Discrete Structures and Graph Theory

Syllabus

Unit : 1 Mathematical logic

Statements and Notations - Connectives (Negation, Conjunction, Disconjunction, conditional and Bi conditional) - Statements Formulas and Truth Tables - well Formed Formulas, Tautologies - Equivalence of Formulas - Duality law - Tautological Implications - Normal Forms (DNF, CNF, PDNF, PCNF) - Theory of Inference for Statement Calculus: Validity using Truth tables - Rules of Inference - Consistency of premises and Indirect method of proof.

Unit : 2 predicate calculus

predicates - The Statement Function - Variables - Quantifiers - predicate Formulas - Free and Bound Variables - The universe of Discourse - Theory of Inference for predicate calculus: Valid Formulas and equivalences - Some valid Formulas over Finite universes - Special Valid Formulas Involving Quantifiers.

Unit : 3 Relations and Functions

Relations : properties of Binary Relations, equivalence - closure of Relations - compatibility and partial ordering Relations - Hasse diagram - Lattices (Basic concepts)

Functions : Inverse function - Composition of Functions - Recursive functions - pigeon Hole principles and its Applications.

Unit : 4 Algebraic Structures

Algebraic Systems - Examples and General properties - Semi Groups - Monoids - Groups and Subgroups - Homomorphism and Isomorphism.

Unit : 5 Graph Theory

Basic Terminology - Multi Graphs - weighted Graphs - Digraphs and Relations - Representations of Graphs (Incidence Matrix, Adjacency Matrix) - operations on Graphs - Isomorphsim and subgraphs - paths and circuits - Graph Traversals (DFS, BFS) - Shortest paths in weighted Graphs - Eulerian paths and circuits - Hamiltonian paths and circuits - planar Graph - Graph coloring - Spanning Trees - Minimum Spanning Trees - Kruskal's Algorithm - prim's Algorithm

Unit:1 Mathematical logic

(1) Statement and Notations

(1) In this section we introduce certain Basic units of our object language called primary statements

(2) By assuming that the object language contains a set of declarative stentences which cannot be further Broken down (or) Analyzed into simpler sentences. This are called primary statements.

(3) In object language we have one and only one of two possible value called "Truth values".

(4) The two truth values are True and False are denoted by the symbols 'T' and 'F' (or) 1 and 0

(5) All the declarative sentences to which it is possible to assign one and only one of the two possible truth values are called statements.

(6) These statements which do not contain any of the connectivils are called atomic statements.

Eg: (1) Canada is a country (True)

(2) $1+101=110$ (False)

(7) Declarative sentences in the object language are called two types

(i) Those sentences which are considered to be primitive statements in the object language.

(ii) Those statements are denoted by distinct symbols that are A, B, C ..... P, Q.

(2) Connectives

(i) Negation

The negation of a statement is generally formed by introducing the word "NOT" at a proper place in the statement. If 'P' denotes the statement then the negation of P is written as '$\neg p$' and read as "NOT P". If the truth value of P is 'T' then the truth value of negation P is 'F'

P	$\neg P$
T	F
F	T

(ii) Conjunction

The conjunction of two statements P and Q is the statement "$P \wedge Q$" which is read as "P and Q". The statement $P \wedge Q$ has the truth value whenever Both $P \wedge Q$ have truth value 'T' otherwise the truth value 'F'

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

(iii) Disconjunction

This Disconjunction of two statements P and P is the statement "$P \vee Q$" which is read as "P or Q". The statement $P \vee Q$ has the truth value 'F' only Both P and Q have a truth value 'F' othere it is True.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

(iv) Conditional

If P and Q are any two statements then the statement $P \to Q$ which is read as "if P then Q" is called a conditional statement. The statement $P \to Q$ has the truth value F when Q has the Truth value F and P truth value T, otherwise it has the Truth value T

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

(v) Bi conditional

If P and Q are any two statements then the statement $P \leftrightarrows Q$ which is read as "P if and only if Q" and abbrivated as "P iff Q" is called Bi conditional.

P	Q	$P \leftrightarrows Q$
T	T	T
T	F	F
F	T	F
F	F	T

(3) Statement Formulas and Truth Tables

(i) $\neg P$ $P \vee Q$ $(P \wedge Q) \vee (\neg P)$ $P \wedge (\neg Q)$

Sol:

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$P \wedge Q$	$(P \wedge Q) \vee (\neg P)$	$P \wedge (\neg Q)$
T	T	F	F	T	T	T	F
T	F	F	T	T	F	F	T
F	T	T	F	T	F	T	F
F	F	T	T	F	F	T	F

(ii) $P \wedge \neg P$

Sol:

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

(iii) $(P \vee Q) \vee (\neg P)$

Sol:

P	Q	$\neg P$	$P \vee Q$	$(P \vee Q) \vee (\neg P)$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

(iv) $P \vee (Q \wedge R)$

Sol:

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
F	T	T	T	T
T	F	F	F	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

(v) $\neg(\neg P \vee \neg Q)$

Sol:

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(\neg P \vee \neg Q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

(vi) $\neg(\neg P \wedge \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

(vii) $P \wedge (Q \wedge P)$

P	Q	$Q \wedge P$	$P \wedge (Q \wedge P)$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

(viii) $\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)$

P	Q	R	$\neg P$	$\neg Q$	$\neg Q \wedge R$	$Q \wedge R$	$P \wedge R$	$\neg P \wedge (\neg Q \wedge R)$	$\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R)$	$\neg P \wedge (\neg Q \wedge R) \vee (Q \wedge R) \vee (P \wedge R)$
T	T	T	F	F	F	T	T	F	T	T
T	T	F	F	F	F	F	F	F	F	F
T	F	T	F	T	T	F	T	F	F	F
F	T	T	T	F	F	T	F	F	T	T
T	F	F	F	T	F	F	F	F	F	F
F	T	F	T	F	F	F	F	F	F	F
F	F	T	T	T	T	F	F	T	T	T
F	F	F	T	T	F	F	F	F	F	F

(ix) $(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge \neg Q$	$(P \wedge Q) \vee (\neg P \wedge Q)$	$(P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$	$(P \wedge Q) \vee (\neg P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \wedge \neg Q)$
T	T	F	F	T	F	F	F	T	F	T
T	F	F	T	F	F	T	F	F	T	T
F	T	T	F	F	T	F	F	T	F	T
F	F	T	T	F	F	F	T	F	T	T

(x) $(P \wedge (P \to Q)) \to Q$

P	Q	$P \to Q$	$P \wedge (P \to Q)$	$(P \wedge (P \to Q)) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(xi) $(P \to Q) \leftrightarrows (\neg P \vee Q)$

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$	$(P \to Q) \leftrightarrows (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(xii) $((P \to Q) \wedge (Q \to R)) \to (P \to R)$

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	$((P \to Q) \wedge (Q \to R))$	$((P \to Q) \wedge (Q \to R)) \to (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	T	F	T
F	T	T	T	T	T	T	T
T	F	F	F	T	F	F	T
F	T	F	T	F	T	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

(xiii) $(P \rightleftarrows Q) \rightleftarrows ((P \wedge Q) \vee (\neg P \wedge \neg Q))$

P	Q	$\neg P$	$\neg Q$	$P \leftrightarrows Q$	$P \wedge Q$	$\neg P \wedge \neg Q$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	$(P \rightleftarrows Q) \rightleftarrows ((P \wedge Q) \vee (\neg P \wedge \neg Q))$
T	T	F	F	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	F	T
F	F	T	T	T	F	T	T	T

(xiii) $(\neg(P \wedge Q) \vee (\neg R)) \vee ((Q \rightleftarrows \neg P) \rightarrow (R \vee \neg S))$

P	Q	R	S	$\neg P$	$\neg R$	$\neg S$	$P \wedge Q$	$Q \rightleftarrows \neg P$	$R \vee \neg S$	$\neg(P \wedge Q)$	$\neg(P \wedge Q) \vee (\neg R)$	$(Q \rightleftarrows \neg P) \rightarrow (R \vee \neg S)$	$(\neg(P \wedge Q) \vee (\neg R)) \vee ((Q \rightleftarrows \neg P) \rightarrow (R \vee \neg S))$
T	T	T	T	F	F	F	T	F	T	F	F	T	T
T	T	T	F	F	F	T	T	F	T	F	F	T	T
T	T	F	T	F	T	F	T	F	F	F	T	T	T
T	F	T	T	F	F	F	F	T	T	T	T	T	T
F	T	T	T	T	F	F	F	T	T	T	T	T	T
T	T	F	F	F	T	T	T	F	T	F	T	T	T
T	F	T	F	F	F	T	F	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	T	T	T	T	T
F	T	F	T	T	T	F	F	T	F	T	T	F	T
F	F	T	T	T	F	F	F	F	T	T	T	T	T
T	F	F	T	F	T	F	F	T	F	T	T	F	T
T	F	F	F	F	T	T	F	T	T	T	T	T	T
F	T	F	F	T	T	T	F	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F	T	T	T	T	T
F	F	F	T	T	T	F	F	F	F	T	T	T	T
F	F	F	F	T	T	T	F	F	T	T	T	T	T

(xiv) $\neg(P \vee (Q \wedge R) \rightleftarrows ((P \vee Q) \wedge (P \vee R))$

P	Q	R	$Q \wedge R$	$P \vee Q$	$P \vee R$	$P \vee (Q \wedge R)$	$\neg(P \vee (Q \wedge R))$	$(P \vee Q) \wedge (P \vee R)$	$\neg(P \vee (Q \wedge R) \rightleftarrows ((P \vee Q) \wedge (P \vee R))$
T	T	T	T	T	T	T	F	T	F
T	T	F	F	T	T	T	F	T	F
T	F	T	F	T	T	T	F	T	F
F	T	T	T	T	T	T	F	T	F
T	F	F	F	T	T	T	F	T	F
F	T	F	F	T	F	F	T	F	F
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	F	F	T	F	F

(xv) $(Q \wedge (P \to Q)) \to P$

P	Q	$P \to Q$	$Q \wedge (P \to Q)$	$(Q \wedge (P \to Q)) \to P$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	F
F	F	T	F	T

(xvi) $(P \rightleftarrows R) \wedge (\neg Q \rightarrow S)$

P	Q	R	S	$\neg Q$	$P \rightleftarrows R$	$\neg Q \rightarrow S$	$(P \rightleftarrows R) \wedge (\neg Q \rightarrow S)$
T	T	T	T	F	T	T	T
T	T	T	F	F	T	T	T
T	T	F	T	F	F	T	F
T	F	T	T	T	T	T	T
F	T	T	T	F	F	T	F
T	T	F	F	F	F	T	F
T	F	T	F	T	T	F	F
F	T	T	F	F	F	T	F
F	T	F	T	F	T	T	T
F	F	T	T	T	F	T	F
T	F	F	T	T	F	T	F
T	F	F	F	T	F	F	F
F	T	F	F	F	T	T	T
F	F	T	F	T	F	F	F
F	F	F	T	T	T	T	T
F	F	F	F	T	T	F	F

(xvii) $(P \vee (Q \rightarrow (R \wedge \neg P))) \rightleftarrows (Q \vee \neg S)$

P	Q	R	S	$\neg P$	$\neg S$	$R \wedge \neg P$	$Q \rightarrow (R \wedge \neg P)$	$P \vee (Q \rightarrow (R \wedge \neg P))$	$Q \vee \neg S$	$(P \vee (Q \rightarrow (R \wedge \neg P))) \rightleftarrows (Q \vee \neg S)$
T	T	T	T	F	F	F	F	T	T	T
T	T	T	F	F	T	F	F	T	T	T
T	T	F	T	F	F	F	F	T	T	T
T	F	T	T	F	F	F	T	T	F	F
F	T	T	T	T	F	T	T	T	T	T
T	T	F	F	F	T	F	F	T	T	T
T	F	T	F	F	T	F	T	T	T	T
F	T	T	F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F	F	T	F
F	F	T	T	T	F	T	T	T	F	F
T	F	F	T	F	F	F	T	T	F	F
T	F	F	F	F	T	F	T	T	T	T
F	T	F	F	T	T	F	F	F	T	F
F	F	T	F	T	T	T	T	T	T	T
F	F	F	T	T	F	F	T	T	F	F
F	F	F	F	T	T	F	T	T	T	T

Logic :

Logic is defined as the science of reasoning. The main aim of logic is to provide values by which one can determine whether any ~~particular argument or~~ reasoning is valid. ~~Any collection of values are in which these values can be stated~~

Types of logic :

(1) propositional logic or Boolean logic or two valued logic

(2) First order logic

(3) Descriptive logic

(4) Horn logic

(5) Model and temporary logic

(6) program verification (or) model checking for verification.

Logic formalises a valid method for reasoning in the process of reasoning one can make infereces in an inferences one can use a collection of statements or premises in order to justify another statement or conclusion.

Types of statement :

Statements are classified into two types

(1) Automatic / primary / primitive statement

(2) compound / Molecular statement

compound statement

A compound statement is a new statements obtained from Automatic statements through the use of logical operators or simply connectivities

Eg:- P: Today is Friday and tomorrow is Saturday

Q: Today I will go to college or I will go to cinema

R: Varshija is reading or Varshiya is writting

S: chittoor is not a city

conjunction:

Example: Find the conjunction of the following statements

P: It is raining today

Q: $2+2=4$

$P \wedge Q$ = It is raining today and $2+2=4$

disjunction:

Example: P: kusuma is reading a Book

Q: Varshiya is writting

$P \vee Q$: kusuma is reading a Book or Varshiya is writting.

Exclusive OR:

Let P and Q be two statements the exclusive of P and Q is denoted by $P \oplus Q$ and defined by a Statement truth value is T. when exactly one of P and Q is true and its truth value is False. It also Represent with $\bar{\vee}$, XOR

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

Converse : For the statement $P \to Q$, the statement $Q \to P$ is called converse.

Inverse : For the statement $P \to Q$, the statement $\bar{P} \to \bar{Q}$ is called inverse.

Contra positive : For the statement $P \to Q$, the statement $\bar{Q} \to \bar{P}$ is called contra positive.

Example (1) Find the converse, Inverse and contrapositive statement in the following statement

If it is raining today then I will go to college tomorrow

Sol: P: It is raining today

Q: I will go to college tomorrow

$P \to Q$: If it is raining today then I will go to college tomorrow

$Q \to P$: If I will go to college tomorrow then it is raining today.

$\bar{P} \to \bar{Q}$: If it is not raining today then I will not go to college tomorrow.

$\bar{Q} \to \bar{P}$: If I will not go to college tomorrow then it is not raining today.

Definition of Truth Table

A Table showing all truth values of a statement formula is called a truth table of the statement formula. If there are n number of discrete variable or components in a statement formula with. Then we require 2^n possible combinations of the truth values in order to construct the truth table

$P \to 2^1 - 2$

$P, Q \to 2^2 - 4$

$P, Q, R \to 2^3 - 8$

(1) Using the statements

R : mark is rich

H : mark is Happy

(1) Mark is poor But Happy ($\neg R \wedge H$)

(2) Mark is rich or unhappy ($R \vee \neg H$)

(3) Mark is nither rich nor Happy ($\neg R \oplus \neg H$)

(4) Mark is poor or he is Both rich and unhappy

$\neg R \vee (R \wedge \neg H)$

Well Formed Formula :

A statement Formula is represent which is a string consist of variables, Connectivity symbols and paranthaesis All the Statetement formulas ~~of~~ are not well defined or all expressions are not statement formulas.

Eg : PV

It is not a Statement Formula.

Definition Recursive

The Recursive Definition of Statement. This is also called well Formed Formula. A well formed formula is generated by following rules. A statement variable standing alone is a well Formed Formula. If A is well formed formula then $\neg A$ is also well-formed Formula. If A and B are well formed Formula then $(A \vee B)$, $(A \wedge B)$.... are well formed Formula.

A string of symbols continuing the statement variable. Logic connectivities and parenthesis is well Formed Formula iff it can be obtained by many applications of rules 1, 2 and 3.

Eg: P → well formed formula (wff)

P, Q (P∧Q) → well formed formula (wff)

Not wff

Eg: ¬P → ¬

P∨ → ∨Q

<u>Tautologies:</u>

The truth value of a statement formula to mean entries in the table lost column are the truth table of the statement Formula.

In general the last column of a truth table of a given statement Formula contains Both True and False. There are some formulas whose truth value Formulas are true or always False regardless of the truth values assignment to the variables.

Eg: P∨¬P

P	¬P	P∨¬P
T	F	T
F	T	T

Eg: P∧¬P

P	¬P	P∧¬P
T	F	F
F	T	F

A Statement Formula is classified into 3 types Based on their truth values the are

(1) Tautology (2) controduction (3) contingency

(1) Tautology :

A Statement Formula which is true regardless of the truth values of the Statement. Which replace the Variables is called an universaly valid Formula or Tautology or logicaly truth.

Eg: $P \vee \neg P$

P	$\neg P$	$P \vee \neg P$
T	F	T
F	T	T

(2) Controduction :

A Statement Formula which is False regardless of the truth values of the Statement which replace the Variables is called a controduction

Eg: $P \wedge \neg P$

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

(3) Contingency

A statement Formula that can Have Both true or False is called Contingency

Eg: $P \vee Q$

P	Q	$P \vee Q$
T	F	T
F	T	T

properties

(1) The conjuction of two tautologies is also Tautology

(2) Let A and B Be two Statement Formulas which are Tautologies then $A \wedge B$ is also a Tautology.

(3) The Negation of a controduction is also Tautology.

Eg: $\neg(P \wedge \neg P)$

P	$\neg P$	$P \wedge \neg P$	$\neg(P \wedge \neg P)$
T	F	F	T
F	T	F	T

(4) The negation of a Tautology is a controduction

P	$\neg P$	$P \vee \neg P$	$\neg(P \vee \neg P)$
T	F	T	F
F	T	T	F

Substitution Instance:

A statement Formula 'P' is called a substitution Instance of and another Formula 'Q', P can be obtained From Q by substituting Formulas for some variables of Q with the condition required is that the same Formula is substituted For the same variable each time occurs.

Eg: A: $P \wedge Q \rightarrow P$ be a given statement Formula,
If P is replaced by $P \leftrightarrow Q$

Eg: B: $P \rightarrow (J \wedge P)$ substitute $R \leftrightarrow S$ For P in B set A

A: $R \leftrightarrow S\,(J \wedge (R \leftrightarrow S))$, A is a substitution instance of B.

Property:

Any Substitution instance is a tautology.

A: $(R \to S) \vee \neg(R \to S)$

R	S	$R \to S$	$\neg(R \to S)$	$(R \to S) \vee \neg(R \to S)$
T	T	T	F	T
T	F	F	T	T
F	T	T	F	T
F	F	T	F	T

C: $((P \vee Q) \wedge R) \vee \neg((P \vee Q) \wedge R)$

P	Q	R	$P \vee Q$	$(P \vee Q) \wedge R$	$\neg((P \vee Q) \wedge R)$	$((P \vee Q) \wedge R) \vee \neg((P \vee Q) \wedge R$
T	T	T	T	T	F	T
T	T	F	T	F	T	T
T	F	T	T	T	F	T
F	T	T	T	T	F	T
T	F	F	T	F	T	T
F	T	F	T	F	T	T
F	F	T	F	F	T	T
F	F	F	F	F	T	T

Method

The Striaght Forward method to determine the given Statement Formula is tautology is Construct to truth table. This process can always ~~be~~ used, But it is difficult to Construct a truth table. If the number of Variables the given Statement formula is large. i.e No of Variables in statement Formula then we have to construct to truth table 2^n No of rows. Pratically it is difficult. So we go for an alternative method using logically equivalence Formulas

① Show that for any propositions or statement Formula P and Q the compond preposition $P \to (P \vee Q)$ is tautology and the compond proposition $P \wedge (\neg P \wedge Q)$

P	Q	$P \vee Q$	$P \to (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

$P \to (P \vee Q)$ is tautology

P	Q	$\neg P$	$\neg P \wedge Q$	$P \wedge (\neg P \wedge Q)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

~~$P \to (P \vee Q$~~ $P \wedge (\neg P \wedge Q)$ is compound proposition

② Show that the truth values of the following compond proposition of indepent of the truth values

(i) $[P \wedge (P \to Q)] \to Q$

P	Q	$P \to Q$	$P \wedge (P \to Q)$	$[P \wedge (P \to Q)] \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

$[P \wedge (P \to Q)] \to Q$ is tautology

(ii) $(P \to Q) \leftrightarrow (\neg P \vee Q)$

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$	$(P \to Q) \leftrightarrow (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

$(P \to Q) \leftrightarrow (\neg P \vee Q)$ is tautology

Logical equivalence

(or)

Equivalence of Statement Formulas

$P_1, P_2, \ldots P_n$, Let A and B be two statements Formula and let $P_1, P_2 \ldots P_n$ denote all the variables occuring in both A and B. Consider an assignment of truth values $P_1, P_2 \ldots P_n$ and the resultant of truth values are A and B.

Equivalence

The statement Formulas are equivalent is said to be truth values of A to the truth values of B. For everyone of the 2^n possible set of truth values assigned to $P_1; P_2, \ldots P_n$.

Assuming that the variable assignment of the Truth values to the variable appeared in the same order in the truth table are A and B. Then the final column in the truth table for A and B are identical. If A and B are equivalent. The equivalence of two statement Formulas A and B are denoted by $A \equiv B$ or $A \Leftrightarrow B$ which is read as "A" is equivalent to B".

Note :-

$A \Leftrightarrow B$ is not a connective

(1) $P \vee (\neg(P \wedge Q))$

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$	$P \vee (\neg(P \wedge Q))$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

(2) $(P \vee Q) \vee \neg P$

P	Q	$\neg P$	$P \vee Q$	$(P \vee Q) \vee \neg P$
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

(3) $P \rightarrow (P \wedge Q)$

P	Q	$P \wedge Q$	$P \rightarrow (P \wedge Q)$
T	T	T	T
T	F	F	F
F	T	F	T
F	F	F	T

(4) $P \rightarrow (P \vee Q)$

P	Q	$P \vee Q$	$P \rightarrow (P \vee Q)$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	T

(5) $((\neg Q) \wedge (P \to Q)) \to \neg P$

P	Q	$\neg Q$	$\neg P$	$P \to Q$	$((\neg Q) \wedge (P \to Q)) \to \neg P$
T	T	F	F	T	T
T	F	T	F	F	T
F	T	F	T	T	T
F	F	T	T	T	T

(6) $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$

P	Q	R	$P \to Q$	$Q \to R$	$P \to R$	$[(P \to Q) \wedge (Q \to R)] \to (P \to R)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	T	T
F	T	T	T	T	T	T
T	F	F	F	T	F	T
F	T	F	T	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

(7) $(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$

P	Q	R	$(Q \to R)$	$(P \to Q)$	$(P \to R)$	$(P \to (Q \to R)) \to ((P \to Q) \to (P \to R))$
T	T	T	T	T	T	T
T	T	F	F	T	F	T
T	F	T	T	F	T	T
F	T	T	T	T	T	T
T	F	F	T	F	F	T
F	T	F	F	T	T	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

(8) $((P \vee Q) \rightarrow R) \leftrightarrow (\neg R \rightarrow \neg(P \vee Q))$

P	Q	R	$\neg R$	$(P \vee Q)$	$\neg(P \vee Q)$	$(P \vee Q) \rightarrow R$	$((P \vee Q) \rightarrow R) \leftrightarrow (\neg R \rightarrow \neg(P \vee Q))$
T	T	T	F	T	F	T	T
T	T	F	T	T	F	F	T
T	F	T	F	T	F	T	T
F	T	T	F	T	F	T	T
T	F	F	T	T	F	F	T
F	T	F	T	T	F	F	T
F	F	T	F	F	T	T	T
F	F	F	T	F	T	F	T

Properties of equivalence

(1) Symmetric: the equivalence of two statement formulas is a symmetric relation. If $A \Leftrightarrow B$ then $B \Leftrightarrow A$

(2) Transitive: The equivalence of three statement Formula is Here Transitive relation i.e If $A \equiv B$ and $B \equiv C$ then $C \equiv A$ (or) $A \equiv C$

(1) For Any two proposition P & Q prove that $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

Ans:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

The Last columns ~~and~~ of the statement Formulas are equal From the above table

$(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$

(2) Prove that For any proposition P and Q. Compound preposition For $P \oplus Q$ and $(P \vee Q) \wedge (\neg P \vee \neg Q)$ are logically equivalent.

Ans:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$(P \vee Q) \wedge (\neg P \vee \neg Q)$
T	T	F	F	T	F
T	F	F	T	T	T
F	T	T	F	T	T
F	F	T	T	F	F

The last column of the statement Formulas are equal From the above table $P \oplus Q$ and $(P \vee Q) \wedge (\neg P \vee \neg Q)$

(3) Prove that

(i) $[P \to (Q \wedge R)] \Leftrightarrow [(P \to Q) \wedge (P \to R)]$

(ii) $[(P \vee Q) \to R] \Leftrightarrow [(P \to R) \wedge (Q \to R)]$

(iii) $[(P \vee Q) \to R] \Leftrightarrow [\neg R \to \neg (P \vee Q)]$

Laws of equivalence :

Law of Logic :-

(1) Double negation law:-

Eg: Eq:1 $\neg(\neg P) = P$

(2) Commulative law:-

Eq2: $P \vee Q \equiv Q \vee P$

Eq3: $P \wedge Q \equiv Q \wedge P$

(3) Assosative law:-

Eq 4: $(P \vee Q) \vee R \equiv P \vee (Q \vee R)$

Eq 5: $(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$

(4) Destributive law:-

Eq6: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

Eq7: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

(5) De-Morgan's law:-

Eq 8: $\overline{P \vee Q} \equiv \bar{P} \wedge \bar{Q}$

Eq 9: $\overline{P \wedge Q} \equiv \bar{P} \vee \bar{Q}$

(6) Independent law:-

Eq 10: $P \vee P \equiv P$

Eq 11: $P \wedge P \equiv P$

Eq 12: $R \vee (P \wedge \bar{P}) \equiv R$

Eq 13: $R \wedge (P \vee \bar{P}) \equiv R$

Eq 14: $R \vee (P \vee \bar{P}) \equiv T_0$

Eq 15: $R \wedge (P \wedge \bar{P}) = F_0$

Eq 16: $P \rightarrow Q \equiv \bar{P} \vee Q$

Eq 17: $\overline{P \rightarrow Q} \equiv P \wedge \bar{Q}$

Eq 18: $P \rightarrow Q \equiv \bar{q} \rightarrow \bar{P}$

Eq 19: $P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$

Eq 20: $\overline{P \leftrightarrow Q} \equiv P \leftrightarrow \bar{Q}$

Eq 21: $P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$

Eq 22: $P \leftrightarrow Q \equiv (P \wedge Q) \vee (\bar{P} \wedge \bar{Q})$

(7) Absorption law:-

Eq 23: $P \vee (P \wedge Q) \equiv P$

Eq 24: $P \wedge (P \vee Q) \equiv P$

(8) Inverse law:-

Eq 25: $P \vee \bar{P} = T_0$ (tautology)

Eq 26: $P \wedge \bar{P} = F_0$ (controduction)

(9) Identity law:-

Eq 27: $(P \vee F_0) \Leftrightarrow P$

Eq 28: $(P \wedge T_0) \Leftrightarrow P$

(10) Domination law:-

Eq 29: $(P \vee T_0) \Leftrightarrow T$

Eq 30: $(P \wedge F_0) \Leftrightarrow F$

(1) Show that $P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R) \Leftrightarrow (P \wedge Q) \to R$

Ans: We know that

$Q \to R \Leftrightarrow \neg Q \vee R$ (eq 16)

$P \quad Q \Leftrightarrow \bar{P} \vee R$

$P \to (Q \to R) \Leftrightarrow P \to (\neg Q \vee R)$

$P \to (\neg Q \vee R) \Leftrightarrow \neg P \vee (\neg Q \vee R)$ (eq 4 Associative)

$\Leftrightarrow (\neg P \vee \neg Q) \vee R$ (eq 8)

$\Leftrightarrow (\neg P \wedge \neg Q) \vee R$

$\Leftrightarrow \neg (P \wedge Q) \vee R$

$\Leftrightarrow (P \wedge Q) \to R$

(2) Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge \underline{R}) \vee (P \wedge \underline{R}) \Leftrightarrow R$

R common

Ans: $\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee (Q \vee P) \wedge R$ (eq 7)

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee (Q \vee P) \wedge R$ (eq 5)

$\Leftrightarrow (\neg (P \vee Q) \wedge R) \vee ((Q \vee P) \wedge R)$ (eq 9)

$\Leftrightarrow (\neg (P \vee Q) \vee (Q \vee P)) \wedge R$ $(\because \bar{P} \vee P = T_0)$

$\Leftrightarrow (\neg (P \vee Q) \vee (P \vee Q)) \wedge R$ eq (25)

$\Leftrightarrow T_0 \wedge R$ eq (28)

$\Leftrightarrow R$

(3) Show that $[(P \vee Q) \wedge \neg (\neg P \wedge (\neg Q \vee \neg R))] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R$ is a Tautology without constructing a truth Table.

Duality law :-

Two statement formulas A and A^* are set to be duals of each other. If either one can be obtained from other by replacing "$\wedge$ by $\vee$" and "$\vee$ by $\wedge$". The connectivities $\wedge$ and $\vee$ are duals of each other.

If the statement formula A contains the special variables ((truth values) /(T/F)) then A^* is obtained by replacing T/F and F/T

eg: $A: P \wedge Q$; $B: P \vee Q$; $A: (P \vee Q) \wedge (\neg P)$

$A^*: P \vee Q$; $B^*: P \wedge Q$; $A^*: (P \wedge Q) \vee (\neg P)$

---

(1) Find the duals of following statement Formulas

(i) $A: (P \vee Q) \wedge R$; $A^*: (P \wedge Q) \vee R$

(ii) $A: (P \wedge Q) \vee T$; $A^*: (P \vee Q) \wedge F$

(iii) $A: \neg(P \vee Q) \wedge (P \vee \neg(Q \wedge \neg S))$; $A^*: \neg(P \wedge Q) \vee (P \wedge \neg(Q \vee \neg S))$

(2) Show that (i) $\underbrace{\neg(P \wedge Q)}_{P} \rightarrow \underbrace{(\neg P \vee (\neg P \vee Q))}_{Q} \Leftrightarrow (\neg P \vee Q)$

$\Leftrightarrow \neg(\neg(P \wedge Q)) \vee (\neg P \vee (\neg P \vee Q))$ eq 16

$\Leftrightarrow (P \wedge Q) \vee (\neg P \vee (\neg P \vee Q))$

$\Leftrightarrow (P \wedge Q) \vee ((\neg P \vee \neg P) \vee Q) \rightarrow$ eq (10) $\quad (\neg P \vee \neg P = \neg P)$

$\Leftrightarrow \underbrace{(P \wedge Q)}_{P} \vee \underbrace{(\neg P}_{Q} \vee \underbrace{Q)}_{R} \rightarrow$ eq 4

$\Leftrightarrow ((P \wedge Q) \vee \neg P) \vee Q$

$\Leftrightarrow ((P \vee \neg P) \wedge (Q \vee \neg P)) \vee Q \rightarrow$ eq 25

$\Leftrightarrow (T_0 \wedge (Q \vee \neg P)) \vee Q \rightarrow$ eq 27

$\Leftrightarrow (Q \vee \neg P) \vee Q$

$\Leftrightarrow (\neg P \vee Q) \vee Q$

$\Leftrightarrow \neg P \vee (Q \vee Q)$

$\Leftrightarrow \neg P \vee Q$

$(A)^* \neg(P \vee Q) \rightarrow (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow \neg P \wedge Q$

Tautological implications

We know that connectivities $\wedge, \vee, \leftrightarrow, \rightarrow$ are Symmetric in the since that P and $Q \Leftrightarrow Q \wedge P$, $P \vee Q \Leftrightarrow Q \vee P$ and $P \leftrightarrow Q \Leftrightarrow Q \leftrightarrow Q$ But $P \rightarrow Q$ is not equivalent to $Q \rightarrow P$

definition:

A Statement Formula A is said to be Tautologically implies A statement Formula B. $A \Rightarrow B$ is a tautology which is denoted by $A \Rightarrow B$ which is read as A is tautologically implies B. One can be determined whether $A \Rightarrow B$ by constructing truth table of $A \cap B$.

(1) Show that the following implications

(i) $(P \wedge Q) \Rightarrow (P \rightarrow Q)$

P	Q	$P \wedge Q$	$P \rightarrow Q$	$(P \wedge Q) \rightarrow (P \rightarrow Q)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	T	T

$(P \wedge Q) \rightarrow (P \rightarrow Q)$ is a Tautology $(P \wedge Q) \Rightarrow (P \rightarrow Q)$

(ii) $P \Rightarrow (Q \rightarrow P)$

P	Q	$Q \rightarrow P$	$P \rightarrow (Q \rightarrow P)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

$P \rightarrow (Q \rightarrow P)$ is a tautology $P \Rightarrow (Q \rightarrow P)$

(3) $(P \to (Q \to R)) \Rightarrow (P \to Q) \to (P \to R)$

P	Q	R	$Q \to R$	$P \to (Q \to R)$	$P \to Q$	$P \to R$	$(P \to Q) \to (P \to R)$	$(P \to (Q \to R)) \Rightarrow (P \to Q) \to (P \to R)$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	F	T
T	F	T	T	T	F	T	T	T
F	T	T	T	T	T	T	T	T
T	F	F	T	T	F	F	T	T
F	T	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

$(P \to (Q \to R)) \to (P \to Q) \to (P \to R)$ is a Tautology, so $(P \to (Q \to R)) \Rightarrow (P \to Q) \to (P \to R)$

Normal Forms

(1) DisJunctive Normal Form (DNF)

(2) conjuctive Normal form (CNF)

(3) principal Disjunctive Normal Form (PDNF)

(4) principal conjuctive Normal Form (PCNF)

A (P_1, P_2 ..... P_n) Be a statement Formula where P_1, P_2 ... P_n are the Variables

If the Statement formula A may have the truth values for all possible Assignment of the truth values prove that the variables $P_1, P_2 \cdots P_n$ then the Statement Formula A is set to be a tolology.

If the Statement Formula A has the truth value of False. For all possible Assignments of truth Values to variable P_1, P_2 ... P_n then the statement Formula A is said to be controduction.

If the statement Formula A has the truth value of True for atleast one Combination of truth values are assigned to the variable P_1, P_2 ... P_n then A is said to be satisfied.

Decision problems

The problem of determining decision in a finite no of steps wheather a given statement formula is a taulogy or a controduction or Atleast a Satisfyable. It is known as decision problems

They are four types of Normal forms

(1) Disjunctive Normal Form

(2) conjuctive Normal Form

(3) PDNF

(4) PCNF

Definition of elementry product :

The product of variable and their negations in a formula is called an elementry product

Eg: Let P and Q be two variable then $P, Q, P \wedge Q$, $P \wedge \neg(Q)$, $\neg P \wedge Q$, $\neg P \wedge Q \wedge \neg P$, $P \wedge Q \wedge R$.

Elementry Sum :

The sum of variables and their negations in a statement formula is called an elementry sum.

Eg: Let P and Q are variable then $P, Q, P \vee Q, P \vee \neg Q$, $\neg P \vee Q$, $\neg P \vee Q \vee \neg P$, $P \vee Q \vee R$.

Definition of DNF

A statement formula which is equivalent to the given statement formula and which consist of some of elementry product is called a Disjunctive Normal Form of the given statement formula.

Method :

Step 1: If the connectivities $\rightarrow$ ~~and~~ $\leftrightarrow$ occurs in the given statement formula then find an equivalent formula by replacing $\rightarrow, \leftrightarrow, \neg, \vee, \wedge$. using laws of equivalence.

Eg: $P \rightarrow Q \equiv \neg P \vee Q$

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$$
$$\equiv (P \vee Q) \wedge (\neg Q \vee P)$$

Step 2: If the negation is applied to the formula then by using demorgans form an equivalent formula can be obtained in which the "$\neg$" is applied to the variables.

Eg: $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$

$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

Step 3: Apply the distributive law to get the DNF

Eg: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

To put conjuctive

(1) Find the Disjunctive normal Form of the following statements.

$(Q \vee (P \wedge R)) \wedge \neg(((P \vee R) \wedge Q))$

P	Q	R	$P\wedge R$	$Q\vee(P\wedge R)$	$P\vee R$	$(P\vee R)\wedge Q$	$\neg(((P\vee R)\wedge Q))$	
T	T	T	T	T	T	T	F	F
T	T	F	F	T	T	T	F	F
T	F	T	T	T	T	F	T	T
F	T	T	F	T	T	T	F	F
T	F	F	F	F	T	F	T	F
F	T	F	F	T	F	F	T	T
F	F	T	F	F	T	F	T	F
F	F	F	F	F	F	F	T	F

$(P \wedge \neg Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$

(2) $\neg(P \vee Q) \Leftrightarrow P \wedge Q$

P	Q	$P\vee Q$	$\neg(P\vee Q)$	$P\wedge Q$	$\neg(P\vee Q) \Leftrightarrow P\wedge Q$
T	T	T	F	T	F
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	T	F	F

$(P \wedge \neg Q) \vee (\neg P \wedge Q)$

$\underbrace{\neg(P \vee Q)}_{P} \leftrightarrow \underbrace{P \wedge Q}_{Q}$

$(\neg(P \vee Q) \wedge (P \wedge Q)) \vee (\neg(\neg(P \vee Q) \wedge \neg(P \wedge Q))$

$\underbrace{(\neg(P \vee Q) \wedge (P \wedge Q))}_{eq\ 8} \vee \underbrace{((P \vee Q) \wedge (\neg P \vee \neg Q))}_{eq\ 9}$

$((\neg P \wedge \neg Q) \wedge (Q \wedge P)) \vee (P \vee Q) \wedge \neg P) \vee ((P \vee Q) \wedge \neg Q))$

$(\neg P \wedge \neg Q \wedge Q \wedge P) \vee ((P \wedge \neg P) \vee (Q \wedge \neg P) \vee ((P \wedge \neg Q) \vee (Q \wedge \neg Q))$

$(\neg P \wedge F_0 \wedge P) \vee (F_0 \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee F_0)$

$(F_0 \wedge F_0) \vee ((Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (F_0 \wedge F_0)$

$\Leftrightarrow F_0 \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee F_0$

$\Leftrightarrow (Q \wedge \neg P) \vee (\neg P \wedge Q)$

(3) $P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$ Find the disjunctive normal Form (DNF):

P	Q	$\neg Q$	$\neg P$	$\neg Q \vee \neg P$	$\neg(\neg Q \vee \neg P)$	$P \rightarrow Q$	$(P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)$	$P \rightarrow ((P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P))$
T	T	F	F	F	T	T	T	T
T	F	T	F	T	F	F	F	F
F	T	F	T	T	F	T	F	T
F	F	T	T	T	F	T	F	T

$(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

Conjuctive normal Form:

A Formula which is equivalent to a given Formula and which consist of a product of elementry sum is called a conjuctive normal Form of the given Statement formula

(1) Find a conjuctive normal Form of the following Statement Formula.

(i) $\neg(P \vee Q) \Leftrightarrow P \wedge Q$

P	Q	$P \vee Q$	$P \wedge Q$	$\neg(P \vee Q)$	$\neg(P \vee Q) \Leftrightarrow P \wedge Q$
T	T	T	T	F	F
T	F	T	F	F	T
F	T	T	F	F	T
F	F	F	F	T	F

$(P \vee Q) \wedge (\neg P \vee \neg Q)$

(i) $\underbrace{(\neg(P \vee Q) \rightarrow (P \wedge Q))}_{\neg(P \vee Q) \Leftrightarrow P \wedge Q} \wedge ((P \wedge Q) \rightarrow \neg(P \vee Q))$ eq 21

$\neg(\neg(P \vee Q) \vee (P \wedge Q)) \wedge (\neg(P \wedge Q) \vee \neg(P \vee Q))$ eq:16

$((\underbrace{P \vee Q}_{P}) \vee (\underbrace{P}_{Q} \wedge \underbrace{Q}_{R})) \wedge ((\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q))$ eq:6

$((P \vee Q) \vee P) \wedge ((P \vee Q) \vee Q) \wedge ((\neg P \vee \neg Q) \vee \neg P) \wedge ((\neg P \vee \neg Q) \vee \neg Q)$

$(P \vee P \vee Q) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg P \vee \neg Q) \wedge (\neg P \vee \neg Q \vee \neg Q)$

$(P \vee Q) \wedge (P \vee Q) \wedge (\neg P \vee \neg Q) \wedge (\neg P \vee \neg Q)$

$(P \vee Q) \wedge (\neg P \vee \neg Q)$

Principe disjunctive Normal Form (PDNF)

Min Term

Let P and Q be two Variables Then Let us construct all possible Fromulas which Consist of Conjunction of P or its negations and Conjunction of Q or its negations then $P \wedge Q$, $\neg P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge \neg Q$. The Above Formulas are called min terms of P and Q.

Definition PDNF :

A Statement Formula whose equivalent Formulas consist of disjunction of min terms only. is known as principle disjunctive normal Form. It is also called as sum of product cononical Form.

~~Conjuct~~

Principle Conjuctive normal Form (PCNF)

Max term :

Let P and Q be two variables the max term of P and Q are given by $P \vee Q$, $\neg P \vee Q$, $P \vee \neg Q$, $\neg P \vee \neg Q$

Definition :

The given Statement Formula an equivalent Formula which consist of conjunction of max terms is known as PCNF. It is also called product of sum conical Form.

(1) $\neg P \vee Q$

P	Q	$\neg P$	$\neg P \vee Q$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

PDNF :- $(P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$

PCNF :- $(P \vee \neg Q)$

(2) Find the PDNF of the following statement Formula

(1) $P \rightarrow [(P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)]$

(2) $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

(3) $Q \wedge (P \vee \neg Q)$

(4) $P \vee (\neg P \rightarrow (Q \vee (\neg Q \rightarrow R)))$

(5) $(Q \rightarrow P) \wedge (\neg Q \wedge Q)$

(6) $P \rightarrow (P \wedge (Q \rightarrow R))$

(7) $(P \rightarrow (Q \wedge R)) \wedge (\neg P \rightarrow (\neg Q \wedge \neg R))$

Theory of Inference

Introduction:

The main function of logic is to provide rules of inference or rules of reasoning or principles of reasoning. The theory associated with rules of inference is known as inference theory. The inference theory is about detuction of conlusion from the set of primeses the process of the relation of conclusion from set of primeses is called detuction or Formal truth.

definition

Consider a set of statement $P_1, P_2, \cdots P_n$ and a Statement 'c' then the compond statement of the form $P_1 \wedge P_2 \wedge P_3 \cdots P_n$ tautological implies 'c' is called as argument where $P_1, P_2 \cdots P_n$ is called set of Primes of the arguments and 'c' is called conclusion of the arguments.

Rules of Inference

It is denoted by 'R' $R_1, R_2 \cdots R_n$ where R_i is an inference rule

Validity using truth table :-

Let A and B be two statement formulas then we say that 'B' logically follows from 'A' i.e $A \Rightarrow B$ (or) B is valid conclusion of the premises of 'A'. If $A \to B$ is a tautology.

Let us consider a set of premises $H_1, H_2, \ldots, H_n$ and A conclusion 'c'. We say that the conclusion 'c' logically followed from A set of primises. If $H_1 \wedge H_2 \cdots \wedge H_n \to C$ is a tautology.

Valid of arguments or determines :-

Let $P_1, P_2 \ldots P_n$ be the statement variables appearing in the premises $H_1, H_2 \ldots H_n$ and conclusion 'c'. Construct a truth table for all possible combination of truth values are assigned to $H_1, H_2 \ldots H_n$ and 'c'. $P_1, P_2 \ldots P_n$ or truth values

Method:

We look for rows in which all $H_1, H_2 \ldots H_n$ have a truth value True. If for every row the truth value true then the 'c' also has the truth value is valid argument $H_1, H_2 \ldots H_n \Rightarrow C$

(1) Find whether the conclusion 'c' follows from set of primises.

$H_1: P \to Q$, $H_2: P$; $C: Q$

P	Q	$P \to Q$	$P \to Q \wedge P$	$(P \to Q \wedge P) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

Rules of Inference

(1) Conjunctive simplification

$IM_1 : P \wedge Q \Rightarrow P$

$IM_2 : P \wedge Q \Rightarrow Q$

(2) Disjunctive simplification

$IM_3 : P \Rightarrow P \vee Q$

$IM_4 : Q \Rightarrow P \vee Q$

$IM_5 : \neg P \Rightarrow P \to Q$

$IM_6 : Q \Rightarrow P \to Q$

$IM_7 : \neg(P \Rightarrow Q) \Rightarrow P$

$IM_8 : \neg(P \Rightarrow Q) \Rightarrow \neg Q$

$IM_9 : P, Q \Rightarrow P \wedge Q$

(3) Disjunctive Syllogism:

$IM_{10} : ((P \vee Q) \wedge \neg P) \Rightarrow Q$

(4) Syllogism:

$IM_{11} : ((P \to Q) \wedge (Q \to R)) \Rightarrow P \to R$

(5) Modus ponens:

$IM_{12} : (P \wedge (P \to Q)) \Rightarrow Q$

(6) Modus Tollens:

$IM_{13} : (P \to Q) \wedge \neg Q \Rightarrow \neg P$

(7) Dilemma:

$IM_{14} : ((P \vee Q) \wedge (P \to R) \wedge (Q \to R)) \Rightarrow R$

(8) Resolution:

$IM_{15} : (P \vee Q) \wedge (\neg P \vee R) \Rightarrow (Q \vee R)$

(1) $H_1: P \to Q$, $H_2: P$, $C: Q$ (valid)

P	Q	$P \to Q$	$(P \to Q) \wedge P$	$((P \to Q) \wedge P) \to Q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(2) $H_1: P \to Q$, $H_2: \neg P$, $C: Q$ (valid)

P	Q	$\neg P$	$P \to Q$	$P \to Q \wedge \neg P$	$(P \to Q \wedge \neg P) \to Q$
T	T	F	T	F	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

(3) $H_1: \neg P$, $H_2: P \leftrightarrow Q$, $C: \neg(P \wedge Q)$ (valid)

P	Q	$\neg P$	$P \leftrightarrow Q$	$\neg(P \wedge Q)$	$\neg P \wedge P \leftrightarrow Q$	$P \wedge Q$	$(\neg P \wedge P \leftrightarrow Q) \to \neg(P \wedge Q)$
T	T	F	T	F	F	T	T
T	F	F	F	T	F	F	T
F	T	T	F	T	F	F	T
F	F	T	T	T	T	F	T

(4) $H_1: P \to Q$, $H_2: Q$, $C: P$ (Invalid)

(1) 24 (b) Show that R is a valid inference from premises $R,\ P \to Q,\ Q \to R$ and P

Ans: (1) $P \to Q$

(2) P

(3) Q — 5th Formula

(4) $Q \to R$

(5) R

(2) 24 (a) Show that $R \wedge (P \vee Q)$ is a valid conclusion from $P \vee Q,\ Q \to R,\ P \to M,\ \neg M$

Ans: (1) $P \to M$

(2) $\neg M$ — 6th Formula I M13

(3) $\neg P$

(4) $P \vee Q$ — 3rd Formula I M10

(5) Q

(6) $Q \to R$ — I M12

(7) $R \wedge (P \vee Q)$

(3) Show that $R \vee S$ follows that logically from premises $C \vee D,\ (C \vee D) \to \neg H,\ \neg H \to (A \wedge \neg B),\ (A \wedge \neg B) \to (R \vee S)$

Ans: (1) $\underset{P}{C \vee D} \to \underset{Q}{\neg H}$ $\quad (P \to Q) \wedge (Q \to R) \Rightarrow P \underset{T}{\to} R$

(2) $\underset{Q}{\neg H} \to \underset{R}{(A \wedge \neg B)}$

(1,2) $\underset{P}{(C \vee D)} \to \underset{Q}{(A \wedge \neg B)}$ $\quad P \wedge (P \to Q) \Rightarrow Q$

(3) $\underset{P}{C \vee D}$

(4) $(A \wedge \neg B)$

(5) $\underset{P}{(A \wedge \neg B)} \to \overset{Q}{(R \vee S)}$

(6) $R \vee S$

Consistancy of premises and Indirect method of proof ($P \vee \neg P = T_0$)

A set of Statement Formulas $A_1, A_2 \ldots A_n$ is Said to be consistancy of their conjunctions $A_1 \wedge A_2 \wedge \ldots \wedge A_n$ as the truth value true for some assignment of the truth value true through the variables appearing in $A_1, A_2 \ldots A_n$. ~~consist~~

Inconsistancy: ($P \wedge \neg P = F_0$).

If for every assignment of truth value atleast one of the variable $A_1, A_2 \ldots A_n$ is False. So that a conjunctions is identically False. Then the Formulas $A_1, A_2 \ldots A_n$ are called inconsistancy

(1) Show that the following system is inconsistance

$P \to Q,\ P \to R,\ Q \to \neg R,\ P$

Ans: ~~(1) $P \to Q$~~

~~(2) P~~

~~(1,2) (3) Q~~

~~(4) $Q \to \neg R$~~

~~(3) $P \to R$~~

~~(2,3) R~~

(1) $P \to Q$

(2) P

(1,2) Q $\quad P \wedge (P \to Q) \Rightarrow Q = I_{m12}$

(3) $Q \to \neg R$ $\longrightarrow$ $I_{m12} : (P \wedge (P \to Q)) \Rightarrow Q$

(4) $\neg R$

(5) $P \to R$ $\quad (P \to Q) \wedge \neg Q \Rightarrow \neg P = I_{m13}$

(6) $\neg P$

(7) $\neg P \wedge P = F_0$

(2) Show that the following system is inconsistance

$R \to \neg Q,\ R \vee S,\ S \to \neg Q,\ P \to Q,\ P$

Ans: (1) $P \to Q \to I_{m12} = P \wedge (P \to Q) \Rightarrow Q$

(2) P

(3) Q

(4) $R \to \neg Q$

(5) $\neg R$

(6) $R \vee S$

(7) S

(8) $S \to \neg Q$

(9) $\neg Q$

(10) $\neg Q \wedge Q = F_0$

The System is inconsistance

$$[(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))] \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$$

$$[(P \vee Q) \wedge \neg(\neg P \wedge \neg(Q \wedge R))] \vee [\neg(P \vee Q) \vee \neg(P \vee R)]$$

$$[(P \vee Q) \wedge \neg(\neg(P \vee (Q \wedge R))] \vee [\neg((P \vee Q) \wedge (P \vee R))]$$

$$[(P \vee Q) \wedge (P \vee (Q \wedge R)] \vee [\neg(P \vee Q) \wedge (P \vee R)]$$

$$[(P \vee Q) \wedge (P \vee (Q \wedge R)] \vee [\neg(P \vee (Q \wedge R)]$$

$$[\because Q \wedge (P \vee \bar{P}) = T_0]$$

Unit-2
Predicate Calculus

Introduction

In the last section we discussed about Statements and statements Formulas. The inference theory was also restricted in a set of primises and a conclusion. The symbols used are $P, Q, R \cdots\cdots$ (or) $P_1, P_2, P_3 \cdots P_n$

The statements what we studied in the last section is about atomic statements "no Analysis" was done on Atomic statement.

Only compound statement was analys and the analysis was done for forming compound Formulas

It is not possible to express the fact of any two atomic statement have some features in common.

predicate calculus definition:

The logic associated with predicates in any statement is called "predicate logic"

predicate calculus:

The domain of a logic that deals with predicate quantifiers is called predicate calculus.

Predicates :

Let us consider two statements

(1) Govindh is a Bachelore

(2) Rama is a Bachalor

If we express these statement by symbols we need two symbols. But they dont say anything about the common features

P: Govind is a Bachelor

Q: Rama is a Bachelor

Both the statements are disscussing about the individuals who are Bachelor. If we introduce some symbols to denotes "is a Bachelor" and a method to join it with symbols denoting the name of individuals then the path "is a Bachelor" is called a predicate.

In general we use captial letters represents the predicate. In lower case letters to represents the objects (Name of objects)

A statement can be return of terms of following by the name of the predicate which the name of the logic

eg: (1) Govind is a Bachelor

(2) Rama is a Bachelor

In a above statements "is a Bachelor" is a predi

A statement can be return of terms of following by the name of the predicate which the name of the logic

Eg:1 (1) Govind is a Bachelor

(2) Rama is a Bachelor

In a above statements "is a Bachelor" is a predicate which is denoted by 'B' the names of individual or object such as govind is denoted by 'g'. Rama is denoted by 'r'. Therefore the statements (1) and (2) can be written as follows.

(1) $B(g)$

(2) $B(r)$

Eg:2 This painting is red

In the above statement "is red" is a predicate which is denoted by 'R' and the object "this painting" is denoted by 'p'. And the statement 2 can be written symoblic form

~~B(p)~~ $R(P)$

Connectives

(1) $B(g) \vee B(r)$: Govid is a Bachelor or Rama is a Bachelor

(2) $B(g) \wedge B(r)$: Govid is a Bachelor and Rama is a Bachelor

(3) $\neg B(g)$:- Govid is not a Bachelor

(4) $B(g) \vee R(P)$: Govid is a Bachelor or This painting is red

(5) $B(g) \rightarrow B(r)$: If govind is a Bachelor then Rama is a Bachelor

(6) $B(g) \rightarrow R(P)$: If govind is a Bachelor then this painting is red

(7) $B(g) \leftrightarrow B(r)$: Govind is a Bachelor if and only if rama is a Bachelor

(8) $B(r) \leftrightarrow R(P)$: Rama is a Bachelor iff this painting is red.

Express the following statements in Symbolic form

(1) Dog is an animal $A(d)$

(2) Cat is an animal $A(c)$

(3) Geetha is a Student $S(g)$

(4) Sweetha is a Student $S(s)$

Let us consider the Statements

(1) Teja is taller than Varshiya

In the above statement "is taller than" is a predicate is called two place predicate

The two place predicate have two objects

The above statement symbolically following the predicate "is taller than" '(T)' and objects are Teja and Varshiya represents as (t, v)

The return of the Answer represented by

$T(t, v)$

Let us consider a statement, Rama sits Between Sita and Laxshmana "sits" is a predicate denoted by 's' objects are denoted by rama, sita, laxshmana (r, s, l)

The above Statement represented as S(r, s, l)

In general an n-place predicate need n number of object name to be fixed positions in the order obtained a Statement.

If 's' represents an n-place predicate and the objects name are represented by $x_1, x_2 \ldots x_n$ then the n-place predicate statement is denoted by $S(x_1, x_2 \ldots x_n)$

The Statement Function, Variable and quantifier

Let 'B' be the predicate is a Bachelor 'r', the name rama, Govid 'g' and s the name Siva then name rama, Govid 'g' and S the following Statement B(r), B(g), B(S) denoted all the following Statement

(1) Rama is a Bachelor

(2) Govid is a Bachelor

(3) Siva is a Bachelor

Infacts these statements have a common feature therefore we write this as B(x): x is Bachelor. Then B(r), B(g), B(S) can be obtained from B(x) by replacing x by and appropriate object name rama, govind, siva.

M(x): x is a man

V(x): x is a vegetarian

Definition of Statement Function:

A simple statement function of one variable is defined to be an expression consistence of predicate symbol, and an individual variable

For example of statement function B(x) variable predicate

M(x), V(x)

A statement functions become a statement when variable is replaced by objects names.

Compound Statement Function:

It can be obtained by combining one or more simple statement function and logical connective

B(x) : x is a Bachelor

V(x) : x is a Vegetarian

$\forall x \rightarrow$ universal quantifier

$\exists x \rightarrow$ existential quantifier

Quantifiers

A statement which involves the univeral or Existential Quantifier is called a Quantificate Statement.

The variable present in Quantifier statement is called "Bound variable". It is bounded by a quantifier.

The symbol $\forall x$ (or) x is called universal Quantifier and represent $\forall x$ (or) For every x (or) For any x

The symbol $\exists x$ or x is called Existenial quantifier and represents their exist some (or) their exist is atleast

(1) For the universe of all integers 'x'. Expree the Following symbolic statement in 'x'.

$P(x) : x > 0$

$Q(x) : x$ is even

$R(x) : x$ is a perfect Square

$S(x) : x$ is divisible by 3

$T(x) : x$ is divisible by 7

(1) $\forall x, [R(x) \rightarrow P(x)]$

(2) $\exists x, [S(x) \rightarrow \neg Q(x)]$

(3) $\forall x, [\neg R(x)]$

(4) $\forall x, [R(x) \vee P(x)]$

Ans: (1) For all x if x is a perfect square, then $x > 0$

(2) Their exist some x if x is divisible by 3 then x is not even

(3) For all x, x is not a perfect Square

(4) For all x, x is a perfect square or $x > 0$

---

(2) write the following Quantifer Statement in a symbolic form

(i) All men are mortal

(ii) Every apple is red

(iii) Any Integer is either positive (or) Negative

(iv) Their exist a man

(v) some men are clever

(vi) some real numbers are rational

(vii) Every one in the I MCA class has a Smart phone

(i) All men are mortal

$M(x)$: x is a men

$B(x)$: x is a mortal

$\forall x, [M(x) \rightarrow B(x)]$

(ii) Every apple is red

$A(x)$: x is a Apple

$R(x)$: x is a Red

$\forall x, [A(x) \rightarrow R(x)]$

(iii) Any Integer is either positive (or) Negative

$I(x)$: x is a Integer

$P(x)$: x is a positive

$\forall x, [I(x) \rightarrow P(x)]$

Alleast, Some, Their exist – $\exists x$

(iv) Their exist a man

Any, All, every – $\forall x$

$M(x)$: x is a man

$\exists x, M(x)$

(v) Some mens are clever

$M(x)$: x is a man

$C(x)$: x is a clever

$\exists x, [M(x) \rightarrow C(x)]$.

(vi) Some real number are rational

$R(x)$: x is a real number

$Q(x)$: x is a rational

$\exists x, [R(x) \rightarrow Q(x)]$

(viii) Every one in the I MCA class has a smart phone

$M(x)$: x is a IMCA

$S(x)$: x is a smart phone

$\forall x, [M(x) \rightarrow S(x)]$

---

(3) x is father of the mother of y

Ans: x is father of z and z is mother of y

$F(x, z) = x$ is father of z

$M(z, y) = z$ is mother of y

$P(z) = z$ is a person

$\exists z\ [F(x, z) \wedge M(z, y) \wedge P(z)]$

Predicate Formula

The captial letters introduced to denote predicate It is also suggested '$\forall$' is used along the predicate such as n place predicate.

* $n=1$, $P(x)$: 1-place predicate
* $n=2$, $P(x_1, x_2)$: 2-place predicate
* $n=3$, $P(x_1, x_2, x_3)$: 3-place predicate
* $P(x_1, x_2 \ldots x_n)$: n-place predicate

In the above Formula 'P' denotes a predicate $x_1, x_2 \ldots x_n$ denotes objects names on variable.

⇒ In general $P(x_1, x_2 \ldots x_n)$ is called as an atomic Formula of predicate calculus

Eg: R, $R(x)$, $R(x, y)$, $R(x, y, z)$ .....

Well formed Formula

A well formed Formula is a predicate caluclus is obtained by using the following rules.

(1) An atomic Formula is well formed Formula.

(2) If 'A' is well Formed Formula then '$\neg A$' is a well Formed Formula

(3) If A and B are well Formed Formulas then $A \land B$, $A \rightarrow B$, $A \leftrightarrow B$ are well Formed Formulas

(4) If A is a well Formed Formula and x is a Variable then $\forall x\ A$ and $\exists x\ A$ is a well formed Formula

Only those Formulas are obtained using the above rules i.e, 1–4 are well formed Formulas

Free and Bound Variables

For any Formula contains a part line $\forall x\ M(x)$ or $\exists x\ M(x)$ then that part is called x Bound part of the Formula.

(1) Any occurence of x in x Bound part is known as a Bound occurence of x.

(2) Any occurence of x which is not a Bound occurence is called Free Occurence

(3) The Formula that immediately Follows the Quantifer is called the Scope of the Quantifier

Ex: $\forall x\ M(x)$ or $\exists x\ M(x)$

In the above Formula $M(x)$ is the Scope of the Quantifier For the Both Formulas.

Example 2: $\exists x\ [M(x) \wedge N(x)]$

In the above Formula $M(x)$ and $N(x)$ is the scope of the Quantifiers

Example 3: $M(x) \wedge \exists x\ N(x)$

In the above Formula $N(x)$ is the scope of the Quantifier

In the above Formala $\exists x\ N(x)$ Both the Variables associated with x is called Bound variable and the occurency of x $M(x)$ is free Variable

(1) write the following predicate Statement in Symbolic Form.

(i) Some one in your class has visited to Lab

$\underbrace{\text{Some one}}_{x}$ in $\underbrace{\text{your class}}_{c}$

$C(x)$: x is in your class

$L(x)$: x is visited to lab

$$\exists x\ [C(x) \wedge L(x)]$$

There exists Somex, x in your class and x is visited to lab

The universe of discourse

Some predicate Statement Representing in Symbolic Form in predicate Caluclus in every much Calculus the complexcity to reduce some extend by restricting in your class object corresponding.

According to this restriction the Quantifier Variable represented only the objects which Belongs a Specific domain or class. ~~The~~

The restricted class or domain is known as universal discourse.

Eg: Everyone in the I MCA has a smart phone

$M(x)$: x is in the 1st MCA class

$S(x)$: x has a smart phone

$\forall x, [m(x) \rightarrow S(x)]$

For all x is x is in the First MCA class then x has a smart phone.

In the above example the universe of ~~dispose~~ course set of all students in the First MCA class

Eg: S{ Vishnu, Hema, Godhari, Teja}

(~~universe of dispose must be defined by the truth Value Statement.~~)

The universe of ~~this~~ discourse must be defined by the truth value of a statement.

---

(1) Consider a predicate statement

(1) $p(x)$. x is greater than 2 and the statement for all x. $p(x)$ and there exist x, ~~p(x) there ex~~

Suppose the universe of ~~this~~ discourse defined as Follow

(1) $\{-5,-3,0,1,2\}$ F

(2) $\{3,5,7,10\}$ T

(3) $\{-1,0,2,6\}$ F

$\forall x$ = All True & False

$\exists x$ = Atleast one element False

(1) $\{-5, -3, 0, 1, 2\}$

$\forall x\ p(x)$ The truth value is False

(2) $\{3, 5, 7, 10\}$

$\forall x\ P(x)$ The truth value is True

(3) $\{-1, 0, 2, 6\}$

$\forall x\ P(x)$ The truth value is False

Inference theory of predicate calculus:

In the section, we first generalize the concept of Valid Formulas and equivalence or predicate calculus.

* we shall use the same terminollogies and symbols that used for Statement calculus

Valid formulas and Equivalence:

A predicate Formula P is said to be a valid Formula in the universe discourse S. If for every assignment ~~if~~ ~~For ever~~ of object name from 's' to corresponding variable in 'P' and For every assignment of the Statement to Statement Variables then the resulting State have the truth Value True. (or)

The quantifier Statement P is said to be logically implies a quantifier Statement 'Q'. If Q is true when ever 'p' is True. This can be written as logically imples

given the Set of quantifier Statements $P_1, P_2 \ldots P_n$ and 'c' we say that 'C' is a valid Conclusion

From premises $P_1, P_2, \dots P_n$. If 'c' is True.
Whenever each of $P_1, P_2 \dots P_n$ is true i.e.

$P_1 \wedge P_2 \wedge P_3 \dots P_n \Rightarrow C$

Equivalence

Let P and Q are two predicate Formulas defined over the common universe denoted by 's'. We define the equivalence of P and Q as Follows.

(1) If For every Assignment of object name universe of discourse S to every variable of P and Q the resulting statement have identical True value which is denoted as $P \Rightarrow Q$ or

Two Quantifier Statement P and Q are set to be equivalence when every they have same truth value in all possible situations.

Examples

(1) $\forall x \; [P(x) \wedge Q(x)] \equiv [\forall x (P(x)) \wedge [\forall x \, Q(x)]$

(2) $\exists x \; [P(x) \vee Q(x)] \equiv \exists x \; P(x) \vee \exists x \; Q(x)$

Rules for Negation:

(1) $\neg [\forall x, P(x)] \equiv [\exists x, \neg P(x)]$

(2). $\neg [\exists x, P(x)] \equiv [\forall x, \neg P(x)]$

Unit - 3 Relations and Functions

Cartesian product

Let A and B are any two sets the cartesian product of A and B is defined by

$A \times B = \{(a,b) / a \in A \text{ and } b \in B\}$

(or)

Let A and B are two sets the set of all order pairs such that the first number of the ordered pair is an element of A and the second number is an element of B is called the cartesian product of A and B

$A \times B = \{(a,b) / a \in A \text{ and } b \in B\}$

Example

Let $A = \{1, 2, 3\}$, $B = \{a, b\}$

$A \times B = \{(1,a), (1,b), (2,a), (2,b), (3,a), (3,b)\}$

$B \times A = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3)\}$

Properties

(1) If A has M elements B has N elements then A×B and B×A we have MN elements

(2) If A and B are disjoint then A×B and B×A are also disjoint

(3) If either A or B is a null set then the set of A×B also Null set

(4) If either A or B is infinite and other one is not infinite set. Then $A \times B$ is also an disjoint set

(5) If $A \subseteq B$ and $C \subset D$ then $A \times C \subset B \times D$

(6) If $A \subseteq B$ then $A \times B \Rightarrow$

Binary Relation

A Binary Relation 'R' from a set A to set B is a subset of $A \times B$ (or) Any set of ordered pairs defines a Binary relation.

Note:

If xRy we have to read as "x is in relation R to y"

If $A = B$ we say that R is a Binary relation on A

Eg: 1) Let 'R' denotes the set of real numbers. Then $R = \{(x \cdot x^2) / x \in R\}$ Define their relation of the square of real numbers

2) The relation x, y $R = \{(x,y) / x$ and $y\}$ are real number and $x^2+y^2<1$

Definition:

Let 'R' be a relation from A to B then domain of R denoted by "dom R" is defined by

$\text{dom}R = \{x / x \in A$ and $(x,y) \in R$ for some $y \in B\}$

Range $R = \{y / y \in B$ and $(x,y) \in R\}$

(1) List all the order pair in the relation $R = \{(a,b) / a$ and $b\}$ on the set $A = \{1,2,3,4\}$

Ans: $R = \{(1,1)(1,2)(1,3)(1,4)$
$(2,1)(2,2)(2,3)(2,4)$
$(3,1)(3,2)(3,3)(3,4)$
$(4,1)(4,2)(4,3)(4,4)\}$

domin $d = \{1,2,3,4\}$

Range $R = \{1,2,3,4\}$

(2) List all the ordered pair in the relation 'R' from $A = \{0,1,2,3,4\}$ $B = \{0,1,2,3\}$ where $(a,b) \in R$ iff $a+b=4$

Sol: $R = \{(2,2)(1,3),(3,1),(4,0)\}$

$d = \{1,2,3,4\}$

Range $= \{0,1,2,3\}$

10marks: Properties of Binary Relation in a set

(1) Reflexive :

A Binary Relation 'R' in a set 'X' is reflexive If for every $x \in X$, xRx, $(x,x) \in R$

Eg: Let 'X' $X = \{1,2,3,4\}$ and the relation $R = \{(1,1),(2,2),(3,3),(4,4)\}$ The relation 'R' is Reflexive.

(2) Symmetric :

A Binary relation 'R' in a set 'X' is a Symmetric Forevery x and y in X whenever 'x' relation 'y' and 'y' relation 'x' (i.e) $xRy \Rightarrow yRx$ $\forall x, y \in X$

Eg: X is a Friend of Y and Y is a friend of X

Friend is a symmetric.

(3) Transitive

A relation 'R' is a set of 'X' is transitive if For every x, y and z in X. whenever xRY and YRZ then ZRX then it is Called Transitive.

Eg: The Relation $<$, $\leq$ and $=$ are transitive in the Set of real numbers.

$X = \{(2,3),(3,4),(4,2)\}$

(4) Irreflexive

A relation R in a set X is irreflexive if For every $x \in X$, $(x,x) \notin R$ i.e $x \not R x$

Eg: The relation in set of {1, 2, 3} which is not reflexive

(5) Anti Symmetric

A relation 'R' in a set 'X' is a Anti Symmetric If For every x, y in X whenever XRY and YRX Then X = Y is Called Anti Symmetric

Eg: The Relation $\leq$ and $\geq$ in 'R' Both are Anti Symmetric

$5 \leq 5$ $5 \geq 5$

---

(1) $X = \{1, 2, 3, 4\}$ and $R = \{(1,1), (1,4), (4,1), (4,4), (2,2), (2,3), (4,2), (3,4), (3,2), (3,3)\}$

Find the properties satisfy XRR in a set X

Sol: Reflexive = {(1,1), (4,4), (2,2), (3,3)}

symmetric = {(1,4), (4,1), (2,3), (3,2)}

Transitive = {(2,3), (3,4), (4,2)}

Equivalence Relations

A Relation 'R' in a Set 'X' is called an equivalence relation if it is reflexive, transitive and symmetric

properties

(1) Equality of number and a set of real numbers

(2) Similarity of triangle on the set of triangles relations of statements be in equivalent in the set of Statements

(3) Relation of living in some person ~~living~~ town all the set of person leaving in the A.P

Operations of relations

Let R and S are relations from A to B.

(i) $R \cap S = \{(a,b) \in A \times B \,/\, (a,b) \in R \ \& \ (a,b) \in S\}$

In the intersection of relation R and S

(ii) $R \cup S = \{(a,b) \in A \times B \,/\, (a,b) \in R \text{ (or) } (a,b) \in S\}$

In the union of relation R and S

(iii) $R - S = \{(a,b) \in A \times B \,/\, (a,b) \in R \text{ and } (a,b) \notin S\}$

In the difference of relation R and S

(1) Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$ the relation $R = \{(1,1),(2,2),(3,3)\}$ and $S = \{(1,1),(1,2),(1,3),(1,4)\}$ Can be obtained

(i) $R \cup S$ (ii) $R \cap S$ (iii) $S - R$

Ans: $R \cup S = \{(1,1),(2,2),(3,3),(1,2),(1,3),(1,4)\}$

$R \cap S = \{(1,1)\}$

$R - S = \{(2,2),(3,3)\}$

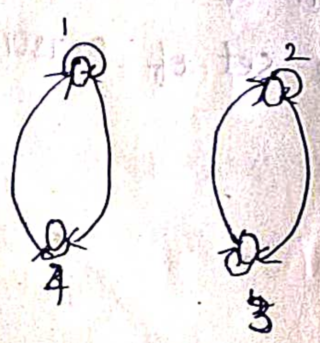
$S - R = \{(1,2),(1,3),(1,4)\}$

(2) Let $x = \{(1,2,3,4)\}$ and $R = \{(1,1),(1,4),(4,1),(4,4),(2,2),(2,3),(3,2),(3,3)\}$

write the matrix of R and S ~~is~~ Statch its graph

Ans:

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$



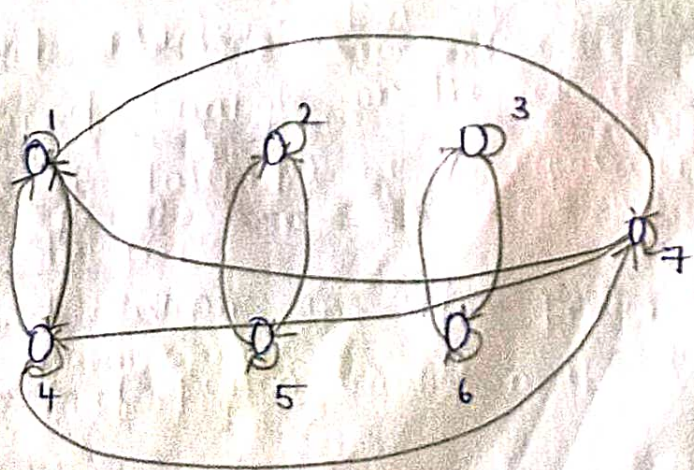
(3) Let $x = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{(x, y) \mid x-y$ is divisible by $3\}$ show that 'R' is an equivalence, relation draw the graph of 'R'.

Ans: $X = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7)\}$

$x-y = \{ (1,1), (1,4), (1,7), (2,2), (2,5), (3,3), (3,6), (4,1), (4,4), (4,7), (5,2), (5,5), (6,3), (6,6), (7,1), (7,4), (7,7)\}$

$$
\begin{array}{c|c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\
\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{matrix} &
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Graph



→ Continue in next page

(4) Let $X = \{1, 2, 3, 4\}$ and $R = \{(x,y) / x > y\}$ to draw the graph for R and give its matrix ?

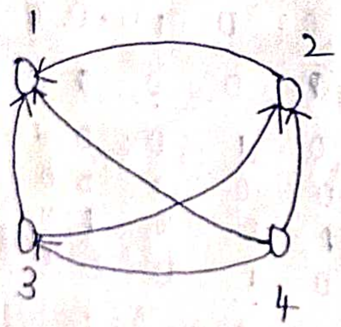
Sol:
$X = \{(1,2), (1,1), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4)$
$(3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$

$x > y = \{(2,1), (3,2), (3,1), (4,1), (4,2), (4,3)\}$

Matrix =

$$\begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Graph:



(3) Continue :

After show R is relation, we have to show that R is reflexive, Symmetric and transitive

we know that, all of ordered pair
{ (1,1), (2,2), (3,3), (4,4), (5,5), (6,6), (7,7)} Belongs to R i.e ordered pair $(a,a) \in R$ For every $a \in X$.

Therefore R is a reflexive.

we consider the ordered pair {(1,4),(4,1)} ∈ R
{(1,7),(7,1)} ∈ R, {(2,5),(5,2)} ∈ R, {(3,6),(6,3)} ∈ R,
{(4,7), (7,4)} ∈ R Therefore whenever $(a,b) \in R$ then ordered pair $(b,a) \in R$ For $(a,b) \in R$. Therefore R is a symmetric.

we consider the ordered pair
{(1,4), (4,7), (7,1)} ∈ R, { (3,3, (3,6), (6,3)} ∈ R
{(2,2), (2,5), (5,2)} ∈ R, {(1,1), (1,4), (4,1)} ∈ R
{(1,1), (1,7), (7,1)} ∈ R

∴ Whenever $(A,B) \in R$, $(B,C) \in R$, $(C,A) \in R$

∴ R is Transition Relation.

∴ From the above explanation, it is called 'R' is equivalence relation.

without Graph :

To show 'R' is equivalence relation without using graph, we can prove that

(1) For every $a \in X$, $a-a/3$
Hence aRa, R is reflexive

(2.) For any $(a,b) \in X$, $a-b/_3$ is divisible by 3 and $b-a/_3$ is divisible by 3

i.e $aRb \Rightarrow bRa$

$\therefore$ R is Symmetric

(3) For any $(a,b,c) \in X$ if aRb, bRc, cRa,

$a-b$ is divisible by 3

$b-c$ is divisible by 3

$c-a$ is divisible by 3

$\therefore$ R is Transitive

$\therefore$ R is equivalence relation

Special case:

The above problem is a special case of more general relation of equality in the modular system. Let 'I' denotes the set of all the integers and Set 'M' is the positive integers For $x \in I$ and $y \in I$ define 'R' as $R = \{(x,y) \mid x-y$ is divisible by $x-y/m$ is equivalent to the ~~systo~~ statement then Both x and y have the sam remainder when each is divided by m.

So we denote 'R' by $\overset{\text{congruent}}{\equiv}$ and write xRy as $x \equiv y$ (or $y \equiv x$.

The relation $\equiv$ is called as congruence

Equivalence class

Let 'R' be an equivalence relation on set 'x'

For any $x \in X$ this sat $[x]_R \subseteq X$ given by

$[x]_R = \{y / y \in X \wedge xRy\}$ is called an 'R' equivalence class generated by $x \in X$

(i.e) The set $[x]_R$ consists of all the 'R' relatives of x in the set 'X'

Note: Sometimes $[x]_R$ is written as x/R

Example: Let 'Z' be the set of integers and let 'R' be the relation called congruence module 3 defined by $R = \{(x, y) / x \in Z \wedge y \in Z \wedge (x-y)$ is divisible by $3\}$

determine the equivalence class generated by the elements of 'Z'

Sol $Z = \{\ldots -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5 \ldots\}$

$[0]R = \{-3, 0, 3, 6 \ldots\}$

$[1]R = \{-5, -2, 4, \ldots\}$

$[2]R = \{-4, -1, 5, 8 \ldots\}$

$[3]R = \{-3, 0, 3, 6 \ldots\}$

$Z/R = \{[0]R, [1]R, [2]R, [3]R\}$

Properties:

For any element $x \in X$ we have xRx Because 'R' is reflexive $x \in [x]_R$

Let $y \in x$ be any other element such that xRy Show that $y \in [x]_R$ Because of the symmetric of 'R' yRx and $x \in [x]_R$ Because of their is an element $z \in [y]_R$ then 'z' must be in $[x]_R$ Because $y \in z$ along with $xRy \Rightarrow xRz$ thus $[y]_R \leq [x]_R$ By Symmetric we must also have

$[x]_R \subseteq [y]_R \Rightarrow [x]_R$

the property to it is shows that xRy then $[x]_R = [y]$ we know that If xRy then $[x]_R$ and $[y]_R$ must be disjoint.

Assume that $z \in [x]_R$ and also $z \in [y]_R$ (i.e) xRz and yRx

Comparitibility relation

A relation 'R' in 'x' is set to be a Compatibity relation: If it is reflexive and Symmetric

Note:

All equivalence relations are compatibity relations But the converse need not be true.

⇒ Hence we discuss the Compatibity relations which are not equivalence relations

Example: Let $X = \{$ball, bed, dog, egg, let$\}$ and Let the relation 'R' is given by

$R = \{(x,y) / x,y) / x,y \in X \cap xRy$ if x and y contains common Letters$\}$

$X = \{$ (Ball, Bed), (Ball, Let), (Ball, Ball), (bed, bed)
(bed, dog), (bed, ball), (bed, egg), (bed, let),
(dog, dog), (dog, bed), (dog, egg), (egg, egg),
(egg, dog), (egg, let), (egg, bed), (Let, Let),
(Let, egg), (Let, bed), (Let, ball) $\}$

Reflexive : For every $x \in X$ (or) xRx (or) $(x, x) \in R$

$R = \{$ (ball, ball), (bed, bed), (dog, dog), (egg, egg), (Let, Let) $\}$

$\therefore$ R is Reflexive

Symmetric:

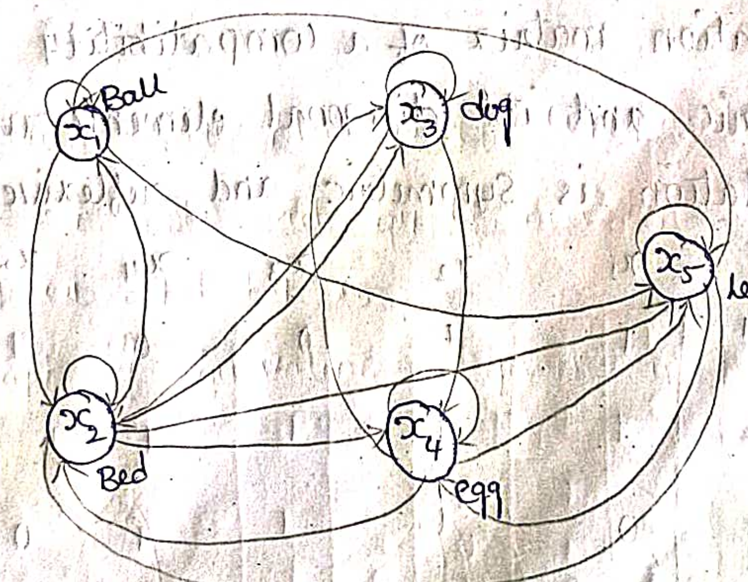
$xRy \Rightarrow yRx \ \forall \ x, y \in X$

$xy = yx$

$= \{$ (Ball, bed), (Ball, Let), (bed, dog), (bed, ball), (bed, egg)
(bed, let), (dog, bed), (dog, egg), (egg, dog), (egg, let),
(egg, bed), (let, egg), (Let, bed), (let, ball) $\}$

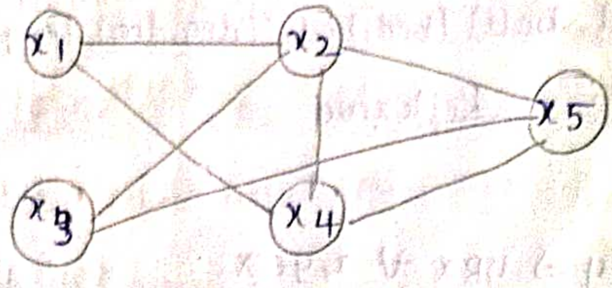
graphical representation of compatibility Relation:

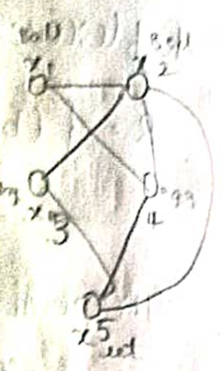
Let we denote Ball by x_1, bed by x_2, dog by x_3, egg by x_4, Let by x_5 the graph of equivalence is given in figure



Since equivalence is a compatibility Relation it is not necessary to draw the loop at each element NOR is it.

* Necessary to draw both xRy & yRx. thus, we can simplify the graph in figure





Note:

The elements in the each of the sets $\{x_1, x_2, x_3\}$ $\{x_2, x_4, x_5\}$ are related to each other i.e the elements are mutually compatible.

* The set $\{x_2, x_4, x_5\}$ also as elements compatibility to each other.

Matrix representation of compatibility Relation

The relation matrix of a compatibility Relation is symmetric and its diagonal elements unity since the relation is symmetric and reflexive.

$$
\begin{array}{ll|c}
 & & \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\begin{array}{l} \text{Ball} \\ \text{Bed} \\ \text{dog} \\ \text{egg} \\ \text{let} \end{array} &
\begin{array}{l} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{array} &
\begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}
\end{array}
$$

Before it is Sufficient to give only the elements of lower triangular path of the relation Matrix in such a case

$$\begin{array}{cccccc}
x_1 & & & & & \\
x_2 & 1 & 0 & 0 & 1 & 0 \\
x_3 & 0 & 1 & 0 & 0 & 0 \\
x_4 & 1 & 1 & 0 & 0 & 0 \\
 & 1 & 1 & 0 & 1 & 0 \\
x_5 & x_1 & x_2 & x_3 & x_4 & x_5
\end{array}$$

Maximal Compatibility Blocks:

X be a Set and equivalence Compatability relation of X. $A \subseteq X$ is called a maximal Compatibility Block. If Any element of 'A' is Compatibility to every other element of 'A'. And No elements of $X - A$ is Compatible to all the elements of A.

$$X = \{x_1, x_2, x_3, x_4, x_5\}$$

$$A = \{x_1, x_2, x_4\}$$

$$X - A = \{x_3, x_5\}$$

Method of finding maximal compatibility Blocks

(i) To find MCB corresponding to the compatability relation and a set X. (i) To draw a Simplified graph of the compatibility relation

(ii) pick from this graph the Largest

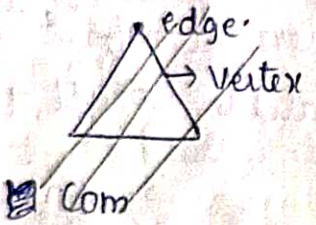
Complete Polygons

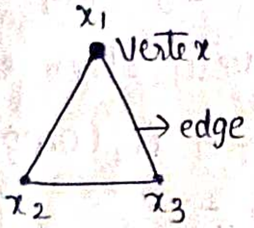
(1) A Largest complete polygons means a polygon which any vertex is connected to every other vertex

Eg: A Traiangle is a complete polygon

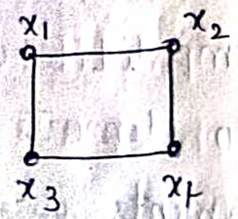
(2) Any element of the set which is related only to itself forms a maximal compatibility Block.

(3) Any two elements which are compatable to one another But no other elements also form a maximal compatability Block.



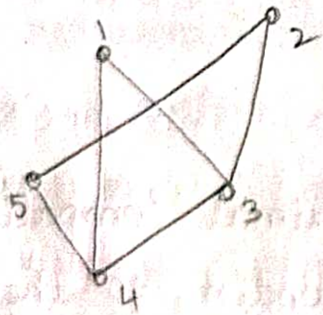


complete polygon



Incomplete polygon

Eg: 1



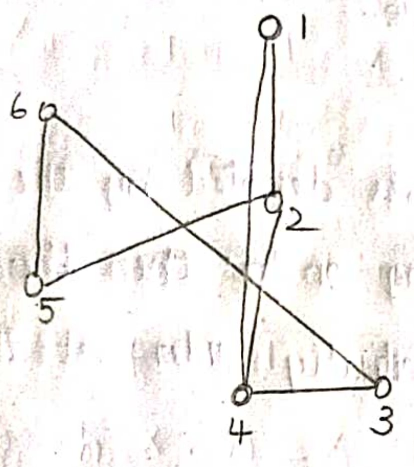
1					
2	0				
3	1	1			
4	1	0	1		
5	0	1	0	1	
	1	2	3	4	5

$X = \{1, 2, 3, 4, 5\}$

Maximal compatibility relation

$\{1, 4, 3\}, \{2, 3\}, \{5, 2\}, \{4, 5\}$

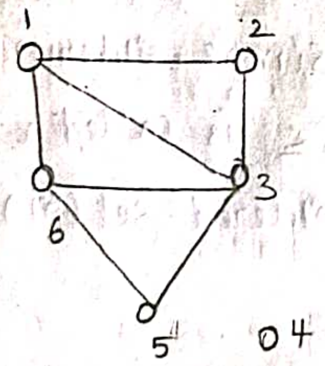
Eg: 2



1	0	0	0	0	0	0
2	1	0	0	0	0	0
3	0	0	0	0	0	0
4	1	1	1	0	0	0
5	0	1	0	0	0	0
6	0	0	1	0	1	0
	1	2	3	4	5	6

MCB = $\{1,2,4\}, \{3,4\}, \{6,5\}, \{5,2\}, \{6,3\}$

Eg: 3



1	0	0	0	0	0	
2	1	0	0	0	0	
3	1	1	0	0	0	
4	0	0	0	0	0	
5	0	0	1	0	0	
6	1	0	1	0	1	
	1	2	3	4	5	6

2	1				
3	1	1			
4	0	0	0		
5	0	0	0	0	
6	1	0	1	0	1
	1	2	3	4	5

MCB = ~~{1,2,3}~~, ~~{1,6}~~, ~~{6,5}~~, ~~{5,3}~~ $\{1,2,3\}, \{1,6\},$

~~{1,2,3}~~, ~~{1,6,3}~~, ~~{6,5,3}~~ $\{3,5,6\}$

Partial ordering

A Binary relation R in a set p is called partial ordering relation (or)

A partial ordering in p if R is reflexive, Antisymmetric transitive.

Note:

A partial ordering is denoted by the symbol '$\leq$'

If $\leq$ is partial ordering on 'p' then the order pair $(P, \leq)$ is called a partially order set or po-set.

Definition:

Let $P, \leq$ be a partially order set. If for every x, y $(x, y \in P)$ we have either $x \leq y \vee y \leq x$ then $\leq$ is called a simple ordering or linear ordering on P and $P, \leq$ is called a Totally ordered (or) simply ordered set (or) change set.

~~Note:~~ properties

It is not necessary to have $x \leq y \vee y \leq x$ For every x and y is a partially ordered set P

* If x is not related to y then we say that x and y are not partially ordered.
* If 'R' is partial ordering on 'P' then it is easy to see that the converse of 'R' Namely $\tilde{R}$ is also partial ordering on 'P'

* If R is denoted by $\leq$ R on R is divided By $\geq$

* If P, $\leq$ is a poset then P, $\geq$ is also a poset

Hasse diagram

A partial ordering $\leq$ on a set P can be represented by mean of a diagram known as Hasse diagram of a partial ordering.

How to draw the Hasse diagram

(1) In a Hasse diagram represented by Small circle or dots

(2) The circle For $x \in P$ is draw Below the circle for $y \in P$

(3) A line is drawn B/w x and y if y covers x

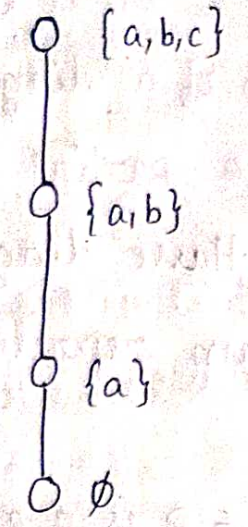
(4) If $x < y$ But y does not cover x. There x and y are not connected directly by a single line However they are connected through one or more elements to P

(1) Let $P = \{1, 2, 3, 4\}$ and $\leq$ be the relation then P, $\leq$ be a po-set. Draw the Hasse diagram of the poset.

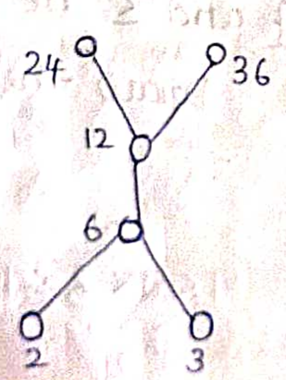
Ans:

$1 \leq 2 \leq 3 \leq 4$

(2) Let us consider $P = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}\}$ and the relation $\leq$ on P then $P, \leq$ is a poset. Now we draw the Hasse diagram of the above set.



(3) Let us consider $X = \{2, 3, 6, 12, 24, 36\}$ and the relation $\leq$ be such that $x \leq y$ iff x divides y. The ordered pair $X, \leq$ is a poset. We draw the Hasse diagram to above poset



Definition

An element of a poset is called Maximal if it is not < any element of the po set.

~~An element of a maximal in this is~~

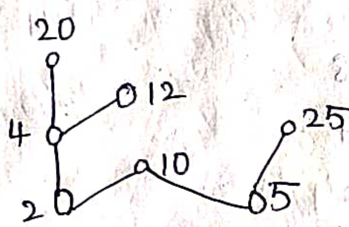
An element of a poset is called Minimal of it is not > any element of the poset.

i.e An element B, B ∈ S such that B < A

Maximal and Minimal elements are easy spot using Hasse diagram.

They are the Top and Bottom elements in the diagram

Eg: Find the maximal and minimal elements of the po set {2, 4, 5, 10, 12, 20, 25}



Maximal = 20, 12, 25

Minimal = 2, 5

Lattices

A Lattices is a partial order set and (L, ≤) in which every pair of elements A, B ∈ L as a Greatest lower Bound and least Upper Bound is called a lattice

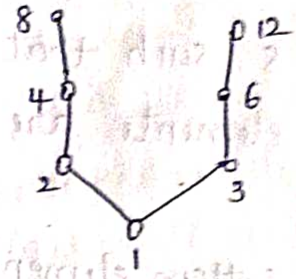
(or)

Definition:

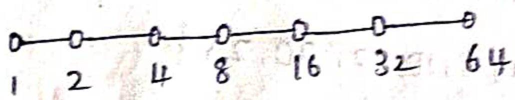
~~A lattice is an~~

(5) Draw the Hasse diagram ~~responding~~ Representing the partial ordering $\{a,b: a/b\}$ on $\{1,2,3,4,6,8,12\}$

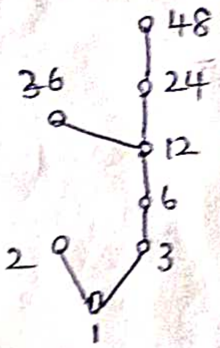
Ans:



(6) Draw the Hasse diagram in the set $\{1,2,4,8,16,32,64\}$



(7) Draw the Hasse diagram for divisibility on the set $\{1,2,3,6,12,24,36,48\}$



(4) Let A be the set of Factors a particular positive integer M and let ≤ be the relation divides i.e $x \leq y$ if x divides y. Then $(A, \leq)$ be the relation x divides y then draw Hasse diagram For $m=2$; $m=6$; $m=30$; $m=210$; $m=12$; $m=45$ For this

Ans: $m=2$

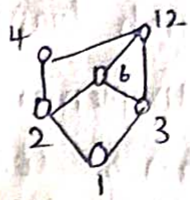
(1, 2)

$m=6$

(1, 2, 3, 6)

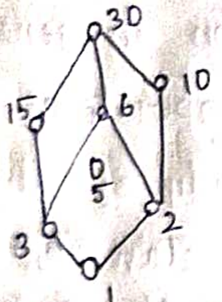
$m=12$

(1, 2, 3, 4, 6, 12)



$m=30$

(1, 2, 3, 5, 6, 10, 15, 30)



$m=210$

(1, 2, 3, 5, 7, 21, 42, 105, 210)

Functions

Introduction

Here we study a particular class of relations called function. We are First concered with discrete functions which transform a finite set into a another finite set. There are Several such transformation involved in the computer implementations of any program Computer output can be Considered as a function of the output. A compiler transform a program into a set of machine language, ~~infi~~ instruction after introducing the concept of function in general we discuss usually and binary operations have important applications in the study of algebraic structures also we discuss special class organizing files on a compiler, along with other technique associated with such organization.

Definition:

Let x and y be any two set A relation f from x to y is called a function if for every $x \in X$ there is a unique $y \in Y$ such that $(x, y) \in F$

ex: let $X = \{1, 2, 3\}$, $y = \{a, b, c\}$

$F = \{(1, a), (2, b), (3, c)\}$ $\therefore$ f is called Functions

Note: The DCF of function requiries that a relation number satisfy two additional conditions in order to quatity as a functions

* Every $x \in X$ must be related to some $y \in Y$ i.e,

(i) domain of 'f' must be X and not a subset of X

(ii) uniquenes $(x,y) \in F \wedge (x,z) \in F \Rightarrow y = z$

* Transformation, map, mapping, correspondance, operation are synonims for function.

* For a Function $f: X \to Y$, if $(x,y) \in F$ then x is called an argument and the corresponding y is called the image of x under f.

* The range of f is defined as $\{y / \exists x \in X \wedge y = f(x)\}$

Eg: Let $X = \{1, 5, P, Jack\}$

$Y = \{2, 5, 7, q, Jill\}$ and

$F = \{(1,2), (5,7), (P,q), (Jack, q)\}$

Ans: $D_f = \{1, 5, P, Jack\}$

$R_f = \{2, 7, q\}$

Eg: Let $X = Y = R$ and $f(x) = x^2 + 2$ $D_f = R$ and $R_f \le R$ the values of F for different values of $x \in R$ all lie on parabola as shown in the figure.

$f(x) = x^2 + 2$

(1) $f(0) = 0 + 2$
$f(0) = 2 \Rightarrow (0,2)$

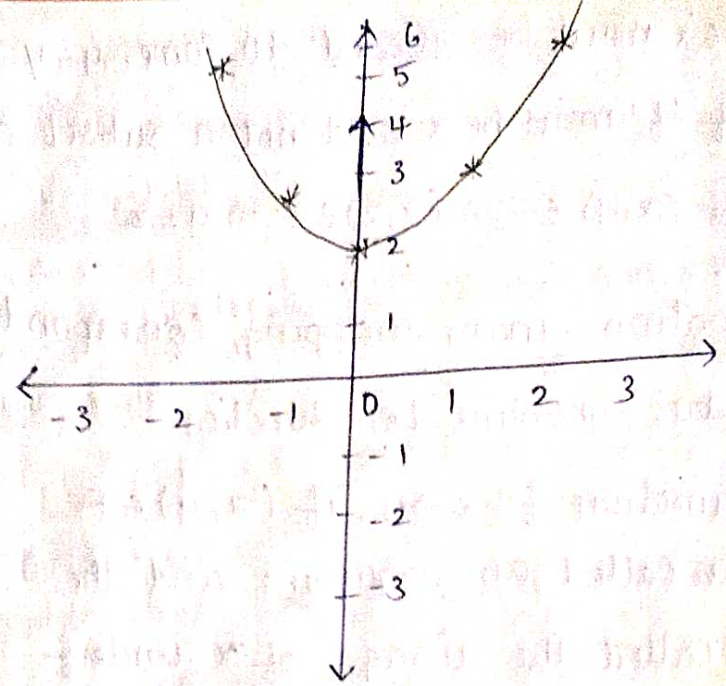
$f(-1) = (-1)^2 + 2$
$= 3 \Rightarrow (-1,3)$

(2) $f(1) = 1 + 2$
$f(1) = 3 \Rightarrow (1,3)$

$f(-2) = (-2)^2 + 2$
$= 6 \Rightarrow (-2,6)$

(3) $f(2) = 4 + 2$
$= 6 \Rightarrow (2,6)$

concerned



$Df = \{0, 1, 2\} = X$

$Rf = \{2, 3, 6\} = Y$

$Df = \{-1, -2, -3\} = X$

~~$Rf = \{3, 6,\} = Y$~~

Definition onto

A map of $f : x \to y$ is called onto. If the range $Rf = y$ otherwise it is called into.

Eg: $f : x \to y$ Let $x = \{x_1, x_2, x_3\}$

$y = \{y_1, y_2, y_3\}$

$f = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$

$Rf = \{y_1, y_2, y_3\} = y$

$f(x) = x+1$

Let $f(x) = y$

~~A Mapping $f : x \to y$ is said to an onto mapping~~

~~If $f(x) = y$~~

$x = 0$	$x = 1$	$x = 2$	$x = -1$	$x = -2$
$f(x) = 0+1$	$f(x) = 1+1$	$f(x) = 2+1$	$f(x) = -1+1$	$f(x) = -2+1$
$f(x) = 1$	$f(x) = 2$	$f(x) = 3$	$= 0$	$f(x) = -1$

$(x, y) = (0,1), (1,2), (2,3)$

Now $Rf = \{1, 2, 3\} = y$ $\therefore$ f is onto function

A Mapping $f : x \to y$ is said to be then mapping onto function Then $f(x) = y$

Into definition:

A mapping $f: X \to Y$ is said to be an into mapping if $f(x)$ is a proper subset of Y

Example

Let $f: Z \to Z$ is defined by $f(x) = 2x$, $x \in Z$

$f(x) \subset y$

$f(x) = 2(x)$ $\quad$ $x = 1$

$x = 0$ $\quad$ $f(x) = 2(1)$

$f(x) = 0$, $\quad$ $= 2$

(x, y)

$df = \{0, 1, 2, 3, 4\} = X$

$rf = \{0, 2, 4, 6, 8\} = Y$

$Y \subset X$

$\therefore$ given function is into

(1) F is a function $Z \to Z$ and the function is

$f(x) = x+1$

Let $f(x) = x+1$

$f(x) = y$

Now $x = 0$ $\quad$ $f(1) = 1+1$ $\quad$ $f(2) = 2+1$ $\quad$ $f(3) = 3+1$

$f(0) = x+1$ $\quad$ $= 2$ $\quad$ $= 3$ $\quad$ $= 4$

$= 0+1$ $\quad$ $f(1) = (1, 2)$ $\quad$ $f(2) = (2, 3)$ $\quad$ $f(3) = (3, 4)$

$f(0) = (0, 1)$

$x = \{0, 1, 2, 3\}$

$y = \{1, 2, 3, 4\}$

$f = \{(0,1), (1,2), (2,3), (3,4)\}$

$df = \{(0, 1, 2, 3)\}$

$Rf = \{1, 2, 3, 4\} = y$

$\therefore$ $\boxed{Rf = y}$

$\therefore$ y is onto

Now findout the into function

$f(x) = x+1$

$x = -1 \Rightarrow -1+1 = 0$

$x = -2 \Rightarrow -2+1 = -1$

$x = -3 \Rightarrow -3+1 = -2$

$x = -4 \Rightarrow -4+1 = -3$

$x = \{-1, -2, -3, -4\}$

$y = \{0, -1, -2, -3\}$

$F(x) = \{(-1,0), (-2,-1), (-3,-2), (-4,-3)\}$

$df = \{-1, -2, -3, -4\}$

$rf = \{0, -1, -2, -3\}$

$Rf \subset x$ is into function

One-one function (or) One-One Mapping:

A Mapping $f: x \rightarrow y$ is called one to one of into distint of x are mapped into distinct elements of y in otherwards f is one-one function

$$\boxed{f(x_1) = f(x_2)}$$

(3) $F(x) = 2x$ prove onto and into

Let $f(x) = 2x$

$f(0) = 2(0) = 0 = (0,0)$

$f(1) = 2(1) = 2 = (1,2)$

$f(2) = 2(2) = 4 = (2,4)$

$f(3) = 2(3) = 6 = (2,6)$

$x = \{0,1,2,3\}$; $y = \{0,2,4,6\}$

$f = \{(0,0), (1,2), (2,4), (3,6)\}$

$df = \{0,1,2,3\} = x$

$Rf = \{0,2,4,6\} = y$

$y = Rf$ is onto

Into function

$F = (2(x))$

$f(-1) = 2(-1)$
$= -2$
$(-1,-2)$

$f(-2) = 2(-2)$
$= -4$
$(-2,-4)$

$f(-3) = 2(-3)$
$= -6$
$(-3,-6)$

$f(-4) = 2(-4)$
$= -8$
$(-4,-8)$

$df = \{-1,-2,-3,-4\} = X$ $\quad x = \{-1,-2,-3,-4\}$

$rf = (-2,-4,-6,-8\} = Y$ $\quad y = \{-2,-4,-6,-8\}$

$rf \subset x$ is into

Prove that the Mapping $f: N \to N$ defined by $f(x) = n^2+n+1$ is one-one but not onto

Ans: Let $n_1, n_2 \in N$

$f(n_1) = n_1^2 + n_1 + 1$

$f(n_2) = n_2^2 + n_2 + 1$

$f(n_1) = f(n_2)$

$n_1^2 + n_1 + 1 = n_2^2 + n_2 + 1$

$n_1^2 - n_2^2 + n_1 - n_2 = 0$

$(n_1 - n_2)(n_1 + n_2) + (n_1 - n_2) = 0$

$(n_1 - n_2)[n_1 + n_2] = 0$

$n_1 - n_2 = 0$

$n_1 = n_2$

$\therefore f$ is one-one

Now $1 \in N$, Let n be pre-image under f then

$f(n) = 1$

$n^2 + n + 1 = 1$

$n^2 + n = 0$

$n(n+1) = 0$

$n + 1 = 0$

$\boxed{n = -1}$

$\exists$ no $n \in \mathbb{N}$ Statement function $f(n) = 1$

$\therefore$ F is not onto

Inverse function

(1) A Mapping $I_x : X \to X$ is called an identify Map

If $I_x = \{(x, x) / x \in X\}$

(2) Observe that for Any function $g : X \to X$ the function

$I_x \circ g$ and $g \circ I_x$ are Both equat to G.

(3) Also for any function $f : X \to Y$ we have $f \circ I_x = f$

(4) This properties of the identity function Can be used in stating the theorem about the inverse of a function

(5) The converse of Relation R from $X \to Y$ is defined to be a relation $\tilde{R}$ from $Y \to X$ such that $(y, x) \in \tilde{R} \Leftrightarrow (x, y) \in R$ i.e the ordered pair of $\tilde{R}$ are obtained from those of R by simply interchanging the mint terms the situation is not quiet the same for functions.

(6) Let $\tilde{F}$ denotes the converse of F where F is consider as a relation from $X \to Y$

(7) Naturally $\tilde{F}$ may not be a function Because

(i) The domain of $\tilde{F}$ may not be Y But only a subset of Y

(ii) $\tilde{F}$ may not be function from ~~of~~ to 'X' Because it may not Satisfy the uniqueness condition.

Example:

Let $x = \{1, 2, 3\}$ $y = \{p, q, r\}$ and $f: x \to y$ be given

By $f = \{(1, p), (2, q), (3, r)\}$ Then $\bar{f} = \{(p, 1), (q, 2), (p, 3)\}$

F is a function $\bar{F}$ is not a function.

Example:

Let R be a set of real numbers and $f: R \to R$

be given by $f = \{(x, x+2) / x \in R\}$

$\bar{f} = \{(x+2, x) / x \in R\}$ is a function

(1) Show that the function $f(x) = x^3$ and $g(x) = x^{\frac{1}{3}}$

For $x \in R$ are inverse of one another

$fog(x) = f(g(x))$

$= f(x^{\frac{1}{3}})$

$= (x^{\frac{1}{3}})^3$

$= x = Ix$

$gof(x) = g(f(x))$

$= g(x^3)$

$= (x^3)^{\frac{1}{3}}$

$= x = Ix$

then $f = g^{-1}$ (or)

$g = f^{-1}$

(2) Let $f: z \to z$ be such that $f(x) = x+1$ If f is invertable

Ans: The function f as an inverse Because it is one to one corresponds the reverse of correspondence y is the image of x. So that $y = x+1$

Then $x = y-1$ This means that $y-1$ is the unique.

$f(x) = x+1$

$f(y) = y+1$

$f(x) = f(y)$

$f(z) = z$ is onto

~~The function f is one one~~

The function f is one one and onto the function is invertable

(3) Let $f: R^+ \rightarrow R^+$ given by $f(x) = x^2$ show that f is invertable.
10M

Ans: The function $f(x) = x^2$ from the set of non negative real numbers.

To the set of non negative real numbers is one to one

$$f(x) = f(y)$$

$$x^2 = y^2$$

$$x^2 - y^2 = a$$

$$(x+y)(x-y) = 0$$

$$x+y = 0, \quad x-y = 0$$

$$x = -y, \quad x = y$$

Because Both x and y are non negative real numbers we must have $x = y$, Therefore f is invertable.

f is one-one.

Composition of functions

Let $F: x \to y$ and $G: y \to z$ be two functions the composite relation gof such that

$gof = \{(x,z) / x \in X \wedge z \in Z \wedge y \in Y \wedge f(x) \wedge z = g(y)\}$

is called the composition of function or relative product of function f and g.

More precisely gof is called the let composition of g with f

Consider three functions $f: x \to y$, $g: y \to z$ and $h: z \to w$ the composite function $gof: x \to z$, can be formed either composite function $hog: y \to w$ can also be function such as $ho(gof)$ and (hog) of

Both of these functions are from x to w

$f: x \to y$; $g: y \to z$; $h: y \to w$

Assuming

$y = f(x)$; $z = g(y)$ & $w = h(z)$ we have $(x,y) \in f$

$(y,z) \in g$, $(z,w) \in h$ &

$(x,z) \in gof$, $(y,w) \in hog$

$(x,w) \in ho(gof)$

we have

$ho(gof) = (hog)of$

(1) Let $x = \{1, 2, 3\}$ and f, g, h and s be function from $X \to x$ given by

$f = \{(1,2), (2,3), (3,1)\}$

$g = \{(1,2), (2,1), (3,3)\}$

$h = \{(1,1), (2,2), (3,3)\}$

$s = \{(1,1), (2,2), (3,3)\}$

find the fog, gof, fohog, gos, sog, sos and fos

Ans: $f \circ h \Rightarrow h = \{(\cancel{1},1), (\cancel{2},2), (\cancel{3},1)\}$

$f = \{(\cancel{1},2), (\cancel{2},3), (\cancel{3},1)\}$

$f \circ h = \{(1,2), (2,3), (1,1)\}$

$g \circ f \Rightarrow f = \{(\cancel{1},2), (\cancel{2},3), (\cancel{3},1)\}$

$g = \{(\cancel{1},2), (\cancel{2},1), (\cancel{3},3)\}$

$g \circ f = \{(2,2), (3,1), (1,3)\}$

$f \circ h \circ g \Rightarrow f \circ h \Rightarrow h = \{(\cancel{1},1), (\cancel{2},2), (\cancel{3},1)\}$

$f = \{(\cancel{1},2), (\cancel{2},3), (\cancel{3},1)\}$

$f \circ h = \{(1,2), (2,3), (3,3)\}$

$f \circ h \circ g \Rightarrow g = \{(\cancel{1},2), (\cancel{2},1), (\cancel{3},3)\}$

$f \circ h = \{(\cancel{1},2), (\cancel{2},3), (\cancel{3},3)\}$

$f \circ h \circ g = \{(2,2), (1,3), (3,3)\}$

$g \circ s \Rightarrow s = \{(\cancel{1},1), (\cancel{2},2), (\cancel{3},3)\}$

$g = \{(\cancel{1},2), (\cancel{2},1), (\cancel{3},3)\}$

$g \circ s = \{(1,2), (2,1), (3,3)\}$

$s \circ g \Rightarrow g = \{(\cancel{1},2), (\cancel{2},1), (\cancel{3},3)\}$

$s = \{(\cancel{1},1), (\cancel{2},2), (\cancel{3},3)\}$

$s \circ g = \{(2,1), (1,2), (3,3)\}$

$s \circ s \Rightarrow s = \{(\cancel{1},1), (\cancel{2},2), (\cancel{3},3)\}$

$s = \{(\cancel{1},1), (\cancel{2},2), (\cancel{3},3)\}$

$s \circ s = \{(1,1), (2,2), (3,3)\}$

$f \circ s \Rightarrow s = \{(\cancel{1},1), (\cancel{2},2), (\cancel{3},3)\}$

$f = \{(\cancel{1},2), (\cancel{2},3), (\cancel{3},1)\}$

$f \circ s \Rightarrow \cancel{s\circ f}\ \{(1,2), (2,3), (3,1)\}$

(2) Consider the function $f: R \to R$ defined by $f(x) = 2x+5$

Let a Reaction $g: R \to R$ is defined by $g(x) = \frac{1}{2}(x-5)$

prove that 'g' is an inverse of f

$$gof(x) = g(2x+5)$$

$$= \frac{1}{2}(2x+\cancel{5}-\cancel{5})$$

$$= \frac{1}{\cancel{2}}(\cancel{2}x)$$

$$= x$$

$$fog(x) = f(\underbrace{\tfrac{1}{2}(x-5)}_{x})$$

$$= \cancel{2}(\tfrac{1}{\cancel{2}}(x-5))+5$$

$$= x-\cancel{5}+\cancel{5}$$

$$= x$$

$\therefore$ g is inverse of f and also f is inverse of g

(3) Let $f: R \to R$ is given by $f(x) = x^2$ and $g: R \to R$ is given by $g(x) = \frac{x}{x-2}$ find fog; gof

$$fog = f(g(x)) = f\left(\frac{x}{x-2}\right)$$

$$= \left(\frac{x}{x-2}\right)^2$$

$$= \frac{x^2}{(x-2)^2}$$

$$= \frac{x^2}{x^2+4-4x}$$

$$gof = g(f(x)) = g(x^2)$$

$$= \frac{x^2}{x-2}$$

$$fog \neq gof$$

(4) If $f(x) = \log_3 x$ and $g(x) = x^4$ find gof and fog

$gof = g(f(x))$

$= g(\log_3 x)$

$= (\log_3 x)^4$

$fog = f(g(x))$

$= f(x^4)$

$= \log_3 x^4$

---

(5) Let $f(x) = x+2$, $g(x) = x-2$, $h(x) = 3x$ for $x \in R$

where R is the set of real numbers. Find

gof, fof, fog, gog, foh, hog, hof and fohog

Ans: (1) $gof \Rightarrow g(f(x))$

$\Rightarrow g(x+2)$

$\Rightarrow (x - \cancel{2} + \cancel{2})$

$gof = \{x \mid x \in R\}$

(2) $fog = f(g(x))$

$= f(x-2)$

$= x + \cancel{2} - \cancel{2}$

$fog = x$

(3) $fof = f(f(x))$

$= f(x+2)$

$= x+2+2$

$= x+4$

(4) $foh \Rightarrow f(h(x))$

$\Rightarrow f(\underset{x}{3x})$

$\Rightarrow 3(x+2)$

$\cancel{\Rightarrow 3x+6}$

$\cancel{\Rightarrow 3x+6}$

(5) $hog \Rightarrow h(g(x))$

$\Rightarrow h(x-2)$

$\Rightarrow 3(x-2)$

$\cancel{\Rightarrow 3x-6} \Rightarrow 3x-6$

$\cancel{3x-6}$

(6) $hof \Rightarrow h(f(x))$

$h(x+2)$

$3(x+2)$

$= 3x+6$

(7) $fohog \Rightarrow foh \Rightarrow f(h(x))$

$\Rightarrow f(3x)$

$= 3(x+2)$

$= 3x+6$

(8) $gog = g(g(x))$

$= g(x-2)$

$= x-2-2$

$= x-4$

(9) $fohog$

$hog = h(g(x))$

$= h(x-2)$

$= 3x-2$

$fohog = f(hog)$

$= f(3x-2)$

$= 3(x+2)-2$

$= 3x+6-2$

$= 3x+4$

Recursive function

Any function $f: N^n \to N$ is called ~~Recursive function~~ Total Because it is defined for every n-Tuples in N^n

Example: $f(x,y) = x+y$ which is defined and Hence is a total function $\forall x,y \in N$

Definition of partial function

If $f: D \to N$, where $D \subseteq N^n$ then f is called a partial function

Example :(1) $g(x, y) = x - y$ which is defined for only those $x, y \in N$ which satisfies $x \geq y$

Hence $g(x, y)$ is partial

(2) $g(2,5) = 2-5 = -3$ Not defined

$g(5,2) = 5-2 = 3$ defined

Note :

A partial function can be made total function of we restrict the domain of the function only to those values for ~~this~~ which function value is defined.

Intial function

A set of three functions call the intial function which are used to defining other function by induction.

Zero function

(i) $Z : z(x) = 0$

(ii) Successor function

$S = S(x) = x+1$

$S(1) = 1+1 = 2$

$S(2) = 3$

$S(3) = 4$

(iii) projection function

$u_i^n : u_i^n(x_1, x_2 \ldots x_n)$ the Projection function is also called $= x_i$ the generalized identity function

Ex : $U_1^2(x, y) = x$, $U_2^3 = (x, y, z) = y$

$U_2^3(2, 4, 6) = 4$ $\quad U^5(1, 2, 3, 4, 5) = 4$

$U_6^{10}(2, 4, 6, 8, 10, 12, 14, 16, 18, 20) = 12$

The operation of composition will be used to generate other functions we have discussed in the eariler section composition of function of one variable the same idea can be extended to function more than one variable

Ex: Let $f_1(x,y)$, $f_2(x,y)$ and $g(x,y)$ be any three function composition of 'g' with f_1 and f_2 is a function 'h' given by $h(x,y) = g(f_1(x,y), f_2(x,y))$

for h to be non-empty it is necessary that the domain of g include $R_{f_1} \times R_{f_2}$ where R_{f_1} and R_{f_2} are ranges f_1 and f_2 respectively.

If f_1 and f_2 and g are total, then h is also total

In general Let $F_1, F_2 \dots F_n$ each be partial functions of an variables and Let g be a partial function of n variables then, the composition of g with $f_1, f_2 \dots f_n$ produces a partial function h given By $h(x_1, x_2 \dots x_m) = g(f_1(x_1, \dots x_m), \dots f_n(x_1, \dots x_n))$

Ex: Let $f_1(x,y) = x+y$, $f_2(x,y) = xy+y^2$

$g(x,y) = xy$ then $h(x,y) = g(f_1(x,y), f_2(x,y))$

$$= g(x+y, xy+y^2)$$

$$= (x+y)(xy+y^2)$$

Hence h is total, Because f_1, f_2 and g are all total.

~~A function $f(x_1,$~~

Recurion function

A function $f(x_1, x_2 \dots x_n y)$ of $(n+1)$ variables of using two another function $g(x_1, x_2 \dots x_n)$ and $h(x_1, x_2 \dots x_m)$ of n and n+2 variables respectively is called recursion $F(x_1, x_2 \dots x_n, 0) = g(x_1, x_2 \dots x_n)$ $f(x_1, x_2 \dots x_n, y+1) = h(x_1, x_2 \dots x_n, y$
$F(x_1, x_2 \dots x_n, y))$

Hence, the variable y is assumed to be inductive variable in the reverse that the value of 'F' at 'y+1' is expressed in terms of the value of 'F' at 'y'. The variables $x_1, x_2 \dots x_n$ are treated as parameters and are assumed to remain fixed throughout the definition also it is assumed that both the functions 'g' and have known

Def: A function is said to be recursive iff it can be obtained from the intial function by a finite no of application of the operations of composition, recurssion and minimization over regular function.

partial function

A function is said to be partial recursive if it can be obtained from intial function by a finite no of applications of the operation composition, recursion and minimization.

Premitive recursion

A function f is called premitive recursive function if it can be obtained from the initial function by a finite no. of operations of composition and recursion

(1) Show that the function $f(x,y) = x+y$ is premitive recursive.

Some of the primitive recursive functions

1: Sign function (or) non-zero test function

Sg: $Sg(0) = 0$, $Sg(y+1) = 1$

(or)

$$Sg(0) = z(0)$$

$$Sg(y+1) = S(z(U_2^2(y, Sg(y))))$$

2: zero test function, $\overline{Sg}$: $\overline{Sg}$ (or) $= 1$, $\overline{Sg}(y+1) = 0$

3: predecessor functions P:

$$P(0) = 0 ; \; P(y+1) = y = U_1^2(y, P(y))$$

Note: $P(0) = 0$; $P(1) = 0$; $P(2) = 1$; $P(3) = 2 \cdots$

4: Odd and even parity function, Pr;

$Pr(0) = 0$, $Pr(y+1) = \overline{Sg}\,(U_2^2(y, Pr(y)))$

$Pr(0) = 0$; $Pr(1) = 1$

$Pr(2) = 0$; $Pr(3) = 1 \cdots$

$Pr(4) = 0$;

5. proper subtraction function, $\dot{-}$:

$$x \dot{-} 0 = x$$

$$x \dot{-} (y+1) = P(x \dot{-} y)$$

Note : $x \dot{-} y = 0$ for $x < y$ and

$x \dot{-} y = x - y$ for $x \geq y$

6. Absolute value function

$$|x - y| = (x - y) + (y - x)$$

7. Min (x, y) = Minimum of x & y

$$\text{Min}(x, y) = x - (x - y)$$

Max (x, y) = Maximum of x & y

$$\text{Max}(x, y) = y + (x - y)$$

8. The square function $f(x) = y^2$

$f(y) = y^2 = V_1^1(y) * V_1^1(y)$

Ackerman's Function:-

The Ackerman's function $A(x,y)$ is defined by

(1) $A(0,y) = y+1$

(2) $A(x+1, 0) = A(x,1)$

(3) $A(x+1, y+1) = A(x, A(x+1, y))$

From the definition it is clear that we can construct the value of $A(x,y)$ for fixed values of (x,y).

Therefore $A(x,y)$ is well defined and total

Note: $A(x,y)$ is not primitive recursive But recursive

Problem

(1) If A denotes Ackerman's function evaluate

(i) $A(1,1)$ (ii) $A(1,2)$ (iii) $A(2,1)$ (iv) $A(2,2)$ (v) $A(1,4)$

(vi) $A(1,3)$

Sol: (i) $A(1,1) = A(\overset{x}{0+1}, \overset{y}{0+1})$ $[\because A(x+1,y+1) = A(x, A(x+1,y))]$

$= A(0, A(0+1,0))$ $[\because A(x+1,0) = A(x,1)]$

$= A(0, A(0,1))$ $[\because A(0,y) = y+1]$

$= A(0,2)$

$= 3$

(ii) $A(1,2) = A(\overset{x}{0+1}, \overset{y}{1+1})$

$= A(0, A(0+1,1))$

$= A(0, A(0,1))$

$= A(0,3)$

$= 4$

(iii) $A(2,1) = A(\overset{x}{1+1}, \overset{y}{0+1})$

$= A(1, A(2,0))$

$= A(1, A(1,1))$

$= A(1,3)$

$= A(\overset{x}{0+1}, \overset{y}{2+1})$

$= A(0, A(0+1,2))$

$= A(0, A(1,2))$

$= A(0,4)$

$= 5$

(iv) $A(2,2) = A(\overset{x}{1+1}, \overset{y}{1+1})$

$= A(1, A(1+1,1))$

$= A(1, A(2,1))$

$= A(1,5)$

$= A(\overset{x}{0+1}, \overset{y}{4+1})$

$= A(0, A(0+1,4))$

$= A(0, A(1,4))$

$= A(0, A(\overset{x}{0+1}, \overset{y}{3+1}))$

$= A(0, A(0, A(0+1,3)))$

$= A(0, A(0, A(1,3)))$

$= A(0, A(0,5))$

$= A(0,6)$

$= 7$

(v) $A(1,4) = A(\overset{x}{0+1}, \overset{y}{3+1})$

$= A(0, A(0+1,3))$

$= A(0, A(1,3))$

$= A(0,5)$

$= 6$

(vi) $A(1,3) = A(0+1, 2+1)$

$= A(0, A(0+1,2))$

$= A(0, A(1,2))$

$= A(0,4)$

$= 5$

pigeonhole principle

A well known proof technique in mathematics is called pigeonable principle are also has the Shoe box argument or dirichlet draws principles

In an informal way the pigeonhole principle says that if there are many pigeon and a few pigeon holes Then there must be some pigeonhole ~~oppuied~~ occupied by two or more pigeons formally.

Let D and R be finite sets if $|D| > [R]$ Then for any function of f from D to R.

There exist $d_1, d_2 \subset D$ Such that

$$f(d_1) = f(d_2)$$

where D = pigeons

R = pigeon holes

$|D|$ = No of pigeons

$|R|$ = No of pigeons holes

Applications of the pigeonhole principles

Among 13 people, there are atleast 2 of them who were Born in the same month

Here the 13 people are the pigeons, and the 12 months are the pigeonholes

Also if 11 shoes are selected from 10 pairs of shoes there must be a pair of method shoes among the selection.

pigeonhole principle can be started in a slightly more general from

For my function f from D to R, there exist i elements $d_1, d_2 \cdots d_i$ in D

$$i = \left[\frac{|D|}{|R|} \right] \text{ Such that}$$

$$f(d_1) = f(d_2) = \dots = f(d_i)$$

Binary and n-array operation:-

- Let 'X' be a set and f be a mapping $f: X * X \rightarrow X$ then 'f' is called a Binary operation on X.
- In general a mapping $f: X^n \rightarrow X$ is called an n-array operations and n is called the order of the operation.
- For n=1, $f: x \rightarrow X$ is called a unary operation.
- If an operation on the members of a set produced images which are also members of the same set then the set is said to be closed under that operation and the property is called closed properlity.

The definition of Binary n-array operation implies that the sets on which such operations are defined are closed under these operations. this property Binary n-array operation from other functions.

- Example :-

The operations of addition, subtraction and multiplication are Binary operations on the set of Integers and also on the set of real numbers.

Definition :- Commutative

A Binary operation $f: X * X \rightarrow X$ is said to be Commutative if for every $x, y \in X$, $f(x,y) = f(y,x)$

Associative :-

A Binary operation $f: X * X \rightarrow X$ is said to be Associative if for every $x, y, z \in X$, $f(f(x,y), z)) = f(x, f(y,z))$

The above definition can be written using '*' to denotes the Binary relations on X.

i.e, $*$ is commutative for any $x, y \in X$, $x * y = y * x$

- $*$ is associative on X, if for any $x, y, z \in X$,

$(x*y)*z = x*(y*z)$

- **Distributive :-**

A Binary operation $f: X * X \to X$ denoted by '$*$' is said to be distributive over the operation $g: X*X$ denoted by '$\circ$'. If for any $x, y, z \in X$,

$x*(y \circ z) = (x*y)\circ(x*z)$

Example :-

The operations of addition and multiplication over the set of real numbers are the set of Commutati and Associative

- **Algebric Sytem :-**

A system consisting of set and one or more n-ary operations on the set is called as Algebric System.

An Algebric system is denoted (by $S, f_1, f_2, ---- f_n$) where 'S' is non-empty set and $f_1, f_2, ---$ are operations of S.

Example :-

$(i, +, \times)$ is algebric system.

- **Algebric structure :-**

The operations and Relations on the set 'S' defined a structure on the elements of 'S' then the Algebric system is called Algebric structure.

Example :-

Let 'i' be the set of Integers Consider the

Algebric system $(I, +, \times)$ where $+$ and $\times$ are the operations of addition and multiplication on 'I'

A list of important properties of these operations.

① for any $a, b, c \in I$ [associative]

$$(a+b)+c = a+(b+c)$$

② for any $a, b \in I$

$$a+b = b+a \quad \text{[Commutative]}$$

③ There exist a distinguished element $0 \in I$ such that for any $a \in I$

$$a+0 = 0+a = a \quad \text{(Identity)}$$

④ for $a \in I$, there exist an element on 'I' denoted by $-a$ and called negative of a such that

$$a+(-a) = 0 \quad \text{(Inverse)}$$

⑤ for any $a, b \in I$

$$a \times b = b \times a \quad \text{(Commutative]}$$

⑥ There exists a distiguished element $1 \in I$ such that for any $a \in I$

$$a \times 1 = 1 \times a = a \quad \text{(Identity]}$$

⑦ for any $a, b, c \in I$

$$a \times (b+c) = (a \times b) + (a \times c) \quad \text{[Distributive]}$$

⑧ for $a, b, c \in I$ and $a \neq 0$ [Cancelation Property]

$$a \times b = a \times c \Rightarrow b = c$$

⑨ for any $a, b, c \in I$ [Associative]

$$(a \times b) \times c = a \times (b \times c)$$

The algebric system $I, +, *$ should have been expressed as $(I, +, *, 0, 1)$ in order to emphasize the fact that 0 and 1 are elements of I.

we shall now give examples of other algebric system with two binary operations which share most of the properties of $(I, +, *)$ listed here.

Example :- 1

Let 'R' be the set of real numbers + and $*$ be the operations of addition and multiple on R. The algebric system $(R, +, *)$ satisfies all the properties given for the system $(I, +, *)$. There are certain other properties which the two system from one another, but we shall not consider these properties.

Example :- 2

In the algebric system $(N, +, X)$ where 'N' is the set of Natural numbers and the operations + and X have the usual meanings all the properties listed for $(I, +, X)$ except (A-4) are satisfied.

Example :- 3

Let 'E' be a universal set and P(E) its power set denoting the union and Intersection on P(E) by + and X, we have an algebric system, $(P(E), +, X)$ with ϕ and E as elements in place of 0 and 1. and this system satisfies all the properties listed except (A4).

Consider the set $B = \{0, 1\}$ and the operations addition and multiplication on B given by following tables.

+	0	+
0	0	1
[illegible]	1	0

X	0	1
0	0	0
1	0	1

The algebric system $(B, +, \times)$ and satisfies all the properties listed for $(I, +, \times)$.

Let $X = \{a, b\}$ and 'S' denotes the set of all mapping from $x \to X$ let us write $S = \{f_1, f_2, f_3, f_4\}$ where

$f_1(a) = a$, $f_2(a) = a$ $\quad f_3(a) = b$, $f_3(b) = b$

$f_1(b) = b$, $f_2(b) = a$ $\quad f_4(a) = b$, $f_4(b) = a$

Then $(S, \times)$ where $\times$ denotes the operation of left Composition of functions S is an algebric system in which the operation is associative. The Composition table for operation '$\times$' is given in table.

o	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_2	f_2	f_2
f_3	f_3	f_3	f_3	f_3
f_4	f_4	f_3	f_2	f_1

Associative ;-

$(f_1 o f_2) o f_3 = f_1 o (f_2 o f_3)$

$f_2 o f_3 = f_1 o f_2$

$f_2 = f_2$

Commutative ;-

$(f_1 o f_2) = (f_2 o f_1)$

$f_2 = f_2$

Identity ;-

$a \in I$, $a + 0$, $0 + a = a$, $(a = f_1)$

$f_1 o 0 = f_1$

$0 o f_1 = f_1$

Inverse :-

$$a + (-a) = 0$$

$$f_1 \circ (-f_1) = 0$$

distributed :-

$$a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$$

$$f_1 \circ (f_2 \circ f_3) = (f_1 \circ f_2) \circ (f_1 \circ f_3)$$

$$f_1 \circ f_2 = f_2 \circ f_3$$

$$f_2 = f_2$$

(2) Let $X = \{1, 2, 3, 4\}$ and 'f' is function from $X \to X$ i given by $f = \{(1,2), (2,3), (3,4), (4,1)\}$ let the operatio defining in the following table.

$\circ$	f^0	f^1	f^2	f^3
f_1^0	f^0	f^1	f^2	f^3
f_2^1	f^1	f^2	f^3	f^0
f_3^2	f^2	f^3	f^0	f^1
f_4^3	f^3	f^0	f^1	f^2

$$f = \{f^0, f^1, f^2, f^3\}$$

then we find algebric system.

Associative :-

$$(f^0 \circ f^1) \circ f^2 = f^0 \circ (f^1 \circ f^2)$$

$$f^1 \circ f^2 = f^0 \circ f^3$$

$$f^3 = f^3$$

Commutative

$$f^0 \circ f^1 = f^1 \circ f^1$$

$f_1 = f_n$

Identity :-

$a \in I$, $a+0$, $0+0=0$, $(a=f^0)$

$$f^0 \circ 0 = f^0$$

$$0 \circ f^0 = f^0$$

Inverse :-

$$a+(-a)=0$$

$$f^0 \circ (-f^0) = 0$$

distributed :-

$$a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$$

$$f^0 \circ (f^1 \circ f^2) = (f^0 \circ f^1) \circ (f^0 \circ f^2)$$

$$f^0 \circ f^3 = f^1 \circ f^2$$

$$f^3 = f^3$$

Cancellation :-

$$a \times b = a \times c \Rightarrow b = c$$

$$f^0 \circ f^1 = f^0 \circ f^2$$

$$f^1 = f^2$$

$$b = c$$

Semigroups and monoids :-

The algebric system $(S, *)$ is called a semi group under the operation $*$ if it the following properties.

(1) Closure property

$\forall a, b \in S, \ a * b \in S$

(2) Associative property

$\forall a, b, c \in S$

$(a \times b) \times c = a \times (b \times c)$

If the binary operation $*$ is commutative then the Semigroup $(S, *)$ is called a Commutative semigroup (or) is a Abelian group.

Monoid :-

The algebric system $(S, *)$ is called a monoid under the operation $*$

(1) closure property :- $\forall a, b \in S, \ a * b \in S$

(2) Associative property :- $(a * b) * c = a * (b * c)$

(3) Identity :-

$\exists$ there exits an element e such that

$e * x = x * e = x \quad \forall x \in S.$

Note :- Every monoid is a semigroup but a semigroup need not be monoid.

Eg:- $(S, *)$ is a monoid

$(S, +)$ is a monoid

* Sub Semigroup :-

Let $(S, *)$ be a Semi group and let T be a subset of S, then $(T, *)$ is called a sub semigroup of $(S, *$

whenever 'T' is closed under '$*$'.

$(S, +)$ - semi group

$(S^+, +)$ - sub semi group.

$*$ Sub monoid:-

If $(S, *)$ is a monoid with e as the Identity element and if 'T' is a subset of S then $(T, *)$ is called a sub monoid of $(S, *)$, whenever 'T' is closed under $*$ $e \in T$.

In each of the following cases of a Binary operations on a set $A = \{a, b\}$ is defined through a multiplication table. Determine whether $(A, *)$ is a semigroup or a monoid or neighther.

(i)

$*$	a	b
a	b	a
b	a	b

(ii)

$*$	a	b
a	a	a
b	b	b

(iii)

$*$	a	b
a	a	a
b	a	a

Since $*$ is a binary operation on A. we have to verify only the associative law and the existence of Identity if any.

(i) $a * (a * a) = a * b = a = b \times a$
$= (a \times a) * a$

(ii) $a * (a * b) = a * a = b = b * b$
$= (a * a) * b$

(i) $a * (b * a) = a * b = a = \cancel{b * b}$ $\quad a * a$
$(b * a) * b \quad (a * b) * a$

(ii) $a * (b * b) = a * b = a = a * b$
$(a * b) * b$

(iii) $b * (a * a) = b * a = b = b * a$
$= (b * (a) * a)$

These relation shows that $*$ is associative

$a*b = a$, $b*a = a$,

$b*b = b$.

There for b is the Identity element in A under $*$. Hence A forms a monoid under $*$.

(ii) Associative law holds good but neither Anor b is the identity element because $a*b = a$ and $b*a =$

since $a*b \neq b*a$

$\therefore (A, *)$ is a non-Commutative semigroup. it is not a monoid.

(iii) we find that

$$b*(a*b) = b*b = a$$

$$(b*a)*b = a*b = b$$

These show that $*$ is not associative.

$\therefore$ A doesnot form a semigroup under $*$.

similarly we can prove (ii) table & (iii)

Imp On the set Q of all Relational numbers the operation $*$ is defined by $a*b = a+b-ab$. show that operation Q forms a commutative monoid.

since $a+b-ab$ is a relational number for all relational numbers a, b.

$\therefore$ The given operation '$*$' is a binary operation on Q.

$\forall a, b, c \in Q$

$$(a*b)*c = (\underbrace{a+b-ab}_{a}) * \overset{}{c}_{b}$$

$$= a+b-ab+c-(a+b-ab)*c$$

$$= a+b-ab+c-ac-bc+abc$$

$$= a+(b+c-bc)-(b+c-bc)*a$$

$$= a*(b+c-bc)$$

$$= a*(b*c)$$

Hence '$*$' is associate. // for any $a \in Q$.

$$a*0 = a+0-a \cdot 0$$
$$= a$$

$$0*a = 0+a-0 \cdot a$$
$$= a$$

$\therefore$ 0 is the Identity element in Q.

Commutative under given $*$ is also $(Q, *)$ is a Commutative monoid with '0' as the identity.

closure law and commutative :-

$\forall a, b \in Q$

$a*b$

$a+b-ab \in Q$

$b*a$

$b+a-ba = a+b-ab$

$\therefore$ closure is satisfied and Commutative

Semigroup Homomorphism and Isomorphism :-

* Consider two Semigroup $(S_1, *_1)$ and $(S_2, *_2)$. let 'f' be a function from S_1 to S_2. Then 'f' is called a Homomorphism from S_1 to S_2. If for all $a, b \in S_1$ we have $f(a *_1 b) = f(a) *_2 f(b)$.

* 'f' is called an Isomorphism from S_1 on to S_2.

  (i) If 'f' is a Homomorphism from S_1 to S_2.

  (ii) If 'f' is one-to-one and on-to.

* If $f : S_1 \to S_2$ is a Homomorphism then $f(S_1)$ which is set of images of elements of S_1 under 'f' is called the Homomorphic image of S_1 under 'f'

* If $f : S_1 \to S_2$ is an Isomorphism then $f(S_1) = S_2$ because 'f' is one-one and on-to.

* we say that A Semigroup $(S_1, *_1)$ is Isomorphism to a semigroup $(S_2, *_2)$. There is an Isomorphism from S_1 on-to S_2. Then we write $S_1 \simeq S_2$

Ex:- $(Z, +)$ & $(E, +)$ define $f : Z \to E$ by $f(x) = 2x$, $\forall x \in Z$, for any $a, b \in Z$

$$f(a+b) = 2(a+b) \qquad \left[\because f(x) = 2x, \quad x = a+b\right]$$
$$= 2a + 2b$$
$$= f(a) + f(b)$$

we take any $y \in E$, then $y = 2x$, for some $x \in Z$

$f(x) = 2x = y$

has a preimage in Z under f

$\therefore$ f is onto

$f(a) = f(b) \Rightarrow 2a = 2b$

$\Rightarrow a = b$; for $a, b \in Z$

Thus 'f' is a one-one correspondance with Z and E and f is homomorphism.

$\therefore$ f is an isomorphism from Z onto E.

* **Groups :-**

Let 'G' be a non-empty set and '*' be a binary operation on G then the algrebic system (G, *) is called a Group under the operation * if the following properties.

(i) Closure :-

$\forall\ a, b \in G \quad , \quad a * b \in G$

(ii) Associative :-

$(a * b) * c = a * (b * c) , \quad \forall\ a, b, c \in G.$

(iii) Identity :-

There is an element $e \in G$ such that

$a * e = e * a = a, \quad \forall\ a \in G.$

(iv) Inverse :-

$\forall\ a \in G$ There is an element $a' \in G$ such that

$a * a' = a' * a = e.$

Note :- Every Group is a monoid and Every Group is a semigroup.

Definition for (Algebric) Abelian system :-

The Algebric system (G, *) is called a Commutative group (or) Abelian group under the operation * of the following operation.

(i) closure :- $\forall a, b \in G$, $a*b \in G$.

(ii) Associative :- $(a*b)*c = a*(b*c)$ $\forall a, b, c \in G$

(iii) Identity :-

There is an element $e \in G$ such that

$$a*e = e*a = a, \quad \forall a \in G.$$

(iv) Inverse :-

$\forall a \in G$ there is an element $a' \in G$ such that

$$a*a' = a'*a = e$$

(v) Commutative :-

$\forall a, b \in G$

$$a*b = b*a.$$

Definition :-

A Group 'G' is said to be a <u>finite group</u>. If G is the finite set of the order of G is in 'n' then $O(G) = n$ with $G = n$.

G is not a finite set is called an <u>Infinite group</u>.

Example :-

(i) $(Z, +)$ is a group

(ii) $(Z, -)$ is not a group

(iii) The set of all $n*n$ non-singular matrix forms an infinite group under matrix multiplication with the unit matrix of order n as the Identity. The group is not Abelian because matrix multiplication is not Commutative.

Let $A = \{0, 1\}$ and the operation $*$ on A be defined by

$*$	0	1
0	0	1
1	1	0

$(A, *)$ is a Abelian group '0' is an Identity element and every element is it own Inverse.

Example :-

$G = \{1, -1, i, -i\}$ the set of all fourth roots of unity

$*$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

$(G, *)$ is an Abelian group. This group is a finite group of order fourth.

v.v Imp ① Let 'G' be a set of all non-zero real numbers and let $a * b = \frac{1}{2} ab$ show that $(G, *)$ is an Abelian group.

for any two non-zero real number a and b.

we note (know) that $\frac{1}{2} ab$ is a non-zero real numbers.

(i) $\forall\, a, b \in G$, $a * b \in G$

$\therefore$ G is closure under $*$

(ii) for any $a, b, c \in G$, we have

$$a * (b * c) = a * (\tfrac{1}{2} bc)$$

$$= \tfrac{1}{2} (\tfrac{1}{2} abc)$$

L.H.S $a * (b * c) = \tfrac{1}{4} (abc)$

$$(a * b) * c = \tfrac{1}{2} ab * c$$

$$= \tfrac{1}{2} (\tfrac{1}{2} abc)$$

R.H.S $(a * b) * c = \tfrac{1}{4} abc$

$$LH.S = R.H.S$$

$$a * (b * c) = (a * b) * c$$

$\therefore$ G is Associative under $*$.

(iii) for any $a \in G$, we have

$$a * b = \tfrac{1}{2} ab$$

$$a * 2 = \cancel{\tfrac{1}{2}} * a \cancel{2} = a \qquad \because b = 2$$

$$2 * a = \cancel{2} * \tfrac{1}{\cancel{2}} a = a$$

$\therefore$ '2' is the Identity element under '$*$'.

(iv) for any $a \in G$, if we set $a' = \frac{4}{a}$

Then $a' \in G$ &

$$a * a' = \tfrac{1}{2} (aa') = \tfrac{1}{\cancel{2}} \cdot \cancel{a} \cdot \frac{\cancel{4}^{2}}{\cancel{a}} = 2$$

$$a' * a = \tfrac{1}{2} (a'a) = \tfrac{1}{2} \tfrac{4}{\cancel{a}} \cdot \cancel{a} = 2$$

$\therefore$ for every $a \in G$ has $a' = 4/a$ has the Inverse in G under $*$.

$\therefore$ $(G, *)$ is a Group.

(v) for any $a, b \in G$

$$a * b = \tfrac{1}{2} ab$$
$$= \tfrac{1}{2} ba$$
$$= b * a$$
$$a * b = b * a$$

$\therefore$ The Group $(G, *)$ is an Abelian group.

(2) If 'o' is an operation on 'Z' defined by $x o y = x + y + 1$ prove that (Z, o) is an Abelian group.

(i) $\forall\ a, x, y \in Z$, ~~$x * y \in Z$~~. $x o y = x + y + 1 \in Z$.

$\therefore$ Z is closure under o.

(ii) for any $x, y, z \in Z$, we have

$$x o (y o z) = x o (y + z + 1)$$
$$= x + y + z + 1 + 1$$
$$x o (y o z) = x + y + z + 2 = L.H.S$$
$$(x o y) o z = (x + y + 1) o z$$
$$= x + y + 1 + z + 1$$
$$(x o y) o z = x + y + z + 2 = R.H.S$$

$\therefore$ L.H.S = R.H.S

$\therefore$ Z is Associative under o.

$x o y = x + y + 1$

$x o (-1) = x - 1 + 1$

(iii) for any $x \in Z$, we have.

$$\cancel{a o} = \cancel{x} -$$
$$x o y = x + y + 1$$
$$x o (-1) = x + (-1) + 1$$
$$x o (-1) = x$$
$$(-1) o x = (-1) + x + 1$$
$$(-1) o x = x$$

$\therefore$ '-1' is the Identity element in 'o'.

(iv) for any $x \in Z$. if we set

$$x' o x = (x' + x + 1) \qquad x o x' = (x + x' + 1)$$

$$= \frac{-2}{x} + x + 1 \qquad = x + \frac{-2}{x} + 1$$

$$= -2 + 1 \qquad = -2 + 1$$

$$= -1 \qquad = -1$$

$\therefore$ for every $x \in Z$ has $x' = -2/x$ has the Inverse in Z under o. (Z, o) is a Group.

(v) for any $x, y \in Z$

$$(x o y) = x + y + 1$$

$$= y + x + 1 = (y o x)$$

$(x o y) = (y o x)$ $\therefore$ for every Z, is a Commutative Group.

$\therefore$ (Z, o) is an Abelian group.

⑥ Prove that a Group G is Abelian. If $(ab)^{-1} = a^{-1} b^{-1}$ $\forall a, b \in G$.

(i) Commutative :-

Suppose that 'G' is Abelian.

(for all) $\forall a, b \in G$.

$$a^{-1} b^{-1} = b^{-1} a^{-1}$$

$$= (ba)^{-1}$$

$$\therefore \quad a^{-1} b^{-1} = (ab)^{-1}$$

Converse the suppose that $(ab)^{-1} = a^{-1} b^{-1}$

for any $\forall a, b \in G$.

$$xy = (x^{-1})^{-1} (y^{-1})^{-1}$$

$$xy = (y^{-1})^{-1}(x^{-1})^{-1}$$

$$\underline{xy = yx}$$

$\therefore$ G is Abelian.

Imp

(1) Theorem :-

In a group there exist only one Identity element in a group is unique.

sol:- Suppose that e_1 and e_2 are Identity element in a group G. since e_1 is an Identity element in G. we have $ae_1 = e_1 a = e_1 \quad \forall a \in G$

Taking $a = e_2$ in this way

$e_2 e_1 = e_1 e_2 = e_1$, since e_2 is an Identity element in G. a similar argument. show that

$$e_1 e_2 = e_2 e_1 = e_2$$

$$e_1 = e_1 e_2 = e_2 e_1 = e_2$$

This shows that e_1 and e_2 are not different.

This means that G has only one identity element.

(2) Theorem :-

In a group, every element has only one Inverse i.e every element has a unique Inverse in G.

sol :- Let 'e' be the Identity element in G. suppose a' and a'' are inverses of an element $a \in G$, Then

$$a' = a'e = a'(aa'') = (a'a)a'' \qquad (e = aa'')$$

$$= ea'' = a'' \qquad e = aa'$$

Thus a' and a'' Cannot be different and hence the theorem

(3) Theorem :-

Imp for any element a, b in G.

(i) $(a^{-1})^{-1} = a$, (ii) $(ab)^{-1} = b^{-1}a^{-1}$

Sol :- (i) Let us set $a' = c$ then

$$ca = a^{-1}a = e \quad \text{and} \quad ac = aa^{-1} = e$$

These shows that a is the Inverse of c.

That is $a = c^{-1}$ Thus $a = (a^{-1})^{-1}$

(ii) we have $(ab)^{-1} = b^{-1}a^{-1} \Rightarrow (ab)(b^{-1}a^{-1})$

$$(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1}$$
$$= aea^{-1}$$
$$= aa^{-1} = e$$

and $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b$
$$= b^{-1}eb$$
$$= b^{-1}b = e$$

$\therefore \quad (ab)^{-1} = b^{-1}a^{-1}$

<u>Unit-5</u>

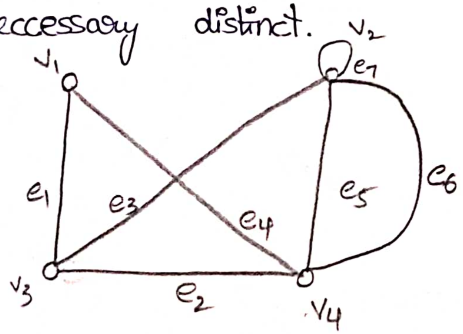
Graph theory

Graph theory :- A Graph 'G' is a 3-tuple (or) Triple denoted by 'G' and is defined by

$G = (V(G), e(G), x(G))$ (or)

$G = (V, e, x)$ where $V(G)$ is a non-empty set of vertices [nodes, points]

* $e(G)$ is a set of edges [arrows, link]
* $x(G)$ is a mapping that associated with each edge of G.
* An unordered pair of vertices of G which are not neccessary distinct.



$V(G) = \{v_1, v_2, v_3, v_4\}$

$e(G) = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

$x(e_1) = \{v_1, v_3\}$, $x(e_2) = \{v_3, v_4\}$, $x(e_3) = \{v_3, v_2\}$

$x(e_4) = \{v_1, v_4\}$, $x(e_5) = \{v_2, v_4\}$, $x(e_6) = \{v_2, v_4\}$,

$x(e_7) = \{v_2, v_2\}$

* Definition :-

* If $e = \{v_1, v_2\}$ is an edge of a graph G then v_1 & v_2 are ends of e. The e joins the vertices v_1 & v_2.
* An edge $e = \{v_1, v_2\}$ where $v_1 \neq v_2$ is called a 'link' but an edge $e = \{v_1, v_1\}$ is called a loop.

Eg:- $v_1 \xrightarrow{\text{link}} v_2$, loop.

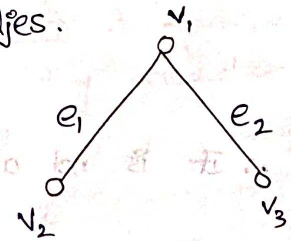
Incident :-

given an edge $e = (v_1, v_2)$ we say that e is incident with v_1 and v_2 we also say that the vertices v_1 and v_2 are incident with 'e'.

Eg:- $v_1 \overset{e}{—} v_2$

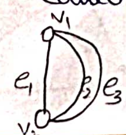
Adjecent :-

if $e_1 = (v_1, v_2)$ and $e_2 = (v_1, v_3)$ are two edges then the edges e_1, e_2 are said to be adjecent the vertices v_1, v_2 and v_3 are adjecent being joined by the edjes.



Parallel edge :-

Two (or) more than two edges join the same vertices then it is called parallel edges.

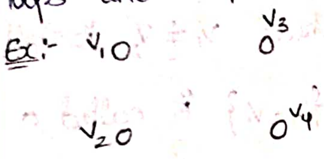


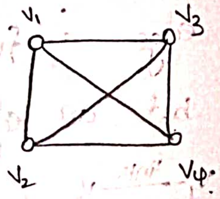
$e_1 = (v_1, v_2)$, $e_2 = (v_1, v_2)$, $e_3 = (v_1, v_2)$

The edges e_1, e_2 & e_3 are parallel edges.

Simple Graph:-
A graph G is called simple graph. If it has no loops and no parallel edges.

Ex:-



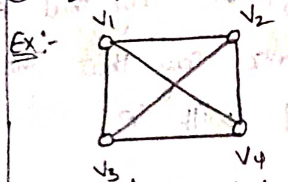


Definition:-

(*) γ (or) p (or) n stands for the no. of vertices

(*) q (or) ε_1 stands for the no. of Edges.

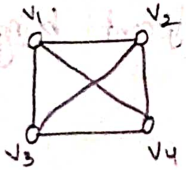
Ex:-

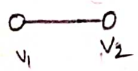


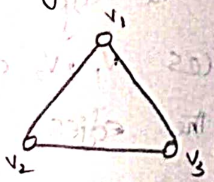
$\gamma = 4$, $\varepsilon_1 = 6$

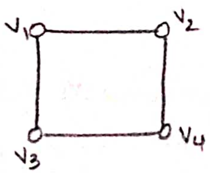
*** Complete graph:-**
A complete graph is a simple graph in which every vertex is adjacent to every other vertices. A Complete graph with n vertices is denoted by K_n.

Ex:-



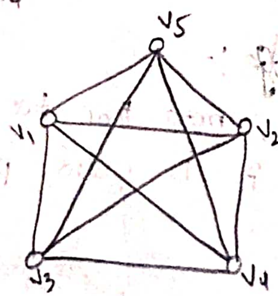






$\therefore$ It is not a complete Graph

5 vertices:-



Null graph or Empty graph:- A graph with no edges is called a null graph.

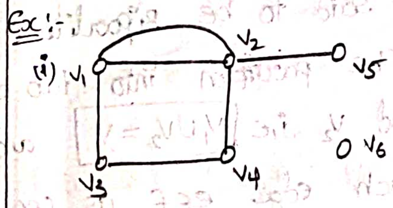
(*) A null graph of order 'n' is denoted by N_n.

Degree of a vertex:- The degree of a vertex 'v' in a graph G is denoted by $d_G(v)$ (or) $d(v)$ (or) $\deg(v)$ and is defined by the equal to the no. of edges incident with 'v'.

(*) each loop will be canted as two edges we denote The maximum of, minimum degree of vertices of a graph G is denoted by $\Delta(G)$ and $\delta(G) \rightarrow$ minimum

maximum

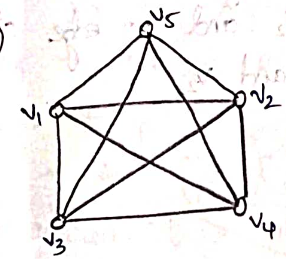
Ex:-
(i)



degree $(v_1) = 3$
$d(v_2) = 4$
$d(v_3) = 2$
$d(v_4) = 2$
$d(v_5) = 1$
$d(v_6) = 0$

maximum $= 4$
minimum $= 0$

(ii)

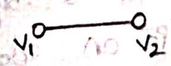


$d(v_1) = 4$
$d(v_2) = 4$
$d(v_3) = 4$
$d(v_4) = 4$
$d(v_5) = 4$

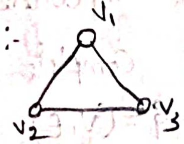
maximum $= 4$
minimum $= 0$.

(*) **Regular graph:-** A regular graph is a graph in which degree of every vertex is equal to 'k'.

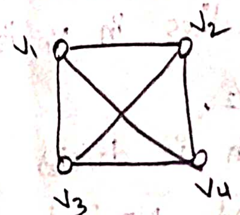
Ex:- 1-regular



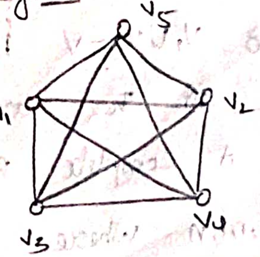
2-regular:-



3-regular



4-regular.

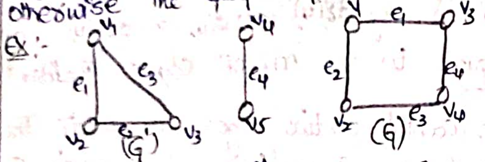


Note:- * A complete graph K_n is a regular graph.

* A complete Bi-parties graph K_n is a regular graph.

* Connected graph:- A graph G is said to be connected if for any two vertices (U,V) of G there is atleast one (U,V) path. otherwise the graph G is disconnected.

Ex:-



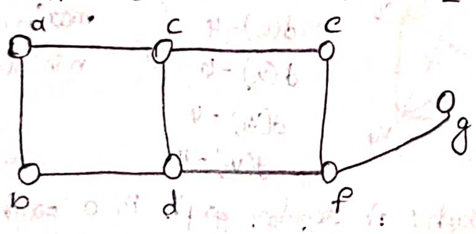
* G is connected graph where as G' is disconnected graph.

* Graph G is said to be connected if every pair of vertices in G are connected.

Bipartite Graph:-

A Graph $G=(V,E)$ is said to be Bipartite if it's vertex set V can be partition into two disjoint subsets. says V_1 and V_2 i.e $\boxed{V_1 \cup V_2 = V}$ and $\boxed{V_1 \cap V_2 = \emptyset}$ such that each edge $e \in E$ as one End in V_1 and another End in V_2 and no edge should have both the ends in V_1 and V_2

Eg:-



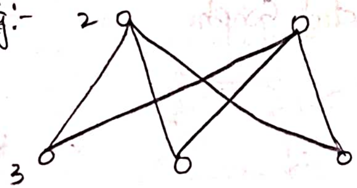
Complete Bipartite Graph:-

A Complete Bipartite graph and a Bipartite Graph $G=(V,E)$ in which their is an edge b/w each pair of vertices (V,V) where $U \in V_1$, $V \in V_2$ and $V_1 \cup V_2 = V$ i.e every vertices in V_1 is adjacent to all the vertices are V_2.

* A Complete Bipartite Graph is denoted by $K_{m,n}$ where $M = |V_1|$ and $N = |V_2|$

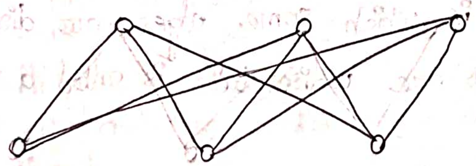
Note:- A Complete Bipartite Graph $K_{m,n}$ have $(m+n)$ no. of vertices and $m.n$ no. of edges.

Eg:-



$\boxed{m \times n = 6}$

Eg:- $K\{3,3\}$



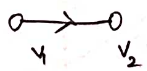
Directed Graph:- [one way connection]

[2M] In a graph G an Edge E which is associated with an ordered pair of $U \times V$ is called a Directed Edge of G

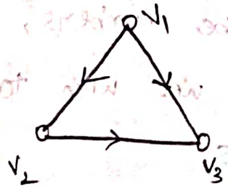
* An Edge E which is associated with an unordered pair of vertices is called an undirected edge.

Definition:-

A Graph G in which every edge is directed is called di-graph (or) directed graph. The directed edges are represented by "arrows" which shows The direction.



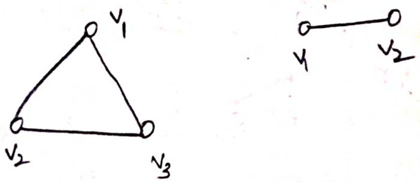
Eg:-



Un-directed definition:-

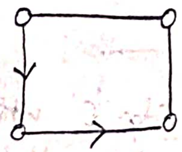
A Graph G in which every edge is undirected is called an un-directed Graph.

Eg:-



<u>Mixed Graph</u>:-

A Graph G in which some edges are directed and some edges are undirected is called a mixed Graph.



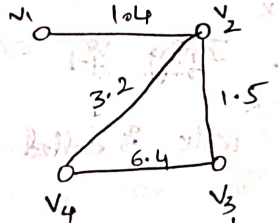
Weighted Graph:-

A weighted Graph is a four-tuple denoted by "Gw" (G_w) and is defined by v, E, f, g and is defined by $G_w = (v, E, f, g)$ where v is a non Empty set of vertices, E is a set of edges, f is a function, whose domain is v and g is a function, whose domain is E. The function 'f' is an assign of weights to the vertices and the function g is assign of it's to the Edges.

<u>Note</u>:- The weights can be numbers, symbols, (or) whatever quantities that we wish to assign to be vertices and edges.

[2M] * A Graph in which weights are assigned to every edge is called a "weighted Graph."

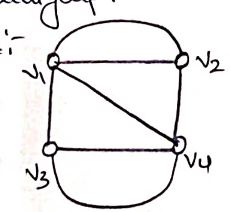
Eg:-

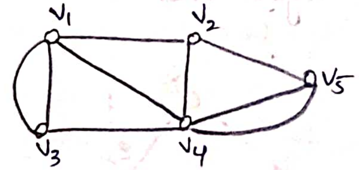


In modeling an abstract graph there are many occasions in which we wish to attach additional information to their vertices and edges of The Graph.

2a) Multi graph:- A Graph G is said to be multiple graph if it contains parallel edges or self loops is called multigraph.

Ex:-



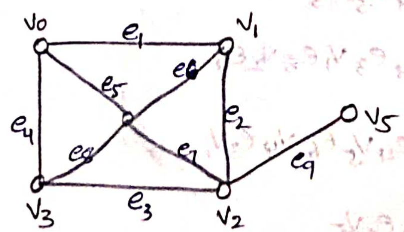


<u>Cycle</u>:- A cycle is a path in which the origin and The terminals are same.

Ex:- $v_0\ e_2\ v_2\ e_3\ v_1\ e_1\ v_4$

<u>Definition</u>:- Two paths in a graph are said to be disjoint if they have no common edges but they have common vertices.

<u>Ex</u>:-



path 1: $v_0\ e_5\ v_4\ e_1\ v_3$

path 2: $v_0\ e_4\ v_3\ e_3\ v_2\ e_2\ v_1\ e_6\ v_4$

∴ The two paths are joint.

Walk:-

Let G be a Graph a walk W in G is a sequence

$W = v_0 e_1 v_1 e_2 v_2 e_3 \ldots v_k e_k$

where e_i is the edge Joining v_{i-1} and v_i

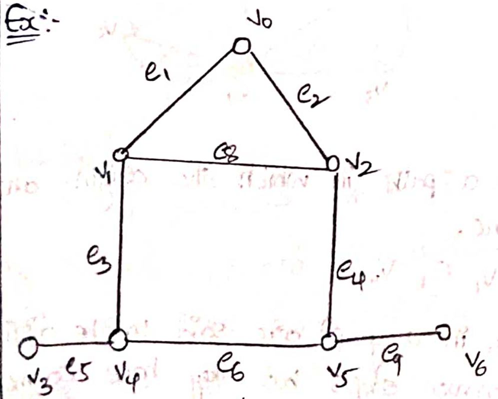
$i = 1, 2, \ldots k$.

The no. of edges in the walk is called the length of the walk.

Trail:- A trail is a walk in which the edge are all distinct. edges only once not repeated

Path:- A path is a walk in which all the vertices and edges are distinct. no repeated vertices & edges.

Ex:-



Walk :- $v_0 e_1 v_1 e_3 v_4 e_5 v_3$

$v_0 e_1 v_1 e_3 v_4 e_3 v_1 e_8 v_2 e_4$

Trail :- $v_0 e_1 v_1 e_8 v_2 e_4 v_5 e_6 v_4 e_3 v_1$

Path :- $v_0 e_1 v_1 e_3 v_4 e_6 v_5$

In a walk $v_0 e_1 v_1 e_2 v_2 e_3 \ldots e_k v_k$ $\hat{v_0}$ is called a origin.

$\therefore v_k$ is called the Terminals of the walk.

Closed walk:-

A closed walk is a walk in which the origin and terminals are same.

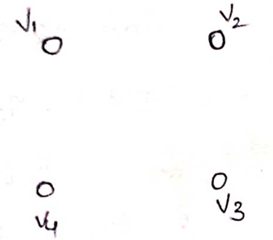
Cycle:-

A cycle is a closed walk in which all the edges and vertex are distinct.

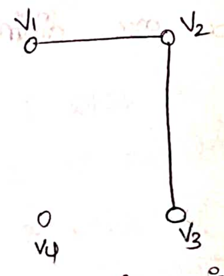
Isolated vertex:-

In graph G a vertex which is not adjecent to any other vertex is called an Isolated vertex.

* A graph containing only isolated vertex is an null Graph.



null graph



$\therefore$ v_4 is a isolated vertex

Representation of a Graph:-

A diagramatic representation of graph as limited usefulness further more such a representation is only possible when the no. of vertices and the edges is resonble small.

* They are many ways to represent graphs in this section we discuss the method of representing graph using matrices. The matrix is the most suitable way for representing graph in computer memory.
* matrix can be entirely manipulated in many application of graph theory.
* matrix can be considered as the natural way of expressing the problem.
* Here we discuss two important matrix representation of Graph.

They are

① Incidence matrix

② Adjacency matrix.

Incidence matrix:-

The incidence matrix of a graph G with "n" no. of vertices and "M" no. of edges and no selfloop in n by m i,e n×m matrix.

$M(G) = [m_{ij}]$

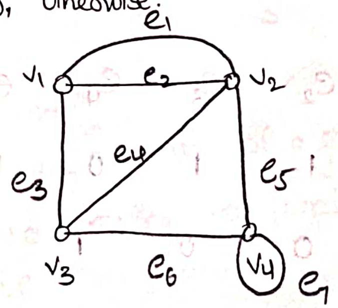
* whose ~~these~~ rows corresponds to it's vertex and column corresponds to it's edges.

i,e m_{ij} is the no. of lines that is v_i and e_j are incident.

Problems:-

The elements of the matrix will have values according to the following rules.

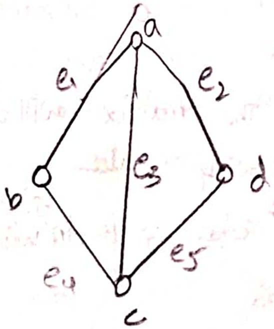
$$m_{ij} = \begin{cases} 1, & \text{if the } j^{th} \text{ edge } e_j \text{ is incident with } i^{th} \text{ vertex } v_i \\ 0, & \text{otherwise} \end{cases}$$



(m_{ij})	e_1	e_2	e_3	e_4	e_5	e_6	e_7
v_1	1	1	1	0	0	0	0
v_2	1	1	0	1	1	0	0
v_3	0	0	1	1	0	1	0
v_4	0	0	0	0	1	1	1

② The elements of the matrix will have values according to the following rules.

$$m_{ij} = \begin{cases} 1, & \text{if} \\ 0, & \text{otherwise} \end{cases}$$

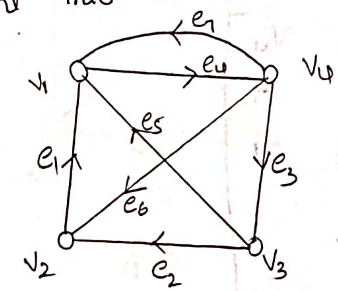


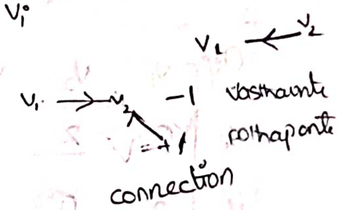
(M_{ij})	e_1	e_2	e_3	e_4	e_5
a	1	1	1	0	0
b	1	0	0	1	0
c	0	0	1	1	1
d	0	1	0	0	1

③ The incidence matrix of a digraph G is defined as $M(G) = [m_{ij}]$ such that

$$m_{ij} = \begin{cases} 1, & \text{if the } j^{th} \text{ edge } e_j \text{ is incident out of the } i^{th} \text{ vertex } v_i \\ 0, & \text{if the } j^{th} \text{ edge } e_j \text{ is not incident on the } i^{th} \text{ vertex } v_i \\ -1, & \text{if the } j^{th} \text{ edge } e_j \text{ is incident into the } i^{th} \text{ vertex } v_i \end{cases}$$

with g: incident into the





$$\begin{array}{c|ccccccc} & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \\ \hline v_1 & -1 & 0 & 0 & 1 & -1 & 0 & -1 \\ v_2 & 1 & -1 & 0 & 0 & 0 & -1 & 0 \\ v_3 & 0 & 1 & -1 & 0 & 1 & 0 & 0 \\ v_4 & 0 & 0 & 1 & -1 & 0 & 1 & 1 \end{array}$$

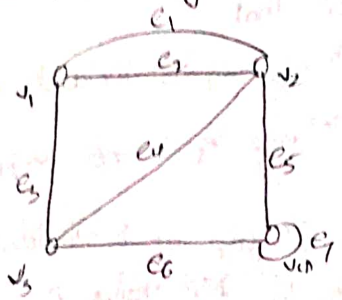
<u>Adjacency matrix</u>:-

A adjacency matrix of a graph G with n no. of vertices and having no parallel edges is an n by m i.e $n \times n$ matrix

Problem: $A(G) = [a_{ij}]$ whose elements are defined as follows.

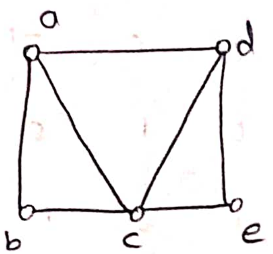
$$a_{ij} = \begin{cases} 1, & \text{if there is an edge b/w } i^{th} \text{ and } j^{th} \text{ vertices.} \\ 0, & \text{otherwise.} \end{cases}$$

a_{ij} = The no. of edges joining v_i and v



$$\begin{array}{c|cccc} a_{ij} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 2 & 1 & 0 \\ v_2 & 2 & 0 & 1 & 1 \\ v_3 & 1 & 1 & 0 & 1 \\ v_4 & 0 & 1 & 1 & 1 \end{array}$$

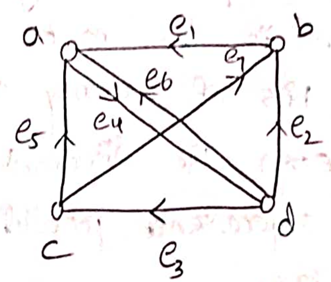
(ii)



$$\begin{array}{c|ccccc} a_{ij} & a & b & c & d & e \\ \hline a & 0 & 1 & 1 & 1 & 0 \\ b & 1 & 0 & 1 & 0 & 0 \\ c & 1 & 1 & 0 & 1 & 1 \\ d & 1 & 0 & 1 & 0 & 1 \\ e & 0 & 0 & 1 & 1 & 0 \end{array}$$

The adjacency matrix of a digraph G is defined as $A(G) = [a_{ij}]$ such that

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge directed from } i^{th} \text{ vertex to } j^{th} \text{ vertex.} \\ 0, & \text{otherwise} \end{cases}$$



$$\begin{array}{c|cccc} a_{ij} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 0 & 0 \\ c & 1 & 1 & 0 & 0 \\ d & 1 & 1 & 1 & 0 \end{array}$$

Properties :- (Adjacency matrix)

* Entries along the principale diagonals matrix $A(G)$ are All 0's if the graph has no loops.
* If there is no loops and no parallel edges then The degree of a vertex is equal to the no. of 1's in the correspondance rows (or) columns of $A(G)$.
* A graph G is disconnected and is in two components G and H if the adjacency matrix $A(G)$ can be partician as follows. $A(G) = \begin{bmatrix} A(g) & 0 \\ 0 & A(h) \end{bmatrix}$

Where $A(g)$ and $A(h)$ are adjacency matrix of Components g and h.

* The advantages of adjacency matrix is that it can represent self loops. As there are n^2 Entries in $A(G)$ it requireds only n^2 bits of memory where n is the no. of vertices. which is less than $n \times e$ bits of memory to represents the graph when $e > n$ The disadvantages is that it can not be represent parallel edges.

digraphs and Relations:-

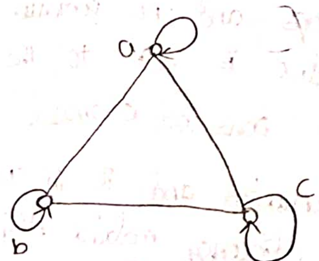
* Reflexive relation:-

for some relations R if every element is in relation R to itself Then R is reflexive.

* A digraph of a Reflexive relation will have a selfloop at each of it's vertex such a digraph is called a reflexive digraph.

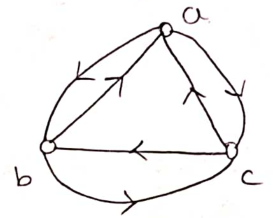
* If a digraph doesn't contain any self loop then it is called an irreflexive digraph.

Eg:-



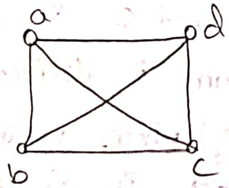
Symmetric Relation:-

A digraph representing a symmetric relation on it's vertex set is called a symmetric digraph. i,e a digraph is called a symmetric digraph. if for every directed edge from vertex a to b there is a directed edge from vertex b to a



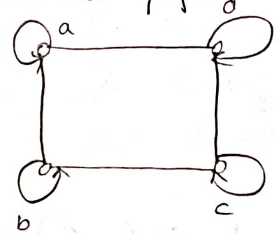
Transitive Relation:-

A Graph representing a Transitive relation on it's vertex set is called a Transitive graph.



Equivalence Relation:-

A graph representing an equivalence relation is call as an Equivalence graph.



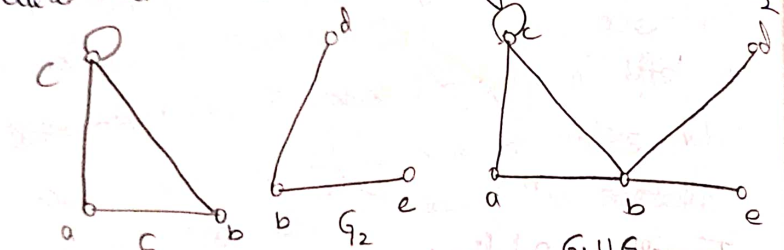
operations on Relation:-Graph:-

Since graphs are defined in terms of vertices and edges so the set of operations can be applied to Graph.

Union:-

let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are Two graphs.

The union of G_1 and G_2 is also a graph. denoted by $G_3 = G_1 \cup G_2$ and is defined by $G_3 = G_1 \cup G_2 = (V_3, E_3)$ whose vertex set is $V_3 = V_1 \cup V_2$ and Edge set is $E_3 = E_1 \cup E_2$

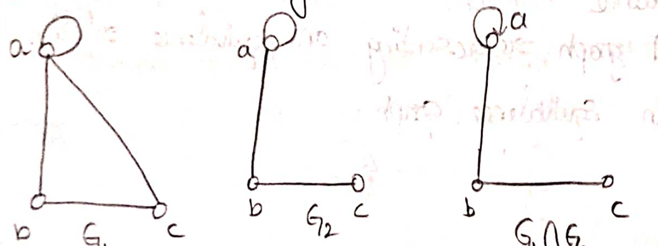


G_1 G_2 $G_1 \cup G_2$

Intersection:-

let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two Graphs. The intersection of G_1 and G_2 is also a Graph is denoted by $G_3 = G_1 \cap G_2$ and is defined by

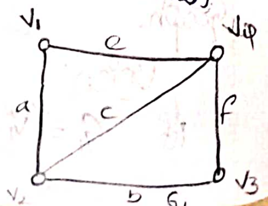
$G_3 = G_1 \cap G_2 = (V_3, E_3)$ whose vertex set is $V_3 = V_1 \cap V_2$ and Edge set is $E_3 = E_1 \cap E_2$.



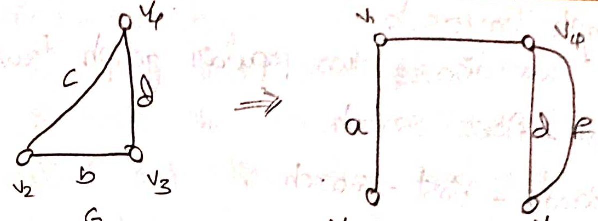
G_1 G_2 $G_1 \cap G_2$

Ring Sum:-

let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs. The ring sum of G_1 and G_2 are denoted by G_3 and is defined by $G_3 = G_1 \oplus G_2 = (V_3, E_3)$ where $V_3 = V_1 \cup V_2$, E_3 is the set of edges that are either in G_1 or G_2 but not in both.



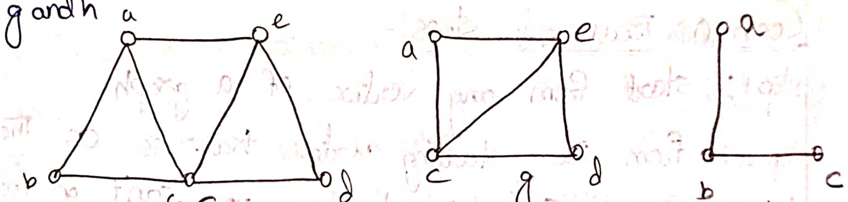
G_1



G_2 $\Rightarrow$ $G_1 \oplus G_2$

decomposition:-

A Graph G is said to have been decomposed into two subgraph g and h if $g \cup h = G$ and $g \cap h = \emptyset$ a nullgraph i,e each edge of G occurs either in g are in h but not in both. But some vertices may occurs both g and h

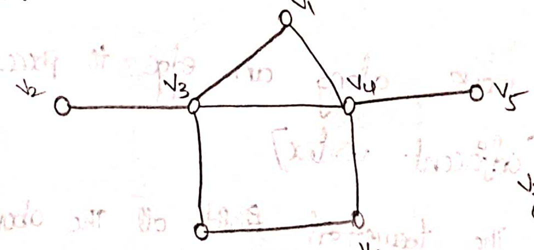


G g h

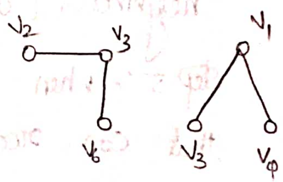
Subgraph:-

If $G = (V(G), E(G))$ and $H = (V(H), E(H))$ are two graphs Then it is said to be a subgraph of G iff.

* $V(H)$ is a subset of $V(G)$ (i,e $V(H) \subseteq V(G)$)
* $E(H)$ is a subset of $E(G)$ (i,e $E(H) \subseteq E(G)$)



G



subgraphs.

Graph Traversals.

We discuss two popular graph traversals techniques

1. Depth – first search
2. Breadth – first – search

Traversing a Graph:-

Traversing a graph means visiting all the verticies of the graph exactly once the standard graph traversal Algorithms are 1. Depth – first search
2. Breadth – first search

Common Traversal steps:-

step 1:- start from any vertex of a graph

step 2:- from the starting vertex traverse as the deep as you can go. whenever you can't go further. then back one vertex and do the same traver-sal from this vertex until we can't traverse further and so on.

step 3:-
process the information contained in that vertex.

step 4:- Then move along an edge to process a neighbour [adjacent vertex]

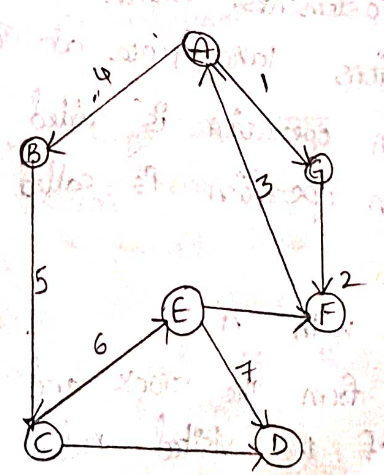
step 5:- when the traversal finish all the above vertices that can reached from the start vertex are processed.

Depth – first – search:- [10M]

The key point that lies behind the DFS traversal is visit the vertex v. first then visit all the vertices that lies along the a path which beings begins at v.

* first visit the vertex v.
* visit the vertex which is immediate adjacent to v [say V_x]
* if V_x as any adjacent vertex [say V_y]
* visit V_y and continue till
* There is a deadend i.e a vertex which does'nt have immediate adjacent vertex (or) immediate adjacent vertex already been visited.
* DFS is also called back tracking. since the algorithm writtens to vertices previously visited to add path.

Eg:- Consider the graph given in the following figure



① we start with vertex A, Then we visit vertex G as it is adjacent to A.

* once can be visited at b instead of G as path of traversal is not unique

② from G we visit to vertex F.

③ Then we back track to A, as there is no path from F, to any vertex otherthan A.

④ from A we visit a next adjacent vertex B Then from B to C Then C to E and finally from E to D

⑤ As there is no path from D to any other vertex which is not visited The algorithm stop at this point.

In order to keep track of The adjacent vertices we use stack of The datastructures

* A stack is an ordered collection of homogeneous dataelements. where The insertion and deletion operations take place at one end only

* The insertion operation is called "push"
* The deletion operation is called "pop".

Traversal steps:-

step 1:- start with 'A'.

step 2:- pop A from the stack and insert in the list of the visited nodes.

step 3:- push A adjacent vertices B and G on The stack.

step 4:- pop G from The stack and The insert in The list of The visited nodes.

step 5:- push G's adjacent vertices F into The stack.

step 6:- pop F from The stack & insert in The list of The visited nodes.

step 7:- since F adjacenty vertex, A is already visited. and The stack as The vertex B. a new search start from this vertex B.

step 8:- pop B from The stack and insert in The list of The visited nodes.

step 9: Now push is B's adjacent vertices C into The stack

step 10:- pop C from The stack and insert in The list of visited nodes.

step 11:- push C's adjacent vertices D and E into The stack.

step 12:- pop E from The stack and insert in The list of visited nodes.

step 13:- push E's adjacent vertices F and D into The stack.

step 14:- pop D from The stack and insert in the list of visited nodes.

step 15:- since D doesn't have any neighbours (or) vertices which have not yet been visited. so pop D out of the stack in ordered to make The stack is empty.

step 16:- get The list of visited nodes are

AGFBCED

1st visited:-

A

A G

A G F

A G F B

A G F B C

A G F B C E

A G F B C E D

Breadth first search:-

The general idea behind the Breadth first search begining at starting node A as follows first we have to examine the starting node [says A] Then we examine all the neighbour's of A x Then we have to examine all the neighbours of a node of A and so on.

* we have to keep track of the neighbour's of a node and care must be taken that no node is visited [OR] processed more than once.
* This can be done by using a datastructure queue to keep track of the node visited.

steps:-

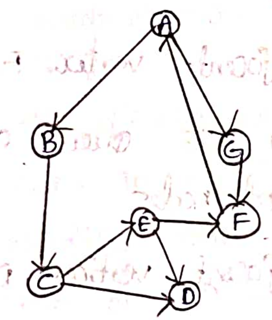
1:- First choose a starting vertex.

2:- find all the vertices which are connected to the starting vertex.

step 3:- Then choose one of the connected vertex and find all the vertices that are connected to that vertex

step 4:- Continue this procedure untill all the vertices are visited.

Eg:- consider the graph given in figure.



* we start with vertex A, Then we visit the vertices in the next level i.e vertices B and G.
* we visit B's adjacent vertex 'C' and G's adjacent vertex F.
* we visit D and E the adjacent vertex of C

In order to keep track of the vertices at a level, we use queue as the datastructure.

Definition of queue:-

A queue is an ordered collection of homogeneous data elements, but in construct to the stack, the insertion and deletion operations take place at two extreme ends.

Traversal steps:-

step 1:- Start with vertex A.

2. visit A and insert in the list of visited nodes

3. Insert A's adjacent vertex B and G into the queue

4. Remove B from queue and insert in the list of visited nodes.

5. Insert B's adjacent vertex C into the Queue

6. remove G from the queue and insert in the list of visited nodes.

7. Insert G's adjacent vertex F into the queue.

8. remove C from the queue and insert in the list of visited nodes

9. Insert C's adjacent vertices D & E into the queue.

10. remove F from queue & insert in the list of visited nodes.

11. Insert F's adjacent vertex 'A' is already since visited. so remove D from the queue and insert in the list of visited nodes.

12. As the D does not have any neighbours so remove E from the queue and insert in the list of visited nodes.

13. stop, since is neighbours F and D are already visited.

14. get the list of visited nodes
A B G C F D E

list visited:-

A
A B
A B G
A B G C
A B G C F
A B G C F D
A B G C F D E

The travelling sales person problem:-

Let $G=(V,E,W)$ be a complete graph of n vertices where W is a function from E, to the set of positive real numbers such that for any three vertices i, j, k in V

$$w(i,j)+w(j,k) \geq w(i,k)$$

we shall refer to $w(i,j)$ as the length of edge $\{i,j\}$ the travelling sales person problem ask for a Hamilton circuit of minimum length.

Eg:- Consider a Graph G as a map of n cities where $w(i,j)$ is the distance b/w cities i and j. A salesperson wants have a vacation of the n cities which start and end at the same city and include visiting each of the remaining of (n-1) cities.

* A travelling sales person problem turns out to be different one in that we know of no efficient procedure for solving the problem.

* We present a procedure known as the nearest neighbour method which gives the reasonable good result for the travelling sales person problem

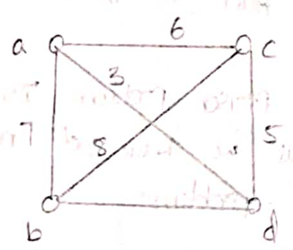
Algorithm:-

1. Start with an arbitrarily chosen vertex and find the vertex that is closest to the starting vertex to form an initial path of one edge, we shall argument this path in a vertex-by-vertex manner as given in step 2.

2. Let x denotes the latest vertex that was added to the path. Among all vertices that are not in the path, pick the one that is closest to x, and added to the path of edge connecting x and this vertex; Repeat this step until all vertices in G are included in the path.

3. From a circuit by adding the edge connecting the starting vertex and the last vertex added.

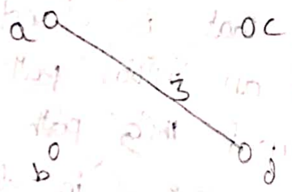
Eg:-
solve the travelling sales man problem for the weighted graph given figure using nearest neighbours method.



Solve:-

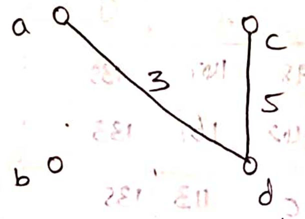
* let the starting vertex 'a' the nearest neighbour of vertex 'a' is vertex 'd' as d is reachable with lowest cost then all other adjacent vertices.

The corresponding Graph is shown in figure.



* Now the current vertex is d and the nearest neighbour of vertex d is vertex c as ~~'c'~~ is re as it as smallest distance from d.

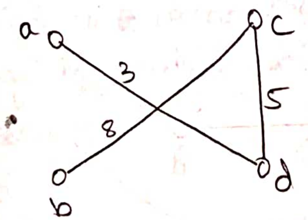
The corresponding graph is shown in figure.



* Now the current vertex is c and the nearest neighbour of vertex of 'c' is vertex a.

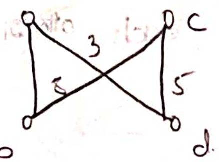
* But vertex 'd' is already included in the path so by considering the other available adjacent vertices are vertex 'c' we get the nearest vertex adjacent to c is the vertex b.

The corresponding Graph is shown in figure.



* Now the current vertex is b and they are no other vertices are available. which have to be included in the path to get the colsest path in the from vertex B to vertex A.

* we add the vertex A to the current path to get a closed path finally we obtained the closed path $\{a, d, c, b, a\}$ as show in below figure.

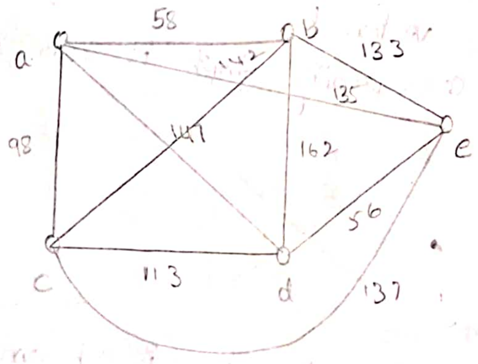


The total distance $= 23$

Solve the travelling salesman problem for the weighted graph whose correspoing ponding adjacent matrix is show below using nearest neighbour method.

	a	b	c	d	e
a	0	58	98	147	135
b	58	0	142	167	133
c	98	142	0	113	135
d	147	167	113	0	56
e	135	133	137	56	0

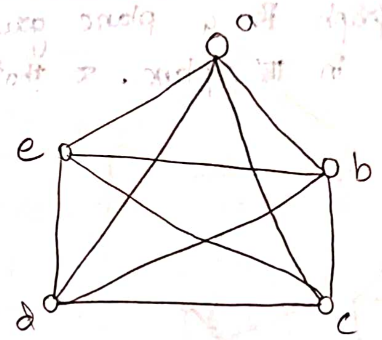
Sol:- The weighted graph corresponding to the adjacent matrix vertex is shown in figure



Planar Graph:-

When drawing a graph on a piece of paper offten find it to intersect at points otherthan at vertices of the graph.

* These points of intersection are called crossover. and the intersecting edges are said to crossover each other.



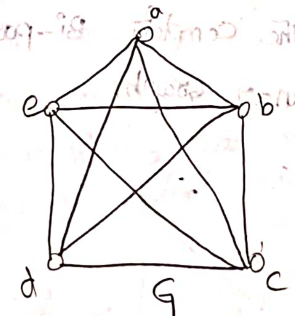
This graph have 3 crossover (b, e), (a, d) & (a, c) & (b, d) crossover (a, c)

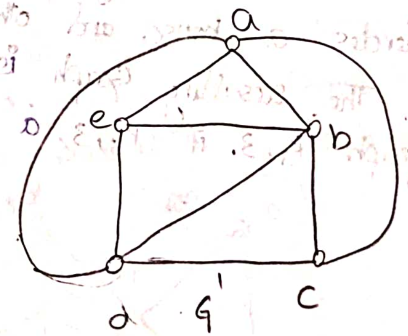
Definition:-

A Graph 'G' is said to be planar if it can be drawn on a plane without any cross overs other -wise G is said to be non-planar.

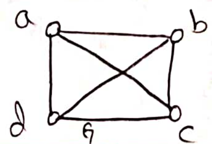
Note:-

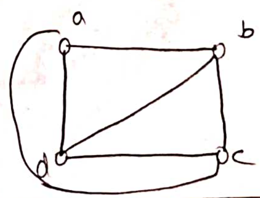
If a graph 'G' has been drawn with crossing edges, this doesn't mean that 'G' is non-planar. There may be another way to draw the graph without crossovers.



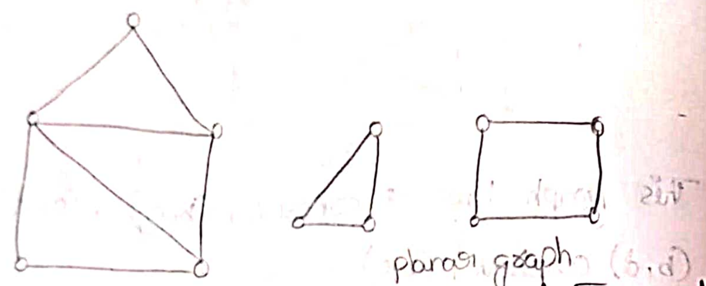


* A Graph 'G' fig(a) can be redrawn as G' in fig(a) without crossovers.





* A planar graph is a plane graph if it is already drawn in the plane. so that no two edge crossover

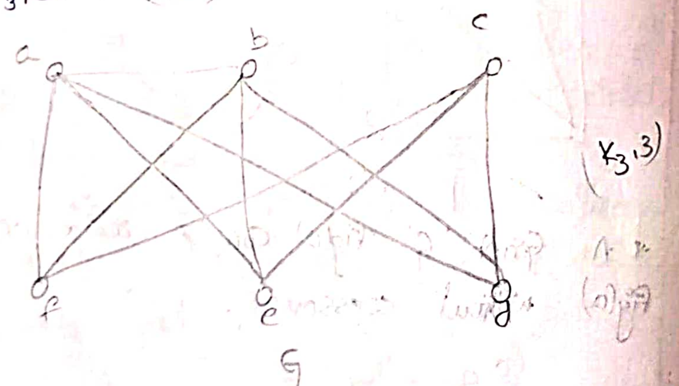


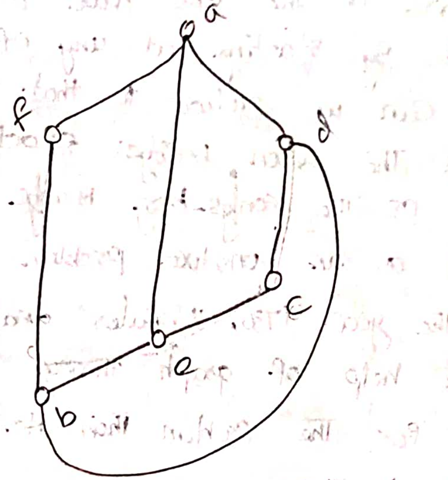
planar graph (b, d)

Eg:- Suppose we have 3 houses and three utility outlets (ele bill, gas & water) situated so that each utility outlets is connected to each house is it possible to connect each utility to each of the 3 houses without lines are main crossing.

sol:- This problem can be represented by a graph whose vertices correspond to the houses & utilities and where an edge join two vertices. if one vertex denotes a house and other vertex denotes utility.

The resulting graph is the complete Bi-partite graph $K_{3,3}$. if $(K_{3,3})$ a planar graph.





We draw the cycle of $K_{3,3}$

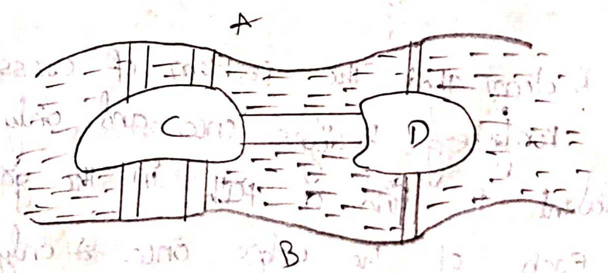
a-d-c-e-b-f-a

* In the above graph (c, f) is missing there is no place to draw this edge without crossover.

∴ $K_{3,3}$ is non-planar.

EULERIAN PATHS AND CIRCUITS

In the eighteenth century a city named Konigsberg in East prussia (Europe), there flowed a river named pregel river which divided the city into four parts. Two of these parts were the banks of the river and two were islands. These (parts) land areas were connected by each other through seven bridges as shown in fig.



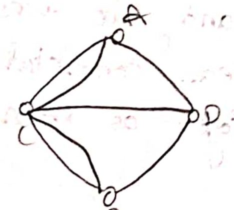
The people of the city have posted a problem that is, by starting at any of the four land areas, can we return to that area after crossing each of the seven bridges exactly once. This problem is known as the Konigs-berg bridge problem and remained as an unsolved problem for several years.

In the year 1736, L. Euler analyzed the problem with the help of graph theory and found the solution for the problem then he became father of the Graph Theory.

His solution is that it was not possible to cross each of the seven bridges on the river exactly once in a walking tour.

The map of the Konigsberg bridges shown in fig. can be represented as a graph.

Let us consider the four land areas as 4 vertices A, B, C & D and the seven bridges connecting them as seven edges. The graph is shown in fig.



It is clear that the problem of crossing each of the Konigsberg bridges once and only once is equivalent to find a path in the graph that travels each of the edges once & only once as it turns out instead of using a brute force trail & error approach which is probably what the people at Konigsberg bridge euler discovered a very simple criterion for determining whether there is a path in a graph that traverses each of the edges once & only once.

* We defined an Eulerian path in a graph to be a path traverses each edge in the graph once and only once. * We defined an Eulerian circuit in a graph to be a circuit that traverses each edge in the graph once & only once.

Definition:-

An "Euler walk" in a multigraph is a walk that includes each edge of the multigraph exactly once and intersect each vertex of the multigraph atleast once.

Euler circuit:-

An Euler circuit is an Euler walk whose end vertices are identical.

* A graph G is said to be Eulerian graph if it containes an Euler circuit

Minimum spanning tree:-

For a given graph one might want to determine a spanning tree of a graph. We consider the more general problems of determing a minimum spanning tree of a weighted graph. where real numbers are assigned to the edges as their weights.

* The weight of the spanning tree is defined

to be the sum of the weights of the branches of the tree.

Definition of minimum spanning tree:-

A minimum spanning tree is one with minimum weight. in a weighted graph G. a spanning tree T is a minimum spanning tree if there exist no other spanning tree at a distance of one from T whose weight is smaller than that of T.

KRUSKAL'S ALGORITHM:-

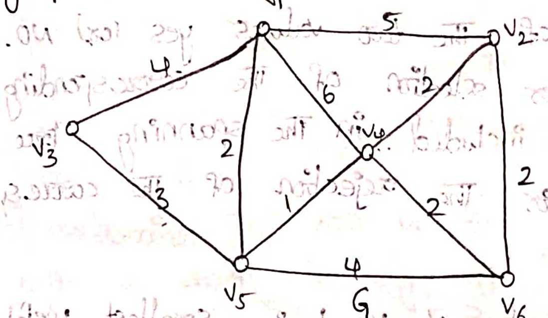
*① list all the edges of Graph 'G' in the increasing order of weights. [we arrange the edges in ascending order]

*② select the smallest edge ['having minimum weight] from the list and added to the spanning tree [which is initially empty] if the inclusion of the edge does't make a circuit. If the selected edge makes a circuit. Then remove it from the list.

*③ Repeat step 1 and step 2 untill the tree contains $n-1$ edges (where n is the no. of vertices) (or) the list is empty.

④ Now if the tree contains less than $(n-1)$ edges and the list happen to be empty then no spanning tree is possible else it gives the minimum spanning tree.

Ⓟ determine the minimum spanning tree of the weighted graph G in the figure.



using kruskal's Algorithm.

Sol:- we arrange all the edges of the graph G in the increasing order of their weights as shown in table.

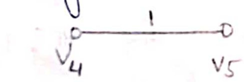
Edges in increasing order of their weights.	weights of edges	selection of the edge.
$V_4 - V_5$	1	yes
$V_1 - V_5$	2	yes
$V_2 - V_4$	2	yes
$V_4 - V_6$	2	yes NO
$V_2 - V_6$	2	NO, yes
$V_3 - V_5$	3	yes
$V_1 - V_3$	4	No
$V_5 - V_6$	4	No
$V_1 - V_2$	5	NO
$V_1 - V_4$	6	NO

By applying kruskal's algorithm to graph G the third column of the table is updated with either one of the two values yes (or) NO. yes stands for selection of the corresponding edges to be included in the spanning tree and NO stands for the rejection of the corresponding edge.

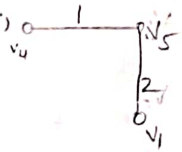
* consider the edge V_3, V_5 having smallest weight of 3. from the remaining edges available in table 3 added to the spanning tree by updating the 3rd column of the 6th row representing the edge V_3, V_5 to yes.

* The algorithm stops on the tree already contains (8−1) no. of edges where '8' is the no. of vertices i.e the no. of vertices in the graph is six and we had already found '5' edges in the tree connecting all vertices without creating a loop

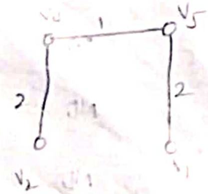
(i)



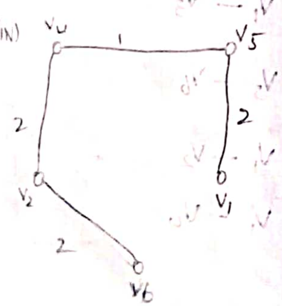
(ii)



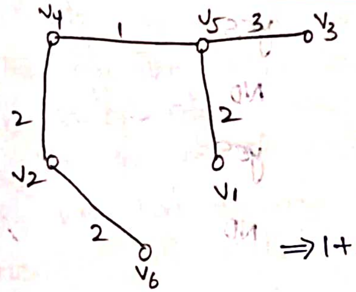
(iii)



(iv)

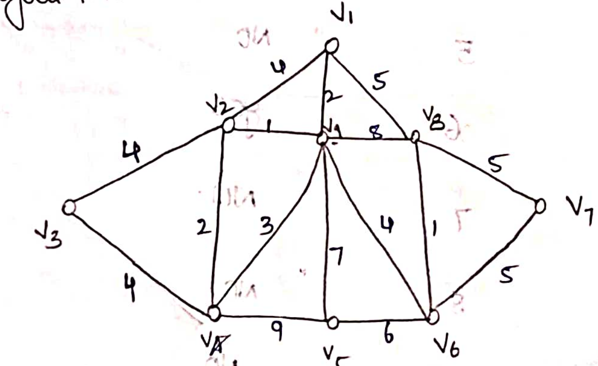


(v)



$\Rightarrow 1+2+2+2+3 = 10$

(P) determine the MST of the weighted connected graph as shown in fig by using kruskal's algorithm.



Sol: we arrange all the edges of the graph G in the increasing order of their weights as shown in table.

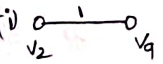
Edges in increasing order of their weights.	weights of edges	selection of the edge.
$V_2 - V_9$		yes
$V_6 - V_8$	1	yes
$V_1 - V_9$	2	yes
$V_2 - V_4$	2	yes
$V_4 - V_9$	3	NO

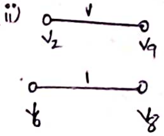
$V_2 - V_3$	4	yes
$V_3 - V_4$	4	NO
$V_6 - V_9$	4	yes
$V_1 - V_2$	4	NO
$V_1 - V_8$	5	NO
$V_7 - V_8$	5	yes
$V_6 - V_7$	5	NO
$V_5 - V_6$	6	yes
$V_5 - V_9$	7	NO
$V_8 - V_9$	8	NO
$V_5 - V_4$	9	NO

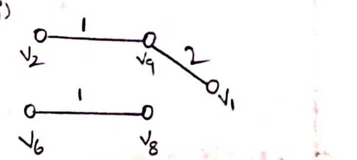
By applying kruskal's algorithm to graph G The third column of The table is updated with either one of The two values yes (or) NO. yes stands for selection of The corresponding edges to be included in The spanning tree and NO stands for The rejection of The corresponding edge.

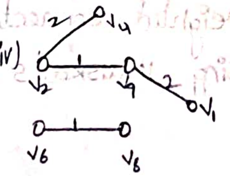
* Consider The edge V_5, V_6 having smallest weight of 6 from The remaining edges available in table added to The spanning tree by updating The 3rd column of The 13th row representing The edge $V_5 V_6$ to yes

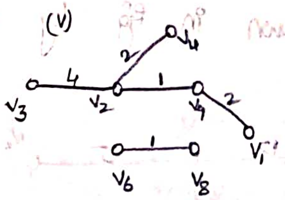
* The algorithm stops on The tree already contains (8-1) no. of edges where '8' is The no. of The vertices i,e The no. of vertices in The graph is nine and we had already found '8' edges in The tree connecting all vertices without creating a loop.

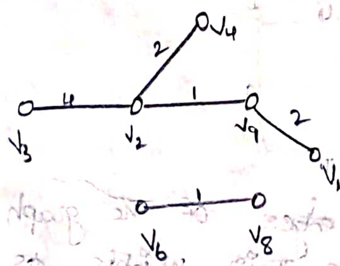
(i) 

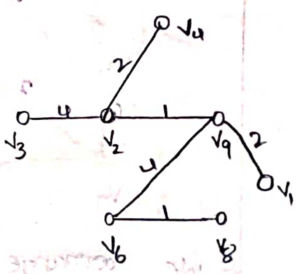
(ii) 

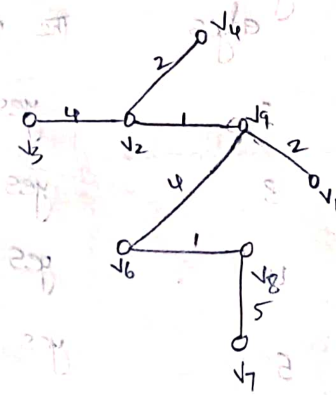
(iii) 

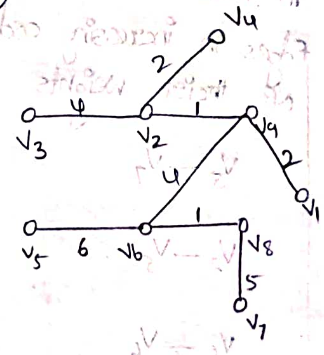
(iv) 

(v) 

(vi) 

(vii) 

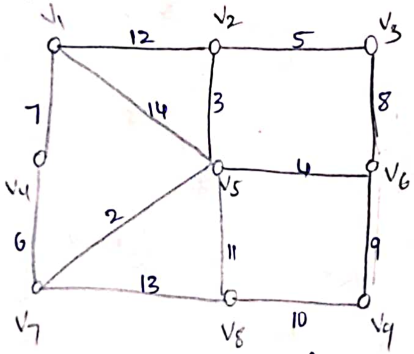
(viii) 

(ix) 

$1+1+2+2+4+6+4+5 = 25$

Empty

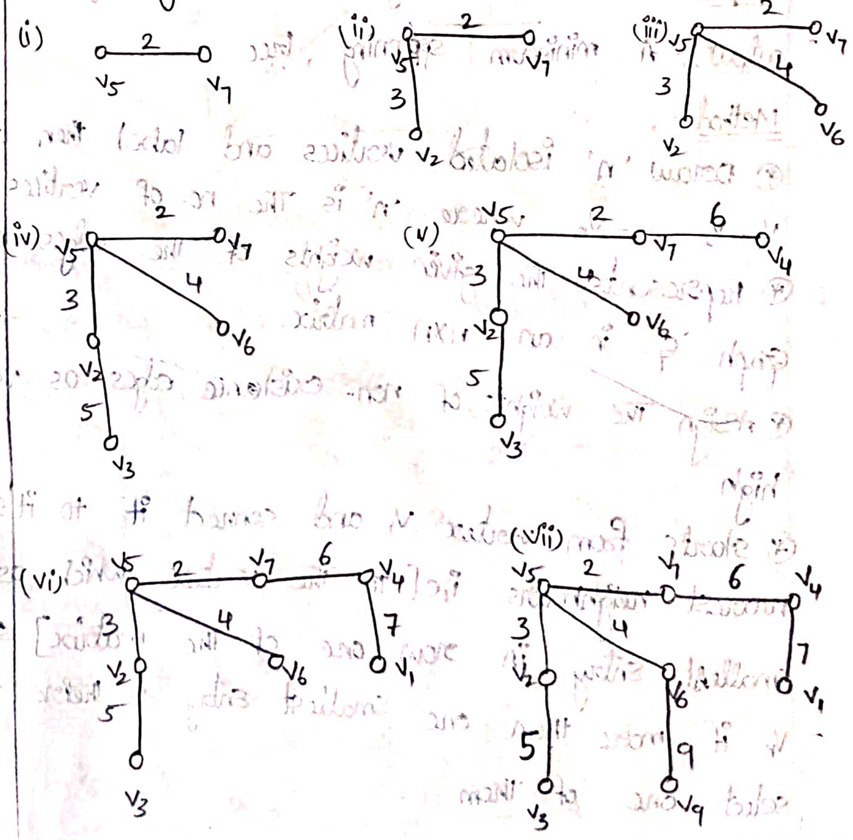
(P) determine The MST of The weighted connected graph as shown in fig by using kruskal's algorithm

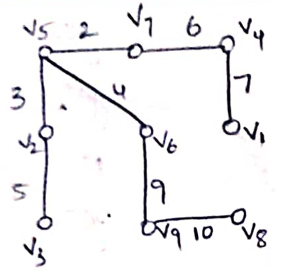


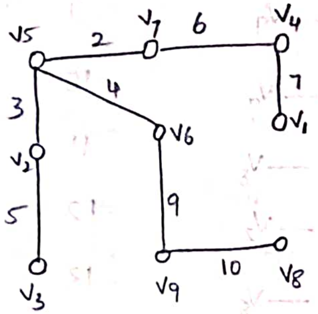
· We arrange all The edges of The graph G in The increasing order of Their weights as shown in table.

Edges in increasing order of their weights	weights of edges	selection of The edge.
$V_5 - V_7$	2	yes
$V_2 - V_5$	3	yes
$V_5 - V_6$	4	yes
$V_2 - V_3$	5	yes
$V_4 - V_7$	6	yes
$V_1 - V_4$	7	yes.
$V_3 - V_6$	8	NO.
$V_6 - V_9$	9	yes
$V_8 - V_9$	10	yes.
$V_5 - V_8$	11	NO.
$V_1 - V_2$	12	~~yes~~. NO
$V_7 - V_8$	13	NO.
$V_1 - V_5$	14	NO.

Theory same as before problem. solution.



(viii) 

(ix) 

$2+3+4+5+6+7+9+10$
$=46$

PRIM'S Algorithm:-

Input :- A simple weighted connected graph

output:- A minimum spanning tree

Method:-

⊛ Draw 'n' isolated vertices and label them as $v_1, v_2 - - - v_n$ where 'n' is the no. of vertices.

⊛ Represents the given weights of the edges of graph 'G' in an $n \times n$ matrix.

⊛ Assign the weights of non-existance edges as very high

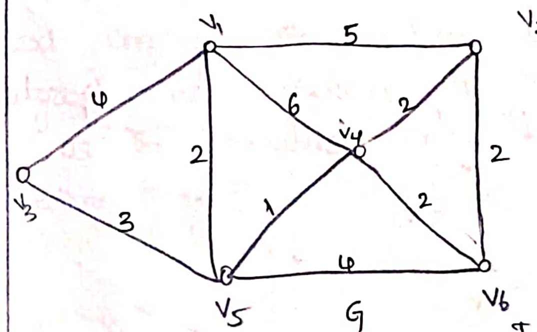
⊛ starts from vertex v_1 and connect it to it's nearest neighbours i,e[to the vertex which as the smallest entry in row one of the matrix] say v_k if more than one smallest entry is there then select one of them.

⑤ Consider v_1 and v_k as one subgraph and connect the subgraph to it's closest neighbour's (i,e to a vertex other than v_1 and v_k which has the smallest entry in row's 1 and v_k] suppose this new vertex be v_i

⑥ Consider the tree with vertices v_1, v_k and v_i as one subgraph and continue the process until all the 'n' vertices have been connected by $(n-1)$ edges

⑦ exit.

let us consider the weighted graph G in the figure.

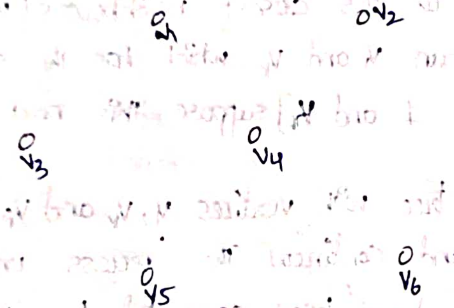


find the minimum spanning tree of the graph using prim's algorithm

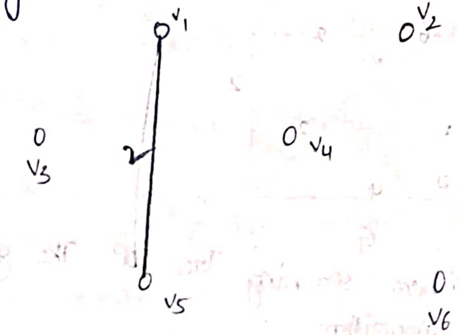
Sol:- we represents the given weighted graph as a 6x6 adjacency matrix.

$$\begin{array}{c|cccccc} & v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ \hline v_1 & - & 5 & 4 & 6 & 2 & - \\ v_2 & 5 & - & - & 2 & - & 2 \\ v_3 & 4 & - & - & - & 3 & - \\ v_4 & 6 & 2 & - & - & 1 & 2 \\ v_5 & 2 & - & 3 & 1 & - & 4 \\ v_6 & - & 2 & - & 2 & 4 & - \end{array}$$

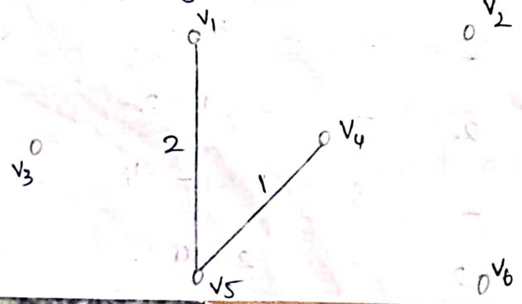
① Draw 6 isolated vertices as show in figure



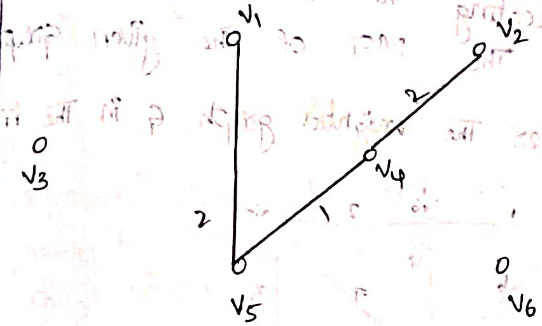
② Start from vertex v_1 and connect it to it's nearest neighbour v_5 as it as the smallest entry row 1.



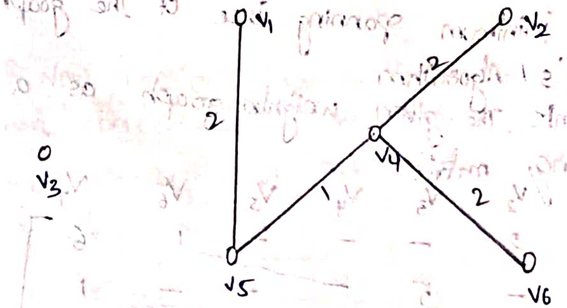
③ Now consider the vertices v_1 and v_5 as one subgraph and connect it to the vertex v_4 as it as the smallest entry in row's 1 & 5. otherthan vertices v_1 & v_5



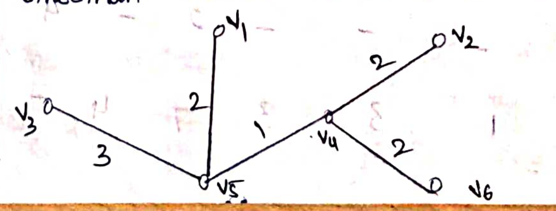
④ Then consider the vertices v_1, v_5, v_4 as one subgraph and connect it to the vertex v_2 as it as the smallest entry in row's 1, 4 & 5. otherthan vertices v_1, v_5, v_4



⑤ next consider the vertices v_1, v_5, v_4 & v_2 as one subgraph and connect it to the vertex v_6 as it as the smallest entry in row's 1, 5, 4, 2 otherthan vertices v_1, v_5, v_4, v_2.



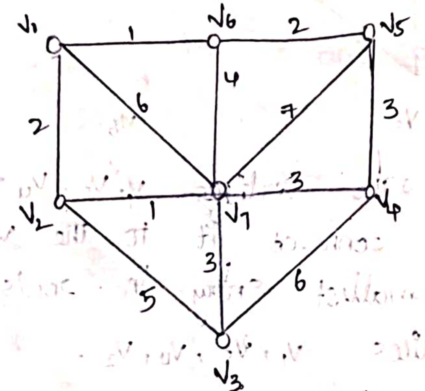
⑥ Now consider the vertices v_1, v_5, v_4, v_2 & v_6 as one subgraph and connect it to the vertex v_3 as it as the smallest entry in row's 1, 5, 4, 2, 6 otherthan vertices v_1, v_5, v_4, v_2, v_6



$\therefore$ Total cost $= 2+1+2+2+3$
$= 10$

(7) Now the algorithm exist as there are already 5 edges connecting the 6 vertices the above figure shows the MST of the given graph.

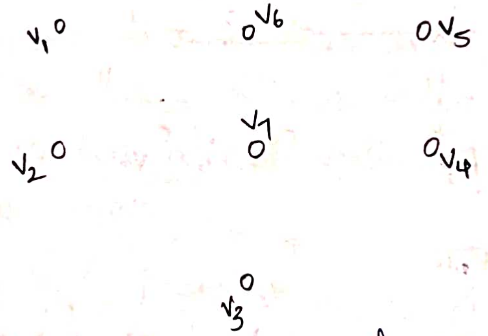
(8) Let us consider the weighted graph G in the figure



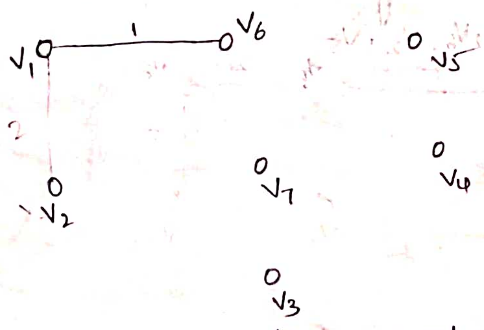
Find the minimum spanning tree of the graph using prim's Algorithm.
we represents the given weighted graph as a 7x7 adjacency matrix.

$$\begin{array}{c|ccccccc} & V_1 & V_2 & V_3 & V_4 & V_5 & V_6 & V_7 \\ \hline V_1 & - & 2 & - & - & - & 1 & 6 \\ V_2 & 2 & - & 5 & - & - & - & 1 \\ V_3 & - & 5 & - & 6 & - & - & 3 \\ V_4 & - & - & 6 & - & 3 & - & 3 \\ V_5 & - & - & - & 3 & - & 2 & 7 \\ V_6 & 1 & - & - & - & 2 & - & 4 \\ V_7 & 6 & 1 & 3 & 3 & 7 & 4 & - \end{array}$$

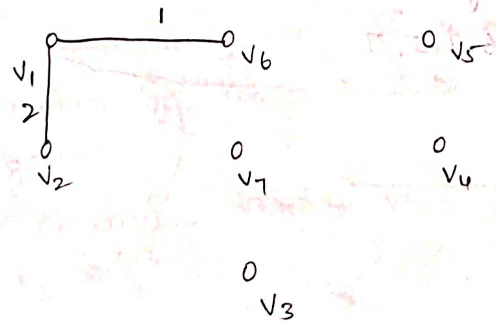
(1) Draw 7 isolated vertices as show in figure



(2) start from vertex V_1 and connect it to it's nearest neighbour V_6 as it as the smallest Entry row 1.

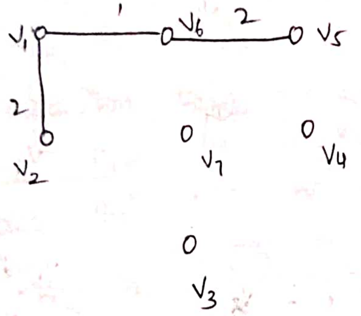


(3) Now consider the vertices V_1 & V_6 as one subgraph and connect it to the vertex V_2 as it as the smallest Entry in row's 1 & 6 other than vertices V_1 & V_6.

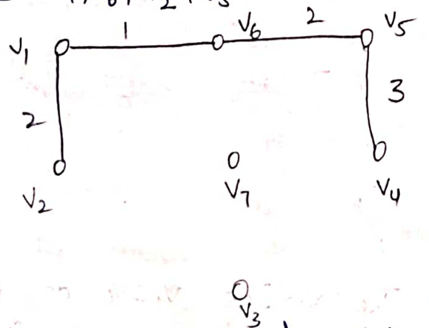


(4) Then consider the vertices V_1, V_6, & V_2 as one subgraph and connect it to the vertex V_5 as it as the smallest Entry in row's 1, 6, 2,

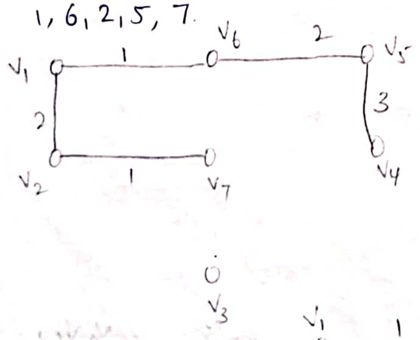
other than vertex v_1, v_6 & v_2



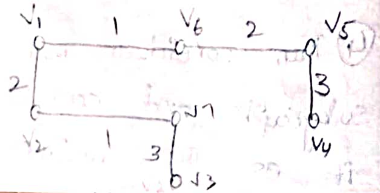
⑤ next consider the vertex v_1, v_6, v_2 & v_5 as one subgraph and connect it to the vertex. v_4 as it as the smallest entry in row's 1, 6, 2, 5 other than vertices v_1, v_6, v_2, v_5



⑥ next consider the vertex v_1, v_6, v_2, v_5 & v_4 as one sub-graph and connect it to the vertex v_7 as it as the smallest entry in row's 1, 6, 2, 5, 4, other that vertices 1, 6, 2, 5, 7.



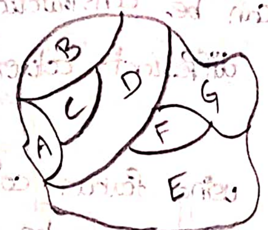
⑦ Same as above theory

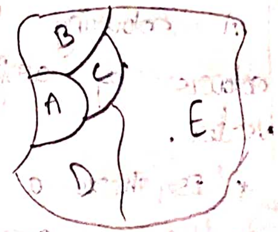


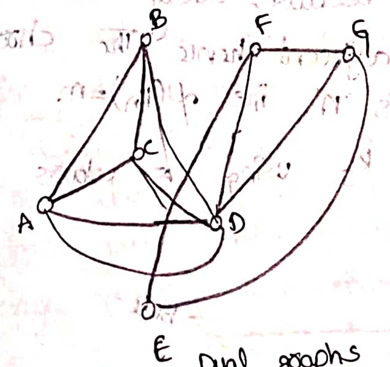
$2+2+2+6=12$

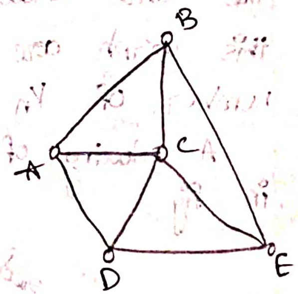
Graph Colouring:-

A colouring of a simple graph is the assignment of the colour to each vertex of the graph so that no two adjacent vertices are assigned the same colour.









Dual graphs of the above maps.

Definition:-

The chromatic number of a graph in the least number of colors needed for a colouring of a graph.

The chromatic number of a graph G is denote by $\chi(G)$

The four colour Theorem:-

The chromatic number of a planar graph is no greater than four

Ex::
What is The chromatic number of K_n ?

Sol:- A colouring of K_n can be constructed using 'n' colours by assigning a different colour to each vertex.

* is there a coloring using fewer colours
* The Answer is no two vertices can be assigned The same colour because Every two vertices of this graph are adjacent hence The chromatic number of K_n is 'n'. i.e $\chi(K_n)=n$.

sai, problem completed

A coloring of K_5 using 5 colours is shown in fig