{n-2}p^{n-1} + \cdots\cdots + x_1 p^2 \oplus x_1 p^2 + x_0 p^1 \oplus x_0 p^1$$

$$= x_{n-1}p^n \oplus x_{n-1} + [x_{n-2} \oplus x_{n-2}]\, p^{n-1} + [x_{n-3} \oplus x_{n-3}]\, p^{n-2} + \cdots + p^2(x_1 \oplus x_1) + p^1(x_0 \oplus x_0)$$

$$= x_{n-1}[p^n \oplus 1] + 0 + 0 + \cdots 0 + 0$$

$$pX(p) \oplus X'(p) = x_{n-1}[p^n \oplus 1]$$

$$\Rightarrow X'(p) = p.[X(p)] \oplus x_{n-1}[p^n \oplus 1]$$

$$\boxed{X'(p) = pX(p) \oplus x_{n-1}[p^n \oplus 1]}$$

The polynomial '$p^n \oplus 1$' and its factors plays major role in cyclic codes.

Cyclic Codes
- non systematic Cyclic codes
- Systematic Cyclic codes

(a) Non Systematic :

Cyclic Code $\boxed{X(P) = M(P) \cdot G(P)}$ $\longrightarrow$ (1)

where $M(P)$ - Message signal polynomial of degree 'K'.

$$\boxed{M(P) = M_{K-1}P^{K-1} + M_{K-2}P^{K-2} + \cdots M_2P^2 + M_1P + M_0}$$

$G(P)$ - Generating polynomial of degree 'q'.

$$\boxed{G(P) = P^q + g_{q-1}P^{q-1} + \cdots\cdots g_2P^2 + g_1P^1 + 1}$$

$C(P)$ - Cyclic code polynomial

$$\boxed{C(P) = C_{q-1}P^{q-1} + C_{q-2}P^{q-2} + \cdots C_2P^2 + C_1P^1 + C_0}$$

(b) Systematic Cyclic Code :

cyclic code $\boxed{X(P) = P^q \cdot M(P) + C(P)}$ $\longrightarrow$ (2)

Equating eqn (1) & (2)

$$M(P) \cdot G(P) = P^q M(P) + C(P)$$

Divide above eqn by $G(P)$.

$$\therefore M(P) = \frac{P^q M(P)}{G(P)} + \frac{C(P)}{G(P)}$$

$\therefore$ Systamatic

$$\boxed{\frac{P^q M(P)}{G(P)} = M(P) \oplus \frac{C(P)}{G(P)}}$$

XOR

C(P)

This equation has the form of $\frac{N}{D} = Q + \frac{R}{D}$

where N - numarator
D - Denominator
Q - Quiscent
R - Remainder

Thus the check bit polynomial is obtained as reminder 'R' is dividing $\frac{P^q \cdot M(P)}{G(P)}$. ie $\boxed{\frac{N}{D} = Q + \frac{R}{D}}$

Encoder (or) Encoder diagram for Cyclic Codes :

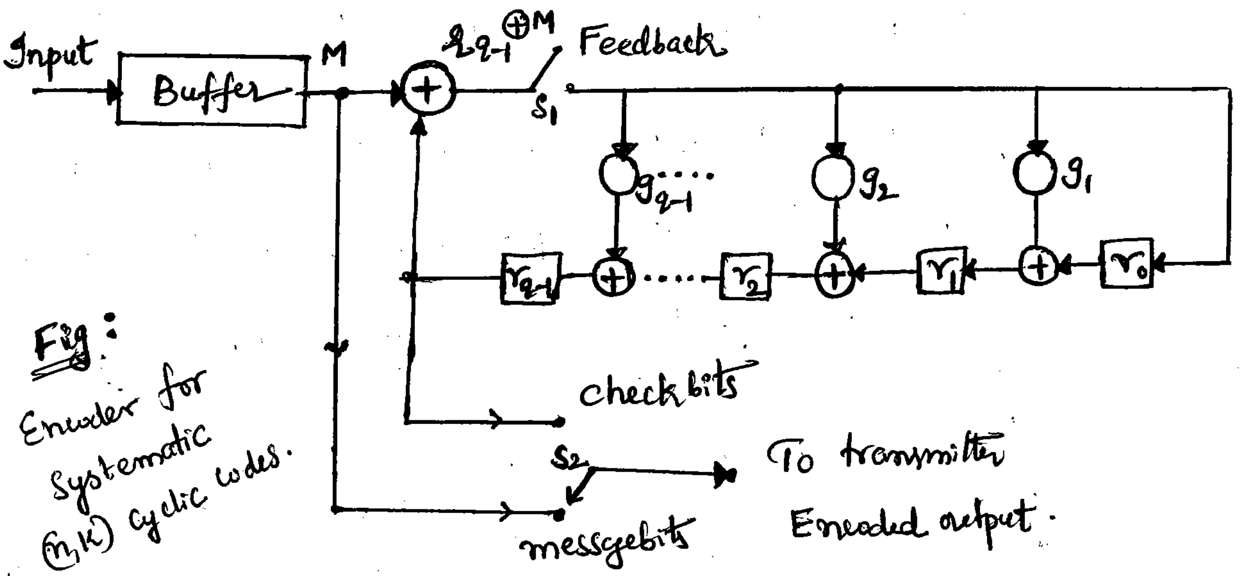


Fig : Encoder for Systematic (n,k) cyclic codes.

where

$r_0, r_1, r_2, \ldots r_{q-1}$ represents flip flops (or) registers they are connected in sequential order to make a shift register.

$g_1, g_2, \ldots g_{q-1}$ represents path establishment.

- ✓ Encoding starts with the feedback switch 'S_1' closed, the output switch is connected to the message bits 'S_2'.
- ✓ All the shift registers are initiated to all zero state.
- ✓ K message bits are shifted to the transmitter as well as shifted into the registers.
- ✓ After the shift of K-message bits, the register contains 'q' check bits.
- ✓ The feedback switch S_1 is now opened and output switch is connected to check bit position.

Thus we can obtain all codewords as cyclic code.

Syndrome decoding for Cyclic Codes :

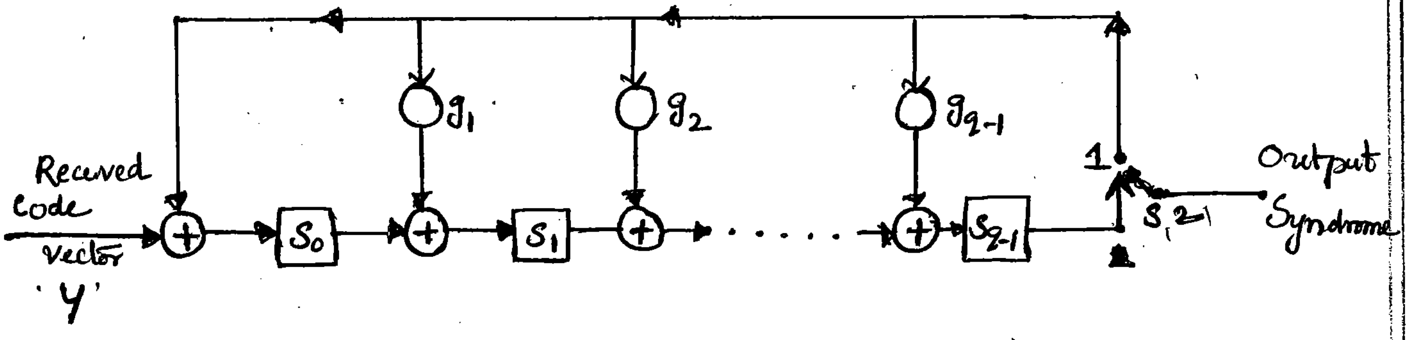


Fig: Syndrome decoding for cyclic codes.

- ✓ Initially all the shift registers are zeros and switch is connected to position ①.
- ✓ The received vector 'y' is shifted bit by bit into the shift registers.
- ✓ After all the bits of 'y' are shifted, 'q' flip flops of shift registers contains 'q' bit syndrome vector.
- ✓ The switch is then closed to position ② & clock (For reset) are applied to the shift registers.
- ✓ The output ~~of~~ is a syndrome vector.

$$S = (S_{q-1}, S_{q-2}, \cdots S_1 \, S_0)$$

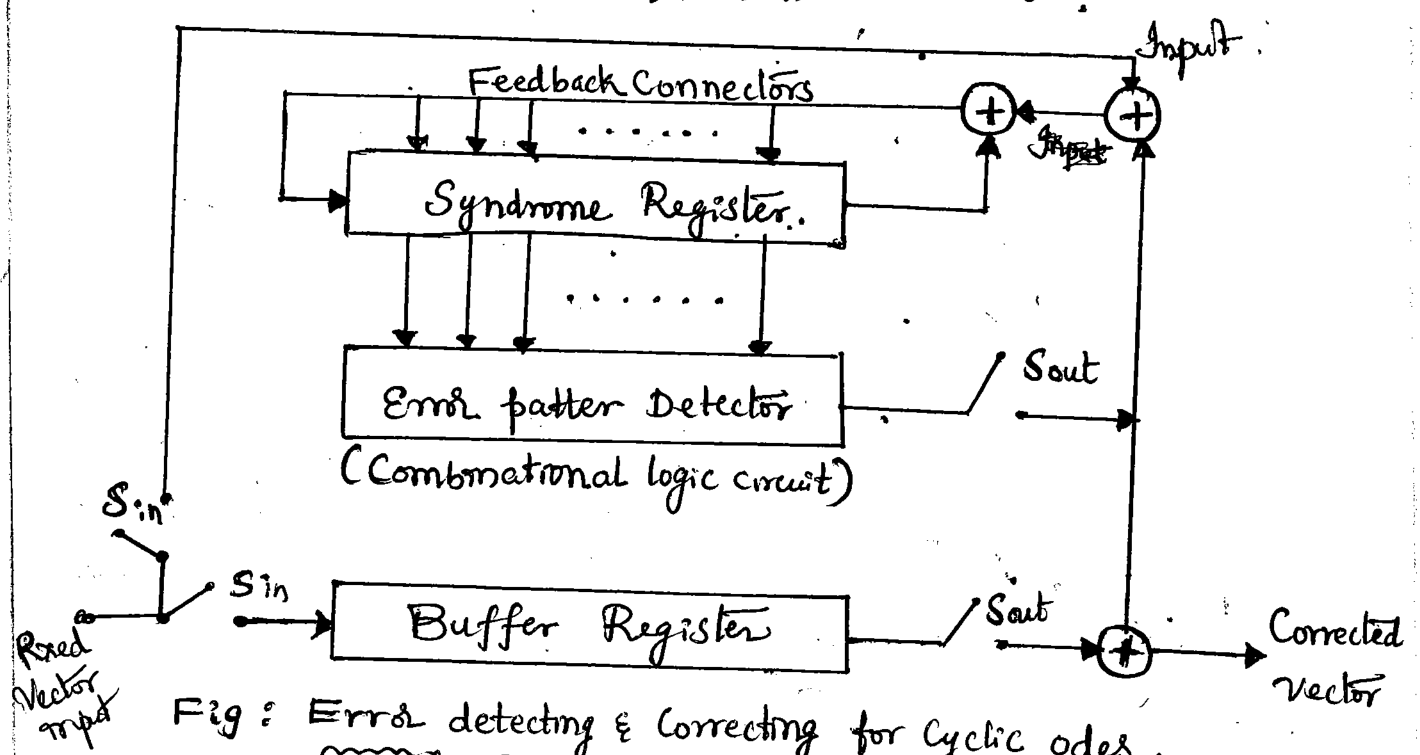


Fig : Error detecting & Correcting for Cyclic odes.

✓ Once the syndrome is calculated, then the error pattern is detected for that particular syndrome vector.

✓ When this error vector is added to the received vector 'Y' then it gives corrected code vector at the output. $\boxed{X = Y \oplus E}$

✓ The switches S_{in} is closed then the bits of the received vector 'Y' are shifted into the buffer register as well as they are shifted into the syndrome register.

✓ The output of the syndrome register is given to the error pattern detector.

✓ A particular syndrome detects a specific error pattern.

✓ The switches S_{in} is opened, S_{out} is closed.

The error pattern is then added bit by bit with the received vector.

* Syndrome decoding can be used to correct the errors.

Let 'Y' represents the received code vector.

The error can be detected as $\boxed{Y = X \oplus E.}$

The error can be corrected as $\boxed{X = Y \oplus E.}$

In the polynomial form $Y(P) = X(P) + E(P)$

We know $X(P) = M(P) \cdot G(P)$

$$\therefore \boxed{Y(P) = M(P) \cdot G(P) + E(P)}$$

Divide with $G(P)$ then

$$\frac{Y(P)}{G(P)} = M(P) + \frac{E(P)}{G(P)}$$

$$\underset{\text{Denominator}}{\overset{\text{numerator}}{\frac{N}{D}}} = \underset{\text{Quisent}}{Q} + \frac{R \rightarrow \text{Remainder}}{D \rightarrow \text{Denominator}}$$

The syndrome vector is obtained by dividing the received vector $Y(P)$ with $G(P)$.

$\therefore$ Syndrome polynomial $\boxed{S(p) = \text{Rem}\left(\frac{Y(P)}{G(P)}\right)}$

Advantages of Cyclic Codes:

- ✓ The error correcting and detecting methods of cyclic codes are simple and easy to implement.
- ✓ It is very simple compared to linear block codes.

Disadvantage of cyclic code:

- ✓ Error detection in cyclic codes is simple but the error correction is complecated in cyclic codes.

Problem:

(1) A (7,4) cyclic code generated by $G(P) = P^3 + P + 1$.

(a) Find all the code vectors for the code in non systematic code.

(b) Find all the code vectors for the code in systematic code.

(c) Design the encoder for given cyclic code and verify its operation for any message vector.

(d) Design a syndrome calculator for a given cyclic code and calculate the syndrome for $Y = 1001101$

(e) Find out the generator matrix by using of cyclic codes.

(f) Find the generator polynomial for the given cyclic code.

Sol. **Solution:**

Given $G(P) = P^3 + P + 1$

(7,4) Cyclic Code $\therefore$ $n = 7$, $K = 4$

Parity bits $q = n - K = 7 - 4 = 3$.

Since $K = 4$, there will be $2^K = 2^4 = 16$ different message code vectors.

Each message vector consists of 4 bits.

For $\underset{M_3\ M_2\ M_1\ M_0}{0\,0\,0\,1} \rightarrow M(P) = M_3 P^3 + M_2 P^2 + M_1 P^1 + M_0 \cdot P^0$

$= 0 \cdot P^3 + 0 \cdot P^2 + 0 \cdot P^1 + 1 \cdot 1$

$= 0 + 1$

$= 1$ $\therefore$ $\boxed{M(P) = 1}$

Ily $0010 \rightarrow P$, $1001 \rightarrow P^3 + 1$ ie.

SL No	Message bits $M_3\ M_2\ M_1\ M_0$	Message Polynomial $M(P)$.
1	0 0 0 0	0
2	0 0 0 1	1
3	0 0 1 0	P
4	0 0 1 1	$P+1$
5	0 1 0 0	P^2
6	0 1 0 1	P^2+1
7	0 1 1 0	P^2+P
8	0 1 1 1	P^2+P+1
9	1 0 0 0	P^3
10	1 0 0 1	P^3+1
11	1 0 1 0	P^3+P
12	1 0 1 1	P^3+P+1
13	1 1 0 0	P^3+P^2
14	1 1 0 1	P^3+P^2+1
15	1 1 1 0	P^3+P^2+P
16	1 1 1 1	P^3+P^2+P+1

(a)

Non Systematic Code: $\boxed{X(P) = M(P) \cdot G(P)}$ $\quad G(P) = P^3+P+1$

For ① (0000) $\rightarrow M(P) = 0$ $\quad \therefore X_1(P) = 0 \cdot (P^3+P+1) = (0000000)$

For ② (0001) $\rightarrow M(P) = 1$. $\quad \therefore X(P) = 1 \cdot (P^3+P+1) = P^3+P+1$

$\therefore X_2(P) = (0001011)$

For ③ (0010) $\rightarrow M(P) = P$ $\quad X(P) = P(P^3+P+1) = P^4+P^2+P.$

$= (0010110)$

For ④ (0011) $M(P) = P+1$ $\quad X(P) = (P+1)(P^3+P+1)$

$= P^4+P^2+P+P^3+P+1$

$= P^4+P^3+P^2+ \underbrace{P \oplus P}_{0} + 1$

$= (0011101)$

For ⑤ (0100) $M(P) = P^2$. $\quad X(P) = P^2(P^3+P+1)$

$= P^5+P^3+P^2 = (0101100)$

For ⑥ (0101) $M(P) = P^2+1$ $\quad X(P) = (P^2+1)(P^3+P+1)$

$= P^5+P^3+P^2+P^3+P+1$

$= P^5+P^3 \oplus P^3+P^2+P+1$

$= P^5+P^2+P+1$

$= (0100111)$

For ⑦ $(0110) \rightarrow M(p) = p^2+p$ $\therefore X(p) = (p^2+p)(p^3+p+1)$

$$= p^5+p^3+\cancel{p^2}+p^4+\cancel{p^2}+p$$
$$= p^5+p^4+p^3+p$$
$$= (0111010)$$

→ 0111 at last.

for ⑨ $(1000) \rightarrow M(p) = p^3$

$\therefore X(p) = p^3(p^3+p+1)$
$$= p^6+p^4+p^3 = (1011000)$$

for ⑩ $(1001) \rightarrow M(p) = p^3+1$

$X(p) = (p^3+1)(p^3+p+1)$
$$= p^6+p^4+p^3+p^3+p+1$$
$$= p^6+p^4+p+1 = (1010011)$$

for ⑪ $(1010) \rightarrow M(p) = p^3+p$

$X(p) = (p^3+p)(p^3+p+1)$
$$= p^6+p^4+p^3+p^4+p^2+p$$
$$= p^6+p^3+p^2+p = (1001110)$$

for ⑫ $(1011) \rightarrow M(p) = p^3+p+1$

$\therefore X(p) = (p^3+p+1)(p^3+p+1)$
$$= p^6+\cancel{p^4}+\cancel{p^3}+\cancel{p^4}+p^2+\cancel{p}+\cancel{p^3}+\cancel{p}+1$$
$$= p^6+p^2+1 = (1000101)$$

for ⑬ $(1100) \rightarrow M(p) = p^3+p^2$ $\therefore X(p) = (p^3+p^2)(p^3+p+1)$
$$= p^6+p^4+\cancel{p^3}+p^5+\cancel{p^3}+p^2$$
$$= (p^6+p^5+p^4+p^2) = (1110100)$$

for ⑭ (1101) $M(p) = p^3+p^2+1$ $\therefore X(p) = (p^3+p^2+1)(p^3+p+1)$
$$= p^6+p^4+p^3+p^5+\cancel{p^3}+p^2+p^3+p+1$$
$$= p^6+p^5+p^4+p^3+p^2+p+1$$
$$= (1111111)$$

for ⑮ (1110) $M(p) = p^3+p^2+p$ $\therefore X(p) = (p^3+p^2+p)(p^3+p+1)$
$$= p^6+\cancel{p^4}+\cancel{p^3}+p^5+\cancel{p^3}+\cancel{p^2}+\cancel{p^4}+\cancel{p^2}+p$$
$$= p^6+p^5+p = (1100010)$$

for ⑯ (1111) $M(p) = p^3+p^2+p+1$ $\therefore X(p) = (p^3+p^2+p+1)(p^3+p+1)$
$$= p^6+\cancel{p^4}+\cancel{p^3}+p^5+\cancel{p^3}+\cancel{p^2}+\cancel{p^4}+\cancel{p^2}+\cancel{p}+p^3+\cancel{p}+1$$
$$= p^6+p^5+p^3+1 = (1101001)$$

→ ⑧ (0111) $M(p) = p^2+p+1$ $X(p) = (p^2+p+1)(p^3+p+1)$
$$= p^5+\cancel{p^3}+\cancel{p^2}+p^4+\cancel{p^2}+\cancel{p}+\cancel{p^3}+\cancel{p}+1$$
$$= (p^5+p^4+1) = (0110001)$$

SL NO	Message bits $M_3 M_2 M_1 M_0$	Non Systematic Cyclic code Vectors	Systematic Cyclic Code Vectors.
1	0000	0000000	0000000
2	0001	0001011	0001011
3	0010	0010110	0010110
4	0011	0011101	0011101
5	0100	0101100	0100111
6	0101	0100111	0101100 ✓
7	0110	0111010	0110001
8	0111	0110001	0111010
9	1000	1011000	1000101
10	1001	1010011	1001110
11	1010	1001110	1010011
12	1011	1000101	1011000
13	1100	1110100	1100010
14	1101	1111111	1101001
15	1110	1100010	1110100
16	1111	1101001	1111111

(b)

Systematic Code : message $\boxed{K \mid q}$ parity.

The check bits can be generated by using

$$C(P) = \text{Rem}\left[\frac{P^q M(P)}{G(P)}\right] \qquad q = n-k = 3.$$

For ① $(0000) \rightarrow M(P) = 0 \Rightarrow P^3(0) = 0$, $\frac{P^2(M(P))}{G(P)} \Rightarrow C(p) = (0\,0\,0)$ $c_2\, c_1\, c_0$

$\therefore X = (M_3 M_2 M_1 M_0\; c_2\, c_1\, c_0)$

$X = (0000\,000)$

For ② $(0001) \rightarrow M(P) = 1$, $P^3(M(P)) = P^3$

$$C(p) = \frac{P^3 M(P)}{G(P)} = \frac{P^3}{P^3+P+1} = 011$$

$$P^3+P+1 \overline{)\; P^3 \;}\; 1$$
$$\cancel{P^3} + P + 1$$
$$P+1 \Rightarrow 011$$

$\therefore X(p) = (0001\,011)$

For ③ $(0010) \rightarrow M(P) = P \rightarrow P^3(M(P)) = P^4$.

$$\frac{P^3 M(P)}{G(P)} = \frac{P^4}{P^3+P+1} = 110$$

$$P^3+P+1 \overline{)\; P^4 \;}\; P$$
$$\cancel{P^4} + P^2 + P$$
$$P^2+P \Rightarrow (1\,1\,0)$$

$\therefore X(P) = (0010\,110)$.

④ for (0011), $M(p) = p+1$, $p^3 M(p) = p^3(p+1) = p^4 + p^3$.

$$\frac{p^3 M(p)}{G(p)} = \frac{p^4+p^3}{p^3+p+1} = (101)$$

$\therefore X(p) = (0011\ 101)$

$$p^3+p+1 \overline{\big) \cancel{p^4}+p^3} \quad p+1$$
$$p^4 + p^2 + p$$
$$\cancel{p^3} + p^2 + p$$
$$p^3 + p + 1$$
$$(101) \leftarrow p^2 + 1$$

⑤ for (0100), $M(p) = p^2$, $p^3 M(p) = p^3 \cdot p^2 = p^5$

$$\frac{p^3 M(p)}{G(p)} = \frac{p^5}{p^3+p+1} = (111)$$

$X(p) = (0100\ 111)$

$$p^3+p+1 \overline{\big) p^5} \quad p^2+1$$
$$\cancel{p^5} + p^3 + p^2$$
$$\cancel{p^3} + p^2$$
$$p^3 + p + 1$$
$$(111) \leftarrow p^2 + p + 1$$

⑥ for (0101), $M(p) = p^2+1$, $p^3 M(p) = p^3(p^2+1) = p^5 + p^3$.

$$\frac{p^3 M(p)}{G(p)} = \frac{p^5+p^3}{p^3+p+1} = (100)$$

$\therefore X(p) = (0101\ 100)$

$$p^3+p+1 \overline{\big) \cancel{p^5}+\cancel{p^3}} \quad p^2$$
$$\cancel{p^5} + \cancel{p^3} + p^2$$
$$(100) \leftarrow p^2$$

⑦ for (0110), $M(p) = p^2+p$, $p^3 M(p) = p^3(p^2+p) = p^5 + p^4$

$$\frac{p^3 M(p)}{G(p)} = \frac{p^5+p^4}{p^3+p+1} = (001)$$

$\therefore X(p) = (0110\ 001)$

$$p^3+p+1 \overline{\big) \cancel{p^5}+p^4} \quad p^2+p+1$$
$$\cancel{p^5} + p^3 + p^2$$
$$\cancel{p^4} + p^3 + \cancel{p^2}$$
$$p^4 + \cancel{p^2} + p$$
$$\cancel{p^3} + \cancel{p}$$
$$p^3 + p + 1$$
$$(001) \leftarrow 1$$

⑧ for (0111), $M(p) = p^2+p+1$, $p^3 M(p) = p^3(p^2+p+1) = p^5 + p^4 + p^3$

$$\frac{p^3 M(p)}{G(p)} = \frac{p^5+p^4+p^3}{p^3+p+1} = (010)$$

$\therefore X(p) = (0111010)$

$$p^3+p+1 \overline{\big) \cancel{p^5}+p^4+\cancel{p^3}} \quad p^2+p$$
$$\cancel{p^5} + p^3 + p^2$$
$$\cancel{p^4} + \cancel{p^2}$$
$$p^4 + p^2 + p$$
$$(010) \leftarrow p$$

⑨ for (1000), $M(p) = p^3$, $p^3 M(p) = p^3 \cdot p^3 = p^6$

$$\frac{p^3 M(p)}{G(p)} = \frac{p^6}{p^3+p+1} = (101)$$

$\therefore X(p) = (1000\ 101)$

$$p^3+p+1 \overline{\big) \cancel{p^6}} \quad p^3+p+1$$
$$\cancel{p^6} + p^4 + p^3$$
$$\cancel{p^4} + p^3$$
$$p^4 + p^2 + p$$
$$\cancel{p^3} + p^2 + \cancel{p}$$
$$p^3 + p + 1$$
$$(101) \leftarrow p^2 + 1$$

⑩ for (1001), $M(p) = p^3+1$, $p^3 M(p) = p^3(p^3+1) = p^6 + p^3$

$$\frac{p^3 M(p)}{G(p)} = \frac{p^6+p^3}{p^3+p+1} = (110)$$

$\therefore X(p) = (1001\ 110)$

$$p^3+p+1 \overline{\big) \cancel{p^6}+\cancel{p^3}} \quad p^3+p$$
$$\cancel{p^6} + p^4 + \cancel{p^3}$$
$$\cancel{p^4}$$
$$p^4 + p^2 + p$$
$$(110) \leftarrow p^2 + p$$

(11) For (1010), $M(p) = p^3+p$ $\therefore p^3M(p) = p^3(p^3+p) = p^6+p^4$

$$\frac{p^3M(p)}{G(p)} = \frac{p^6+p^4}{p^3+p+1} = (011)$$

$$\therefore X(p) = (1010011)$$

$$\begin{array}{r} p^3+1 \\ p^3+p+1 \overline{)\, p^6+p^4 \qquad} \\ p^6+p^4+p^3 \\ \hline p^3 \\ p^3+p+1 \\ \hline (011) \leftarrow p+1 \end{array}$$

(12) For (1011), $M(p) = p^3+p+1$, $p^3M(p) = p^3(p^3+p+1)$
$= p^6+p^4+p^3$.

$$\therefore \frac{p^3M(p)}{G(p)} = \frac{p^6+p^4+p^3}{p^3+p+1} = (000)$$

$$\therefore X(p) = (1011000)$$

$$\begin{array}{r} p^3 \\ p^3+p+1 \overline{)\, p^6+p^4+p^3} \\ p^6+p^4+p^3 \\ \hline (000) \leftarrow 0 \end{array}$$

(13) For (1100), $M(p) = p^3+p^2$, $p^3M(p) = p^3(p^3+p^2)$
$= p^6+p^5$.

$$\therefore \frac{p^3M(p)}{G(p)} = \frac{p^6+p^5}{p^3+p+1} = (010)$$

$$\therefore X(p) = (1100010).$$

$$\begin{array}{r} p^3+p^2+p \\ p^3+p+1 \overline{)\, p^6+p^5 \qquad\qquad} \\ p^6+p^4+p^3 \\ \hline p^5+p^4+p^3 \\ p^5+p^3+p^2 \\ \hline p^4+p^2 \\ p^4+p^2+p \\ \hline (010) \leftarrow p \end{array}$$

(14) For (1101), $M(p) = p^3+p^2+1$, $p^3M(p) = p^3(p^3+p^2+1)$
$= p^6+p^5+p^3$

$$\frac{p^3M(p)}{G(p)} = \frac{p^6+p^5+p^3}{p^3+p+1} = (001)$$

$$\therefore X(p) = (1101001)$$

$$\begin{array}{r} p^3+p^2+p+1 \\ p^3+p+1 \overline{)\, p^6+p^5+p^3 \qquad} \\ p^6+p^4+p^3 \\ \hline p^5+p^4 \\ p^5+p^3+p^2 \\ \hline p^4+p^3+p^2 \\ p^4+p^2+p \\ \hline p^3+p \\ p^3+p+1 \\ \hline (001) \leftarrow 1 \end{array}$$

(15) For (1110), $M(p) = p^3+p^2+p$
$p^3M(p) = p^3(p^3+p^2+p)$
$= p^6+p^5+p^4$

$$\frac{p^3M(p)}{G(p)} = \frac{p^6+p^5+p^4}{p^3+p+1} = (100)$$

$$\therefore X(p) = (1110100)$$

$$\begin{array}{r} p^3+p^2 \\ \therefore\ p^3+p+1 \overline{)\, p^6+p^5+p^4 \qquad} \\ p^6+p^4+p^3 \\ \hline p^5+p^3 \\ p^5+p^3+p^2 \\ \hline (100) \leftarrow p^2 \end{array}$$

(16) For (1111), $M(p) = p^3+p^2+p+1$
$p^3M(p) = p^3(p^3+p^2+p+1)$
$= p^6+p^5+p^4+p^3$

$$\frac{p^3M(p)}{G(p)} = \frac{p^6+p^5+p^4+p^3}{p^3+p+1} = (111)$$

$$\therefore X(p) = (1111111).$$

$$\begin{array}{r} p^3+p^2+1 \\ \therefore\ p^3+p+1 \overline{)\, p^6+p^5+p^4+p^3} \\ p^6+p^4+p^3 \\ \hline p^5 \\ p^5+p^3+p^2 \\ \hline p^3+p^2 \\ p^3+p+1 \\ \hline (111) \leftarrow p^2+p+1 \end{array}$$

(C) The given generator polynomial is $G(P) = P^3 + P + 1$

$\therefore G(p) = p^q + g_{q-1}p^{q-1} + g_{q-2}p^{q-2} + \cdots\cdots + g_2 p^2 + g_1 p' + 1.$

$\underline{q=3}$ $\quad \therefore G(p) = p^3 + g_2 p^2 + g_1 p^1 + 1.$

Compare with $G(p) = p^3 + p + 1$ (given)

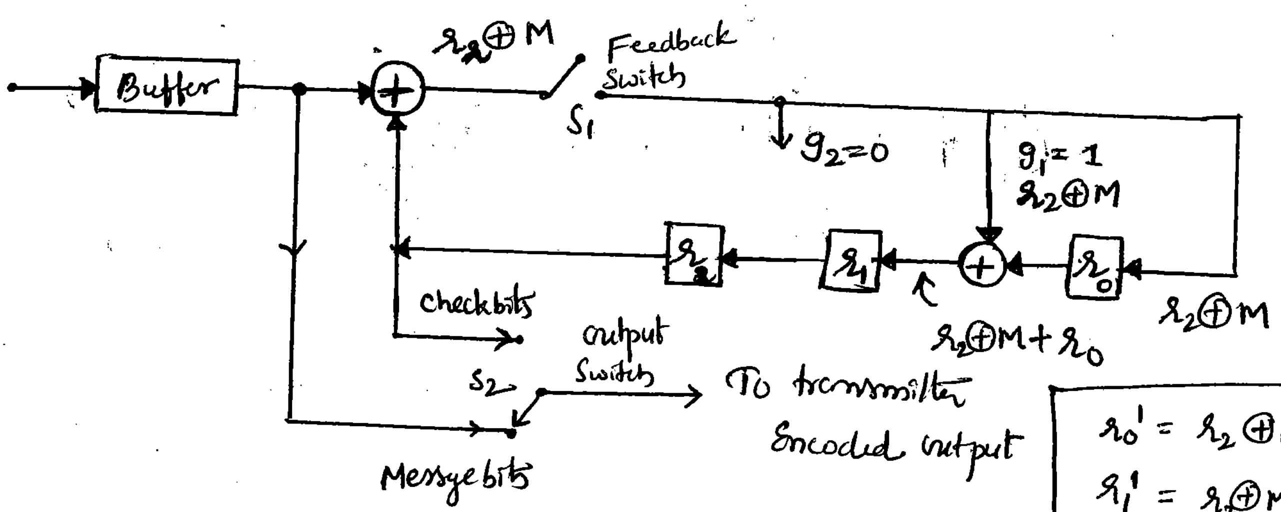
$\therefore \quad g_2' = 0$ (Not existing path).
$\quad g_1 = 1$ (Existing path).

Since $q = 3$. there are 3 flip flops in shift registers.

$q = n - k$
$= 7 - 4$
$= 3$

Since $g_2 = 0$ its link is not connected and $g_1 = 1$ its link is connected.

__Encoder diagram :__



$r_0' = r_2 \oplus M$
$r_1' = r_2 \oplus M + r_0$
$r_2' = r_1.$

- After cyclic shift of 4 times $(K=4)$ the register hold the check bits.

For Message (0101)

(a) Input Message bits	(b) Register bit input before shift			(c) Register bit output after shift.		
M	r_2	r_1	r_0	$r_2' = r_1$	$r_1' = r_0 \oplus r_2 \oplus M$	$r_0' = r_2 \oplus M$
—	0	0	0	0	0	0
0	0	0	0	0	0	0
1	0	0	0	0	1	1
0	0	1	1	1	1	0
1	1	1	0	$\boxed{1}$	$\boxed{0}$	$\boxed{0}$

$\therefore X = [M_3 M_2 M_1 M_0 C_2 C_1 C_0] \Rightarrow X = [\underbrace{0\,1\,0\,1}_{\text{msg bits}}\ \underbrace{1\,0\,0}_{\text{Check bits}}]$

~~Hence~~ Verified

(d) Syndrome decoder : For the given code $n=7$, $K=4$, $q=n-k=3$.

Method ①: The given polynomial of the generator $G(p)=p^3+p+1$

$$G(p) = p^q + g_{q-1}p^{q-1} + g_{q-2}p^{q-2} + \dots + g_1 p + 1.$$

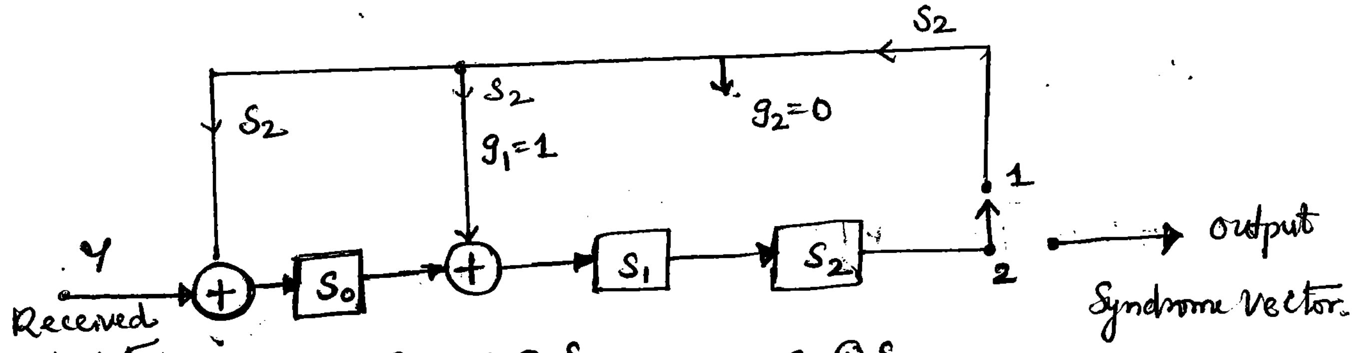
$q=3$ $\quad G(p) = p^3 + g_2 p^2 + g_1 p^1 + 1.$

Compare with $G(p) = p^3+p+1$

$\therefore g_2 = 0$ (No path)

$g_1 = 1$ (path established).

$Y = [1001101]$



$S_0 = Y \oplus S_2$ $\quad S_1 = S_0 \oplus S_2$ $\quad S_2 = S_1$

Shift	Received Vector ie bits in 'Y'	Content of flipflop in Shift registers		
		$S_0 = Y + S_2$	$S_1 = S_0 \oplus S_2$	$S_2 = S_1$
–	–	0	0	0
1	1	1	0	0
2	0	0	1	0
3	0	0	0	1
4	1	0	1	0
5	1	1	0	1
6	0	1	0	0
7	1	1	1	0
		S_0	S_1	S_2

(previous bits operation)

The end of the last shift registers contents are. $S_0S_1S_2 = '110'$

$\therefore$ Hence the Calculated Syndrome $\boxed{S=(S_2S_1S_0) = 011}$

Method ②: Syndrome Vector is obtained by dividing received vector $r(p)$ by $G(p)$.

$$\therefore \quad S(p) = \text{Rem}\left[\frac{r(p)}{G(p)}\right]$$

Given. Rxed Vector $\quad r = [1\ 0\ 0\ 1\ 1\ 0\ 1]$

$$\therefore \quad r(p) = 1\cdot p^6 + 0\cdot p^5 + 0\cdot p^4 + 1\cdot p^3 + 1\cdot p^2 + 0\cdot p^1 + 1$$

$$\therefore \quad r(p) = p^6 + p^3 + p^2 + 1$$

Generator polynomial $\quad G(p) = p^3 + p + 1$.

$$\therefore \frac{r(p)}{G(p)} \Rightarrow \frac{p^6 + p^3 + p^2 + 1}{p^3 + p + 1}$$

$$\therefore \quad p^3 + p + 1 \,\big)\, \cancel{p^6} + \cancel{p^3} + p^2 + 1 \quad \left(p^3 + p \right)$$

$$\cancel{p^6} + p^4 + \cancel{p^3}$$

$$\cancel{p^4} + \cancel{p^2} + 1$$

$$p^4 + p^2 + p$$

$$\text{Rem}: 011 \leftarrow \underline{p + 1}$$

$\therefore$ Syndrome vector

$$\boxed{S = 011}$$

$\qquad S_2\ S_1\ S_0$.

(e) <u>Generator Matrix by using Cyclic Codes:</u>

Given $\quad n = 7,\ K = 4,\ \quad q = n - k = 7 - 4 \Rightarrow q = 3$.

$$G(p) = p^3 + p + 1 \quad \Rightarrow 1\cdot p^3 + 0\cdot p^2 + 1\cdot p + 1.$$

multiply with 'p^i' on both sides ie

$$p^i \cdot G(p) = p^i \cdot p^3 + p^i \cdot p + p^i \cdot 1$$

$$= p^{3+i} + p^{1+i} + p^i$$

Since $K = 4$, the no. of rows of generated matrix is given as $K - 1 = 3$, $\quad i = 3, 2, 1, 0$.

For row 1, $i = 3$, $\quad p^3 \cdot G(p) = p^6 + p^4 + p^3 \rightarrow 1\ 0\ 1\ 1\ 0\ 0\ 0$

For row 2, $i = 2$, $\quad p^2\, G(p) = p^5 + p^3 + p^2 \Rightarrow 0\ 1\ 0\ 1\ 1\ 0\ 0$

For row 3, $i = 1$, $\quad p^1 \cdot G(p) = p^4 + p^2 + p \Rightarrow 0\ 0\ 1\ 0\ 1\ 1\ 0$

For row 4, $i = 0$, $\quad p^0 \cdot G(p) = p^3 + p + 1 \Rightarrow 0\ 0\ 0\ 1\ 0\ 1\ 1$

$\therefore$ Generator Matrix

$$G_{K\times n} = \begin{array}{c} \begin{array}{ccccccc} p^6 & p^5 & p^4 & p^3 & p^2 & p^1 & p^0 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{4\times 7} \end{array}$$

Note : Since the generator matrix is non systematic form Hence parity check matrix can not be obtained using direct method ($H = [P^T \vdots I]$).

(f) For (7,4) Cyclic Codes.

$n = 7$, $K = 4$, $q = n-k = 7-4 = 3$ $\therefore q = 3$.

The generator polynomial is a factor of $[p^n + 1]$

$\therefore$ $p^7 + 1 = [p+1]\,[p^3+p^2+1]\,[p^3+p+1]$

$\therefore$ $\boxed{p^7+1 = (p+1)\,(p^3+p^2+1)\,(p^3+p+1)}$.

The valid generating polynomial is given by $G(P)$ as

$G(p) = p^q + g_{q-1}\,p^{q-1} + \dots + g_2 p^2 + g_1 p^1 + 1$.

$\underline{q=3}$ $\therefore G(P) = p^3 + g_2 p^2 + g_1 p^1 + 1$.

The degree of generating polynomial should be 'q'

For this problem $q = 3$

$\therefore$ The valid generator polynomials for 'p^7+1' will be $(p^3 + p^2 + 1)$ and $(p^3 + p + 1)$

'$p+1$' will not be a generated polynomial since its degree is not q ($q=3$).

$\therefore$ The generator polynomials for given (7,4) Cyclic Codes are

$$\boxed{\begin{array}{l} G_1(P) = p^3 + p^2 + 1 \\ G_2(P) = p^3 + p + 1 \end{array}}$$

The characteristics of five Important classes of Cyclic Codes :

① Cyclic Redundancy Check codes (CRC Codes) :

✓ Cyclic Codes are extremely well suited for error detection.
ie "A cyclic code used for error detection is referred to as Cyclic redundancy check (CRC) code".

✓ A CRC error burst of length B in an n-bit received word as a contiguous sequence of B bits in which the first & last bits of any no. of intermediate bits are received in error.

* Binary (n, K) CRC codes are capable of detecting the following error patterns.

(i) All CRC error bursts of length $n-K$ or less.
(ii) A fraction of CRC error bursts of length equal to $n-K+1$, the fraction equals $1-2^{-(n-K-1)}$.
(iii) A fraction of CRC error bursts of length greater than $n-K+1$, the fraction equals $1-2^{-(n-K)}$.
(iv) All combinations of $d_{min}-1$ (or fewer) errors.
(v) All error patterns with an odd no. of errors if the generator polynomial $g(x)$ for the code has an even no. of non zero coefficients.

Ex: CRC-12 Code : $1+p+p^2+p^3+p^{11}+p^{12}$, $n-K=12$
CRC-16 code : $1+p^2+p^{15}+p^{16}$, $n-K=16$.

② Maximum length codes :

For positive integer $K \geq 3$, there exists a maximum length code with the parameters as

Block length $n = 2^K - 1$

Min. distance $d_{min} = 2^{K-1}$

∴ The max. length codes are generated by polynomials of the form

$$g(p) = \frac{1+p^n}{h(p)}$$

where $h(P)$ = any primitive polynomial of degree 'k'.

ie The maximum length codes are the dual of Hamming Codes.

✓ The maximum length codes are also referred to as Pseudo-Noise (PN) codes.

ie The name "Pseudo Noise" is derived from the fact that these codes have correlation & spectral characteristics that resemble those of a white noise sequence.

(3) Golay Codes: $(n, k) \rightarrow (23, 12)$

✓ A Golay code is a very special binary code that is capable of correcting any combinations of three & fewer random errors in a block of 23 bits.

✓ The code has minimum distance of '7'.

ie Indeed the (23,12) Golay code is the only known three correcting binary perfect cyclic code.

Ex: The (23,12) Golay code is generated either by the polynomial

$$g_1(p) = 1 + p^2 + p^4 + p^6 + p^{10} + p^{11}.$$

(or)

$$g_2(p) = 1 + p + p^5 + p^6 + p^7 + p^9 + p^{11}.$$

∴ Both $g_1(p)$ & $g_2(p)$ are factor of '$1+D^{23}$'

ie

$$1 + D^{23} = (1+D)[G_1(P) \cdot G_2(P)]$$

∴ The golay codes does not generalize to other combinations of parameters n & K.

(4) BCH (Bose-Chaudhuri-Hocquenghem) Codes:

✓ One of the most important & powerful classes of linear block codes are "BCH" codes.

✓ Specifically, for any positive integers K (equal to or >3) & t (less than $(2^K-1)/2$) there exists a binary BCH code with parameters

Block length $n = 2^K - 1$

no of msg bits $K \geq n -$ ..

Minimum distance $d_{min} \geq 2t + 1$.

✓ Each BCH code is a t-error correcting code

ie It can detect & Correct upto 't' random errors per codeword.

The hamming single error correcting codes can be described BCH codes.

✓ BCH codes offer flexibility in the choice of code parameters ie Code rate and block length.

✓ Generator polynomials for binary block BCH codes of length upto $2^5 - 1$.

(5) Reed-Solomon (RS) Codes:

✓ The Reed solomon codes are an important sub class of non binary BCH codes.

✓ The encoder for an RS code differs from a binary encoder in that it operates on multiple bits rather than individual bits

* The encoder for an RS (n,K) code on m-bit symbols groups the incoming binary data stream into blocks, each 'km' bits long. Each block is treated as K-symbols, with each symbol having m-bits.

✓ The m-bit symbols are called bytes, m is integer power of two'

∴ 8 bit **RS** codes are extremely powerful.

A t-error correcting RS code has

✓ Block length : $n = 2^K - 1$ symbols

✓ Message size : K-symbols

✓ Parity check size : $n - K = 2t$ symbols

✓ Minimum distance : $d_{min} = 2t + 1$ symbols.

✓ Every RS code is a maximum distance seperable code.

* RS-Codes → Highly efficient use of redundancy, & block lengths & Symbol sizes can be adjusted readily & efficient decoding.

Convolutional Codes :

✓ Convolutional codes were first introduced by Elias (1995) as an alternative to block codes.

* In block coding, the encoder accepts a K-bit message blocks and generates an n-bit code word. Thus code words are produced on a block-by block basic.

However, where the messages bits come in serially rather than in large blocks, in which case the use of a buffer may be undesirable. In such situations, use 'convolutional coding'.

* The main differences between block Codes and Convolutional Codes as follows

① In a block codes, a block of n-bits generated by the encoder in a perticular time unit depends only on the block of K-input message bits within that time during.

② In a Convolutional code, the block of n-code bits generated by the encoder on the time during depends on not only the block of k-message bits but also on the previous (N-1) block of messages (N>1).

③ Like block codes, convolutional codes can be designed both for detecting and Correcting of the errors.

④ Block Codes are better suitable for error detection and Convolutional Codes are mainly used for error correction.

⑤ A Convolutional encoder operates on the incoming message sequence continuously in a serial manner.

⑥ Convolutional Codes are well suited for mobile communications and satellite communications etc.

⑦ The use of non systematic codes is ordinarily preferred over systematic codes in Convolutional Codes.

Encoder for Convolutional Codes:

- A convolutional code can be generated by using of shift register and X-OR operation.
- The general representation of convolutional code is

$$(n, K, M)$$

where $n \rightarrow$ No. of X-OR adders.

$K \rightarrow$ At a time no. of bits given to the register

$M \rightarrow$ No. of Shift Registers.

let

$L \rightarrow$ No. of bits present in the message sequence.

$L+M \rightarrow$ Total no. of shifts required to reset the register.

$n(L+M) \rightarrow$ Total no. of bits in the Encoder output Sequence.

- The code rate for the convolutional code is given by

$$r_c = \frac{L}{n(L+M)} \qquad \left(\because\ r_c = \frac{K}{n} \text{ for block codes}\right)$$

If $L >> M$ then

$$r_c = \frac{L}{n \cdot L} = \frac{1}{n}$$

Code rate $\boxed{r_c = \frac{1}{n} \text{ bits/symbol}}$

- The Constraint length (N) of convolutional codes is defined as the no. of shift over which a single message bit can influence the encoder output.

In an encoder with an M-stage shift registers the memory of the encoder equals, M message bits and $N = M+1$ shifts are required for a message bit to enter the shift register and finally come out.

Hence the constraint length of the encoder is $\boxed{N = M+1}$

Example: (2,1,3) Convolutional Codes. Find the output of the encoder for an input data stream of 1011.

Sol: Given (n, K, M) convolutional codes as $(2,1,3)$

$n = 2$ → No. of X-OR adders.

$K = 1$ → At a time no. of bits given to the register

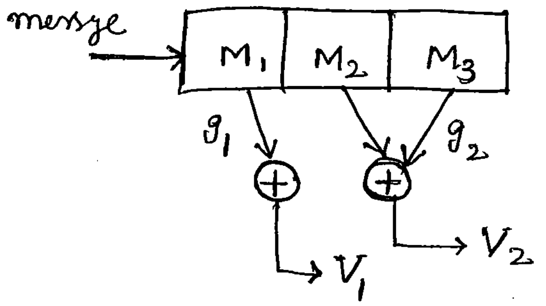
$M = 3$ → No. of shift registers.

Given message: 1011.

∴ $L = 4$ → No. of bits present in the message sequence.

$L+M = 3+4 = 7$ → shifts are required to reset the registers.

$n(L+M) = 2(7) = 14$ → The total no. of bits in the encoder output sequence.



(∵ let $g_1 = 100$
$g_2 = 011$).

$V_1 = M_1$

$V_2 = M_2 \oplus M_3$.

∴ Assume initially all shift registers are reset position

Now $M = \underset{MSB}{1}011$ is entered in the shift register from MSB bit

First bit of the input data stream is enter into M_1, the bit is initially present in M_1 is shifted to M_2 in next state ... this process continuous until the last bit of the message has been entered and comes out.

Shift	Input	Registers M_1	M_2	M_3	Outputs $V_1 = M_1$	$V_2 = M_2 \oplus M_3$
-	-	0	0	0	0	0
1	1	1	0	0	1	0
2	0	0	1	0	0	1
3	1	1	0	1	1	1
4	1	1	1	0	1	1
5	0	0	1	1	0	0
6	0	0	0	1	0	1
7	0	0	0	0	0	0

$L+M = 7$
so 7 shifts.

where
V_1 - First Code Symbol
V_2 - Second Code Symbol
in Encoded output.

∴ The Encoder output Sequence $X = (\overset{V_1 V_2}{10}\ \overset{V_1 V_2}{01}\ 11\ 11\ 00\ 01\ \overset{V_1 V_2}{00})$.

Encoding methods of Convolutional Codes:

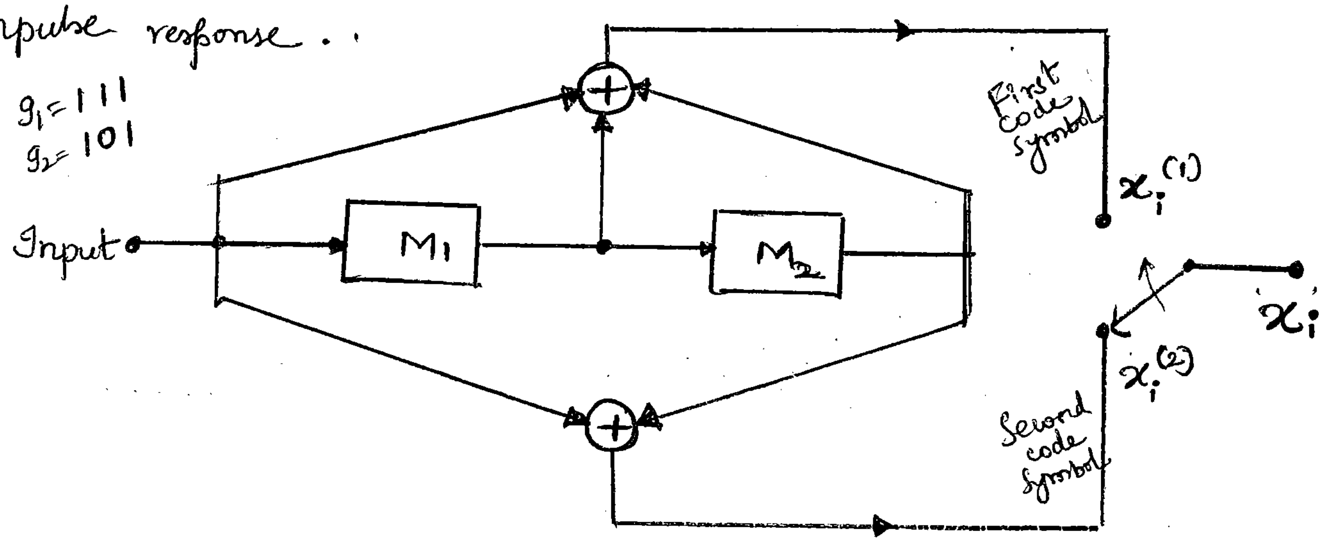
There are five methods for encoding of convolutional codes.

1. Time domain approach (or) Impulse approach (or) Connection diagram
2. Frequency domain approach (or) Transfer domain approach (or) Connection polynomial (or) Connection representation.
3. Tree diagram
4. State diagram
5. Trellis diagram.

1 Time domain approach:

The time domain behaviour of convolutional codes with a code rate of '$\frac{1}{n}$' may be defined in terms of a set of 'n' impulse response..

let $g_1 = 111$
$g_2 = 101$



- This consists of two modulo-2 adders and the code rate is $\frac{1}{2}$
- This Encoder are non systematic codes.
- For a given encoder two impulse responses are needed to characterised its behaviour in time domain

Ex: $g_1 = (1\ 1\ 1)$ $g_2 = (1\ 0\ 1)$

up

down → connection doesnot exist.

* Let the sequence $[g_0^{(1)}, g_1^{(1)}, g_2^{(1)} \ldots g_M^{(1)}]$ denotes the impulse response of the input <u>top-adder</u> output path of the encoder.

$[g_0^{(2)}, g_1^{(2)}, g_2^{(2)} \ldots g_M^{(2)}]$ denotes the impulse response of the input <u>bottom-adder</u> output path of the encoder.

✓ The impulse response is so defined as the generated sequence of the code.

* The <u>top output sequence</u> is defined by the convolutional sum as

$$\boxed{x_i^{(1)} = \sum_{l=0}^{M} g_l^{(1)} \cdot M_{i-l}} \quad \text{where } i = 0, 1, 2, \ldots$$

* The <u>bottom output sequence</u> is defined as

$$\boxed{x_i^{(2)} = \sum_{l=0}^{M} g_l^{(2)} \cdot M_{i-l}} \quad , \text{where } i = 0, 1, 2, \ldots$$

The two <u>output sequences</u> $x_i^{(1)}$ and $x_i^{(2)}$ are combined by the multiplexer to produce the encoder output sequence (x_i)

$$\therefore \boxed{\{x_i\} = (x_0^{(1)} x_0^{(2)} \quad x_1^{(1)} x_1^{(2)} \quad x_2^{(1)} x_2^{(2)} \ldots\ldots)}$$

Problem ①

A convolutional encoder has a single shift register with two stages (ie Constraint length 3), two X-OR adders and an output ~~sequ~~ multiplexer. The generated sequences of the encoder as

$$g_1 = (\underset{g_0}{1}\ \underset{g_1}{1}\ \underset{g_2}{1}) \quad \& \quad g_2 = (\underset{g_0}{1}\ \underset{g_1}{0}\ \underset{g_2}{1}).$$

Draw the block diagram of encoder and find the output of the encoder for an input data stream of '$\underset{M_0}{1}\ \underset{M_1}{0}\ \underset{M_2}{0}\ \underset{M_3}{1}\ \underset{M_4}{1}$' by using time domain approach.

(PTO) →

Sol

Given generated sequences

$$g_1 = \underset{g_0\ g_1\ g_2}{(1\ 1\ 1)} \quad \& \quad g_2 = \underset{g_0\ g_1\ g_2}{(1\ 0\ 1)}$$

Constraint length $N = M+1 = 3$

$$\boxed{M=2}$$

$\therefore$ So. no-of shift registers are '2' M_0, M_1. $\boxed{M=2}$

Given two x-OR adders. $\boxed{n=2}$

At a time no-of bits given to the register
ie single shift register | $K=1 \Rightarrow \boxed{K=1}$.

$(n, K, M) = (2, 1, 2)$ Convolutional Codes.

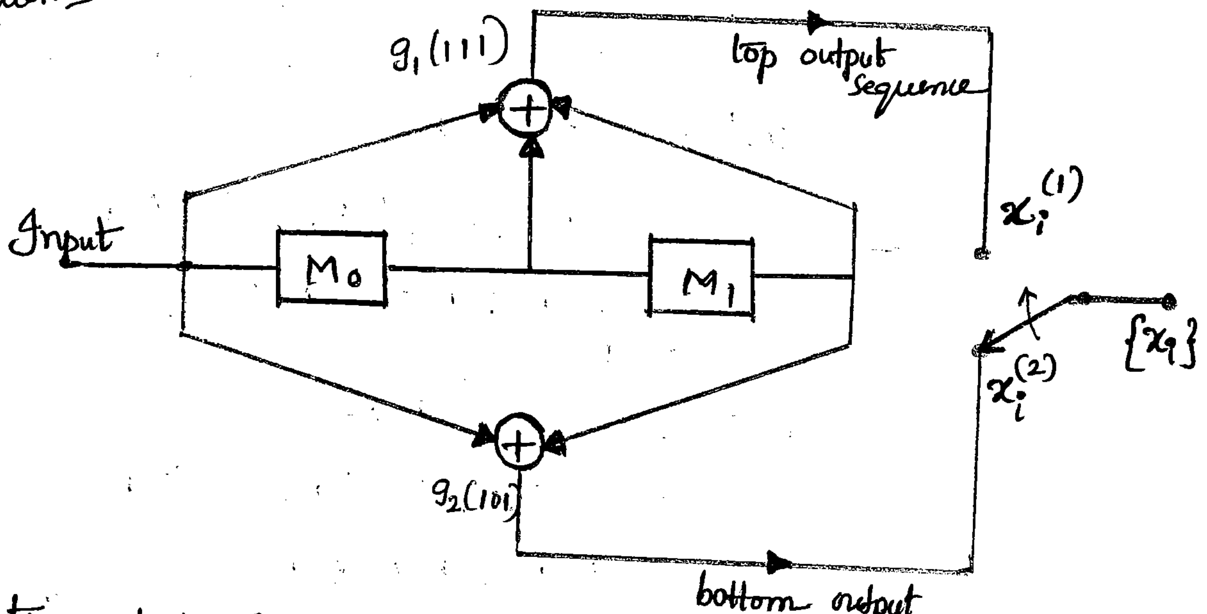
Given Input data stream 1 0 0 1 1

$\therefore \boxed{L=5}$ ie no of bits in message.

$L+M = 5+2 = 7 \Rightarrow \boxed{L+M=7}$

Total no-of bits in Encoded o/p sequence. $n(L+M) = 2(7) = 14 \Rightarrow \boxed{n(L+M)=14}$.

Thus.



* The top output Sequence can be generated by using.

$$x_i^{(1)} = \sum_{l=0}^{M} g_l^{(1)} M_{i-l}. \quad \text{when } i = 0, 1, 2, 3, 4, 5, 6. \ (L+M-1)$$

* The bottom output sequence can be generated by using

$$x_i^{(2)} = \sum_{l=0}^{M} g_l^{(2)} M_{i-l}, \quad i = 0, 1, 2, 3, 4, 5, 6.$$

✓ The top output sequence

$$x_i^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{i-l} \quad , i = 0,1,2,3,4,5,6.$$

For $i=0$, $x_0^{(1)} = \sum_{l=0}^{2} g_l^{(1)} . M_{-l}$

$= g_0^{(1)} M_0 + g_1^{(1)} . M_{-1} + g_2^{(1)} . M_{-2}$

M

$\underset{M_0}{1}\ \underset{M_1}{0}\ \underset{M_2}{0}\ \underset{M_3}{1}\ \underset{M_4}{1}$

$= 1(1) + 1(0) + (1)(0)$

$= 1 \Rightarrow \boxed{x_0^{(1)} = 1}$

$g_1 = \underset{g_0}{1}\ \underset{g_1}{1}\ \underset{g_2}{1}$

For $i=1$, $x_1^{(1)} = \sum_{l=0}^{2} g_l^{(1)} . M_{1-l}$

$= g_0^{(1)} . M_1 + g_1^{(1)} . M_0 + g_2^{(1)} . M_{-1}$

$= 1(0) + 1(1) + 1(0)$

$= 1 \quad \therefore \boxed{x_1^{(1)} = 1}$

For $i=2$, $x_2^{(1)} = \sum_{l=0}^{2} g_l^{(1)} . M_{2-l}$.

$= g_0^{(1)} M_2 + g_1^{(1)} . M_1 + g_2^{(1)} . M_0$.

$= 1(0) + 1(0) + 1(1)$

$= 1 \quad \therefore \boxed{x_2^{(1)} = 1}$

For $i=3$, $x_3^{(1)} = \sum_{l=0}^{2} g_l^{(1)} . M_{3-l}$

$= g_0^{(1)} M_3 + g_1^{(1)} M_2 + g_2^{(1)} M_1$

$= 1(1) + 1(0) + 1(0) \quad \therefore \boxed{x_3^{(1)} = 1}$

$= 1$

For $i=4$, $x_4^{(1)} = \sum_{l=0}^{2} g_l^{(1)} . M_{4-l}$

$= g_0^{(1)} M_4 + g_1^{(1)} . M_3 + g_2^{(1)} . M_2$

$= 1(1) + 1(1) + 1(0)$

$= 1 \oplus 1 \quad \therefore \boxed{x_4^{(1)} = 0}$

$= 0$

For $i=5$, $x_5^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{5-l}$

$= g_0^{(1)} M_5 + g_1^{(1)} M_4 + g_2^{(1)} M_3$

$= 1(0) + 1(1) + 1(1)$

$= 1 \oplus 1 \quad \therefore \boxed{x_5^{(1)} = 0}$

$= 0$

For $i=6$, $x_6^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{6-l}$

$= g_0^{(1)} M_6 + g_1^{(1)} M_5 + g_2^{(1)} M_4$

$= 1(0) + 1(0) + 1(1)$

$= 1$ $\quad \therefore \boxed{x_6^{(1)} = 1}$

$\therefore$ The top output sequence

$$\boxed{x_i^{(1)} = (1\ 1\ 1\ 1\ 0\ 0\ 1)}$$

✓ The bottom output sequence

$$x_i^{(2)} = \sum_{l=0}^{2} g_l^{(2)} \cdot M_{i-l}, \quad i = 0,1,2,3,4,5,6.$$

For $i=0$, $x_0^{(2)} = \sum_{l=0}^{2} g_l^{(2)} \cdot M_{-l}$

$= g_0^{(2)} \cdot M_0 + g_1^{(2)} \cdot M_{-1} + g_2^{(2)} \cdot M_{-2}$.

$= (1)(1) + 0(0) + 1(0)$

$= 1$ $\quad \therefore \boxed{x_0^{(2)} = 1}$.

$\underset{M_0}{1}\ \underset{M_1}{0}\ \underset{M_2}{0}\ \underset{M_3}{1}\ \underset{M_4}{1}$

$g_2 = \underset{g_0}{1}\ \underset{g_1}{0}\ \underset{g_2}{1}$

For $i=1$, $x_1^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{1-l}$

$= g_0^{(2)} M_1 + g_1^{(2)} \cdot M_0 + g_2^{(2)} M_{-1}$

$= 1(0) + 0(1) + 1(0)$

$= 0$ $\quad \therefore \boxed{x_1^{(2)} = 0}$

For $i=2$, $x_2^{(2)} = \sum_{l=0}^{2} g_l^{(2)} \cdot M_{2-l}$.

$= g_0^{(2)} \cdot M_2 + g_1^{(2)} M_1 + g_2^{(2)} M_0$

$= 1(0) + 0(0) + 1(1)$

$= 1$. $\quad \therefore \boxed{x_2^{(2)} = 1}$.

For $i=3$, $x_3^{(2)} = \sum_{i=0}^{2} g_l^{(2)} M_{3-l}$

$= g_0^{(2)} M_3 + g_1^{(2)} M_2 + g_2^{(2)} M_1$

$= 1(1) + 0(0) + 1(0)$

$= 1$ $\quad \therefore \boxed{x_3^{(2)} = 1}$.

For $i=4$, $x_4^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{4-l}$

$$= g_0^{(2)} M_4 + g_1^{(2)} M_3 + g_2^{(2)} M_2$$

$$= 1(1) + 0(1) + 1(0)$$

$$= 1 \qquad \therefore \boxed{x_4^{(2)} = 1}$$

For $i=5$, $x_5^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{5-l}$

$$= g_0^{(2)} M_5 + g_1^{(2)} M_4 + g_2^{(2)} M_3$$

$$= 1(0) + 0(1) + 1(1)$$

$$= 1 \qquad \therefore \boxed{x_5^{(2)} = 1}$$

For $i=6$, $x_6^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{6-l}$

$$= g_0^{(2)} M_6 + g_1^{(2)} M_5 + g_2^{(2)} M_4$$

$$= 1(0) + 0(0) + 1(1)$$

$$= 1 \qquad \therefore \boxed{x_6^{(2)} = 1}$$

$\therefore$ The bottom output sequence

$$\boxed{x_i^{(2)} = (1\ 0\ 1\ 1\ 1\ 1\ 1)}$$

$\therefore$ The Encoded output sequence is given by.

$$\{x_i\} = (x_0^{(1)}\ x_0^{(2)}\ \ x_1^{(1)} x_1^{(2)}\ \ldots\ldots\ x_6^{(1)}\ x_6^{(2)})$$

$$x_i^{(1)} = (1\ 1\ 1\ 1\ 0\ 0\ 1)$$

$$x_i^{(2)} = (1\ 0\ 1\ 1\ 1\ 1\ 1)$$

$$x_i = (11\ 10\ 11\ 11\ 01\ 01\ 11)$$

$\therefore$ The Encoded output Sequence

$$\boxed{x_i = (1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1)}$$

(2) Frequency domain approach :

- Transfer domain response is represented in the form of polynomials.

* The input top adder - output path of the encoder

$$g^{(1)}(D) = g_0^{(1)} + g_1^{(1)} D + g_2^{(1)} D^2 + \cdots\cdots + g_M^{(1)} D^M.$$

* The input - bottom adder - output path of the encoder

$$g^{(2)}(D) = g_0^{(2)} + g_1^{(2)} D + g_2^{(2)} D^2 + \cdots\cdots + g_M^{(2)} D^M.$$

where $g_0^{(1)}, g_1^{(1)}, \cdots g_M^{(1)}$ and $g_0^{(2)}, g_1^{(2)} \cdots g_M^{(2)}$ are the elements of the transfr domain response of the path.

The message polynomial is given as

$$M(D) = M_0 + M_1 D + M_2 D^2 + \cdots\cdots + M_{L-1} D^{L-1}.$$

where L - length of the message sequence.

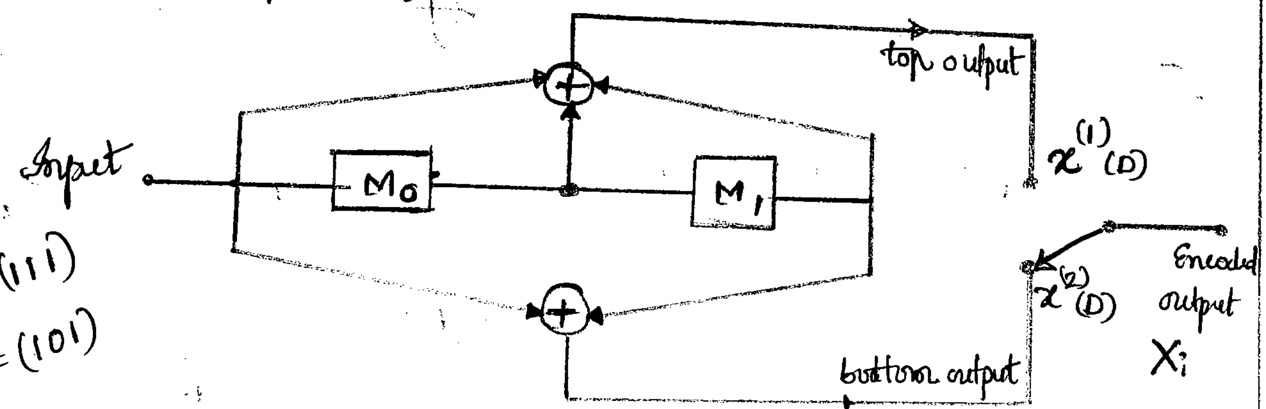
∴ The top-output polynomial is given as

$$\boxed{X^{(1)}(D) = g^{(1)}(D) \cdot M(D)}$$

The bottom output polynomial is given as

$$\boxed{X^{(2)}(D) = g^{(2)}(D) \cdot M(D).}$$

∴ The Encoded output Sequence $X_i = [x^{1}(D)\ x^{(2)}(D) \cdots\cdots]$



let $g_1 = (111)$
$g_2 = (101)$

Problem (2)

A convolutional encoder has a single shift register with two stages (ie Constraint length 3), two X-OR adders and an output multiplexer. The generated polynomials of the encoder as follows $g^{(1)}(D) = 1+D+D^2$ and $g^{(2)}(D) = 1+D^2$.

Draw the block diagram of encoder and find the output of the encoder for an input data stream of (1 0 0 1 1) by using frequency / transfer domain approach.

Sol

Given data $g^{(1)}(D) = 1+D+D^2$

$g^{(2)}(D) = 1+D^2$

Input data stream ie message as $\underset{M_0\ M_1\ M_2\ M_3\ M_4}{1\ 0\ 0\ 1\ 1}$.

Given single shift register $\boxed{K=1}$

Constraint length $N = M+1 = 3 \Rightarrow \boxed{M=2}$

$\therefore$ No-of shift register 2

No-of X-OR adders 2 ; $\boxed{n=2}$

$\boxed{L=5}$, $L+M = 5+2=7$ $\therefore \boxed{L+M=7}$

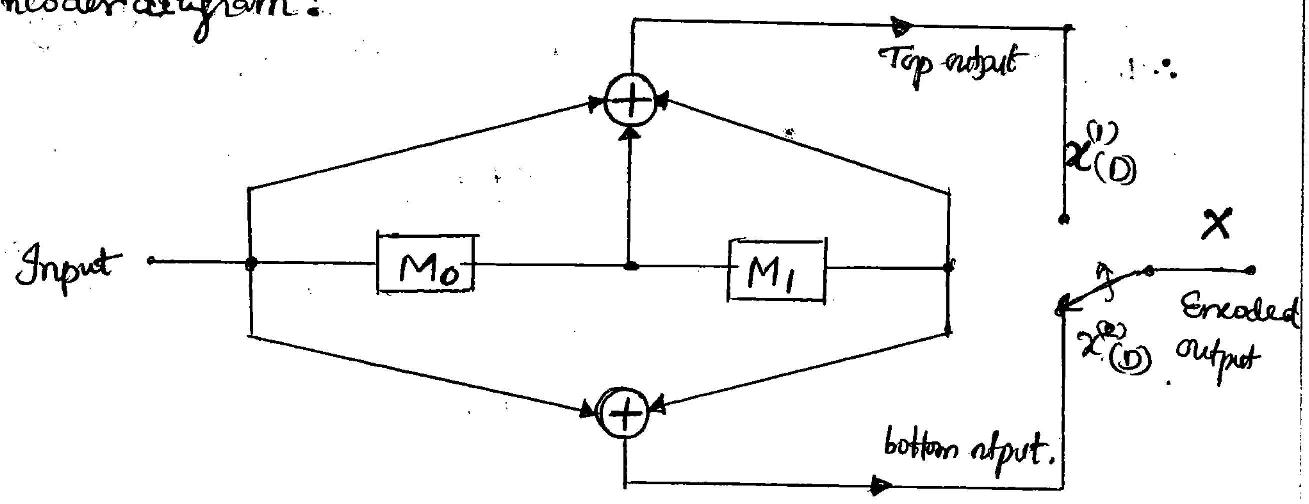
$n(L+M) = 2(7) = 14$ $\therefore \boxed{n(L+M)=14}$

Message polynomial as

$$M(D) = 1\cdot D^0 + 0\cdot D^1 + 0\cdot D^2 + 1\cdot D^3 + 1\cdot D^4$$

$$\therefore \boxed{M(D) = 1+D^3+D^4}$$

Encoder diagram:



✓ The top output polynomial is given as

$$x^{(1)}(D) = g^{(1)}(D) \cdot M(D)$$
$$= (1+D+D^2)(1+D^3+D^4)$$
$$= 1+D^3+D^4+D+D^4+D^5+D^2+D^5+D^6$$
$$= 1+D+D^2+D^3+D^6.$$

(XOR operat
$1+1=0$
$0+0=0$
$0+1=1$
$1+0=1$)

$$\therefore \boxed{X^1(D) = (1\ 1\ 1\ 1\ 0\ 0\ 1)}$$

✓ The bottom output polynomial is given as

$$x^2(D) = g^{(2)}(D) \cdot M(D)$$
$$= (1+D^2)(1+D^3+D^4)$$
$$= 1+D^3+D^4+D^2+D^5+D^6$$
$$= 1+D^2+D^3+D^4+D^5+D^6$$

$$\therefore \boxed{X^{(2)}(D) = (1\ 0\ 1\ 1\ 1\ 1\ 1)}$$

∴ The Encoded output polynomial in bits.

$$X^1(D) = (1\ 1\ 1\ 1\ 0\ 0\ 1)$$
$$X^2(D) = (1\ 0\ 1\ 1\ 1\ 1\ 1)$$

$$\therefore \boxed{X_i = (1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 1\ 1)}$$

HW
3. A convolutional encoder has a single shift register with two stages, 3 modulo-2 adders & an output multiplexer. The generated sequence of the encoder as follows.

$$g_1 = (1\,0\,1), \quad g_2 = (1\,1\,0) \quad \& \quad g_3 = (1\,1\,1)$$

Draw the block diagram of encoder & the output of the encoder using time and frequency domain approach with message data $(1\,0\,0\,1\,1)$.

Sol Given data Input message (10011).

$L = 5 \rightarrow$ no of bits in the input data

$K = 1 \rightarrow$ single shift register.

$M = 2 \rightarrow$ No. of shift registers.

$L + M = 5 + 2 = 7 \rightarrow$ total no of shift required to reset the register.

$n = 3 \rightarrow$ no. of modulo-2 adders.

$n(L+M) = 3(7) = 21 \rightarrow$ Total no of bits in the encoder output sequence.

Transfer / Frequency domain approach:

✓ Message polynomial $M(D) = 1 + D^3 + D^4$

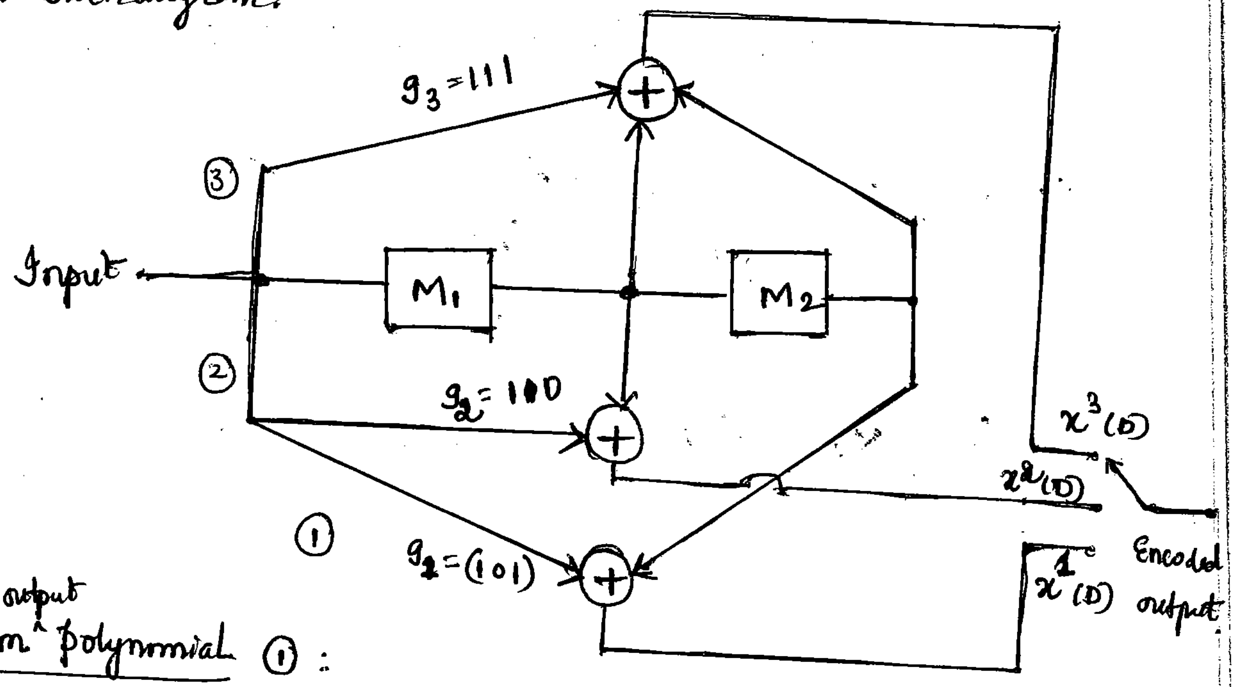
✓ Generated polynomial

$$g^{(1)}(D) = (1\,0\,1) = 1 + D^2$$

$$g^{(2)}(D) = (1\,1\,0) = 1 + D.$$

$$g^{(3)}(D) = (1\,1\,1) = 1 + D + D^2.$$

∴ Encoder block diagram:



The bottom output polynomial ①:

$$X^1(D) = g^{(1)}(D) \cdot M(D)$$

$$= (1 + D^2)(1 + D^3 + D^4)$$

$$= 1 + D^3 + D^4 + D^2 + D^5 + D^6$$

$$= 1 + D^2 + D^3 + D^4 + D^5 + D^6$$

$$\boxed{X^1(D) = (1\;0\;1\;1\;1\;1\;1)}$$

The bottom output polynomial ② :

$$X^{(2)}(D) = g^{(2)}(D) \cdot M(D)$$
$$= (1+D)(1+D^3+D^4)$$
$$= 1+D^3+\cancel{D^4}+D+\cancel{D^4}+D^5$$
$$= 1+D+D^3+D^5$$

$$\boxed{X^{(2)}(D) = 1\ 1\ 0\ 1\ 0\ 1\ 0}$$

The top output polynomial ③ :

$$X^{(3)}(D) = g^{(3)}(D) \cdot M(D)$$
$$= (1+D+D^2)(1+D^3+D^4)$$
$$= 1+D^3+D^4+D+\cancel{D^4}+\cancel{D^5}+D^2+\cancel{D^5}+D^6$$
$$= 1+D+D^2+D^3+D^6.$$

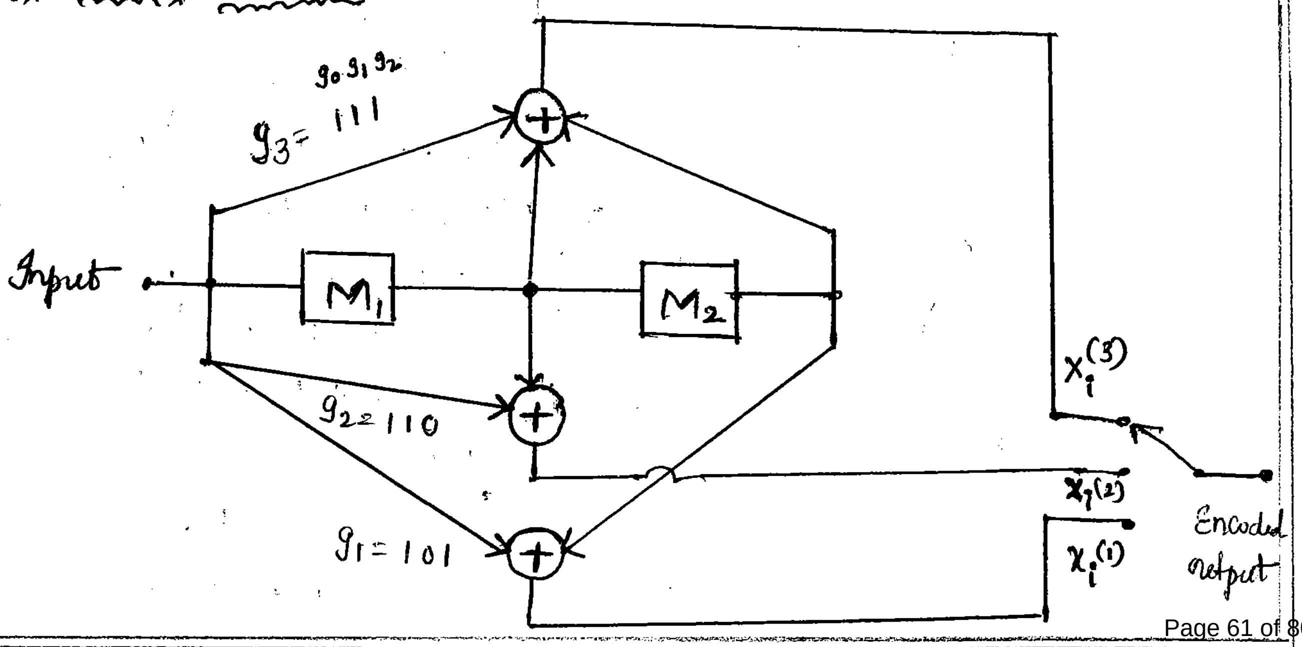
$$\boxed{X^{(3)}(D) = (1\ 1\ 1\ 1\ 0\ 0\ 1)}$$

∴ The Encoder output polynomial is given by $\{X\}$.

$X^1 = (1\ 0\ 1\ 1\ 1\ 1\ 1)$
$X^2 = (1\ 1\ 0\ 1\ 0\ 1\ 0)$
$X^3 = (1\ 1\ 1\ 1\ 0\ 0\ 1)$

$$\therefore \boxed{\{X\} = (1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1)}$$

Time domain approach :



✓ Message bits $\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \end{pmatrix}$
$\quad M_0 \; M_1 \; M_2 \; M_3 \; M_4$

~ Given generated Sequences

$$g_1 = \underset{g_0\; g_1\; g_2}{(1\;0\;1)} \qquad g_2 = \underset{g_0\; g_1\; g_2}{(1\;1\;0)} \;, \quad g_3 = \underset{g_0\; g_1\; g_2}{(1\;1\;1)}$$

✓ The bottom output sequence ① can be generated by using

$$x_i^{(1)} = \sum_{l=0}^{M} g_l^{(1)} M_{i-l} \; ; \quad i = 0, 1, 2, 3, 4, 5, 6.$$

For $i=0$, $x_0^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{-l}$

$$= g_0^{(1)} M_0 + g_1^{(1)} M_{-1} + g_2^{(1)} M_{-2}$$

$M = \underset{M_0\, M_1\, M_2\, M_3\, M_4}{(1\;0\;0\;1\;1)}$

$$= 1(1) + 0(0) + 1(0)$$
$$= 1 \quad \therefore \boxed{x_0^{(1)} = 1}$$

$g_1 = \underset{g_0\, g_1\, g_2}{(1\;0\;1)}$

For $i=1$, $x_1^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{1-l}$

$$= g_0^{(1)} M_1 + g_1^{(1)} M_0 + g_2^{(1)} M_{-1}$$
$$= 1(0) + 0(1) + 1(0) = 0 \quad \therefore \boxed{x_1^{(1)} = 0}$$

For $i=2$, $x_2^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{2-l}$

$$= g_0^{(1)} M_2 + g_1^{(1)} M_1 + g_2^{(1)} M_0$$
$$= 1(0) + 0(0) + 1(1) = 1 \quad \therefore \boxed{x_2^{(1)} = 1}$$

for $i=3$, $x_3^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{3-l}$

$$= g_0^{(1)} M_3 + g_1^{(1)} M_2 + g_2^{(1)} M_1$$
$$= 1(1) + 0(0) + 1(0) = 1 \quad \therefore \boxed{x_3^{(1)} = 1}$$

For $i=4$, $x_4^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{4-l}$

$$= g_0^{(1)} M_4 + g_1^{(1)} M_3 + g_2^{(1)} M_2$$
$$= 1(1) + 0(1) + 1(0) = 1 \quad \therefore \boxed{x_4^{(1)} = 1}$$

for $i=5$, $x_5^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{5-l}$

$$= g_0^{(1)} M_5 + g_1^{(1)} M_4 + g_2^{(1)} M_3$$
$$= 1(0) + 0(1) + 1(1) = 1 \quad \therefore \boxed{x_5^{(1)} = 1}$$

for $i=6$, $x_6^{(1)} = \sum_{l=0}^{2} g_l^{(1)} M_{6-l}$

$$= g_0^{(1)} M_6 + g_1^{(1)} M_5 + g_2^{(1)} M_4$$
$$= 1(0) + 0(0) + 1(1) = 1 \quad \therefore \boxed{x_6^{(1)} = 1}$$

The bottom output Sequence ② can be generated by using

$$x_i^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{i-l} \quad ; \; i = 0, 1, 2, 3, 4, 5, 6.$$

$$\begin{array}{c} M_0 M_1 M_2 M_3 M_4 \\ M = 1\;0\;0\;1\;1 \end{array} \qquad \begin{array}{c} g_0\; g_1\; g_2 \\ g_2 = 1\;1\;0 \end{array}$$

For $i=0$

$$x_0^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{-l}$$
$$= g_0^{(2)} M_0 + g_1^{(2)} M_{-1} + g_2^{(2)} M_{-2}$$
$$= 1(1) + 1(0) + 0(0) = 1 \quad \therefore \boxed{x_0^{(2)} = 1}$$

For $i=1$

$$x_1^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{1-l}$$
$$= g_0^{(2)} M_1 + g_1^{(2)} M_0 + g_2^{(2)} M_{-1}$$
$$= 1(0) + 1(1) + 0(0) = 1 \quad \therefore \boxed{x_1^{(2)} = \_\_}$$

For $i=2$,

$$x_2^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{2-l}$$
$$= g_0^{(2)} M_2 + g_1^{(2)} M_1 + g_2^{(2)} M_0$$
$$= 1(0) + 1(0) + 0(1) = 0 \quad \therefore \boxed{x_2^{(2)} = 0}$$

For $i=3$,

$$x_3^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{3-l}.$$
$$= g_0^{(2)} M_3 + g_1^{(2)} M_2 + g_2^{(2)} M_1$$
$$= 1(1) + 1(0) + 0(0) = 1 \quad \therefore \boxed{x_3^{(2)} = 1}$$

For $i=4$,

$$x_4^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{4-l}$$
$$= g_0^{(2)} M_4 + g_1^{(2)} M_3 + g_2^{(2)} M_2$$
$$= 1(1) + 1(1) + 0(0) = 0 \quad \therefore \boxed{x_4^{(2)} = 0}$$

For $i=5$,

$$x_5^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{5-l}$$
$$= g_0^{(2)} M_5 + g_1^{(2)} M_4 + g_2^{(2)} M_3$$
$$= 1(0) + 1(1) + 0(1) = 1 \quad \therefore \boxed{x_5^{(2)} = 1}$$

For $i=6$,

$$x_6^{(2)} = \sum_{l=0}^{2} g_l^{(2)} M_{6-l}$$
$$= g_0^{(2)} M_6 + g_1^{(2)} M_5 + g_2^{(2)} M_4$$
$$= 1(0) + 1(0) + 0(1) = 0 \quad \therefore \boxed{x_6^{(2)} = 0}$$

∴ The bottom output sequences ① & ②

$$\boxed{x_i^{(1)} = (1\;0\;1\;1\;1\;1\;1)}$$
$$\boxed{x_i^{(2)} = (1\;1\;0\;1\;0\;1\;0)}$$

∴ The top output sequence ③ can be generated by using

$$x_i^{(3)} = \sum_{l=0}^{M} g_l^{(3)} M_{i-l}, \quad i = 0, 1, 2, 3, 4, 5, 6.$$

$M_0\ M_1\ M_2\ M_3\ M_4$

$M = (1\ 0\ 0\ 1\ 1)$

$g_3 = (1\ 1\ 1)$
$\quad\ \ g_0\ g_1\ g_2$

For $i=0$, $x_0^{(3)} = \sum_{l=0}^{2} g_l^{(3)} M_{-l}$

$= g_0^{(3)} M_0 + g_1^{(3)} M_{-1} + g_2^{(3)} M_{-2}$

$= 1(1) + 1(0) + 1(0) = 1$ $\quad \therefore \boxed{x_0^{(3)} = 1}$

For $i=1$, $x_1^{(3)} = \sum_{l=0}^{2} g_l^{(3)} M_{1-l}$

$= g_0^{(3)} M_1 + g_1^{(3)} M_0 + g_2^{(3)} M_{-1}$

$= 1(0) + 1(1) + 1(0) = 1$ $\quad \therefore \boxed{x_1^{(3)} = 1}$

For $i=2$, $x_2^{(3)} = \sum_{l=0}^{2} g_l^{(3)} M_{2-l}$

$= g_0^{(3)} M_2 + g_1^{(3)} M_1 + g_2^{(3)} M_0$

$= 1(0) + 1(0) + 1(1) = 1$ $\quad \therefore \boxed{x_2^{(3)} = 1}$

For $i=3$, $x_3^{(3)} = \sum_{l=0}^{2} g_l^{(3)} M_{3-l}$

$= g_0^{(3)} M_3 + g_1^{(3)} M_2 + g_2^{(3)} M_1$

$= 1(1) + 1(0) + 1(0) = 1$ $\quad \therefore \boxed{x_3^{(3)} = 1}$

For $i=4$, $x_4^{(3)} = \sum_{l=0}^{2} g_l^{(3)} M_{4-l}$

$= g_0^{(3)} M_4 + g_1^{(3)} M_3 + g_2^{(3)} M_2$

$= 1(1) + 1(1) + 1(0) = 0$ $\quad \therefore \boxed{x_4^{(3)} = 0}$

For $i=5$, $x_5^{(3)} = \sum_{l=0}^{2} g_l^{(3)} M_{5-l}$

$= g_0^{(3)} M_5 + g_1^{(3)} M_4 + g_2^{(3)} M_3$

$= 1(0) + 1(1) + 1(1) = 0$ $\quad \therefore \boxed{x_5^{(3)} = 0}$

For $i=6$, $x_6^{(3)} = \sum_{l=0}^{2} g_l^{(3)} M_{6-l}$

$= g_0^{(3)} M_6 + g_1^{(3)} M_5 + g_2^{(3)} M_4$

$= 1(0) + 1(0) + 1(1) = 1$ $\quad \therefore \boxed{x_6^{(3)} = 1}$

$\therefore$ The top output sequence ③ as

$$\therefore \boxed{x_i^{(3)} = (1\ 1\ 1\ 1\ 0\ 0\ 1)}$$

$\therefore$ $x_i^{(1)} = (1\ 0\ 1\ 1\ 1\ 1\ 1)$

$x_i^{(2)} = (1\ 1\ 0\ 1\ 0\ 1\ 0)$

$x_i^{(3)} = (1\ 1\ 1\ 1\ 0\ 0\ 1)$

$\therefore$ The encoder output sequence can be generated by using

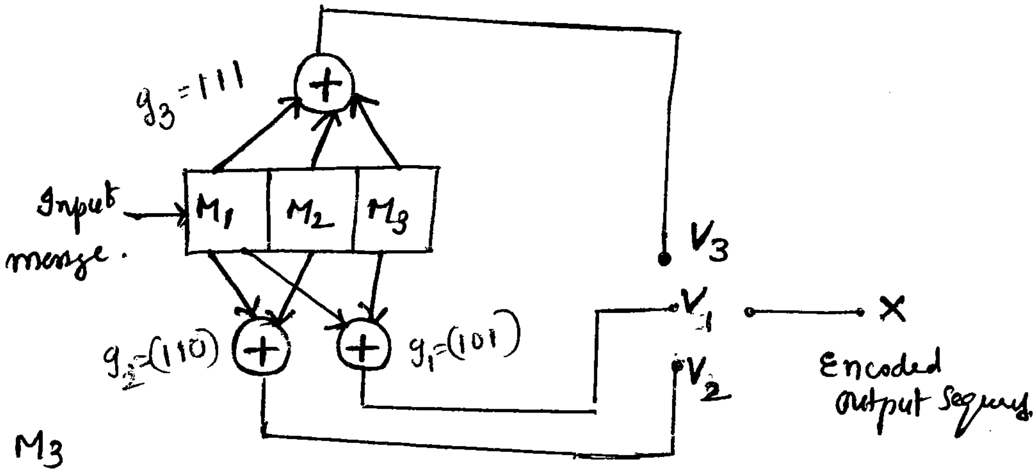
$$X = \{ x_0^{(1)} x_0^{(2)} x_0^{(3)}\ x_1^{(1)} x_1^{(2)} x_1^{(3)} \ldots\ldots x_6^{(1)} x_6^{(2)} x_6^{(3)} \}.$$

$$\therefore \boxed{X = (1\ 1\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 1\ 0\ 1)}$$

II. Method:

$g_1 = (1\,0\,1),\ g_2 = (1\,1\,0),\ g_3 = (1\,1\,1)$.

Message Sequence $(1\,0\,0\,1\,1)$.



$\therefore$ $V_1 = M_1 \oplus M_3$

$V_2 = M_1 \oplus M_2$

$V_3 = M_1 \oplus M_2 \oplus M_3$.

Shift No	Input M	Shift registers M_1	M_2	M_3	Output sequences $V_1 = M_1 \oplus M_3$	$V_2 = M_1 \oplus M_2$	$V_3 = M_1 \oplus M_2 \oplus M_3$
-	-	0	0	0	0	0	0
1	1	1	0	0	1	1	1
2	0	0	1	0	0	1	1
3	0	0	0	1	1	0	1
4	1	1	0	0	1	1	1
5	1	1	1	0	1	0	0
6	0	0	1	1	1	1	0
7	0	0	0	1	1	0	1

$\therefore$ $v_1 = (1\,0\,1\,1\,1\,1\,1)$

$V_2 = (1\,1\,0\,1\,0\,1\,0)$

$V_3 = (1\,1\,1\,1\,0\,0\,1)$.

$\therefore$ The Encoded output $\boxed{X = (111\,011\,101\,111\,100\,110\,101)}$

HW:

④ Consider the code rate is 1/2, Constraint length '2', Find encoded output sequence for the given input data $(1\,0\,1\,1\,1)$ & $g_1 = (1\,1)$, $g_2 = (0\,0)$, using time & transfer domain approach methods. $\therefore$ $x_1 = (1\,1\,1\,0\,0\,1)$

$x_2 = (0\,0\,0\,0\,0\,0)$

$\therefore$ $X = (1\,0\,1\,0\,1\,0\,0\,0\,0\,0\,1\,0)$.

***

(5) A (2,1,3) Convolutional Code is described by

$$g_1 = (1\,1\,1) \quad \& \quad g_2 = (1\,0\,1)$$

(a) Draw the Encoder diagram for this code.

(b) Find the encoded output sequence for an input message sequence 1 1 0 1 1.

(c) Draw the Code tree diagram for this code and find the encoded output sequence for the given message sequence.

(d) Draw the State diagram & find encoded output sequence for the given message sequence.

(e) Draw the trellis diagram & find encoder output sequence for the given message sequence.

Sol. Given (n, K, M) convolutional code $\rightarrow (2, 1, 3)$.

$\therefore$ $n = 2 \rightarrow$ no. of X-OR adders.

$K = 1 \rightarrow$ At a time no. of bits given to the register.

$M = 3 \rightarrow$ no. of shift registers.

Given message sequence (1 1 0 1 1).

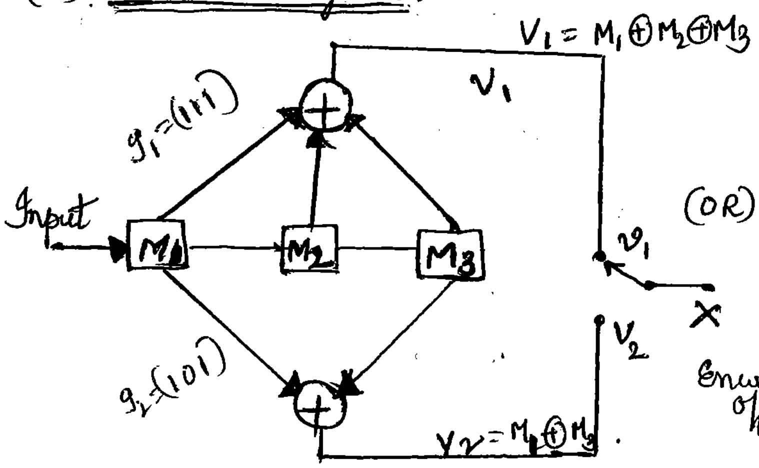
$\therefore$ $L = 5 \rightarrow$ no. of bits present in message sequence.

$L+M = 5+3 = 8 \rightarrow$ no. of shift required to reset the registers.

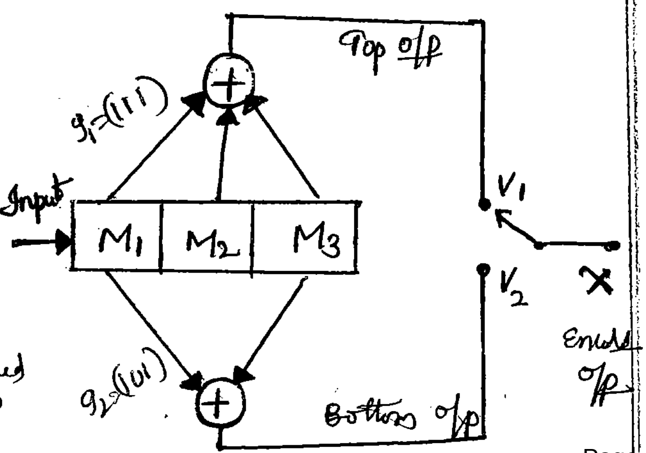
$n(L+M) = 2(8) = 16 \rightarrow$ Total no. of bits in the encoded output sequence.

Given $g_1 = (1\,1\,1)$ & $g_2 = (1\,0\,1)$.

(a) Encoder diagram:



(OR)



(b) <u>Encoded Output Sequence:</u>

Given message '11011'.

Shift	Input	Registers			top & bottom output sequences	
		M_1	M_2	M_3	$V_1 = M_1 \oplus M_2 + M_3$	$V_2 = M_1 \oplus M_3$
–	–	0	0	0	0	0
1	1	1	0	0	1	1
2	1	1	1	0	0	1
3	0	0	1	1	0	1
4	1	1	0	1	0	0
5	1	1	1	0	0	1
6	0	0	1	1	0	1
7	0	0	0	1	1	1
8	0	0	0	0	0	0

∴ The top output sequence $v_1 = (1\,0\,0\,0\,0\,0\,1\,0)$

The bottom output sequence $v_2 = (1\,1\,1\,0\,1\,1\,1\,0)$

∴ The Encoded output Sequence is given by:

$$\boxed{X = (1\,1\,0\,1\;\;0\,1\,0\,0\;\;0\,1\,0\,1\;\;1\,1\,0\,0)}$$

(OR) <u>Frequency domain approach:</u>

Message 11011 ∴ $M(D) = 1 + D + D^3 + D^4$

$g_1 = (1\,1\,1)$ $\quad g^{(1)}(D) = 1 + D + D^2$

$g_2 = (1\,0\,1)$ $\quad g^{(2)}(D) = 1 + D^2$

∴ Top o/p Sequence is

$$x^{(1)}(D) = g^{(1)}(D) \cdot M(D)$$

$$= (1+D+D^2)(1+D+D^3+D^4)$$

$$= 1 + \cancel{D} + \cancel{D^3} + \cancel{D^4} + \cancel{D} + \cancel{D^2} + \cancel{D^4} + \cancel{D^5} + \cancel{D^2} + \cancel{D^3} + \cancel{D^5} + D^6$$

$$= 1 + D^6$$

$$x^1(D) = (1\,0\,0\,0\,0\,0\,1\,0)$$

Bottom o/p Sequence

$$x^{(2)}(D) = g^{(2)}(D) \cdot M(D)$$

$$= (1+D^2)(1+D+D^3+D^4)$$

$$= 1 + D + \cancel{D^3} + D^4 + D^2 + \cancel{D^3} + D^5 + D^6$$

$$= 1 + D + D^2 + D^4 + D^5 + D^6$$

$$x^{(2)}(D) = (1\,1\,1\,0\,1\,1\,1\,0)$$

∴ O/p Encoded sequence $\boxed{X = (1\,1\,0\,1\,0\,1\,0\,0\,0\,1\,0\,1\,1\,1\,0\,0)}$

(c) Code tree diagram:

The number of states is given by $2^{K(M-1)}$

where K – At a time no. of bits given to the register
M – no. of shift registers.

$(n, K, M) = (2, 1, 3)$

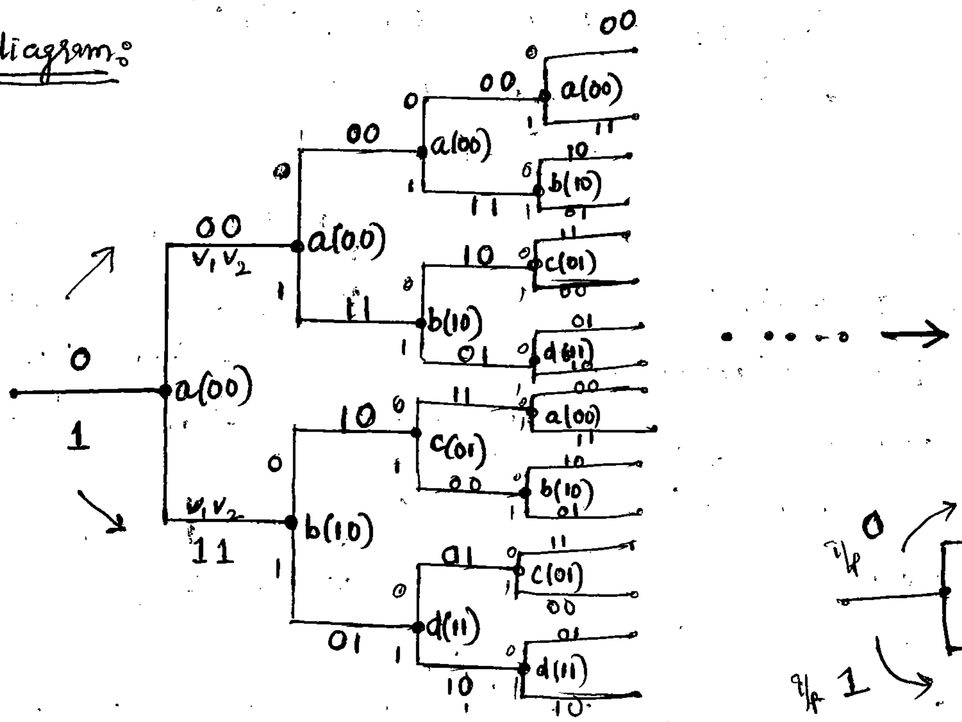
∴ no. of states $2^{1(3-1)} = 2^2 = 4$ ie 00, 01, 10, 11

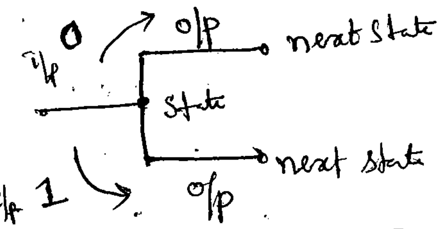
ie two states S_1 and S_2.

Let $a = 00$, $b = 10$, $c = 01$, $d = 11$.

Input M_1	Present States S_1 M_2	S_2 M_3		Outputs $V_1 = M_1 \oplus M_2 \oplus M_3$	$V_2 = M_1 \oplus M_3$	Next states S_1 M_1	S_2 M_2	
0	0	0	a	0	0	0	0	a
1	0	0	a	1	1	1	0	b
0	0	1	c	1	1	0	0	a
1	0	1	c	0	0	1	0	b
0	1	0	b	1	0	0	1	c
1	1	0	b	0	1	1	1	d
0	1	1	d	0	1	0	1	c
1	1	1	d	1	0	1	1	d

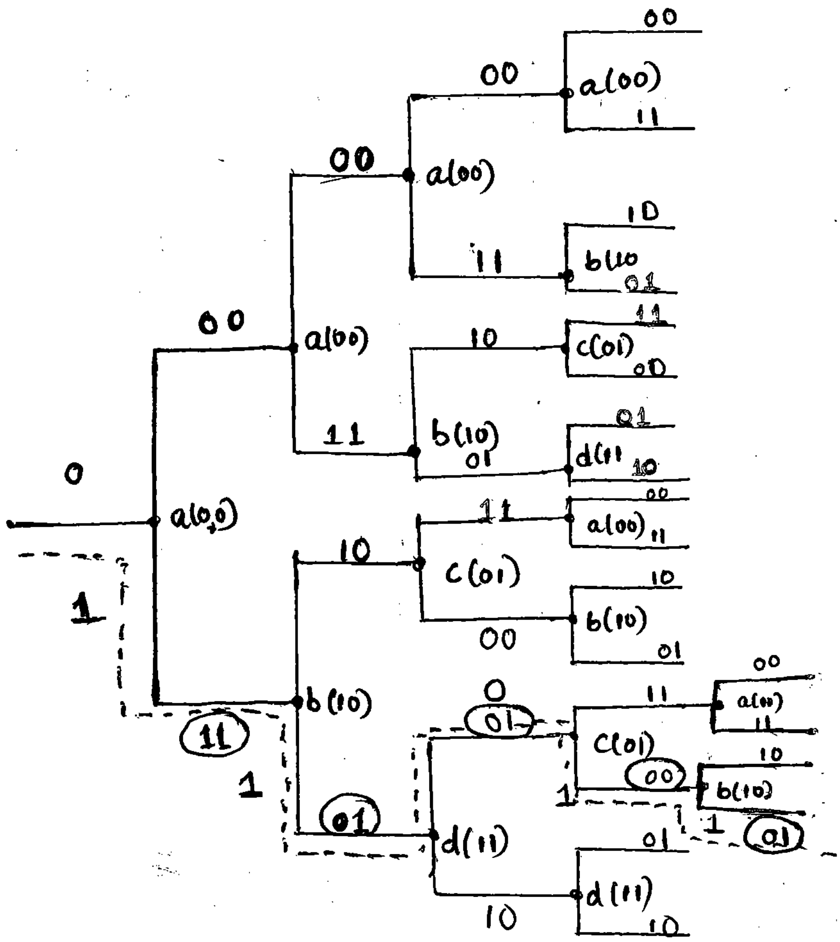
Tree diagram:





The given message sequence 1 1 0 1 1.

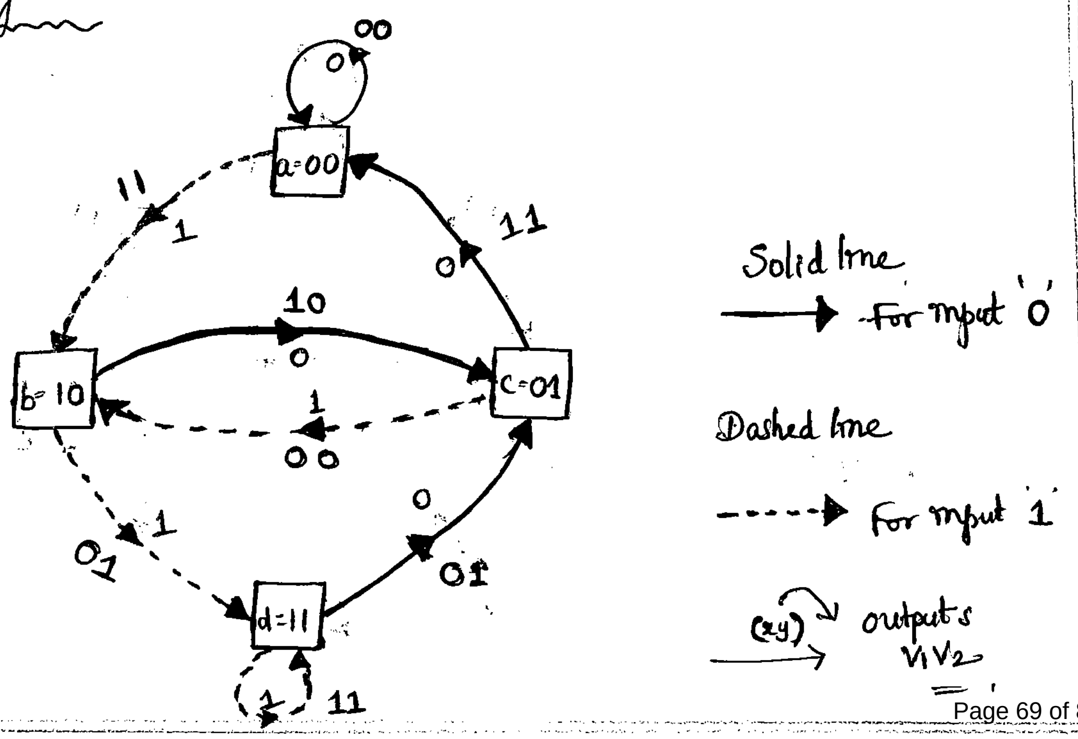
The tree diagram



The Encoded output for the given message 1 1 0 1 1.

$$\boxed{(1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 0\ 1)}$$

(d) State diagram:



For the given message sequence 1 1 0 1 1, the state table is given by.

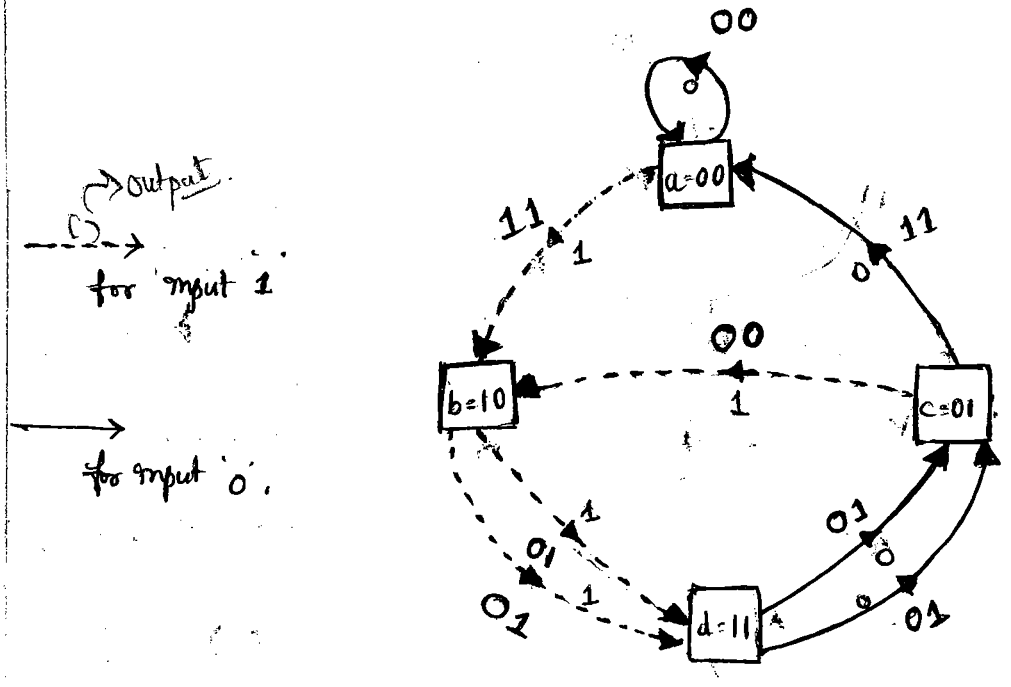
	Input	Registers $M_1\ M_2\ M_3$	Present States $M_2\ M_3$	Future/next state $M_1\ M_2$	outputs $V_1 = M_1 \oplus M_2 \oplus M_3$	$V_2 = M_1 + M_3$
	-	0 0 0	0 0	0 0	0	0
1.	1	1 0 0	0 0 a	1 0 b	1	1
2.	1	1 1 0	1 0 b	1 1 d	0	1
3.	0	0 1 1	1 1 d	0 1 c	0	1
4.	1	1 0 1	0 1 c	1 0 b	0	0
5.	1	1 1 0	1 0 b	1 1 d	0	1
6.	0	0 1 1	1 1 d	0 1 c	0	1
7.	0	0 0 1	0 1 c	0 0 a	1	1
8.	0	0 0 0	0 0 a	0 0 a	0	0

The Encoded output for the given message sequence is given by

(11 01 01 00 01 01 11 00).

<u>State diagram</u> for given message sequence.

from above state stable.



(e) Trellis diagram:

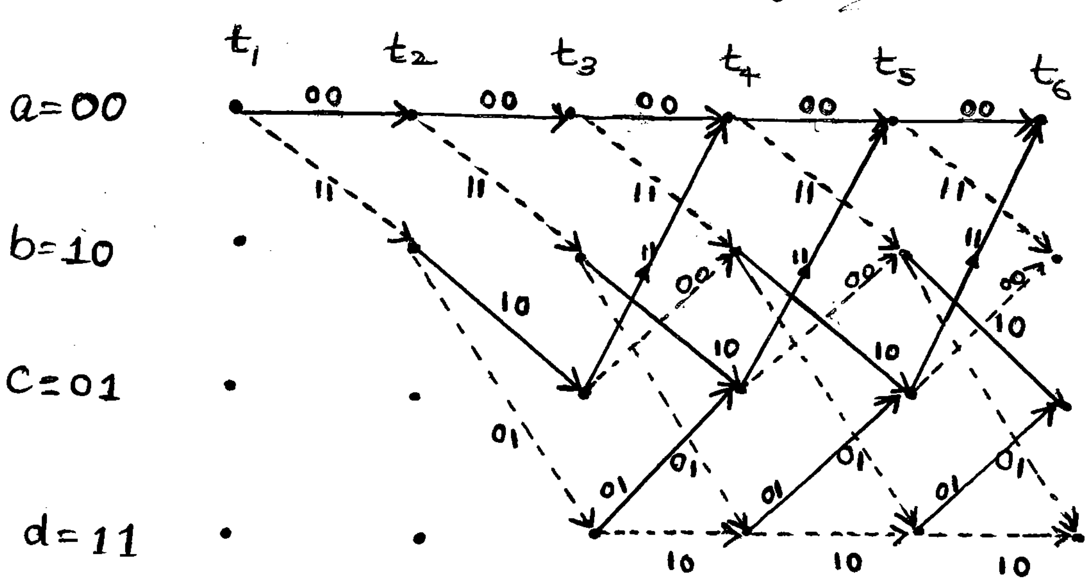
The number of time axis length is given as

$$t_1, t_2, t_3, \ldots\ldots t_{L+1}.$$

where L - no-of bits present in the message sequence.

Trellis diagram can be obtained from state diagram.

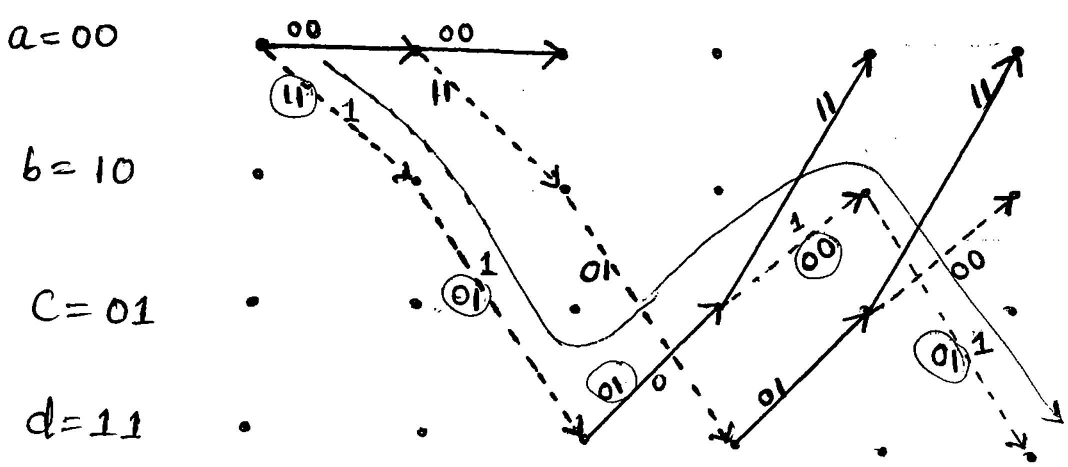
$$2^{K(M-1)} = 2^{1(3-1)} = 2^2 = 4 \text{ stages.}$$



Solide line : ⟶ The output generated by an input bit '0'

Dashed line : - - - -▶ The output generated by an input bit '1'.

For the given message sequence 11011 the trellis diagram using state diagram for the message sequence.



∴ The Encoded output sequence : (11 01 01 00 01)

Decoding Methods of Convolutional Codes :

There are two methods for decoding of convolutional codes

1. Viterbi decoding.
2. Sequential decoding.

(1) Viterbi Decoding :

✓ The viterbi algorithm are currently used in about one million cellphones, which is probably the largest numbers in any application.

✓ The largest current consumer of viterbi algorithm processor cycle is probably digital video broadcasting.

✓ Now being decoded by the viterbi algorithm in digital TV sets around the world every second of every day.

✓ Now a days they are also used in bluetooth implementations.

✓ Viterbi algorithm essentially performs max-likelihood decoding.

* The algorithm involves calculating a measure of similarity or distance between the received signal at time 't' and all the trellis paths entering each state at time 't'.

✓ When two paths enter the same state, the one having the best metric is choosen. This path is called the "Serviving Path".

* The hamming distance between two code vectors is called "Metric".

ie The smallest path from transmitter to receiver is called "metric path" (or) "Serviving path".

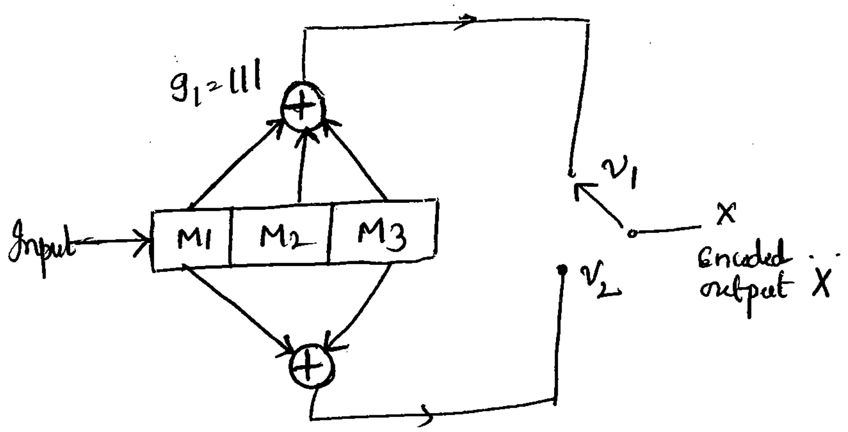
(1) A (2,1,3) Convolutional Code is described by $g_1 = (111)$ & $g_2 = (101)$. The output of the detector is '1101011001' using the viterbi algorithm. Find the transmitted data?

Sol Given $(2,1,3) = (n,K,M)$ Convolutional Code.

$n = 2, \quad K = 1, \quad M = 3.$

Given $g_1 = (111)$, $g_2 = (101)$

The encoder diagram as



$v_1 = M_1 \oplus M_2 \oplus M_3$

$v_2 = M_1 \oplus M_3.$

The state stable is given by.

The no. of states $2^{K(M-1)} = 2^2 = 4$ states

let $a = 00$, $b = 10$, $c = 01$, $d = 11$.

In general.

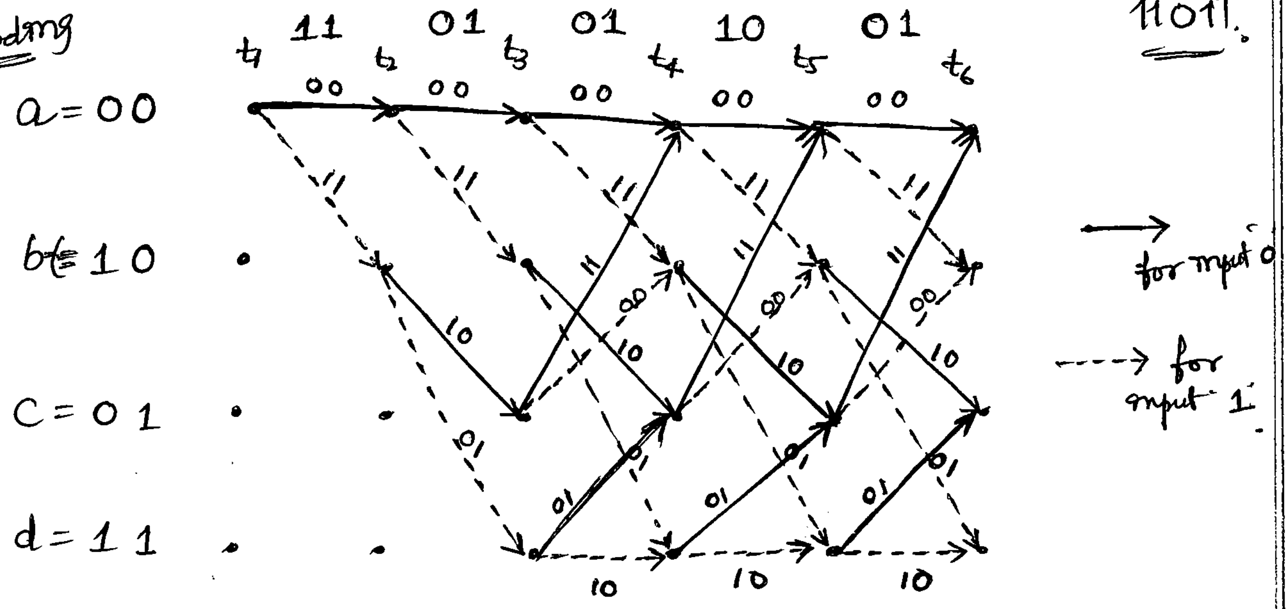
Input M_1	Present states M_2 M_3	Next states M_1 M_2	Outputs $v_1 = M_1 \oplus M_2 \oplus M_3$	$v_2 = M_1 \oplus M_3$.
0	0 0 a	0 0 a	0	0
1	0 0 a	1 0 b	1	1
0	0 1 c	0 0 a	1	1
1	0 1 c	1 0 b	0	0
0	1 0 b	0 1 c	1	0
1	1 0 b	1 1 d	0	1
0	1 1 d	0 1 c	0	1
1	1 1 d	1 1 d	1	0

(PTO)

Viterbi decoding can be obtained by using trellis diagram.

Trellis diagram: Let the no of time axis $t_1, t_2, \ldots t_{L+1}$. i/p 11011.

Encoding

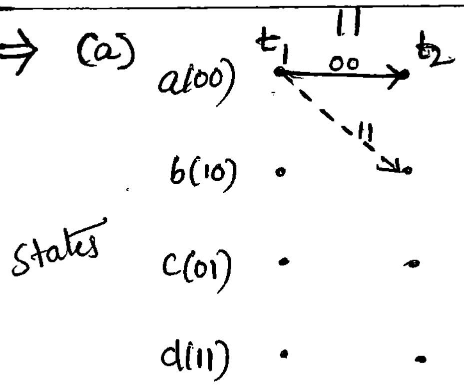


Viterbi decoding:

The output of the decoder is 11 01 01 10 01

The smallest path from transmitter to receiver is metric path.

⇒ (a)



$\gamma_a = 2$

$\gamma_b = 0$

State metric path.

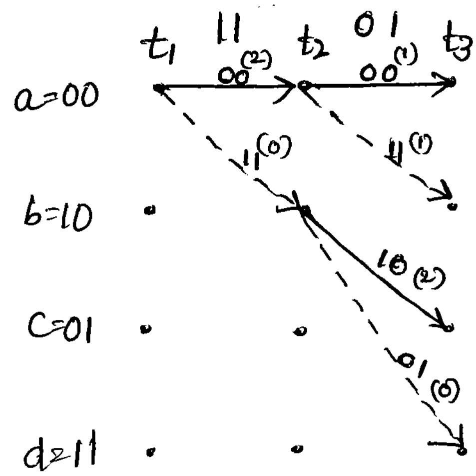
(∵ $\gamma_a \to$ hamming distance

$\begin{matrix}11\\ \downarrow\downarrow \\ 00\end{matrix} \to \gamma_a = 2$

$\gamma_b \to \begin{matrix}11\\ \downarrow\downarrow \\ 11\end{matrix} \to 0 \Rightarrow \gamma_b = 0$.

Fig (a) Survivors at t_2

⇒ (b)



$\gamma_a = 2 + 1 = 3$

$\gamma_b = 2 + 1 = 3$

$\gamma_c = 0 + 2 = 2$

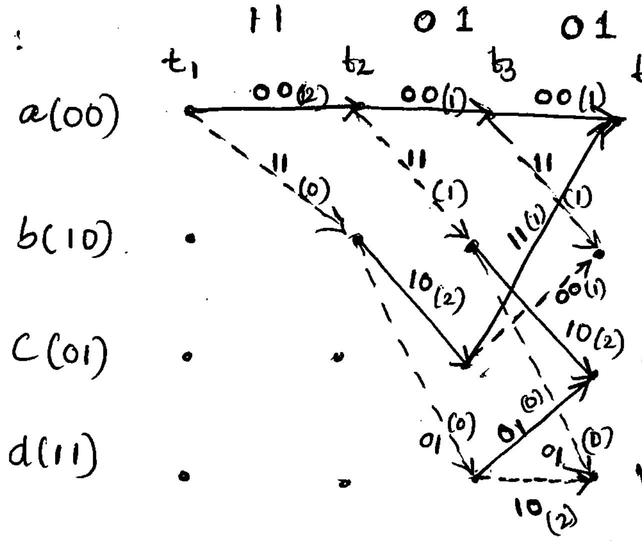
$\gamma_d = 0 + 0 = 0$

Fig: Survivors at t_3.

(38)

(c) :

⇒



$\gamma_a = 2+1+1 = 4$ (or)
$0+2+1 = 3$ ✓

$\gamma_b = 2+1+1 = 4$ (or)
$0+2+1 = 3$ ✓

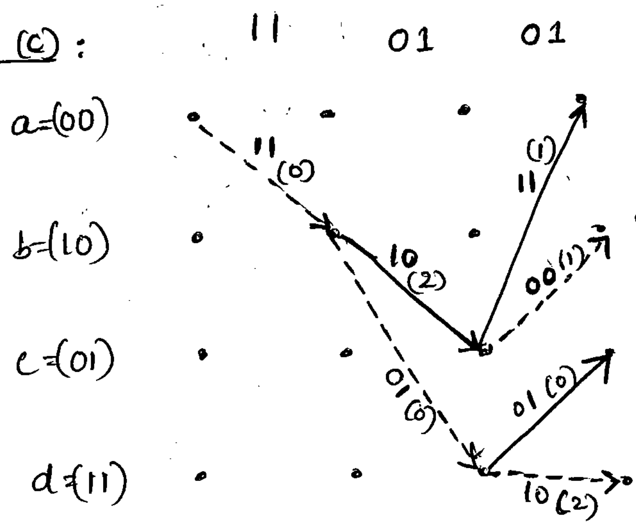
$\gamma_c = 2+1+2 = 5$ (or)
$= 0+0+0 = 0$ ✓

$\gamma_d = 2+1+0 = 3$
$= 0+0+2 = 2$ ✓

Fig: Metric components/comparison at t_4.

∴ Modified (c) :

⇒



$\gamma_a = 0 + 2+1 = 3$

$\gamma_b = 0+2+1 = 3$

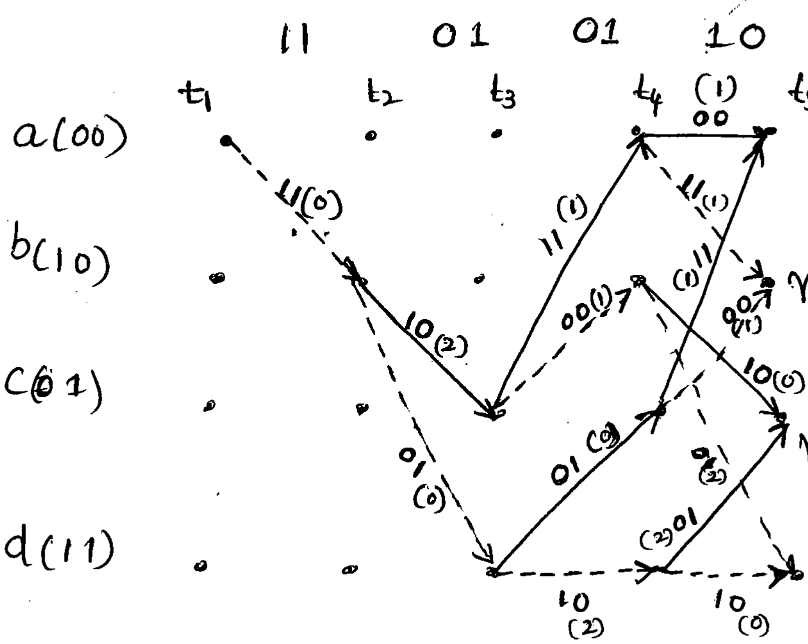
$\gamma_c = 0+0+0 = 0$

$\gamma_d = 0+0+2 = 2$

Fig: Survivors at t_4.

(d) :

⇒



$\gamma_a = 0+2+1+1 = 5$
(or) $0+0+0+1 = 1$ ✓

$\gamma_b = 0+2+1+1 = 4$
(or) $0+0+0+1 = 1$ ✓

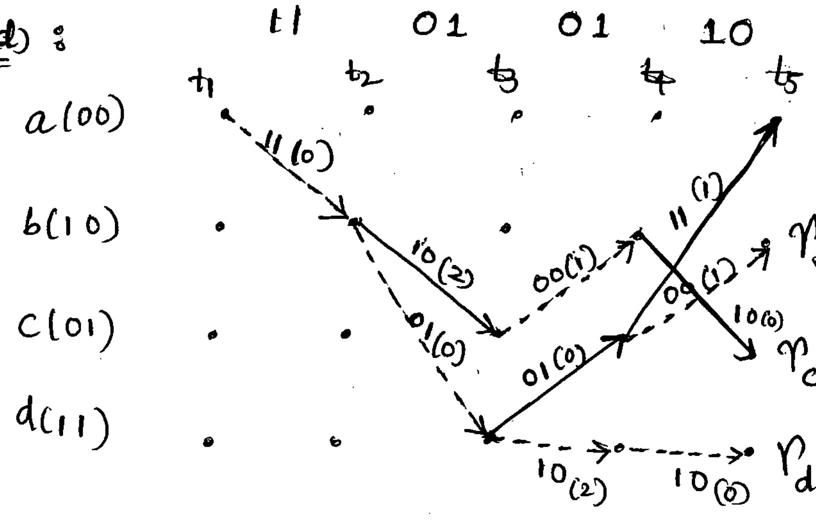
$\gamma_c = 0+2+1+0 = 3$ ✓
(or) $0+0+2+2 = 4$

$\gamma_d = 0+2+1+2 = 5$
(or) $0+0+2+0 = 2$ ✓

fig: Metric Component / Comparison at 't_5'

Modified (d) :

⇒



metric path

$\gamma_a = 0+0+0+1 = 1$

$\gamma_b = 0+0+0+1 = 1$

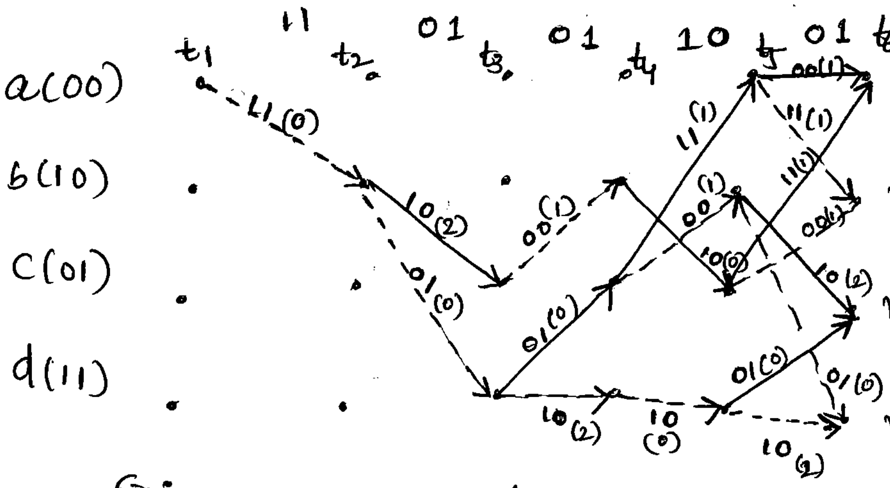
$\gamma_c = 0+2+1+0 = 3$

$\gamma_d = 0+0+2+0 = 2$

Fig : Survivors at t_5.

(e).

⇒



$\gamma_a = 0+0+0+1+1 = 2$ ✓
(or) $0+2+1+0+1 = 4$

$\gamma_b = 0+2+1+0+1 = 4$
(or) $0+0+0+1+1 = 2$ ✓

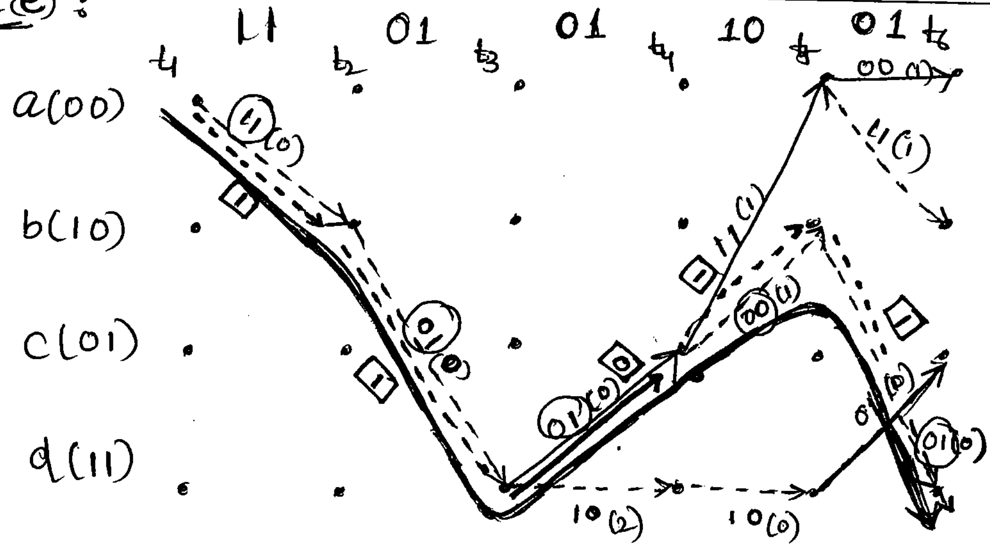
$\gamma_c = 0+0+0+1+2 = 3$
(or) $0+0+2+0+0 = 2$ ✓

$\gamma_d = 0+0+0+1+0 = 1$ ✓
(or) $0+0+2+0+2 = 4$

Fig: Metric Components / Comparison at t_6

Modified (e) :

⇒



$\gamma_a = 2$

$\gamma_b = 2$

$\gamma_c = 2$

$\gamma_d = 1$

Shortest path

Fig: Survivors at t_6.

✓ Among four state metric path values $\gamma_d = 1$ as a minimum length. So, The path from t_1 to t_6 through γ_d is selected for decoding the message.

Message **<u>11011</u>**

∴ Transmitted Data : 1 1 0 1 0 1 0 0 0 1
Received Data : 1 1 0 1 0 1 1 0 0 1

Message 11011

(39)

② Sequential Decoding :

- ✓ Viterbi decoding algorithm for a Convolutional code is maximum likelihood, the error performance of the algorithm over a discrete memoryless Channel is Optimum.
- ✓ The Computational complexity of the algorithm is large.
- ✓ The constraint length 'N=M+1' is limited, so error correcting capability of the code is restricted.

Hence a decoding algorithm that avoids computing the likeli-hood & metric, of every path in the trellis, there by reducing Computational complexity and allowing the constraint length 'N' to take on very large values. there is a suboptimum class of such algorithm known as "<u>Sequential decoding algorithm</u>".

- ✓ The complexity of a sequential decoder is essentially ~~in~~ indepen-dent of the constraint length 'N' so that very large values of 'N' can be employed.
- ✓ Although sequential decoding algorithms are not quite as good as maximum likelihood decoding / viterbi decoding algorithms.

* Sequential decoding is an intuitive trial-and-error technique for searching out the correct path in a Code tree.

- ✓ During the search, the decoder moves forward & backward in the code tree, one node at a time.
- ✓ The decision whether to move forward & backward is determined by the manner in which the metric of the algorithm varies along the path followed by the decoder.

(a) Fano Matric

(b) Fano Algorithm.

(a) Fano Metric :

Let us consider constraint length N
Code rate $r = K/n$

convolutional code over a discrete memory less channel.

let $c_{ij} \rightarrow$ i^{th} bit of binary label on the j^{th} branch of the code tree.

let $y_{ij} \rightarrow$ The corresponding bit of received sequence at the channel output.

∴ The bit metric is defined by

$$y_{ij} = \log_2 \left(\frac{p(y_{ij}/c_{ij})}{p(y_{ij})} \right) - r \longrightarrow (1)$$

where $p(y_{ij}/c_{ij}) \rightarrow$ a transition probability of the channel.

$p(y_{ij}) \rightarrow$ The ~~nominal~~ (nominal) probability of the channel output.

If the binary label on each branch of the code tree has 'n' bits.

Hence bit metric becomes branch metric as

$$\therefore \quad y_j = \sum_{i=1}^{n} y_{ij}$$

$$y_j = \sum_{i=1}^{n} \left[\log_2 \left(\frac{p(y_{ij}/c_{ij})}{p(y_{ij})} \right) - r \right] \longrightarrow (2)$$

The Sequential decoder has followed a path with 'l' branches, starting from the origin of the code tree.

∴ The path metric (or) 'Fano metric' as

$$\gamma(l) = \sum_{j=1}^{l} y_j$$

$$\boxed{\gamma(l) = \sum_{i=1}^{n} \sum_{j=1}^{l} \left[\log_2 \left[\frac{p(y_{ij}/c_{ij})}{p(y_{ij})} \right] - r \right]} \longrightarrow (3)$$

where $l = 1, 2, \ldots\ldots$

(b) Fano Algorithm :

✓ In this the decoder moves forward or backward through the code tree always one node at a time.

✓ The decision whether or not the decoder favours a node is made by comparing the path metric at a node with a →

running threshold maintained by the decoder.

* The running threshold 'T' is defined as an integer multiple of the threshold spacing 'Δ', which is a design parameter.

- when 'Δ' is too small, the decoder frequently back tracks even when it is on the correct path.

- when 'Δ' is too large, the decoder incorrect turns are not identified quickly.

Thus a compromise choice of 'Δ' is required to minimize computation.

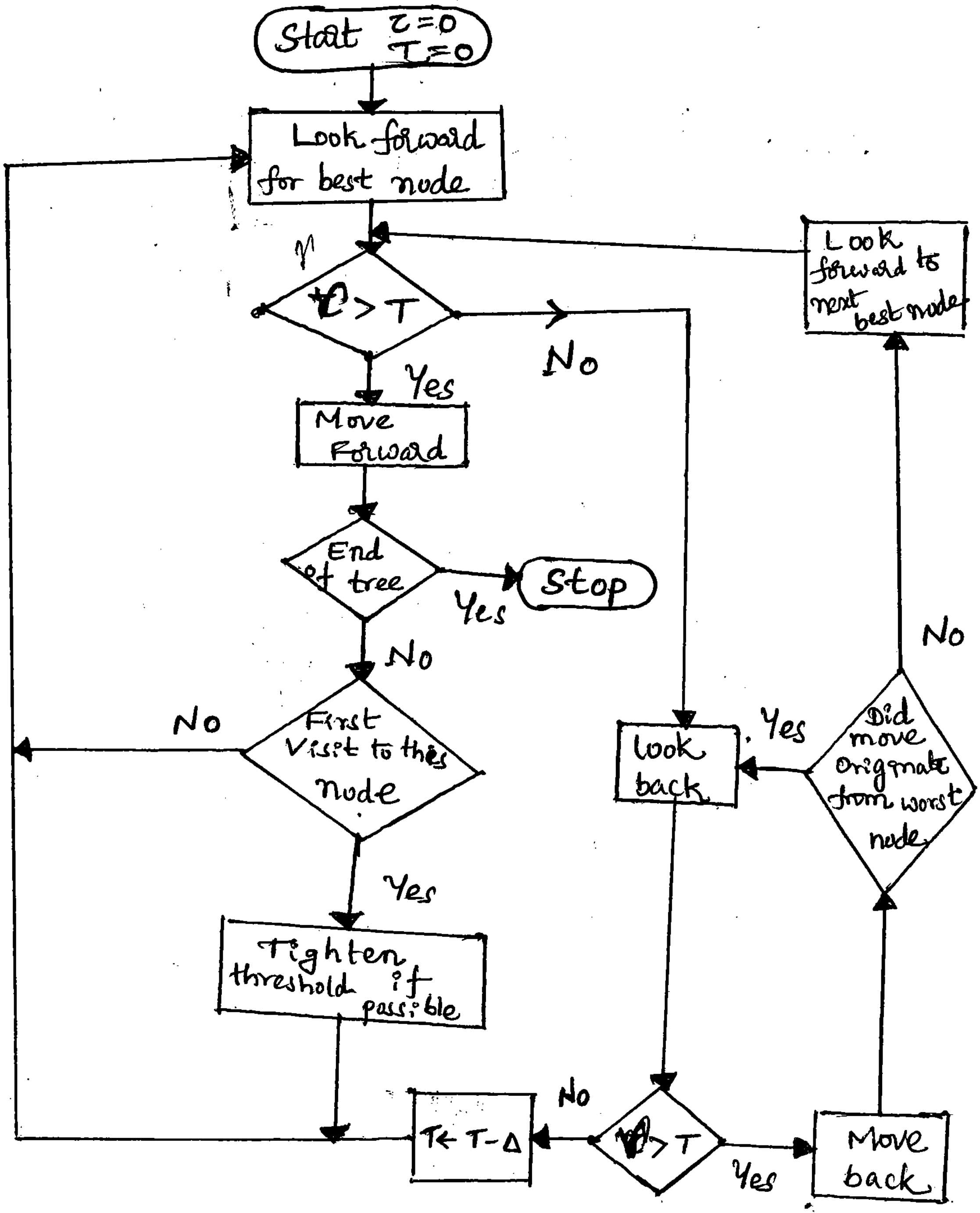


Fig: Flow chart for the Fano Algorithm.

Example :

Let us consider code tree diagram for Code rate = $1/2$ & Constraint length '7' of Convolutional code.

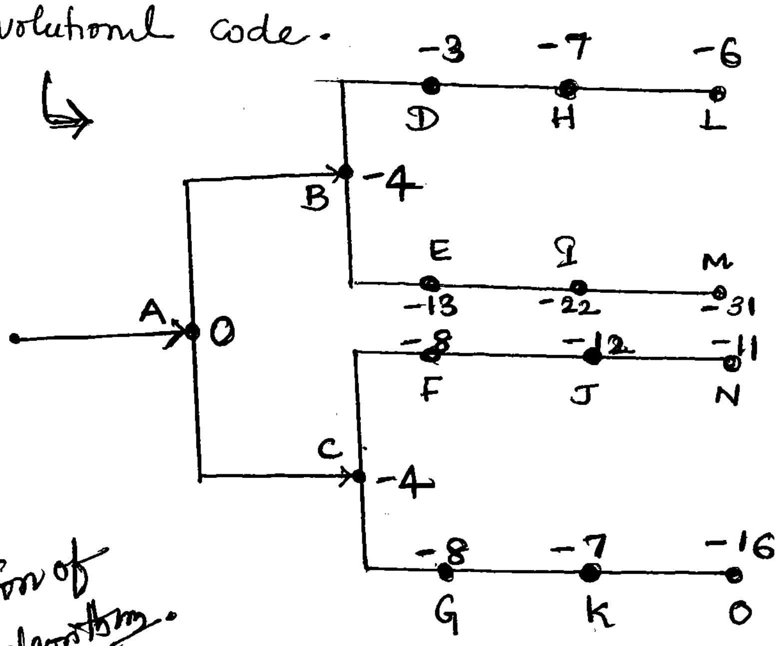


Illustration of Fano algorithm.

Step	Node	Fano Metric '$\Gamma(e)$'	Threshold 'T'
1	A	0	0
2	A	0	-4
3	C	-4	-4
4	A	0	-4
5	B	-4	-4
6	D	-3	-4
7	B	-4	-4
8	A	0	-4
9	A	0	-8
10	C	-4	-8
11	G	-8	-8
12	K	-7	-8
13	G	-8	-8
14	C	-4	-8
15	A	0	-8
16	B	-4	-8
17	D	-3	-8
18	H	-7	-8
19	L	-6	-8
20	← STOP →		

* The End

UNIT-5

Band Pass Data Transmission

Syllabus: Introduction ASK modulator, bandwidth and frequency spectrum of ASK, Coherent ASK detector, noncoherent ASK detector, Signal space representation & probability error for ASK. FSK - bandwidth & frequency Spectrum of FSK, Signal Space representation & probability of error for FSK, non coherent FSK detector, Coherent FSK detector, FSK detection using PLL. BPSK - bandwidth & frequency spectrum of BPSK, signal space representation & probability of error for BPSK, Coherent PSK detection, Differential Binary Phase shift Keying (DPSK), Quadrature phase Shift Keying (QPSK), QPSK demodulator, Introduction to M-ary signalling - GMSK and QAM.

Introduction:

* The output of any digital modulator like PCM, DM, DPCM, ADM will be 0's and 1's.
* The 0's & 1's can be transmitted through wire by assigning electrical voltage & it is called "Baseband digital data transmission" because it contains zero frequency components.
* In order to transmit 0's & 1's through free space, electrical energy should converted into electromagnetic energy.
* The device which converts electrical energy into electromagnetic energy called "Antenna" provided that the input to the antenna should be in continuous form.

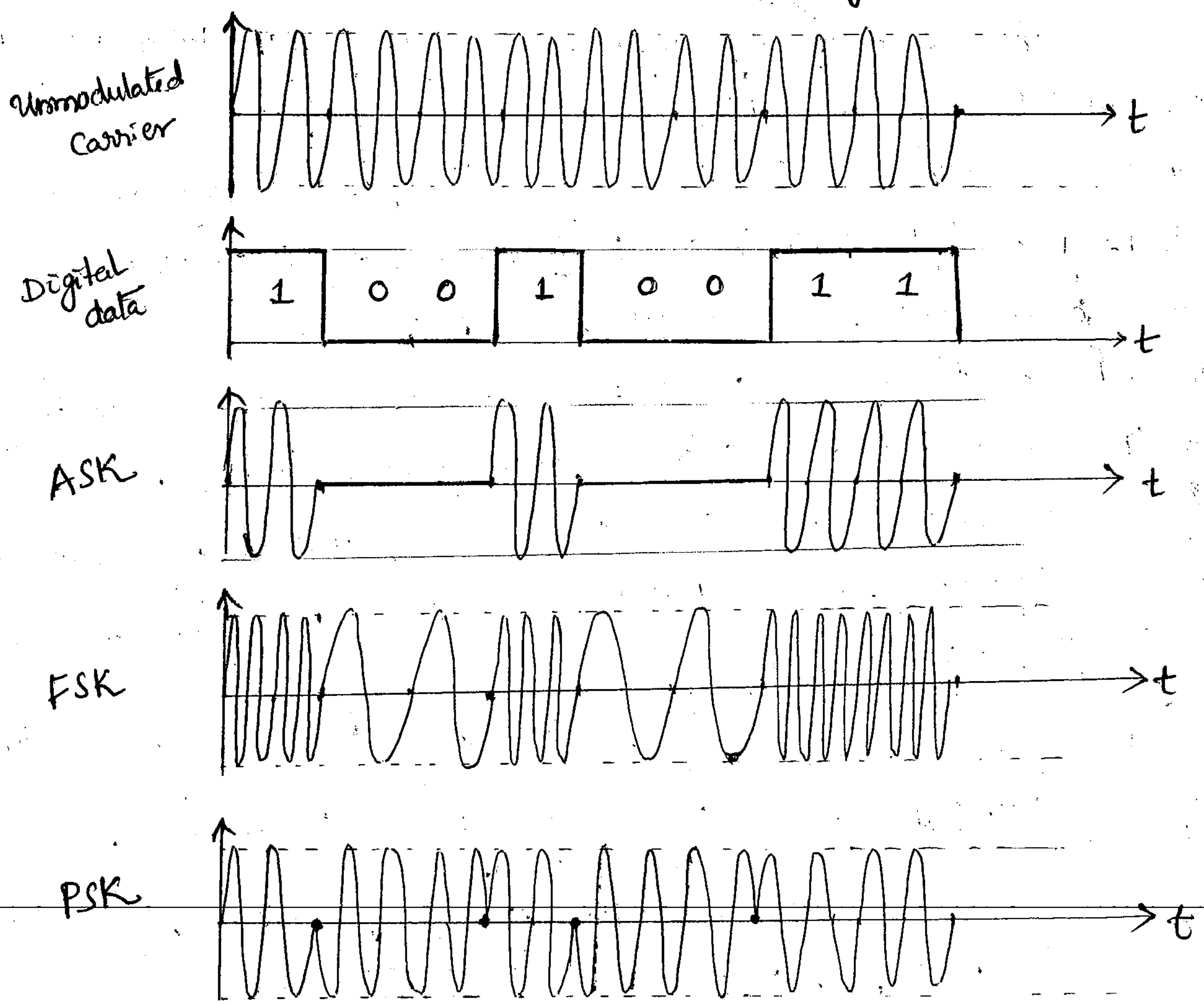
  So, to transmit 0's & 1's through free space it should be converted into continuous form.
* The modulation scheme used to convert discrete to continuous signals is called "Digital carrier modulation".
* The output of digital carrier modulation is a bandpass signal, hence it is also called "Bandpass digital data transmission".

Digital Modulation Techniques

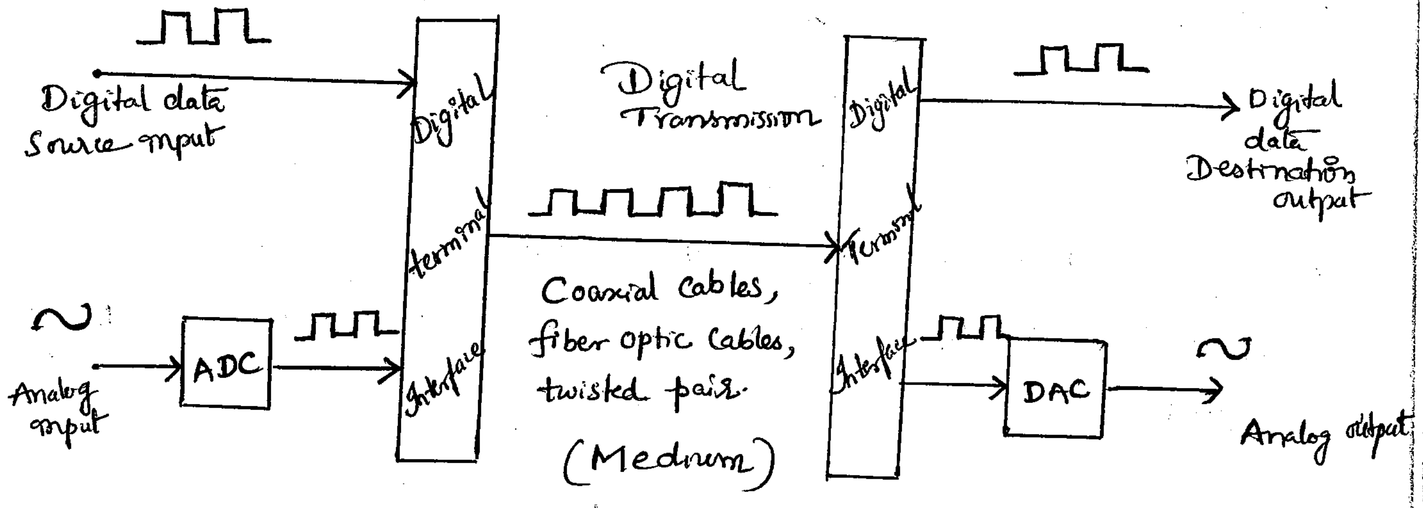
- **Binary** (Encodes only one bit at a time)
  - ASK
    - Coherent
    - Noncoherent
  - FSK
    - Coherent
    - Non Coherent
  - PSK → DPSK, QPSK
    - Coherent
    - Non Coherent.
- **M-ary** (Encodes more than two bits at a time)
  - M-ary ASK
  - M-ary FSK
  - M-ary PSK
  - M-ary QAM → Quadrature Amplitude Modulation

ASK – Amplitude Shift Keying
FSK – Frequency Shift Keying.
PSK – Phase shift Keying

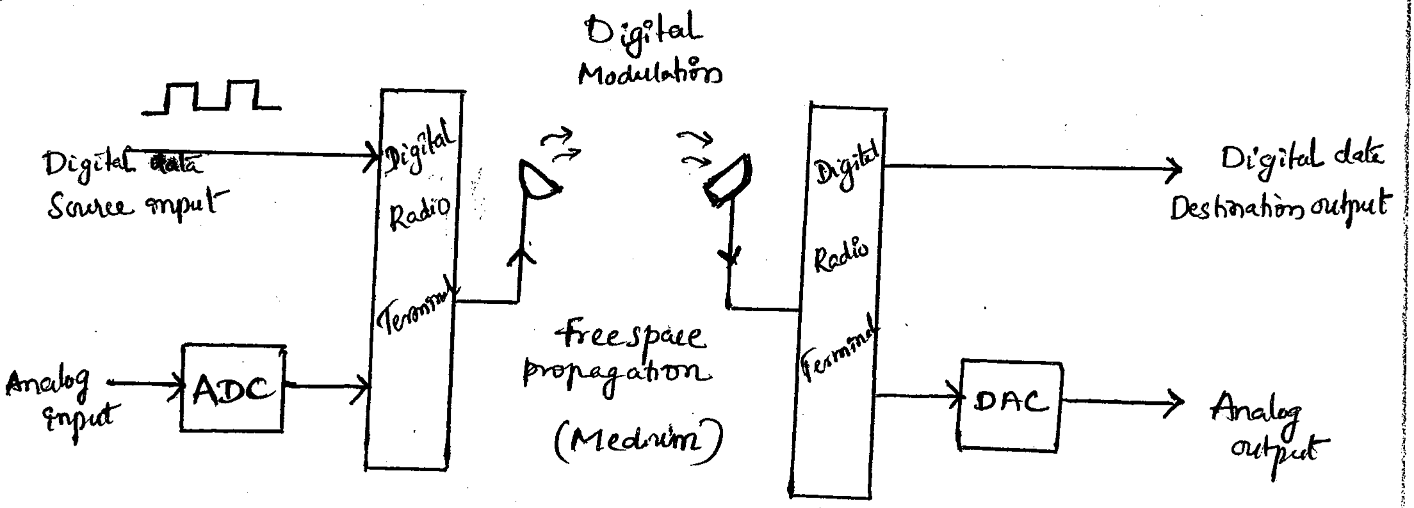
Goals:
1. Maximum data rate
2. Minimum probability of symbol error
3. Minimum Transmitted power.
4. Minimum channel bandwidth
5. Maximum resistance to interfering signals.
6. Minimum Circuit complexity.



1. Digital Transmission:



2. Digital Radio:



Note:

To perform demodulation at the receiver, we have the choice of Coherent (or) Non coherent detection.

✓ Coherent detection is performed by cross-correlating the received signal with each one of the replicas & then making a decision based on comparisons with preselected thresholds, Exact knowledge of the carrier wave's phase reference is required.

✓ Non coherent detection, knowledge of the carrier wave's phase is not required, the complexity of the receiver is thereby reduced but at the expense of an inferior error performance, compared to a coherent system.

Amplitude Shift Keying (ASK) : (or) OOK
ON-OFF Keying.

- ✓ The binary ASK system is the simplest form of digital modulation
- ✓ ASK is used in Wireless telegraphy.
- ✓ It is no longer used widely in digital communication.

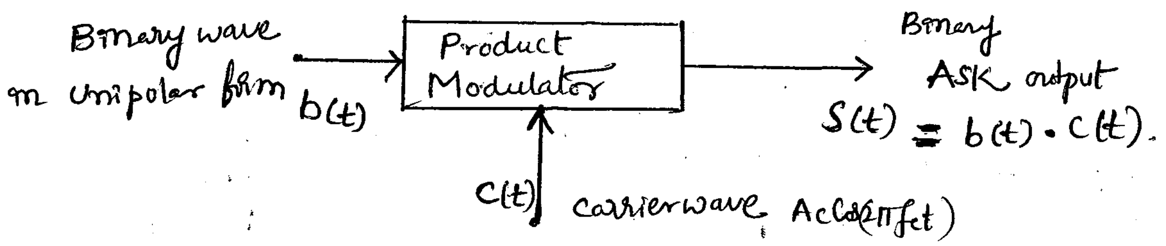
* Symbol 1 represented by transmitting sinusoidal carrier with fixed amplitude A_c and fixed frequency f_c for the bit duration T_b sec.

Symbol '0' represented by switching off the carrier of T_b sec.

∴ The signal can be generated simply by turning the carrier of a sinusoidal oscillator ON & OFF for the prescribed periods indicated by modulating the pulse train.

For this reason ASK is also known as ON-OFF Keying (OOK).

Generation of ASK & ASK Modulator:



let the Sinusoidal carrier be represented by

$$c(t) = A_c \cdot \cos(2\pi f_c t)$$

∴ The binary ASK signal can be represented by $s(t)$ given by

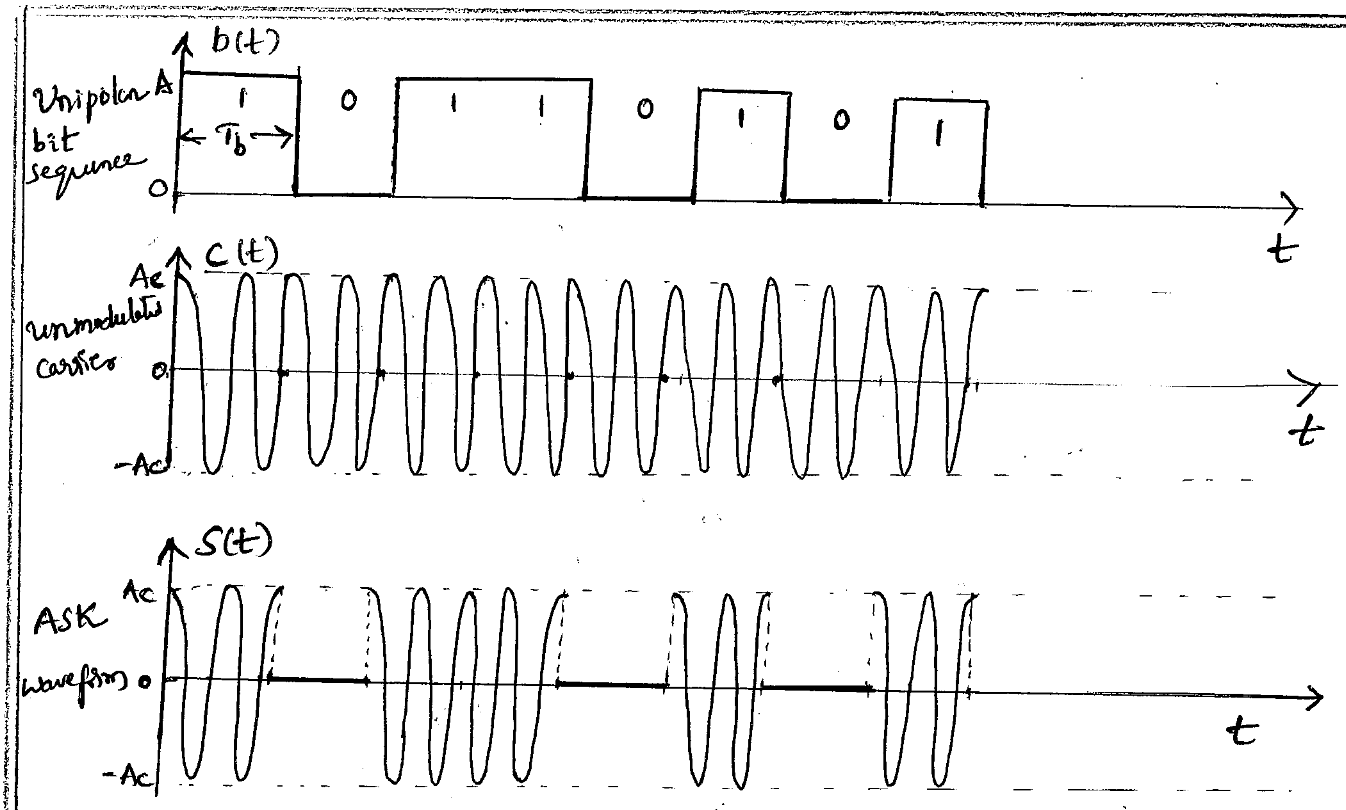
$$s(t) = b(t) \cdot (A_c \cos(2\pi f_c t))$$

where $b(t) = 1$; binary 1

$b(t) = 0$ binary '0'

$$\therefore \boxed{s(t) \Rightarrow \begin{cases} A_c \cos(2\pi f_c t) & ; \text{binary } 1 \\ 0 & ; \text{binary } '0' \end{cases}}$$

Let us consider the binary data $b(t) = \{1\ 0\ 1\ 1\ 0\ 1\ 0\ 1\}$

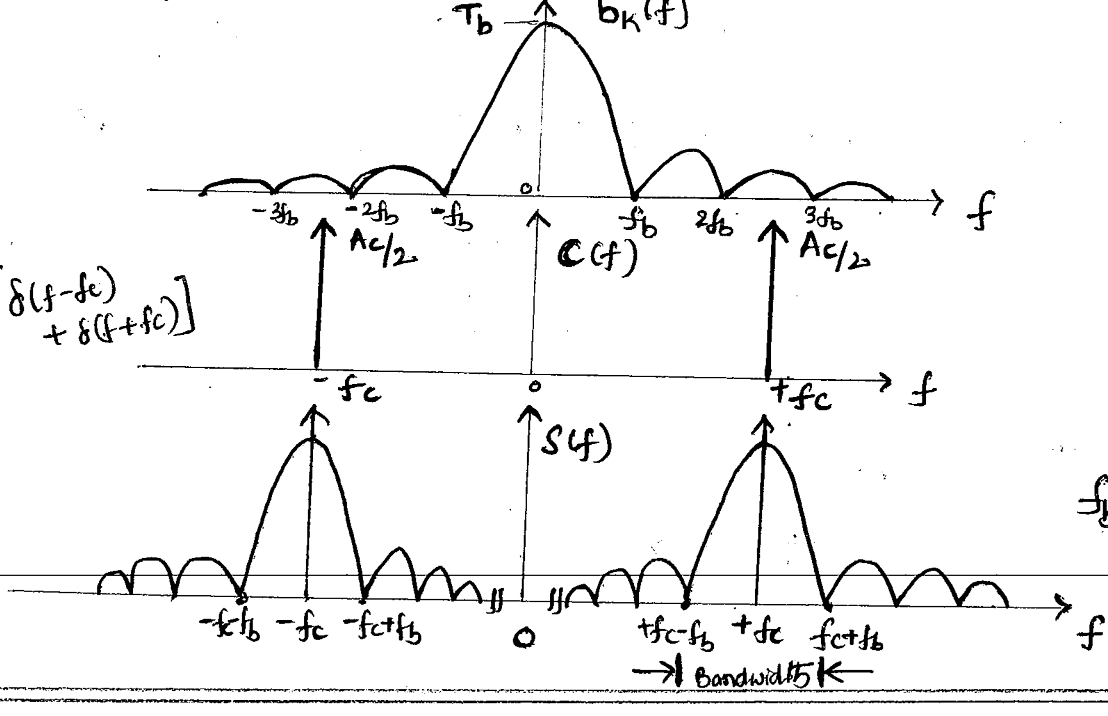


- ASK signal can be generated by applying the incoming binary data and the sinusoidal carrier to the two input of a product modulator (or) Balanced modulator.
- Modulator causes a shift of the baseband signal spectrum.
- The ASK signal which is basically the product of binary sequence & carrier signal has PSD same as that of base band ON-OFF signal but shifted in frequency domain by $\pm f_c$

  ie two impulses will occur at $\pm f_c$.

Bandwidth & Frequency Spectrum of ASK:

$$C(f) = \frac{A_c}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right]$$



$f_b = 1/T_b$

$$\text{The bandwidth of ASK} = f_2 - f_1$$
$$= (f_c + f_b) - (f_c - f_b)$$
$$= 2f_b \;\&\; 2/T_b .$$

$\therefore$ Bandwidth $\boxed{B \cdot W = 2f_b \;\&\; 2/T_b \text{ Hz}}$ for ASK

Signal Space representation of ASK :

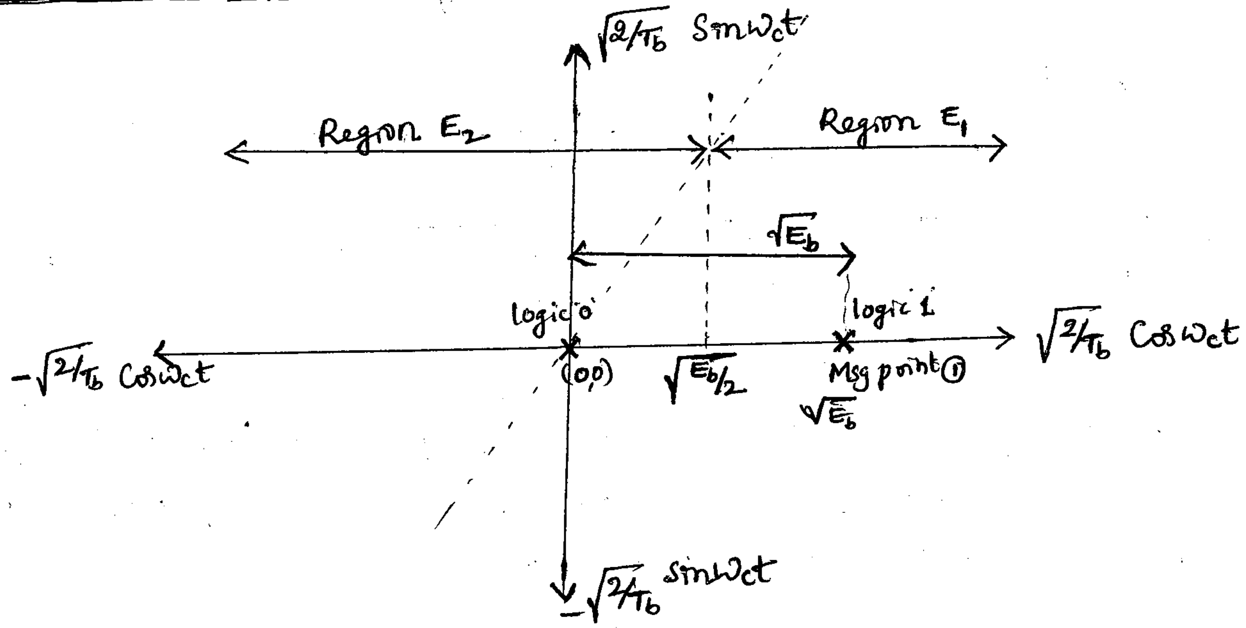


fig. One dimensional signal space with two messages.

✓ ASK output can be represented in power and Energy as follows

In power $S(t) = \begin{cases} \sqrt{2P_c}\,\cos w_c t & ;\ \text{logic } 1 \\ 0 & ;\ \text{logic } 0 \end{cases}$

$v = A_C \quad R = 1$

$P_c = \dfrac{V_{rms}^2}{R} = \dfrac{(V/\sqrt{2})^2}{R} = \dfrac{A_c^2}{2R} = \dfrac{A_c^2}{2}$

$\therefore A_c^2 = 2P_c \Rightarrow \boxed{A_c = \sqrt{2P_c}}$

in Energy. $S(t) = \begin{cases} \sqrt{E_b}\cdot\sqrt{2/T_b}\,\cos w_c t & ;\ \text{logic } 1 \\ 0 & ;\ \text{logic } 0 \end{cases}$

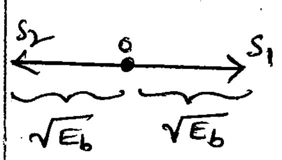
$E = P \cdot T_b$

$\Rightarrow A_c = \sqrt{2P_c} \times \dfrac{\sqrt{T_b}}{\sqrt{T_b}} = \sqrt{P_c T_b}\cdot\sqrt{2/T_b}$

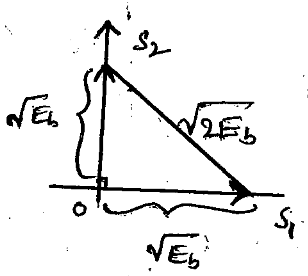
$\boxed{A_c = \sqrt{E_b}\cdot\sqrt{2/T_b}}$

Note : Basics, Consider two signals S_1 & S_2.

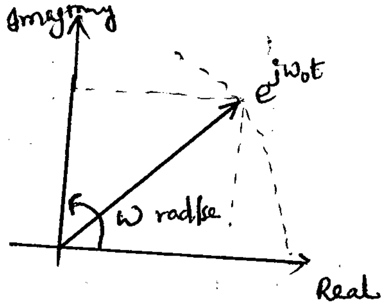
(a) Antipolar binary signal vectors



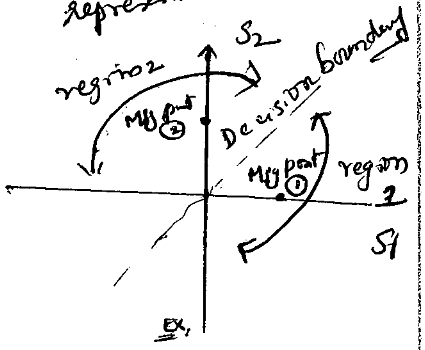
(b) Orthogonal binary signal vectors



(c) phasor representation of



(d) Signal Space representation.



Detection of ASK:

(a) Coherent detection
(b) Non coherent detection.

(a) Coherent Detection of ASK:

Coherent & Synchronous detector.

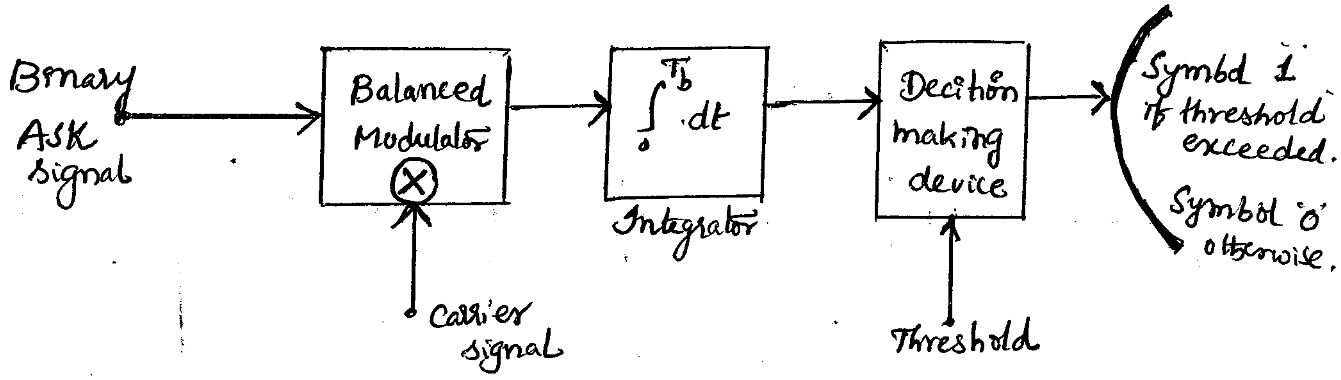


Fig: Coherent detection of ASK.

✓ Coherent detection consists of product modulator followed by integrator and decision device.

* The incoming ASK signal is applied to one input of the product modulator & second input is sinusoidal carrier.

✓ The output of product modulator goes to an integrator, the integrator operates on the output of multiplexer for successive bit intervals and essentially performs Low pass filtering.

* The output of Integrator goes to input of decision making device. The decision making device compares the output of integrator with a preset threshold level then produces symbol 1 if threshold is exceeded and symbol '0' if otherwise.

✓ Coherent detection involves the use of linear operation. here the local carrier is in perfect synchronization with the carriers used in the transmitter.

ie The frequency and phase of the locally generated carrier is same as those carriers used in the transmitter.

(b) Non coherent detection :

- ✓ It is also called asynchronous or Envelop detection of ASK.
- ✓ Binary ASK signal can also be demodulated non coherently using Envelop detector.

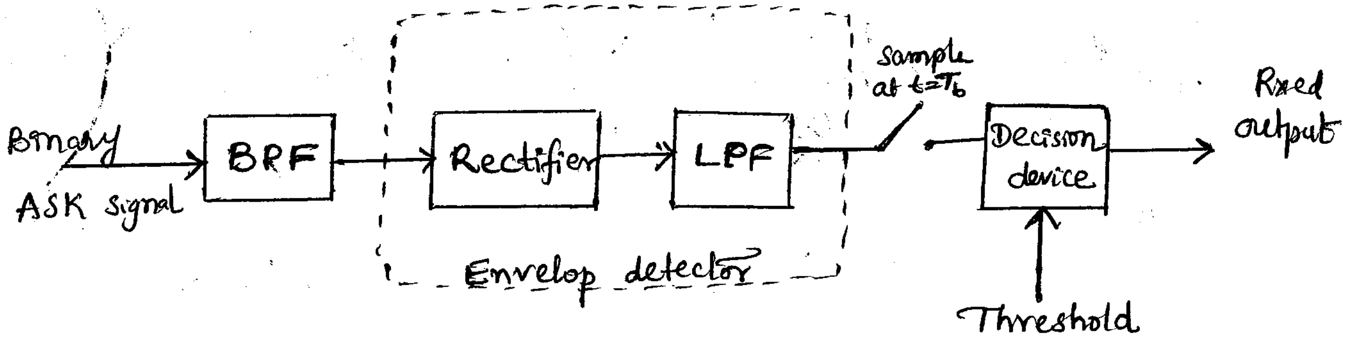
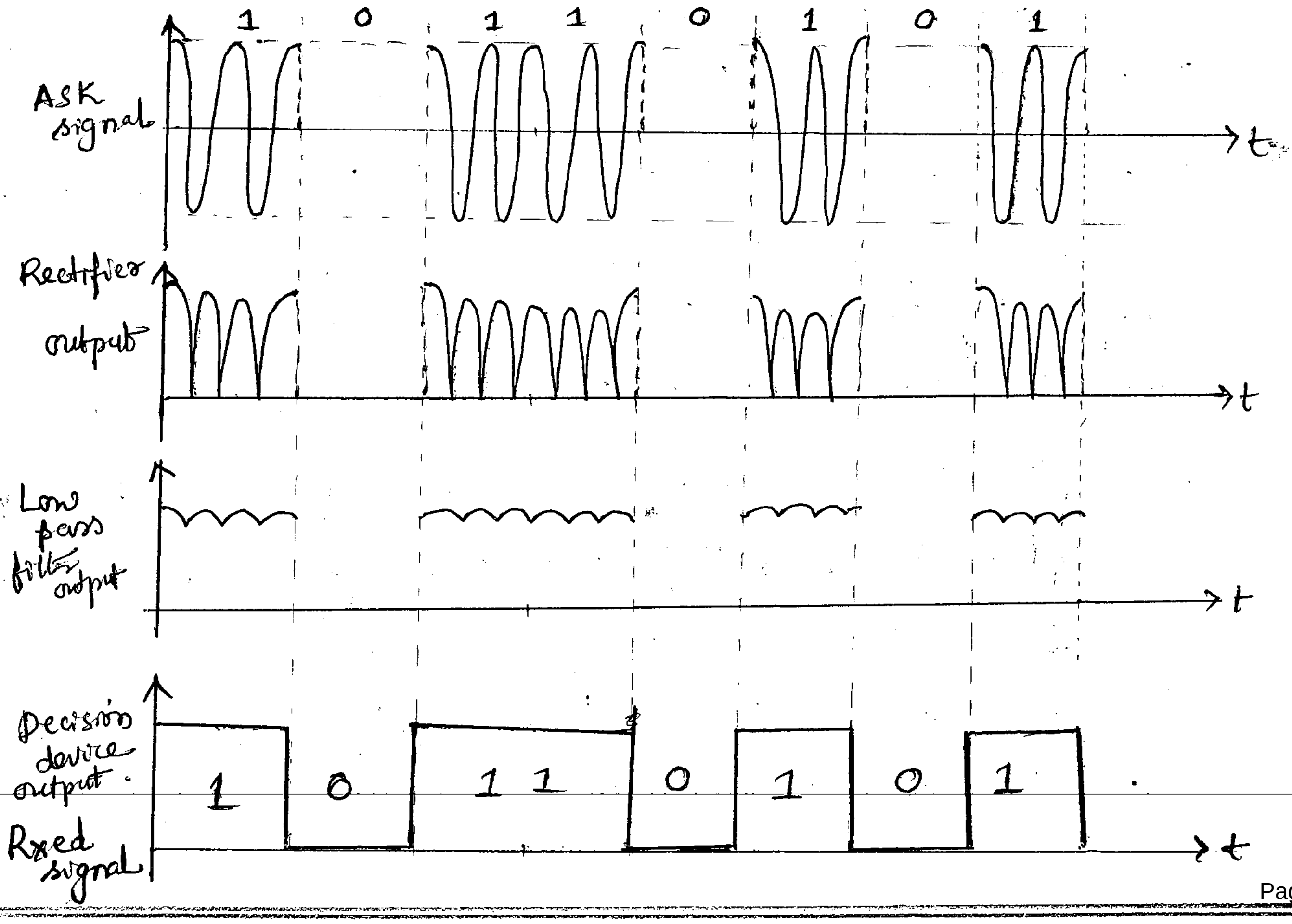


Fig: Non-coherent detection of ASK Signal.

- ✓ This greatly simplifies the design consideration needed in detector. (in synchronous detection)
- ✓ Non coherent detection schemes do not require a phase-coherent local oscillator.
- ✓ This schemes involves some form rectification and low pass filtering at the receiver.



Frequency Shift Keying (FSK):

In binary FSK system, two sinusoidal carrierwaves of the same amplitude A_c but different frequencies f_{c_1} & f_{c_2} are used to represent binary symbol '1' and '0' respectively.

The binary FSK wave $S(t)$ may be written as

$$\boxed{S(t) = \begin{cases} A_c \cdot \cos(2\pi f_{c_1} t) & ; \text{ binary '1'} \\ A_c \cos(2\pi f_{c_2} t) & ; \text{ binary '0'} \end{cases}}$$

(OR)

$$S(t)_{FSK} = \begin{cases} A_c \cos(\omega_c + \omega_d)t; & \text{logic 1} \\ A_c \cos(\omega_c - \omega_d)t; & \text{logic 0} \end{cases} \quad \text{(in voltage)}$$

$$S(t)_{(FSK)} = \begin{cases} \sqrt{2P_c} \cos(\omega_c + \omega_d)t; & \text{logic 1} \\ \sqrt{2P_c} \cos(\omega_c - \omega_d)t; & \text{logic 0} \end{cases} \quad \text{(in power)}$$

$P = A_c^2/2$

$A_c = \sqrt{2P_c}$.

In Energy.

$$S(t)_{FSK} = \begin{cases} \sqrt{E_b}\sqrt{2/T_b} \cos(\omega_c + \omega_d)t; & \text{logic 1} \\ \sqrt{E_b} \cdot \sqrt{2/T_b} \cos(\omega_c - \omega_d)t; & \text{logic 0} \end{cases}$$

$E = P.T_b$

$A_c = \sqrt{2P_c} = \sqrt{2 \cdot P_c} \cdot \frac{\sqrt{T_b}}{\sqrt{T_b}}$

$= \sqrt{P_c \cdot T_b} \cdot \sqrt{2/T_b}$

$A_c = \sqrt{E_b} \cdot \sqrt{2/T_b}$.

where ω_d = a constant frequency offset, from a normalized carrier frequency ω_c.

Generation of FSK:

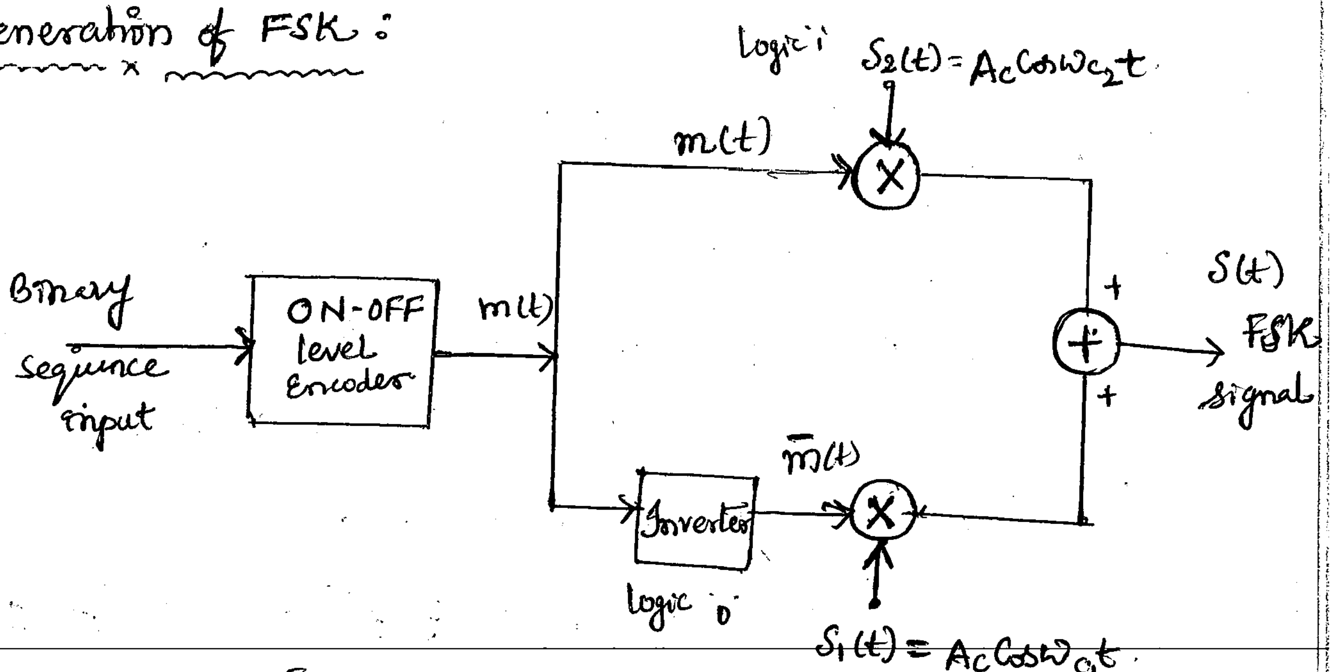
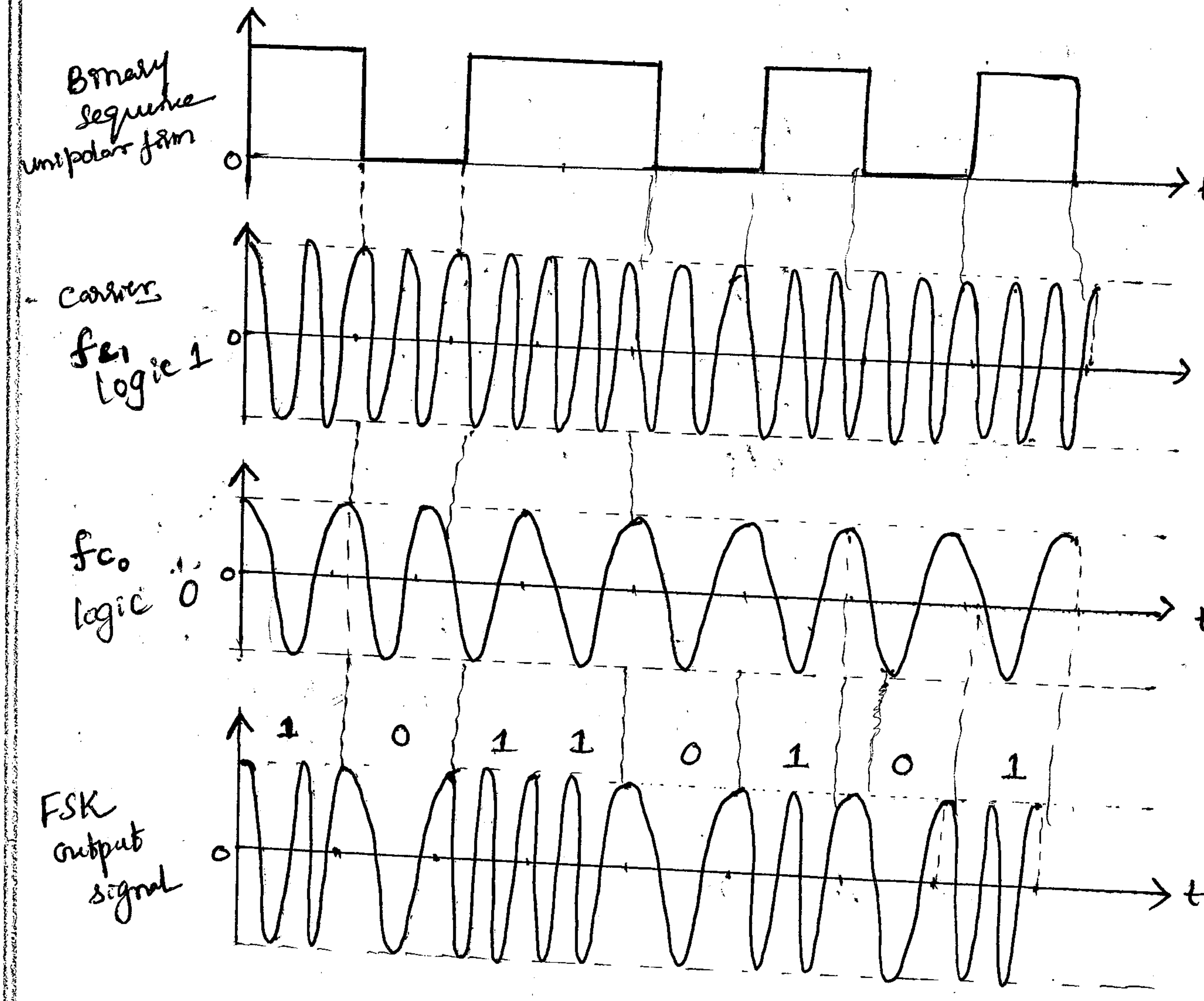


Fig: FSK Modulator.

- A binary FSK transmitter has an incoming binary data sequence is applied to on-off level encoder.
- The output of encoder is $\sqrt{E_b}$ volts for symbol 1 & 0 for symbol 0.

* Let us consider the binary sequence {1 0 1 1 0 1 0 1}.



Bandwidth and Frequency Spectrum of FSK:

- Data signal in frequency domain the spectrum is sinc function

$$S_1(t) = A_c \cos \omega_{c_1} t = A_c \cos[\omega_c - \omega_d]t$$

For logic 0

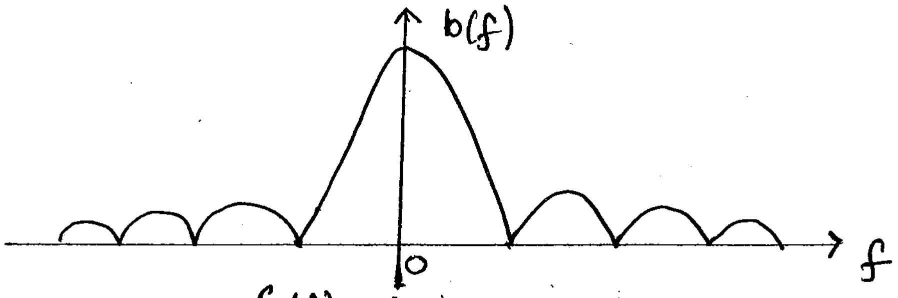
$$S_1(f) = \frac{A_c}{2}\left[\delta(f-(f_c-f_d)) + \delta[f+(f_c-f_d)]\right]$$

lly $S_2(t) = A_c \cos \omega_{c_2} t = A_c \cos[\omega_c + \omega_d]t$

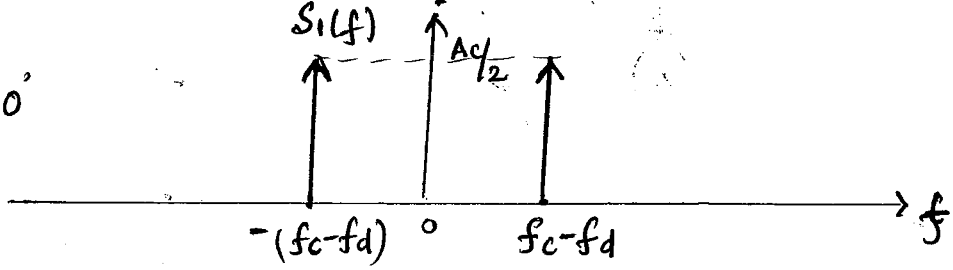
For logic 1.

$$S_2(f) = \frac{A_c}{2}\left[\delta[f-(f_c+f_d)] + \delta[f+(f_c+f_d)]\right]$$

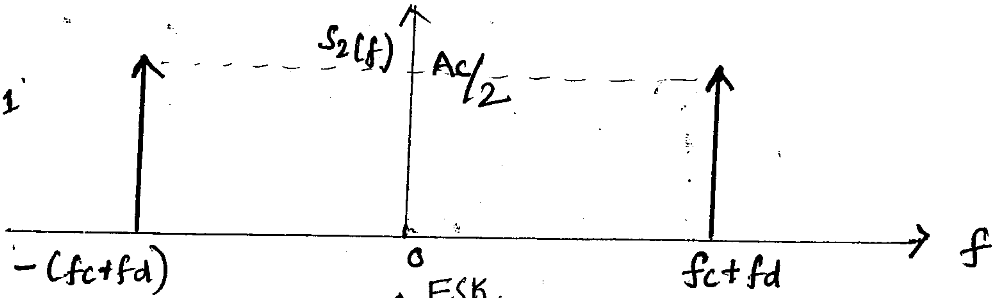
data



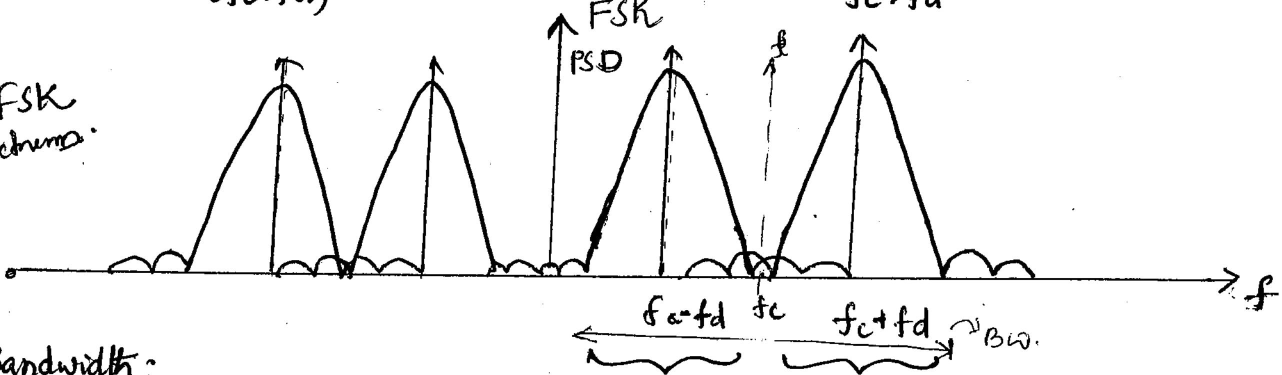
for logic '0'



for logic '1'



FSK spectrum.



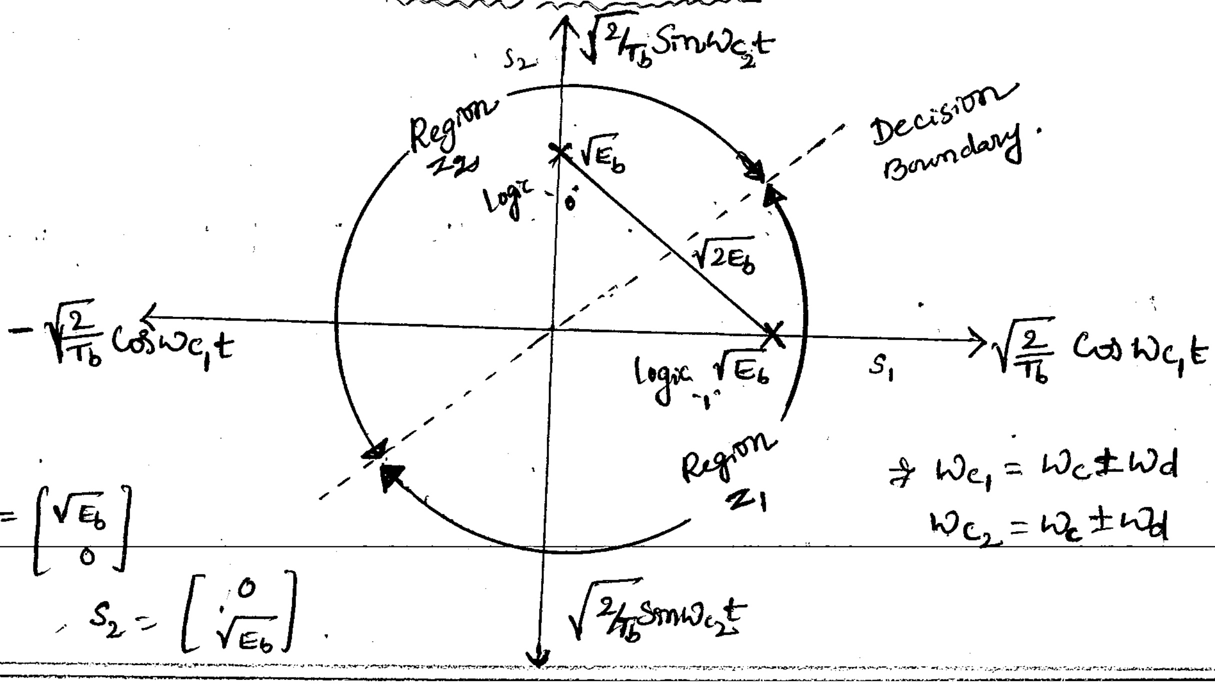
Bandwidth:

Bandwidth of FSK $= 2f_b + 2f_b = 4f_b$ & $4/T_b$.

$\therefore$ $\boxed{B.W = 4f_b \text{ \& } 4/T_b}$ Hz.

BW is very high compared to ASK & PSK.

Signal Space & Vector Representation of FSK:



$S_1 = \begin{bmatrix} \sqrt{E_b} \\ 0 \end{bmatrix}$

$S_2 = \begin{bmatrix} 0 \\ \sqrt{E_b} \end{bmatrix}$

$\because \omega_{c_1} = \omega_c \pm \omega_d$

$\omega_{c_2} = \omega_c \pm \omega_d$

Detection of FSK :

(a) Coherent Detection.
(b) Non coherent detection.

(a) Coherent detection of FSK :

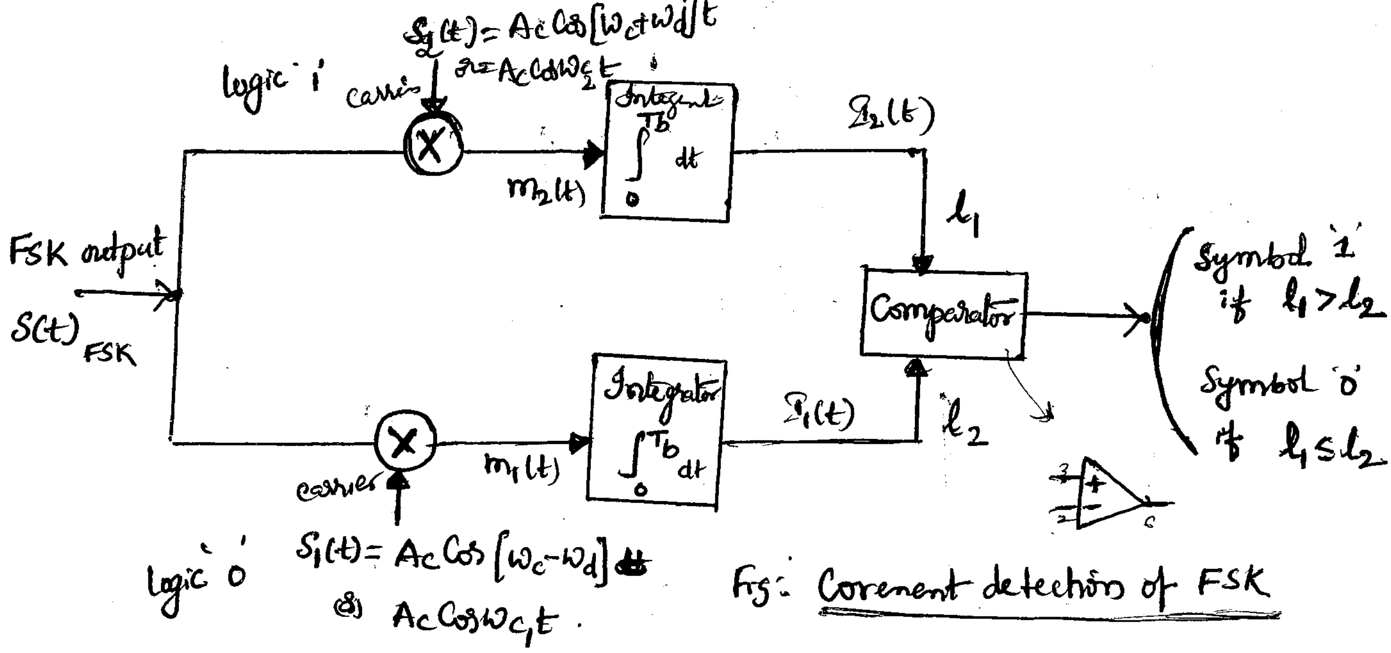


Fig: Corenent detection of FSK

- The incoming FSK signal is multiplied by a received carrier signal that has the exact same frequency and phase as the transmitter reference.

* The multiplexed output of each multiplier is subsequently passed through integrators generating outputs l_1 & l_2 in the two paths.

✓ The output of the two integrators are then fed to the decision making device.

* The decision making device is essentially a comparator which compares the output l_1 (in the upper path) & Output l_2 (in the lower path)

✓ If the output $l_1 >$ output l_2, the comparator output is Symbol '1'
If the output $l_1 \le$ output l_2, the comparator output is Symbol '0'.

* This type of digital communication receivers are also called Correlation receiver.

- The coherent detection requires phase & timing synchronization.

* FSK coherent detection is rarely used.

(b) Non Coherent detection of FSK :

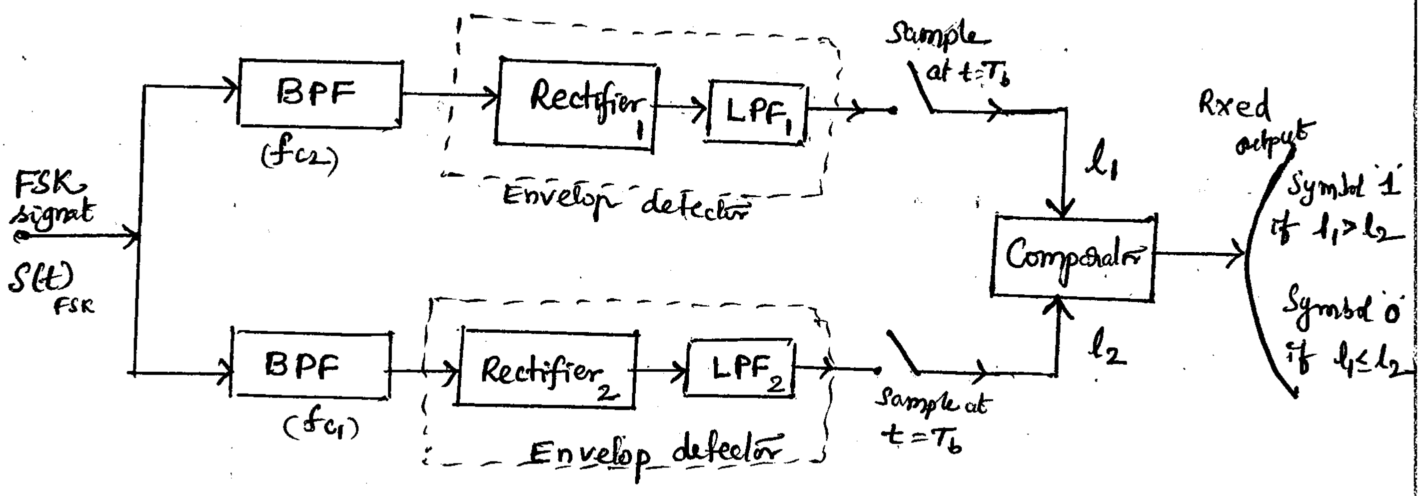
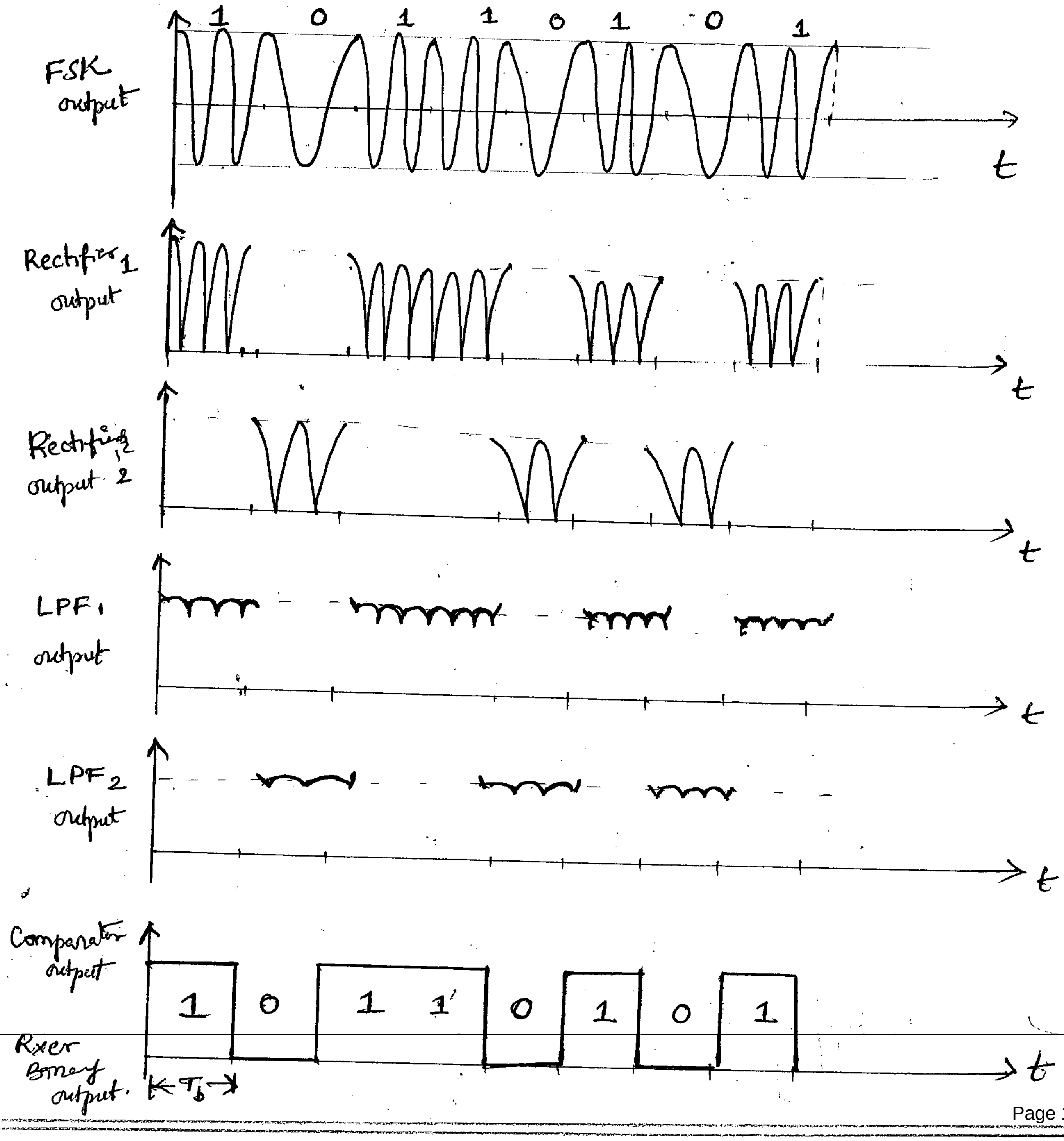


Fig. Non coherent detection of FSK.



Phase Shift Keying (PSK) :

In a binary PSK system, a sinusoidal carrier wave of fixed amplitude A_c and fixed frequency 'f_c' is used to represent both symbol 1 and '0' except that the carrier phase of each symbol differs by a phase of 180°.

Let the unmodulated carrier as

$$c(t) = A_c \cos(2\pi f_c t).$$

$\therefore$ The binary PSK signal $s(t)$ can be written as

$$s(t) = \begin{cases} A_c \cos(2\pi f_c t) & ; \text{Symbol is binary } 1 \\ A_c \cos(2\pi f_c t + \pi) & \text{symbol is binary '0'}. \end{cases}$$

(or)

$$\boxed{s(t) = \begin{cases} A_c \cos(2\pi f_c t) & ; \text{logic } 1 \\ -A_c \cos(2\pi f_c t) & ; \text{logic '0'} \end{cases}} \quad \text{(In voltage).}$$

$$s(t) = \begin{cases} \sqrt{2P_c} \cos(2\pi f_c t) & ; \text{logic } 1 \\ -\sqrt{2P_c} \cos(2\pi f_c t) & ; \text{logic } 0 \end{cases} \quad \text{(In power).}$$

$$s(t) = \begin{cases} \sqrt{2E_b/T_b} \cos(2\pi f_c t) & ; \text{logic } 1 \\ -\sqrt{2E_b/T_b} \cos 2\pi f_c t & ; \text{logic } 0 \end{cases} \quad \text{(In Energy).}$$

Generation of PSK :

- ✓ PSK signal can be generated by using the same scheme as used in the generation of 'ASK'.
- * The only difference is that the incoming binary data should be in the Bipolar form.
- ✓ A binary PSK may be also viewed as a DSB-SC (Double-Sideband Suppressed Carrier) modulated wave.

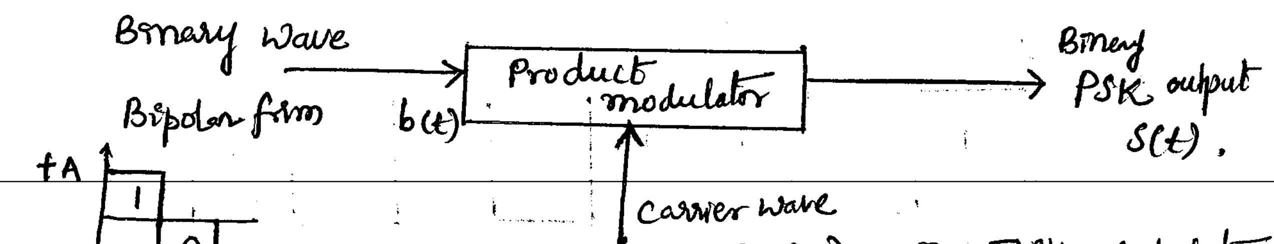
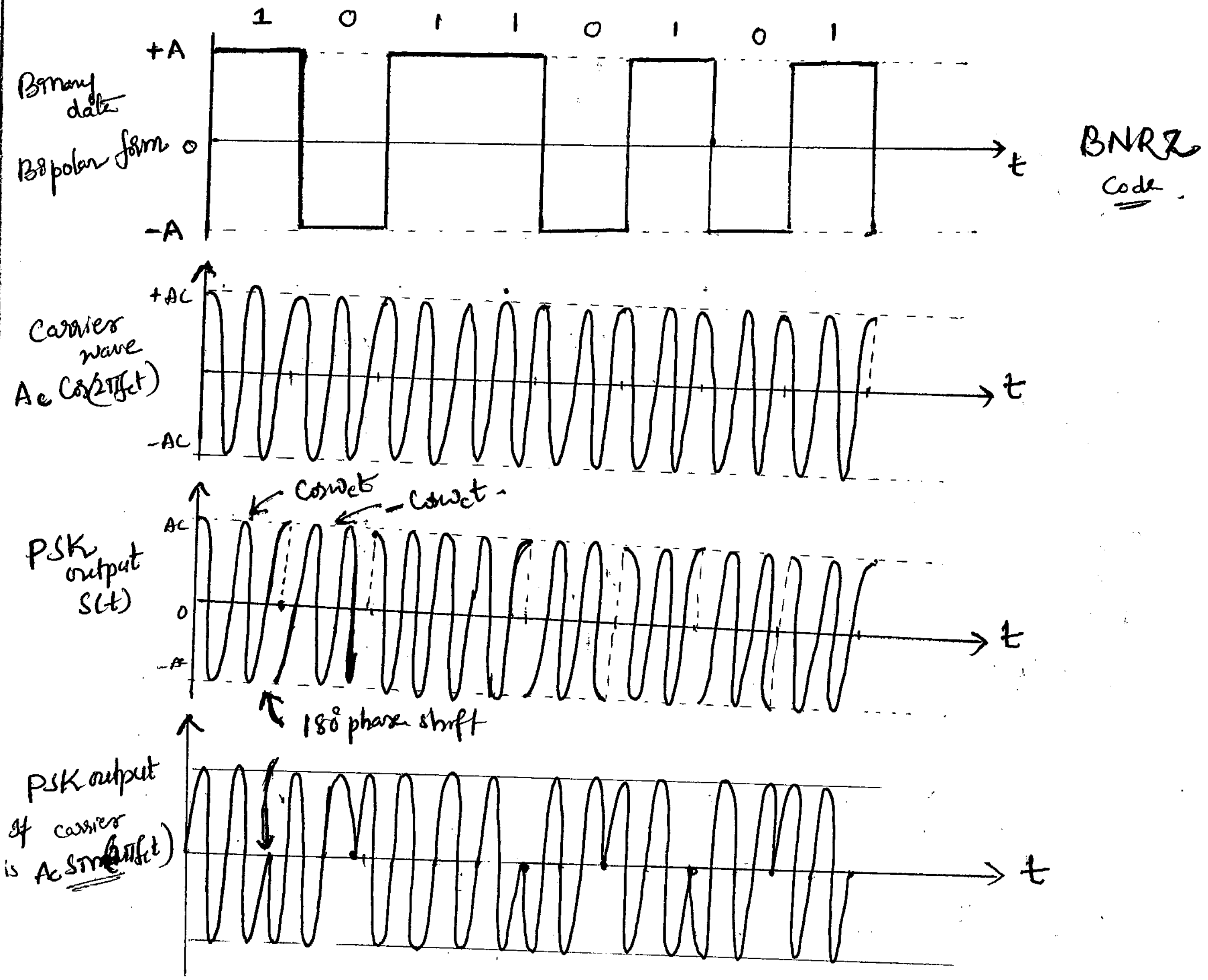
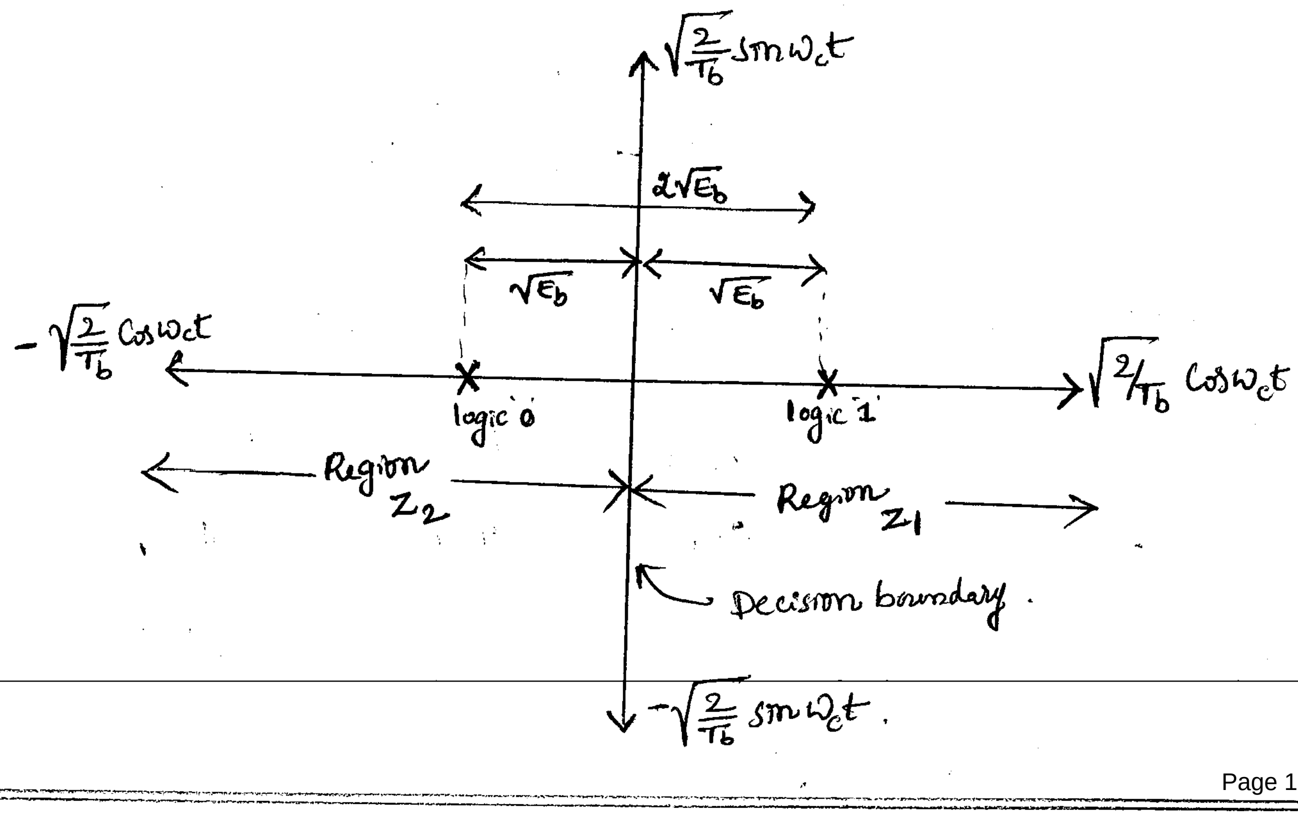


Fig. PSK Modulator.

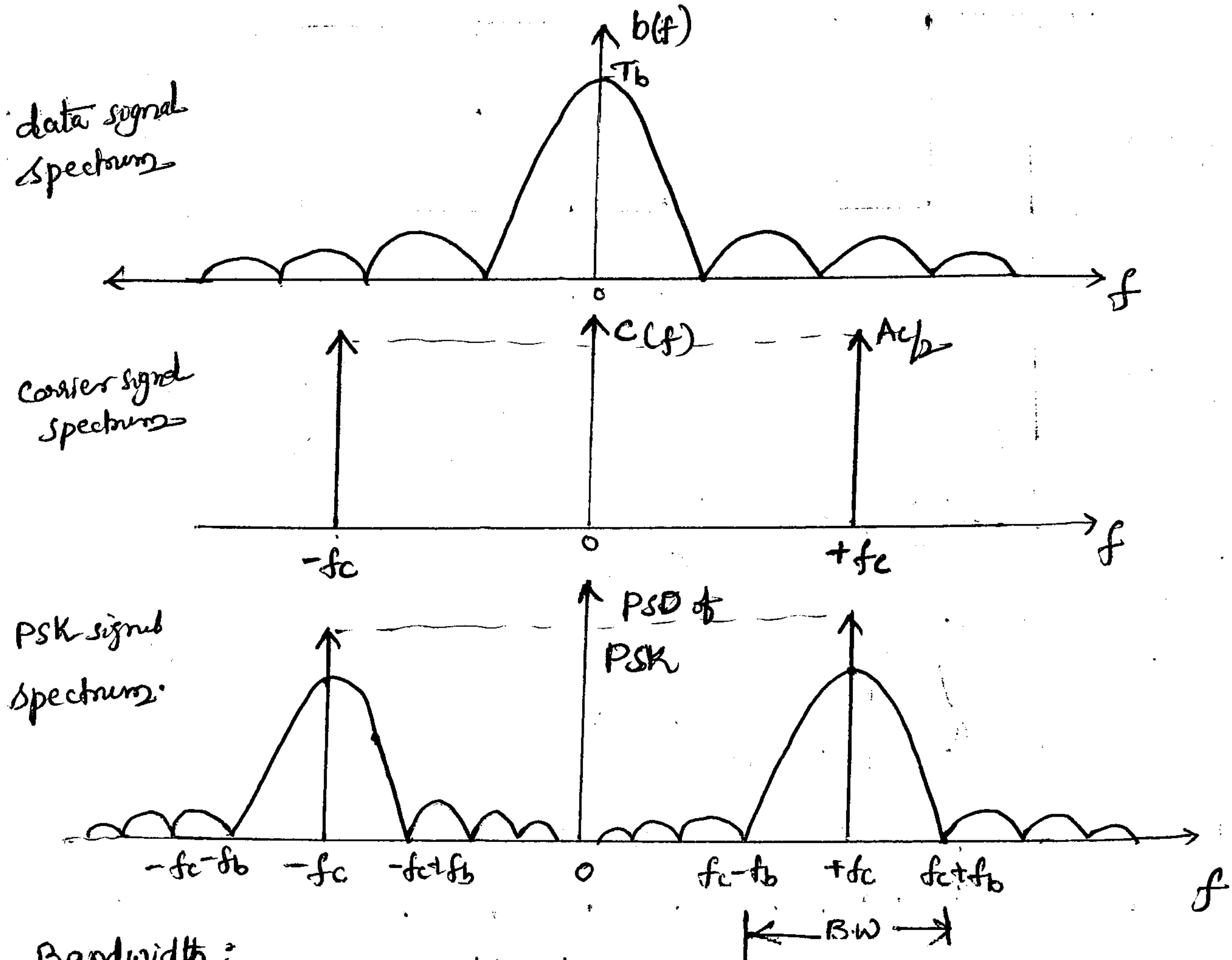
Let us consider a binary sequence {1 0 1 1 0 1 0 1}



Signal Space & Vector Representation of PSK :



Bandwidth & Frequency Spectrum of PSK:



Bandwidth: The bandwidth of binary PSK system is

$$
\begin{aligned}
B.W &= f_2 - f_1 \\
&= (f_c + f_b) - (f_c - f_b) \\
&= \cancel{f_c} + f_b - \cancel{f_c} + f_b \\
&= 2f_b \ \& \ 2/T_b
\end{aligned}
$$

$\therefore \boxed{B.W \text{ of PSK} = 2f_b \ \& \ 2/T_b}$

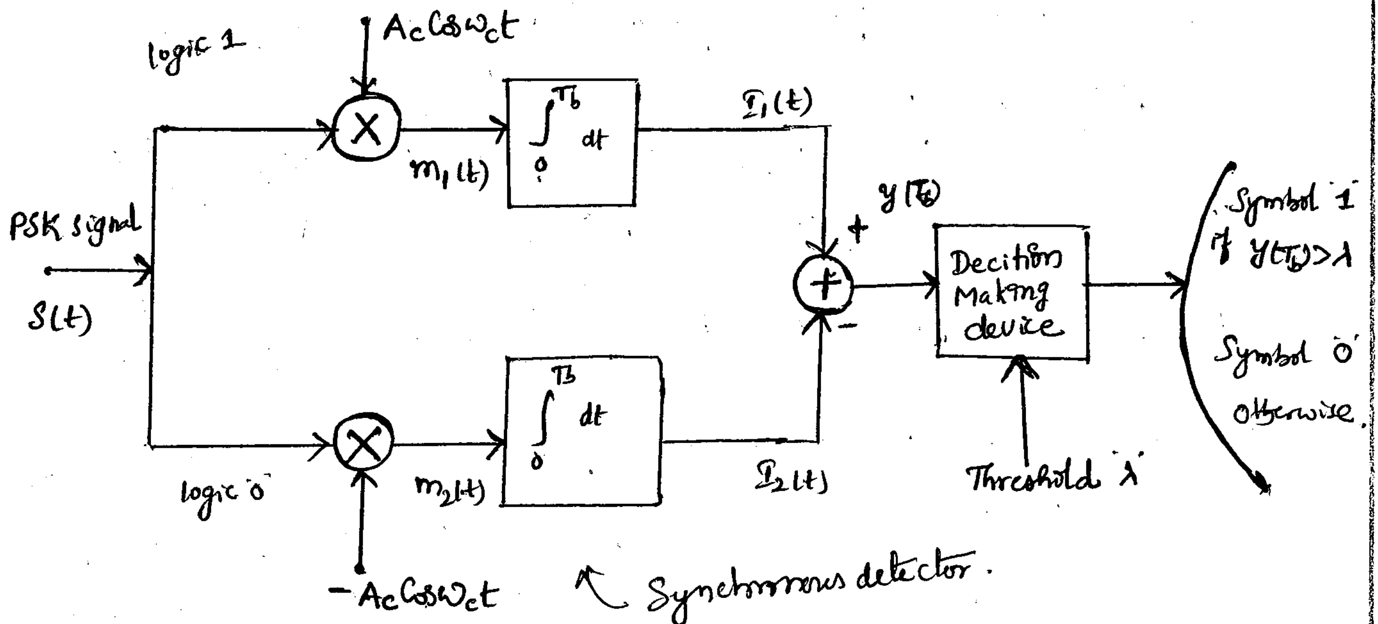
Note: Bandwidth & noise immunity for the digital modulation techniques

	ASK	FSK	PSK
Bandwidth :	$2f_b$	$\geq 4f_b$	$2f_b$
Noise immunity :	Less	high	Very high
~~H/W Complexity~~ :			

Detection of PSK:

(a) Coherent detection
(b) Non-Coherent detection (DPSK).

(a) Coherent detection of PSK:



✓ Synchronous detector output gets only when phase & frequency must be matches otherwise gets zero.

Logic 1: input as $A_c \cos\omega_c t = s(t)$

$$m_1(t) = (A_c \cos\omega_c t)\cdot(A_c \cos\omega_c t)$$

$$m_1(t) = A_c^2 \cos^2\omega_c t$$

$$\Rightarrow m_2(t) = 0$$

$$I_1(t) = \int_0^{T_b} A_c^2 \cos^2\omega_c t$$

$$I_1(t) = \frac{A_c^2}{2} \cdot T_b .$$

& $I_2(t) = 0$

logic '0' input as $-A_c \cos\omega_c t = s(t)$

$$\therefore m_2(t) = (-A_c \cos\omega_c t)(-A_c \cos\omega_c t) = A_c^2 \cos^2\omega_c t$$

$$m_1(t) = 0$$

$$\Rightarrow I_2(t) = \int_0^{T_b} A_c^2 \cos^2\omega_c t = \frac{A_c^2}{2} T_b .$$

$$I_1(t) = 0 .$$

$\therefore$ Threshold value $\lambda = \dfrac{I_1(t) - I_2(t)}{2} = \dfrac{0-0}{2} = 0 .$ $\boxed{\lambda = 0}$

$\therefore$ The decision making device detected as
If $y(T_b) > 0$ it generates output as symbol '1'
If $y(T_b) \leq 0$ it detects output as symbol '0'.

Non coherent detection of PSK :

There is no non-coherent detection for PSK because there should be phase synchronization since information exists in phase for PSK.

So, we use differential phase shift keying (DPSK) for non coherent detection of PSK.

Differential Phase Shift Keying (DPSK) :

- It is a noncoherent detection of PSK.
- It eliminates the need for a coherent reference signal at the receiver by combining two basic operations at the transmitter.
  - (a) Differential encoding of the input binary wave.
  - (b) Phase-shift keying

Hence the name, differential phase-shift keying (DPSK).

- In effect, to send symbol '1' we leave the phase of the current signal waveform unchanged and to send symbol '0', we phase advance the current signal waveform by 180°.
- The receiver is equipped with a storage capability, so that it can measure the relative phase difference between the waveforms received during two successive bit intervals.
- Provided that the unknown phase θ contained in the received wave varies slowly. The phase difference between waveforms received in two successive bit intervals will be independent of θ.

DPSK Transmitter :

$d_K = b_K \odot d_{K-1}$ (EX NOR)

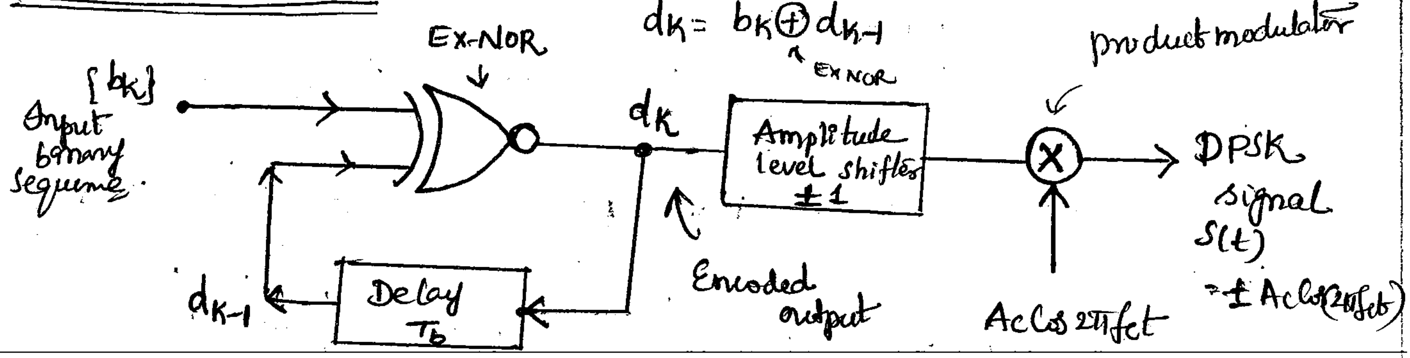


Fig: Block diagram of DPSK transmitter.

DPSK Receiver:

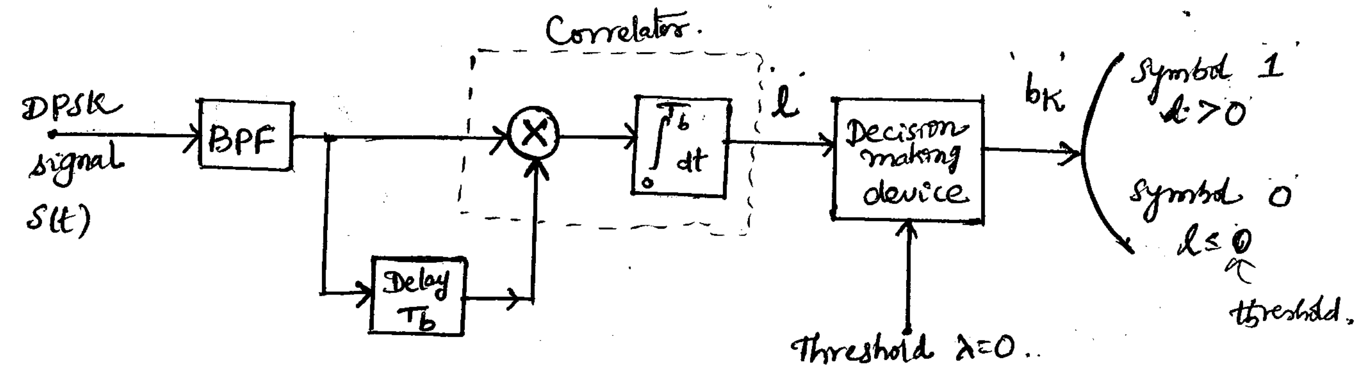


Fig: Block diagram of DPSK receiver.

✓ EX-NOR operation $X_{out} = AB \oplus A'B'$ ← Modulo-2 operation

EX-NOR output

A	B	out
0	0	1
0	1	0
1	0	0
1	1	1

* Let us consider the binary sequence $\{b_K\} = \{0\,0\,1\,0\,0\,1\,0\,0\,1\,1\}$

let an extra bit - Symbol 1 has been arbitrarily added as an initial bit.

*

(a)	Binary data $\{b_K\}$	Ex-NOR 0 0 1 0 0 1 0 0 1 1
(b)	differentially Encoded data at transmitter $\{d_K\}$	1* 0 1 1 0 1 1 0 1 1 1
(c)	Phase of DPSK 1 → no change 0 → 180° phase shift	0 π 0 0 π 0 0 π 0 0 0
(d)	Shifted differentially Encoded data at Rx $\{d_{K-1}\}$	1 0 1 1 0 1 1 0 1 1
(e)	Phase of shifted DPSK.	0 π 0 0 π 0 0 π 0 0
(f)	Phase Comparison output Between (c) & (e).	− − + − − + − − + +
(g)	Detected binary sequence at Receiver $\{b_K\}$	0 0 1 0 0 1 0 0 1 1

where 1 → 0' phaseshift
0 → 180° π phaseshift

comparison π & 0 = 0
0 & π = 0
π & π = 1
0 & 0 = 1

− → Symbol 0
+ → Symbol 1

Let an extra bit as symbol (0) added as an initial bit.

00 -1
01 0
10 -1
11

Ex-NOR

(a)	Binary data $\{b_K\}$		0	0	1	0	0	1	0	0	1	1
(b)	Differentially Encoded data $\{d_K\}$	0*	1	0	0	1	0	0	1	0	0	0
(c)	phase of DPSK	π	0	π	π	0	π	π	0	π	π	π
(d)	Shifted differential Encoded data $\{d_{K-1}\}$		0	1	0	0	1	0	0	1	0	0
(e)	phase of shifted DPSK.		π	0	π	π	0	π	π	0	π	π
(f)	phase Comparison output.		−	−	+	−	−	+	−	−	+	+
(g)	Detected binary data at Receiver $\{b_K\}$		0	0	1	0	0	1	0	0	1	1.

Thus, It is verified that the extra chosen bit 0 changes the phase of the DPSK sequence but the detected sequence remains invariant.

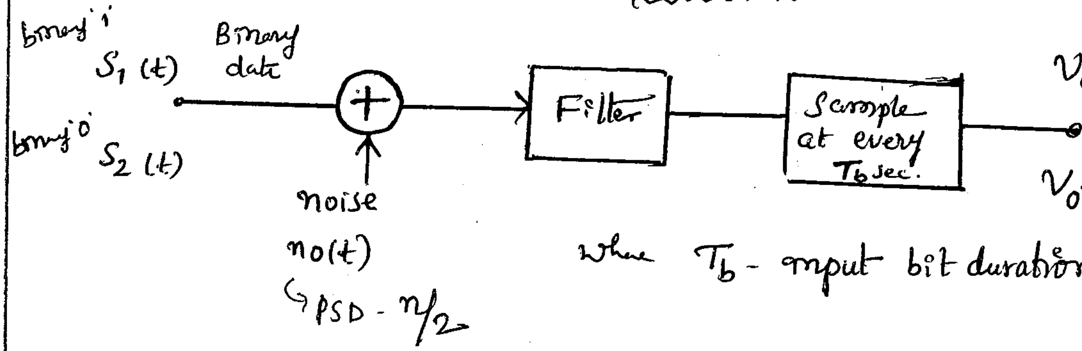
If any error is present

(a)	Binary data $\{b_K\}$	Let	1	1	0	1	1	0	0	1	1	0
(b)	Differentially Encoded data $\{d_K\}$	1*	1	1	0	1 (Error)	0	1	0	0	0	1
(c)	phase of DPSK	0	0	0	π	0	π	0	π	π	π	0
(d)	Shifted differential Encoded data		1	1	1	0	1	0	1	0	0	0
(e)	phase of shifted dPSK		0	0	0	π	0	π	0	π	π	π
(f)	phase comparison output		+	+	−	−	−	−	−	+	+	−
(g)	Detected binary data at Rxer.		1	1	0	$\boxed{0}$	$\boxed{0}$	0	0	1	1	0

Dibit Error

Thus. If one error is generated at transmitter, then two errors will be present at the output of receiver. This is called "Dibit Error." & If two error is transmitter then four error to Receiver.

Calculation of probability of Error (P_e):



$V_0(T_b) = S_{o_1}(T_b) + n_o(T_b)$

$V_0 T_b = S_{o_2}(T_b) + n_o(T_b)$.

where T_b - input bit duration.

- ✓ Error will results at $P(t) = S_1(t) - S_2(t)$.
- ✓ The probability of error $\boxed{P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{\gamma^2_{max}}{8}\right)^{1/2}}$ ← Complementary error function

where $\gamma_{max} = \frac{P_0(T_b)}{\sigma_0}$

$\Rightarrow \gamma^2_{max} = \frac{P_0^2(T_b)}{\sigma_0^2}$.

(OR) $\boxed{\gamma^2_{max} = \frac{2}{\eta} \int_0^{T_b} [P(t)^2]\, dt}$

In terms of Q-function

probability error $\boxed{P_e = Q\left(\frac{\gamma_{max}}{2}\right)}$

Error probability of ASK:

In Amplitude shift keying $S(t) = \begin{cases} A_c \cos 2\pi f_c t & ; \text{logic } 1 \\ 0 & ; \text{logic } 0 \end{cases}$

$\therefore S_1(t) = A_c \cos 2\pi f_c t = A_c \cos \omega_c t$.

$S_2(t) = 0$

$\therefore P(t) = S_1(t) - S_2(t)$.

$= A_c \cos 2\pi f_c t - 0 \Rightarrow P(t) = A_c \cos \omega_c t$.

Probability error $P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{\gamma^2_{max}}{8}\right)^{1/2}$

where $\gamma^2_{max} = \frac{2}{\eta} \int_0^{T_b} [P(t)]^2 dt$.

(or) $\frac{P_0^2(T_b)}{\sigma_0^2} \Rightarrow \gamma^2_{max} = P_0^2(T_b) / (\eta/2)$ → Signal power → noise power

$$\therefore \gamma^2_{max} = \frac{2}{\eta}\int_0^{T_b}[A_c \cos\omega_c t]^2 dt$$

$$= \frac{2}{\eta}\cdot A_c^2 \int_0^{T_b} \cos^2\omega_c t\, dt$$

$$= \frac{2A_c^2}{\eta}\int_0^{T_b}\left[\frac{1+\cos 2\omega_c t}{2}\right]dt$$

$$= \frac{A_c^2}{\eta}\left[\int_0^{T_b} 1\, dt + \int_0^{T_b}\cos 2\omega_c t\, dt\right]$$

$$= \frac{A_c^2}{\eta}\left[t\Big|_0^{T_b} + \underbrace{\frac{\sin 2\omega_c t}{2\omega_c}\Big|_0^{T_b}}_{\Downarrow\, 0}\right] \qquad \omega_c = 2\pi f_c$$

$$= \frac{A_c^2}{\eta}\cdot T_b$$

$$\therefore \boxed{\gamma^2_{max} = \frac{A_c^2 T_b}{\eta}}$$

$$\therefore P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{1}{8}\cdot\frac{A_c^2 T_b}{\eta}\right)^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{1}{4\eta}\cdot\left(\frac{A_c^2 T_b}{2}\right)\right]^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_b}{4\eta}} \qquad \text{where} \quad E_b = P_c T_b = \frac{A_c^2 T_b}{2}$$

where $P_c = \frac{A_c^2}{2}$

$$\therefore \boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_b}{4\eta}}} \quad \text{For ASK}$$

In terms of Q-function

$$P_e = Q\left(\frac{\gamma_{max}}{2}\right) = \int_{\frac{\gamma_{max}}{2}}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \qquad \left(\because Q(x) = \frac{1}{\sqrt{2\pi}}\int_x^{\infty} e^{-x^2/2}\,dx\right)$$

$$\gamma^2_{max} = \frac{A_c^2 T_b}{\eta} \Rightarrow \gamma_{max} = \sqrt{\frac{A_c^2 T_b}{\eta}}$$

$$\frac{\gamma_{max}}{2} = \sqrt{\frac{A_c^2 T_b}{4\eta}}$$

$$\therefore P_e = Q\left(\frac{\gamma_{max}}{2}\right) = Q\left(\sqrt{\frac{A_c^2 T_b}{4\eta}}\right).$$

$$Q\left(\sqrt{\frac{E_b}{2\eta}}\right)$$

Thus. Error probability for ASK system

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{4\eta}}\right) \quad \text{(or)} \quad P_e = Q\left(\sqrt{\frac{A_c^2 T_b}{4\eta}}\right).}$$

Error probability in PSK :

In Phase Shift Keying $S(t) = \begin{cases} A_c \cos\omega_c t & ; \text{logic } 1 \\ -A_c \cos\omega_c t & ; \text{logic } 0 \end{cases}$

$$S_1(t) = A_c \cos\omega_c t$$
$$S_2(t) = -A_c \cos\omega_c t \Rightarrow P(t) = S_1(t) - S_2(t)$$
$$= A_c\cos\omega_c t + A_c \cos\omega_c t$$
$$\therefore P(t) = 2A_c \cos\omega_c t .$$

Probability error $P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\gamma^2_{max}}{8}\right)^{1/2}$

where

$$\gamma^2_{max} = \frac{2}{\eta}\int_0^{T_b} [P(t)]^2 dt$$
$$= \frac{2}{\eta}\int_0^{T_b}[2A_c\cos\omega_c t]^2 dt$$
$$= \frac{4A_c^2 \cdot 2}{\eta}\int_0^{T_b}\cos^2\omega_c t\, dt$$
$$= \frac{4A_c^2}{\eta}\times\cancel{2}\int_0^{T_b}\left[\frac{1+\cos 2\omega_c t}{\cancel{2}}\right]dt$$
$$= \frac{4A_c^2}{\eta}\cdot\left[(T_b) + \underbrace{\frac{\sin 2\omega_c t}{2\omega_c}\Big|_0^{T_b}}_{0}\right]$$
$$= \frac{4A_c^2 T_b}{\eta} \quad \therefore \boxed{\gamma^2_{max} = \frac{4A_c^2 T_b}{\eta}}$$

Probability error
$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{1}{\cancel{8}_2}\cdot\frac{\cancel{4}\cdot A_c^2\cdot T_b}{\eta}\right)^{1/2}$$
$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{A_c^2}{2}\cdot T_b\cdot\frac{1}{\eta}\right]^{1/2}$$
$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{E_b}{\eta}\right]^{1/2} \quad \text{where } E_s = P_c \cdot T_b = \frac{A_c^2 T_b}{2}$$
$$P_c = A_c^2/2$$
$$\therefore P_e = \frac{1}{2}\,\mathrm{erfc}\sqrt{(E_b/\eta)}.$$

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_b}{\eta}}} \quad \text{For PSK}$$

In terms of Q-function

probability error $P_e = Q\left(\frac{\gamma_{max}}{2}\right)$. $\quad \gamma^2_{max} = \frac{4A_c^2 T_b}{\eta}$

$$\gamma_{max} = \sqrt{\frac{4A_c^2 T_b}{\eta}} = 2\sqrt{\frac{A_c^2 T_b}{\eta}}$$

$$\therefore \frac{\gamma_{max}}{2} = \sqrt{\frac{A_c^2 T_b}{\eta}}.$$

Probability Error $Q\left(\frac{\gamma_{max}}{2}\right) = Q\left(\sqrt{\frac{A_c^2 T_b}{\eta}}\right).$

Thus, the probability error in PSK systems as

$$\boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{\eta}}\right) \quad \text{or} \quad P_e = Q\left(\sqrt{\frac{A_c^2 T_b}{\eta}}\right) \ \text{ (or) } \ Q\left(\sqrt{\frac{2E_b}{\eta}}\right)}$$

Error probability in FSK:

In Frequency Shift Keying $\quad S(t) = \begin{cases} A_c\cos(\omega_c + \Omega)t \\ A_c\cos(\omega_c - \Omega)t \end{cases}$ ← constant offset frequency.

$$\therefore S_1(t) = A_c\cos(\omega_c+\Omega)t$$
$$S_2(t) = A_c\cos(\omega_c-\Omega)t \qquad P(t) = S_1(t) - S_2(t)$$

$$\therefore P(t) = A_c\left[\cos(\omega_c+\Omega)t - \cos(\omega_c-\Omega)t\right]$$

Probability error $\quad P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\gamma_{max}^2}{8}\right)^{1/2}$

where $\gamma_{max}^2 = \frac{2}{\eta}\int_0^{T_b}[p(t)]^2\,dt$

$$\therefore \gamma_{max}^2 = \frac{2}{\eta}\int_0^{T_b} A_c^2\left[\cos(\omega_c+\Omega)t - \cos(\omega_c-\Omega)t\right]^2 dt$$

$$= \frac{2}{\eta}\int_0^{T_b} A_c^2\left\{\cos^2(\omega_c+\Omega)t + \cos^2(\omega_c-\Omega)t - 2\cos(\omega_c+\Omega)t\cdot\cos(\omega_c-\Omega)t\right\}dt$$

$$= \frac{2A_c^2}{\eta}\left\{\int_0^{T_b}\cos^2(\omega_c+\Omega)t\,dt + \int_0^{T_b}\cos^2(\omega_c-\Omega)t\,dt - 2\int_0^{T_b}\cos(\omega_c+\Omega)t\cdot\cos(\omega_c-\Omega)\,dt\right\}$$

$$= \frac{2A_c^2}{\eta}\left[\int_0^{T_b}\left(\frac{1+\cos 2(\omega_c+\Omega)t}{2}\,dt\right) + \int_0^{T_b}\left[\frac{1+\cos 2(\omega_c-\Omega)t}{2}\right]dt\right.$$
$$\left. - \int_0^{T_b}\left[\cos\left[(\omega_c+\cancel{\Omega}+\omega_c-\cancel{\Omega})t\right] + \cos\left[\cancel{\omega_c}+\Omega-\cancel{\omega_c}+\Omega)t\right]\right]dt\right.$$

∵ Since $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$2\cos A \cos B = \cos(A+B) + \cos(A-B)$.

$$\therefore \gamma^2_{max} = \frac{2A_c^2}{\eta}\left\{ \int_0^{T_b} \frac{1}{2}\,dt + \int_0^{T_b} \frac{\cos 2(\omega_c+\Omega)t}{2}dt + \int_0^{T_b}\frac{1}{2}dt + \int_0^{T_b} \frac{\cos 2(\omega_c-\Omega)t}{2} - \int_0^{T_b} \cos(2\omega_c t)dt - \int_0^{T_b} \cos(2\Omega)t\,dt \right\}$$

$$= \frac{2A_c^2}{\eta}\left[\frac{T_b}{2} + 0 + \frac{T_b}{2} + 0 - 0 - \frac{\sin 2\Omega t}{2\Omega}\Big|_0^{T_b} \right]$$

$$= \frac{2A_c^2}{\eta}\left[T_b - \frac{\sin(2\Omega T_b)}{2\Omega} \right]$$

$$\gamma^2_{max} = \frac{2A_c^2 T_b}{\eta}\left[1 - \frac{\sin(2\Omega T_b)}{2\Omega T_b} \right]$$

the quantity γ^2_{max} attains its largest value 'Ω' is selected such that $2\Omega T_b = 3\pi/2$

$$\gamma^2_{max} = \frac{2A_c^2 T_b}{\eta}\left[1 - \frac{\sin(3\pi/2)}{3\pi/2} \right] \qquad \left[\sin 270^\circ = -1\right]$$

$$= \frac{2A_c^2 T_b}{\eta}\left[1 + \frac{2}{3\pi} \right]$$

$$= 2\left(\frac{3\pi+2}{3\pi}\right) \cdot \frac{A_c^2 T_b}{\eta}$$

$$= 2.42 \cdot \frac{A_c^2 T_b}{\eta} \qquad \therefore \boxed{\gamma^2_{max} = 2.42 \cdot \frac{A_c^2 T_b}{\eta}}$$

Probability Error $P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{\gamma^2_{max}}{8}\right)^{1/2}$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{1}{8} \cdot 2.42 \cdot \frac{A_c^2 T_b}{\eta}\right]^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{1}{4} \cdot \frac{(2.42)}{\eta} \cdot \frac{A_c^2 T_b}{2}\right]^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{0.6}{\eta} \cdot E_b\right]^{1/2} \qquad \left[\because E_b = \frac{A_c^2 T_b}{2} = P_e T_b\right]$$

$$= \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{0.6 E_b}{\eta}} \qquad \therefore \boxed{P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{0.6 E_b}{\eta}}\right)} \text{ for FSK}$$

Interms of Q-function Probability error $P_e = Q\left(\frac{\gamma_{max}}{2}\right)$.

$$\therefore \gamma^2_{max} = 2.42 \cdot \frac{A_c^2 \cdot T_b}{\eta}$$

$$\gamma_{max} = \sqrt{2.42 \cdot \frac{A_c^2 \cdot T_b}{\eta}}$$

$$\frac{\gamma_{max}}{2} = \sqrt{\frac{2.42}{2} \cdot \frac{A_c^2 \cdot T_b}{2} \cdot \frac{1}{\eta}}$$

$$= \sqrt{1.21 \cdot \frac{E_b}{\eta}}$$

$$\therefore \text{Probability error } P_e = Q\left(\sqrt{\frac{1.2 E_b}{\eta}}\right).$$

Thus the probability of error in FSK systems is

$$\boxed{P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{(0.6) E_b}{\eta}}\right) \quad (\text{or}) \quad P_e = Q\left(\sqrt{\frac{(1.2) \cdot E_b}{\eta}}\right)}$$

*** Error probability in QPSK :

In Quadrature Phase shift keying.

$$S(t) = \begin{cases} A_c \cos\omega_c t & ; \text{ logic } 10 \\ -A_c \cos\omega_c t & ; \text{ logic } 00 \\ A_c \sin\omega_c t & ; \text{ logic } 11 \\ -A_c \sin\omega_c t & ; \text{ logic } 01 \end{cases}$$

→ $S_1(t) = A_c \cos\omega_c t$, $S_2(t) = -A_c \cos\omega_c t$ $\quad \therefore P_1(t) = S_1(t) - S_2(t) = 2A_c \cos\omega_c t$.

→ $S_3(t) = A_c \sin\omega_c t$, $S_4(t) = -A_c \sin\omega_c t$ $\quad \therefore P_2(t) = S_3(t) - S_4(t) = 2A_c \sin\omega_c t$.

$\therefore$ Probability Error $P_e = P_{e_1} + P_{e_2}$

$$P_{e_1} = \frac{1}{2} \operatorname{erfc}\left(\frac{\gamma_1^2{}_{max}}{8}\right)^{1/2}, \quad P_{e_2} = \frac{1}{2} \operatorname{erfc}\left(\frac{\gamma_2^2{}_{max}}{8}\right)^{1/2}.$$

✓ $$\gamma_1^2{}_{max} = \frac{2}{\eta} \int_0^{T_b/2} [P_1(t)]^2 dt = \frac{2}{\eta} \int_0^{T_b/2} [A_c^2 \cos\omega_c t]^2 dt.$$

✓ $$\gamma_2^2{}_{max} = \frac{2}{\eta} \int_0^{T_b/2} [P_2(t)]^2 dt = \frac{2}{\eta} \int_0^{T_b/2} [2A_c \sin\omega_c t]^2 dt$$

$$\therefore \gamma_{1\,max}^2 = \frac{2}{\eta}\int_0^{T_b/2}\left[2A_c\cdot\cos w_c t\right]^2 dt$$

$$= \frac{8A_c^2}{\eta}\int_0^{T_b/2}\cos^2 w_c t\, dt$$

$$= \frac{8A_c^2}{\eta}\int_0^{T_b/2}\left(\frac{1+\cos 2w_c t}{2}\right) dt$$

$$= \frac{4A_c^2}{\eta}\cdot\left[\frac{T_b}{2} + \underbrace{\frac{\sin 2w_c t}{2w_c}\Bigg|_0^{T_b/2}}_{0}\right]$$

$$= \frac{4A_c^2}{\eta}\cdot\frac{T_b}{2}$$

$$= \frac{2A_c^2 T_b}{\eta} \qquad \therefore \boxed{\gamma_{1\,max}^2 = \frac{2A_c^2 T_b}{\eta}}$$

Probability Error $\quad P_{e_1} = \frac{1}{2}\,\mathrm{erfc}\left[\frac{\gamma^2_{max}}{8}\right]^{1/2}$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{2A_c^2\cdot T_b}{8\cdot\eta}\right]^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{A_c^2\cdot T_b}{2}\cdot\frac{1}{2\eta}\right]^{1/2}$$

$$= \frac{1}{2}\,\mathrm{erfc}\left[\frac{E_b}{2\eta}\right]^{1/2} \qquad \left(\because E_b = P_c\cdot T_b = \frac{A_c^2}{2}\cdot T_b.\right)$$

$$P_{e_1} = \frac{1}{2}\,\mathrm{erfc}\sqrt{\frac{E_b}{2\eta}}$$

Illy

$$\gamma_{2\,max}^2 = \frac{2}{\eta}\int_0^{T_b/2}\left[2A_c \sin w_c t\right]^2 dt$$

$$= \frac{8A_c^2}{\eta}\int_0^{T_b/2}\sin^2 w_c t\, dt$$

$$= \frac{8A_c^2}{\eta}\int_0^{T_b/2}\left[\frac{1-\cos 2w_c t}{2}\right] dt$$

$$= \frac{4A_c^2}{\eta}\left[\frac{T_b}{2} - \underbrace{\frac{\sin 2w_c t}{2w_c}\Bigg|_0^{T_b/2}}_{0}\right]$$

$$= \frac{4A_c^2}{\eta}\cdot\frac{T_b}{2}$$

$$= \frac{2A_c^2 T_b}{\eta} \qquad \therefore \boxed{\gamma_{2\,max}^2 = \frac{2A_c^2 T_b}{\eta}}$$

Similarly Probability error $\boxed{P_{e_2} = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2\eta}}\right)}$

∴ Total probability $P_e = P_{e_1} + P_{e_2}$

$$= \frac{1}{2}\,\text{erfc}\left(\sqrt{E_b/2\eta}\right) + \frac{1}{2}\,\text{erfc}\left(\sqrt{E_b/2\eta}\right)$$

$$= \text{erfc}.\sqrt{E_b/2\eta}$$

∴ $\boxed{P_e = \text{erfc}\left(\sqrt{\frac{E_b}{2\eta}}\right)}$ For QPSK.

In terms of Q-function. Probability of error $P_e = Q\left(\frac{\gamma_{1max}}{2}\right) + Q\left(\frac{\gamma_{2max}}{2}\right)$

ie $P_e = P_{e_1} + P_{e_2}$.

∴ $$\gamma_{1max}^2 = \frac{2A_c^2 \cdot T_b}{\eta}$$

$$\gamma_1 max = \sqrt{\frac{2A_c^2 \cdot T_b}{\eta}}$$

$$\Rightarrow \frac{\gamma_1 max}{2} = \sqrt{\frac{\cancel{2} A_c^2 \cdot T_b}{\cancel{2}4\eta}} = \sqrt{\frac{A_c^2 \cdot T_b}{2} \cdot \frac{1}{\eta}} = \sqrt{E_b/\eta}$$

$$\text{lly} \quad \frac{\gamma_2 max}{2} = \sqrt{E_b/\eta}$$

∴ Probability of error $P_{e_1} = Q\left(\frac{\gamma_{1}max}{2}\right) = Q\left(\sqrt{\frac{E_b}{\eta}}\right)$

$$P_{e_1} = Q\left(\frac{\gamma_2 max}{2}\right) = Q\left(\sqrt{\frac{E_b}{\eta}}\right).$$

∴ Total probability of error

$$P_e = P_{e_1} + P_{e_2}$$

$$= Q\left(\sqrt{\frac{E_b}{\eta}}\right) + Q\left(\sqrt{\frac{E_b}{\eta}}\right).$$

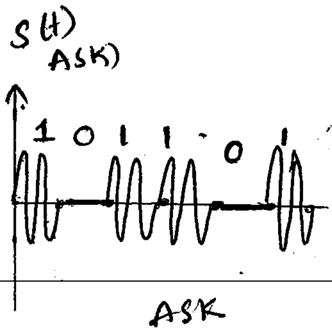
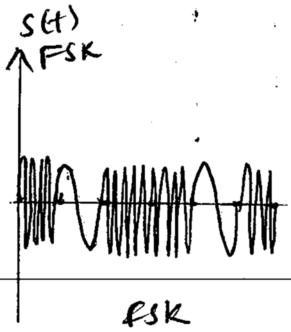
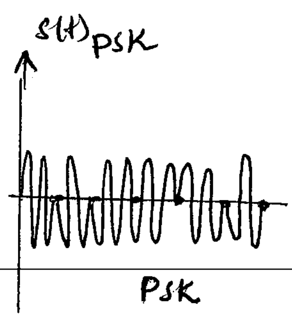
$$= 2Q\left(\sqrt{\frac{E_b}{\eta}}\right)$$

∴ $\boxed{P_e = 2Q\left(\sqrt{\frac{E_b}{\eta}}\right)}$.

Thus the probability of error in <u>QPSK</u> system is

$$\boxed{P_e = \text{erfc}\left(\sqrt{\frac{E_b}{2\eta}}\right) \quad (\text{or}) \quad P_e = 2Q\left(\sqrt{E_b/\eta}\right).}$$

Comparison of ASK, FSK & PSK systems:

Parameter	ASK	FSK	PSK
1. Expansion	Amplitude shift Keying & OOK ON-OFF Keying	Frequency shift Keying	Phase shift Keying.
2. Definition 1 - Symbol '1' 0 - Symbol '0'	$S(t) = \begin{cases} A_c \cos \omega_c t ; & 1 \\ 0 ; & 0 \end{cases}$	$S(t) = \begin{cases} A_c \cos \omega_1 t ; & 1 \\ A_c \cos \omega_2 t ; & 0 \end{cases}$	$S(t) = \begin{cases} A_c \cos \omega_2 t ; & 1 \\ -A_c \cos \omega_2 t ; & 0 \end{cases}$
3 Bandwidth (Hz)	$2f_b$ & $2/T_b$ T_b - Bit duration.	$\geq 4f_b$ & $\geq 4/T_b$	$2f_b$ & $2/T_b$.
4. Distance between '0' & '1'	$\sqrt{E_b}$	$\sqrt{2E_b}$	$2\sqrt{E_b}$.
5 Power	Moderate	Less	More.
6. Noise immunity	less	high	very high
7. Probability error P_e	$\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{4\eta}}\right)$ (or) $Q\left(\sqrt{\frac{E_b}{2\eta}}\right)$.	$\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{0.6 \cdot E_b}{\eta}}\right)$ (or) $Q\left(\sqrt{\frac{(1.2) E_b}{\eta}}\right)$.	$\frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{\eta}}\right)$ (or) $Q\left(\sqrt{\frac{2E_b}{\eta}}\right)$
8. Hardware Complexity ✓ Coherent Non coherent	High Low	High Low	Very High Low.
9 let data 101101 Representation of Digital modulation techniques	 ASK	 FSK	 PSK

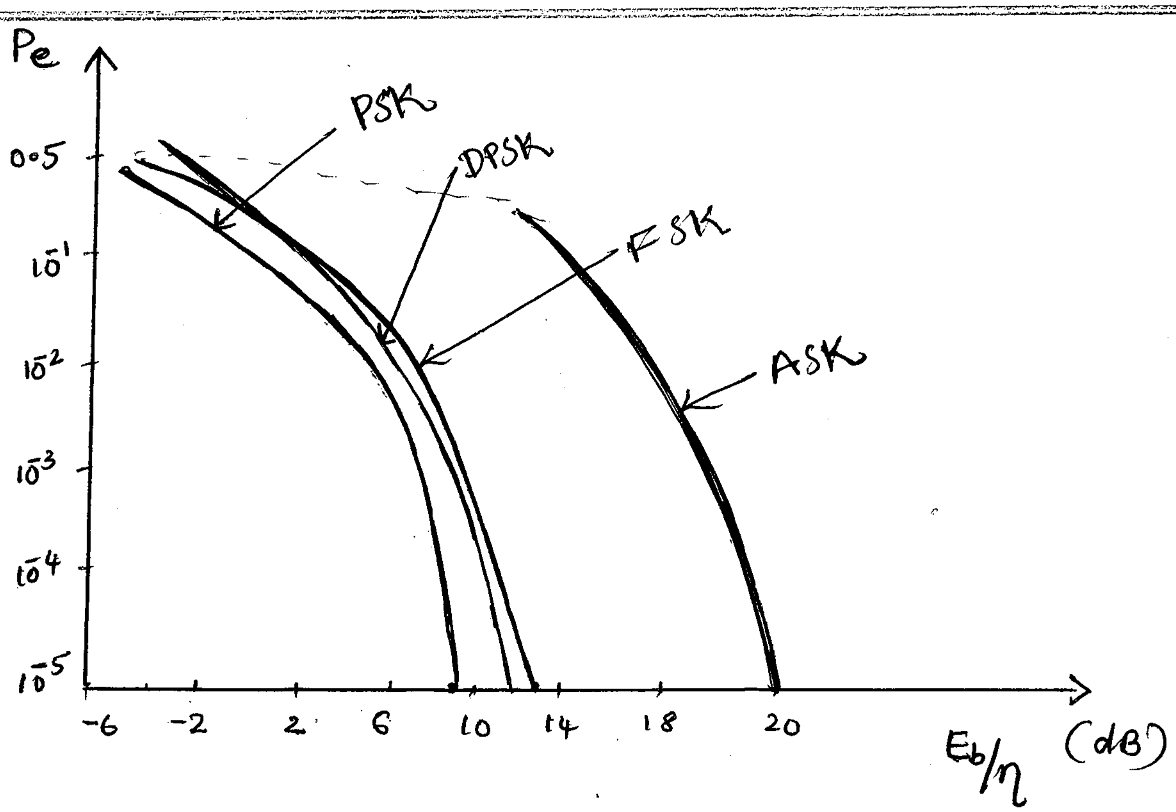


Fig: Probability of error for ASK, FSK, PSK & DPSK.

FSK detection using PLL: PLL - phase Locked Loop.

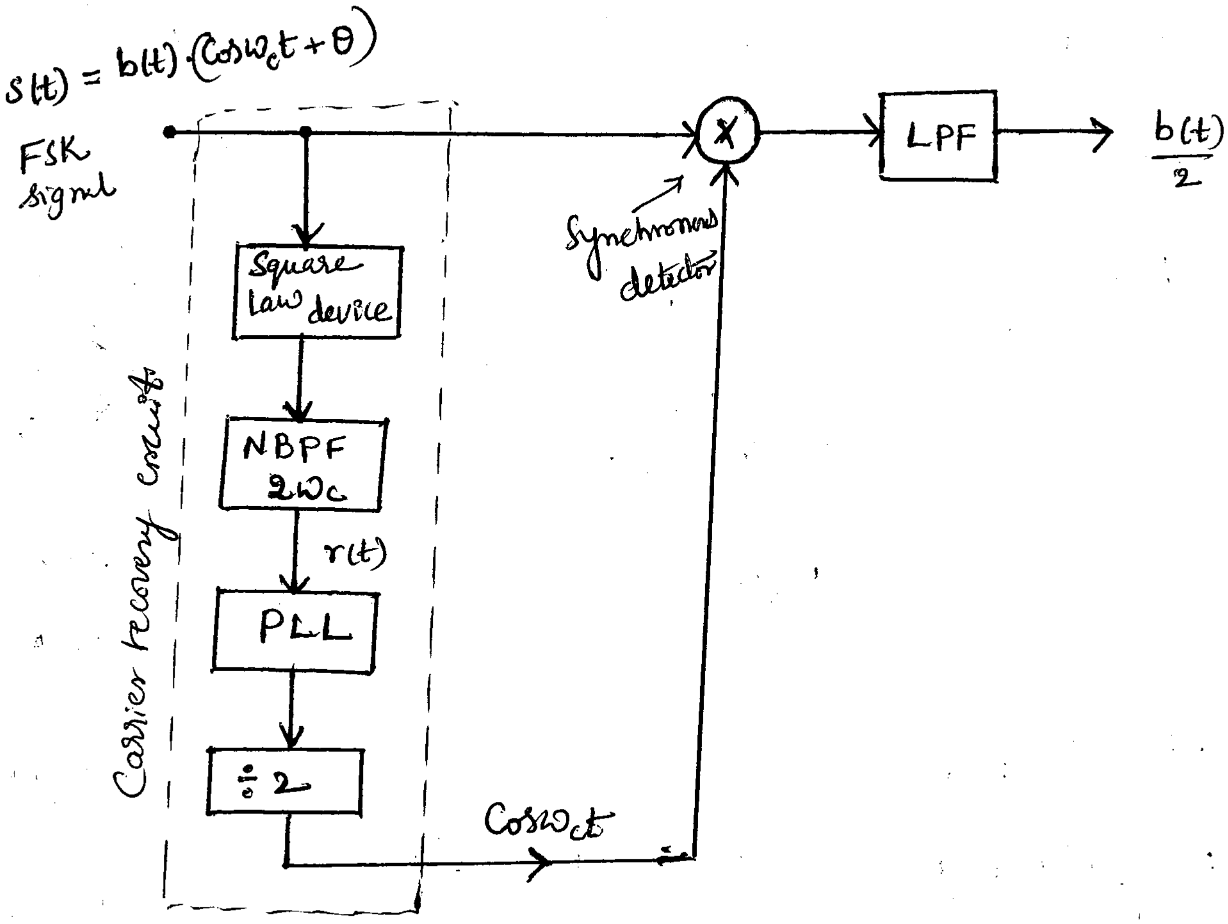


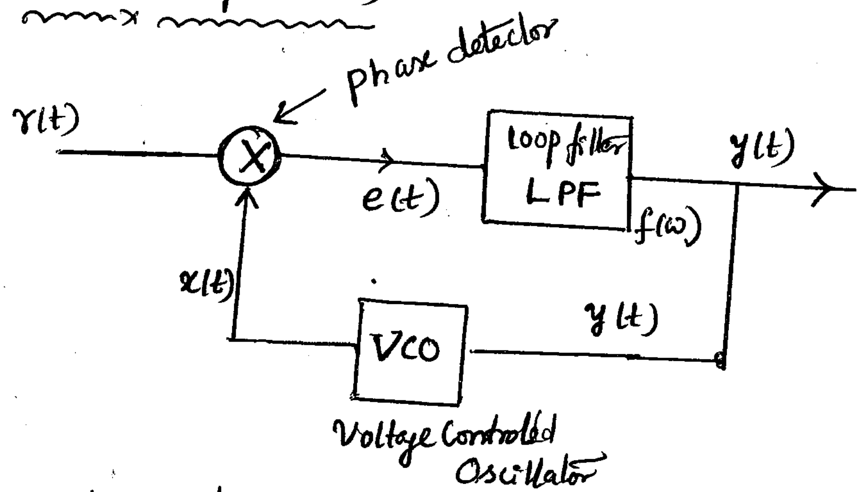
Fig: FSK detection using PLL.

- First the received signal is squared to generate the signal.

$$\cos^2(\omega_c t+\theta) = \frac{1+\cos 2(\omega_c t+\theta)}{2} = \frac{1}{2}+\frac{1}{2}\cos 2(\omega_c t+\theta)$$

- The dc component is removed by the NBPF, whose pass band centered around $2f_c$.
- The frequency divider (Composed of a flipflop & NBPF tuned to f_c) is used to generate the waveform $\cos(\omega_c t+\theta)$.

Phase Locked loop (PLL):



- It is a heart of the all synchronization circuits.
- VCO generated signal is the phase of a locally generated replica of the incoming signal.
- It is nearly used in frequency demodulators.
- PLL have three basic components
  - (a) Phase detector or multiplier
  - (b) Loop filter & LPF
  - (c) Voltage Controlled Oscillator
- multiplier which multiplies FM signal & FSK signal and VCO output.
- LPF, the function of which is to remove high frequency components contained in the multiplier output signal, these variations in the error signal.
- VCO is a device that produce the carrier replica.

VCO is an Oscillator whose output frequency is a linear function of its input voltage over some range of input & output.

Introduction to M-ary Signalling:

- The main requirements in the communication system is less bandwidth, less transmission power & less hardware complexity.
- In digital modulation scheme, to reduce the bandwidth in binary signalling scheme, M-ary signalling scheme is used.

* General Expression for PSK

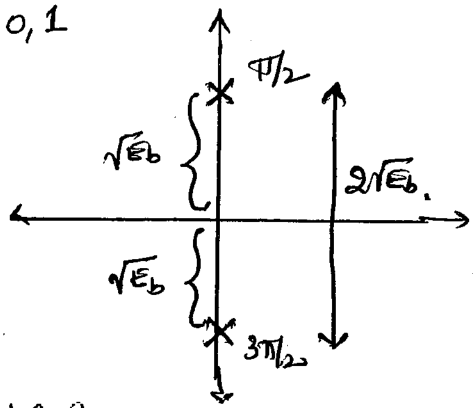
$$S(t)_{M\text{-}PSK} = A_c \cos(\omega_c t + \phi_m)$$

where $\phi_m = -(2m+1)\frac{\pi}{M}$; $m = 0, 1, 2, \ldots M-1$

⇒ Let $M = 2$, $\phi_m = -(2m+1)\pi/2$ $m = 0, 1$

BPSK ∴ $m=0$, $\phi_0 = -\pi/2$

$m = 1$, $\phi_1 = -3\pi/2$



when $M=4$ ie **QPSK**.

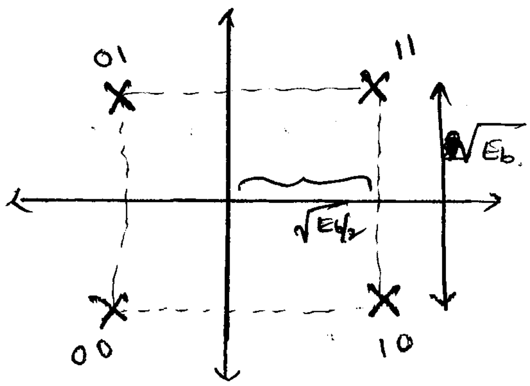
$\phi_m = -(2m+1)\pi/4$; $m = 0, 1, 2, 3$.

$m=0$, $\phi_0 = -\pi/4$

$m=1$, $\phi_1 = -3\pi/4$

$m=2$, $\phi_2 = -5\pi/4$

$m=3$, $\phi_3 = -7\pi/4$.



Ily **M=8** → 8-PSK

$\phi_m = -(2m+1)\pi/8$

$m=0$, $\phi_0 = -\pi/8$

$m=1$, $\phi_1 = -3\pi/8$

$m=2$, $\phi_2 = -5\pi/8$

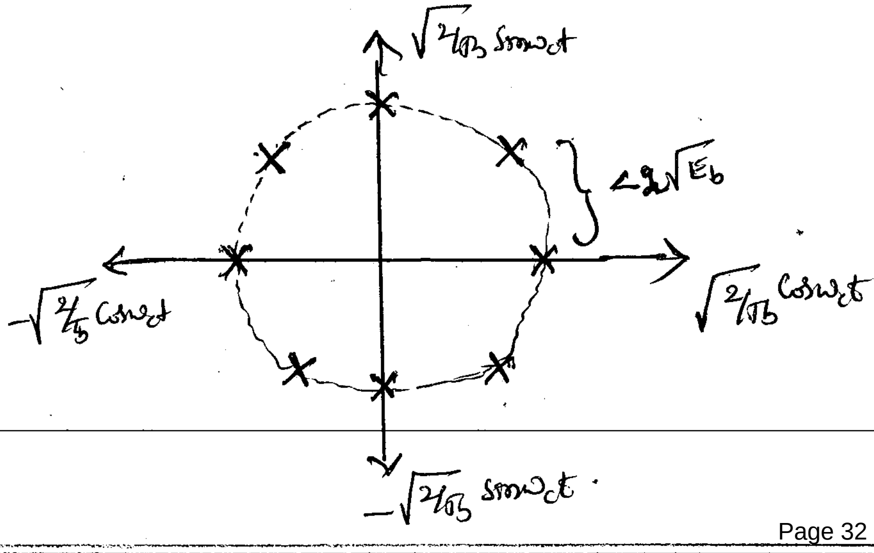
$m=3$, $\phi_3 = -7\pi/8$

$m=4$, $\phi_4 = -9\pi/8$

$m=5$, $\phi_5 = -11\pi/8$

$m=6$, $\phi_6 = -13\pi/8$

$m=7$, $\phi_7 = -15\pi/8$.



Quadrature Phase Shift Keying (QPSK):

- ✓ QPSK is an extension of binary PSK.
- ✓ In binary data transmission, we transmit only one of two possible signals during each bit interval T_b,
- ✓ In M-ary data transmission, it is possible to send any one of M-possible signals, during each signalling interval T

ie The no. of possible signal is $M = 2^n$

where n is an integer.

The signalling interval is $T = nT_b$.

* QPSK is an example of M-ary data transmission with $M=4$.
- ✓ In QPSK, one of four possible signals is transmitted during each signalling interval, with each signal uniquely related to a <u>dibit</u>.

ie. The four dibits may be 00, 01, 10, 11 in natural coded form (or) 10, 00, 11, 01 in gray encoded form.

∴ In QPSK system we may represent the four possible dibits in Gray encoded form by transmitting a sinusoidal carrier with one of the four values as

$$S(t) = \begin{cases} A_c\cos(2\pi f_c t - 3\pi/4) & ; \text{ dibit } 00^{(--)} \\ A_c\cos(2\pi f_c t - \pi/4) & ; \text{ dibit } 10^{(+-)} \\ A_c\cos(2\pi f_c t + \pi/4) & ; \text{ dibit } 11^{(++)} \\ A_c\cos(2\pi f_c t + 3\pi/4) & ; \text{ dibit } 01^{(-+)} \end{cases}$$

(OR)

$$S(t) = A_c\cos\left[\omega_c t - (2m+1)\pi/4\right] \quad ; m = 0, 1, 2, 3, \quad M=4$$

$$= A_c\left[\cos\omega_c t \cos(2m+1)\pi/4 + \sin\omega_c t \cdot \sin(2m+1)\pi/4\right]$$

$$s(t) = b_e(t)\cdot A_c\cdot\cos(2\pi f_c t) + b_o(t)\cdot A_c\sin(2\pi f_c t)$$

$$s(t) = b_e(t)\cdot\sqrt{E_b/T_s}\cos 2\omega_c t + b_o(t)\sqrt{\frac{E_b}{T_s}}\sin\omega_c t \qquad (\because T_s = 2T_b)$$

where $b_e(t) = \sqrt{2} \cdot \cos(2m+1)\pi/4$

$b_o(t) = \sqrt{2} \sin(2m+1)\pi/4$.

ie

$b_e(t) = \sqrt{2} \cos(2m+1)\pi/4$

$m=0,\ b_e(t) = \sqrt{2} \times \frac{1}{\sqrt{2}} = +1$

$m=1,\ b_e(t) = \sqrt{2} \times \cos 3\pi/4 = -1$

$m=2;\ b_e(t) = \sqrt{2} \cos 5\pi/4 = -1$

$m=3,\ b_e(t) = \sqrt{2} \cos \frac{7\pi}{4} = +1$

$\Rightarrow$ 00 → − − Third Q_3

11 → + + First Q_1

$b_o(t) = \sqrt{2} \sin(2m+1)\pi/4$.

$m=0 .\ b_o(t) = \sqrt{2} \sin \pi/4 = +1$

$m=1\ \ b_o(t) = \sqrt{2} \sin 3\pi/4 = +1$

$m=2\ \ b_o(t) = \sqrt{2} \sin 5\pi/4 = -1$

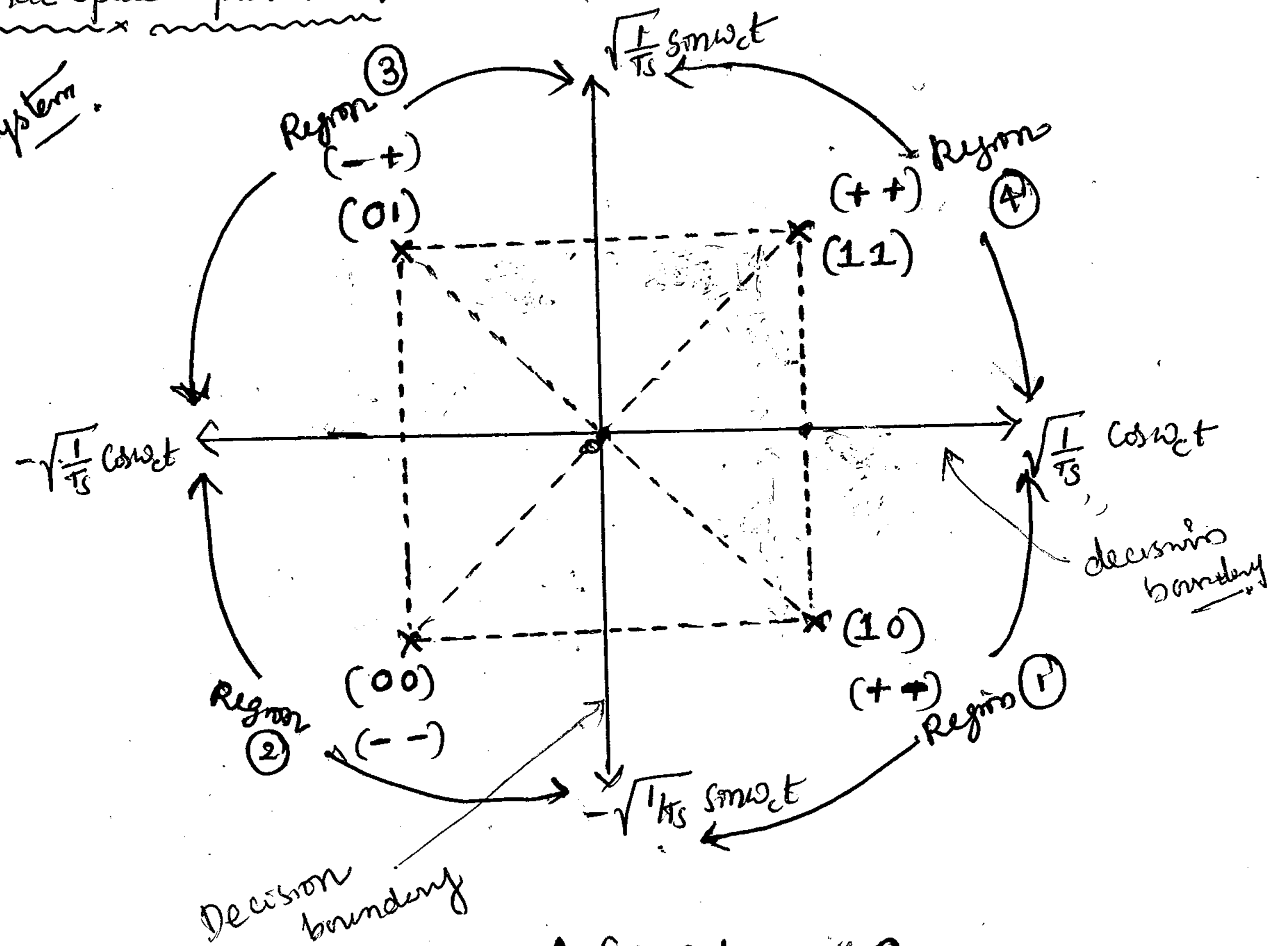
$m=3\ \ b_o(t) = \sqrt{2} \sin 7\pi/4 = -1$.

01 → − + → Second Q_2

10 → + − → Fourth Q_4.

Signal Space representation:

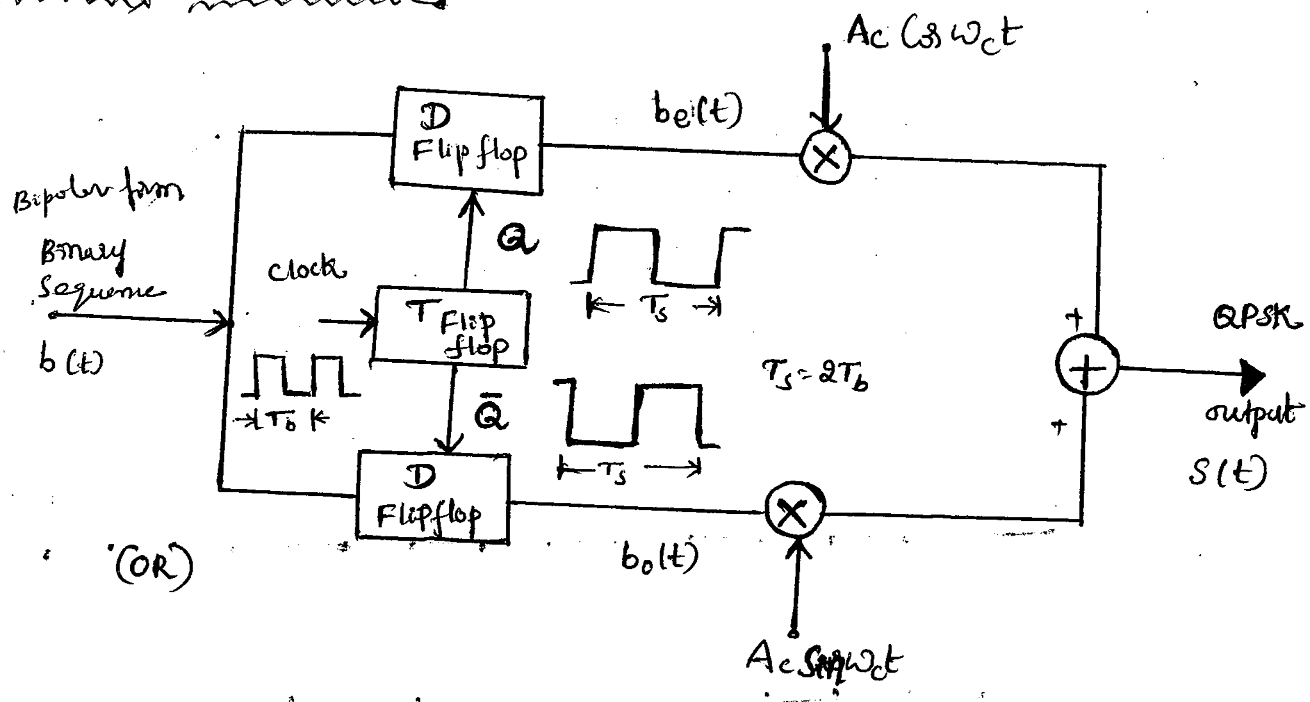
for QPSK system.



∴ QPSK

$$s(t) = \begin{cases} A_c \cos\omega_c t \ ; & 10 \\ -A_c \cos\omega_c t \ ; & 00 \\ A_c \sin\omega_c t \ ; & 11 \\ -A_c \sin\omega_c t \ ; & 01 \end{cases}$$

QPSK Transmitter:



(OR)

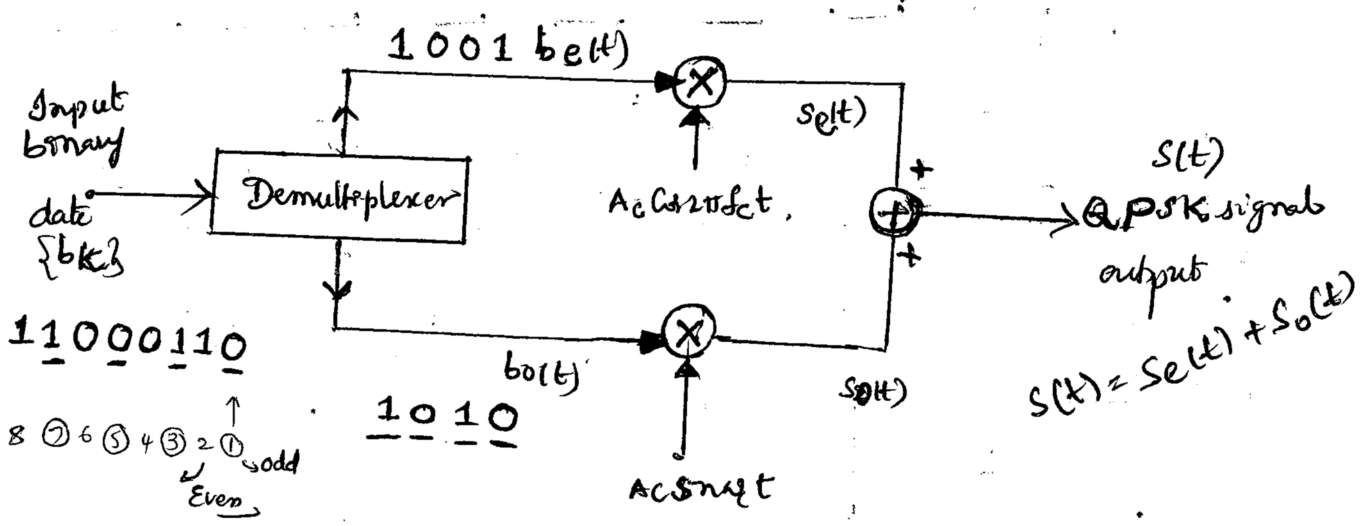


Fig: Block diagram of QPSK transmitter

Operation:

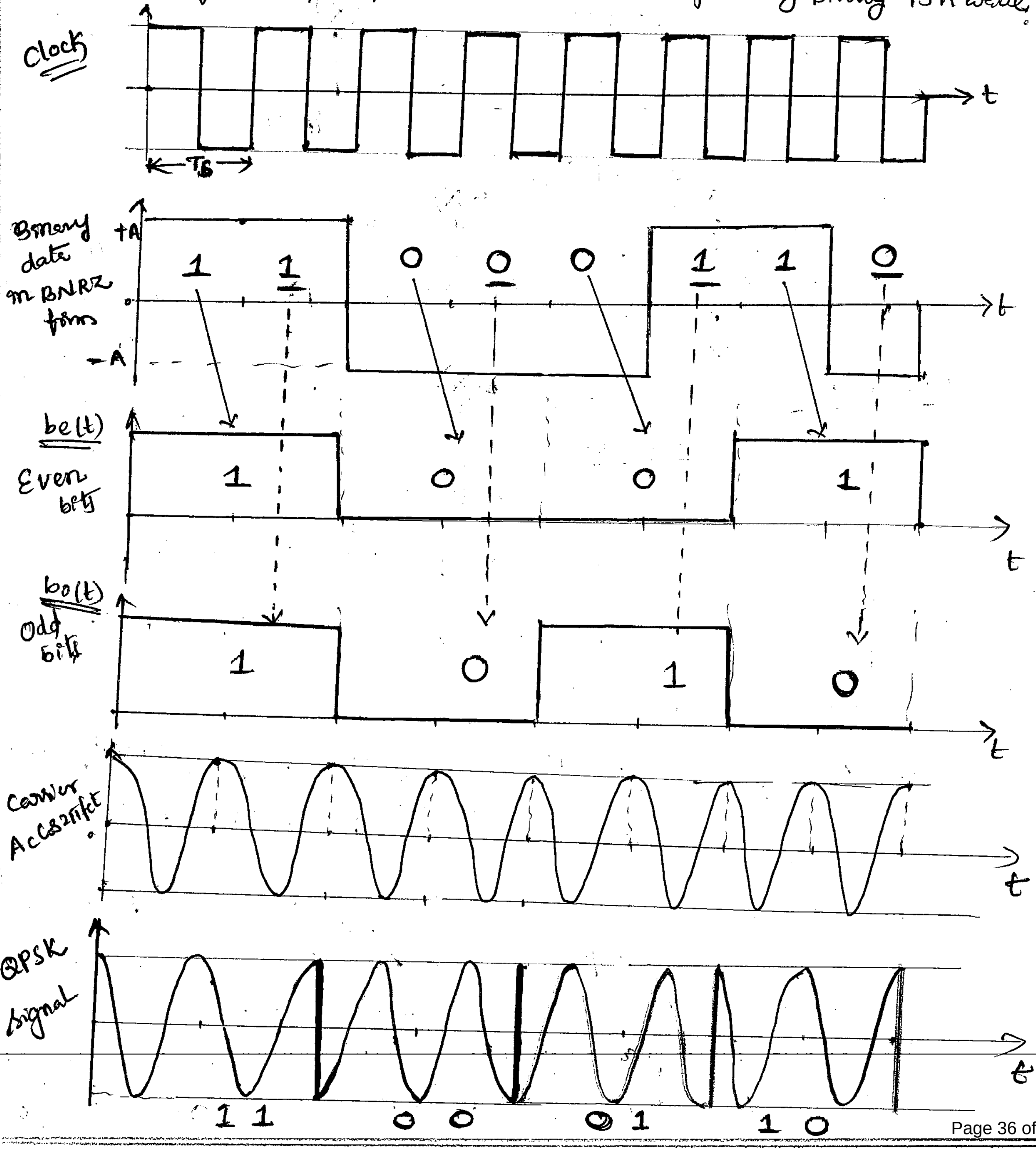
- ✓ The binary sequence b(t) is represented in bipolar form with symbol '0' and symbol '1'.
- ✓ The binary wave is divided by means of a demultiplexer into two seperate binary waves consisting of the odd & Even numbered input bits.
- ✓ These two binary waves are denoted by $b_e(t)$ and $b_o(t)$ are used to modulate a pair of quadrature carrier or orthogonal basis functions $A_c \cos 2\pi f_c t$ and $A_c \sin \omega_c t$.
- ✓ The result is a pair of binary psk waves which may be detected independently due to the orthogonality of $A_c \cos \omega_c t$ & $A_c \sin \omega_c t$.

✓ Finally the two binary PSK waves are added to produce the desired QPSK wave.

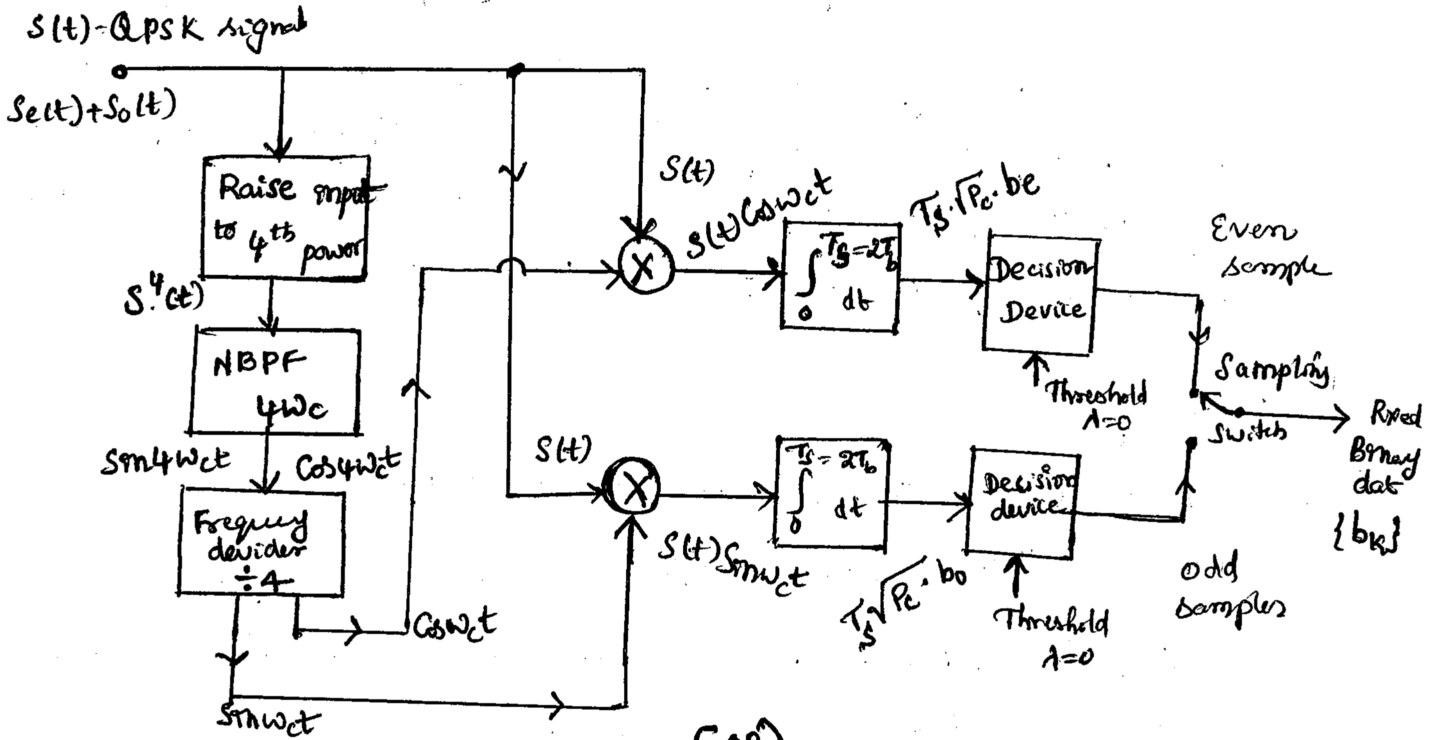
* The symbol duration T_s of a QPSK wave is twice as long as the bit duration T_b of the input binary wave $\therefore T_s = 2T_b$.

ie For a given bit rate $1/T_b$ a QPSK wave requires half the transmission bandwidth of the corresponding binary PSK wave.

(or) For a given transmission bandwidth, a QPSK wave carries twice as many bits of information as the corresponding binary PSK wave.



QPSK Receiver :



(OR)

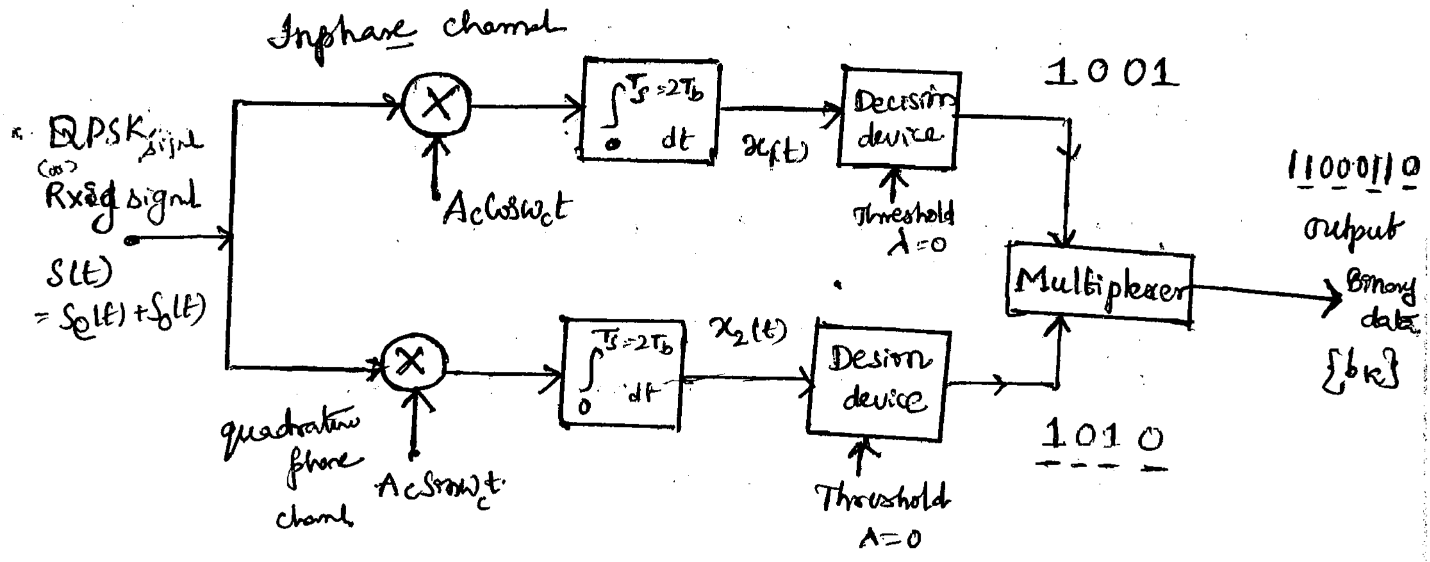


fig: Block diagram of Coherent QPSK Receiver.

Operation:

- ✓ The QPSK receiver consists of a pair of correlators with a common input '$S_o(t)+S_e(t)=S(t)$' and supplied with a locally generated pair of reference carrier signals $A_c\cos\omega_c t$ & $A_c\sin\omega_c t$.
- ✓ The correlator outputs $x_1(t)$ & $x_2(t)$ are each compared with a threshold of zero.

ie If $x_1(t) > 0 \rightarrow$ Symbol 1

$x_1(t) < 0 \rightarrow$ Symbol 0 at Inphase channel.

& upper channel.

&

$x_2(t) > 0 \rightarrow$ Symbol '1'

$x_2(t) < 0 \rightarrow$ Symbol '0'

at Quadrature phase channel & Lower channel.

✓ These two binary sequences at the inphase and quadrature channel outputs are combined in a multiplexer to produce the original binary sequence at the transmitter input with the minimum probability of symbol error.

Summary:

✓ QPSK transmits two bits at a time by assigning four (4) phases $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ i.e. $45°, 135°, 225°, 315°$.

✓ QPSK divides bit duration 'T_b' into '$2T_b$' bit duration such that even bit, odd bit with '$2T_b$' duration.

✓ PSK & DPSK transmits with duration T_b & the bandwidth is '$2f_b$' Hz

✓ QPSK transmits with duration $2T_b$ & the bandwidth $\frac{2f_b}{2} = f_b$ Hz

✓ The probability of error in QPSK system is

$$\boxed{P_e = \text{erfc}\left(\sqrt{\frac{E_b}{2\eta}}\right)}$$

where $E_b = \frac{A_c^2 T_b}{2}$

$$\gamma_{1\,max}^2 = \gamma_{2\,max}^2 = \frac{2A_c^2 T_b}{\eta}$$

In terms of Q-function the probability of error

$$\boxed{P_e = 2Q\left(\sqrt{\frac{E_b}{\eta}}\right)}$$

where $E_b = \frac{A_c^2 T_b}{2}$

✓ The main disadvantage of QPSK system is it requires more hardware complex circuits.

✓ $$S(t) = A_c \cos\omega_c t\,(b_e) + A_c \sin\omega_c t\,(b_o)$$

QPSK definition

$$S(t) = \begin{cases} A_c \sin\omega_c t & b_o = 1 & 11 \\ -A_c \sin\omega_c t & b_o = -1 & 01 \\ A_c \cos\omega_c t & b_e = 1 & 10 \\ -A_c \cos\omega_c t & b_e = -1 & 00 \end{cases}$$

Minimum Shift Keying : 'MSK' → (CPFSK).

In the coherent detection of binary FSK, the phase information contained in the received signal was not fully exploited other than to provide for synchronization of the receiver to the transmitter.

By proper utilization of the phase when performing detection, it is possible to improve the noise performance of the Rx significantly. It is acheived by a "continuous-phase frequency shift keying" (CPFSK)

ie Minimum Shift Keying is a special form of binary CPFSK with the change in the carrier frequency from symbol '0' to symbol '1' & symbol '1' to symbol '0' is equal to one half the bit rate of the incoming data

ie The CPFSK signal $S(t)$ can be expressed as

$$\boxed{S(t) = A_c \cos[2\pi f_c t + \theta(t)]}$$

where $\theta(t)$ - phase - continuous function of time.

f_c - nominal carrier frequency is choosen as the arithmetic mean of two frequencies f_1 & f_2

$$\therefore f_c = \frac{f_1+f_2}{2} \qquad \begin{array}{l} f_1 - \text{for logic 1} \\ f_2 - \text{for logic 0} \end{array}$$

The phase $\theta(t)$ increases or decreases linearly with time during each bit period of T_b sec.

ie $$\theta(t) = \theta(0) \pm \frac{\pi h}{T_b} t \qquad 0 \le t \le T_b.$$

(+ → logic '1', − → logic '0')

where the parameter $h = T_b(f_1 - f_2)$ ↳ 'deviation ratio'

$\therefore$ at $t = T_b$

$$\theta(T_b) - \theta(0) = \begin{cases} +\pi h \ ; & \text{logic '1'} \\ -\pi h \ ; & \text{logic '0'} \end{cases}$$

ie The sending of symbol 1 increases the phase of the CPFSK signal $s(t)$ by πh radians, where as the sending of symbol '0' reduces by πh radians.

$$\therefore \quad S(t) = A_c \cos\left[2\pi f_c t + \theta(t)\right]$$

$$S(t) = A_c\{\underbrace{\cos[\theta(t)]\cos(2\pi f_c t)}_{\text{Inphase Component}} - \underbrace{\sin[\theta(t)]\cdot\sin(2\pi f_c t)}_{\text{Quadrature component}}\}$$

The deviation ratio $h = 1/2$

then $\boxed{\theta(t) - \theta(0) = \pm\frac{\pi}{2T_b}t}$

The phase tree is as follows.

ie the phase of the CPFSK signal is an odd or even multiple of πh radians at odd or even multiples of the bit duration T_b

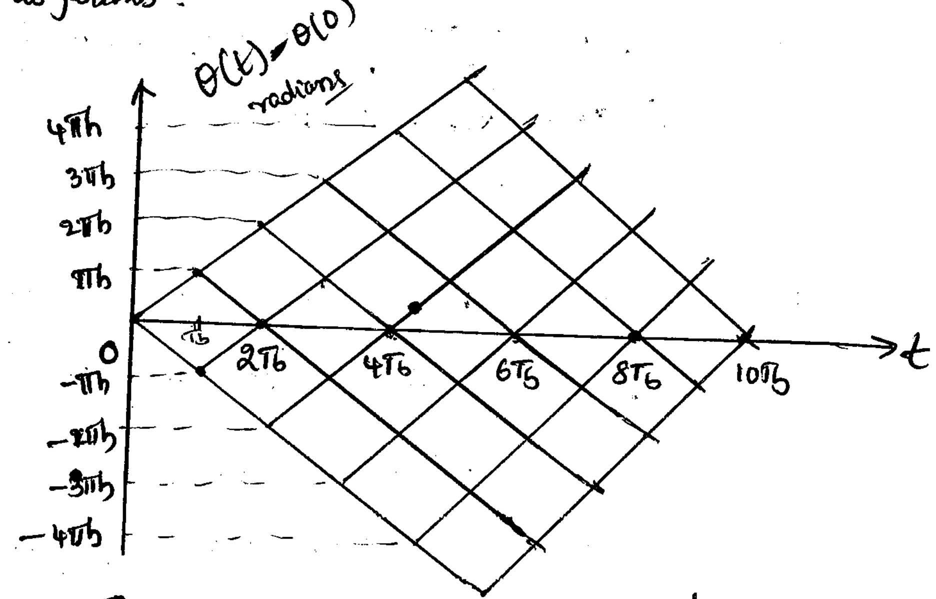


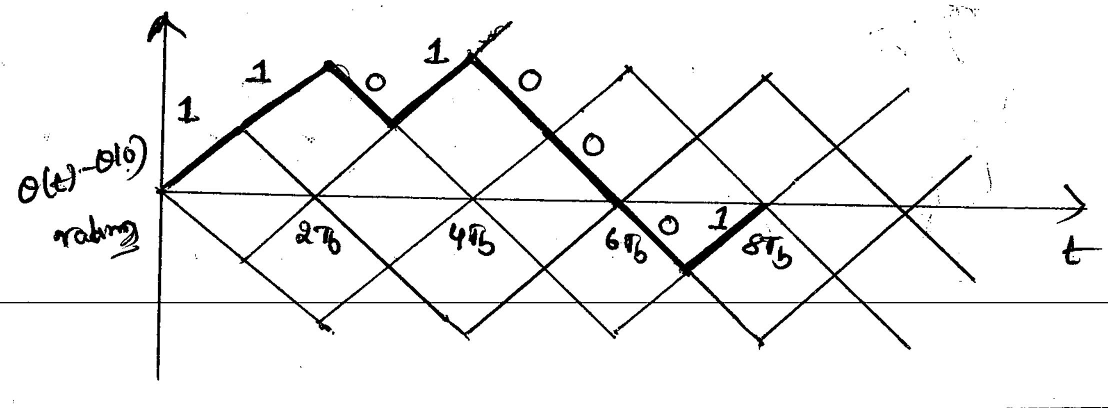
Fig: Phase tree

$$\therefore \quad S_I(t) = A_c[\cos\theta(t)] \qquad \theta(0)=0.$$

Inphase $\boxed{S_I(t) = \pm A_c \cos\left(\frac{\pi}{2T_b}t\right)}$ $\quad + \rightarrow \theta(0)=0$, $\; - \rightarrow \theta(0)=\pi$ $\quad$ 0 to T_b

lly, $\boxed{S_Q(t) = \pm A_c \sin\left(\frac{\pi}{2T_b}\right)t}$ $\quad + \rightarrow \theta(T_b) = \pi/2$, $\; - \rightarrow \theta(T_b) = -\pi/2$ $\quad$ T_b to $2T_b$.

Example: the binary sequence 1101000 1, with $\theta(0)=0$
$h=1/2$



ie A CPFSK signal with a deviation ratio of one-half is reffered as Minimum Shift Keying (MSK)

It is also referred as fast FSK.

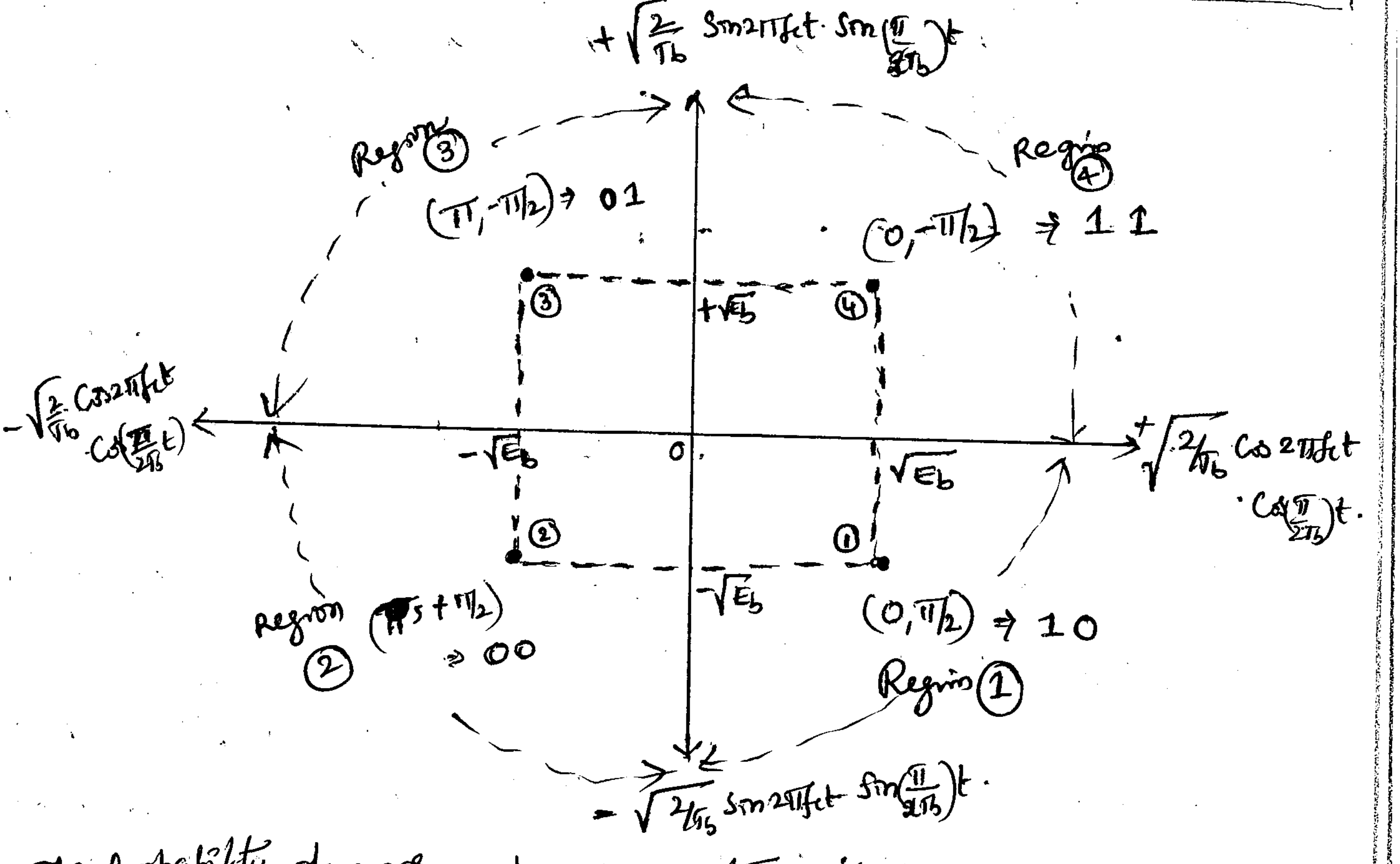
ie The coordinates of the message points for the QPSK signal is expressed in terms of signal energy per symbol 'E'.

But for the MSK signal expressed in terms of the signal energy per bit E_b with $E_b = E/2$.

Signal Space Representation:

ie

Txed Binary Symbol $0 \le t \le T_b$	phase states (radians)		Coordinates of message points		gray coded form
	$\theta(0)$	$\theta(T_b)$	S_1	S_2	
1	0	$+\pi/2$	$+\sqrt{E_b}$	$-\sqrt{E_b}$	1 0
0	π	$+\pi/2$	$-\sqrt{E_b}$	$-\sqrt{E_b}$	0 0
1	π	$-\pi/2$	$-\sqrt{E_b}$	$+\sqrt{E_b}$	0 1
0	0	$-\pi/2$	$+\sqrt{E_b}$	$+\sqrt{E_b}$	1 1



The probability of error of MSK system is

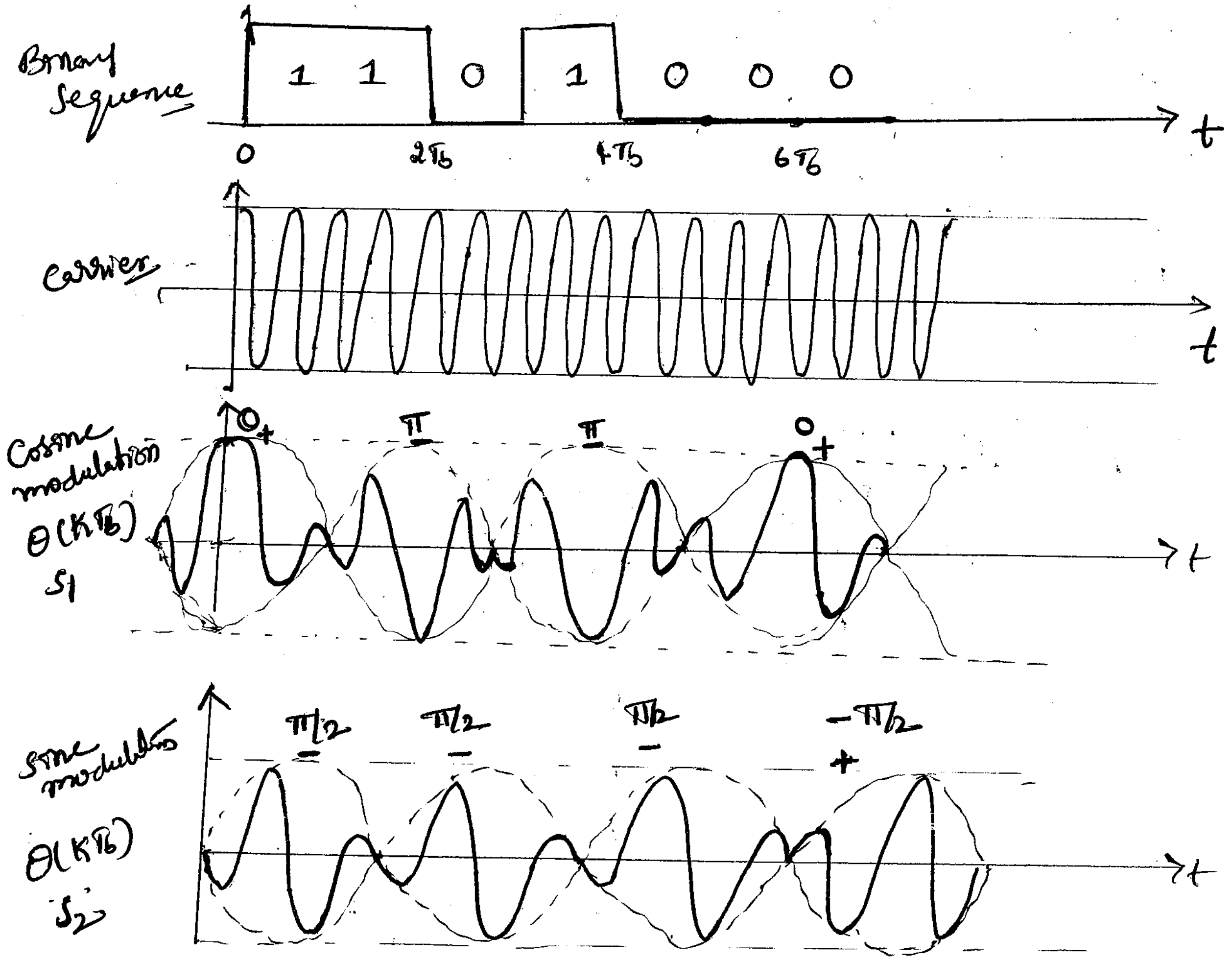
$$P_e = \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) - \frac{1}{4}\operatorname{erfc}^2\left(\sqrt{\frac{E_b}{N_0}}\right) \Rightarrow E_b/N_0 >> 1 \text{ then}$$

$$\boxed{P_e = \operatorname{erfc}\sqrt{\frac{E_b}{N_0}}}$$

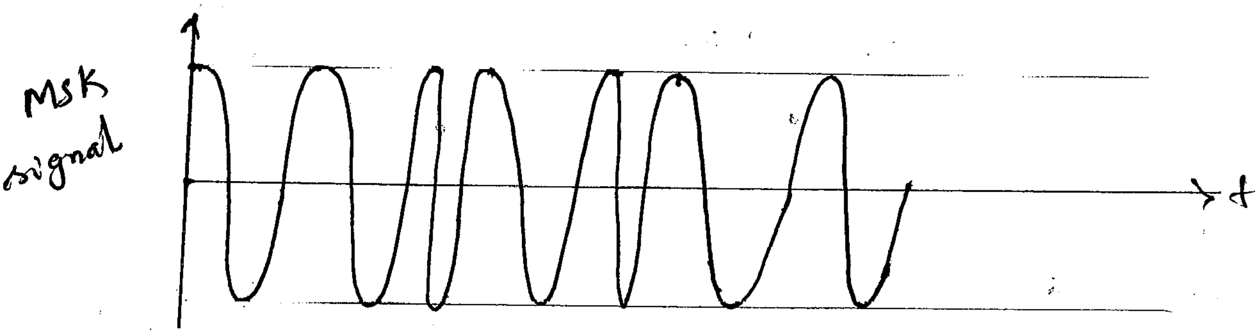
$N_0 = \eta/2$

Same as PSK

Waveform Representation:



The sum of the signals of S_1 and S_2 we will get MSK signal $S(t)$.



Thus, MSK may be viewed as a quadrature multiplexed frequency modulated wave.

The basic difference between QPSK and MSK is that in QPSK the phase shift $\theta(t)$ assume a constant distinct value for a entire duration of symbol depending on dibit Txed. Where as MSK the phase shift $\theta(t)$ varies with time along a distinct straight path depending on the debit being ~~sent~~ Txed.

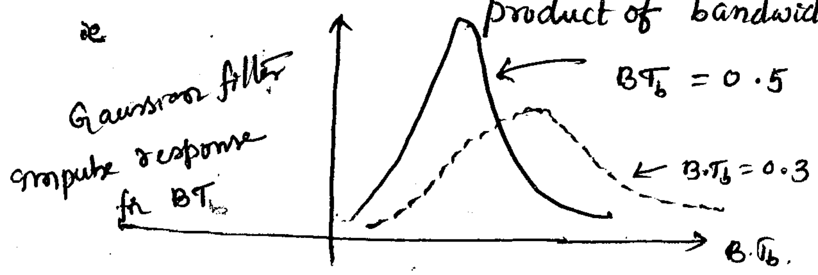
Gaussian Minimum Shift Keying (GMSK):

- The modified MSK using a gaussian filter is called "Gaussian filtered MSK & simply Gaussian MSK ie GMSK.
- In GMSK, a premodulation low pass Gaussian filter is used as a pulse shaping filter to reduce the bandwidth of the baseband signal before it is applied to the MSK modulator.
- The use of a filter with gaussian characteristics with MSK are
  - * Reduction in the transmitted bandwidth of the signal
  - * Uniform Envelop
  - * Reduction of side lope levels of the power spectrum.
  - * Reduction of adjacent channel interference
  - * Suppression of out-of band noise.
- The relationship between the premodulation filter bandwidth $f_{3\text{-dB}}$ and bit period T_b defines the bandwidth of the system.

$$B \cdot T_b = \frac{f_{3\text{-dB}}}{\text{Bit rate}} \rightarrow \text{*Change the Shape of the response of pulse shaping filter}$$

product of bandwidth and bit duration

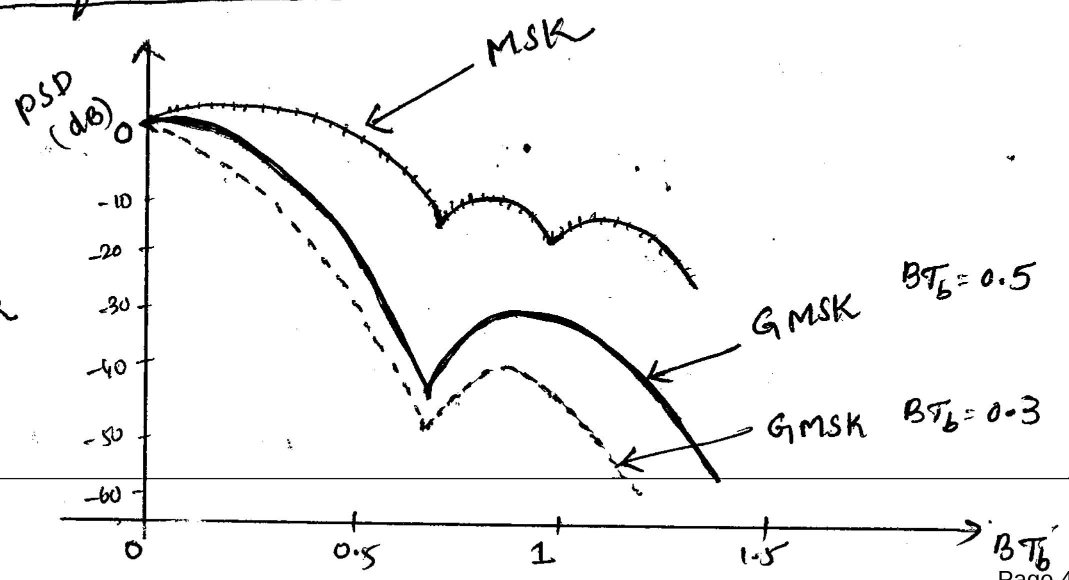
ie



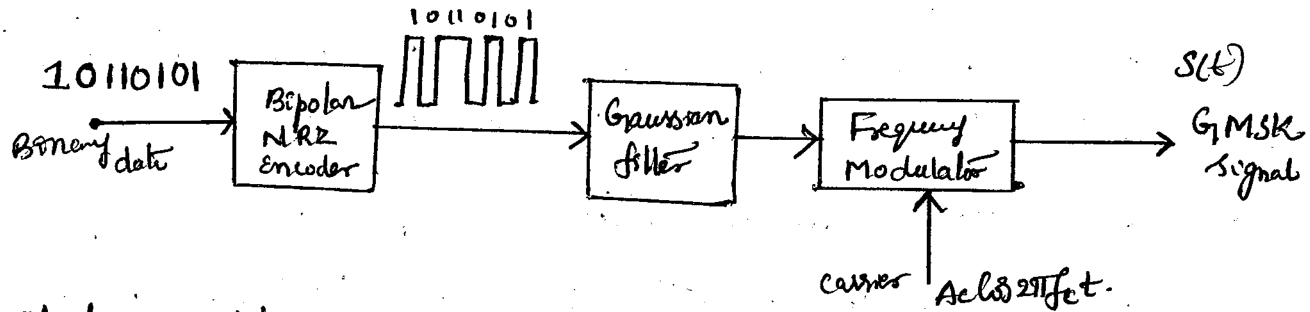
If $f = 2880$ Hz
$R_b = 9.6$ Kbps
$B \cdot T_b = 0.3$

PSD Comparison of MSK & GMSK:

ie
The channel spacing can be tighter for GMSK when compared to MSK for same adjacent channel interference.



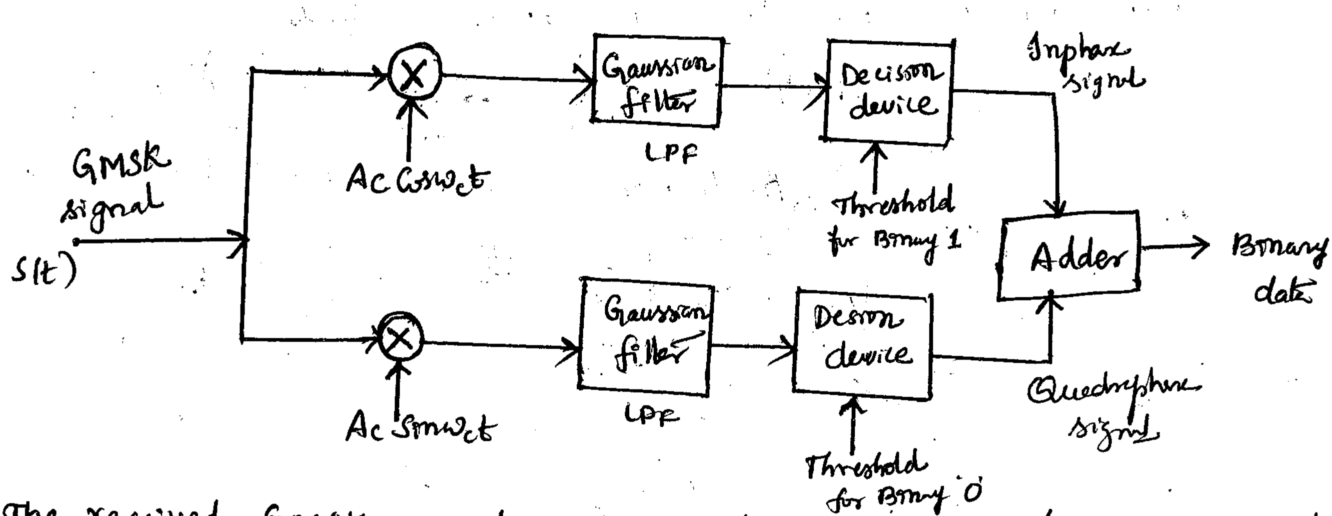
Generation of GMSK:



- ✓ The binary data sequence is encoded using a bipolar NRZ encoder
- ✓ The resulting data stream is then applied through a gaussian lowpass filter whose characteristics are gaussian in nature.
- ✓ The filtered signal acts as modulating signal which modulates the carrier signal in frequency modulator.
- ✓ The output of the frequency modulator is a GMSK signal.

Degeneration & Detection of GMSK:

- ✓ GMSK can be non-coherently detected as in FSK demodulator or coherently detected as in MSK demodulator.



- ✓ The received GMSK signal is applied to two product / balanced modulators whose carrier signals have a phase shift of 90° with each other.
- ✓ The output of product modulator is applied to Gaussian Low pass filter.
- ✓ The detection of binary data is done by the decision device.

Hence GMSK provides high spectrum efficiency, excellent power efficiency and a constant amplitude envelope.

* GMSK is widely used in the GSM cellular radio and PCS systems.

↳ Global System for Mobile Communication

M-ary Quadrature Amplitude Modulation : (QAM)

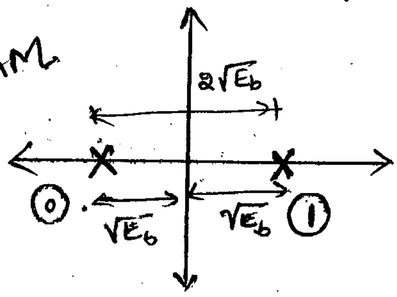
It is also called as Quadrature Amplitude Shift Keying (QASK).

(or) QAM = ASK + PSK.

- In this modulation scheme the carrier experiences amplitude as well as phase modulation.
- QAM is logical extension of QPSK, since it also consist of two independently amplitude modulated carriers in quadrature.
- QAM is also called as Amplitude phase shift Keying (APK).

(a) 2-QAM

ie BPSK



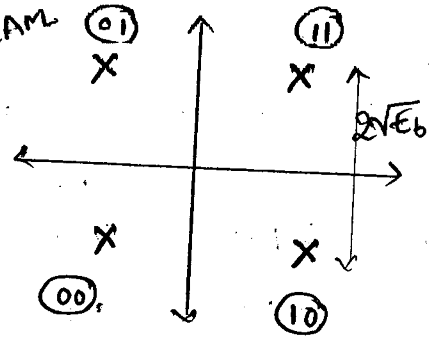
$B.W = \frac{2f_b}{N}$

1 block = 1 bit

$\boxed{B.W = 2f_b}$

(b) 4-QAM

ie. QPSK



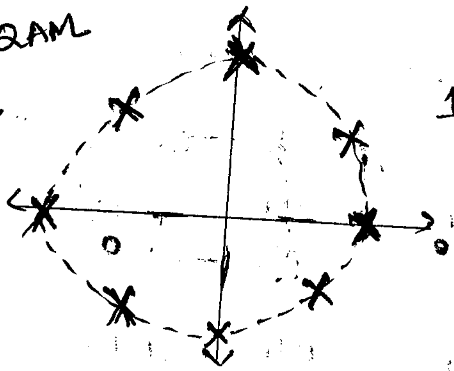
1 block = 2 bits

$B\,W = \frac{2f_b}{2} = f_b$

$\therefore \boxed{B.W = f_b}$

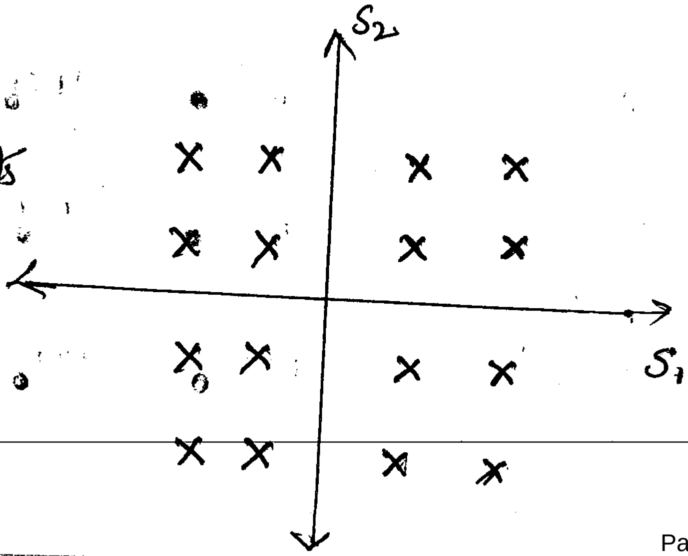
(c) 8-QAM

& 8-PSK



1 block = 3 bits $\therefore \boxed{B.W = \frac{2f_b}{3}}$

(d) 16-QAM $\Rightarrow$



1 block = 4 bits

Bandwidth $= \frac{2f_b}{4} = f_b/2$

$\therefore \boxed{B.W = \frac{f_b}{2}}$

The general form of M-ary QAM is defined by the TX'ed sig.

$$S(t) = a_i \cdot A_c \cos(2\pi f_c t) + b_i \cdot A_c \sin(2\pi f_c t)$$

(or)

$$\boxed{S(t) = \sqrt{\frac{2E_b}{T_b}}\, a_i \cos(2\pi f_c t) + \sqrt{\frac{2E_b}{T_b}}\, b_i \sin(2\pi f_c t)} ; \quad 0 \le t \le T_b$$

where E_b → the energy of the signal

a_i, b_i – a pair of independent integers whose chosen in accordance with the location of the message points.

T_b – bit duration.

* $S(t)$ consists of two phase quadrature carriers, each of which is modulated by a set of discrete amplitudes, hence the name called Quadrature amplitude Modulation (QAM).

$\phi_1(t) = \sqrt{2/T_b} \cos 2\pi f_c t \qquad \phi_2(t) = \sqrt{2/T_b} \sin 2\pi f_c t.$

The coordinates of message points $a_i\sqrt{E_b}$ and $b_i\sqrt{E_b}$.

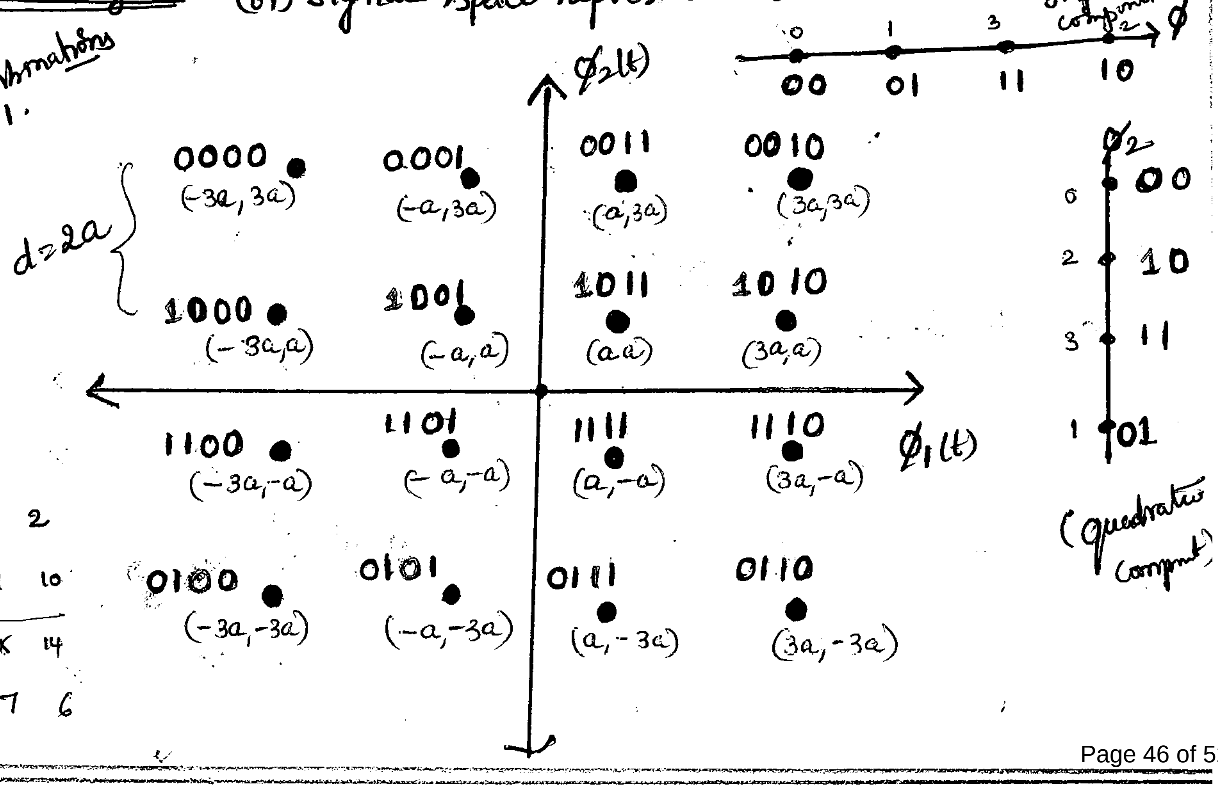
let $L=4$, thus $M = L^2 = 16$ QAM. & $M = \sqrt{L}$

$$[a_i\ b_i] = \begin{bmatrix} (-3a, 3a) & (-a, +3a) & (+a, 3a) & (3a, 3a) \\ (-3a, a) & (-a, a) & (a, a) & (3a, a) \\ (-3a, -a) & (-a, -a) & (a, -a) & (3a, -a) \\ (-3a, -3a) & (-a, -3a) & (a, -3a) & (3a, -3a) \end{bmatrix}$$

<u>Constellation diagram:</u> (or) Signal Space Representation.

L=4
$2^4 = 16$ combinations
0000 → 1111.

16 QAM.
M=16.



0	1	3	2
8	9	11	10
12	13	15	14
4	5	7	6

✓ The average normalized energy

$$E_s = \frac{1}{4}\left\{ (a^2+a^2) + (9a^2+a^2) + (a^2+9a^2) + (9a^2+9a^2) \right\}$$

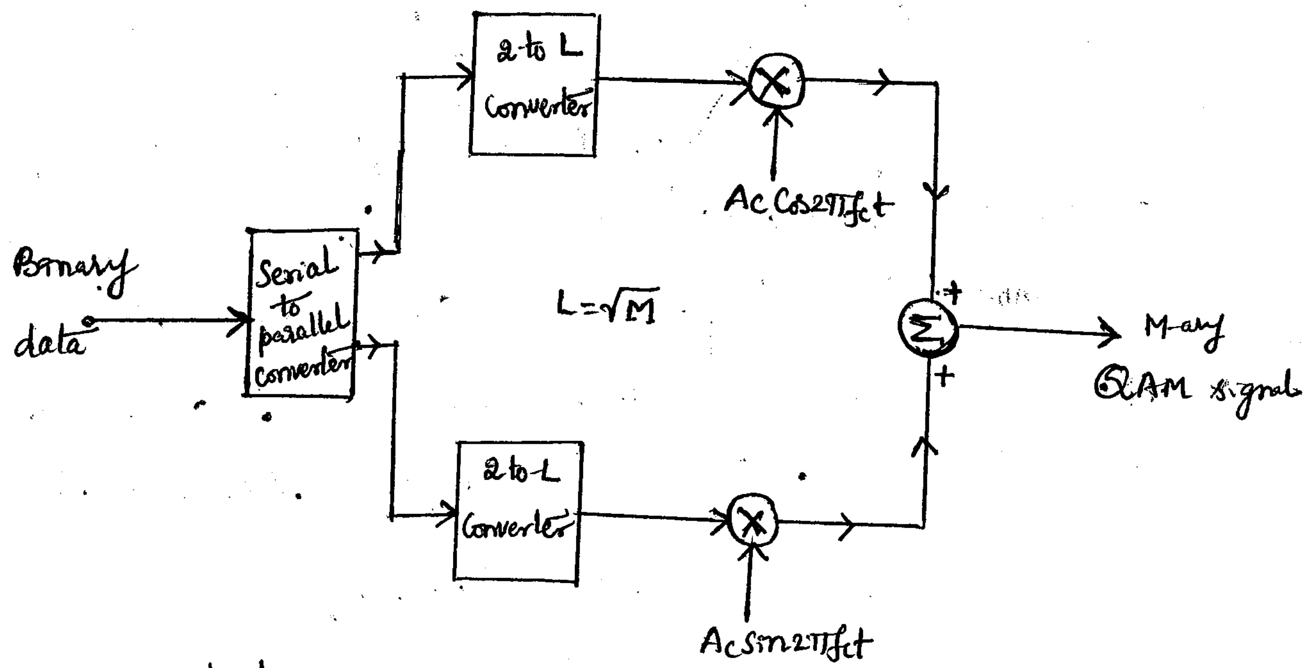
$$= \frac{1}{4}\, 40a^2 = 10a^2$$

$$E_s = 10a^2 \Rightarrow a^2 = 0.1 E_s \Rightarrow a = \sqrt{0.1 E_s}$$

the signal represents 4-bits so $E_s = 4 E_b$

$$\therefore \boxed{a = \sqrt{0.4 E_b}}, \quad d = 2a \Rightarrow \boxed{d = 2\sqrt{0.4 E_b}}$$

QAM Transmitter:

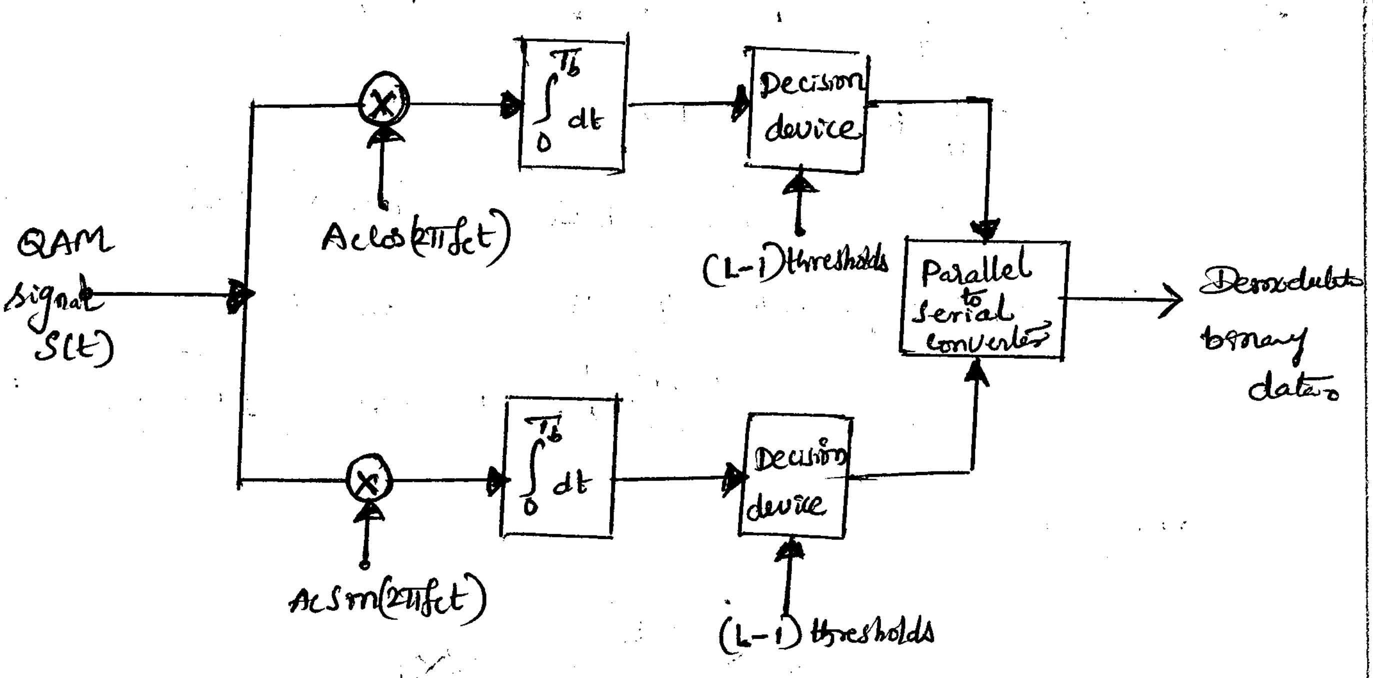


✓ The serial to parallel converter accepts a binary sequence at a bit rate $R_b = 1/T_b$ and produces two parallel binary sequences whose bit rates are $R_b/2$ each.

✓ The 2-L level converters, where $L = \sqrt{M}$, generate polar L-level signals in response to the respective in-phase and quadrature channel inputs.

✓ Quadrature carrier multiplexing of the two polar L-level signals so generated produces the desired M-ary QAM signal.

QAM Receiver :



- The decoding of each baseband channel is accomplished at the output of decision device circuit, which is designed to compare the L-levels signals against (L-1) decision thresholds.
- The two binary sequences so detected are combined in the paralled-to-serial converter to reproduce the original binary sequence.

* The probability of error in QAM system is

$$P_e = 2\left(1 - \frac{1}{L}\right) \operatorname{erfc} \sqrt{\frac{E_b}{N_0}} \quad \& \; N_0 = \eta/2.$$

$$L = \sqrt{M}$$

$$\boxed{P_e = 2\left(1 - \frac{1}{\sqrt{M}}\right) \operatorname{erfc}\left(\sqrt{\frac{2E_b}{\eta}}\right).}$$

Ily we have

More bandwidth efficient and used high data rate transmission in Terestrial micro wave digital radio

32-QAM

64-QAM ✓

128 QAM

256 QAM ✓

digital video broadcast cable & modems.

Pe The waveform MSK signal exhibits phase continuity, that is, there are no abrupt phase change as in QPSK.

Problems:

(1) Binary data is transmitted over an RF bandpass channel with a usable bandwidth of 10 MHz at a rate of 4.8×10^6 bits/sec using a ASK signalling method. The carrier amplitude at the receiver antenna is 1 mV and the noise PSD at the receiver input is 10^{-15} watts/Hz. Find the error probability of the receiver.

Sol Given data ASK system

$$B\cdot W = 10\,MHz$$
$$r_b = 4.8\times10^6 \text{ bits/sec}$$
$$A_c = 1\,mV$$
$$N_0 = \eta/2 = 10^{-15} \text{ watts/Hz} \quad \Rightarrow \quad \eta = 2\times10^{-15} \text{ watts/Hz}$$

$$\therefore T_b = \frac{1}{f_b} = \frac{1}{r_b} = \frac{1}{4.8\times10^6} = 0.208\,\mu\text{sec.}$$

$\therefore$ The probability of error in ASK is

$$P_e = \frac{1}{2}\,\text{erfc}\sqrt{\frac{E_b}{4\eta}} \quad \text{or} \quad P_e = Q\left(\sqrt{\frac{E_b}{2\eta}}\right)$$

where $E_b = \frac{A_c^2 T_b}{2} = \frac{(1\times10^{-3})^2\times 0.208\times10^{-6}}{2}$

$$E_b = 1.04\times10^{-3}$$

$$P_e = Q\left(\sqrt{\frac{1.04\times10^{-3}}{2\times 2\times10^{-15}}}\right) = Q(\sqrt{26}) \quad \therefore \boxed{P_e = Q(\sqrt{26})}$$

(or) $\boxed{P_e = 2\times10^{-7}}$

(2) A bandpass data transmitter scheme uses a PSK signalling scheme with

$S_1(t) = A_c\cos\omega_c t \quad ; 0\le t\le T_b \quad ; T_b = 0.2\,\text{msec.}$

$S_2(t) = -A_c\cos\omega_c t \quad ; 0\le t\le T_b \quad ; \omega_c = 10\pi/T_b.$

The carrier amplitude at the receiver input is 1 mV and the PSD of AWGN at input is 10^{-11} watts/Hz. Calculate the average bit error rate of the receiver.

Sol Given PSK system

$$S_1(t) = A_c\cos\omega_c t$$
$$S_2(t) = -A_c\cos\omega_c t$$

$$P(t) = S_1(t) - S_2(t) = 2A_c\cos\omega_c t$$

$T_b = 0.2\text{ m sec}$

$\omega_c = \frac{10\pi}{T_b}$

$2\pi f_c = \frac{10 \times \pi}{0.2 \times 10^{-3}}$

$\Rightarrow \boxed{f_c = 25\text{ KHz}}$

& AWGN PSD.

$N_0 = \eta/2 = 10^{-11}$ watts/Hz

$\therefore \eta = 2 \times 10^{-11}$.

& $f_b = \frac{1}{T_b} = r_b = \frac{1}{0.2 \times 10^{-3}} = 5000$

$\therefore \boxed{r_b = 5000 \text{ bits/sec.}}$

$\therefore$ Probability of error in PSK scheme is

$$P_e = Q\left(\frac{\gamma_{max}}{2}\right)$$

where $\gamma_{max}^2 = \frac{2}{\eta}\int_0^{T_b} p^2(t)\cdot dt$

$$= \frac{2}{\eta}\int_0^{T_b}[2A_c \cos\omega_c t]^2 dt$$

$$= \frac{2}{\eta} \times 4A_c^2 \cdot \int_0^{T_b} \cos^2\omega_c t\, dt$$

$$= \frac{8A_c^2}{\eta}\cdot\int_0^{T_b}\left[\frac{1+\cos 2\omega_c t}{2}\right] dt$$

$$= \frac{4A_c^2}{\eta}\cdot\left[\int_0^{T_b}1\cdot dt + \int_0^{T_b}\cos 2\omega_c t\, dt\right]$$

$$= \frac{4A_c^2}{\eta}T_b + \underbrace{\frac{\sin 2\omega_c t}{2\omega_c}\Big|_0^{T_b}}_{0}$$

$\gamma_{max}^2 = \frac{4A_c^2 T_b}{\eta} \Rightarrow \gamma_{max} = \sqrt{\frac{4A_c^2 T_b}{\eta}}$

$$\frac{\gamma_{max}}{2} = \sqrt{\frac{4A_c^2 T_b}{4\eta}} = \sqrt{\frac{A_c^2 T_b}{\eta}}$$

$$= \sqrt{\frac{(1 \times 10^{-3})^2 \times (0.2 \times 10^{-3})}{2 \times 10^{-11}}}$$

$$= \sqrt{\frac{\cancel{2} \times 10^{-10}}{\cancel{2} \times 10^{-11}}}$$

$$= \sqrt{10}$$

$\therefore$ Probability of error $\boxed{P_e = Q(\sqrt{10}) = 0.0008}$

- The average bit error rate $= r_b \times P_e = 5000 \times 0.0008 = 4$

$\therefore \boxed{\text{Average bit error rate} = 4 \text{ bits/sec.}}$

③ Binary data has to be transmitted over a telephone link that has a usable bandwidth of 3000Hz and a maximum achievable signal to noise ratio of 6 dB at its output.

(a) Determine the maximum signalling rate and P_e if a coherent ASK scheme is used for tx'ing binary data through the channel.

(b) If the data rate is maintained at 300 bits/sec, calculate the error probability.

Sol: Given data ASK system

let ASK signal requires a bandwidth of $3r_b$ Hz r bits/sec.

ie $B.W = 3000\,Hz$

$3r_b = 3000$

$r_b = 1000$ bits/sec $\qquad (SNR)_o = 6\,dB.$

(a) $T_b = 1/r_b = 1\,msec.$

ASK:

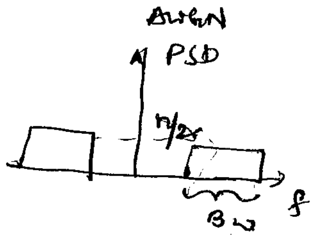
$$S(t) = \begin{cases} A_c \cos w_c t & ; \text{logic } 1 \\ 0 & ; \text{logic } 0 \end{cases}$$

∴ Average signal power $S_o = \dfrac{A_c^2/2}{2} = \dfrac{A_c^2}{4}$

Average Noise power $N_o = 2 \times PSD \times B.W$

$= 2 \times \dfrac{\eta}{2} \times 3000$

$\boxed{N_o = 3000\eta}$



∴ Signal to noise ratio $(SNR)_o = 6\,dB$

$$\Rightarrow \frac{\text{Avg signal power}}{\text{Noise power}} = 6\,dB \qquad 6\,dB = 10\log_{10}\frac{P_o}{P_i}$$

$\log_{10}\dfrac{P_o}{P_i} = 0.6$

$\dfrac{P_o}{P_i} = 10^{0.6}$

$= 3.98$

$\simeq 4$

$$\Rightarrow \frac{\frac{A_c^2}{4}}{3000\eta} = 4$$

$$\Rightarrow \frac{A_c^2}{12000\eta} = 4$$

$$\Rightarrow \frac{A_c^2}{\eta} = 48000$$

$$\Rightarrow \frac{A_c^2}{4\eta} \times T_b = \frac{48000}{4} \times 1\times10^{-3} = \frac{48}{4} = 12$$

∴ The probability of error in ASK is

$$P_e = Q\left(\sqrt{\frac{E_b}{2\eta}}\right) \;\&\; Q\left(\sqrt{\frac{A_c^2 T_b}{4\eta}}\right)$$

$$\therefore P_e = Q(\sqrt{12}) = Q(3.464) \cong 0.0003$$

$$\boxed{P_e = 0.0003} \Rightarrow \boxed{P_e = 3\times10^{-4}}$$

The maximum signalling rate $r_b = 1000$ bits/sec.

(b) If the bit rate is reduced to 300 bits/sec.

$\therefore r_b = 300$ bits/sec

$$T_b = \frac{1}{r_b} = 3.33 \text{ msec.}$$

$$\frac{A_c^2 T_b}{4\eta} = \frac{48000}{4} \times 3.33\times10^{-3}$$

$$= 12 \times 3.33$$

$$\cong 40$$

∴ Probability of error $P_e = Q\sqrt{\frac{A_c^2 T_b}{4\eta}}$

$$P_e = Q(\sqrt{40})$$

$$= Q(6.326)$$

$$\cong 10^{-10}$$

$$\therefore \boxed{P_e = 10^{-10}.}$$

All The Best
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Digital Communications

THE END

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