ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE

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Differential equations have wide applications in various engineering and science disciplines. In general, modeling of the variation of a physical quantity, such as temperature, pressure, displacement, velocity, stress, strain, current, voltage, or concentration of a pollutant, with the change of time or location, or both would result in differential equations. Similarly, studying the variation of some physical quantities on other physical quantities would also lead to differential equations. In fact, many engineering subjects, such as mechanical vibration or structural dynamics, heat transfer, or theory of electric circuits, are founded on the theory of differential equations. It is practically important for engineers to be able to model physical problems using mathematical equations, and then solve these equations so that the behavior of the systems concerned can be studied.

1.1 DIFFERENTIAL EQUATION:

A differential equation is an equation which contains derivatives of dependent variables with respect to independent variables, either ordinary derivatives or partial derivatives.

Ex: 1. $\frac{dy}{dx} + 2y = cos\ x$ 2. $\frac{d^2y}{dx^2} - 2\ \frac{dy}{dx} + 3y = 0$ 3. $\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \boldsymbol{2y}$

1.2 TYPES OF DIFFERENTIAL EQUATIONS:

ORDINARY DIFFERENTIAL EQUATION:

A differential equation is called an ordinary differential equation, abbreviated by ode, if it has ordinary derivatives in it.

Ex: $\frac{d^2y}{dx^2} - 2\ \frac{dy}{dx} + 3y = 0$

PARTIAL DIFFERENTIAL EQUATION:

A differential equation is called a partial differential equation, abbreviated by pde, if it has partial derivatives in it.

Ex: $\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = \boldsymbol{2y}$

1.3 ORDER OF DIFFERENTIAL EQUATION:

The order of the derivative of the highest order present in a differential equation is called the order of the differential equation

Ex: $\frac{d^{2y}}{dx^2} + y = 0$; Here order is 2

1.4 DEGREE OF DIFFERENTIAL EQUATION:

The degree of the derivative of the highest order present in a differential equation, when the equation is made free from radical signs and fractions, is called the degree of the differential equation.

$Ex: \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + y = 0;$ Here degree is 1

1.5 FAMILY OF CURVES

Sometimes a family of curves can be represented by a single equation. In such a case the equation contains an arbitrary constant c. By assigning different values for c, we get a family of curves. In this case c is called the parameter or arbitrary constant of the family.

Examples

(i) $y = mx$ represents the equation of a family of straight lines through the origin , where m is the parameter.
(ii) $x^2 + y^2 = a^2$ represents the equation of family of concentric circles having the origin as centre, where a is the parameter.
(iii) $y = mx + c$ represents the equation of a family of straight lines in a plane, where m and c are parameters

1.6 FORMATION OF DIFFERENTIAL EQUATIONS

Each family of curves has a differential equation.
To obtain the differential equation of the family of curves differentiate with respect to x , treating the parameter as a constant. If the derived equation is free from parameter then the derived equation is the differential equation of the family.

Note:

(i) The differential equation of a one parameter family is obtained by differentiating the equation of the family one time and by eliminating the parameters.
(ii) In general, the order of the differential equation to be formed is equal to the number of arbitrary constants present in the equation of the family of curves.

PROBLEMS

1. Find the differential equation corresponding to $y=Ax^2$ where A is arbitrary constant.

Sol: Given $y = Ax^2$ ---(1)

$y^1 = A(2x)$ ---(2)

From (1) & (2); $xy^1 = 2y,$ which is first order O.D.E

2. Find the differential equation corresponding to y=mx+f(m), m is arbitrary constants.

Sol: Given y = mx + f(m) ---(1)

$y^1 = m$ ---(2)

From (1) & (2); $y = y^1 x + f(y^1)$, which is first order O.D.E

3. Find the differential equation corresponding to $y = a\,x + b\,x^2$, where a and b are arbitrary constants

Sol: Given $y = a\,x + b\,x^2$ ---(1)

$y^1 = a + b\,(2x)$ ---(2)

$y^{11} = b(2)$ ---(3)

From (2) & (3), $b=y^{11}/2$ and $a = y^1 - y^{11}$ ---(4)

Substituting a and b in (1); $y = (y^1 - y^{11})x + x^2(y^{11}/2)$; which is the second order O.D.E

4. Find the differential equation corresponding to $y=Ae^x + Be^{2x}$ where A&B are arbitrary constants

Sol: Given $y=Ae^x + Be^{2x}$

$\Longrightarrow\ y^1 = Ae^x + B(2e^{2x}) = y + B(2e^{2x})$

& $y^{11} = y^1 + B(4e^{2x})$

$Then\ y^{11} = y^1 + y^1 - y \Longrightarrow y^{11} - 2y^1 + y = 0$, which is second order O.D.E

5. Find the differential equation corresponding to $y = a\,x^2 + b\,x + c$ where a,b and c are arbitray constants

Sol: Given $y = a\,x^2 + b\,x + c$ ---(1)

$y^1 = a(2x) + b$ ---(2)

$y^{11} = a(2)$ ---(3)

$y^{111} = 0$ ---(4), which is fourth order O.D.E

6.Find the differential equation corresponding to $y = ae^x + be^{2x} + ce^{3x}$ where a, b, c and arbitrary constants

Sol: $y = ae^x + be^{2x} + ce^{3x}$

$\Longrightarrow y^1 = y + be^{2x} + 2ce^{3x}$

& $y^1 - y = be^{2x} + 2ce^{3x}$

$\Longrightarrow y^{11} - y^1 = 2be^{2x} + 6ce^{3x} = 2\left(y^1 - y\right) + 2ce^{3x}$

$\Longrightarrow y^{11} - 3y^1 + 2y = 2ce^{3x}$

$\Longrightarrow y^{11} - 6y^{11} + 11y^1 - 6y = 0$ Which is third order O.D.E

7. Find the differential equation of a family of curves given by y = a cos (mx+b), a and b being arbitrary constants.

Sol:

$y = a\,\cos(mx + b)$ __________(1)

$\Rightarrow \dfrac{dy}{dx} = -m\,a\sin(\,mx + b$

$\Rightarrow \dfrac{d^2y}{dx^2} = -m^2 a\cos(mx + b) = -m^2\,y \quad [u\sin\,g\,(1)]$

$\therefore \dfrac{d^2y}{dx^2} + m^2\,y = 0\,is\ the\ required\ differential\ equation$

EXERCISE

(1) Form the differential equation by eliminating constants log(y/x)=cx
(2) Form the differential equation by eliminating A and B from $Ax^2 + By^2 = 1$
(3) Find the differential equation of the family of straight lines y=mx+(a/m), m is parameter
(4) Find the differential equation of the curve $xy = a e^x + b e^{-x}$
(5) Find the differential equation by eliminating a and b from $y = a e^{3x} + b e^{5x}$
(6) Form the differential equation from $y = c x + c - c^3$
(7) Form the differential equation from $y = k \sin^{-1} x$

ANSWERS

(1) $\log(y/x)=\frac{y}{x}\frac{dy}{dx} - 1$
(2) $x y_1^2 + xyy_2 - yy_1 = 0$
(3) $x(y^1)^2 - y y^1 + a = 0$
(4) $x y^{11} + 2 y^1 - xy = 0$
(5) $y^{11} - 8 y^1 + 15 y = 0$
(6) $y = x y^1 + y^1 - (y^1)^3$
(7) $(1-x^2)(\sin^{-1})^2(y^1)^2 = y^2$

1.7 FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS

A Differential Equation of order 1 and degree 1 is said to be First Order and First Degree Differential Equation

The general form of first order differential equation is $\frac{dy}{dx} = f(x, y)$

Solution: A solution of a differential equation is an explicit or implicit relation between the variables which satisfies the given differential equation and does not contain any derivatives.
Types of Solutions:

General Solution: If the solution of a differential equation contains as many arbitrary constants of integration as its order, then the solution is said to be the general solution of the differential equation.

Particular Solution: The solution obtained from the general solution by assigning particular values for the arbitrary constants, is said to be a particular solution of the differential equation.

1.8 SOLUTIONS OF FIRST ORDER AND FIRST DEGREE DIFFERENTIAL EQUATIONS:

The first order and first degree ODE has certain standard types for which solutions can be readily obtained by standard methods such as

(i) Variable-separable
(ii) Homogeneous differential equation
(iii) Non-homogenous differential equation reducible to homogenous differential equation
(iv) Linear first-order differential equation (Leibnitz's equation)
(v) Bernoulli's differential equation
(vi) Exact differential equation
(vii) Non-exact differential equations that can be made exact with the help of integrating factors

1.9 VARIABLES SEPARABLE

If it is possible to re-arrange the terms of the first order and first degree differential equation in two groups, each containing only one variable, is said to be variable separable differential equation.

METHOD OF SOLVING

Consider the O.D.E. $\frac{dy}{dx} = f(x, y)$ --- (1)

Step1: When variables are separated, the differential equation takes the form $\frac{dy}{dx} = \frac{f(x)}{g(y)}$

$\Rightarrow f(x)dx - g(y)dy = 0$, *Where* $f(x)$ is a function of x only and $g(y)$ is a function of y

Step2: Integrating, we get $\Rightarrow \int f(x)dx - \int g(y)dy = c$ which is the solution of (1)

PROBLEMS

1. Solve the differential equation x dy + y dx = 0

Sol: Given x dy + y dx = 0

$$\Rightarrow \frac{dy}{y} + \frac{dx}{x} = 0$$

$$\Rightarrow \int \frac{dy}{y} + \int \frac{dx}{x} = 0$$

$$\Rightarrow \log y + \log x = c$$

2. Solve the differential equation $(1 + y^2)dx = (1 + x^2)dy$

Sol: Given $(1 + y^2)dx = (1 + x^2)dy$

$$\Rightarrow \frac{dy}{1 + y^2} = \frac{dx}{1 + x^2}$$

Integrating on both sides $\Rightarrow \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2} + tan^{-1}c$

$$\Rightarrow tan^{-1}y = tan^{-1}x + tan^{-1}c$$

3. Solve the differential equation $\frac{dy}{dx} = e^{2x-y} + x^3e^{-y}$

Sol: Given $\frac{dy}{dx} = e^{2x-y} + x^3e^{-y}$

$$\Rightarrow \frac{dy}{dx} = (e^{2x} + x^3)e^{-y}$$

$$\Rightarrow e^y dy = (e^{2x} + x^3)dx$$

Integrating on both sides $\Rightarrow \int e^y dy = \int (e^{2x} + x^3)dx + c$

$$\Rightarrow e^y = \frac{1}{2}e^{2x} + \frac{x^4}{4} + c$$

4.Solve the differential equation x$\sqrt{1+y^2dx}+y\sqrt{1+x^2}dy = 0$

Sol: Given x$\sqrt{1+y^2dx}+y\sqrt{1+x^2}dy = 0$

$$\Rightarrow \frac{x}{\sqrt{1+x^2}}dx+\frac{y}{\sqrt{1+y^2}}dy = 0$$

Integrating on both sides $\Rightarrow \int \frac{x}{\sqrt{1+x^2}}dx+\int \frac{y}{\sqrt{1+y^2}}dy = 0$

$$\Rightarrow \sqrt{1+x^2}+\sqrt{1+y^2} = c$$

5. Solve $(1+e^x)ydy = (1+y)e^x dx$

Sol: Given $(1+e^x)ydy = (1+y)e^x dx$

$$\Rightarrow \frac{y}{1+y}dy = \frac{e^x}{1+e^x}dx$$

Integrating on both sides

$$\Rightarrow \int \frac{y}{1+y}dy = \int \frac{e^x}{1+e^x}dx + logc$$

$$\Rightarrow \int \left[1-\frac{1}{1+y}\right]dy = \int \frac{e^x}{1+e^x}dx + logc$$

$$\Rightarrow y-\log(1+y) = \log(1+e^x)+\log c$$

6. Solve the differential equation $\frac{dy}{dx}=e^{3x+y}$

Sol: Given $\frac{dy}{dx}=e^{3x+y}$

$$\Rightarrow \frac{dy}{dx}=e^{3x}\ .e^{y}$$

$$\Rightarrow \frac{dy}{e^y}=e^{3x}\ dx$$

$$\Rightarrow \int e^{-y}\ dy=\int e^{3x}\ dx+c$$

$$\Rightarrow -e^{-y}\ =\frac{e^{3x}}{3}+c$$

7. Solve (sin x + cos x) dy + (cos x – sin x) dx = 0

Sol: Given (sin x + cos x) dy + (cos x – sin x) dx = 0

$$\Rightarrow dy+\frac{\cos x-\sin x}{\sin x+\cos x}=0$$

$$\Rightarrow \int dy+\int\frac{\cos x-\sin x}{\sin x+\cos x}\ dx = c$$

$$\Rightarrow y+\log(\sin x+\cos x)=c$$

8. Solve $\frac{dy}{dx}=e^{x-y}+x^2e^{-y}$

Sol: Given $\frac{dy}{dx}=e^{-y}\left(x^2+e^x\right)$

$$e^y\,dy = (x^2+e^x)dx$$

Integrating on both sides

$$\Rightarrow \int e^y\,dy = \int (x^2 + e^x)dx$$

$$\Rightarrow e^y = \frac{x^3}{3} + e^x + c$$

9. Solve $(y - yx)dx + (x + xy)dy = 0$

Sol: Given $y(1-x)dx + x(1+y)dy = 0$

Integrating on both sides

$$\Rightarrow \int \frac{1-x}{x}dx + \int \frac{1+y}{y}dy = 0$$

$$\Rightarrow \log x - x + \log y + y = c$$

EXERCISE

(1) Solve $3e^x \tan y + (1-e^x)\sec^2 y\,dy = 0$

(2) Solve $\frac{dy}{dx} = e^{x+y} + x^2 e^{x^3+y}$

(3) Solve $(x^2 - yx^2)\frac{dy}{dx} + (y^2 + x^2y^2) = 0$

(4) Solve $\tan x \cdot \sin^2 y\,dx + \cos^2 x.\cot y = 0$

(5) Solve $(xy+x)dx = (x^2y^2+x^2+y^2+1)dy$

(6) Solve $(1-x^2)(1-y)\,dx = xy(1+y)dy$

(7) Solve $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$

(8) Solve $\frac{dy}{dx} = \frac{1+y}{1-x}$

ANSWERS

(1) $\tan y = c\,(1-e^x)^3$ (2) $e^x + e^{-y} + 1/3(e^{x^3}) = c$

(3) $x-(1/x)-(1/y)-\log y = c$ (4) $\tan^2 x - \cot^2 y = c$

(5) $\log(x^2+1) = y^2 - 2y + 4\log[c(y+1)]$

(6) $\log[x(1-y)^2] = x^2/2 - y^2/2 - 2y + c$

(7) $y\sin y + x^2\log x = c$ (8) $(1+y)(1-x)=c$

1.10 HOMOGENEOUS DIFFERENTIAL EQUATION

A differential equation in *x* and *y* is said to be homogeneous if it can be defined in the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree in *x* and *y*.

METHOD OF SOLVING

Consider the homogeneous differential equation

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

Step1: If we put $y = vx$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and the differential equation reduces to variables separable form

Step2: Use the method of solving of variable separable form

Step3: Replacing $v = \frac{y}{x}$, we get the solution of the homogeneous differential equation

PROBLEMS

1. Solve $\left(x^2 + y^2\right)dx = 2xydy$

Sol: The given equation can be written as $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ ---(1)

Here the RHS is homogenous function of x and y

Put y=vx $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

(1)Becomes as $x\frac{dv}{dx} = \frac{1-v^2}{2v}$

By variable separable method $\frac{dx}{x} = \frac{2v}{1-v^2}dv$

After Integration we get $x^2 - y^2 = cx$ as solution

2. Solve $(x^3 + y^3)\, dx = (x^2 y + x y^2)\, dy$

Sol: The given equation can be written as

$$\Rightarrow \frac{dy}{dx} = \frac{x^3+y^3}{x^2 y + x y^2}$$

$$put\ \ y = vx \ \therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v^3}{v+v^2}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{1+v^3}{v+v^2} - v = \frac{1-v^2}{v(v+1)} = \frac{(1-v)(1+v)}{v(v+1)}$$

$$\Rightarrow \int \frac{v}{1-v}dv = \int \frac{1}{x}dx + c$$

$$\Rightarrow \int \frac{-v}{1-v}dv = -\int \frac{1}{x}dx + c$$

$$\Rightarrow v + \log(1-v) = -\log x + c$$

Replacing v by y/x , we get y/x + log(x-y) = c

EXERCISE

(1) Solve $(x^2 + y^2)dx – 2xy\, dy=0$

(2) Solve $(x^2 – 2xy + 3y^2)dx + (y^2 + 6xy – x^2)dy = 0$

(3) Solve $x \sin(y/x)\frac{dy}{dx} = y\sin(y/x) – x$

ANSWERS: (1) $x=c(x^2 – y^2)$ (2) $y^3 – 9xy^2 + 3x^2y – x^3=c$ (3) $\cos(y/x) = \log x + c$

1.11 NON-HOMOGENEOUS DIFFERENTIAL EQUATIONS OF THE FIRST DEGREE IN x & y

If a_1,b_1,c_1 and a_2,b_2,c_2 are constants and atleast one of c_1,c_2 is not zero then $\frac{dy}{dx} = \frac{a_1 x+b_1 y+c_1}{a_2 x+b_2 y+c_2}$ is called non-homogenous differential equation of first degree in x & y.

METHOD OF SOLVING

Step1: Convert the Non homogenous differential equation into homogeneous form by substituting x=X +h, y=Y+k where h& k such that $a_1h+b_1k+c_1 = 0$ and $a_2h+b_2k+c_2=0$

Step2: Use the method of solving of Homogenous differential equation

1.12 EXACT DIFFERENTIAL EQUATION

A differential equation of the form Mdx +Ndy=0 is said to be exact differential equation if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

METHOD OF SOLVING

To solve the exact differential equation M(x,y) dx+N(x,y)dy=0

Step1: Find $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$

If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then the given equation is an exact DE

Step2: The general solution of the given equation is $\int_{y constant} M dx + \int (terms\ in\ N\ which\ do\ not\ contain\ x) dy = C$

PROBLEMS

1. Solve $(2x-y+1)dx+(2y-x-1)dy=0$

Sol: Given $(2x-y+1)dx+(2y-x-1)dy=0$---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

Here $M(x,y)=2x-y+1,\ N(x,y)=2y-x-1$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = -1$$

So the given differential equation is exact hence the general solution is

$$\int_{y constant} M dx + \int (terms\ in\ N\ which\ do\ not\ contain\ x) dy = C$$

$$\Rightarrow \int (2x-y+1)dx + \int (2y-1)dy = c$$

$$\Rightarrow x^2+y^2-xy+x-y=c$$

2. Solve $\left(e^{y}+1\right)\cos x dx+e^{y}\sin x dy=0$

Sol: Given $\left(e^{y}+1\right)\cos x dx+e^{y}\sin x dy=0$ ---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

Here M(x,y)=$\left(e^{y}+1\right)\cos x,$ $N(x,y)=e^{y}\sin x$

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=e^{y}\cos x$$

Hence the given differential equation is exact. The general solution can be written as

$$\int_{y constant} M dx+\int(terms\ in\ N\ which\ do\ not\ contain\ x)dy=C$$

$$\Rightarrow\int\left(e^{y}+1\right)\cos x dx+\int 0 dy=c$$

$$\Rightarrow\left(e^{y}+1\right)\sin x=c$$

3. Solve $(\text{x}\, e^{xy}+y)\frac{dy}{dx}+\ y\, e^{xy}=0$

Sol: Given equation can be written as

$(x\, e^{xy}+\ 2y)dy+ye^{xy}dx=0$ ---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

Here M = $(x\, e^{xy}+\ 2y)$ N= ye^{xy}

$\frac{\partial M}{\partial y}=\ ye^{xy}x+\ e^{xy}$ & $\frac{\partial N}{\partial x}=xe^{xy}y+\ e^{xy}$

So that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Hence the given differential equation is exact. The general solution can be written as

$$\int_{y constant} M dx+\int(terms\ in\ N\ which\ do\ not\ contain\ x)dy=C$$

$$\Rightarrow\int_{(y\ constant)} y\, e^{xy}dx+\ \int 0dy=\ c$$

$$\Rightarrow y\ \frac{e^{xy}}{y}=\ c$$

$\Rightarrow e^{xy}=\ c$, This is the general solution

4. Solve e^y dx + (x e^y+2y)dy=0

Sol: Given e^y dx + (x e^y+2y)dy=0 ---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

Here M = e^y N= x e^y + 2y

$\frac{\partial M}{\partial y}=\ e^{y}$ & $\frac{\partial N}{\partial x}=e^{y}$

So that $\frac{\partial M}{\partial y}=\ \frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

The general solution can be written as

$$\int_{y constant} M dx + \int (terms\ in\ N\ which\ do\ not\ contain\ x) dy = C$$

$$\Rightarrow \int_{(y\ constant)} e^y dx + \int 2y dy = c$$

$\Rightarrow x\, e^y + y^2 = c$; This is the general solution

5. Solve $\left(3x^2y + \frac{y}{x}\right) dx + (x^3 + \log x\, dy) = 0$

Sol: Given equation is $\left(3x^2y + \frac{y}{x}\right) dx + (x^3 + \log x) dy = 0$---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

Here M = $\left(3x^2y + \frac{y}{x}\right)$ N= $(x^3 + \log x)$

$\frac{\partial M}{\partial y} = 3x^2 + \frac{1}{x}$ & $\frac{\partial N}{\partial x} = 3x^2 + \frac{1}{x}$

So that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

The general solution can be written as

$$\int_{y constant} M dx + \int (terms\ in\ N\ which\ do\ not\ contain\ x) dy = C$$

$$\Rightarrow \int_{(y\ constant)} \left(3x^2 + \frac{y}{x}\right) dx + \int 0\, dy = c$$

$\Rightarrow x^3y + y log\, x = c$; This is the general solution

6. Solve (cos x – x cos y)dy – (sin y + y sinx)dx=0

Sol: Given (cos x – x cos y)dy – (sin y + y sinx)dx=0---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

Here M = -sin y – y sin x N= cos x – x cos y

$\frac{\partial M}{\partial y} = -\cos y - \sin x$ & $\frac{\partial N}{\partial x} = -\sin x - \cos y$

So that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

The general solution can be written as

$$\int_{y constant} M dx + \int (terms\ in\ N\ which\ do\ not\ contain\ x) dy = C$$

$$\Rightarrow \int_{(y\ constant)} (-\sin y - y \sin x)\, dx + \int 0\, dy = c$$

$\Rightarrow -x \sin y + y \cos x = c$; This is the general solution

7. Solve $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\,siny)dy=0$

Sol: Given equation is

$$\left[y\left(1+\frac{1}{x}\right)+\cos y\right]dx+(x+\log x-x\,siny)dy=0$$

Let M = $\left[y\left(1+\frac{1}{x}\right)+\cos y\right]$ N=$(x+\log x-x\,siny)$

$\frac{\partial M}{\partial y}=1+\frac{1}{x}-\sin y$ & $\frac{\partial N}{\partial x}=1+\frac{1}{x}-\sin y$

So that $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Hence the given differential equation is exact.

The general solution can be written as

$$\int_{yconstant} Mdx+\int(terms\ in\ N\ which\ do\ not\ contain\ x)dy=C$$

$$\Rightarrow \int_{(y\ constant)}\left(\left(1+\frac{1}{x}\right)y+\cos y\right)dx+\int 0\,dy=c$$

$\Rightarrow (x+\log x)y+xcos\,y=c$; This is the general solution

EXERCISE

(1) Solve $\left(y^2-2xy\right)dx=\left(x^2-2xy\right)dy$

(2) Solve $(2x^2+6xy-y^2)dx+(3x^2-2xy+y^2)dy=0$

(3) Solve $\frac{dy}{dx}+\frac{y\cos x+\sin y+y}{\sin x+x\cos y+x}=0$

(4) Solve $\left(1+e^{\frac{x}{y}}\right)dx+\left(1-\frac{x}{y}\right)e^{\frac{x}{y}}$ dy = 0

(5)Solve $(x^2+y^2-a^2)xdx+(x^2-y^2-b^2)ydy=0$

(6) Solve $xdx+ydy+\frac{xdy-ydx}{x^2+y^2}=0$

(7)Solve $(\sin x.\sin y - x\,e^y)dy=(e^y+\cos x.\cos y)dx$

(8) Solve $(1+4xy+2y^2)dx+(1+4xy+2x^2)dy=0$

ANSWERS

(1) $y^2x-x^2y=c$

(2) $2x^3-9x^2y-3xy^2+y^3=k$

(3) $y\sin x+(\sin y+y)x=c$

(4) $x+ye^{\frac{x}{y}}=c$

(5) $x^4+2x^2y^2-y^4-2a^2x^2-2b^2y^2=c$

(6) $x^2-2tan^{-1}\left(\frac{x}{y}\right)+y^2=c$

(7) $xe^y+\sin x.\cos y=c$

(8) $x+2x^2y+2xy^2+y=c$

1.13 NON EXACT DIFFERENTIAL EQUATION

A differential equation which is not exact is said to be Non exact differential equation.

METHOD OF SOLVING

Step1: Convert Non Exact equation into Exact equation through Integrating Factor.
Step2: Use the method of solving of Exact Equation.

INTEGRATING FACTOR (I.F)

A given differential equation may not be integrable as such.But it may become integrable when it is multiplied by a function.Such a function is called the integrating factor (I.F).
Hence an integrating factor is one which changes a differential equation into one which is directly integrable.

Methods to find an Integrating Factor:

(1) If $Mdx+Ndy$ =0 is homogeneous then I.F.= $\frac{1}{Mx+Ny}$

(2) If $Mdx+Ndy=0$ is of the form y f(xy)dx+x g(xy)dy=0 then I.F.= $\frac{1}{Mx-Ny}$

(3) If $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x) then I.F=e^{\int f(x)dx}$

(4) If $\dfrac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=g(y) then I.F=e^{\int g(y)dy}$

PROBLEMS

1. Solve $x^2ydx-\left(x^3+y^3\right)dy=0$

Sol: Given $x^2ydx-\left(x^3+y^3\right)dy=0$ ---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

$Let M(x,y)=x^2y, N(x,y)=-\left(x^3+y^3\right)$

$$\frac{\partial M}{\partial y}=x^2 \quad \& \quad \frac{\partial N}{\partial x}=-3x^2$$

Then we have $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$

So the given differential equation is not exact
But the differential equation is homogeneous so

$$\text{I.F.}=\frac{1}{Mx+Ny}=-\frac{1}{y^4}$$

Now multiplying the given differential equation throughout with $-\frac{1}{y^4}$ we get

$-\left(\frac{x^2}{y^3}\right)dx+\left(\frac{x^3+y^3}{y^4}\right)dy=0$ ---(2), which is exact differential equation

Compare (2) with M(x,y) dx+N(x,y)dy=0

Here M=$-\frac{x^2}{y^3}$ & N=$\frac{x^3+y^3}{y^4}$

Hence the solution is

$$\int_{y constant} Mdx+\int(terms\ in\ N\ which\ do\ not\ contain\ x)dy=C$$

$$\Rightarrow -\int\frac{x^2}{y^3}dx+\int\frac{1}{y}dy=c$$

$\Rightarrow -\frac{x^3}{3y^3}+\log y=c$ is required solution

2. Solve $y\left(x^2y^2+2\right)dx+x\left(2-2x^2y^2\right)dy=0$

Sol: Given $y\left(x^2y^2+2\right)dx+x\left(2-2x^2y^2\right)dy=0$ ---(1)

Compare (1) with M(x,y) dx+N(x,y)dy=0

Here $M\left(x,y\right)=y\left(x^2y^2+2\right)$ $N\left(x,y\right)=x\left(2-2x^2y^2\right)$

$$\frac{\partial M}{\partial y}=3x^2y^2+2\neq\frac{\partial N}{\partial x}=2-6x^2y^2$$

So the given differential equation is not exact, but it is of the form

$yf\left(xy\right)dx+xg\left(xy\right)dy=0 Hence$

$$I.F.=\frac{1}{Mx-Ny}=\frac{1}{3x^3y^3}$$

Now multiplying the given differential equation throughout with I.F. we get

$\frac{x^2y^2+2}{3x^3y^2}dx+\frac{2-2x^2y^2}{3x^2y^3}dy=0$ ---(2)

Clearly we observe that it is exact differential equation.

Compare (2) with M(x,y) dx+N(x,y)dy=0

Here M=$-\frac{x^2y^2+2}{3x^3y^2}$ & N=$\frac{2-2x^2y^2}{3x^2y^3}$

Hence the solution is

$$\int_{y constant} Mdx+\int(terms\ in\ N\ which\ do\ not\ contain\ x)dy=C$$

$$\Rightarrow\left(\frac{1}{3}\right)\int\frac{1}{x}dx+\frac{2}{3y^2}\int\frac{1}{x^3}dx-\frac{2}{3}\int\frac{1}{y}dy=c$$

$$\Rightarrow\frac{1}{3}\log x-\frac{1}{3x^2y^3}-\frac{2}{3}\log y=c$$

3.Solve $(x^2y^2+xy+1)ydx+(x^2y^2-xy+1)xdy=0$

Sol: Let M(x,y)=$(x^2y^2+xy+1)y$ & N(x,y)=$(x^2y^2-xy+1)x$

$$\frac{\partial M}{\partial y}=3x^2y^2+2xy+1\neq\frac{\partial N}{\partial x}=3x^2y^2-2xy+1$$

So the given differential equation is not exact , but it is of the form $yf(xy)dx+xg(xy)dy=0$ *Hence*

$$I.F.=\frac{1}{Mx-Ny}=\frac{1}{2x^2y^2}$$

Now multiplying the given differential equation throughout with I.F. we get

$$\frac{(x^2y^2+xy+1)y}{2x^2y^2}dx+\frac{(x^2y^2-xy+1)x}{2x^2y^2}dy=0$$

$$\Rightarrow\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\left(\frac{x}{2}-\frac{1}{2y}+\frac{1}{2xy^2}\right)dy=0$$

Clearly we observe that it is exact differential equation.

Hence the solution is

$$\int_{y\,constant}\left(\frac{y}{2}+\frac{1}{2x}+\frac{1}{2x^2y}\right)dx+\int\left(-\frac{1}{2y}\right)dy=c$$

$$\Rightarrow\frac{yx}{2}+\frac{1}{2}\log x-\frac{1}{2xy}-\frac{1}{2}\log y=c$$

4. Solve $2xydy-\left(x^2+y^2+1\right)dx=0$

Sol: Let M $=-\left(x^2+y^2+1\right)$ $\quad N=2xy$

$$\frac{\partial M}{\partial y}=-2y\neq\frac{\partial N}{\partial x}=2y$$

So the given differential equation is not exact

But $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=-\dfrac{2}{x}=f(x)$

so I.F = $e^{-\int\frac{2}{x}dx}=\dfrac{1}{x^2}$

Now multiply the given differential equation through out with I.F. we get

$$\frac{2y}{x}dy-\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx=0$$

Observe that it is exact,hence the solution is

$$-\int\left(1+\frac{y^2}{x^2}+\frac{1}{x^2}\right)dx+\int 0\ dy=c$$

$$y^2-x^2-cx+1=0$$

5. Solve $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$

Sol: Let $M(x,y)=xy^3+y$ & $N(x,y)=2x^2y^2+2x+2y^4$

$$\frac{\partial M}{\partial y}=3xy^2\neq\frac{\partial N}{\partial x}=4xy^2+2$$

So the given differential equation is not exact and

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{M}=-\frac{(xy^2+1)}{xy^3+y}=-\frac{1}{y}=g(y)\ \text{Hence}\ I.F.=e^{-\int g(y)dy}=e^{\int\left(\frac{1}{y}\right)dy}=e^{\log y}=y$$

Now multiplying the given differential equation throughout with I.F. we get

$(xy^4+y^2)dx+(2x^2y^3+2xy+2y^5)dy=0$

Clearly we observe that it is exact differential equation. Hence the solution is

$$\int_{y constant}(xy^4+y^2)dx+\int(2y^5)dy=c$$

$$\Rightarrow y^4\left(\frac{x^2}{2}\right)+y^2(x)+2\left(\frac{y^6}{6}\right)=c$$

6. Solve $\left(y^4+2y\right)dx+\left(xy^3+2y^4-4x\right)dy=0$

Sol: Let $M\left(x,y\right)=y^4+2y \quad N\left(x,y\right)=xy^3+2y^4-4x$

$$\frac{\partial M}{\partial y}=4y^3+2\neq\frac{\partial N}{\partial x}=y^3-4$$

So that given differential equation is not exact but

$$\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M}=-\frac{3}{y}=f\left(y\right)$$

$$\text{IF}=e^{-\int\frac{3}{y}dy}=\frac{1}{y^3}$$

Multiply the given differential equation thought out with $\frac{1}{y^3}$ we get

$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0$, which is exact DE

Now the solution is

$$\int_{y\ constant}\left(y+\frac{2}{y^2}\right)dx+\int 2ydy=c$$

$\Rightarrow\left(y+\frac{2}{y^2}\right)x+y^2=c$ is the required solution

EXERCISE

(1) Solve $(x^4+y^4)dx-xy^3dy=0$
(2) Solve $y(1+xy)dx+x(1-xy)dy=0$
(3) Solve $y(2xy+e^x)dx=e^xdy$
(4) Solve $(xy^3+y)dx+2(x^2y^2+x+y^4)dy=0$
(5) Solve $(xy^2-e^{1/x^3})dx-x^2ydy=0$
(6) Solve $(x^2y^2+x)dy+(x^2y^3-y)dx=0$
(7) Solve $(y+xy^2)dx-xdy=0$
(8) Solve $(x^2y^3+1)dx+x^4y^2dy=0$

ANSWERS

(1) $y^4=4x^4\log x+cx^4$ (2) $\log(x/y)-(1/xy)=c$
(3) $x^2+\frac{e^x}{y}=c$ (4) $\frac{x^2y^4}{2}+y^2x+\frac{y^6}{3}=c$
(5) $2e^{\frac{1}{x^3}}-\frac{3y^2}{x^2}=c$ (6) $(x^2y^2)/2+\log(y/x)=c$
(7) $(x/y)+(x^2/2)=c$ (8) $(xy)^3/3+\log x=c$

1.14 LINEAR DIFFERENTIAL EQUATION

A first order differential equation is said to be linear when the dependent variable and its derivatives occur only in first degree and no product of these occur.

Note:
1. The general form of first order linear differential equation with y as dependent and x as independent valuables is $\frac{dy}{dx}+p(x)y=Q(x)$,where P and Q are functions of x only
2. The general form of first order linear differential equation with x as dependent and y as independent valuables is $\frac{dx}{dy}+P(y)x=Q(y)$,where P and Q are functions of y only

METHOD OF SOLVING

Step1: Express the given linear DE in the standard form $\frac{dy}{dx}+p(x)y=Q(x)$ or

$\frac{dx}{dy}+P(y)x=Q(y)$

Step2: a) The general solution of $\frac{dy}{dx}+p(x)y=Q(x)$ is

$y(I.F)=\int Q(x).(I.F)dx+c$,where I.F. $=e^{\int p(x)dx}$

b)The general solution of $\frac{dx}{dy}+P(y)x=Q(y)$is

$x(I.F)=\int Q(y)(I.F)dy+c$,where I.F. $=e^{\int P(y)dy}$

PROBLEMS

1. Solve $(x^2-1)\frac{dy}{dx}+2xy=1$

Sol: The given differential equation can be written as

$$\frac{dy}{dx}+\left(\frac{2x}{x^2-1}\right)y=\frac{1}{x^2-1} \text{ ---(1)}$$

Compare (1) with $\frac{dy}{dx}+p(x)y=Q(x)$

Here P = $\frac{2x}{x^2-1}$ and Q = $\frac{1}{x^2-1}$

I.F.=$e^{\int Pdx}=e^{\int\frac{2x}{x^2-1}dx}=e^{\log(x^2-1)}=x^2-1$

The Solution of the given equation is

$$y(IF)=\int Q(IF)dx+c$$

$$\Rightarrow y(x^2-1)=\int\frac{1}{(x^2-1)}(x^2-1)dx+c \quad \Rightarrow$$

$$y(x^2-1)=\int dx+c \Rightarrow y(x^2-1)=x+c$$

2. Solve $cos^2x\frac{dy}{dx}+y=\tan x$

Sol: The given differential equation can be written as

$$\frac{dy}{dx}+sec^2x\,y=\tan x\,sec^2x \text{ ---(1)}$$

Compare (1) with $\frac{dy}{dx}+p(x)y=Q(x)$

Here P=$\sec^2$x and Q=tan x $\sec^2$x

I.F.=$e^{\int Pdx}=e^{\int sec^2xdx}=e^{\tan x}$

The Solution of the given equation is

$$y(IF)=\int Q(IF)dx+c$$

$$\Rightarrow y(e^{\tan x})=\int(\tan x\sec^2x)e^{\tan x}dx+c$$

[Putting tan x = t, $\sec^2$x dx = dt]

$= \int te^t dt+c$

$= te^t - e^t + c$

$= (\tan x - 1)e^{\tan x}+c$

$y=(\tan x - 1)+c\, e^{-\tan x}$

3. Solve $(1+y^2)dx=(Tan^{-1}y-x)dy$

Sol: The given differential equation can be written as

$$\frac{dx}{dy}+\frac{1}{1+y^2}x=\frac{\tan^{-1}y}{1+y^2} \text{ ---(1)}$$

Compare (1) with $\frac{dx}{dy}+P(y)x=Q(y)$

Here P(y)=$\frac{1}{1+y^2}$ and Q(y)=$\frac{tan^{-1}y}{1+y^2}$

$$\text{I.F.} = e^{\int p(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1} y}$$

The general solution is $x(I.F) = \int Q(y)(I.F)dy + c$

$$\Rightarrow xe^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} e^{\tan^{-1} y} dy + c$$

$$= e^{\tan^{-1} y}\left(\tan^{-1} y - 1\right) + c$$

i.e, $x = \left(\tan^{-1} y - 1\right) + ce^{-\tan^{-1} y}$

4. Solve: $x\left(x-1\right)\frac{dy}{dx} - y = x^2\left(x-1\right)^2$

Sol: The given differential equation can be written as

$$\frac{dy}{dx} - \frac{y}{x\left(x-1\right)} = x\left(x-1\right) \;;$$

$$\text{I.F} = e^{-\int \frac{1}{x(x-1)}dx} = e^{\int\left(\frac{1}{x}-\frac{1}{x-1}\right)dx} = e^{\log\left(\frac{x}{x-1}\right)} = \frac{x}{x-1}$$

The general solution is $y\frac{x}{x-1} = \int x\left(x-1\right)\frac{x}{\left(x-1\right)}dx + c = \frac{x^3}{3} + c$

5. Solve $(1-x^2)\frac{dy}{dx} - xy = 1$

Sol: The given equation is $(1-x^2)\frac{dy}{dx} - xy = 1$

$$\Rightarrow \frac{dy}{dx} - \frac{x}{1-x^2}y = \frac{1}{1-x^2}$$

This is of the form $\frac{dy}{dx} + Py = Q$

$$P = \frac{-x}{1-x^2} \;;\; Q = \frac{1}{1-x^2}$$

$$\Rightarrow I.F = e^{\int P\,dx} = e^{\int \frac{-x}{1-x^2}dx} = \sqrt{1-x^2}$$

The general solution is ,

$$y(I.F) = \int Q(I.F)\,dx + c$$

$$y\sqrt{1-x^2} = \int \frac{1}{1-x^2}\sqrt{1-x^2}\,dx + c$$

$$= \int \frac{dx}{\sqrt{1-x^2}} + c$$

$$y\sqrt{1-x^2} = \sin^{-1} x + c$$

6. Solve $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin\ 2x$

Sol: The given differential equation can be written as

$$\frac{dy}{dx}+y\cos x=\frac{1}{2}\sin\ 2x \text{ ---(1)}$$

Compare (1) with $\frac{dy}{dx}+p(x)y=Q(x)$

Here P = cos x and Q = ½ sin 2x

I.F.=$e^{\int Pdx}=e^{\int \cos x dx}=e^{\sin x}$

The general solution is ,

$$y(I.F)=\int Q(I.F)\,dx+c$$

$$=\int \frac{1}{2}\sin\ 2x(e^{\sin x})\,dx+c$$

$$=\int \sin\ x\cos x\ \ e^{\sin x}\ \ dx\ +\ c \qquad\qquad Let\ \sin\ x=t$$

$$=\int t\,e^{t}\ dt\ +c \qquad\qquad \cos x\,dx=dt$$

$$=e^{t}\ (t-1)+c$$

$$=e^{\sin x}\ (\sin\ x-1)+c$$

7. Solve $\cos x\frac{dy}{dx}+y\sin\ x=1$

Sol: The given equation can be reduced to

$$\frac{dy}{dx}+y\frac{\sin\ x}{\cos\ x}=\frac{1}{\cos x} \quad or \quad \frac{dy}{dx}+\ y\ \tan x=\sec\ x$$

$$Here\ P=\tan x \qquad ; \qquad Q=\sec x$$

$$I.F\ =e^{\int \tan x\,dx}=e^{\log \sec x}=\sec x$$

The general solution is ,

$$y(I.F)=\int Q(I.F)\,dx+c$$

$$y(\sec x)=\int \sec^{2} x\,dx+c$$

$$=\tan\ x+\ c$$

EXERCISE

(1) $Solve\ \ \frac{dy}{dx}=-\frac{(x+y\cos x)}{1+\sin x}$

(2) Solve $\frac{dy}{dx}-\frac{2y}{x}=\frac{5x^{2}}{(2+x)(3-2x)}$

(3) $Solve\ \ (x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$

(4) $Solve\ \ \frac{dy}{dx} + y.\cot x = 4x\ cosec\ x\ \ given\ \ y = 0\ when\ x = \frac{\pi}{2}$

X

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(5) Solve $x\ cosx\frac{dy}{dx} + y(xsinx + cosx) = 1$

(6) Solve $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right)\frac{dy}{dx} = 1$

(7) Solve $\frac{dy}{dx} + y = e^{e^x}$

(8) Solve $\frac{dy}{dx} + \frac{y}{x}\sin x^2$

(9) Solve $\left(x+2y^3\right)\frac{dy}{dx} = y$

(10) Solve $y^2dx + (xy - 2y^2 - 1)dy = 0$

ANSWERS

y

(1) $y(1+\sin x) = -(x^2/2)+c$ (2) $y = \frac{5x^2}{7}\log\left(\frac{x+2}{3-2x}\right)c$

(3) $y/(x+1) = e^{3x}/3+c$ (4) $y\sin x = 2x^2 \quad -\frac{\pi^2}{2}$

(5) xy sec x=tan x+c (6) $ye^{2\sqrt{x}} = 2\sqrt{x+c}$

(7) $ye^x = e^{e^x} + c$ (8) $xy = \frac{1}{2}\cos x^2 + c$

(9) $x = y^3 + cy$ (10) $xy = y^2 + \log y + c$

1.15 NONLINEAR EQUATION REDUCIBLE TO LINEAR FORM [BERNOULLI'S D. E.]

An equation of the form $\frac{dy}{dx} + p(x)y = Q(x)y^n$ is called Bernoulli's Differential equation which is nonlinear.

METHOD OF SOLVING

Step1: Rewrite the given ODE into standard form of Bernoulli's equation

Step2: Dividing both sides by y^n

Step3: Put $z = y^{1-n}$ then the given ODE becomes as linear differential equation of 'z'

Step4: Solve linear differential equation in z by the method discussed previously and replace z by y^{1-n}

PROBLEMS

1. Solve $x\frac{dy}{dx} + y = x^3y^6$

Sol: The given differential equation is $\frac{dy}{dx} + \frac{y}{x} = x^2y^6$ ---(1)

Divide with y^6 we get; $\frac{1}{y^6}\frac{dy}{dx}+\frac{1}{y^5}\frac{1}{x}=x^2$

Put $\frac{1}{y^5}=z \Rightarrow \frac{1}{y^6}\frac{dy}{dx}=-\frac{1}{5}\frac{dz}{dx}$

$\therefore$ (1) Because as $\frac{dz}{dx}-\frac{5}{x}z=-5x^2$,which is LDE in z

$$\text{IF } = e^{-\int \frac{5}{x}dx} = \frac{1}{x^5}$$

Hence the general solution is $z\left(\frac{1}{x^5}\right)=\int -\frac{5}{x^3}dx+c$

$\frac{1}{x^5y^5}=\frac{5}{2x^2}+c$ is solution of given DE

2. Solve $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$

Sol: The given differential equation can be written in the standard form $\frac{dy}{dx}+\frac{x}{1-x^2}y=x\sqrt{y}$

Dividing by $\sqrt{y}$; $y^{-\frac{1}{2}}\frac{dy}{dx}+\frac{x}{1-x^2}y^{\frac{1}{2}}=x$ ---(1)

Put $z=\sqrt{y}$

$$\therefore \frac{dz}{dx}=\frac{1}{2}y^{-1/2}\frac{dy}{dx}$$

$$\Longrightarrow 2.\frac{dz}{dx}=y^{-1/2}\frac{dy}{dx}$$

$$(1)\Longrightarrow\ 2.\frac{dz}{dx}+\frac{x}{1-x^2}z=x$$

$$\Longrightarrow\ \frac{dz}{dx}+\frac{x}{2(1-x^2)}z=x/2$$

$$I.F=e^{\int\left[\frac{x}{2(1-x^2)}\right]dx}=e^{-1/4\log(1-x^2)}=(1-x^2)^{-1/4}$$

Hence the general solution is

$$z(1-x^2)^{-1/4}=\int\frac{x}{2}(1-x^2)^{-1/4}dx+c$$

$$\Longrightarrow z(1-x^2)^{-1/4}=-\frac{1}{3}(1-x^2)^{3/4}+c$$

$$\Longrightarrow z=-\frac{1}{3}(1-x^2)\quad+c(1-x^2)^{1/4}$$

$$\therefore \sqrt{y}=-\frac{1}{3}(1-x^2)\quad+c(1-x^2)^{1/4}$$

3. Solve $\frac{dy}{dx}-y\,tanx\ =\frac{\sin x\,cos^2x}{y^2}$

Sol: The given differential equation can be written in the standard form

$y^2\frac{dy}{dx}-y^3\,tanx\ =\sin x\,cos^2x$ ---(1)

Put $z=y^3$ $\Longrightarrow \frac{dz}{dx}=3y^2\frac{dy}{dx}$

$$\Rightarrow \frac{1}{3}\frac{dz}{dx} = y^2\frac{dy}{dx}$$

$$(1)\Rightarrow \frac{1}{3}\frac{dz}{dx} - ztan\, x = \sin x\, cos^2 x$$

$$\Rightarrow \frac{dz}{dx} - 3ztan\, x = 3\sin x\, cos^2 x$$

P=-3 tan x;Q=sin x $\cos^2$x

$$I.F = e^{-3\int \tan x dx} = e^{3\log\cos x} = cos^3 x$$

Hence the general solution is

$$z.cos^3 x = \int 3\sin x\, cos^2 x.cos^3 x dx + c$$

$$= -\int cos^5 x(-\sin x)\, dx + c$$

$$\Rightarrow z.cos^3 x = -\frac{cos^6 x}{2} + c$$

$$\Rightarrow y^3.cos^3 x = -\frac{cos^6 x}{2} + c$$

4. Solve $\frac{dy}{dx} + xsin\, 2y = x^3\, cos^2 y$

Sol: Given equation is $\frac{dy}{dx} + xsin\, 2y = x^3\, cos^2 y$ ---(1)

$$\Rightarrow \frac{1}{cos^2 y}\frac{dy}{dx} + \frac{2\sin y\cos y}{cos^2 y}x = x^3$$

$$\Rightarrow sec^2 y\frac{dy}{dx} + (2\tan y)\, x = x^3 \text{ ---(2)}$$

Put tan y = u so that $sec^2 y\frac{dy}{dx} = \frac{du}{dx}$

Substituting in (2), we get $\frac{du}{dx} + 2u\, x = x^3$, which is a linear equation of first order

I.F.= $I.F. = e^{\int p\, dx} = e^{\int 2x dx} = e^{x^2}$

Hence the general solution is;

u$e^{x^2} = \int x^3 e^{x^2} dx + c\ (put\ x^2 = t\ so\ that\ 2xdx = dt)$

$= \int \frac{t}{2} e^t dt + c = \frac{1}{2} e^t(t-1) + c = \frac{1}{2}\, e^{x^2}(x^2-1) + c$

Substituting u = tan y, we get the general solution of (1) as

$(\tan y)\, e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c$

5. Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\, secy$

Sol: Given equation is $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\, secy$

Dividing secy we get; $\cos y\frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)e^x$ ---(1)

Put sin y = u so that $\cos y\ \frac{dy}{dx} = \frac{du}{dx}$

Now (1) become $\frac{du}{dx} - \frac{u}{1+x} = (1+x)e^x$ ---(2), which is linear in u

Here P = $\frac{-1}{1=X}$, $Q = (1+x)e^x$

I.F.= $e^{\int P\, dx} = e^{-\int\frac{dx}{1+x}} = e^{-\log(1+x)} = \frac{1}{x+1}$

The solution of (2) is $u(I.F) = \int Q x I.F.dx + c$

i.e., $\frac{u}{1+x} = \int (1+x)e^x . \frac{1}{1+x} dx + c = \int e^x dx + c$

i.e., $u = (1+x)(e^x + c)$

or $\sin y = (1+x)(e^x + c)$, which is the required solution

6. Solve $2y\cos y^2 \frac{dy}{dx} - \frac{2}{1+x}\sin y^2 = (x+1)^3$

Sol: The given equation

$$2y\cos y^2 \frac{dy}{dx} - \frac{2}{1+x}\sin y^2 = (x+1)^3 \qquad \text{---(1)}$$

Put $z = \sin y^2$

$$\therefore \frac{dz}{dx} = \cos y^2 . 2y \frac{dy}{dx}$$

(1)$\Longrightarrow \frac{dz}{dx} - \frac{2}{x+1} z = (x+1)^3$

Here P= $-\frac{2}{x+1}$ and Q=$(x+1)^3$

$$I.F = e^{-\int \frac{2}{x+1}dx} = e^{-2\log(x+1)} = \frac{1}{(x+1)^2}$$

Hence the general solution is

$$z\frac{1}{(x+1)^2} = \int (1+x)^3 . \frac{1}{(x+1)^2} dx + c$$

$$\Longrightarrow z\frac{1}{(x+1)^2} = \frac{(x+1)^2}{2} + c$$

$$\Longrightarrow \frac{\sin y^2}{(x+1)^2} = \frac{(x+1)^2}{2} + c$$

EXERCISE

(1) Solve $\frac{dy}{dx} = 2y\tan x + y^2 \tan^2 x$

(2) Solve $\frac{dy}{dx} + y\tan x = y^2 \sec x$

(3) Solve $\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y}\frac{1}{x} = 1$

(4) Solve $x^2 - y^2 + 2xy\frac{dy}{dx} = 0$

(5) Solve $dx - (x^2y^3 + xy)dy = 0$

(6) Solve $\frac{dy}{dx} + \left(\frac{y}{x}\right)\log y = \frac{y}{x}(\log y)^2$

(7) Solve $x\frac{dy}{dx} + y = y^2 \log x$

(8) Solve $y^2 dy = (x^3 + y^3)dx$

(9) Solve $2xy\frac{dy}{dx} = y^2 - 2x^3; y(1) = 2$

(10) Solve $\frac{dy}{dx} + y + y^2(\sin x - \cos x) = 0$

ANSWERS

(1) $\frac{sec^2x}{y} = c - \frac{tan^3x}{3}$ (2) $\frac{1}{y}\cos x = -x + c$

(3) $\frac{1}{yx} = c - \log x$ (4) $y = 3 - 2\,sinx + c\,e^{-\sin x}$

(5) $x(2 - y^2) + cxe^{-\frac{y^2}{2}} = 1$ (6) $(1 + cx)\log y = 1$

(7) y(cx+1+log x)=1 (8) $y^3 = -x^3 - x^2 - \frac{2}{3}x - \frac{2}{9} + ce^{3x}$

(9) $y^2 = x(5 - x^2)$ (10) $y(ce^x - \sin x) = 1$

APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

1.16 OTHOGONAL TRAJECTORIES

A curve which cuts every member of a given family of curves at a right angle is an orthogonal trajectory of the given family.

I. Orthogonal trajectories in Cartesian Co-ordinates:

Working Rule:

To find the family of orthogonal trajectories in Cartesian form i.e., f(x,y,c) = 0 ---(1), be the equation of given family of curves in Cartesian form

Step1: Eliminate c from (1) and obtain the differential equation F(x,y,y^1) = 0 ---(2) of the given family of curves

Step2: Replace $\frac{dy}{dx}$ $by - \frac{dx}{dy}$ in (2). Then the DE of the family of orthogonal trajectories is

F(x,y,- $\frac{dx}{dy}$) $= 0$---(3)

Step3: Solving the equation (3) to get the equation of the family of orthogonal trajectories of (1)

PROBLEMS

1. Find the orthogonal trajectories of families of semi-cubical parabolas $ay^2 = x^3$, where a is the parameter.

Sol: The given family of semi-cubical parabola is $ay^2 = x^3$ ---(1)

Differentiating w.r.t. x, we get, a.2y. $\frac{dy}{dx} = 3\,x^2$ ---(2)

Eliminating 'a' from (1) and (2), we get 3y = $2x\frac{dy}{dx}$ --- (3),

is the differential equation of a family of curves

To get the differential equation of family of orthogonal trajectories, we replace $\frac{dy}{dx}$ $by - \frac{dx}{dy}$

Hence y= $-\frac{2}{3}x\ \frac{dx}{dy}$ --- (4)

Solving this we get the differential equation of family of orthogonal trajectories.

Separating the variables, $ydy = -\frac{2}{3}$xdx

Integrating , we get $\frac{y^2}{2} = -\frac{x^3}{3} + c \rightarrow \frac{x^3}{3c} + \frac{y^2}{2c} = 1$, which is the equation of the family of orthogonal trajectories

2. Find the orthogonal trajectories of hyperbola xy=c.

Sol: The given curve is xy = c ---(1)

Differentiating w.r.t. x, we get, x. $\frac{dy}{dx} + y = 0$ ---(2)

is the DE of the given family

To get the differential equation of family of orthogonal trajectories, we replace $\frac{dy}{dx}$ $by -\frac{dx}{dy}$

Hence $x\left(-\frac{dx}{dy}\right) + y = 0$ --- (3)

Solving this we get the differential equation of family of orthogonal trajectories.

Separating the variables, $ydy = x\,dx$

Integrating , we get $\frac{y^2}{2} = \frac{x^2}{2} + c \Rightarrow x^2 - y^2 = a^2$, which is the equation of the family of orthogonal trajectories

3. Find the orthogonal trajectories of the family of circles passing through origin and centre on x-axis

Sol: Given family of circles is $x^2 + y^2 + 2gx = 0$ ---(1)

$$\Rightarrow 2x + 2y\frac{dy}{dx} + 2g = 0 \Rightarrow x + y\frac{dy}{dx} + g = 0$$

$$\Rightarrow g = -x - y\frac{dy}{dx} \quad \text{---(2)}$$

Substituting in (1), we get $x^2 + y^2 + 2x(-x - y\frac{dy}{dx}) = 0$

$$\Rightarrow y^2 - x^2 - 2xy\frac{dy}{dx} = 0 \quad \text{---(3)}$$

Replacing $\frac{dy}{dx}$ $with -\frac{dx}{dy}$, we get the differential equation of the corresponding orthogonal trajectories as

$$y^2 - x^2 + 2xy\frac{dx}{dy} = 0$$

$$\Rightarrow 2xy\frac{dx}{dy} = x^2 - y^2$$

Taking $x^2 = u$, we get $2x\frac{dx}{dy} = \frac{du}{dy}$

The differential equation of orthogonal trajectory becomes $\frac{du}{dy} - \frac{u}{y} = -y$, which is a linear eqn

I.F. = $e^{-\int\frac{1}{y}dy} = e^{-\log y} = \frac{1}{y}$

General solution is $\frac{1}{y}\, u = \int -y.\frac{1}{y}dy = -y + c$

$$\Rightarrow \frac{1}{y}.x^2 = -y + c$$

4. Prove that system of parabolas $y^2 = 4a(x+a)$ is self orthogonal

Sol: Given parabola is $y^2 = 4a(x+a)$

$$\Rightarrow y^2 = 4ax + 4a^2 \qquad \text{---(1)}$$

Differentiating (1) with respect to x, we get $2yy_1 = 4a$

$$\Rightarrow a = \frac{yy_1}{2} \qquad \text{---(2)}$$

Substituting (2) in (1), $y^2 = 4.\frac{yy_1}{2} + 4.\frac{y^2{y_1}^2}{4}$

or $y^2 = 2xyy_1 + y^2y_1^2$ ---(3)

Equation (3) is the differential equation of the given system of parabolas. Replacing y_1 with -$1/y_1$, we get equation of the orthogonal trajectories as

$y^2 = 2xy(-1/y_1) + y^2(-1/y_1)^2$

$\Rightarrow y^2 = 2xy/y_1 + y^2/y_1^2$

$\Rightarrow y^2y_1^2 = -2xyy_1 + y^2$ ---(4),

which is differential equation of the orthogonal trajectories of the given family .

Equations (3) and (4) are same, hence the given system is self orthogonal. \

II.Orthogonal trajectories in polar co-ordinates:

Working Rule:

To find the family of orthogonal trajectories in polar form i.e., $f(r,\theta,c) = 0$ --- (1), be the equation of given family of curves in polar form

Step1: Eliminate c from (1) and obtain the differential equation $F(r,\theta,\frac{dr}{d\theta}) = 0$ ---(2) of the given family of curves

Step2: Replace $\frac{dr}{d\theta}$ by $-r^2\frac{dr}{d\theta}$ in (2). Then the differential equation of the family of Orthogonal trajectories is $F(r,\theta,-r^2\frac{dr}{d\theta}) = 0$ ---(3)

Step3: Solve the equation (3) to get the equation of the family of orthogonal trajectories of (1)

PROBLEMS

1. Find the orthogonal trajectories of the family of cardioids $r=a(1-\cos\theta)$,where a is parameter

Sol: Given equation of the family of cardiods is $r=a(1-\cos\theta)$ ---(1)

Differentiating with respect to θ we get

$$\frac{dr}{d\theta} = a\,sin\theta \Rightarrow a = \frac{1}{sin\theta}\frac{dr}{d\theta} \qquad \text{---(2)}$$

Eliminating a from (1) and (2), we get $r = \frac{1}{\sin\theta}(1-\cos\theta)\frac{dr}{d\theta}$

$$\Rightarrow \frac{dr}{d\theta} = \frac{r\,sin\theta}{1-\cos\theta} \Rightarrow \frac{dr}{d\theta} = r\cot\frac{\theta}{2} \qquad \text{---(3),}$$

which is the differential equation of family of given curves.

To get the differential equation of the family of orthogonal trajectories replace $\frac{dr}{d\theta}\ by - r^2\ \frac{d\theta}{dr}$ in (3).

Then $-r^2\frac{d\theta}{dr} = r\cot\frac{\theta}{2} \Rightarrow \frac{dr}{r} = -\tan(\frac{\theta}{2})d\theta$

Integrating, log r = 2 log cos $(\frac{\theta}{2})$ + log (2c) = log (2c $\cos^2\theta/2$)

$\Rightarrow r = 2c\,cos^2\left(\frac{\theta}{2}\right) \Rightarrow r = c(1+\cos\theta)$, which is the equation of family of orthogonal trajectories.

2. Find the orthogonal trajectories of the family of curves $r^n=a^n \cos n\theta$,where a is parameter

Sol: Given equation of the family of curves $r^n=a^n \cos n\theta$ ---(1)

Taking logarithms on both sides n log r = n log a + log cos nθ

Differentiating with respect to θ we get

$$n\ \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cos n\theta}n(-\sin n\,\theta) \Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta \text{---(2),}$$

which is the differential equation of family of given curves.

To get the differential equation of the family of orthogonal trajectories replace $\frac{dr}{d\theta}\ by - r^2\ \frac{d\theta}{dr}$ in (2).

Then $\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\tan n\theta \Rightarrow r\ \frac{d\theta}{dr} = \tan n\theta \Rightarrow \frac{dr}{r} = \frac{d\theta}{\tan n\theta}$

Integrating, we get log r = (1/n) log sin nθ +(1/n) log c ⟹ log r^n = log sin nθ+log c

$\Rightarrow r^n = c\sin n\,\theta$, which is the equation of family of orthogonal trajectory

EXERCISE

(1)Find the orthogonal trajectories of the given family of curve $e^x+e^{-y}=c$

(2) Find the orthogonal trajectories of the given family of curve $x^2+y^2=c^2$

(3) Find the orthogonal trajectories of the given family of curve $y^2=4ax$

(4)Show that the system of confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1, where\ \lambda\ is\ the\ parameter$, is self orthogonal

(5) Find the orthogonal trajectories of the given family of curve $r^2 = a\,sin\,2\theta$

(6) Find the orthogonal trajectories of the given family of curve $r = a(1+cos\,\theta)$

(7) Find the orthogonal trajectories of the given family of curve $r = a\, cos^2\theta$

(8) Find the orthogonal trajectories of the given family of curve $r^2 = a^2\, cos\, 2\theta$

ANSWERS

(1) $e^y - e^{-x} = k$ (2) $y = kx$

(3) $2x^2 + y^2 = k$ (5) $r^2 = c\, cos\, 2\theta$

(6) r= $c\,(1 - cos\,\theta)$ (7) $r^2 = c\, sin\,\theta$

(8) $r^2 = c^2\, sin\, 2\theta$

1.17 NEWTON'S LAW OF COLLING

The rate of change (in time) of the temperature is proportional to the difference between the temperature T of the object and the temperature T_S of the environment surrounding the object.

$$\frac{dT}{dt} = -k(T - T_s)\ ,k>0$$

Let $x = T - T_s$

sothat $\frac{dx}{dt} = \frac{dT}{dt}$ and

The above differential equation becomes d x / d t = - k x

The solution to the above differential equation is given by $x = A\, e^{-kt}$ substitute $x = T - T_s$ so that $T - T_s = A\, e^{-kt}$

PROBLEMS

1. A body is originally at 80^0C and cools down to 60^0C in 20 minutes. If the temperature of the air is 40^0C, find the temperature of the body after 40 minutes.

Sol: Let T be the temperature of the body at time t.

By Newton's law of cooling, we have $\frac{dT}{dt} = -k(T - T_s)$

$$\Rightarrow \frac{dT}{dt} = -k(T - 40) \quad ---(1)$$

Integrating on both sides, we get $\int \frac{dT}{T-40} = -k \int dt$

$$\Rightarrow log\,(T - 40) = -kt + logc$$

$$\Rightarrow log\,(\frac{T-40}{c}) = -kt$$

$$\Rightarrow \frac{T-40}{C} = e^{-kt}$$

$$\Rightarrow T - 40 = c\, e^{-kt} \quad ---(2)$$

When t=0, T= 80^0 and when t=20,T=60^0

Substituting in (2) we get c=40 and 20 = c e^{-20k}

$$\Rightarrow \frac{20}{40} = e^{-20k}$$

$$\Rightarrow k = \frac{1}{20} log\ 2$$

Then (2) becomes $T - 40 = 40\ e^{-\left(\frac{1}{20}log\ 2\right)t}$

When t=40, $T = 40 + 40\ e^{-\left(\frac{1}{20}log\ 2\right)40} = 40 + 40e^{-2log2} = 50^0C$

2. If the temperature of a body is changing from 100^0C to 70^0C in 15 minutes, find when the temperature will be 40^0 C, if the temperature of air is 30^0C.

Sol: By Newton's law of cooling, we have $\frac{dT}{dt} = -k(T - T_s)$

$$\Rightarrow \frac{dT}{dt} = -k(T - 30) \qquad \text{---(1)}$$

Integrating on both sides, we get $\int \frac{dT}{T-30} = -k \int dt$

$$\Rightarrow log\ (T - 30) = -kt + c$$

At t=0, T=100^0C , Then c= log 70^0

Substituting in (2) we get log (T-30^0) = -kt + log 70 ---(2)

Then kt = log (70) – log (T-30)

Again when t=15, T = 70^0 ;

15 k = log 70^0 – log 40^0 --- (3)

Dividing (2) with (3), we get $\frac{t}{15} = \frac{log\ 70 - log\ (T-30)}{log\ 70 - log\ 40}$

When T= 40^0; $\frac{t}{15} = \frac{log\ 70 - log\ 10}{log\ 70 - log\ 40} = \frac{log\ 7}{log\ \frac{7}{4}} = 3.48$

Then t=52.20

The temperature will be 40^0C after 52.2 minutes

3. If a substance cools from 370K to 330K in 10 minutes,when the temperature of the surrounding air is 290K, find the temperature of the substance after 40 minutes.

Sol: Let T be the temperature of the body at time t.

By Newton's law of cooling, we have $\frac{dT}{dt} = -k(T - T_s)$

$$\Rightarrow \frac{dT}{dt} = -k(T - 290) \qquad \text{---(1)}$$

Integrating on both sides, we get $\int \frac{dT}{T-290} = -k \int dt$

$$\Rightarrow log\ (T - 290) = -kt + logc$$

$$\Rightarrow log\ (\frac{T-290}{c}) = -kt$$

$$\Rightarrow \frac{T-290}{C} = e^{-kt}$$

$$\Rightarrow T - 290 = c\ e^{-kt} \qquad \text{---(2)}$$

When t=0, T= 370^0 and when t=10,T=330^0

Substituting in (2) we get c=80 and 40 = c e^{-10k}

$$\Rightarrow \frac{40}{80} = e^{-10k}$$

$$\Rightarrow k = \frac{1}{10} log\ 2$$

Then (2) becomes $T - 290 = 80\ e^{-\left(\frac{1}{10}log\ 2\right)t}$

When t=40,

$$T = 290 + 80\ e^{-\left(\frac{1}{10}log\ 2\right)40} = 290 + 80e^{-4log2} = 295^0C$$

4. A cup of tea at temperature 90^0C is placed in a room with temperature as 25^0C and it cools to 60^0C in 5 minutes. Find its temperature after a interval of 5 minutes. Also find the time at which the temperature of tea will come down further by 20^0C.

Sol: Let T be the temperature of the body at time t.

By Newton's law of cooling, we have $\frac{dT}{dt} = -k(T - T_s)$

$$\Rightarrow \frac{dT}{dt} = -k(T - 25) \quad \text{---(1)}$$

Integrating on both sides, we get $\int \frac{dT}{T-25} = -k \int dt$

$$\Rightarrow log\ (T - 25) = -kt + logc$$

$$\Rightarrow log\ (\frac{T-25}{c}) = -kt$$

$$\Rightarrow \frac{T-25}{C} = e^{-kt}$$

$$\Rightarrow T - 25 = c\ e^{-kt} \quad \text{---(2)}$$

When t=0, T= 90^0 and when t=5,T=60^0

Substituting in (2) we get c=65 and 35 = c e^{-5k}

$$\Rightarrow \frac{35}{65} = e^{-5k}$$

$$\Rightarrow k = \frac{1}{5} log\ \frac{13}{7}$$

Then (2) becomes $T - 25 = 65\ e^{-\left(\frac{1}{5}log\ \frac{13}{7}\right)t}$

When t=10, $T = 25 + 18.8 = 43.8^0C$

Temperature of tea after a further interval of 5 minutes=43.8^0C

Now when T=60-20=40, we get 15=65e^{-kt}

$$e^{kt}=65/15$$

$$\Rightarrow t = \frac{1}{k} log\ \frac{13}{3} = \frac{5\ \log(13/3)}{log(13/7)} = 11.8\ min$$

5. An object whose temperature is 75^0C cools in an atmosphere of constant temperature 25^0C at the rate of kθ, θ being the excess temperature of the body over that of the temperature. If after 10 minutes, the temperature of the object falls to 65^0, find the temperature after 20 minutes. Also find the time required to cool down to 55^0.

Sol: We will take one minute as unit of time. It is given that $\frac{d\theta}{dt} = -k\theta$ ---(1)

Solution of (1) is $\theta = ce^{-kt}$ ---(2)

Initially when t=0, θ=75-25=50. Hence c=50

(2)$\Longrightarrow \theta = 50e^{-kt}$ ---(3)

When t=10, θ=65-25=40. Hence 40 = 50 e^{-10t} i.e.,e^{-10t}=4/5 ---(4)

Putting t=20 in (3), we get

$$\theta = 50e^{-20t} = 50(e^{-10k})^2 = 50(\tfrac{4}{5})^2$$

Hence the temperature after 10 minutes =32+25=57

EXERCISE

(1) A body is originally at 140^0C and cools down to 80^0C in 20 minutes. If the temperature of the air is 30^0C, find when the temperature of water will become 35^0c.

(2) Water at temperature 100^0c cools in 10 minutes to 80^0C in a room of temperature 30^0C. Find i)the temperature of water after 20 minutes ii)the time when the temperature is 40^0C

ANSWERS

(1)60 minutes
(2)(i)58.5^0C (ii)57 minutes

1.18 L-R CIRCUIT

Let us consider the RL (resistor R and inductor L) circuit. At t = 0 the switch is closed and current passes through the circuit. Electricity laws state that

i) The voltage across a resistor of resistance R is equal to R i and

ii) The voltage across an inductor L is given by L di/dt (i is the current).

By Kirchhoff's law, the total potential drop(or voltage drop) in the circuit is equal to the applied voltage(emf) i.e.,

$$L\frac{di}{dt} + iR = E$$

Where E is a constant voltage.

By voltage law,

$$L\frac{di}{dt} + iR = E$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}, \qquad \textit{which is a linear equation}$$

Its solution is given by i = $\frac{E}{R} + ce^{\frac{Rt}{L}}$

At t = 0, i = 0, c= -$\frac{E}{R}$

$\therefore i = \frac{E}{R}\left(1 - e^{\frac{Rt}{L}}\right)$

PROBLEMS

1. A voltage E e^{-at} is applied at t=0 to a LR circuit. Find the current at any time t.

Sol: The differential equation governing the LR circuit is

$$\frac{di}{dt}+\frac{R}{L}i=\frac{e.m.f}{L}=\frac{Ee^{-at}}{L} \text{---(1)}$$

Equation (1) is linear equation; Then IF=$e^{\int\frac{R}{L}dt}=e^{\frac{Rt}{L}}$

Therefore, the solution is

$$i(IF)=\int\frac{Ee^{-at}}{L}(\text{IF})\,dt+c$$

$$\text{i.}e^{\frac{Rt}{L}}=\int\frac{Ee^{-at}}{L}\left(e^{\frac{RT}{L}}\right)dt+c$$

$$=\frac{E}{L}\int e^{\frac{RT}{L}-at}dt+C$$

$$=\frac{E}{L}\frac{e^{\frac{RT}{L}-at}}{\frac{R}{L}-a}+C$$

$$=\frac{E}{R-aL}e^{\frac{RT}{L}-at}+C$$

And so $\quad i=\frac{E}{R-aL}e^{-at}+Ce^{-\frac{RT}{L}}$

Using the initial condition i(0)=0, we get C=$-\frac{R}{R-aL}$. Hence i = $\frac{E}{R-aL}\left[e^{-at}-e^{-\frac{RT}{L}}\right]$

2. A simple electrical circuit containing a resistance R, inductance L, and capacitance C in series. If the electromotive force E=v sin wt. Find the current i.

Sol: By Kirchhoff's law,

$$L\frac{di}{dt}+iR=E=v\sin wt$$

$$\frac{di}{dt}+\frac{R}{L}i=\frac{v}{L}\sin wt \qquad \text{---(1)}$$

Equation (1) is linear equation; Then

$$\text{IF}=e^{\int\frac{R}{L}dt}=e^{\frac{Rt}{L}}$$

$$\therefore i(IF)=\int\frac{V}{L}\sin wt(IF)dt+c$$

$$\text{i.}e^{\frac{Rt}{L}}=\int\frac{V}{L}\sin wt.e^{\frac{Rt}{L}}dt+c$$

$$\text{i.}e^{\frac{Rt}{L}}=\frac{V}{l}\left[\frac{R\sin wt-wL\cos wt}{R^2+W^2L^2}\right]\text{L}e^{\frac{Rt}{L}}+c$$

or $\quad$ i=v$\left[\frac{R\sin wt-wL\cos wt}{R^2+W^2L^2}\right]+ce^{-\frac{Rt}{L}}$

Initially, when t=0,i=i_0

$$c=i_0+\frac{VwL}{R^2+W^2L^2}$$

$$\therefore i=v\left[\frac{R\sin wt-wL\cos wt}{R^2+W^2L^2}\right]+\left[i_0+\frac{VwL}{R^2+W^2L^2}\right]e^{-\frac{Rt}{L}}$$

EXERCISE

(1) When a resistance R is connected in series with an inductance L henries, an emf of E volts, the current I amperes at time t is given by $L\frac{di}{dt} + iR = E$.If E=10 sin t volts and i=0, when t=0, find i as a function of t.

(2) Find the solution of the equation $L\frac{di}{dt} + iR = 200.\cos 300\, t$,when R = 100,L=0.05, and find i given that i=0 when t=0, what value does I approach after a long time.

(3) When a switch is closed in a circuit containing a battery E, a resistance R, and an inductance L, the current i builds up at rage given be $L\frac{di}{dt} + iR = E$,find i in terms of t.How long will it be before the current has reached one-half its maximum value if E=6 volts, R=100 ohms and L=0.1 henry.

ANSWERS

(1) i =$\frac{10}{L^2+R^2}\left(R\sin t - L\cos t + Le^{-R/L}\right)$

(2) i =$\frac{40}{409}(20\cos 300t + 3\sin 300t) - \frac{800}{409}e^{-200t}$; and $\frac{40}{\sqrt{409}}$

(3) 0.000693 seconds

LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER

DEFINITION:

An equation of the form $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + ... + P_n(x)y = Q(x)$ is called a linear differential equation of order n. where $P_1(x)$,$P_2(x)$,...$P_n(x)$ and Q(x) are all continuous and real valued functions of x .

2.1 LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Def. An equation of the form $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + ... + P_n y = Q(x)$

.... (A)

is called as an ordinary Linear Differential Equation of order n with constant coefficients.. Where P_1, P_2....P_n are real constants and Q(x) is a continuous function of x

Complete solution: To find the complete solution of the differential equation of n^{th} order with constant coefficients.

$$\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + ... + a_n y = X \qquad .....(B)$$

F(D) y = X

y=complementary function +Particular integral $y = C.F. + P.I$

2.2 AUXILIARY EQUATION:

Consider the differential equation,

$$(D^n + a_1 D^{n-1} + a_2 D^{n-1} + ... + a_n)y = 0 \qquad ....(1)$$

Let $y=e^{mx}$ is solution of (1)

Then $Dy=me^{mx}$

$D^2y=m^2e^{mx}$

$D^ny=m^ne^{mx}$

Substituting above values in (1) we get , $(m^n + a_1m^{n-1} + a_2m^{n-2} + ... + a_n)e^{mx} = 0$

$m^n + a_1m^{n-1} + a_2m^{n-1} + ... + a_n = 0$ ....(2),$e^{mx} \neq 0$.

Equation (2) is called auxiliary equation (1)

Operator D: Let us denote $\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}$ ... with D, D^2, D^3 ... so that

$$Dy=\frac{d}{dx}(y), D^2y=\frac{d^2}{dx^2}(y), D^3y=\frac{d^3}{dx^3}(y)......$$

The equation (1) can now be written in the symbolic form as

$(D^n+P_1D^{n-1}+P_2D^{n-2}+.....+P_n)y=Q(x)$

$i.e. f(D)y=Q(x)$

Where $f(D)=D^n+P_1D^{n-1}+P_2D^{n-2}+....+P_n$ is a polynomial in D . The symbol D stands for the operation of differentiation. $f(D)$ follows the usual rules of algebra .

Procedure to find complementary function of $f(D)y=X$:

- ✓ **Form the Auxiliary equation f(m)=0**
- ✓ **Solve the Auxiliary equation f(m)=0 to get the roots m_1, m_2.....m_n .**
- ✓ **Corresponding to these roots write the terms in C.F using the following table.**

S .No	Roots of A.E f(m)=0	C.F. (Complementary Function)
1	$m_1,m_2,m_3,....m_n$ *i.e.*, all roots are real and distinct	$C_1e^{m_1x}+C_2e^{m_2x}+....+C_ne^{m_nx}$
2.	$m_1,m_2,m_3,....m_n$ (i.e., two roots are real and equal and rest are real and different).	$(C_1+C_2x)e^{m_1x}+C_3e^{m_3x}+C_ne^{m_3x}+.....+C_ne^{m_nx}$
3.	$m_1,m_2,m_3\ m_4\ ,...m_n$ (*i.e.*, three roots are real and equal and rest are real and different)	$(C_1+C_2x+C_3x^2)e^{m_1x}+C_4e^{m_4}+.....+C_ne^{m_nx}$
4.	Two roots of A.E are complex say $\alpha\pm i\beta$ and the remaining roots are real and different.	$e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)+C_3e^{m_3x}+....+c_ne^{m_nx}$
5.	A pair of conjugate complex roots $\alpha\pm i\beta$ are repeated thrice and the remaining roots are real and different.	$e^{\alpha x}[(C_1+C_2x+C_3x^2)\cos\beta x+(C_4+C_5x+C_6x^2)\sin\beta x]+C_7e^{m_7x}+Cge^{m_gx}+....+C_ne^{m_nx}$

1. Solve $\frac{d^4y}{dt^4}+4y=0$**.**

Sol: Writing the operator form $(D^4+4)x=0$

A.E is $m^4+4=0 \Rightarrow (m^4+4m^2+4)-4m^2=0$

$\Rightarrow (m^2+2)^2-(2m)^2=0$

$\Rightarrow (m^2+2+2m)(m^2+2-2m)\ [\because a^2-b^2=(a+b)(a-b)]$

$$\Rightarrow m = \frac{-2 \pm \sqrt{-4}}{2}, \frac{2 \pm \sqrt{-4}}{2}$$

$$\Rightarrow m = -1 \pm i \ ,1 \pm i$$

$$y = e^{-t}[c_1 \cos t + c_2 \sin t] + e^{t}[c_3 \cos t + c_4 \sin t]$$

2. Solve $[D^2 - (a+b)D + ab]y = 0$ **a and b being real and unique.**

Sol: A.E is $m^2 - (a+b)m + ab = 0$

$m(m-a) - b(m-a) = 0$

$(m-a)(m-b) = 0 \Rightarrow m = a, b$

$y = c_1 e^{ax} + c_2 e^{bx}$

3. Solve $\dfrac{d^2 y}{dx^2} - a^2 y = 0, a \neq 0$

Sol: Given equation in the operator form is

$(D^2 - a^2)\ y = 0$ ....(1)

Let $f(D) = D^2 - a^2$.

Auxiliary equation (A.E) is *f(m)*=0

$\Rightarrow m^2 - a^2 = 0 \therefore m = \pm a.$

The roots are real and different.

$\therefore$ The general solution of (1) is $y = C_1 e^{ax} + C_2\ e^{-ax}$

Where C_1, C_2 are arbitrary constants.

4.Solve $\dfrac{d^3 y}{dx^3} - 9\dfrac{d^2 y}{dx^2} + 23\dfrac{dy}{dx} - 15y = 0$

Sol: Given equation in the operator form is

$(D^3 - 9D^3 + 23D - 15)y = 0$ ....(1)

Let $f(D) = D^3 - 9D^2 + 23D - 15$

Auxiliary equation is *f(m)*=0

$\Rightarrow m^3 - 9m^2 + 23m - 15 = 0$ ...(2)

$\Rightarrow (m-1)(m-3)(m+5) = 0$

$\therefore m = 1, 3, -5$

The roots are 1, 3, -5.

The roots are real and different and hence the general solution is

$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} = c_1 e^{x} + c_2 e^{3x} + c_3 e^{-5x}$, where c_1, c_2, c_3 are arbitrary constants.

5. Solve $\frac{d^3x}{dt^3}-2\frac{d^2x}{dt^2}-3\frac{dx}{dt}=0$

Sol: Given equation can be written as $(D^3-2D^2-3D)x=0$ ....(1)

Where $D=\frac{d}{dt}$.Let $f(D)=D^3-2D^2-3D$

Auxiliary equation is $m^3-2m^2-3m=0$ ....(2)

$\Rightarrow m(m^2-2m-3)=0 \Rightarrow m(m-3)(m+1)=0$

The roots are m=0,3, and -1.The general solution of (1) is $x=c_1+c_2e^{3t}+c_3e^{-x}$

6. Solve $(D^4-2D^3-3D^2+4D+4)y=0$.

Sol: Given equation is

$(D^4-2D^3-3D^2+4D+4)y=0$ .... (1)

Let $f(D)=D^4-2D^3–3D^2+4D+4$

The AE is $f(m)=0$

i.e., $m^4-2m^3-3m^2+4m+4=0$ .....(2)

$\Rightarrow (m+1)(m^3-3m^2+4)=0$

$\Rightarrow (m+1)(m+1)(m^2-4m+m)=0$

$\Rightarrow (m+1)^2(m-2)^2=0$

The roots are m= -1,-,2, 2. Hence the general solution of (1) is

$y= (c_1+c_2x)e^{-x}+(c_3+c_4x)e^{2x}$.

7.Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=0$

Sol: Given equation in operator form is $(D^2+D+1)y=0$ .....(1)

Let $f(D)=D^2+D+1$

A.E. is $f(m)$=0 i.e.,$m^2+m+1=0 \Rightarrow m=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm i\sqrt{3}}{2}=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}=\alpha\pm i\beta$

The roots are complex and conjugate.

$\therefore$ The general solution of (1) is $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

i.e., $y=e^{\frac{-x}{2}}\left(c_1\cos\frac{\sqrt{3}x}{2}+c_2\sin\frac{\sqrt{3}x}{2}\right)$

8. Solve $(D^4+8D^2+16)y=0.$

Sol: Given equation $(D^4+8D^2+16)y=0.$ ....(1)

Let $f(D)=D^2+8D^2+16$

The AE is $f(m)=0$ .....(2)

(i.e.,)$m^4+8m^2+16=0 \Rightarrow (m-2i)^2(m+2i)^2=0$

The roots of (2) are m=2i,2i,-2i,-2i where 2i,-2i, are occurring twice.

$\therefore$ The general solution of (1) is $y= (c_1+c_2x)\cos 2x+ (c_1+c_4x)\sin 2x$.

Note: If $\alpha+\sqrt{\beta}$ is a real irrational root of $f(m)$ =0, $\alpha-\sqrt{\beta}$ is also a root of the equation. The part of the

complementary function corresponding to these rots can also be put in the form.

$$e^{\alpha x}(c_1\cosh\sqrt{\beta}x+d_1\sinh\sqrt{\beta}x).$$

9. Find the general solution of $y''+2y'=0$

Sol: Given equation in the symbolic form is $(D^2+2D)y=0$

The A.E. is $m^2+2m=0$

i.e., $m(m+2)=0 \Rightarrow m=0,-2$

$\therefore$ The general solution is

$y=c_1e^{0.x}+c_2e^{-2x}=c_1+c_2e^{-2x}$ where c_1,c_2 are constants.

10. Solve $y''+6y'+9y=0, y(0)=-4, y'(0)=14.$

Sol: Given equation in the standard form is $(D^2+6D+9)y=0$

The A.E. is $(m+3)^2=0$

$\therefore$ $m=-3,-3$. Roots are real and equal.

$\therefore$ The general solution is $y=(c_1+c_2x)e^{-3x}$ ...(1)

Diff.(1) w.r.t.x, we get

$y'=(c_1+c_2x)(-3e^{-3x})+e^{-3x}(c_2)$

Given $y'(0)=14$

$\Rightarrow 14=-3c_1+c_2$ ....(2)

And y(0)=-4

$\Rightarrow -4=c_1$ ....(3)

From (2) and (3), we get c_1 and c_2 in(1), we get $y=(-4+2x)e^{-3x}$ which is the required solution.

11.Solve $y''-y'-2y=0$.

Sol: Given D.E. can be written in operator form as $(D^2-D-2)y = 0$

Auxiliary equation is $f(m)=0$.

$\Rightarrow m^2-m-2=0 \Rightarrow m^2+m-2=0$

$\Rightarrow (m+1)(m-2)=0$

$\therefore m=2,-1$

$\therefore$ General solution is $y=c_1e^{2x}+c_2e^{-x}$ where c_1 and c_2 are constants.

EXERCISE

1. solve $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y=0$

2) solve $\frac{d^3y}{dx^3}-5\frac{d^2y}{dx^2}+8\frac{dy}{dx}-4y=0$

3) solve $\frac{d^3y}{dx^4}-8y=0$

4) Solve $\frac{d^4y}{dx^4}-\frac{d^3y}{dx^3}-9\frac{d^2y}{dx^2}-11\frac{dy}{dx}-4y=0$

5) Solve $\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}-8\frac{dy}{dx}=0$

ANSWERS

1. $y=c_1e^{x}+c_2e^{2x}+c_3e^{3x}$
2. $y=(c_1x+c_2)e^{2x}+c_3e^{x}$
3. $y=e^{-x}(c_1+c_2\cos\sqrt{3}x+c_2\sin\sqrt{3}x)+c_3e^{2x}$
4. $y=(c_1+c_2x+c_3x^2)e^{-x}+c_4e^{4x}$
5. $y=c_1+c_2e^{2x}+c_3e^{-4x}$

2.3 INVERSE OPERATION

The operator D^{-1} is called inverse of the differential operator D.

Def.1: If X is any function of x then $D^{-1}X$ or $\frac{1}{D}X$ is called the integral of X.

We write $\frac{1}{D}X=\psi$ to mean D ψ =X

e.g. $\frac{1}{D}(\cos 3x)=\int\cos 3x\,dx=\frac{\sin 3x}{3}$ $\quad$ *Since* $D\left(\frac{\sin 3x}{3}\right)=\cos 3x$

Definition: Inverse Operator $\frac{1}{f(D)}$:

$\frac{1}{f(D)}$ is a function of x which when operated upon by f(D) gives x is

$$f(D)\frac{1}{f(D)}X=X$$

$\therefore (D), \frac{1}{(D)}$ are inverse operators. Then the P.I can be written as y= $\frac{1}{f(D)}$ X

Formula for $\frac{1}{D-a}X$:

Let $\frac{1}{D-a}X = y$

Operating both sides by (D-a)

$$(D-a)\frac{1}{D-a}X = (D-a)y$$

$$X = (D-a)y \Rightarrow \frac{dy}{dx} - ay = X$$

Note:

$$\frac{1}{D-a}X = e^{ax}\int Xe^{-ax}dx$$

Similarly $\frac{1}{D+a}X = e^{-ax}\int Xe^{ax}dx$

Def.: If $\frac{1}{D-\beta}, \frac{1}{D-\alpha}$ are two inverse operators , then we define

$$\frac{1}{(D-\beta)(D-\alpha)}X = \frac{1}{(D-\beta)}\left[\frac{1}{D-\alpha}X\right]$$

When α and β are constants and Q is a function of x.

$$i.e., \frac{1}{(D-\beta)(D-\alpha)}X = \frac{1}{(D-\beta)}\left[e^{ax}\int Xe^{-\alpha x}dx\right] = e^{\beta s}\int e^{-\beta x}\left\{e^{\alpha x}\int Xe^{-\alpha x}dx\right\}dx$$

1. Find(i)$\frac{1}{D}(x^2)$ (ii) $\frac{1}{D^3}(cosx)$.

Sol: (i) $\frac{1}{D}(x^2) = \int x^2 dx = \frac{x^3}{3}$

(ii) $\frac{1}{D^3}(\cos x) = \frac{1}{D^2}\left(\frac{1}{D}\cos x\right) = \frac{1}{D^2}\left(\int \cos x dx\right) = \frac{1}{D^2}(\sin x)$

$$= \frac{1}{D}\left(\frac{1}{D}\sin x\right) = \frac{1}{D}\left(\int \sin x dx\right) = \frac{1}{D}(-\cos x) = -\int \cos x dx = -\sin x$$

2. Find the particular value of $\frac{1}{D+1}(x)$.

Sol: $$\frac{1}{D+1}(x) = e^{-x}\int xe^{x}dx = e^{-x}(xe^{-x} - e^{x}) = x-1$$

3. Find particular value of $\frac{1}{(D-2)(D-3)}e^{2x}$.

Sol: $$\frac{1}{(D-2)(D-3)}e^{2x} = \frac{1}{(D-2)}\left[\frac{1}{D-3}e^{2x}\right]$$

Now $\frac{1}{D-3}(e^{2x}) = e^{3x}\int e^{2x}e^{-3x}dx = e^{3x}(-e^{-x}) = -e^{2x}$

$$\therefore \frac{1}{D-2}\left[\frac{1}{D-3}e^{2x}\right] = \frac{1}{D-2}(-e^{2x}) = -e^{2x}\int e^{2x}e^{-2x}dx = -e^{2x}\int dx = -xe^{2x}$$

Ex : $\frac{1}{D^2+3D+2}(e^{4x}) = \frac{e^{4x}}{30}$

Since $(D^2+3D+2)\left(\frac{e^{4x}}{30}\right) = \frac{16e^{4x}}{30} + \frac{12e^{4x}}{30} + \frac{2e^{4x}}{30} = \frac{30e^{4x}}{30} = e^{4x}$

Ex: $\frac{1}{D+2}(cos3x)$=$e^{-2x}\int e^{2x}cos3x\,dx$ since *(D+2) sin 3x= 3cos 3x+2sin3x* and is not equal to cos3x.

Note: The above method to find particular integral (P.I) is a general method and it will be useful when X is not the form tan ax, cot ax, sec ax, cosec ax.

Examples

1. Solve $\frac{1}{D-3}x = e^{3x}\int xe^{-3x}dx$

$$= e^{3x}\left[-x\frac{e^{-3x}}{3} - \frac{e^{-3x}}{9}\right] = \frac{-x}{3} - \frac{1}{9}$$

2. Find the particular integral of $(D^2-4D+3)y = e^{2x}$

$$P.I : \frac{1}{D^2-4D+3}e^{2x} = \frac{1}{2}\left[\frac{1}{D-3} - \frac{1}{D-1}\right]e^{2x}$$

$$= \frac{1}{2}\left[\frac{1}{D-3}e^{2x} - \frac{1}{D-1}e^{2x}\right] = \frac{1}{2}\left[e^{3x}\int e^{-3x}e^{2x}dx - e^{x}\int e^{-x}e^{2x}dx\right]$$

$$= \frac{1}{2}\left[e^{3x}\int e^{-x}dx - e^{x}\int e^{x}dx\right] = \frac{1}{2}\left[e^{3x}(-e^{-x}) - e^{x}e^{x}\right] = -e^{2x}$$

2.4 GENERAL SOLUTION OF *f(D)y=Q(x)*

We know that if $y=y_p$ is a particular solution of $f(D)y=Q(x)$ containing no arbitrary constant and $y=y_c$ is the general solution of $f(D)y=0$ then $y=y_c + y_p$ is the general solution $f(D)y=Q(x)$.

We have previously discussed the methods to find the general solution of $f(D)y=0$.

If $y=y_c$ is the general solution of $f(D)y=0$ then we know that y_c is the C.F. $f(D)y=Q(x)$.

- **Particular Integral of *f(D)y=Q(x)*.**

Given equation is

$f(D)\, y = Q(x)$. .....(1)

operating (1) by $\frac{1}{f(D)}$, we get

$$\frac{1}{f(D)}[f(D)y] = \frac{1}{f(D)}Q(x) \Rightarrow y = \frac{1}{f(D)}Q(x)$$

Clearly (1) is satisfied ,if we take $y = \frac{1}{f(D)}Q(x)$

Thus particular integral =P.I = $\frac{1}{f(D)}Q(x)$

Note1. To find the P.I of $f(D)y=Q(x)$, we find the value of $\frac{1}{f(D)}Q(x)$

Note2. P.I of $f(D)y=Q(x)$ *contains no arbitrary constants.*

GENERAL METHOD OF FINDING PARTICULAR INTEGRAL:

- **P.I of*f(D) y=Q(x)* , *when* $\frac{1}{f(D)}$ is expressed As Partial Fractions.**

Let $f(D) = (D-\alpha_1)(D-\alpha_2)....(D-\alpha_3)$

P.I =

$$\frac{1}{f(D)}Q = \frac{1}{(D-\alpha_1)(D-\alpha_2)....(D-\alpha_3)}Q = \left[\frac{A_1}{D-\alpha_1} + \frac{A_2}{D-\alpha_2} + ... + \frac{A_n}{D-\alpha_n}\right]Q$$

(resolving into partial fraction)

$$= A_1 e^{\alpha_1 x}\int Q e^{-\alpha_1 x}dx + A_2 e^{\alpha_2 x}\int Q e^{-\alpha_x x}dx + ... + A_n e^{\alpha_n x}\int Q e^{-\alpha_n x}dx$$

1. Solve $(D^2-5D+6)y = xe^{4x}$.

Sol: Given equation is $(D^2-5D+6)y=xe^{4x}$ ....(1)

Let $f(D)=D^2-5D+6$

The AE is $f(m)=0$.i.e., $m^2-5m+6=0$ ....(2)

The roots of (2) are $m=2,3$. $\therefore y_c=C.F\ of\ (1)=c_1e^{2x}+c_2e^{3x}$

$$y_p=P.I.\ of\ (1)=\frac{xe^{4x}}{D^2-5D+6}=\left(\frac{1}{D-3}-\frac{1}{D-2}\right)xe^{4x}$$

$$=\frac{1}{D-3}xe^{4x}-\frac{1}{D-2}xe^{4x}=e^{3x}\int xe^{4x}e^{-3x}dx-e^{2x}\int xe^{4x}e^{-2x}dx$$

$$=e^{3x}\int xe^{x}dx-e^{2x}\int xe^{2x}dx \text{ [Integration by parts]}$$

$$=e^{3x}(xe^x-e^x)-e^{2x}\left(\frac{xe^{2x}}{2}-\frac{e^{2x}}{4}\right)=e^{4x}\frac{(2x-3)}{4}$$

$\therefore$ The general solution of (1) is

$$y=y_c+y_p=c_1e^{2x}+c_2e^{3x}+\frac{1}{4}e^{4x}(2x-3)$$

2. Solve $(D^2-4D+3)y=e^{e^x}$.

Sol: Given equation is $(D^2+4D+3)y=e^{e^x}$

A.E. is $m^2+4m+3\Rightarrow(m+3)(m+1)=0$

$\therefore m=-3,-1$

$\therefore$ Roots are real and different.

Hence C.F. is $y_c=c_1e^{-3x}+c_2e^{-x}$

Now P.I. is $y_p=\frac{1}{(D+3)(D+1)}(e^{e^x})=\frac{1}{2}\left[\frac{1}{D+1}-\frac{1}{D+3}\right](e^{e^x})$

$$\therefore y_p=\frac{1}{2}\left[\frac{1}{D+1}(e^{e^x})-\frac{1}{D+3}(e^{e^x})\right]=\frac{1}{2}\left[P.I_1-P.I_2\right]$$

We know that $\frac{1}{D-\alpha}(e^{e^x})=e^{-x}\int(e^{e^x})e^xdx$ [Put $e^x=t:e^xdx=dt$]

$$=e^{-x}\int e^t\,dt=e^{-x}e^t=e^{-x}e^{ex}$$

And P.I.$_2$ $\frac{1}{D+3}(e^{e^x})=e^{-3x}\int(e^{e^x})e^{3x}dx=e^{-3x}\int e^tt^2dt$ [Put $e^x=t:e^xdx=dt$]

Hence $y_p = \frac{1}{2}[P.I_1 - P.I_2] = e^{-3x} e^{t}(t^2 - 2t + 2) = e^{-3x} e^{e^x} (e^{2x} - 2e^x + 2)$

$= \frac{1}{2} e^{e^x} [e^{-x} - e^{-x} + 2e^{-2x} - 2e^{-3x}]$

$\therefore$ The general solution is $y = y_c + y_p$

$\therefore y = c_1 e^{-3x} + c_2 e^{-x} + e^{e^x} (e^{-2x} - e^{-3x})$

2.5 RULES FOR FINDING PARTICULAR INTEGRAL IN SOME SPECIAL CASES

- P.I. of $f(D)\, y = \phi(x)$ *when* $\phi(x) = e^{ax}$, *where 'a' is constant.*

  Case I: Let $f(D)y = e^{ax}$. Then

  $y_p = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ if $f(a) \neq 0$. Since

  $f(D)\left\{\frac{1}{f(a)} e^{ax}\right\} = \frac{1}{f(a)} f(D)(e^{ax}) = \frac{f(a)}{f(a)} e^{ax} = e^{ax}$,

$$\therefore \frac{1}{f(D)}(e^{ax}) = \frac{1}{f(a)} e^{ax} \quad \text{if } f(a) \neq 0$$

$$\therefore \frac{1}{f(D)}(e^{ax}) = \frac{1}{f'(a)} x e^{ax} \text{ if } f(a) = 0,\ f'(a) \neq 0 \quad \text{etc}$$

Case II: Let *f(a) =0.then (D-a) is a factor of f(D) .If a is a root repeated K times for f(a)=0 then* $f(D) = (D-a)^k \phi(D)$ *where* $\phi(a) \neq 0$. *Then we have.*

$$\frac{1}{f(D)} e^{ax} = \frac{1}{(D-a)^k} \frac{1}{\phi(D)} (e^{ax}) = \frac{1}{\phi(a)} \frac{e^{ax}}{(D-a)^k} = \frac{1}{\phi(a)} e^{ax} . \frac{x^k}{k!}$$

The reason is as follows:

$$\frac{1}{(D-a)} e^{ax} = x e^{ax} \quad \text{since} \quad (D-a)(x e^{ax}) = e^{ax}$$

$$\frac{1}{(D-a)^k} e^{ax} = \frac{x^k}{k!} \quad \text{since} \quad (D-a)^k \left(\frac{x^k e^{ax}}{k!}\right) = e^{ax}$$

Hence $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{\phi(a)} . \frac{x^k}{k!}$ *If f(a) =0 and* $\phi(a) \neq 0$.

Working Rule

(1) In f (D), put D = a and P.I. will be a calculated.

$\therefore$ P.I. $= \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ provided $f(a) \neq 0$

(2) If $f(a)=0$, then above method fails. Now proceed as below.

If $f(D)=(D-a)^m \phi(D)$ (i.e., 'a' is a repeated root m times), then

P.I.$= \frac{e^{ax}}{\phi(a)} . \frac{x^m}{m!}$, $\phi(a) \neq 0$

Note: In order to find the P.I. if sinh ax or cosh ax express them as $\frac{1}{2}(e^{ax} - e^{-ax})$ and $\frac{1}{2}(e^{ax} + e^{-ax})$ respectively.

1.Solve $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4y = e^{3x}$**.**

Sol: Given equation can be written in operator form as $(D^3 - 3D^2 + 4)y = e^{3x}$

A.E is $m^3+3m^2+4=0$

$(m+1)(m-2)(m-2)=0$

$m=-1,2,2$

$y_c = C.F : c_1e^{-x} + (c_2 + c_3x)e^{2x}$

$y_p = P.I = \frac{1}{f(D)} X = \frac{1}{D^3 - 3D^2 + 4} e^{3x}$ Put D=3

$\frac{1}{3^3 - 3x3^2 + 4} e^{3x} = \frac{e3x}{4}$

Complete solution $y = yc + yp$

$y = c_1e^{-x} + (c_2 + c_3x)e^{2x} + \frac{e^{3x}}{4}$

2. Solve $(D^3 + 3D^2 + 3D + 1)y = e^{-x}$**.**

Sol: A.E: $m^3 + 3m^2 + 3m + 1 = 0$

$(m+1)^3 = 0 \Rightarrow m = -1,-1,-1$

$y_c = C.F = (c_1 + c_2x + c_3x^2)e^{-x}$

$y_p = P.I = \frac{1}{f(D)} X = \frac{1}{D^3 + 3D^2 + 3D + 1} e^{-x}$ Put D=-1 [$\therefore$ f(-1)=0]

$= \frac{x}{3D^2 + 6D + 3} e^{-x}$ Put D=-1

$$= \frac{x^2}{6D+6} e^{-x} \quad \text{Put D=-1}$$

$$= \frac{x^3}{6} e^{-x}$$

$$y = y_c + y_p = (c_1 + c_2 x + c_3 x^2) e^{-x} + \frac{x^3}{6} e^{-x}$$

3. Solve $(D^3 + 6D^2 + 11D - 6)y = 1 + e^{-2x}$**.**

Sol: A.E is $m^3 + 6m^2 + 11m - 6 = 0$

$$(m-1)(m^2 - 5m + 6) = 0$$

$$= (m-1)(m-2)(m-3) = 0$$

$$y_c = C.F = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

$$y_p = P.I = \frac{1}{f(D)} X = \frac{1}{(D-1)(D-2)(D-3)} (1 + e^{-2x})$$

$$= \frac{1}{(D-1)(D-2)(D-3)} e^{0x} + \frac{1}{(D-1)(D-2)(D-3)} e^{-2x}$$

$$= \frac{1}{(-1)(-2)(-3)} e^{0x} + \frac{1}{(-3)(-4)(-5)} e^{-2x} = -\frac{1}{6} \frac{e^{-2x}}{60}$$

$\therefore$ General solution is,

$$y = y_c + y_p = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{e^{-2x}}{360}$$

4. Solve (i) $\frac{d^2 y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{2x}$ **(ii)** $(D^2 - 3D + 2)y = e^{5x}$**.**

Sol: (i) Given equation can be written in operator form as $(D^2 - 3D + 2)y = e^{2x}$ **....(1)**

Let $f(D) = D^2+4D+3$

The A.E is $f(m)=0$ *i.e.* $m^2+4m+3=0$ .....(2)

i.e., $(m+3)(m+1)=0. \therefore m=-3,-1$ the roots are real and distinct.

$\therefore$C.F.=y_c=$c_1 e^{-x} + c_2 e^{-3x}$

Now $y_p = P.I. = \frac{e^{2x}}{D^2 + 4D + 3} = \frac{e^{2x}}{2^2 + 4(2) + 3} = \frac{e^{2x}}{15}$ [Put D=2]

$\therefore$ The general solution of (1) is $y = yc + yp$

i.e., $y = c_1 e^{-x} + c_2 e^{-3x} + \frac{e^{2x}}{15}$, where c_1 and c_2 are constants.

(ii) Auxiliary Equation is $m^2-3m+2=0$

i.e., $(m-1)(m-2)=0 \therefore m=1,2.$

C.F. $=c_1e^x+c_2e^{2x}$

$$\text{P.I.} = \frac{1}{D^2-3D+2}e^{5x} \qquad \text{[Put D=5]}$$

$$\frac{1}{(5)^2-3(5)+2}e^{5x} = \frac{1}{25-15+2}e^{5x} = \frac{e^{5x}}{12}$$

$\therefore$ The general solution is

y=C.F.+P.I.=$c_1e^x+c_2e^{2x}+\dfrac{e^{5x}}{12}$, where c_1 and c_2 are constants.

5. Solve (i)$(4D^2-4D+1)y=100$. (ii) $(D^2-5D+6)y=4^{ex}+5$.

Sol: i) Auxiliary equation is

$4m^2-4m+1=0 \Rightarrow (2m-1)^2=0 \Rightarrow m=\frac{1}{2},\frac{1}{2}$

$\therefore$ C.F. $=(C_1+C_2\,x)e^{x/2}$

$$\text{Now P.I} = \frac{100}{4D^2-4D+1} = \frac{100.e^{0.x}}{(2D-1)^2} = \frac{100}{(0-1)^2} = 100$$

Hence the general solution is

y=C.F.+P.I.=$(C_1+C_2x)e^{x/2}+100$.

(ii) A.E. is $m^2-5m+6=0$

i.e., $(m-2)(m-3)=0 \therefore m=2,3$

Hence C.F.=$C_1e^{2x}+C_2e^{3x}$

$$\text{Now P.I.}=\frac{1}{D^2-5D+6}(4e^{5x}+5)$$

$$=4.\frac{1}{D^2-5D+6}e^{5x}+5\frac{1}{D^2-5D+6}e^{0.x}$$

$$=4.\frac{1}{25-25+6}e^{5x}+5.\frac{1}{0-0+6}e^{0.x}$$

$$=\frac{4}{6}e^{5x}+\frac{5}{6}=\frac{1}{6}(4e^{5x}+5)$$

Hence the general solution is

y=C.F.+P.I

=$C_1e^{2x}+C_2e^{3x}+\frac{1}{6}(4e^{5x}+5)$ where c_1 and c_2 are constants.

6. Solve $(D^3-5D^2+8D-4)y = e^{2x}$.

Sol: Given equation is $(D^3-5D^2+8D-4)y=e^{2x}$ ....(1)

Let $f(D)=D^3-5D^2+8D-4$

The A.E. is f(m) =0

i.e., $m^3-5m^2+8m-4=0$ or $(m-1)(m^2-4m+4)=0$

$\Rightarrow (m-1)(m-2)^2=0$ ....(2)

$\therefore$ the roots of (2) are m=1,2,2.

Thus C.F $=y_c=c_1 e^x+(c_2+c_3x)e^{2x}$.

Here $f(2)=0$. Let $\phi(D)=D-1$. Then $\phi(2)=2-1\neq 0$ and $m=2$.

$\therefore y_p = \frac{1}{(2-2)}.\frac{x^2}{2!}e^{2x} = \frac{x^2e^{2x}}{2}$ [$\therefore$ Case of failure f(2)=0]

$\therefore$ The general solution of (1)is $y = yc + yp$.

i.e., $y = c_1e^x + (c_2 + c_3x)e^{2x} + \frac{x^2e^{2x}}{2}$, where c_1,c_2,c_3 are constants.

7. Solve $(D^2 - 3D + 2)y = \cosh x$

Sol: Given equation is $(D^2 - 3D + 2)y = \cosh x$ **...(1)**

Let $f(D) = D^2 + 3D + 2$

The A.E is $f(m)=0$ i.e., $m^2-3m+2=0$ ......(2)

The roots are *m=1 and m=2 which are real and distinct.*

C.F. is $y_c = c_1e^x + c_2e^{2x}$

$$P.I. = y_p = \frac{1}{(D-2)(D-1)}\cosh x = \frac{1}{(D-2)(D-1)}\left[\frac{e^x + e^{-x}}{2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(D-2)(D-1)}e^x + \frac{1}{(D-2)(D-1)}e^{-x}\right] \quad ....(3)$$

Now $\frac{1}{(D-2)(D-1)}e^x = \frac{1}{1-2}xe^x = -xe^x$ ....(4)

Again $\frac{1}{(D-2)(D-1)}e^{-x} = \frac{e^{-x}}{(-1-2)(-1-1)} = \frac{e^{-x}}{6}$ .....(5)

$\therefore y_p = \frac{1}{2}\left[-xe^x + \frac{e^{-x}}{6}\right]$ [from (3),(4) and (5)]

The general solution of(1) is

$y = y_c + y_p \Rightarrow y = c_1e^x + c_2e^{2x} - \frac{1}{2}xe^x + \frac{1}{12}e^{-x}$, where C_1 and C_2 are constants.

8. Solve *$(D+2)(D-1)^2y=e^{-2x}+2\sinh x$.*

Sol: The given equation is

$(D+2)(D-1)^2y=e^{-2x}+2\sinh x$ ....(1)

This is of the form $f(D)y=e^{-2x}+2\sinh x$

A.E. is $f(m)=0 \Rightarrow (m+2)(m-1)^2=0$ $\therefore m=-2,1,1$

The roots are real and one root is repeated twice.

$\therefore$ C.F. is $y_c = c_1e^{-2x} + (c_2 + c_3x)e^x$

$$P.I. = \frac{e^{-2x} + 2\sinh x}{(D+2)(D-1)^2} = \frac{e^{-2x} + e^x - e^{-x}}{(D+2)(D-1)^2} \qquad \left[\therefore \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

$= y_{p1} + y_{p2} + y_{p3}$

Now $y_{p_1} = \dfrac{e^{-2x}}{(D+2)(D-1)^2}$

Hence $f(-2)=0$. Let $\phi(D)=(D-1)^2$. Then $\phi(2)\neq 0$ and $m=1$

$\therefore y_{p_1} = \dfrac{e^{-2x}x}{9} = \dfrac{xe^{-2x}}{9}$

$y_{p_2} = \dfrac{e^x}{(D+2)(D-1)^2}$ .Here $f(1)=0$

$= \dfrac{e^x x^2}{(3)2!} = \dfrac{x^2e^x}{6}$ [$\therefore \phi(D)=D+2; \phi(1)=3$ and $m=2$]

And $y_{p_3} = \dfrac{e^{-x}}{(D+2)(D-1)^2}$

Putting D= -1, we get $y_{p_3} = \dfrac{e^{-x}}{(D+2)(D-1)^2} = \dfrac{e^{-4}}{4}$

$\therefore$ General solution is $y = y_c + y_{p1} + y_{p2} + y_{p3}$

i.e., $y = c_1e^{-2x} + (c_2 + c_3x)e^x + \dfrac{xe^{-2x}}{9} + \dfrac{x^2e^x}{6} + \dfrac{x^{-x}}{4}$

9. Solve $(D^2+5D+6)y = e^x$

Sol: A.E is $m^2+5m+6=0$

$\Rightarrow (m+3)(m+2)=0$

$\therefore m=-3$ or $m=-2$

The roots are real and different.

$\therefore$ C.F $= y_c = c_1e^{-3x} + c_2e^{-2x}$

Now P. I. $= y_p = \dfrac{e^x}{D^2 + 5D + 6}$ [Put D=1]

$$\frac{e^x}{1+5+6} = \frac{e^x}{12}$$

General solution is $y = yc + yp$

$$\therefore y = c_1 e^{-3x} + c_2 e^{-2x} + \frac{e^x}{12}.$$

10. Solve $y'' - 4y' + 3y = 4e^{3x}, y(0) = -1, y'(0) = 3$.

Sol: Writing the given differential equation in operator form,

$(D^2 - 4D + 3)y = 4.e^{3x}$ .....(1)

A.E is $m^2-4m+3=0$

$\Rightarrow (m-3)(m-1)=0 \Rightarrow m=3 \text{ or } 1$

$\therefore$The roots are real and different.

C.F. $= y_c = c_1 e^{3x} + c_2 e^x$ .....(2)

P.I= $y_p = \frac{4e^{3x}}{D^2 + 4D + 3} = \frac{4e^{3x}}{(D-1)(D-3)} = \frac{1}{D-3}.\frac{4e^{3x}}{D-1} [\because f(3) = 0]$

$$\therefore y_p = \frac{4xe^{3x}}{3-1} = 2xe^{3x}$$

$\therefore$ The general solution is $y = yc + yp$

i.e., $y=c_1e^{3x}+c_2e^x+2xe^{3x}$ ...(3)

Differentiating (3) w.r.t 'x' , we get

$y' = 3c_1 e^{3x} + c_2 e^x + 2e^{3x} + 6xe^{3x}$ ....(4)

Given $y(0) = -1, y'(0) = 3$

Form (3), $-1 = c_1+c_2$ ....(5)

Form (4), $3=3c_1+c_2+2 \Rightarrow 3c_1+c_2=1$ ....(6)

Solving (5) and (6) , we get $C_1=1$, $C_2=-2$

$\therefore y=-2e^x+(1+2x)e^{3x}$

This is the required solution.

11. Solve $(D^2+6D+9)y = 2e^{-3x}$.

Sol: Given equation is $(D^2+6D+9)y=2e^{-3x}$ ....(1)

Auxiliary equation is $m^2+6m+9=0$

i.e., $(m+3)^2=0$

$\Rightarrow$ *$m=-3,-3$. Roots are real and equal.*

Hence C.F. $=(c_1+c_2x)e^{-3x}$

Now P.I.= $\frac{2e^{-3x}}{D^2+6D+9} = \frac{2e^{-3x}}{(D+3)^2} = 2\frac{x^2}{2!}e^{-3x}$ [$\therefore$ *f(-3)=0]*

$\therefore$ The general solution of (1) is

$y = C.F. + P.I. = (c_1 + c_2 x)e^{-3x} + x^2 e^{-3x}$.

12. Solve $y'' - y' - 2y = 3e^{2x}, y(0) = 0, y'(0) = -2$.

Sol: Writing the given equation in the symbolic form , we have

$(D^2+D-2)y=3e^{2x}$ ....(1)

Auxiliary Equation is $m^2-m-2=0$

i.e., $(m-2)(m+1)=0$

$\Rightarrow m=2$ *or* $m=-1$

Hence C.F.=$c_1e^{2x}+c_2^{e-x}$

Now P.I. = $\frac{1}{D^2-D-2}3e^{2x}$

$= \frac{3}{(D-2)(D+1)}e^{2x} = \frac{3}{(D-2)(2+1)}e^{2x}$

$= \frac{1}{D-2}e^{2x} = xe^{2x}$ [$\therefore$ f(2)=0]

$\therefore$ The general solution of(1) is

$y = C.F. + P.I. = c_1e^{2x} + c_2e^{-x} + xe^{2x}$ ....(2)

Given y(0) =0 $\Rightarrow 0=c_1+c_2$ .....(3)

Differentiating (2) w.r.t. 'x' , we get

$y' = 2c_1e^{2x} - c_2e^{-x} + 2xe^{2x} + e^{2x}$

Given $y'(0) = -2 \Rightarrow -2 = 2c_1 - c_2 + 1 \Rightarrow 2c_1 - c_2 = -3$ ....(4)

(3)+(4) gives $3c_1=-3$ or $c_1=-1$

From (3), $C_2=-c_1=1$

Thus the required solution is $y = -e^{2x} + e^{-x} + xe^{2x}$.

13. Solve the differential equation $\frac{d^2y}{dx^2} + 4.\frac{dy}{dx} + 5y = -2\cosh x.$ **Given** $y = (0) = 0, y'(0) = 1.$

Sol: The operator form of the given equation is $(D^2+4D+5)y=-2\cosh x$.

The A.E. is $m^2+4m+5=0$

$\therefore m = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm i2}{2} \Rightarrow m = -2 \pm i = \alpha \pm i\beta$ (say)

Hence C.F. is $y_c = e^{-2x}(c_1 \cos x + c_2 \sin x)$

Now P.I. $= y_p = -\left(\frac{e^x + e^{-x}}{D^2+4D+5}\right) = -\frac{e^x}{D^2+4D+5} - \frac{e^{-x}}{D^2+4D+5}\left[\therefore \cosh x = \frac{e^x+e^{-x}}{2}\right]$

$-\frac{e^x}{1+4+5} - \frac{e^{-x}}{1-4+5} = \frac{e^x}{10} - \frac{e^{-x}}{2}$

$\therefore$ General solution is $y = yc + yp$

$i.e., y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{e^x}{10} - \frac{e^{-x}}{2}$ .....(1)

Given y(0)=0

$\Rightarrow 0 = c_1 - \frac{1}{10} - \frac{1}{2} \Rightarrow c_1 = \frac{3}{5}$

Differentiating (1) $y' = r^{-2x}(-c_1 \sin x + c_2 \cos x) - 2e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{e^x}{10} + \frac{e^{-x}}{2}$

Given y (0)=0

$\Rightarrow 0 = c_1 - \frac{1}{10} - \frac{1}{2} \Rightarrow c_1 = \frac{3}{5}$

Differentiating (1), $y' = e^{-2x}(-c_1 \sin x + c_2 \cos x) - 2e^{-2x}(c_1 \cos x + c_2 \sin x) - \frac{e^x}{10} + \frac{e^{-x}}{2}$

Given $y'(0)=1$

$\Rightarrow 1 = c_1 - 2c_1 - 1/10 + 1/20 \Rightarrow c_2 = 1 + 1/10 - 1/2 + 2c_1 = 1 + 1/10 - 1/2 + 6/5 = 9/5.$

Particular solution is

$y = e^{-2x}\left(\frac{3}{5}\cos x + \frac{9}{5}\sin x\right) - \frac{e^x}{10} - \frac{e^{-x}}{2}.$

14. Solve the differential equation (D^3-1) y = (e^x+1)^2.

Sol: Auxiliary Equation is m^3-1=0

i.e., *(m-1=)(m*^{2}+m+1)=0

$\Rightarrow m-1=0$ or $m^2+m+1=0$ i.e., $m = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore m = 1, \frac{-1 \pm i\sqrt{3}}{2}$

$\therefore$ Complementary function, $y_c = c_1 e^x + e^{-x\backslash 2}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right)$

Particular Integral , $y_p = \frac{1}{D^3-1}(e^x+1)^2$

$$=\frac{1}{D^3-1}(e^{2x}+2e^x+1)$$

$$=\frac{1}{D^3-1}(e^{2x}+2e^x+1)$$

$$=\frac{1}{D^3-1}e^{2x}+2\frac{1}{D^3-1}e^x+\frac{1}{D^3-1}e^{0.x}$$

$$=\frac{1}{8-1}e^{2x}+2.\frac{1}{(D-1)(D^2+D+1)}e^x+\frac{1}{0-1}e^{0.x}$$

$$=\frac{1}{7}e^{2x}+\frac{2x}{3}-1$$

$\therefore$ The general solution is $y = yc + yp$

$$=c_1e^x+e^{-x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)+\frac{1}{7}e^{2x}+\frac{2x}{3}-1$$

15. Solve $(D^3-3D^2+4)y=(1+e^{-x})^3$**.**

Sol: $(D^3-3D^2+4)y=(1+e^{-x})^3$

A.E. is $(m^2-3m^2+4)=0$

i.e.,$(m+1)(m^2-4m+4)=0$

or $(m+1)(m-2)^2=0$ $\therefore$ m=-1,2,2

$\therefore$ One root is real and other two roots are real and equal.

Hence C.F. is $y_c=c_1e^{-x}+(c_2+c_3x)e^{2x}$ ....(1)

Now P.I is $y_p=\frac{1+e^{-3x}+3e^{-2x}+3e^{-x}}{D^3-3D^2+4}$

$$=\frac{1}{D^3-3D^2+4}+\frac{e^{-3x}}{D^3-3D^2+4}+\frac{3e^{-2x}}{D_3-3D^2+4}$$

$y_{p1}+y_{p2}+y_{p3}+y_{p4}$

Now $y_{p1}=\frac{1}{D^3-3D^2+4}$ [Put D=0]= $\frac{1}{4}$

$y_{p2}=\frac{e^{-3x}}{D^3-3D^2+4}$ [Put D=-3]

$$=\frac{e^{-3x}}{-27+27+4}=\frac{e^{-3x}}{4}$$

$y_{p3}=\frac{3e^{-2x}}{D^3-3D^2+4}$ [Put D=-2]

$$=\frac{3e^{-2x}}{-8-12+4}=\frac{3e^{-2x}}{-16}$$

And $y_{p4}=\frac{3e^{-x}}{(D+1)(D^2-4D+4)}$ [Put D=-1]

$$=\frac{xe^{-x}}{1+4+4}=\frac{xe^{-3}}{3}$$

$\therefore$ General solution is $y=y_c+y_{p1}+y_{p2}+y_{p3}+y_{p4}$

$$\therefore y=c_1e^{-x}+(c_2+c_3x)e^{2x}+\frac{1}{4}+\frac{e^{-3x}}{4}-\frac{3e^{-2x}}{16}+\frac{xe^{-x}}{3}$$

16. Solve (D^3+3D^2-4)y=sinh 2x+7.

Sol: Given *(D3+3D2-4)y=sinh 2x+7*

A.E. is $m^3+3m^2-4=0$

$\therefore$ The roots are $m=-1,2,2$

C.F.is $y_c=c_1e^{-x}+(c_2+c_3x)e^{2x}$

P.I is $y_p=\frac{\frac{e^{2x}-e^{-2x}}{2}+7}{D^3+3D^2-4}=\frac{e^{2x}}{2(D^3+3D^2-4)}-\frac{e^{-2x}}{2(D^3+3D^2-4)}+\frac{7}{D^3+3D^2-4}$

$y_{p1}=\frac{1}{2}\frac{e^{2x}}{(D+1)(D-2)^2}$ [Put D=2]

$$=\frac{1}{2}.\frac{\frac{x^2}{2}.e^x}{(3)}=\frac{x^2e^{2x}}{12}$$

$y_{p2}=\frac{-\frac{1}{2}e^{-2x}}{(D+1)(D-2)^2}$ [Put D=-2]

$$=-\frac{1}{2}.\frac{e^{-2x}}{(-1)(16)}=\frac{e^{-2x}}{32}$$

$=y_{p3}=\frac{7}{D^3+3D^2-4}=7.\frac{1}{D^3+3D^2-4}e^{0.x}$ [Put D=0]

$$=-\frac{7}{4}$$

$\therefore$ General solution is $y=y_c+y_{p1}+y_{p2}+y_{p3}$

$$\therefore y=c_1e^{-x}+(c_2+c_3x)e^{2x}+\frac{x^2e^{2x}}{12}+\frac{e^{-2x}}{32}-\frac{7}{4}$$

EXERCISE

1. Solve $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{-x}$.

2. Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=e^{x}$

3. . Solve $4\frac{d^2y}{dx^2}+4\frac{dy}{dx}-3y=e^{2x}$.

4. Solve $(D^3+1)\,y=(e^x+1)^2$.

ANSWERS

1. $y=c_1e^{-x}+c_2e^{-2x}+xe^{-2}$

2. $y=e^{-x/2}(c_1\cos\frac{x\sqrt{3}}{2}+c_2\sin\frac{x\sqrt{3}}{3}e^{x}$

3. $y=c_1e^{x/2}+c_2e^{\frac{-3x}{2}}+\frac{1}{21}e^{2x}$

4. $y=c_1e^{-x}+e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x\right)+\frac{1}{9}e^{2x}+e^{x}+1$

To evaluate $\frac{1}{f(D^2)}$ sin ax or $\frac{1}{f(D^2)}$ cos ax:

1. P.I = $\frac{\sin ax}{f(D^2)}=\frac{\sin ax}{f(-a^2)}$, **Provided** $f(-a^2)\neq 0$

2. P.I= $\frac{\cos ax}{f(D^2)}=\frac{\cos ax}{f(-a^2)}$**, Provided** $f(-a^2)\neq 0$

Case of Failure:

3. (i)P.I.= $\frac{\cos ax}{D^2+a^2}=\frac{x}{2a}\sin ax, iff(-a^2)=0$

(ii) P.I.= $\frac{1}{f(D^2)}$ **cos ax =** $\frac{1}{\phi(-a^2)}\left(\frac{x}{2a}\sin ax\right), if\phi(-a^2)\neq 0$

4. (i) P.I.= $\frac{\sin ax}{D^2+a^2}=\frac{-x}{2a}\cos ax, iff(-a^2)=0$

(ii) P.I.= $\frac{1}{f(D^2)}\sin ax=\frac{1}{\phi(-a^2)}\left(-\frac{x}{2a}\cos ax\right), if\phi(-a^2)\neq 0$

1. Solve $(D^3+D)y=\cos x$.

Sol: A.E is $m^3+m=0\Rightarrow m(m^2+1)=0\Rightarrow m=0,-i$

$$y_c = C.F = c_1 + (c_2 \cos x + c_3 \sin x)$$

$$y_p = P.I = \frac{1}{f(D)} X = \frac{1}{D^3 + D}\cos x = \frac{1}{D}\frac{1}{D^2+1}\cos x$$

$$= \frac{1}{D}\frac{x}{2D}\cos x = \frac{x}{2D}\int \cos x dx = \frac{x}{2D}\sin x$$

$$= \frac{x}{2}\int \sin x dx = -\frac{x}{2}\cos x$$

Complete solution is $y = y_c + y_p$

$$y = c_1 + (c_2 \cos x + c_3 \sin x) - \frac{x}{2}\cos x$$

2. Solve $(D^2+4)y = \cos x$.

Sol: A.E: $m^2 + 4 = 0 \Rightarrow m^2 = 4i^2 \Rightarrow m = \pm 2i$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

$$y_p = \frac{1}{f(D)} = \frac{1}{D^2+4}\cos x = \frac{1}{3}\cos x \quad \therefore y = y_c + y_p$$

3. Solve $(D-1)(D^2+2)^2 y = \sin 2x \sin x$.

Sol: A.E is $(m-1)(m^2+1)^2 = 0$

m=1,i,i,-i,-i

$$y_c = c.f = c_1 e^x + (c_2 x + c_3)\cos x + (c_4 x + c5)\sin x$$

$$y_p = P.I = \frac{1}{f(D)} X = \frac{1}{(D-1)(D^2+1)^2}\sin 2x \sin x$$

$$\frac{1}{(D-1)(D^2+1)^2}\left(\frac{\cos x - \cos 3x}{2}\right) = \frac{1}{2}\frac{1}{(D-1)(D^2+1)^2}\cos x \frac{-1}{2}\frac{1}{(D-1)(D^2+1)^2}\cos 3x \quad ...(1)$$

Consider , $\frac{1}{(D-1)(D^2+1)^2}\cos x = \frac{1}{2(D^2+1)^2}\frac{(D+1)}{(D^2-1)}\cos x$ Put D^2=7

$$= \frac{1}{2(D^2+1)^2}(D+1)\frac{1}{(-1)-1}\cos x \frac{1}{2(D^2+1)^2}(D+1)\frac{1}{(-2)}\cos x$$

$$= -\frac{1}{4}\frac{1}{(D^2+1)^2}(-\sin x + \cos x)$$

$$= +\frac{1}{4}\frac{1}{(D^2+1)}\sin x - \frac{1}{4}\frac{1}{(D^2+1)^2}\cos x$$

$$= \frac{1}{4}\frac{x^2}{2}(-\sin x) - \frac{1}{4}\frac{x^2}{2}(-\cos x)$$

$$=\frac{x^2}{8}\sin x+\frac{x^2}{8}\cos x \qquad ...(2)$$

Consider $\frac{1}{2}\frac{1}{(D-1)(D^2+1)^2}\cos 3x=\frac{1}{2(D-1)(-a+1)^2}\cos 3x$

$$=\frac{1}{128(D-1)}\cos 3x=\frac{(D+1)}{128(D^2-1)}\cos 3x=\frac{(D+1)}{128(-9-1)}\cos 3x$$

$$=\frac{1}{1280}(D+1)\cos 3x=\frac{1}{1280}(-3\sin 3x+\cos 3x) \qquad ....(3)$$

From (1),(2) &(3) $y_p=\frac{-x^2}{8}\sin x+\frac{x^2}{8}\cos x+\frac{1}{1280}(\cos 3x-\sin 3x)$

Complete solution is , $y=y_c+y_p$

$$y=c_1e^x+(c_2x+c_3)\cos x+(c_4x+c_5)\sin x+\frac{x^2}{8}[\cos x-\sin x)+\frac{1}{128}(\cos 3x-\sin 3x)$$

4.(i) Solve (D^2+3D+2) y = sin 3x (ii) (D^2-4D+3) y = cos 2x.

Sol: (i) Auxiliary Equation (A.E) is $m^2+3m+2=0$ *i.e.*, $(m+1)(m+2)=0$

$\Rightarrow m+1=0$ *or* $m+2=0$

$\therefore$ m=-1,-2. The roots are real and different.

Thus complementary Function (C.F) is $y=c_1e^{-x}+c_2e^{-2x}$

Now particular integral (P.I.)= $\frac{1}{D^2+3D+2}\sin 3x$

$$=\frac{\sin 3x}{-9+3D+2}\left[Put D^2=-3^2=-9\right]$$

$$=\frac{\sin 3x}{3D-7}=\frac{(3D+7)\sin 3x}{9D^2-49}[put D^2=-3^2=-9]$$

$$=\frac{(3D+7)\sin 3x}{9(-9)-49}=\frac{-1}{130}\left[3\frac{d}{dx}(\sin 3x)+7\sin 3x\right]$$

$$=-\frac{1}{130}[9\cos 3x+7\sin 3x]$$

Hence the general solution is

$y=C.F.+P.I.=c_1e^{-x}+c_2e^{-2x}-\frac{1}{130}(9\cos 3x+7\sin 3x)$ where C_1 and $C_2$are constant.

(ii)A.E.is $m^2-4m+3=0$ *i.e.*, $(m-1)(m-3)=0$ $\Rightarrow$m=1,3. The roots are real and different.

$\therefore$ C.F.=$C_1e^x+C_2e^{3x}$

P.I.= $\frac{\cos 2x}{D^2-4D+3} = \frac{\cos 2x}{-2^2-4D+3}$

$= \frac{\cos 2x}{-(4D+1)} = -\frac{(4D-1)\cos 2x}{16D^2-1}$ [Put $D^2=-2^2=-4$]

$= \frac{(1-4D)\cos 2x}{16(-4)-1} = \frac{1}{65}(1-4D)\cos 2x = \frac{-1}{65}\left[\cos 2x - 4\frac{d}{dx}(\cos 2x)\right]$

$= -\frac{1}{65}[\cos 2x - 4(-2\sin 2x)] = -\frac{1}{65}(\cos 2x + 8\sin 2x)$

Hence the general solution is

$y = C.F. + P.I. = c_1 e^x + c_2 e^{3x} - \frac{1}{65}(\cos 2x + 8\sin 2x)$ Where c_1, c_2 are constants.

5. Solve $(D^2-4)y=2\cos^2 x$.

Sol: Given equation is $(D^2-4)y=2\cos^2 x$ ....(1)

Let $f(D)=D^2-4$.A.E. is $f(m)=0$ i.e., $m^2-4=0$

The roots are m=2,-2.The roots are real and different.

C.F.=$y_c = c_1 e^{2x} + c_2 e^{-2x}$

P.I.= $y_p = \frac{1}{D^2-4}(2\cos^2 x) = \frac{1}{D^2-4}(1+\cos 2x)$

$= \frac{e^{ox}}{D^2-4} + \frac{\cos 2x}{D^2-4} = PI_1 + PI_2$

$P.I._1 = y_{p1} = \frac{e^{0.x}}{D^2-4}[put D=0] = \frac{e^{0.x}}{-4} = -\frac{1}{4}$

$P.I._2 = y_{p2} = \frac{\cos 2x}{D^2-4} = \frac{\cos 2x}{-8}[put D^2 = -2^2 = -4]$

$\therefore$ General solution of (1)is $y = y_c + y_{p1} + y_{p2}$

i.e., $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4} - \frac{\cos 2x}{8}$, where C_1 and C_2 are constants.

6. Solve $(D^2+4)y=e^x+\sin 2x+\cos 2x$.

Sol: Given equation is $(D^2+4)y=e^x+\sin 2x+\cos 2x$.

Let $f(D)=D^2+4$

A.F. is $f(m)=0 (i.e.) m^2+4=0$

The roots are m=2i,-2i. The roots are complex and conjugate.

$\therefore$ C.F.= $y_c = c_1\cos 2x + c_2\sin 2x$.

P.I. $= y_p = \frac{1}{D^2+4}(e^x + \sin 2x + \cos 2x)$

$$=\frac{1}{D^2+4}e^x+\frac{\sin 2x}{D^2+4}+\frac{\cos 2x}{D^2+4}=PI_1+PI_2+PI_3$$

P.I$_1$= $\frac{1}{D^2+4}e^x\,(putD=1)\therefore P.I_1=\frac{1}{5}e^x$

P.I$_2$= $\frac{1}{D^2+4}(\sin 2x)$.

$P.I_2.=\frac{-cox2x}{4}$ [$\therefore$ Case of failure $f(-a^2)=0$], using $\frac{\sin ax}{D^2+a^2}=-\frac{-x}{2a}\cos ax$

Similarly ,P.I$_3$= $\frac{\cos 2x}{D^2+4}=\frac{x\sin 2x}{4}$

General solution is $y=y_c+y_{p1}+y_{p2}+y_{p3}$

i.e., $y=c_1\cos 2x+c_2\sin 2x+\frac{1}{5}e^x-\frac{x}{4}+\cos 2x+\frac{x}{4}\sin 2x$

7.Solve $(D^2+4D+3)y=\sin 3x\cos 2x.$

Sol: Given equation is $(D^2+4D+3)y=\sin 3x\cos 2x.$ **....(1)**

Let $f(D)=D^2-4D+3$. A.E is f(m) =0 i.e., $m^2-4m+3=0$

The roots are m=1,3. The roots are real and different.

C.F. is $y_c+c_1e^x+c_2e^{3x}$

PI= $y_p=\frac{\sin 3x\cos 2x}{D^2-4D+3}=\frac{1}{2}\frac{(\sin 5x+\sin x)}{D^2-4D+3}$ [$\therefore$ 2 sin A cos B =Sin (A+B)+sin(A-B)]

$$=\frac{1}{2}.\frac{\sin 5x}{D^2-4D+3}+\frac{1}{2}.\frac{\sin x}{D^2-4D+3}=PI_1+PI_2$$

$$PI_1=\frac{1}{2}.\frac{\sin 5x}{D^2-4D+3}(putD^2=-5^2=-25)$$

$$=\frac{1}{2}.\frac{\sin 5x}{-25-4D+3}=\frac{1}{2}.\frac{\sin 5x}{-4D-22}=-\frac{1}{4}.\frac{(2D-11)\sin 5x}{(2D+11)(2D-11)}$$

$$=\frac{-1}{4}.\frac{(2D-11\sin 5x}{4D^2-121}(putD^2=-5^2=-25)$$

$$=\frac{-1}{4}.\frac{(2D-11)\sin 5x}{-100-121}=\frac{1}{884}(10\cos 5x-11isn5x)$$

$$PI_2=\frac{1}{2}.\frac{\sin x}{D^2-4D+3}(putD^2=-1)$$

$$=\frac{1}{2}.\frac{\sin x}{-1-4D+3}=\frac{1}{2}.\frac{\sin x}{2-4D}=\frac{1}{4}.\frac{(1+2D)\sin x}{(1-2D)(1+2D)}$$

$$=\frac{1}{4}.\frac{(1+2D)\sin x}{1-4D^2}(PutD^2=-1)=\frac{1}{4}\frac{(1+2D)\sin x}{1+4}=\frac{1}{20}(\sin x+2\cos x)$$

$\therefore$ General solution is $y = y_c + y_{p1} + y_{p2}$

i.e., $y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{884}(10\cos 5x - 11isn5x) + \frac{1}{20}(\sin x + 2\cos x)$

8.Solve the differential equation (D^2-2D+2)y = cos9x.

Sol: A.E is $m^2 - 2m + 2 = 0$

$$\therefore m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm i$$

The roots are conjugate complex numbers.

C.F. is $y_c = e^x(c_1\cos x + c_2 \sin x)$

Now $y_p = P.I. = \frac{\cos 9x}{D^2 - 2D + 2} (put D^2 = -81)$

$$= \frac{\cos 9x}{-81 + 2 - 2D} = \frac{-\cos 9x}{-2D + 79}$$

$$= \frac{-(2D - 79)\cos 9x}{4D^2 - 6241} = \frac{18\sin 9x + 79\cos 9x}{4D^2 - 6241} (PutD^2 = -81)$$

$$= \frac{18\sin 9x + 79\cos 9x}{4(-81) - 6241} = \frac{18\sin(9x) + 79\cos(9x)}{-6565}$$

$\therefore$ The general solution is $y = y_c + y_p$

i.e., $y = e^x(c_1 \cos x + c_2 \sin x) + \frac{18\sin 9x + 79\cos 9x}{-6565}$

9.Solve the differential equation (D^3+4D)y=5+sin2x.

Sol: A.E: $m^3 + 4m = 0 \Rightarrow m(m^2+4) = 0$

$\therefore m=0, m=\pm 2i$ are the roots.

One root is real, other two roots are complex conjugate numbers.

Hence C.F.= $y_c + c_1 + c_2\cos(2x) + c_3\sin(2x)$

$PI_1 = y_{p1} = \frac{5}{D^3 + 4D} = \frac{5}{D(D^2+4)} e^{0.x} = \frac{5}{D}\left[\frac{1}{D^2+4} e^{0.x}\right]$ (put D=0)

$$= \frac{5x}{4}$$

$PI_2 = y_{p2} = \frac{\sin 2x}{D(D^2+4)} = \frac{-\cos 2x}{2(D^2+4)}$

$$= \frac{-1}{2} \cdot \frac{1}{4} x \sin(2x)$$

$$= -\frac{1}{8} x \sin(2x)$$

$\therefore$ The general Solution is $y = y_c + y_{p1} + y_{p2}$

i.e., $y = c_1 + c_2 \cos(2x) + c_3 \sin(2x) + \dfrac{5x}{4} - \dfrac{1}{8} x \sin(2x)$

10 .Solve the differential equation $(D^2 + 5D + 4)y =$ 2 sin ax.

Sol: The given equation is $(D^2 + 5D + 4)y = 2 \sin ax.$

A.E. is m_2+5m+4=0

i.e.,$(m+4)(m+1)=0$

$\therefore$ $m=-4,-1$. The roots are real and different.

Hence C.F. is $y_c = c_1 e^{-4x} + c_2 e^{-x}$

Now P.I $= P.I. = y_p = \dfrac{2\sin ax}{D^2 + 5D + 4} (Put D^2 = -a^2)$

$$= \frac{2\sin ax}{5D + (4 - a^2)}$$

$$= \frac{2[5D - (4 - a^2)]\sin(a.x)}{25D^2 - (4 - a^2)^2} (put D^2 = -a^2)$$

$$= \frac{2[5a\cos(ax) - (4 - a^2)\sin(ax)]}{-25a^2 - (16 - a^4 - 8a^2}$$

$$= \frac{2[5a\cos ax - (4 - a^2)\sin(ax)}{-17a^2 - 16 - a^4}$$

14. Solve $(D^2 + 2D + 10)y = -37\sin 3x$.

Sol: A.E is $m^2 + 2m + 10 = 0$

$m=-1\pm 3i.$

$y_c : C.F = e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$

$y_p = P.I = \dfrac{1}{f(D)} X = \dfrac{1}{D^2 + 2D + 10}(37\sin 3x)$ Put $D^2 = -3^2$

$$= -37 \frac{1 - 2D}{1 - 4D^2} \sin 3x = -37 \frac{1 - 2D}{1 - 4(-3^2)} \sin 3x$$

$$= -(\sin 3x - 6\cos 3x)$$

Complete solution $y = y_c + y_p = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) - (\sin 3x - 6\cos 3x)$

15. Solve $y'' + 4y' + 20y = 23\sin t - 15\cos t, y(0) = 0, y'(0) = -1.$

Sol: Given equation in operator form is $(D^2 + 4D + 20)y = 23$ sin t-15 cos t ....(1)

Where D stands for $\frac{d}{dt}$ and D^2 stands for $\dfrac{d^2}{dt^2}$.

Auxiliary Equation is $m^2+4m+20=0$. *Here* $a=1, b=4, c=20$

$$\therefore m=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{16-80}}{2}=\frac{-4\pm i8}{2}=-2\pm i4$$

Hence C.F is $y_c=e^{-2t}(c_1\cos 4t+c_2\sin 4t)$

Now P.I.= $y_p=\frac{1}{D^2+4D+20}(23\sin t-15\cos t)$

$$=\frac{1}{-1+4D+20(23\sin t-15\cos t)}$$ (Replacing D^2 with -1)

$$=\frac{1}{4D+19}(23\sin t-15\cos t)=\frac{(4D-19)}{16D^2-361}$$ (23 sin t -15 cos t)

$$=\frac{4D-19}{16(-1)-361}(23\sin t-15\cos t)$$

$$=\frac{-1}{377}(4D-19)(23\sin t-15\cos t)$$

$$=\frac{-1}{377}[92\cos t+60\sin t-437\sin t+285\cos t]$$

$$=\frac{-1}{377}(377\cos t-377\sin t)=\sin t-\cos t.$$

$\therefore$ General solution of(1) is

$y=C.F.+P.I.=yc+yp$

$=e^{-2t}(c_1\cos 4t+c_2\sin 4t)+\sin t-\cos t$ ...(2)

Given that y(0)=0

$\Rightarrow 0=c_1-1 \Rightarrow c_1=1.$

Differentiating (2) with respect to ‘t’ , we get

$$y'(t)=-2e^{-2t}(c_1cps4t+c_2\sin 4t)+e^{-2t}(-4c_1,\sin 4t+4c_2\cos 4t)+\cos t+\sin t.$$

Also given that $y'(0)=-1$

$$\Rightarrow -1=-2(c_1+0)+(0+4c_2)+1+0$$

$$\Rightarrow -1=-2c_1+4c_2+1=-2+4c_2+1(\because c_1=1)$$

$$\Rightarrow c_2=0$$

Hence the required solution is $y=c^{-2t}\cos 4t+\sin t-\cos t$

[Substituting the values of C_1 and C_2 in (2)]

16. Solve $(D^2+4)y=\sin t+\frac{1}{3}\sin 3t+\frac{1}{5}\sin 5t, y(0)=1, y'(0)=\frac{3}{35}.$

Sol: Given equation is $(D^2+4)y=\sin t+\frac{1}{3}\sin 3t+\frac{1}{5}\sin 5t$

Auxiliary Equation is $m^2+4=0 \Rightarrow m=\pm i2$

$\therefore$ C.F.=$c_1 \cos 2t + c_2 \sin 2t$

$$\text{P.I.}= \frac{1}{D^2+4}\left(\sin t + \frac{1}{3}\sin 3t + \frac{1}{5}\sin 5t\right)$$

$$= \frac{1}{D^2+4}\sin t + \frac{1}{3}.\frac{1}{D^2+4}\sin 3t + \frac{1}{5}.\frac{1}{D^2+4}\sin 5t$$

$$= \frac{1}{-1+4}\sin t + \frac{1}{3}.\frac{1}{-9+4}\sin 3t + \frac{1}{5}.\frac{1}{-25+4}\sin 5t$$

$$= \frac{1}{3}\sin t - \frac{1}{15}\sin 3t - \frac{1}{105}\sin 5t$$

$\therefore$ General solution of the given equation is

y=C.F.+P.I.

i.e., $y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{3}\sin t - \frac{1}{15}\sin 3t - \frac{1}{105}\sin 5t$ .....(1)

Given that y(0)=1

$$\Rightarrow 1 = c_1 + 0 + \frac{1}{3}(0) - \frac{1}{15}(0) - \frac{1}{105}(0)$$

$\Rightarrow c_1=1$

Differentiating (1) w.r.t 't' , we get

$$y'(t) = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{1}{3}\cos t - \frac{1}{5}\cos 3t - \frac{1}{21}\cos 5t$$

Also given that $y'(0) = \frac{3}{35}$

$$\Rightarrow \frac{3}{35} = 0 + 2c_2 + \frac{1}{3} - \frac{1}{5} - \frac{1}{21}$$

$$\Rightarrow 2c_2 = \frac{3}{35} - \frac{1}{3} + \frac{1}{5} + \frac{1}{21} = \frac{1}{105}(9-35+21+5) = 0$$

$\Rightarrow$ $c_2=0$.

Substituting the values of c_1 and c_2 in (1), we get

$$y = \cos 2t + \frac{1}{3}\sin t - \frac{1}{15}\sin 3t - \frac{1}{105}\sin 5t$$

Which is the required solution.

17. Solve the differential equation: $(D^3+1)y = \cos(2x-1)$**.**

Sol: Given differential equation is

$$(D^3+1)y = \cos(2x-1)$$

Auxiliary Equation is $m^3+1=0$

$\Rightarrow (m+1)(m^2-m+1)=0$ [$\because$ $a^3+b^3 = (a+b)(a^2-ab-b^2)$]

$\Rightarrow m=-1, \frac{1\pm i\sqrt{3}}{2}$

Hence C.F.= $c_1e^{-x}+e^{x/2}\left[c_2\cos\frac{\sqrt{3x}}{2}+c_3\sin\frac{\sqrt{3x}}{2}\right]$

Now P.I. $=\frac{1}{D^3+1}\cos(2x-1)$

Putting $D^2=-4,$ we have

P.I.= $\frac{1}{1-4D}\cos(2x-1)=\frac{1+4D}{1-16D^2}[\cos(2x-1)$

Again putting D^2=-4,

P.I., $=\frac{1}{65}[\cos(2x-1)-8\sin(2x-1)]$

$\therefore$ General solution is

y=C.F+P.I

$=c_1e^{-x}+e^{x/2}\left[c_2\cos\frac{\sqrt{3x}}{2}+c_3\sin\frac{\sqrt{3x}}{2}\right]+\frac{1}{65}[\cos(2x-1)-8\sin(2x-1)].$

18. Solve $(D^4-2D^3+2D^2-2D+1)y=\cos x$.

Sol: The auxiliary equation is $m^4-2m^3+2m^2-2m+1=0$

i.e., $(m^4+2m^2+1)-(2m^3+2m)=0$

i.e., $(m^2+1)^2-2m(m^2+1)=0$ or $(m^2+1)(m^2-2m+1)=0$ or $(m^2+1)(m-1)^2=0$

$\Rightarrow m^2+1=0$ or $(m-1)^2=0$

$\therefore m=\pm i,1,1.$

$\therefore$ The complementary function is

$y_c=c_1\cos x+c_2\sin x+(c_3+c_4x)e^x$

$P.I.=\frac{1}{D^4-2D^3+2D^2-2D+1}\cos x=\frac{1}{(D^2+1)(D^2-2D+1)}\cos x$ [Put $D^2=-1^2=-1$]

$=\frac{\cos x}{(-2D)(D^2+1)}=\frac{-1}{2(D^2+1)}\int\cos x\,dx=\frac{-1}{2}\frac{\sin x}{D^2+1}$

$==\left(\frac{-1}{2}\right)\left(\frac{-x\cos x}{2}\right)$ [$\because f(-1)^2=0$]

$=\frac{1}{4}x\cos x$

Hence the general solution is

y=C.F.+P.I.=($C_1\cos x+C_2\sin x$)+ $(C_3+C_4x)e^x+\frac{1}{4}$ x cos x.

19.Solve the differential equation $(D^3-1)y = e^x + \sin^3 x + 2.$

Sol: Given D.E. is $(D^3-1)y=e^x+\sin^3 x+2$ ....(1)

A.E. is $m^3-1=0 \Rightarrow (m-1)(m^2+m+1)=0$

$$\therefore m=1, \frac{-1\pm i\sqrt{3}}{2}$$

Thus one roots is real and other two roots are complex and conjugate.

$$\text{C.F.}= y_c = c_1e^x + e^{-x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) \quad ...(2)$$

$$\text{Now P.I} = \frac{1}{D^3-1}(e^x+\sin^3 x+2) = \frac{e^x}{D^3-1} + \frac{\sin^3 x}{D^3-1} + \frac{2}{D^3-1} = y_{p1}+y_{p2}+y_{p3}$$

$$y_{p1} = \frac{e^x}{(D-1)(D^2+D+1)} = \frac{x}{1!}.\frac{e^x}{D^2+D+1}[\because f(1)=0]$$

$$= \frac{x}{1!}.\frac{e^x}{1+1+1}[PutD=1] = \frac{xe^x}{3} \quad ....(3)$$

$$y_{p2} = \frac{\sin^3 x}{(D^3-1)}$$

$$= \frac{3}{4}.\frac{\sin x}{D^3-1} - \frac{1}{4}.\frac{\sin 3x}{D^3-1}\left[\because \sin^3 A = \frac{1}{4}(3\sin A - \sin 3A)\right]$$

$$= \frac{3}{4}.\frac{\sin x}{-D-1} - \frac{1}{4}.\frac{\sin 3x}{-9D-1}[PutD^2=-1 andD^2=-9]$$

$$= \frac{-3}{4}\frac{(D-1)\sin x}{(D+1)(D-1)} + \frac{1}{4}\frac{(9D-1)\sin 3x}{(9D-1)(9D+1)}$$

$$= \frac{-3}{4}\frac{\cos x - \sin x}{-2} + \frac{1}{4}\frac{27\cos 3x - \sin 3x}{-729-1}[PutD^2=-1andD^2=-9]$$

$$= \frac{3}{8}(\cos x - \sin x) - \frac{1}{1920}(27\cos 3x - \sin 3x) \quad ....(4)$$

and $y_{p3} = \frac{2}{D^3-1}[PutD=0] = -2$ .....(5)

Hence the general solution is $y = y_c + y_{p1} + y_{p2} + y_{p3}$

$$y = c_1e^{-x} + e^{-x/2}\left(c_2\cos\frac{\sqrt{3x}}{2} + c_3\sin\frac{\sqrt{3x}}{2}\right) + \frac{xe^x}{3}$$

$$+\frac{3}{8}(\cos x - \sin x) - \frac{1}{2920}(27\cos 3x - \sin 3x) - 2$$

EXERCISE

1. Solve $(D^3+1)y=\cos 2x$.

2. Solve $(D^3+1)y=\sin(2x+1)$

3. Solve $(D^2+D+1)y=\sin 2x$.

4. Solve $(D^2-5D+6)\ y = \sin 3x$

ANSWERS

1. $y=c_1e^{-x}+e^{\frac{x}{2}}(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3x}}{2})+\frac{1}{65}(\cos 2x-8\sin 2x)$

2. $y=c_1e^{-x}+e^{\frac{x}{2}}(c_2\cos\frac{\sqrt{3}}{2}x+c_3\sin\frac{\sqrt{3}}{2}x)\frac{1}{65}[\sin(2x+1)+8\cos(2x+1)$

3. $y=e^{\frac{-x}{2}}\left[c_1\cos\frac{x\sqrt{3}}{2}+c_2\frac{\sin x\sqrt{3}}{2}\right]-\frac{1}{13}(2\cos 2x+3\sin 2x)$

4. $y=c_1e^{2x}+c_2e^{3x}+\frac{1}{78}(5\cos 3x-\sin 3x)$

P.I. of *f(D)y=φ(x) when* $\phi(x)=x^k$*, where K is a Positive Integer.*

Let f(D)y $=x^k$. *Operating by* $\frac{1}{f(D)}$, we get $y=\frac{1}{f(D)}x^k$

$\therefore$ P.I. $=\frac{1}{f(D)}x^k$

TO evaluate P.I. reduce $\frac{1}{f(D)}$ to the form $\frac{1}{1\pm\phi(D)}$ by taking out the lowest degree term from *f(D)*.Now

Write $\frac{1}{f(D)}$ as $[1\pm\phi(D)]^{-1}$ and expand it in ascending powers of D using Binomial theorem upto the term containing D^k. Then operate x^k with the terms of the expansion of $[1\pm\phi(D)]^{-1}$

If f(D) is resolvable into factors then split up $\frac{1}{f(D)}$ into partial factors and proceed.

We frequently use the following rules.

- $\frac{1}{1-D}=(1-D)^{-1}=1+D+D^2+D^3+.....$
- $\frac{1}{1+D}=(1+D)^{-1}=1-D+D^2-D^3+.....$

- $\frac{1}{(1-D)^2} = (1-D)^{-2} = 1+2D+3D^2+4D^3+.....$
- $\frac{1}{(1+D)^2} = (1+D)^{-2} = 1-2D+3D^2-4D^3+.....$
- $\frac{1}{(1-D)^3} = (1-D)^{-3} = 1+3D+6D^2+10D^3+....$
- $\frac{1}{(1+D)^3} = (1+D)^{-3} = 1-3D+6D^2-10D^3+....$

1. Solve $(D^2+5D+4)y = x^2-3x+2$

Sol: **A.E is** $m^2+5m+4=0$

$(m+1)(m+4)=0 \Rightarrow m=-1,-4$

$y_c = C.F = c_1e^{-x}+c_2e^{-4x}$

$$y_p = P.I = \frac{1}{f(D)}X = \frac{1}{D^2+5D+4}(x^2-3x+2)$$

$$= \frac{1}{4\left(1+\frac{5D+D^2}{4}\right)}(x^2-3x+2) = \frac{1}{4}\left(1+\frac{5D+D^2}{4}\right)^{-1}(x^2-3x+2)$$

$$= \frac{1}{4}\left(1-\frac{5D+D^2}{4}+\left(\frac{5D+D^2}{4}\right)^2\right)(x^2-3x+2)$$

$$= \frac{1}{4}\left(1-\frac{5D}{4}-\frac{D^2}{4}+\frac{25}{16D^2}\right)(x^2-3x+2)$$ (Neglecting 3rd and higher terms.)

$$= \frac{1}{4}\left(x^2-3x+2-\frac{5}{2}x+\frac{15}{4}-\frac{1}{2}+\frac{25}{8}\right)$$

$$= \frac{1}{4}\left(x^2-\frac{11x}{2}+\frac{67}{8}\right) \quad \therefore y = y_c+y_p$$

$$y = c_1e^{-x}+c_2e^{-4x}+\frac{1}{4}\left(x^2-\frac{11x}{2}+\frac{67}{8}\right)$$

2. Solve $D^2(D^2+4)+y = 320(x^3+2x^2)$**.**

Sol: **A.E** $m^2(m^2+4)=0$

m=0,0,2i,-2i

$y_c = C.F : c_1+c_2x+c_3\cos 2x+c_4\sin 2x$

$$y_p = P.I = \frac{1}{f(D)} X = \frac{1}{D^2(D^2+4)} 320(x^3+2x^2)$$

$$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1} 320(x^3+2x^2)$$

$$= \frac{1}{4D^2}\left(1-\frac{D^2}{4}+\frac{D^4}{16}-\frac{D^6}{64}+......\right)320(x^3+2x^2)$$

$$= \frac{320}{4}\left(\frac{1}{D^2}-\frac{1}{4}+\frac{D^2}{16}.......\right)(x^3+2x^2)$$

$$= 80\left(\frac{x^5}{20}-\frac{1}{4}(x^3+2x^2)+\frac{1}{16}(6x+4)\right)$$

$$= 4x^5 - 20x^3 - 40x^2 + 30x + 20$$

Complete solution = $y = y_c + y_p$

$$y = c_1 + c_2 x + c_3 \cos 2x + c_4 \sin 2x + 4x^5 - 20x^3 - 40x^2 + 30x + 20$$

3. Solve (i) $(D^2+D+1)y = x^3$ **(ii)** $(D^3+2D^2+D)y = x^3$.

Sol: (i) Given equation is $(D^2+D+1)y = x^3$

Auxiliary Equation is $m^2+m+1=0$

$$\therefore m = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1\pm i\sqrt{3}}{2}$$

The roots are $\frac{-1}{2} \pm \frac{\sqrt{3}i}{2} = \alpha \pm i\beta$ (say)

$\therefore$ Roots are complex and conjugate

Hence C.F. = $y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) = e^{\frac{-1}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$

Now

$$\text{P.I.} = y_p = \frac{x^3}{D^2+D+1} = [1+(D^2+D)]^{-1}(x)^3$$

$$= [1-(D^2+D)+(D^2+D)^2-(D^2+D)^3+......](x^3)$$

$[\because (1+D)^{-1} = -1 - D + D^2 - D^3 + ...]$

$=(1-D+D^3)x^3$(neglecting the terms D^4, D^5,.....) $[\because D^4(x^3)=D^5(x^3)=........=0]$

$=x^3-D(x^3)+D^3(x^3)=x^3-3x^2+6$

$\therefore$ The general solution is given by $y=y_c+y_p$

i.e., $y = e^{\frac{-x}{2}}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + x^3 - 3x^2 + 6$

(ii) Given D.E. is $(D^3+2D^2+D)y=x^3$

A.E. is $m^3+2m^2+m=0$

$\Rightarrow m(m^2+2m+1)=0 \Rightarrow m(m+1)^2=0.$

$\therefore$ $m=0,-1,-1$ are the roots

Hence C.F. $= y_c = c_1 + e^{-x}(c_2 + c_3x)$

Now P.I. $= y_p = \dfrac{x_3}{D^3+2D+D} = \dfrac{x^3}{D(D+1)^2} = \dfrac{1}{(D+1)^2}\int x^3 dx$

$$= \frac{x^4}{4(1+D)^2} = \frac{1}{4}(1+D)^{-2}x^4$$

$$= \frac{1}{4}(1-2D+3D^2-4D^3+5D^4-.....)x^4$$

$$= \frac{1}{4}(x^4-8x^3+36x^2-96x+120) = \frac{x^4}{4} - 2x^3 + 9x^2 - 24x + 30$$

General solution is $y=y_c + y_p$

$$y = c_1 + e^{-x}(c_2+c_3x) + \frac{x^4}{4} - 2x_3 + 9x^2 - 24x + 30$$

4. Solve $(D^3+3D^2+4D-2)y = e^x + \cos x + x$ **.**

Sol: Auxiliary equation is $m^3-3m^2+4m-2=0$ i.e., $(m-1)(m^2-2m+2)=0$

$\Rightarrow m-1=0$ or $m^2-2m+2 = 0.$

$\therefore m=1, 1\pm i$

Complementary function $c_1e^x + e^x(c_2\cos x + c_3\sin x) = e^x(c_1 + c_2\cos x + c_3\sin x)$

Particular Integral (P.I)$= \dfrac{1}{D^3-3D^2+4D-2}(e^x + \cos x + x)$

$$= \frac{1}{D^3-3D^2+4D-2}e^x + \frac{1}{D^3-3D^2+4D-2}\cos x + \frac{1}{D^3-3D^2+4D-2}.x$$

$$= \frac{1}{(D-1)(D^2-2D+2)}e^x + \frac{1}{-D+3+4D-2}.\cos x - \frac{1}{2\left(1-\frac{D^3-3D^2+4D}{2}\right)}x$$

$$= \frac{1}{(D-1)(1-2+2)}e^x + \frac{1}{3D+1}\cos x - \frac{1}{2}\left(1-\frac{D^3-3D^2+4D}{2}\right)^{-1}x$$

$$= xe^x - \frac{1}{10}(-3\sin x - \cos x) - \frac{1}{2}(x+2) = xe^x + \frac{1}{10}(3\sin x + \cos x) - \frac{1}{2}(x+2)$$

$\therefore$ The general solution is y=C.F.+P.I.

$$= e^x(c_1 + c_2\cos x + c_3\sin x) + xe^x + \frac{1}{10}(3\sin x + \cos x) - \frac{1}{2}(x+2)$$

Where C_1, C_2 and C_3 are constants.

5. Solve the differential equation *$(D^3$-$3D$-$2)y = x^2$*.

Sol: The auxiliary equation is $m^3-3m-2=0$ *i.e.,* $(m+1)(m^2-m-2)=0$

i.e., $(m+1)(m+1)(m-2)=0$ *or* $(m+1)^2(m-2)=0$

$\therefore m = -1,-1,2.$ Hence C.F.= $y_c = (c_1 + c_2x)e^{-x} + c_3e^{2x}$

$$\text{P.I.} = \frac{1}{D^3 - 3D - 2}x^3 = -\frac{1}{2\left(1 - \dfrac{D^3 - 3D}{2}\right)}x^3$$

$$= -\frac{1}{2}\left(1 - \frac{D^3 - 3D}{2}\right)^{-1} x^3$$

$$= -\frac{1}{2}\left[1 + \frac{D^3 + 3D}{2} + \left(\frac{D^3 + 3D}{2}\right)^2 + \left(\frac{D^3 + 3D}{2}\right)^3 + \ldots\right]x^3$$

$$= -\frac{1}{2}\left[1 + \frac{D^3 + 3D}{2} + \frac{9}{4}D^2 + \frac{27}{8}D^3 + \ldots.\right]x^3$$

$$= -\frac{1}{2}\left[x^3 + \frac{1}{2}(6) + \frac{3}{2}(3x^2) + \frac{9}{4}(6x) + \frac{27}{8}(6)\right]$$

$$= -\frac{1}{2}\left[x^3 + \frac{9}{2}x^2 + \frac{27}{2}x + \frac{93}{4}\right]$$

$$= -\frac{1}{8}[4x^3 + 18x^2 + 54x + 93]$$

Hence the general solution is

$$y = C.F. + P.I = (c_1 + c_2x)e^{-x} + c_3e^{2x} - \frac{1}{8}[4x^3 + 18x^2 + 54x + 93]$$

EXERCISE

1) Solve $D^3 - 3D^2 + 3D - 1)y = \sin x + x^3$.

2) Solve $(D^2 + 3D + 2)y = 2\cos(2x+3) + 2e^x + x^2$

3) Solve $y''' + 2y'' - y' - 2y = 1 - 4x^3$

4) Solve $(D^3 - 3D + 2)y = x$

ANSWERS

1. $y = e^{x}(c_1 + c_2x + c_3x^2) + \frac{1}{4}(\sin x - \cos x) - (x^3 + 9x^2 + 36x + 69)$

2. $y = c_1e^{-x} + c_2e^{-2x}\ \frac{1}{10}[\cos(2x+3) - 3\sin(2x+3)] + \frac{1}{3}e^{x} + \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]$

3. $y = c_1e^{x} + c_2e^{-x} + c_3e^{-2x} + 2x^3 - 3x^2 + 15x - 8$

4. $y = (c_1 + c_2x)e^{x} + c_3e^{-2x} + \frac{1}{2}\left[x + \frac{3}{2}\right]$

P.I of *f(D)y=ϕ(x) when ϕ (x)=e^ax V where 'a' Is a Constant And V Is a Function of X.*

We will use the method to find P.I. When V is sin bx or x^k or a polynomial of degree k.

In this case , P.I= $\frac{1}{f(D)}\left(e^{ax}V\right) = e^{ax}\frac{1}{f(D+a)}(V)$

Working Rule: To find P.I. for e^{ax} V , take out e^{ax} to the left of *f(D)* and replace every D with (D+a) so that *f(D)* becomes *f(D-a)* and now operate $\frac{1}{f(D+a)}$ with 'V' alone by the previous methods.

1. **Solve** $(D^3 - 7D - 6)y = \cosh x \cos x$

Sol: A.E $m^3 - 7D - 6 = 0$

$(m+1)(m^2 - m - 6) = 0 \Rightarrow m = -1, -2, 3$

C.F= $c_1e^{-x} + c_2e^{-2x} + c_3e^{3x}$

P.I= $\frac{1}{D^3 - 7D - 6}\cosh x \cos x$

$= \frac{1}{D^3 - 7D - 6}\left(\frac{e^{x} + e^{-x}}{2}\right)\cos x$

$= \frac{1}{2}.\frac{1}{D^3 - 7D - 6}e^{x}\cos x + \frac{1}{2}.\frac{1}{D^3 - 7D - 6}e^{-x}\cos x$

Now $\frac{1}{D^3 - 7D - 6}e^{x}\cos x = e^{x}\frac{1}{(D+1)^3 - 7(D+1) - 6}\cos x$

$= e^{x}\frac{1}{D^3 - 3D^2 - 4D - 12}\cos x$

$= e^{x}\frac{1}{-D - 3 - 4D - 12}\cos x$ Replace D^2 by -1^2

$$= \frac{-1}{5} e^x \frac{1}{D+3} \cos x = \frac{-1}{5} e^x \frac{D-3}{D^2-9} \cos x$$

$$= \frac{1}{5} e^x \frac{1}{(-1-9)} (D-3) \cos x = \frac{e^x}{50} (-\sin x - 3\cos x)$$

In the same way $\frac{1}{D^3-7D-6} e^{-x} \cos x = \frac{e^{-x}}{34} (3\cos x - 5\sin x)$ complete solved on

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x} - \frac{e^x}{100} (\sin x + 3\cos x) + \frac{1}{68} e^{-x} (3x\cos x - 5\sin x)$$

2. Solve $(D^2 - 4D + 4)y = xe^{2x}$.

Sol: To find C.F,

The A.E is $m^2 - 4m + 4 = 0$ and the roots are 2,2

To find P.I:

$$\text{P.I} = \frac{1}{D^2 - 4D + 4} (e^{2x} - x)$$

$$= \frac{1}{(D-2)^2} e^{2x}.x = e^{2x}. \frac{1}{(D+2-2)^2} x$$

$$= e^{2x} \frac{1}{D^2} x = e^{2x} \frac{1}{D} \left(\frac{x^2}{2} \right) = e^{2x}. \frac{x^3}{6}$$

Hence the general solution is $y = (c_1 + c_2 x) e^{2x} + \frac{x^3 e^{2x}}{6}$

EXERCISE

1. $(D^2 + 2D + 4)y = e^x \sin 2x$
2. $(D^3 - 3D^2 + 3D - 1)y = (x+1)e^x$
3. $(D^3 - 2D + 4)y = e^x \sin x$
4. $y'' - 2y' + 2y = 1 + xe^{-x}$
5. $\frac{d^2 y}{dx^2} - 4y = x \sinh x$

ANSWERS

1. $y = [c_1 \cos(x\sqrt{3}) + c_2 \sin(x\sqrt{3})]e^{-x} - (8\cos 2x - 3\sin 2x) \frac{e^x}{73}$
2. $y = (c_1 + c_2 x + c_3 x^2) e^x + \frac{e^x}{24} (x+1)^4$
3. $y = c_1 e^{-2x} + (c_2 \cos x + c_3 \sin x) e^x + (\sin x + 3\cos x) \frac{xe^x}{-20}$

4. $y = e^{x}\left(c_1 \cos x + c_2 \sin x\right) + \frac{1}{2} + xe^{x}$

5. $y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

When $X = x^m v$ where m is a positive integer ,V is any function of x.

Working rules to find P.I of *f(D)* $y = x^m \sin ax \, or \, x^m \cos ax$

(i) P.I $= \dfrac{1}{f(D)} x^m \sin ax = \text{imaginary part of } \dfrac{1}{f(D)} x^m e^{iax}$

(ii) P.I $= \dfrac{1}{f(D)} x^m \cos ax = \text{real part of } \dfrac{1}{f(D)} x^m e^{iax}$

Case (VI): P.I= $\dfrac{1}{f(D)} xv = \left[x - \dfrac{1}{f(D)} f'(D)\right]\dfrac{1}{f(D)} v$

1. **Solve :*(D²-5D+6)y = xe⁴ˣ*.**

Sol: Auxiliary Equation is $m^2-5m+6=0$ *i.e.,* $(m-2)(m-3)=0 \therefore m=2,3.$

$\therefore$ The complementary function is $y_c = c_1 e^{2x} + c_2 e^{3x}$

Particular Integral, $y_p = \dfrac{xe^{4x}}{D^2 - 5D + 6} e^{4x} . \dfrac{1}{(D+4)^2 - 5(D+4) + 6}$

$$\therefore P.I. = \left[\frac{1}{f(D)} e^{ax} V = e^{ax} . \frac{1}{f(D+a)} V\right]$$

$$\therefore y_p = e^{4x} . \frac{1}{D^2 + 3D + 2} x = e^{4x} . \frac{1}{2}\left[1 + \frac{D^2 + 3D}{2}\right]^{-1} x$$

$$= \frac{1}{2} e^{4x}\left[1 - \frac{D^2 + 3D}{2}\right] x = \frac{1}{2} e^{4x}\left[x - \frac{1}{2}(0+3)\right]$$

$$= \frac{1}{2} e^{4x}\left(x - \frac{3}{2}\right) = \frac{1}{4} e^{4x}(2x-3)$$

Hence the general solution is y=C.F.+P.I. $= yc + yp$

i.e., $y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{4} e^{4x}(2x-3)$ where c_1 and c_2 are constants.

2. **Solve** $\dfrac{d^2 y}{dx^2} - 6\dfrac{dy}{dx} + 13y = 8e^{3x} \sin 2x$**.**

Sol: Given equation in operator form is $(D^2 - 6D + 13)y = 8e^{3x} \sin 2x$ ....(1)

Let $f(D)=D^2-6D+13$ $\therefore$ A.E. is *f(m)=0*

i.e., $m^2-6m+13=10 \Rightarrow m=3+2i.3-2i.$

The roots are complex and conjugate.

$\therefore$ C.F. is $y_c = e^{3x}(c_1 \cos 2x + c_2 \sin 2x)$

$$\text{P.I.} = y_p = 8\frac{e^{3x}\sin 2x}{D^2 - 6D + 13} = 8e^{3x}\frac{\sin 2x}{(D+3)^2 - 6(D+3) + 13}$$

$$8e^{3x}\frac{\sin 2x}{D^2+4} = 8e^{3x}\left(\frac{-x\cos 2x}{4}\right) \qquad \left[\therefore f(-a^2) = 0, \frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos as\right]$$

$= -2x \cos 2x\ e^{3x}$

Hence the general solution is $y = yc + yp$

i.e., $y = e^{3x}(C_1 \cos 2x + c_2 \sin 2x) - 2xe^{3x} \cos 2x$.

3. **Solve** $(D^3 - 3D^2 + 3D - 1)y = x^2.e^x$

Sol: Given D.E, is $(D^3 - 3D^2 + 3D - 1)y = x^2.e^x$

A.E. is $m^3 - 3m^2 + 3m - 1 = 0 \Rightarrow (m-1)^3 = 0$

$\therefore$ *m=1,1,1. The roots are real and equal.*

$\therefore C.F. = y_c = (c_1 + c_2 x + c_3 x^2)e^x$

$$\text{P.I.} = y_p = \frac{x^2.e^x}{(D-1)^3}[Put D = D+1]$$

$$= e^x.\frac{x^2}{(D+1-1)^3} = e^x\frac{x^2}{D^3} = e^x.\frac{1}{D^3}(x^2)$$

$$= e^x.\frac{1}{D^2}\left(\frac{x^3}{3}\right) = \frac{e^x}{3}.\frac{1}{D}\left(\frac{x^4}{4}\right) = e^x.\frac{x^5}{60}$$

$\therefore$ General solution is $y = yc + yp$

i.e., $y = (c_1 + c_2 x + c_3 x^2)e^x + e^x.\dfrac{x^5}{60}$

4. **Solve** $\boldsymbol{(D^2+1)y = x^2 \cosh x}$.

Sol: The roots are complex and conjugate numbers

Thus C.F. is $y_c = c_1 \cos x + c_2 \sin x$

$$\text{Now P.I. } \frac{1}{D^2+1}(x^2 \cosh x) = \frac{1}{D^2+1}\left[x^2\left(\frac{e^x + e^{-x}}{2}\right)\right]$$

$$= \frac{1}{2(D^2+1)}x^2 e^x + \frac{1}{2(D^2+1)}x^2 e^{-x} = P.I_1 + P.I_2$$

$$\therefore P.I_1 = \frac{x^2 e^x}{2(D^2+1)} = \frac{e^x}{2}\frac{x^2}{(D+1)^2+1} = \frac{e^x}{2}.\frac{x^2}{D^2+2D+2}$$

$$= \frac{e^x}{2} \cdot \frac{1}{2}\left(1 + \frac{D^2 + 2D}{2}\right)^{-1}(x^2)$$

$$= \frac{e^x}{4}\left[1 - \left(\frac{D^2 + 2D}{2}\right) + \left(\frac{D^2 + 2D}{2}\right)^2 + ...\right](x^2)$$

$$= \frac{e^x}{4}[x^2 - 1 - 2x + 2] = \frac{e^x}{4}(x^2 - 2x - 1)$$

$$P.I_2 = \frac{x^2 e^{-x}}{2(D^2+1)} = \frac{e^{-x}}{2}\frac{x^2}{(D+1)^2+1} = \frac{e^{-x}}{2} \cdot \frac{x^2}{D^2 + 2D + 2}$$

$$= \frac{e^{-x}}{4} \cdot \frac{x^2}{1 + \frac{D^2 - 2D}{2}} = \frac{e^{-x}}{4}\left(1 + \frac{D^2 - 2D}{2}\right)^{-1}(x^2)$$

$$= \frac{e^{-x}}{4} \cdot \left(1 - \frac{D^2 - 2D}{2} + \left(\frac{D^2 - 2D}{2}\right)^2 .....\right)(x^2)$$

$$= \frac{e^{-x}}{4}\left[x^2 - \frac{1}{2}(2) + (2x) + 2\right] = \frac{e^x}{4}(x^2 + 2x - 1)$$

$$\therefore P.I. = \frac{e^x}{4}(x^2 - 2x - 1) + \frac{e^{-x}}{4}(x^2 + 2x - 1)$$

Hence the general solution is

$$y = C.F. + P.I. = c_1 \cos x + c_2 \sin x + \frac{e^x}{4}(x^2 - 2x - 1) + \frac{e^{-x}}{4}(x^2 + 2x - 1)$$

5. Solve the equation $(D^2 - 2D + 2)y = e^x \tan x.$

Sol: A.E is m^2-$2m+2=0$

$$\therefore m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i2}{2} = 1 + i$$

$$\therefore C.F. = e^x(c_1 \cos x + c_2 \sin x)$$

$$P.I. = \frac{e^x \tan x}{D^2 - 2D + 2} e^x\left[\frac{1}{(D+1)^2 - 2(D+1) + 2}\right]\tan x$$

$$= e^x \cdot \frac{1}{D^2+1}\tan x = e^x \cdot \frac{1}{D^2 - i^2}\tan x = e^x\left[\frac{1}{(D+1)(D-1)}\right]\tan x$$

$$= \frac{e^x}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\tan x \qquad .....(1)$$

Now $\frac{1}{D-i}\tan x = e^{ix}\int e^{-ix}\tan x dx \left[\because \frac{1}{D-\alpha}X = e^{\alpha x}\int X.e^{-\alpha x}dx\right]$

$= e^{ix}\int(\cos x - i\sin x)\frac{\sin x}{\cos x}dx$

$= e^{ix}\int\left(\sin x - i\frac{1-\cos^2 x}{\cos x}\right)dx$

$= e^{ix}\left[-\cos x - i\int(\sec x - \cos x)dx\right]$

$= e^{ix}\left[-\cos x - i\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right) + i\sin x\right]$

$= -e^{ix}\left[(\cos x - i\sin x) + i\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$

$= -e^{ix}\left[e^{-ix} + i\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$

$= -\left[1 + ie^{ix}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$ ...(2)

Replacing i by -i in(2) , we get

$\frac{1}{D+i}\tan x = -\left[1 - ie^{-ix}\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right]$ ....(3)

By (1),(2) and (3)

$P.I. = \frac{e^x}{2i}\left[(-i)\left((e^{-ix}+e^{ix}\right)\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right] = -e^x\cos x\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

$\therefore$ The required general solution is

$y = C.F. + P.I = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$

6. Solve the equation $(D^2-4D+4)y = e^{2x} + \cos 2x + e^x\sin 2x$**.**

Sol: Auxiliary equation is

$m^2-4m+4=0$ i.e., $(m-2)^2=0$ $\therefore$ *m=2,2. Roots are real and equal*

C.F is $y_c = (c_1 + c_2 x)e^{2x}$

Now P.I.= $\frac{e^{2x}+\cos 2x + e^x\sin 2x}{D^2-4D+4} = \frac{e^{2x}}{(D-2)^2} + \frac{\cos 2x}{D^2-4D+4} + \frac{e^x\sin 2x}{D^2-4D+4}$

$= \frac{x^2}{2!}e^{2x} + \frac{\cos 2x}{-4-4D+4} + e^x.\frac{1}{(D+1)^2-4(D+1)+4}\sin 2x$

$$=\frac{x^2}{2}e^{2x}-\frac{1}{4D}(\cos 2x)+e^x.\frac{1}{D^2-2D+1}\sin 2x$$

$$=\frac{x^2}{2}e^{2x}-\frac{1}{4}\int \cos 2x dx+e^x.\frac{1}{-4-2D+1}\sin 2x$$

$$=\frac{x^2}{2}e^{2x}-\frac{1}{4}\left(\frac{\sin 2x}{2}\right)-e^x\frac{1}{2D+3}\sin 2x$$

$$=\frac{x^2}{2}e^{2x}-\frac{1}{8}\sin 2x-e^x\frac{(2D-3)\sin 2x}{4D^2-9}$$

$$=\frac{x^2}{2}e^{2x}-\frac{1}{8}\sin 2x-e^x\frac{(2D-3)\sin 2x}{4(-4)-9}$$

$$=\frac{x^2}{2}e^{2x}-\frac{1}{8}\sin 2x+\frac{1}{25}e^x(4\cos 2x-3\sin 2x)$$

∴ The general solution is

y=C.F.+P.I.

$$=(c_1+c_2x)e^{2x}+\frac{x^2}{2}e^{2x}-\frac{1}{8}\sin 2x+\frac{e^x}{25}(4\cos 2x-3\sin 2x)$$

7. Solve $(D^2+4D+3)y=e^x\cos 2x-\cos 3x-3x^3$**.**

Sol: Auxiliary equation is

$m^2+4m+3=0\Rightarrow(m+1)(m+3)=0$

∴ $m=-1,-3$. The roots are real and different.

∴ C.F. $c_1e^{-x}+c_2e^{-3x}$

$$\text{P.I.}=\frac{e^x\cos 2x-\cos 3x-3x^3}{D^2+4D+3}$$

$$=\frac{e^x\cos 2x}{D^2+4D+3}-\frac{\cos 3x}{D^2+4D+3}-\frac{3x^3}{D^2+4D+3}$$

$$=e^x.\frac{1}{(D+1)^2+4(D+1)+3}\cos 2x-\frac{\cos 3x}{-9+4D+3}-\frac{3x^3}{3\left(1+\frac{(D^2+4D)}{3}\right)}$$

$$=e^x.\frac{1}{D^2+6D+8}\cos 2x-\frac{\cos 3x}{4D-6}-\left(1+\frac{D^2+4D}{3}\right)^{-1}x^3$$

$$=e^x.\frac{1}{-4+6D+8}\cos 2x-\frac{(4D+6)\cos 3x}{16(-9)-36}-\left(1-\left(\frac{D^2+4D}{3}\right)+\left(\frac{D^2+4D}{3}\right)-\frac{64}{27}D^3\right)x^3$$

$$= e^x.\frac{1}{6D+4}\cos 2x - \frac{(4D+6)\cos 3x}{16(-9)-36} - \left[1-\left(\frac{D^2+4D}{3}\right)+\frac{16}{9}D^2+\frac{8}{9}D^3-\frac{64}{27}D^3\right]x^3$$

$$= e^x.\frac{(6D-4)(\cos 2x)}{36D^2-16} - \frac{(4D+6)\cos 3x}{-180} - \left[1-\frac{4}{3}D+\frac{13}{9}D^2-\frac{40}{27}D^3\right]x^3$$

$$= e^x.\frac{(6D-4)\cos 2x}{36(-4)-16} + \frac{1}{180}(-12\sin 3x + 6\cos 3x) - \left[1-\frac{4}{3}(3x^2)+\frac{13}{9}(6x)-\frac{40}{27}(6)\right]$$

$$= \frac{-e^x}{160}(-12\sin 2x - 4\cos 2x) + \frac{1}{90}[-6\sin 3x + 3\cos 3x] - \left[1-4x^2+\frac{26x}{3}-\frac{80}{9}\right]$$

$$= \frac{e^x}{160}.4(\cos 2x + 3\sin 2x) + \frac{1}{90}(3\cos 3x - 6\sin 3x) - \left[1-4x^2+\frac{26x}{3}-\frac{80}{9}\right]$$

$$= \frac{e^x}{40}(\cos 2x + 3\sin 2x) + \frac{1}{30}(\cos 3x - 2\sin 3x) - \left(1-4x^2+\frac{26x}{3}-\frac{80}{9}\right)$$

$\therefore$ The general solution is

y=C.F.+P.I.

$$= c_1e^{-x} + c_2e^{-2x} + \frac{e^x}{40}(\cos 2x + 3\sin 2x) + \frac{1}{30}(\cos 3x - 2\sin 3x) - \left[1-4x^2+\frac{26x}{3}-\frac{80}{9}\right]$$

Where c_1, c_2 are constants

8. Solve $(D^2-2D+1)y = x^2e^{3x} - \sin 2x + 3$.

Sol: Auxiliary equation is $m^2+2m+1= 0 \Rightarrow (m-1)^2=0$ $\therefore m=1,1$

The roots are real and equal.

$\therefore$ C.F.$=(c_1+c_2x)e^x$, where c_1, c_2 being arbitrary constants.

$$P.I = \frac{1}{D^2-2D+1}(x^2e^{3x} - \sin 2x + 3)$$

$$= \frac{1}{D^2-2D+1}x^2e^{3x} - \frac{1}{D^2-2D+1}\sin 2x + \frac{1}{D^2-2D+1}(3)$$

$$= \frac{1}{(D-1)^2}x^2e^{3x} - \frac{1}{D^2-2D+1}\sin 2x + \frac{e^{0.x}}{D^2-2D+1}(3)$$

$$= e^{3x}\frac{1}{(D+3-1)^2}x^2 - \frac{1}{-4-2D+1}\sin 2x + \frac{3}{0-0+1}$$

$$= e^{3x}\frac{1}{(D+2)^2}x^2 + \frac{1}{2D+3}\sin 2x + 3$$

$$= e^{3x}.\frac{1}{4\left(1+\frac{D}{2}\right)^2}x^2 + \frac{2D-3}{4D^2-9}\sin 2x + 3$$

$$= \frac{e^{3x}}{4}\left(1+\frac{D}{2}\right)^2 x^2 + \frac{2D-3}{4(-4)-9}.\sin 2x + 3$$

$$= \frac{e^{3x}}{4}\left(1 - D + \frac{3}{4}D^2 - ....\right)x^2 - \frac{1}{25}(2D-3)\sin 2x + 3$$

$$= \frac{e^{3x}}{4}\left(x^2 - 2x + \frac{3}{4}.2\right) - \frac{1}{25}(4\cos 2x - 3\sin 2x) + 3$$

$$= \frac{e^{3x}}{8}(2x^2 - 4x + 3) - \frac{1}{25}(4\cos 2x - 3\sin 2x) + 3$$

∴ Required solution by y=C.F.+P.I.

$$= (c_1 + c_2 x)e^x + \frac{1}{8}e^{3x}(2x^2 - 4x + 3) - \frac{1}{25}(4\cos 2x - 3\sin 2x) + 3$$

9. Solve $(D^2 + 4D + 4)y = e^{-x}\sin 2x$**.**

Sol: Auxiliary equation is $m^2+4m+4=0 \Rightarrow (m+2)^2=0$

The roots are real and equal

C.F.= $(c_1 + c_2 x)e^{-2x}$

$$P.I. = \frac{e^{-x}\sin 2x}{D^2 + 4D + 4} = e^{-x}\frac{1}{(D-1)^2 + 4(D-1) + 4}\sin 2x$$

$$= e^{-x}\frac{1}{D^2 + 2D + 1}\sin 2x = e^{-x}.\frac{1}{-4 + 2D + 1}\sin 2x$$

$$= e^{-x}.\frac{2D+3}{(2D-3)(2D+3)}\sin 2x = e^{-x}.\frac{2D+3}{4D^2 - 9}\sin 2x$$

$$= e^{-x}.\frac{2D+3}{4(-4)-9}\sin 2x = -\frac{e^{-x}}{25}(4\cos 2x + 3\sin 2x)$$

∴ The general solution is

$y = C.F + P.I. = (c_1 + c_2 x)e^{-2x} - \frac{e^{-x}}{25}(4\cos 2x + 3\sin 2x)$ where c_1 and c_2 are constants.

10. Solve $(D^2 + 9)y = (x^2 + 1)e^{3x}$**.**

Sol: Auxiliary equation is $m^2+9=0 \Rightarrow m^2=(3i)^2$ ∴ $m=\pm i3$

The roots are complex and conjugate numbers.

Hence C.F. $=c_1\cos 3x+c_2\sin 3x$

Now P.I. $= \frac{(x^2 + 1)e^{3x}}{D^2 + 9} = \frac{x^2 e^{3x}}{D^2 + 9} + \frac{e^{3x}}{D^2 + 9}$

$$= e^{3x}.\frac{1}{(D+3)^2+9}x^2 + \frac{e^{3x}}{3^2+9} = e^{3x}.\frac{1}{D^2+6D+18}x^2 + \frac{e^{3x}}{18}$$

$$= e^{3x}.\frac{1}{18\left(1+\frac{D^2+6D}{18}\right)}x^2 + \frac{e^{3x}}{18} = \frac{e^{3x}}{18}\left(1+\frac{D^2+6D}{18}\right)^{-1}x^2 + \frac{e^{3x}}{18}$$

$$= \frac{e^{3x}}{18}\left[1-\frac{D^2+6D}{18}+\left(\frac{D^2+6D}{18}\right)^2+...\right]x^2 + \frac{e^{3x}}{18}$$

$$= \frac{e^{3x}}{18}\left[1-\frac{D^2+6D}{18}+\frac{36D^2}{(18)^2}\right]x^2 + \frac{e^{3x}}{18}$$

$$= \frac{e^{3x}}{18}\left[x^2-\frac{1}{18}(2+12x)+\frac{36}{(18)^2}(2)\right] + \frac{e^{3x}}{18} = \frac{e^{3x}}{18}\left[x^2-\frac{1}{9}+\frac{2}{3}x+\frac{2}{9}\right] + \frac{e^{3x}}{18}$$

$$= \frac{e^{3x}}{18}\left(x^2+\frac{2}{3}x+\frac{1}{9}\right) + \frac{e^{3x}}{18} = \frac{e^{3x}}{18}\left(x^2+\frac{2}{3}x+\frac{10}{9}\right)$$

$\therefore$ The general solution is y=C.F.+P.I.= $c_1\cos 3x + c_2\sin 3x + \frac{e^{3x}}{18}\left(\frac{x^2}{2}+\frac{2}{3}x+\frac{10}{9}\right)$

Where c_1 and c_2 are constants.

11. Solve $\frac{d^4y}{dx^4} - y = \cos x\cosh x + 2x^4 + x - 1$**.**

Sol: Auxiliary equation is $m^4-1=0 \Rightarrow (m^2-1)(m^2+1)=0$

$\Rightarrow m^2-1=0$ or $m^2+1=0$ $\therefore m=1,-1,\pm i$.

Two roots are real and different other two roots are complex, conjugate numbers.

$$\therefore C.F. = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x$$

$$\text{P.I.} = \frac{1}{D^4-1}(\cos x\cosh x + 2x^4 + x - 1)$$

$$= \frac{1}{D^4-1}.\frac{1}{2}(e^x+e^{-x})\cos x + \frac{1}{D^4-1}(2x^4+x+1)$$

$$= \frac{1}{2}\left[\frac{1}{D^4-1}e^x\cos x + \frac{1}{D^4-1}e^{-x}\cos x\right] - \frac{1}{1-D^4}(2x^4+x-1)$$

$$=\frac{1}{2}\left[e^x.\frac{1}{(D+1)^4-1}\cos x+e^{-x}.\frac{1}{(D-1)^4-1}\cos x\right]-(1-D^4)-1(2x^4+x-1)$$

$$=\frac{1}{2}\left[e^x.\frac{1}{(D^4+4D^3+6D^2+4D+1)-1}\cos x+e^{-x}.\frac{1}{(D^4+4D^3+6D^2-4D+1)-1}\cos x\right]$$
$$-(1+D^4+D^8+...)(2x^4+x-1)$$

$$=\frac{1}{2}\left[e^x.\frac{1}{(-1)^2+4D(-1)+6(-1)+4D}\cos x+e^{-x}.\frac{1}{(-1)^2-4(D)(-1)+6(-1)-4D}\cos x\right]$$
$$-[(2x^4+x-1)+(48)]$$

$$=\frac{1}{2}\left[e^x.\frac{1}{5}\cos x+e^{-x}.\frac{1}{-5}\cos x\right]-[2x^4+x+47]$$

$$=-\frac{1}{5}\cos x\left(\frac{e^x+e^{-x}}{2}\right)-(2x^4+x+47)=\frac{1}{5}\cos x\cosh x-(2x^4+x+47)$$

$\therefore$ The general solution is

$$y=C.F+P.I.=c_1e^x+c_2e^{-x}+c_3\cos x+c_4\sin x-\frac{1}{5}\cos\cosh x-(2x^4+x+47)$$

Where c_1, c_2, c_3, c_4 are constants.

P.I. of *f(D)y=φ(x) when φ(x) = x^m V,m being a positive integer and V is any function of x.*

Here V is either sin ax or con ax only. It should not be of the form x^n or e^{ax}.

Working Rule for finding P.I. of f(D)y=x^m sin ax or x^m cos ax

(i) $P.I=\frac{1}{f(D)}x^m\sin ax=$ ***Imaginary Part(I.P.)*** $\frac{1}{f(D)}x^m(\cos ax+i\sin ax)$

$I.P.of\ \frac{1}{f(D)}x^me^{-iax}$

(ii) P.I.= $\frac{1}{f(D)}x^m\cos ax=R.P(realpart)of\ \frac{1}{f(D)}x^me^{iax}$

Alternative Method for Finding P.I of *f(D)y=φ(x) when φ(x)=xV(when m=1) where V is a function of x.*

Let *f(D)y=x V where V is a function of x. Operating with* $\frac{1}{f(D)}$*, we get* $y=\frac{1}{f(D)}(xV)$

$\therefore P.I.=\frac{1}{f(D)}(xV)$

Consider $D(xV)=xDV+V;D^2(xV)=xD^2V+2DV$

Similarly $D^n(xV) = xD^nV + nD^{n-1}V$

$$\therefore (D^n + P_1D^{n-2} + \ldots P_n)xV = x[D^n + P_1D^{n-1} + P_2D^{n-2} + \ldots + P_n]V + [nD^{n-1} + P_1(n-1)D^{n-2} + \ldots + P_{n-1}]V$$

$$\Rightarrow f(D)xv = xf(D)V + f'(D)V \qquad \ldots(1)$$

Let $f(D)V = V_1 \Rightarrow V = \dfrac{1}{f(D)}V_1 \qquad \ldots(2)$

$$\therefore f(D)x\frac{1}{f(D)}V_1 = xV_1 + f'(D)\frac{1}{f(D)}V_1 \qquad \text{[From (1) \& (2)]}$$

Operating with $\dfrac{1}{f(D)}$ on both sides, we get

$$\frac{1}{f(D)}(xV_1) = x\frac{1}{f(D)}V_1 - \frac{1}{f(D)}f'(D)\frac{1}{f(D)}V_1$$

i.e., $\dfrac{1}{f(D)}(xV_1) = \left[x - \dfrac{1}{f(D)}f'(D)\right]\dfrac{1}{f(D)}V_1$

$$\therefore P.I. = \frac{1}{f(D)}(xV) = \left[x - \frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}V.$$

1. Solve $(D^2 - 2D + 1)y = xe^x \sin x$

Sol: A.E is $m^2 - 2m + 1 = 0 \Rightarrow m = 1,1$

$$y_c = C.F = (c_1 + c_2x)e^x$$

$$y_p = P.I = \frac{1}{f(D)}X = \frac{1}{(D-1)^2}e^x x\sin x = e^x x\sin x = e^x\frac{1}{(D+1-1)^2}x\sin x$$

$$= e^x\frac{1}{D^2}x\sin x = e^x[I.Pof\,\frac{1}{D^2}xe^{ix}]$$

$$= e^x\left[I.Pofe^{ix}\frac{1}{(D+i)^2}x\right] = e^x\left[I.Pofe^{ix}\frac{1}{D^2 + 2iD - 1}x\right]$$

$$= e^x\left[I,Pofe^{ix}(-1)\right]\quad\left[\{1 - (D^2 + 2iD\}^{-1}x\right]$$

$= e^x[I.Pofe^{ix}(-1)\{1 + D^2 + 2iD\}x]$ expanding up to terms containing D only

$$= e^x[I.Pofe^{ix}(-1)\{x + 2iDx\}]$$

$$= e^x[I.Pofe^{ix}(-1)\{x + 2i\}]$$

$$= e^x[I.Pof(\cos x + i\sin x)(-x - 2i)]$$

$$= -e^x(2\cos x + x\sin x) \qquad \therefore y = y_c + y_p$$

2. Solve $\frac{d^2y}{dx^2} - y = x^2\cos^2 x$ **.**

Sol: Given equation in operating form is, $(D^2 - D)y = x^2\cos^2 x$

A.E is $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$y_c = C.F = c_1e^x + c_2e^{-x}$

$$y_p = P.I = \frac{1}{f(D)}X = \frac{1}{D^2-1}x^2\cos^2 x = \frac{x^2(1+\cos 2x)}{2(D^2-1)}$$

$$= \frac{x^2}{2(D^2-1)} + \frac{x^2\cos 2x}{2(D^2-1)} = y_{p1} + y_{p2} \qquad ....(1)$$

$$y_{p1} = \frac{x^2}{2}(1-D^2)^{-1} = (1+D^2)\left(\frac{x^2}{2}\right) = \frac{-x^2}{2} - x = -\left(\frac{x^2}{2}+1\right)$$

$$y_{p2} = \frac{x^2 R.Pof}{2(1-D)^2}e^{i2x} = R.Pof\ \frac{e^{i2x}}{2}\left[\frac{x^2}{1-(D+2i)^2}\right]$$

$$=\text{-R.P of } \frac{e^{i2x}}{2}\left[\frac{x^2}{1-D^2-4i^2-4Di}\right]$$

$$=\text{-R.P of } \frac{e^{i2x}}{2}\left[\frac{x^2}{5-D^2-4Di}\right]$$

$$=\text{-R.P of } \frac{e^{i2x}}{10}\left[\frac{x^2}{1-\frac{D^2+4iD}{5}}\right]$$

$$=\text{-R.P of } \frac{e^{i2x}}{10}\left[x^2+\frac{2}{5}+\frac{8ix}{5}\right]$$

$$=\text{R.P of } \left(\frac{\cos 2x + i\sin 2x}{10}\right)\left(x^2+\frac{2}{5}+\frac{8}{5}ix\right)$$

$$= -\frac{x^2\cos 2x}{10} - \frac{\cos 2x}{25} + \frac{8x\sin 2x}{50}$$

$$= \frac{4x\sin 2x}{25} - \frac{x^2\cos 2x}{10} - \frac{\cos 2x}{25}$$

$\therefore y_p = y_{p1} + y_{p2}$

Complete solution is , $y = c_1e^x + c_2e^{-x} - \left(\frac{x^2}{2}+1\right) + \frac{4x\sin 2x}{25}$

3. Solve $(D^2+3D+2)=\sin e^x$

Sol: A.E is $m^2+3m+2=0 \Rightarrow (m+1)(m+2)=0$

$m=-1,-2$

$y_c = C.F = c_1e^{-x}+c_2e^{-2x}$

$y_p = P.I = \frac{1}{f(D)}X = \frac{1}{D^2+3D+2}\sin x = \frac{1}{(D+1)(D+2)}\sin e^x$

$= \frac{1}{D+1}\sin e^x - \frac{1}{D+2}\sin e^x$

$= e^{-x}\int e^x \sin e^x dx - e^{-2x}\int e^{2x}\int e^{2x}\sin e^x dx$

Put $e^x = t \Rightarrow e^x dx = dt$

$P.I = \bar{e}^t\int \sin t dt - \bar{e}^{2t}\int t\sin t dt$

$= \bar{e}^t(-\cos t) - \bar{e}^{2t}[t(-\cos t)+(1)(\sin t)]$

$= -\bar{e}^t\cos t - \bar{e}^{2t}[-e^x\cos e^x + \sin e^x] = -e^{2x}\sin e^x$

Complete solution $y = y_c + y_p$

$y = c_1e^{-x}+c_2e^{-2x}-e^{2x}\sin e^x$

4..Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x.$

Sol: Given equation $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x.$

Let $f(D)=D^2\text{-}4D+4.$

A.E is $f(m)=0 \Rightarrow m^2\text{-}4m+4=0 \Rightarrow (m\text{-}2)^2=0$ $\therefore$ m=2,2

Hence C.F is $y_c = (c_1+c_2x)e^{2x}$

Here P.I can be found out using the above case twice which is laborious. We will find P.I. in another way.

$$P.I. = \frac{8e^{2x}x^2\sin 2x}{D^2-4D+4} = 8e^{2x}\left[\frac{x^2\sin 2x}{(D+2)^2-4(D+2)+4}\right]$$

$$\therefore y_p = 8e^{2x}\left[\frac{x^2\sin 2x}{D^2}\right] \text{ imaginary part of } 8e^{2x}\left\{\frac{1}{D^2}x^2e^{2ix}\right\}$$

$$=8e^{2x}I.P.of e^{2ix}\frac{x^2}{(D+2i)^2}\left[\because P.I.\frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$$

$$=8e^{2x,}I.P.of e^{2ix}\frac{x^2}{4i^2\left(1+\frac{D}{2i}\right)^2} = 8e^{2x}I.P.of\frac{e^{2ix}}{-4}\left(1+\frac{D}{2i}\right)^{-2}(x^2)$$

$$= 8e^{2x} I.P.of \frac{e^{2ix}}{-4}\left[1 - \frac{2D}{2i} + 3\frac{D^2}{4i^{2\ldots\ldots}}\right](x^2) = 8e^{2x} I.Pof \frac{e^{2ix}}{-4}\left[x^2 - \frac{2x}{i} + \frac{3}{2i^2}\right]$$

$$= 8e^{2x} I.P.of \frac{e^{2ix}}{-4}\left[x^2 + 2ix - \frac{3}{2}\right] = 2e^{2x} I.P.of\,(\cos 2x + i\sin 2x)\left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= 2e^{2x}\left[2x\cos 2x + \left(x^2 - \frac{3}{2}\right)\sin 2x\right]$$

The general solution is $y = y_c + y_p$

$$i.e., y = (c_1 + c_2 x)e^x - 2e^{2x}\left[2x\cos 2x + \left(x^2 - \frac{3}{2}\right)\sin 2x\right]$$

5. **Solve** $(D^2 - 4)y = x\sin \lambda x$.

Sol: Auxiliary equation is *m2-4=0* ⇒m= ±2. The roots are real and different

C.F.= $c_1e^{2x}+c_2e^{-2x}$ Where c_1 and $c_2$are constants.

$$\text{P.I.} \frac{1}{D^2 - 4} x\sin \lambda x = P.I.of \frac{1}{D^2 - 4} xe^{i\lambda x}$$

$$= I.P.of e^{i\lambda x} \frac{1}{(D + i\lambda)^2 - 4} x = I.P.of e^{i\pi x} \frac{1}{D^2 + 2i\lambda D - (4 + \lambda^2)} x$$

$$=\text{P.I. of } e^{i\lambda x}\left(-\frac{1}{4 + \lambda^2}\right)\left[1 - \frac{D^2 + 2i\lambda D}{4 + \lambda^2}\right]^{-1} x$$

$$=\text{I.P. of } -\frac{e^{i\lambda x}}{4 + \lambda^2}\left[1 + \frac{D^2 + 2i\lambda D}{4 + \lambda^2}\right]x = I.Pof -\frac{e^{i\lambda x}}{4 + \lambda^2}\left[x + \frac{2i\lambda}{4 + \lambda^2}\right]$$

$$\text{I.P. of} - \frac{1}{4 + \lambda^2}(\cos \lambda x + i\sin \lambda x)\left(x + \frac{i2\lambda}{4 + \lambda^2}\right)$$

$$= -\frac{x\sin \lambda x}{4 + \lambda^2} - \frac{2\lambda\cos \lambda x}{(4 + \lambda^2)^2} = -\frac{1}{(4 + \lambda^2)^2}[(4 + \lambda^2)x\sin \lambda x + 2\lambda\cos \lambda x)]$$

∴ The general solution is

$$y = C.F + P.I = c_1e^{2x} + c_2e^{-2s} - \frac{1}{(4 + \lambda^2)^2}[(4 + \lambda^2)x\sin \lambda x + 2\lambda\cos \lambda x]$$

6. Solve $(D^2 + 2D + 1)y = x\cos x$**.**

Sol: Given equation is $(D^2 + 2D + 1)y = x\cos x$

Let *f(D)=D2+2D+1. Then AE is f(m)=0*

⇒$m^2+2m+1=0$ ⇒$(m+1)^2=0$

∴ The roots are -1,-1 which are real and equal.

Thus y_c=C.F. $=(c_1+c_2x)e^{-x}$

Now P.I. $y_p = \frac{x\cos x}{D^2+2D+1} = \left[x - \frac{1}{f(D)} f'(D)\right]\frac{1}{f(D)}\cos x$

$$=\left[x - \frac{1}{D^2+2D+1}2(D+1)\right]\frac{1}{D^2+2D+1}\cos x$$

$$=\left[x - \frac{1}{(D+1)^2}2(D+1)\right]\frac{1}{-1+2D+1}\cos x$$

$$=\left[x - \frac{2}{(D+1)}\right]\frac{1}{2D}\cos x = \left[x - \frac{2}{D+1}\right]\frac{\sin x}{2}$$

$$=\frac{x}{2}\sin x - \frac{\sin x}{D+1} = \frac{x}{2}\sin x - \frac{D-1}{D^2-1}\sin x = \frac{x\sin x}{2} - \frac{D-1}{-1-1}\sin x$$

$$=\frac{x\sin x}{2} - \frac{(\cos x - \sin x)}{-2} = \frac{x\sin x}{2} + \frac{\cos x - \sin x}{2}$$

Hence the general solution is given by $y = y_c + y_p$

i.e., $y = (c_1 + c_2 x)e^{-x} + \frac{x}{2}\sin x + \frac{1}{2}(\cos x - \sin x)$

7. **Solve** $(D^2+1)x = t\cos t \;\; given x = 0, \frac{dx}{dt} = 0$ **at t=0**.

Sol: A.E is $m^2+1=0 \Rightarrow$ m = $\pm$ i

$\therefore$ The roots are conjugate complex numbers.

C.F. $x_c = c_1\cos t + c_2\sin t$

P.I.= $x_p = \frac{t\cos t}{D^2+1}$

$$=\left(t - \frac{2D}{D^2+1}\right)\frac{\cos(t)}{D^2+1} = \left(t - \frac{2D}{D^2+1}\right)\left(\frac{t}{2}\sin t\right)$$

$$=\frac{t^2}{2}\sin t - \frac{D}{D^2+1}(t\sin t) = \frac{t^2}{2}\sin t - \frac{|t\cos t + \sin t|}{D^2+1}$$

$$=\frac{t^2}{2}\sin t - \frac{t\cos t}{D^2+1} - \frac{\sin t}{D^2+1} = \frac{t^2}{\sin t} + \frac{t\cos t}{2} - \frac{t\cos t}{D^2+1}$$

$$=\frac{t^2}{\sin t} + \frac{t\cos t}{2} - x_p$$

$$\Rightarrow 2x_p = \frac{t^2}{\sin t} + \frac{t\cos t}{2} \Rightarrow x_p = \frac{t^2}{2\sin t} + \frac{t\cos t}{4}$$

G.S is $x = x_c + x_p = c_1\cos t + c_2\sin t + \frac{t^2}{2\sin t} + \frac{t\cos t}{4}$

By data when $t=0 \Rightarrow x=0 \Rightarrow c_1=0$

$$\frac{dx}{dt} = -c_1 \sin t + c_2 \cos t + \frac{\sin t(2t) - t^2(\cos t)}{2\sin^2 t} + \frac{t(-\sin t) + \cos t}{4}$$

By data $t = 0 \Rightarrow \frac{dx}{dt} = 0 \Rightarrow 0 = 0 + c_2 + 0 + \frac{1}{4}$

$\therefore c_2 = -\frac{1}{4}$

$\therefore$ General solution is $x = -\frac{1}{4}\sin t + \frac{t^2}{2\sin t} + \frac{t\cos t}{4}$.

8. **Solve the differential equation** $(D^2+4)y = x\sin x$.

Sol: Auxiliary Equation is $m^2+4=0 \Rightarrow m^2=(2i)^2$

$\therefore m=\pm i2$. The roots are complex and conjugate.

Hence Complementary Function, $y_c = c_1\cos 2x + c_2 \sin 2x$

Particular Integral, $y_p = \frac{1}{D^2+4} x\sin x$

$= I.P.of \frac{1}{D^2+4} xe^{ix}$

$=$I.P. of $e^{ix} \frac{1}{(D+i)^2+4} x = I.P.of e^{ix} \frac{1}{D^2+2Di+3} x$

$=$I.P. of $\frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}\right)^{-1} x$

$=$I.P. of $\frac{e^{ix}}{3}\left(1+\frac{D^2+2Di}{3}+....\right)x$

$=$I.P. of $\frac{e^{ix}}{3}\left(1-\frac{2}{3}Di\right)x[\because D^2(x)=0, etc.]$

$=$I.P. of $\frac{1}{3}(\cos x + i\sin x)\left(x - i\frac{2}{3}\right)$

$=\frac{1}{3}\left(-\frac{2}{3}\cos x + x\sin x\right)$

Hence the general solution is

$$y = y_c + y_p = c_1\cos 2x + c_2 \sin 2x + \frac{1}{3}\left(x\sin x - \frac{2}{3}\cos x\right)$$

9 .**Solve** $\frac{d^2y}{dx^2}+9y=x\sin 2x$.

Sol: Given equation is $(D^2+9)y=x\sin 2x$

Here $f(D)=D^2+9$ then A.E is $f(m)=0\Rightarrow m^2+9=0$

The roots are $m=\pm 3i$ *which are complex conjugate numbers.*

$\therefore C.F.=y_c=c_1\cos 3x+c_2\sin 3x$

P.I.= $y_p=\frac{x\sin 2x}{D^2+9}=\left[x-\frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}\sin 2x$

$=\left[x-\frac{2D}{D^2+9}\right]\frac{1}{D^2+9}\sin 2x$ [Put $D^2=-4$]

$=\left[x-\frac{2D}{D^2+9}\right]\frac{\sin 2x}{-4+9}=\frac{x\sin 2x}{5}-\frac{2D\sin 2x}{5(D^2+9)}$

$=\frac{x\sin 2x}{5}-\frac{4\cos 2x}{5(-4+9)}=\frac{x\sin 2x}{5}-\frac{4}{25}\cos 2x$

General solution is $y=y_c+y_p$

i.e., $y=c_1\cos 3x+c_2\sin 3x+\frac{x\sin 2x}{5}-\frac{4}{25}\cos 2x$

10. **Solve** $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=xe^x\sin x$.

Sol: Given equation is the operator form is

$(D^2+3D+2)y=xe^x\sin x$ ....(1)

A.E. is $f(m)=0\Rightarrow m^2+3m+2=0\Rightarrow(m+2)(m+1)=0$

The roots are -2 and -1 which are real and different.

$\therefore C.F.=y_c=c_1e^{-2x}+c_2e^{-x}$

Now P.I.= $y_p=\frac{xe^x\sin x}{D^2+3D+2}=e^x\frac{x\sin x}{(D+1)^2+3(D+1)+2}$

$=e^x\frac{x\sin x}{D^2+5D+6}=e^x\left[x-\frac{2D+5}{D^2+5D+6}\right]\frac{\sin x}{D^2+5D+6}$

$=e^x\left[x-\frac{2D+5}{D^2+5D+6}\right]\frac{\sin x}{-1+5D+6}=e^x\left[x-\frac{2D+5}{D^2+5D+6}\right]\frac{\sin x}{5(D+1)}$

$\therefore y_p=\frac{e^x}{5}\left[x-\frac{2D+5}{D^2+5D+6}\right]\frac{(D-1)\sin x}{D^2-1}$

$=\frac{e^x}{5}\left[x-\frac{2D+5}{D^2+5D+6}\right]\frac{\cos x-\sin x}{-2}$

$$=\frac{e^x x(\sin x-\cos x)}{10}+\frac{e^x}{10}\frac{(2D+5)(\cos x-\sin x)}{D^2+5D+6}$$

$$=\frac{e^x x(\sin x-\cos x)}{10}+\frac{e^x}{10}\left[\frac{-2\sin x-2\cos x+5\cos x-5\sin x}{D^2+5D+6}\right]$$

$$=\frac{e^x x(\sin x-\cos x)}{10}+\frac{e^x}{10}\left[\frac{3\cos x-7\sin x}{D^2+5D+6}\right]$$

$$=\frac{e^x x(\sin x-\cos x)}{10}+\frac{e^x}{10}\frac{3\cos x-7\sin x}{-1+5D+6}$$

$$=\frac{e^x x(\sin x-\cos x)}{+10}+\frac{e^x}{10}\frac{(D-1)}{5(D+1)(D-1)}(3\cos x-7\sin x)$$

$$=\frac{e^x x(\sin x-\cos x)}{10}+\frac{e^x}{50}(D-1)\left(\frac{3\cos x-7\sin x}{-2}\right)$$

$$=\frac{e^x x(\sin x-\cos x)}{10}-\frac{e^x}{100}(-3\sin x-7\cos x-3\cos x+7\sin x)$$

$$=\frac{e^x x(\sin x-\cos x)}{10}-\frac{e^x}{100}(4\sin x-10\cos x)$$

General solution is $y=y_c+y_p$

i.e., $y=c_1e^{-x}+c_2e^{-2x}+e^x\left[\frac{x}{10}(\sin x-\cos x)-\frac{1}{25}\sin x+\frac{1}{10}\cos x\right]$

11. **Solve** $\frac{d^2y}{dx^2}-y=x\sin x+1(1+x^2)e^x$.

Sol: Given equation in operator form is $(D^2-1)y=x\sin x+(1+x^2)e_x$. ....(1)

This is differential equation with constant coefficients .

A.E. is $m^2-1=0$. The roots re $m=\pm 1$. They are real and different.

C.F. is $y_c=c_1e^x+c_2e^{-x}$

P.I.= $y_{p1}=\frac{1}{D^2-1}(x\sin x)=I.P.of\ \frac{1}{D^2-1}xe^{ix}$

$=I.P.of e^{ix}\frac{1}{(D+i)^2-1}x=I.P.of e^{ix}\frac{1}{D^2+2i-2}x$

=I.P. of $e^{ix}\left(\frac{-1}{2}\right)\left(1-\frac{D^2+2iD}{2}\right)^{-1}x$

=I.P. of $e^{ix}\left(\frac{-1}{2}\right)\left[1+\frac{D^2+2iD}{2}\right](x)$

$$=\text{I.P.}\,(\cos x + i\sin x)\left(\frac{-1}{2}\right)[x+1] = \frac{-1}{2}(x\sin x + \cos x)$$

$$\text{P.I}_2 = y_{p2} = \frac{1}{D^2-1}(1+x^2)e^x = e^x\frac{1}{(D+1)^2-1}(1+x^2)$$

$$= e^x\frac{1}{D^2+2D}(1+x^2) = e^x\frac{1}{2D}\frac{1}{1+\frac{D}{2}}(1+x^2) = e^x.\frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= e^x\frac{1}{2D}\left(1-\frac{D}{2}+\frac{D^2}{4}-\frac{D^3}{8}+......\right)(1+x^2) = \frac{e^x}{2}\left(\frac{1}{D}-\frac{1}{2}+\frac{D}{4}-\frac{D^2}{8}\right)(1+x^2)$$

$$= \frac{e^x}{2}\left(x+\frac{x^3}{3}-\frac{1}{2}-\frac{x^2}{2}+\frac{2x}{4}-\frac{2}{8}\right) = \frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}-\frac{3}{4}\right)$$

$\therefore$ General solution is given by

$$y = y_c + y_{p1} + y_{p2}$$

i.e. $y = c_1 ex + c_2 e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{3x}{2}-\frac{3}{4}\right)$

2.6 Method of variation of parameters:

General form is $\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = R(x)$ Where P_1, P_2, X are functions of X.

Working rule:

1. Reduce the given equation to the standard form.

2. Find complementary function of $\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = R(x)$ and let the solution be $y_c = c_1 u(x) + c_2 v(x)$.

3.P.I =Au+Bv where A and B are functions of x.

4. Find w(u,v) $= u\frac{dv}{dx} - v\frac{du}{dx}$ where w(u,v) is wronskian of u, v

5. Find A and B using $A = -\int \frac{vR}{w(u,v)}dx$

$$= -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}}$$

$$B = \int \frac{uRdx}{u\frac{dv}{dx} - v\frac{du}{dx}}$$

Complete solution is $y = c_1u(x) + c_2v(x) + Au(x) + Bv(x)$ where C_1, C_2 are constants.

1. $Solve\ y'' - 6y'' + 9y = \frac{e^{3x}}{x^2}$

A) $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

A.E: $m^2 - 6m + 9 = 0 \Rightarrow (m-3)^2 = 0 \Rightarrow m = 3,3$

$y_c = C.F = (c_1 + c_2x)e^{3x} = c_1e^{3x} + c_2xe^{3x}$

$= c_1u(x) + c_2v(x)$

Where $u(x) = e^{3x}, v(x) = xe^{3x}$

$u'(x) = 3e^{3x}, v'(x) = e^{3x} + x3e^{3x}$

$w = uv' - vu' = e^{3x}(e^{3x} + 3xe^{3x}) - xe^{3x}3e^{3x}$

$= e^{6x} + 3xe^{6x} - 3xe^{6x} = e^{6x}$

$$A = -\int \frac{vRdx}{w} = -\int \frac{\frac{xe^{3x}e^{3x}}{x^2}dx}{e^{6x}} = -\int \frac{1}{x}dx = -\log x$$

$$B = \int \frac{uRdx}{w} = \int \frac{\frac{e^{3x}e^{3x}}{x^2dx}}{e^{6x}} = \int \frac{1}{x^2}dx = -\frac{1}{x}$$

$$y = P.I = -e^{3x}\log x - x^{e3x}\frac{1}{x} = -(e^{3x}\log x + e^{3x})$$

Computing solution $y = y_c + y_p$

$y = (c_1 + c_2x)e^{3x} - (e^{3x}\log x + e^{3x})$

2. **Solve** $(D^2 - 3D + 2)y = e^{4x}$ **by the method of variation of parameters.**

Sol: A.E: $m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m = 1,2$

$y_c = C.F : c_1e^x + c_2e^{2x} = c_1u(x) + c_2v(x)$

Where $u(x) = e^x and v(x) = e^{2x}$

$u'(x) = e^x and v'(x) = 2e^{2x}$

$w = uv' - vu' = e^x2e^{2x} - e^{2x}e^x = 2e^{3x} - e^{3x} = e^{3x}$

$$A = -\int \frac{vRdx}{w} = -\int \frac{e^{2x}e^{4x}}{e^{3x}}dx = -\int e^{3x}dx = -\frac{e^{3x}}{3}$$

$B = \int \frac{uRdx}{w} = \int \frac{e^xe^{4x}}{e^{3x}} = \int e^{2x}dx = \frac{e^{2x}}{2}$ Let P.I=Au+Bv

$$y_p = P.I = \frac{-e^{3x}}{3}e^{x} + \frac{e^{2x}}{2}e^{2x} = \frac{-e^{4x}}{3} + \frac{e^{4x}}{2} = \frac{e^{4x}}{6}$$

$\therefore$ Complete solution is $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{2x} + \frac{e^{4x}}{6}$$

3. Solve $(D^2+1)x = t\cos 2t$ given $x=0 + \frac{dx}{dt}=0$ at $t=0$.

Sol: Given D.E is $(D^2+1)x = t\cos 2t$.

A.E is $m^2+1=0 \Rightarrow m=\pm i$

Thus C.F.= $x_c = c_1\cos t + c_2 \sin t$

Now P.I.= $x_p = \frac{t\cos 2t}{D^2+1} = \left(t - \frac{2D}{D^2+1}\right)\frac{\cos 2t}{D^2+1}$ [Put $D^2=-4$]

$$= \left(t - \frac{2D}{-3}\right)\frac{\cos 2t}{-3} = \frac{t\cos 2t}{-3} - \frac{2D\cos 2t}{9}$$

$$\therefore\ x_p = \frac{t\cos 2t}{-3} + \frac{4\sin 2t}{9}$$

Hence the general solution is

$$x = (c_1\cos t + c_2\sin t) - \frac{t\cos 2t}{3} + \frac{4\sin 2t}{9} \qquad ....(1)$$

By data, x=0 at t=0

$\therefore$ (1) $\Rightarrow$ 0=c1-0+0 $\Rightarrow$ c1=0

Differentiating (1) w.r.t. ‘t’ we get

$$\frac{dx}{dt} = c_2\cos t + \frac{2t\sin 2t}{3} - \frac{\cos 2t}{3} + \frac{8}{9}\cos 2t (\because c_1 = 0) \qquad .....(2)$$

By data $\frac{dt}{dx} = 0$ at t=0

$$\therefore (2) \Rightarrow 0 = 0 + c_2 + 0 - \frac{1}{3} + \frac{8}{9} \Rightarrow c_2 = \frac{1}{3} - \frac{8}{9} = \frac{-5}{9}$$

Substituting the values of c_1 and c_2 in (1) , we get required solution as

$$x = \frac{-5}{9}\sin t - \frac{t\cos 2t}{3} + \frac{4\sin 2t}{9}$$

4. Apply the method of variation of parameters of solve $\frac{d^2y}{dx^2} + y = \cos ecx.$

Sol: Given equation in the operator form is $(D^2+1)y$= cosec x ....(1)

Here P=0, Q=1 and R=cosec x

A.E. is $m2+1=0 \Rightarrow m=\pm i$. *The roots are complex conjugate numbers.*

$\therefore$ C.F is $y_c = c_1 \cos x + c_2 \sin x$

Let $y_p = A\cos x + B\sin x$ be the P.I of (1) .....(2)

Here u=cosx ,v=sinx

$$\therefore u\frac{dv}{dx} - v\frac{du}{dx} = \cos^2 + \sin^2 x = 1$$

A and B are given by

$$A = -\int \frac{vR}{\frac{udv}{dx} - \frac{vdu}{dx}} dx = -\int \frac{\sin x \cos ecx}{1} dx = -\int dx = -x$$

$$B = \int \frac{uR}{\frac{udv}{dx} - \frac{vdu}{dx}} dx = \int \cos x.\cos ecx dx = \int \cot x dx = \log|\sin x|$$

$$\therefore y_p = -x\cos x + \sin(x)\log|\sin x|$$

Hence the general solution is given by $y = y_c + y_p$

i.e., $y = c_1 \cos x + c_2 \sin x - x\cos x + \sin s \log(\sin x)$

5. Solve $(D^2 + a^2)y = \tan ax,$ **by the method of variation of parameters.**

Sol: Given equation is $(D^2 + a^2)y = \tan ax,$

Here P=0,Q=a^2 and R = tan ax

A.E. is $m2+a2+0 \Rightarrow m=\pm ai$. The roots are complex conjugate numbers.

$\therefore y_c = C.F = c_1 \cos ax + c_2 \sin ax$

Let $y_p = P.I = A\cos ax + B\sin ax$

Let u= cos ax and v=sin ax

Now $u\frac{dv}{dx} - v\frac{du}{dx} = a$

A and B are given by

$$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} dx = -\int \frac{\sin ax.\tan ax}{a} dx$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax} dx = -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax} dx = -\frac{1}{a}\left[\int \sec ax dx - \int \cos ax dx\right]$$

Or $A = \frac{1}{a^2}\log|\sec ax + \tan ax| + \frac{1}{a^2}\sin ax$

And $B=\int \frac{uRdx}{u\frac{dv}{dx}-v\frac{du}{dx}}=\int \frac{\cos ax \tan ax}{a}dx=\frac{1}{a}\int \sin ax dx=-\frac{1}{a^2}\cos ax$

The general solution is given by $y=y_c+y_p$

i.e.,

$$y=c_1\cos ax+c_2\sin ax+\left[-\frac{1}{a^2}\log|\sec ax+\tan ax|+\frac{1}{a^2}\sin ax\right]\cos ax-\frac{1}{a^2}\cos ax.\sin ax$$

Which can be written by $y=c_1\cos ax+c_2\sin ax-\frac{\cos ax}{a^2}\log|\sec ax+\tan ax|$

6. Solve $\frac{d^2y}{dx^2}+y=x\cos x$ by the method of variation of parameters.

Sol: Given equation in the operator form is

$(D^2+1)y=x\cos x$ .....(1)

Here P=0,Q=1 and R=x cosx

A.E is $m2+1=0 \Rightarrow m=\pm i$. the roots are complex conjugate numbers.

$C.F.=y_c=c_1\cos x+c_2\sin x. Let u=\cos x, v=\sin x, R=x\cos x$

Then $u\frac{dv}{dx}-v\frac{du}{dx}=\cos^2 x+\sin^2 x=1$

Let y_p=P.I.=A cos x+B sin x, where A and B are functions of x.

A is given by

$$A=-\int \frac{vR}{u\frac{dv}{dx}-v\frac{du}{dx}}dx=-\int \frac{\sin x.x\cos x}{1}dx=-\frac{1}{2}$$

$$-\frac{1}{2}\left[-\frac{x}{2}\cos 2x+\frac{1}{2}\int \cos 2x dx\right]=\frac{x}{4}\cos 2x-\frac{1}{8}\sin 2x$$

$$B=\int \frac{uRdx}{u\frac{dv}{dx}-v\frac{du}{dx}}=\int \frac{\cos x.\cos x}{1}dx=\frac{1}{2}\int (x+x\cos 2x)dx$$

$$=\int \frac{x(1+\cos 2x)}{2}dx=\frac{x^2}{4}+\frac{x\sin 2x}{4}+\frac{\cos 2x}{8} \text{ (integration by parts)}$$

The general solution is given by= $y=y_c+y_p$

i.e., $y=c_1\cos x+c_2\sin x+\left(\frac{x}{4}\cos 2x-\frac{1}{8}\sin 2x\right)\cos x+\left[\frac{x^2}{4}+\frac{x\sin 2x}{4}+\frac{\cos 2x}{8}\right]\sin x$

7. Solve $(D^2+4)y=\sec 2x$, by the method of variation of parameters.

Sol:Given equation is $(D^2+4)y=\sec 2x$ ....(1)

$\therefore$ A.E is $m^2+4=0 \Rightarrow m=\pm 2i$

The roots are complex conjugate numbers.

$\therefore y_c = C.F. = c_1 \cos 2x + c_2 \cos 2x$

Let y_p=P.I.A=cos 2x+B sin 2x

Here u=cos 2x, v=sin2x and R =Sec 2x.

$$\therefore \frac{du}{dv} = -2\sin 2x and \frac{dv}{dx} = 2\cos 2x$$

$$\therefore u.\frac{du}{dx} - v\frac{du}{dx} = (\cos 2x)(2\cos 2x) - \sin 2x(-2\sin 2x)$$

$=2\cos^2 2x+2\sin^2 2x=2(\cos^2 2x+\sin^2 2x)=2$

A and B are given by:

$$A = -\int \frac{vR}{u.\frac{dv}{dx} - v\frac{du}{dx}} dx = -\int \frac{(\sin 2x)(\sec 2x)}{2} dx = -\frac{1}{2}\int \tan 2x dx = \frac{1}{2}\frac{\log|\cos 2x|}{2}$$

$$\Rightarrow A = \frac{\log|\cos 2x|}{4}$$

$$B = \int \frac{uRdx}{u.\frac{dv}{dx} - v\frac{du}{dx}} = \int \frac{\cos 2x.\sec 2x}{2} dx = \frac{1}{2}\int dx = \frac{x}{2}$$

$$\therefore y_p = P.I = \frac{\log|\cos 2x|}{4}(\cos 2x) + \frac{x}{2}(\sin 2x)$$

$\therefore$ The general solution is given by:

$$y = y_c + y_p = C.F + P.I$$

i.e., $y = c_1 \cos 2x + c_2 \sin 2x + \frac{\cos 2x}{4} \log|\cos 2x| + \frac{x}{2}\sin 2x$

8. Solve $(D^2 - 2D + 2)y = e^x \tan x$ by the method of variation of parameters.

Sol: A.E is m^2-$2m+2=0$

$$\therefore m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i2}{2} = 1 \pm i$$

We have $y_c = e^x(c_1 \cos x + c_2 \sin x) = c_1 ex \cos x + c_2 e^x \sin x$

Where u=e^x cos x ,v=e^x sinx

$$\therefore \frac{du}{dx} e^x(-\sin x) + e^x \cos x, \frac{dv}{dx} = e^x \cos x + e^x \sin x$$

Now $u\frac{dv}{dx} - v\frac{du}{dx} = e^x \cos x(e^x \cos x + e^e \sin x) - e^x \sin x(e^x \cos x - e^x \sin x)$

$$= e^{2x}(\cos^2 x + \cos x \sin x - \sin x \cos x + \sin^2 x) = e^{2x}$$

Using variation of parameters,

$$A = -\int \frac{vR}{u\frac{dv}{dx} - v\frac{du}{dx}} dx = -\int \frac{e^x \tan x}{e^{2x}} (e^x \sin x) dx$$

$$= -\int \tan x \sin x dx = \int \frac{\sin^2 x}{\cos x} dx = \int \frac{(1-\cos^2 x)}{\cos x} dx$$

$$= \int (\sec x - \cos x) dx = \log(\sec x + \tan x) + \sin x$$

$$B = \int \frac{uR}{u\frac{dv}{dx} - v\frac{du}{dx}} dx$$

$$= \int \frac{e^x \cos x . e^x \tan x}{e^{2x}} dx = \int \sin x dx = -\cos x$$

General solution id given by $y=y_c+Au+Bv$

i.e., $y = c_1 e^x \cos x + c_2 e^x \sin x + [\log(\sec x + \tan x) - \sin x] e^x \cos x - e^x \cos x \sin x$

or $y = c_1 e^x \cos x + c_2 \sin x + [\log(\sec + \tan x) - 2\sin x] e^x \cos x$

EXERCISE

1. Solve $y'' + 4y$ =tan 2x by the method of variation of parameters.

2. Solve $\frac{d^2 y}{dx^2} + 9y = \tan 3x$ by the method of variation of parameters.

3. Solve $\frac{d^2 y}{dx^2} + y = x \sin x$ by the method of parameters.

ANSWERS

1) $y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4} \cos 2x \log[\sec 2x + \tan 2x]$

2) $y = c_1 \cos 3x + c_2 \sin 3x + \frac{x}{2} \sin x - \frac{x^2}{4} \cos x$

3) $y = c_1 \cos x + c_2 \sin x + \frac{x}{2} \sin x \frac{-x^2}{4} \cos x$

LAPLACE TRANSFORM

LAPLACE TRANSFORM

INTRODUCTION

The knowledge of Laplace transforms has in recent years become an essential part of mathematical background required of engineers and scientists. This is because the transform methods provide an easy and effective means for the solution of many problems arising in engineering. This subject originated from the operational methods applied by the English engineer Oliver Heaviside (1850-1925), to problems in electrical engineering. Unfortunately, Heaviside's treatment was unsystematic and lacked rigour, which was placed on sound mathematical footing by Bromwich and Carson during 1916-17. It was found that Heaviside's operational calculus is best introduced by means of a particular type of definite integrals called Laplace transforms. The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants. Moreover, the ready tables of Laplace transforms reduce the problem of solving differential equations to mere algebraic manipulation.

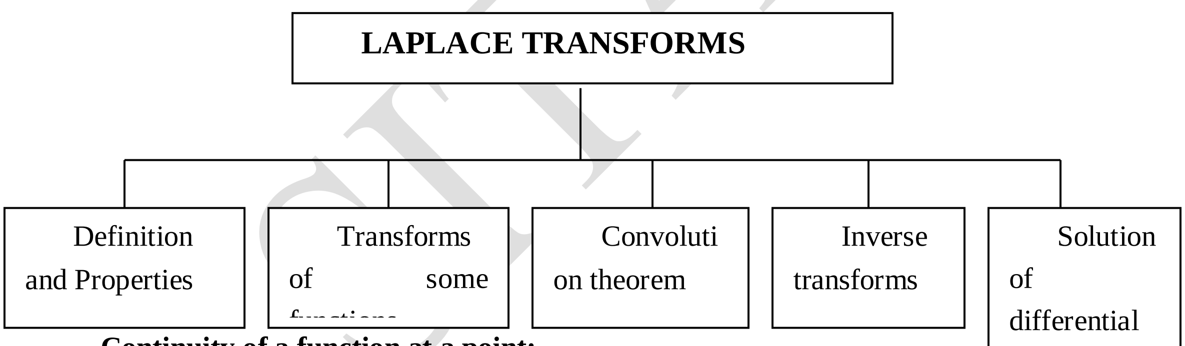
- Laplace transform is an integral transform employed in solving physical problems.
- Many physical problems when analyzed assumes the form of a differential equation subjected to a set of initial conditions or boundary conditions.
- By initial conditions we mean that the conditions on the dependent variable are specified at a single value of the independent variable.
- If the conditions of the dependent variable are specified at two different values of the independent variable, the conditions are called boundary conditions.
- The problem with initial conditions is referred to as the Initial value problem.
- The problem with boundary conditions is referred to as the Boundary value problem.

Example 1 : The problem of solving the equation $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=x$ with conditions y(0) = y' (0) = 1 is an initial value problem

Example 2 : The problem of solving the equation $3\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\cos x$ with y(1)=1, y(2)=3 is called Boundary value problem.

Laplace transform is essentially employed to solve initial value problems. This technique is of great utility in applications dealing with mechanical systems and electric circuits. Besides the technique may also be employed to find certain integral values also. The transform is named after the French Mathematician P.S. de' Laplace (1749 – 1827).

The subject is divided into the following sub topics.



Continuity of a function at a point:

$let\ f: S \to R,\ a \in s,$ we say that f is continuous at ' a ' if given $\varepsilon > 0, \exists\ \delta > 0, \ni\ |x-a| < \delta$ $\Rightarrow |f(x)-f(a)| < \varepsilon, x\epsilon s$

$i.e., \lim_{x\to a} f(x) = f(a)$

Continuity of a function in [a, b]

A function $f(x)$ is said to be continuous in [a, b] if it is continuous at every point of the interval [a, b]

Sectionally continuous (or) piece wise continuous :

Let a function $f(t)$ be continuous in [a, b] then the function $f(t)$ is said to be sectionally continuous (or) piece wise continuous if $f(t)$ is continuous in every subinterval when then [a, b] is divided into a finite number of subintervals

Ex: 1. $F(t) = sint$ is sectionally continuous in $[0, 2\pi]$

2. $F(t) = cost$ is sectionally continuous in $[0, 2\pi]$

Function of exponential order:

Let $f(t)$ be a real valued function $\forall t \geq 0$ then $f(t)$ is said to be a function of exponential order if $\lim_{t\to\infty} f(x)\ e^{-st} = 0 = a\ finite\ quantity$

Function of Class A

A real valued function $f(t)$ is said to be a function of Class A if

1. $f(t)$ is Sectionally continuous
2. $f(t)$ is function of Exponential Order

Ex: 1. Examine whether t^n is function of exponential order

Sol:: $let\ f(t) = t^n$

By the definition of function of exponential order

$$\lim_{t\to\infty} f(x)\ e^{-st} = \lim_{t\to\infty} t^n e^{-st}$$

$$\lim_{t\to\infty} \frac{n!}{s^n e^{st}} = \frac{n!}{s^n e^{\infty}} = 0 \quad (applying\ L\ Hospitials\ rule\ \ n\ times)$$

$\Rightarrow t^n\ \ is\ a\ function\ of\ exponential\ order$

4.1 DEFINITION OF LAPLACE TRANSFORM:

Let $F(t)$ be a function of Class A then the Laplace Transform of $F(t)$ is denoted by $L\{F(t)\}$ and is defined as

$$L\{F(t)\} = \int_0^\infty F(t)\ e^{-st}\ dt = f(s)\ or\ \overline{f(s)},$$ where s is a parameter

4.2 PROPERTIES OF LAPLACE TRANSFORM:

1. If 'C' is any constant and $f(s)$ is the laplace transform of $F(t)$ then show that $L\{CF(t)\} = C\ l\{F(t)\} = C\ f(s)$

Proof:: Given $L\{F(t)\} = f(s)$

By the definition Laplace Transfrom

$$L\{F(t)\} = \int_0^\infty F(t)e^{-st}dt = f(s)$$

$$L\{CF(t)\} = \int_0^\infty CF(t)e^{-st}dt = C\int_0^\infty F(t)e^{-st}dt = CL\{F(t)\} = Cf(s)$$

LINEARITY PROPERTY : For any two functions f(t) and ϕ(t) (whose Laplace transforms exist) and any two constants a and b, we have

$$L\{aF(t) + b\,\phi(t)\} = a\,L\{f(t)\} + b\,L\{\phi(t)\}$$

Proof :- By definition, we have

$$L\{F(t)\} = \int_0^\infty F(t)e^{-st}dt = f(s)$$

$$L\{aF(t) + b\phi(t)\} = \int_0^\infty e^{-st}\left[aF(t) + b\phi(t)\right]dt = a\int_0^\infty e^{-st}F(t)dt + b\int_0^\infty e^{-st}\phi(t)dt$$

$= \text{a L}\{\text{F(t)}\} + \text{b L}\{\phi(\text{t})\}$

This is the desired property.

In particular, $for\ a = b = 1$, we have

$$L\{f(t) + \phi(t)\} = L\{f(t)\} + L\{\phi(t)\}$$

and $for\ a = -b = 1$, we have

$$L\{f(t) - \phi(t)\} = L\{f(t)\} - L\{\phi(t)\}$$

4.3 LAPLACE TRANSFORMS OF SOME FUNCTIONS

Let a be a constant. Then

1. $L\{e^{at}\} = \int_0^\infty e^{-st}e^{at}dt = \int_0^\infty e^{-(s-a)t}dt$

$$= \left.\frac{e^{-(s-a)t}}{-(s-a)}\right|_0^\infty = \frac{1}{s-a}, \quad s > a$$

Thus,

$$L\{e^{at}\} = \frac{1}{s-a}$$

In particular, when $a = 0$, we get

$$L\{1\} = \frac{1}{s}, \quad s > 0$$

2. $\text{L}\{cosh\ at\} = L\left(\frac{e^{at} + e^{-at}}{2}\right) = \frac{1}{2}\int_0^\infty e^{-st}\left[e^{at} + e^{-at}\right]dt$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(s-a)t} + e^{-(s+a)t}\right] dt$$

$Let\ \ s > |a|.\ Then,$

$$L\{\cosh at\} = \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} + \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty = \frac{s}{s^2 - a^2}$$

Thus,

$$L\{cosh\ at\} = \frac{s}{s^2 - a^2}, \quad s > |a|$$

3. $L\{sinh\ at\} = L\left(\frac{e^{at} - e^{-at}}{2}\right) = \frac{a}{s^2 - a^2}, \quad s > |a|$

Thus,

$$L\{sinh\ at\} = \frac{a}{s^2 - a^2}, \quad s > |a|$$

4. $L\{sin\ at\} = \int_0^\infty e^{-st} \sin at\ dt$

Here we suppose that s > 0 and then integrate by using the formula

$$\int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

Thus,

$$. L\{sinat\} = \int_0^\infty e^{-st} \sin at\ dt = \left[\frac{e^{-st}}{(-s)^2+a^2}[(-s)sinat - acosat]\right]_{t=0}^{\infty}$$

$$L\{sin\ at\} = \frac{a}{s^2+a^2},\ s > 0$$

5. $L\{cos\ at\} = \int_0^\infty e^{-st} \cos atdt$

Here we suppose that s>0 and integrate by using the formula

$$\int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx]$$

Thus,

$$L\{cos\ at\} = \int_0^\infty e^{-st} \cos atdt = \left[\frac{e^{-st}}{(-s)^2+a^2}[(-s)cosat + a\ sinat]\right]_{t=0}^{\infty}$$

$$L\{cos\ at\} = \frac{s}{s^2+a^2},\ s > 0$$

6. Let n be a constant, which is a non-negative real number or a negative non-integer. Then

$$L\{t^n\} = \int_0^\infty e^{-st} t^n dt \qquad L\{t^n\} = \int_0^\infty e^{-x}\left(\frac{x}{s}\right)^n \frac{dx}{s} = \frac{1}{s^{n+1}}\int_0^\infty e^{-x} x^n dx$$

Let $s > 0$ *and set* $st = x,$ then

The integral $\int_0^\infty e^{-x} x^n dx$ is called gamma function of (n+1) denoted by $\Gamma(n+1)$. Thus

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$$

In particular, if n is a non-negative integer then $\Gamma(n+1) = n!$. Hence

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

TABLE OF LAPLACE TRANSFORMS

f(t)	$L\{F(t)\} = f(s)$
1	$\frac{1}{s}$, s > 0
e^{at}	$\frac{1}{s-a}$, s > a
e^{-at}	$\frac{1}{s+a}$, s > a
$coshat$	$\frac{s}{s^2-a^2}$, s > |a|
$sinhat$	$\frac{a}{s^2-a^2}$, s > |a|
$sinat$	$\frac{a}{s^2+a^2}$, s > 0
$cosat$	$\frac{s}{s^2+a^2}$, s > 0
$t^n, \quad n = 0,1,2, \ldots..$	$\frac{n!}{s^{n+1}}$, s > 0
	$\frac{\Gamma(n+1)}{s^{n+1}}$, s > 0

4.4 CHANGE OF SCALE PROPERTY

If $L\{F(t)\} = f(s)$, then $L\{f(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$, where a is a positive constant.

PROOF :- By definition, we have

$$L\{F(t)\} = \int_0^\infty F(t)e^{-st}dt = f(s)$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)dt \qquad (1)$$

Let us set $at = x$. Then expression (1) becomes,

$$L\{F(at)\} = \frac{1}{a}\int_0^\infty e^{-\left(\frac{s}{a}\right)x} f(x)dx = \frac{1}{a} f\left(\frac{s}{a}\right)$$

This is the desired property.

4.5 FIRST SHIFTING THEOREM

If $L\{F(t)\} = f(s)$ then $L\{e^{at}F(t)\} = f(s-a)$

PROOF :- By the definition of Laplace Transform we have

$L\{F(t)\} = \int_0^\infty F(t)e^{-st}dt = f(s)$

$$L\{e^{at}\,F(t)\} = \int_0^\infty e^{-st}\left[e^{at} f(t)\right]dt \quad = \int_0^\infty e^{-(s-a)} f(t)dt$$

$$= f(s-a)$$

ALITER::

if $L\{F(t)\} = f(s)$ $then$ $L\{e^{-at}F(t)\} = f(s+a)$

PROOF :- By the definition of Laplace Transform we have

$L\{F(t)\} = \int_0^\infty F(t)e^{-st}dt = f(s)$

$$L\{e^{-at}\,F(t)\} = \int_0^\infty e^{-st}\left[e^{-at} f(t)\right]dt \quad = \int_0^\infty e^{-(s+a)} f(t)dt$$

$$= f(s+a)$$

This is the desired property.

Here we note that the Laplace transform of e^{at} f(t) can be written down directly by changing s to s-a in the Laplace transform of f(t).

APPLICATION OF FIRST SHIFTING THEOREM

The shifting property is

If $L\{f(t)\} = f(s)$, then $L\{e^{at} f(t)\} = f(s-a)$

Application of this property leads to the following results :

1. $L\{e^{at}\cosh bt\} = \left[L\{\cosh bt\}\right]_{s\to s-a} = \left(\dfrac{s}{s^2-b^2}\right)_{s\to s-a} = \dfrac{s-a}{(s-a)^2-b^2}$

Thus,

$$L\{e^{at}\,coshbt\} = \frac{s-a}{(s-a)^2-b^2}$$

2. $L\{e^{at}\sinh bt\} = \left[L\{\sinh bt\}\right]_{s\to s-a} = \left[\dfrac{b}{s^2-b^2}\right]_{s\to s-a} = \dfrac{a}{(s-a)^2-b^2}$

3. $L\{e^{at}\cos bt\} = \dfrac{s-a}{(s-a)^2+b^2}$

4. $L\{e^{at}\sin bt\} = \dfrac{b}{(s-a)^2-b^2}$

5. $L\{e^{at}t^n\} = \dfrac{\Gamma(n+1)}{(s-a)^{n+1}}$ or $\dfrac{n!}{(s-a)^{n+1}}$ as the case may be

6. Find $L\{e^{-3t}(2\cos 5t - 3\sin 5t)\}$

Sol:: Given $= 2L\{e^{-3t}\cos 5t\} - 3L\{e^{-3t}\sin 5t\}$

$$= 2\frac{(s+3)}{(s+3)^2+25} - \frac{15}{(s+3)^2+25}\text{, by using shifting property}$$

$$= \frac{2s-9}{s^2+6s+34}\text{ , on simplification}$$

7. Find $L\{coshat\ sinat\}$

Sol:: Here

$$L\{coshat\ sinat\} = L\left\{\frac{(e^{at}+e^{-at})}{2}\sin at\right\}$$

$$= \frac{1}{2}\left[\frac{a}{(s-a)^2+a^2} + \frac{a}{(s+a)^2+a^2}\right]\text{ by first shifting property in L.T.}$$

$$= \frac{a(s^2+2a^2)}{[(s-a)^2+a^2][(s+a)^2+a^2]}\text{, on simplification}$$

8. Find $L\{cosht\ sin^3 2t\}$

Sol:: Given

$$L\left\{\left(\frac{e^t+e^{-t}}{2}\right)\left(\frac{3\sin 2t - \sin 6t}{4}\right)\right\}$$

$$= \frac{1}{8}\left[3\cdot L\{e^t \sin 2\} - L\{e^t \sin 6t\} + 3L\{e^{-t}\sin 2t\} - L\{e^{-t}\sin 6t\}\right]$$

$$= \frac{1}{8}\left[\frac{6}{(s-1)^2+4} - \frac{6}{(s-1)^2+36} + \frac{6}{(s+1)^2+4} - \frac{6}{(s+1)^2+36}\right]$$

$$= \frac{3}{4}\left[\frac{1}{(s-1)^2+4} - \frac{1}{(s-1)^2+36} + \frac{1}{(s+1)^2+24} - \frac{1}{(s+1)^2+36}\right]$$

9. Find $L\{e^{-4t}t^{-5/2}\}$

Sol:: We have

$$L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}\quad Put\ n = -5/2.\ Hence$$

$$L\{t^{-5/2}\} = \frac{\Gamma(-3/2)}{s^{-3/2}} = \frac{4\sqrt{\pi}}{3s^{-3/2}}$$ Change s to s+4 by first shifting Theorem in L.T.

Therefore, $$L\{e^{-4t}t^{-5/2}\} = \frac{4\sqrt{\pi}}{3(s+4)^{-3/2}}$$

PROBLEMS

1. Find $L\{f(t)\}$ *given* $f(t) = \begin{cases} t, & 0 < t < 3 \\ 4, & t > 3 \end{cases}$

Sol:: Here

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^3 e^{-st}t dt + \int_3^\infty 4e^{-st}dt$$

Integrating the terms on the RHS, we get

$$L\{f(t)\} = \frac{1}{s}e^{-3s} + \frac{1}{s^2}(1-e^{-3s})$$

This is the desired result.

2. Find $Lf(t)$ *given* $f(t) = \begin{cases} \sin 2t, & 0 < t \le \pi \\ 0, & t > \pi \end{cases}$

Sol:: Here

$$L\{f(t)\} = \int_0^\pi e^{-st} f(t)dt + \int_\pi^\infty e^{-st} f(t)dt = \int_0^\pi e^{-st} \sin 2t dt$$

$$= \left[\frac{e^{-st}}{s^2+4}\{-s\sin 2t - 2\cos 2t\}\right]_0^\pi = \frac{2}{s^2+4}\left[1-e^{-\pi s}\right]$$

This is the desired result.

3. Evaluate : (i) L{sin3t sin4t}

(ii) $L\{\cos^2 4t\}$

(iii) $L\{\sin^3 2t\}$

Sol:: (i) Here

$$L\{\sin 3t \sin 4t\} = L\{\frac{1}{2}(\cos t - \cos 7t)\}$$

$$= \frac{1}{2}\left[L\{\cos t\} - L\{\cos 7t\}\right], \text{ by using linearity property}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+49}\right] = \frac{24s}{(s^2+1)(s^2+49)}$$

(ii) Here

$$L\{\cos^2 4t\} = L\left\{\frac{1}{2}(1+\cos 8t)\right\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+64}\right]$$

(iii) We have

$$\sin^3\theta = \frac{1}{4}\left(3\sin\theta - \sin 3\theta\right)$$

For θ=2t, we get

$$\sin^3 2t = \frac{1}{4}\left(3\sin 2t - \sin 6t\right)$$

so that

$$L\{\sin^3 2t\} = \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right] = \frac{48}{(s^2+4)(s^2+36)}$$

This is the desired result.

4. Find $L\{cost\ cos2t\ cos3t\}$

Sol:: Here

$$cos2t\ cos3t = \frac{1}{2}[\cos 5t + \cos t]$$

so that

$$cost\ cos2t\ cos3t = \frac{1}{2}[\cos 5t \cos t + \cos^2 t]$$

$$= \frac{1}{4}[\cos 6t + \cos 4t + 1 + \cos 2t]$$

Thus

$$L\{cost\ cos2t\ cos3t\} = \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]$$

5. Find $L\{cosh^2 2t\}$

Sol:: We have

$$\cosh^2\theta = \frac{1+\cosh 2\theta}{2}$$

For $\theta = 2t$, we get

$$\cosh^2 2t = \frac{1+\cosh 4t}{2}$$

Thus,

$$L\{\cosh^2 2t\} = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2-16}\right]$$

6. Evaluate (i) $L\{\sqrt{t}\}$ (ii) $L\left\{\frac{1}{\sqrt{t}}\right\}$ (iii) $L\{t^{-\frac{3}{2}}\}$

Sol:: We have $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

(i) For $n = \frac{1}{2}$, we get

$$L\{t^{1/2}\} = \frac{\Gamma(\frac{1}{2}+1)}{s^{3/2}}$$

Since $\Gamma(n+1) = n\Gamma(n)$, we have $\Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2}$

Thus, $L\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}$

(ii) For $n = -\frac{1}{2}$, we get

$$L\{t^{-1/2}\} = \frac{\Gamma\left(\frac{1}{2}\right)}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}}$$

(iii) For $n = -\frac{3}{2}$, we get

$$L\{t^{-3/2}\} = \frac{\Gamma\left(-\frac{1}{2}\right)}{s^{-1/2}} = \frac{-2\sqrt{\pi}}{s^{-1/2}} = -2\sqrt{\pi s}$$

7. Evaluate : (i) $L\{t^2\}$ (ii) $L\{t^3\}$

Sol:: We have,

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

(i) For $n = 2$, we get

$$L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

(ii) For $n = 3$, we get

$$L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

EXERCISE

I Find $L\{F(t)\}$ in each of the following cases :

1. $F(t) = \begin{cases} e^t & ,0 < t < 1 \\ 0 & ,t > 1 \end{cases}$

2. $F(t) = \begin{cases} \frac{t}{\tau} & ,0 < t < \tau \\ 1 & ,t > \tau \end{cases}$

3. $F(t) = \begin{cases} \frac{t}{a}, & 0 < t \le a \\ 0 & ,t > a \end{cases}$

II. Find the Laplace transforms of the following functions :

4. cos(3t + 4) 5. Sin2t sin3t 6. Cos5t cos2t 7. Sin4t cost

8. sint sin2t sin3t 9. $\sin^2 5t$ 10. $(sint - cost)^2$ 11. $\cos^3 2t$

12. $\sinh^3 2t$ 13. $t^{5/2}$ 14. $\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^3$ 15. $\frac{1}{\sqrt{\pi t}}$

16. sinhat-sinat 17. $e^{3t} \sin^2 t$ 18. e^{2t} sin3t cost 19. $(1 + te^{-t})^3$

20. $t^2 e^{3t} \sin t$ 21. $\frac{1-e^{-2t}}{t}$ 22. $\frac{e^{-at}-e^{-bt}}{t}$ 23. $\frac{\sin^2 t}{t}$

24. $\frac{\cos 4t \sin 2t}{t}$

ANSWERS

(I) 1. $\frac{e^{1-s}-1}{1-s}$ 2. $\frac{1-e^{-st}}{\tau s^2}$ 3. $\frac{1-e^{-as}}{s^2}$

(II) 4. $\frac{s\cos 4-3\sin 4}{s^2+9}$ 5. $\frac{12s}{(s^2+1)(s^2+25)}$ 6. $\frac{s(s^2+29)}{(s^2+9)(s^2+49)}$ 7. $\frac{1}{2}\left[\frac{5}{s^2+25}+\frac{3}{s^2+9}\right]$

8. $\frac{1}{4}\left[\frac{2}{s^2+2^2}-\frac{6}{s^2+6^2}+\frac{4}{s^2+4^2}\right]$ 9. $\frac{50}{s(s^2+100)}$ 10. $\frac{1}{s}-\frac{2}{s^2+4}$ 11. $\frac{1}{4}\left[\frac{S}{S^2+36}+\frac{3S}{S^2+4}\right]$

12. $\frac{48}{(S^2-4)(S^2-36)}$ 13. $\frac{15}{8S^{7/2}}\sqrt{\pi}$ 14. $\frac{\sqrt{\pi}}{4}\left[\frac{3}{S^{5/2}}-\frac{6}{S^{3/2}}+\frac{12}{\sqrt{S}}+8\sqrt{S}\right]$ 15. $\frac{1}{\sqrt{s}}$ 16. $\frac{2a^3}{s^4-a^4}$

17. $\frac{1}{2}\left[\frac{1}{s-3}-\frac{s-3}{(s-3)^2+4}\right]$ 18. $\frac{2}{(s-2)^2+16}+\frac{1}{((s-2)^2+4)}$ 19. $\frac{1}{s}+\frac{3}{(s+1)^2}+\frac{6}{(s+2)^3}+\frac{6}{(s+3)^4}$

20. $\frac{6(s-3)^2-2}{((s-3)^2+1)^3}$ 21. $\log\left(\frac{s+2}{s}\right)$ 22. $\log\left(\frac{s+b}{s+a}\right)$ 23. $\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$ 24. $\frac{1}{2}\left[\tan^{-1}\frac{s}{2}+\tan^{-1}\frac{s}{6}\right]$

4.6 LAPLACE TRANSFORM OF $t^n F(t)$

Show that $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$

Proof::

Here we suppose that n is a positive integer. By definition, we have

$$L\{F(t)\} = f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

Differentiating 'n' times on both sides w.r.t. s, we get

$$\frac{d^n}{ds^n} f(s) = \frac{\partial^n}{\partial s^n} \int_0^\infty e^{-st} F(t)\,dt$$

Performing differentiation under the integral sign, we get

$$\frac{d^n}{ds^n}\left[f(s)\right] = \int_0^\infty (-t)^n e^{-st} F(t)\,dt$$

Multiplying on both sides by $(-1)^n$, we get

$$(-1)^n \frac{d^n}{ds^n}\left[f(s)\right] = \int_0^\infty (t^n F(t) e^{-st} dt = L\{t^n F(t)\}\text{, by definition}$$

Thus,

$$\boxed{L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)}$$

In particular, we have

$$\boxed{L\{t\, f(t)\} = -\frac{d}{ds} F(s),} \quad \text{for n=1}$$

$$\boxed{L\{t^2\, f(t)\} = \frac{d^2}{ds^2} F(s),} \quad \text{for n=2, etc.}$$

4.7 LAPLACE TRANSFORM OF $\frac{\text{F(t)}}{\text{t}}$

Show that $L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)ds$

Proof:: We have , $f(s) = \int_0^\infty e^{-st} F(t) dt$

Therefore,

$$\int_s^\infty f(s)ds = \int_s^\infty \left[\int_0^\infty e^{-st} F(t)dt\right] ds \qquad = \int_0^\infty F(t)\left[\int_s^\infty e^{-st} ds\right] dt$$

$$= \int_0^\infty F(t)\left[\frac{e^{-st}}{-t}\right]_s^\infty dt \qquad = \int_0^\infty e^{-st}\left[\frac{F(t)}{t}\right] dt = L\left\{\frac{F(t)}{t}\right\}$$

Thus,

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)ds$$

PROBLEMS

1. Find $L\{te^{-t}\, sin4t\}$

Sol:: We have,

$$L\{e^{-t}\sin 4t\} = \frac{4}{(s+1)^2+16}$$

So that,

$$L\{te^{-t}\, sin4t\} = 4\left[-\frac{d}{ds}\left\{\frac{1}{s^2+2s+17}\right\}\right]$$

$$= \frac{8(s+1)}{(s^2+2s+17)^2}$$

2. Find $L\{t^2\, sin3t\}$

Sol:: We have

$$L\{sin3t\} = \frac{3}{s^2+9}$$

So that,

$$L\{t^2\, sin3t\} = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right) \quad = -6\frac{d}{ds}\frac{s}{(s^2+9)^2} \quad =$$

2. **Find** $L\{e^{-t}\sin t\}$

Sol:: We have

$$L\{e^{-t}\sin t\} = \frac{1}{(S+1)^2+1}$$

Hence $L\left\{\frac{e^{-t}\sin t}{t}\right\} = \int_0^{\infty}\frac{ds}{(s+1)^2+1} = \left[\tan^{-1}(s+1)\right]_s^{\infty}$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) \quad = \cot^{-1}(s+1)$$

4. Find $L\left\{\frac{\sin t}{t}\right\}$. Using this, evaluate $L\left\{\frac{\sin at}{t}\right\}$

Sol:: We have

$$L\{sint\} = \frac{1}{s^2+1}$$

So that

$$L\{f(t)\} = L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1} s\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s = f(s)$$

Consider

$$L\left\{\frac{\sin at}{t}\right\} = a\,L\left\{\left(\frac{\sin at}{at}\right)\right\}L = aL\{F(at)\}$$

$$= a\left[\frac{1}{a} f\left(\frac{s}{a}\right)\right],\ \text{in view of the change of scale property}$$

$$= \cot^{-1}\left(\frac{s}{a}\right)$$

5. Find $L\left\{\frac{\cos at - \cos bt}{t}\right\}$

Sol:: We have

$$L\{cosat - cosbt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

So that

$$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right] ds \qquad = \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\underset{s\to\infty}{Lt} \log\left(\frac{s^2+a^2}{s^2+b^2}\right) - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0 + \log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right] \quad = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

6. Prove that $\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$

Sol:: We have

$$\int_0^\infty e^{-st} t \sin t dt = L\{t \sin t\} \quad = -\frac{d}{ds}\left[L\{\sin t\}\right] \quad = -\frac{d}{ds}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{2s}{(s^2+1)^2}$$

Putting s = 3 in this result, we get

$$\int_0^\infty e^{-3t} t \sin t dt = \frac{3}{50}$$

This is the result as required.

EXERCISE

I Evaluate the following integrals using Laplace transforms :

1. $\int_0^\infty te^{-2t} \cos t\, dt$ 2. $\int_0^\infty e^{-t} t^3 \sin t\, dt$ 3. $\int_0^\infty \left(\frac{e^{-at} - e^{-bt}}{t} \right) dt$

4. $\int_0^\infty \frac{e^{-at} \sin^2 t}{t} dt$

ANSWERS

1. $\frac{3}{25}$ 2. 0 3. $\log(b/a)$ 4. $\frac{1}{4} log\left(\frac{a^2+4}{a^2}\right)$

4.8 LAPLACE TRANSFORMS OF THE DERIVATIVES OF F(t)

Show that $L\{F'(t)\} = sL\{F(t)\} - F(0)$ and

$$L\{F^n(t)\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - .... - F^{n-1}(0)$$

Proof::

Consider

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t) dt$$

$$= \left[e^{-st}F(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st}F(t)dt$$, by using integration by parts

$$= \left[\underset{t\to\infty}{Lt}(e^{-st}F(t) - F(0)\right] + sLF(t)$$

$$= 0 - F(0) + s\, L\{F(t)\}$$

Thus

$$L\{F'(t)\} = s\, L\{F(t)\} - F(0)$$

Similarly,

$$L\{F''(t)\} = s^2 L\{F(t)\} - sF(0) - F'(0).$$

In general, we have

$$L\{F^n(t)\} = s^n L\{F(t)\} - s^{n-1}F(0) - s^{n-2}F'(0) - .... - F^{n-1}(0)$$

4.9 LAPLACE TRANSFORM OF $\int_0^t F(t)dt$

Show That $L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}L\{F(t)\}$

Proof:: Let $\phi(t) = \int_0^t f(t)dt$. Then $\phi(0) = 0$ and $\phi'(t) = f(t)$

Now,

$$L\{\phi(t)\} = \int_0^\infty e^{-st}\phi(t)dt$$

$$= \left[\phi(t)\frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \phi'(t)\frac{e^{-st}}{-s}dt$$

$$= (0-0) + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

Thus,

$$L\left\{\int_0^t f(t)dt\right\} = \frac{1}{s}L\{F(t)\}$$

PROBLEMS

1. By using the Laplace transform of $sinat$, find the Laplace transform of $cosat$.

Sol:: Let

F(t) = $sin\,at$, then L{F(t)} = $\frac{a}{s^2+a^2}$

We note that

$$F'(t) = a\cos at$$

Taking Laplace transforms, we get

$$L\{F'(t)\} = L\{a\cos at\} = aL\{\cos at\}$$

or L{cosat} = $\frac{1}{a}L\{F'(t)\} = \frac{1}{a}\left[sL\{F(t)\} - F(0)\right]$

$$= \frac{1}{a}\left[\frac{sa}{s^2+a^2} - 0\right]$$

Thus

$$L\{cosat\} = \frac{s}{s^2+a^2}$$

This is the desired result.

2. Given $L\left\{2\sqrt{\frac{t}{\pi}}\right\}=\frac{1}{s^{3/2}}$, show that $L\left\{\frac{1}{\sqrt{\pi t}}\right\}=\frac{1}{\sqrt{s}}$

Sol:: Let F(t) = $2\sqrt{\frac{t}{\pi}}$, given L{F(t)} = $\frac{1}{s^{3/2}}$

We note that, $F'(t)=\frac{2}{\sqrt{\pi}}\frac{1}{2\sqrt{t}}=\frac{1}{\sqrt{\pi t}}$

Taking Laplace transforms, we get

$$L\{F'(t)\}=L\left[\frac{1}{\sqrt{\pi t}}\right]$$

Hence

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\}=L\{F'(t)\}=sL\{F(t)\}-F(0)$$

$$= s\left(\frac{1}{s^{3/2}}\right)-0$$

Thus

$$L\left\{\frac{1}{\sqrt{\pi t}}\right\}=\frac{1}{\sqrt{s}}$$

This is the result as required.

3. Find $L\left\{\int_0^t\left(\frac{\cos at-\cos bt}{t}\right)dt\right\}$

Sol:: Here

L{F(t)} = $L\left\{\left(\frac{\cos at-\cos bt}{t}\right)\right\}=\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$

Using the result

$L\left\{\int_0^t F(t)dt\right\}=\frac{1}{s}L\{F(t)\}$

We get,

$$L\left\{\int_0^t\left(\frac{\cos at-\cos bt}{t}\right)dt\right\} = \frac{1}{2s}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

4. Find $L\left\{\int_0^t te^{-t}\sin 4t\ dt\right\}$

Sol:: Here

$$L\{e^{-t}\sin 4t\} = \frac{8(s+1)}{(s^2+2s+17)^2}$$

Thus

$$L\left\{\int_0^t te^{-t}\sin 4t dt\right\} = \frac{8(s+1)}{s(s^2+2s+17)^2}$$

EXERCISE

1. Find (i) $L\left\{\int_0^t \frac{\sin t}{t} dt\right\}$ (ii) $L\left\{\int_0^t te^{-t}\sin 5t dt\right\}$ (iii) $L\left\{\int_0^t \frac{e^{-t}\sin t}{t} dt\right\}$

(v) $L\left\{\int_0^t \frac{\sin^3 t}{t} dt\right\}$

ANSWERS

(i) $\frac{\pi}{2s} - \frac{1}{s}\tan^{-1} s$ (ii) $\frac{10(s+1)}{s((s+1)^2+25)^2}$ (iii) $\frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right]$

(iv) $\frac{1}{s}\left[\frac{\pi}{4} - \frac{3}{4}\tan^{-1} s + \frac{1}{4}\tan^{-1}\left(\frac{s}{3}\right)\right]$

4.10 INVERSE LAPLACE TRANSFORM

Def:: Let $L\{F(t)\} = f(s)$. Then F(t) is defined as the inverse Laplace transform of f(s) and is denoted by $L^{-1}\{f(s)\}$. Thus $L^{-1}\{f(s)\} = F(t)$.

4.11 LINEARITY PROPERTY

Let $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$ and a and b be any two constants. Then $L^{-1}\{af(s) + bg(s)\} = a\,L^{-1}\{f(s)\} + bL^{-1}\{g(s)\}$

<u>TABLE OF INVERSE LAPLACE TRANSFORMS</u>

f(s)	$F(t) = L^{-1}\{f(s)\}$
$\frac{1}{s}, s>0$	1
$\frac{1}{s-a}, s>a$	e^{at}
$\frac{1}{s+a}, s>a$	e^{-at}
$\frac{s}{s^2+a^2}, s>0$	Cos at
$\frac{1}{s^2+a^2}, s>0$	$\frac{\text{Sin at}}{\text{a}}$
$\frac{1}{s^2-a^2}, s>\lvert a\rvert$	$\frac{\text{Sin h at}}{\text{a}}$

$\frac{s}{s^2-a^2},\ s>\lvert a\rvert$	Cos h at
$\frac{1}{s^{n+1}},\ s>0$ $n = 0, 1, 2, 3, \ldots$	$\frac{t^n}{n!}$
$\frac{1}{s^{n+1}},\ s>0$ $n > -1$	$\frac{t^n}{\Gamma(n+1)}$

PROBLEMS

1. Find the inverse Laplace transforms of the following:

$$(i)\frac{1}{2s-5} \qquad (ii)\frac{s+b}{s^2+a^2} \qquad (iii)\frac{2s-5}{4s^2+25}+\frac{4s-9}{9-s^2}$$

Sol:: Here

$$(i)\ L^{-1}\left\{\frac{1}{2s-5}\right\}=\frac{1}{2}L^{-1}\left\{\frac{1}{s-5/2}\right\}=\frac{1}{2}e^{\frac{5t}{2}}$$

$$(ii)\ L^{-1}\left\{\frac{s+b}{s^2+a^2}\right\}=L^{-1}\left\{\frac{s}{s^2+a^2}\right\}+b\,L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\cos at+\frac{b}{a}\sin\ at$$

$$(iii)\ L^{-1}\left\{\frac{2s-5}{4s^2+25}+\frac{4s-8}{9-s^2}\right\}=\frac{2}{4}L^{-1}\left\{\frac{s-5/2}{s^2+\frac{25}{4}}\right\}-4L^{-1}\left\{\frac{s-9/2}{s^2-9}\right\}$$

$$=\frac{1}{2}\left[\cos\frac{5t}{2}-\sin\ \frac{5t}{2}\right]-4\left[\cos h3t-\frac{3}{2}\sin\ h3t\right]$$

4.12 FIRST SHIFTING THEOREM OF INVERSE LAPLACE TRANSFORM

Statement:: if $L^{-1}\{f(s)\}=F(t)$ then $L^{-1}\{f(s-a)\}=e^{at}L^{-1}\{f(s)\}$

Proof::

By First Shifting Theorem of Laplace Transform we have

If $L\{F(t)\}=f(s)$, then $L\{e^{at}F(t)\}=f(s-a)$, and so

$$L^{-1}\{f(s-a)\}=e^{at}F(t)=e^{at}L^{-1}\{f(s)\}$$

PROBLEMS

1. Evaluate : $L^{-1}\left\{\dfrac{3s+1}{(s+1)^4}\right\}$

Sol:: Given $= L^{-1}\left\{\dfrac{3(s+1-1)+1}{(s+1)^4}\right\} = 3\,L^{-1}\left\{\dfrac{1}{(s+1)^3}\right\} - 2\,L^{-1}\left\{\dfrac{1}{(s+1)^4}\right\}$

$$= 3e^{-t}\,L^{-1}\left\{\frac{1}{s^3}\right\} - 2e^{-t}\,L^{-1}\left\{\frac{1}{s^4}\right\}$$

Using the formula

$L^{-1}\left\{\dfrac{1}{s^{n+1}}\right\} = \dfrac{t^n}{n!}$ *and taking* $n = 2$ *and* 3, *we get*

$$Given = \frac{3e^{-t}t^2}{2} - \frac{e^{-t}t^3}{3}$$

2. Evaluate : $L^{-1}\left\{\dfrac{s+2}{s^2-2s+5}\right\}$

Sol::

Given $= L^{-1}\left\{\dfrac{s+2}{(s-1)^2+4}\right\} = L^{-1}\left\{\dfrac{(s-1)+3}{(s-1)^2+4}\right\} = L^{-1}\left\{\dfrac{s-1}{(s-1)^2+4}\right\} + 3L^{-1}\left\{\dfrac{1}{(s-1)^2+4}\right\}$

$$= e^t\,L^{-1}\left\{\frac{s}{s^2+4}\right\} + 3\,e^t\,L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

$$= e^t\cos 2t + \frac{3}{2}e^t\sin 2t$$

3. Evaluate : $L^{-1}\left\{\dfrac{2s+1}{s^2+3s+1}\right\}$

Sol:: Given $= 2L^{-1}\left\{\dfrac{\left(s+\frac{3}{2}\right)-1}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right\} = 2\left[L^{-1}\left\{\dfrac{\left(s+\frac{3}{2}\right)}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right\} - L^{-1}\left\{\dfrac{1}{\left(s+\frac{3}{2}\right)^2-\frac{5}{4}}\right\}\right]$

$$= 2\left[e^{\frac{-3t}{2}}L^{-1}\left\{\frac{s}{s^2-\frac{5}{4}}\right\} - e^{\frac{-3t}{2}}L^{-1}\left\{\frac{1}{s^2-\frac{5}{4}}\right\}\right]$$

$$= 2e^{\frac{-3t}{2}}\left[\cos h\frac{\sqrt{5}}{2}t\,\frac{2}{\sqrt{5}}\sin\,h\frac{\sqrt{5}}{2}t\right]$$

4. Evaluate : $L^{-1}\left\{\dfrac{2s^2+5s-4}{s^3+s^2-2s}\right\}$

Sol:: *we have*

$$\frac{2s^2+5s-4}{s^3+s^2-2s}=\frac{2s^2+5s-4}{s\left(s^2+s-2\right)}=\frac{2s^2+5s-4}{s(s+2)(s-1)}=\frac{A}{s}+\frac{B}{s+2}+\frac{C}{s-1}$$

Then

$2s^2+5s-4 = A(s+2)(s-1) + Bs(s-1) + Cs(s+2)$

For s = 0, we get A = 2, for s = 1, we get C = 1 and for s = -2, we get B = -1. Using these values in (1), we get

$$\frac{2s^2+5s-4}{s^3+s^2-2s}=\frac{2}{s}-\frac{1}{s+2}+\frac{1}{s-1}$$

Hence

$$L^{-1}\left\{\frac{2s^2+5s-4}{s^2+s^2-25}\right\}=L^{-1}\left\{\frac{2}{s}-\frac{1}{s+2}+\frac{1}{s-1}\right\}=2-e^{-2t}+e^{t}$$

5. Evaluate $:L^{-1}\left\{\frac{4s+5}{(s+1)^2+(s+2)}\right\}$

Sol:: Let us take

$$\frac{4s+5}{(s+1)^2+(s+2)}=\frac{A}{(s+1)^2}+\frac{B}{s+1}+\frac{C}{s+2}$$

Then

$4s + 5 = A(s + 2) + B(s + 1)(s + 2) + C(s + 1)^2$

For s = -1, we get A = 1, for s = -2, we get C = -3

Comparing the coefficients of s^2, we get B + C = 0, so that B = 3. Using these values in (1), we get

$$\frac{4s+5}{(s+1)^2+(s+2)}=\frac{1}{(s+1)^2}+\frac{3}{(s+1)}-\frac{3}{s+2}$$

Hence

$$L^{-1}\left\{\frac{4s+5}{(s+1)^2+(s+2)}\right\}=e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}+3e^{-t}L^{-1}\left\{\frac{1}{s}\right\}-3e^{-2t}L^{-1}\left\{\frac{1}{s}\right\}$$

$$=te^{-t}+3e^{-t}-3e^{-2t}$$

5. Evaluate $:L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\}$

Sol:: *Let*

$$\frac{s^3}{s^4-a^4}=\frac{A}{s-a}+\frac{B}{s+a}+\frac{Cs+D}{s^2+a^2} \qquad (1)$$

Hence

$s^3 = A(s + a)(s^2 + a^2) + B(s-a)(s^2+a^2)+(Cs + D)(s^2 – a^2)$

For s = a, we get A = ¼; for s = -a, we get B = ¼; comparing the constant terms, we get

D = a(A-B) = 0; comparing the coefficients of s^3, we get

1 = A + B + C and so C = ½. Using these values in (1), we get

$$\frac{s^3}{s^4-a^4}=\frac{1}{4}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]+\frac{1}{2}\frac{s}{s^2+a^2}$$

Taking inverse transforms, we get

$$L^{-1}\left\{\frac{s^3}{s^4-a^4}\right\}=L^{-1}\left\{\frac{1}{4}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]+\frac{1}{2}\frac{s}{s^2+a^2}\right\}=\frac{1}{4}\left[e^{at}+e^{-at}\right]+\frac{1}{2}\cos at$$

$$=\frac{1}{2}\left[\cos hat+\cos at\right]$$

6. Evaluate $:L^{-1}\left\{\dfrac{s}{s^4+s^2+1}\right\}$

Sol:: Consider

$$\frac{s}{s^4+s^2+1}=\frac{s}{\left(s^2+s+1\right)+\left(s^2-s+1\right)}=\frac{1}{2}\left[\frac{2s}{\left(s^2+s+1\right)\left(s^2-s+1\right)}\right]$$

$$=\frac{1}{2}\left[\frac{\left(s^2+s+1\right)-\left(s^2-s+1\right)}{\left(s^2+s+1\right)\left(s^2-s+1\right)}\right]=\frac{1}{2}\left[\frac{1}{\left(s^2-s+1\right)}-\frac{1}{\left(s^2+s+1\right)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}-\frac{1}{\left(s+\frac{1}{2}\right)^2+\frac{3}{4}}\right]$$

Therefore

$$L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}=\frac{1}{2}\left[e^{\frac{1}{2}t}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}-e^{-\frac{1}{2}t}L^{-1}\left\{\frac{1}{s^2+\frac{3}{4}}\right\}\right]$$

$$=\frac{1}{2}\left[e^{\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}-e^{-\frac{1}{2}t}\frac{\sin\frac{\sqrt{3}}{2}t}{\frac{\sqrt{3}}{2}}\right]$$

$$=\frac{2}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}t\right)\sin h\left(\frac{t}{2}\right)$$

EXERCISE

I.Find Inverse Laplace Transform of the following

1. $\frac{4}{S-4}$ 2. $\frac{4S}{S^2-4}$ 3. $\frac{5S+1}{S^2+16}$ 4. $\frac{S^2-3S+4}{S^3}$ 5. $\frac{\sqrt{S}+9}{S^{\frac{7}{2}}}$

6. $\frac{3S-10}{4S^2-9}+\frac{9S-50}{16-S^2}$ 7. $\frac{1}{2S^2+3}$

II. Find the Inverse Laplace Transform of the following

1. $\frac{1}{(s+1)(s-2)}$ 2. $\frac{1}{(s+1)(s+2)(s+3)}$ 3. $\frac{1}{(s+1)(s^2+1)}$ 4. $\frac{s}{(s+2)(s^2+1)}$

5. $\frac{4s+5}{(s-1)^2(s+2)}$ 6. $\frac{s^2}{(s-1)(s^2+1)}$ 7. $\frac{s^2+10s+13}{(s-1)(s^2-5s-6)}$

ANSWERS

I.

1. $4e^{4t}$ 2. $4cosh2t$ 3. $5cos4t+\frac{1}{4}sin4t$ 4. $1-3t+2t^3$

5. $\frac{1}{2}t^2+\frac{24}{5}\sqrt{\pi}t^{5/2}$ 6. $\frac{3}{4}cosh\frac{3t}{2}-\frac{5}{3}sinh\frac{3t}{2}-9cosh4t+\frac{25}{2}sinh4t$

7. $\frac{1}{\sqrt{6}}sin\sqrt{\frac{3}{2}}t$

II.

1. $\frac{1}{3}e^{2t}-\frac{1}{3}e^{-t}$ 2. $\frac{1}{2}[e^{-t}-2e^{-2t}+e^{-3t}]$ 3. $\frac{1}{2}[e^{-t}-cost+sint]$

4. $-\frac{2}{5}e^{-2t}+\frac{2}{5}cost+\frac{1}{5}sint$ 5. $\frac{2}{3}e^{t}+3te^{t}+e^{-2t}$

6. $\frac{1}{2}e^{t}+\frac{1}{2}(cost+sint)$ 7. $-\frac{12}{5}e^{t}+\frac{2}{7}e^{-t}+\frac{109}{35}e^{6t}$

4.13 EVALUATION OF $L^{-1}[e^{-as}\ f(s)]$

We have, if $L\{F(t)\} = f(s)$, then $L\{F(t-a)\,H(t-a)\} = e^{-as}f(s)$, and so $L^{-1}[e^{-as}F(s)] = f(t-a)\,H(t-a)$

PROBLEMS

(1)*Evaluate*: $L^{-1}\frac{e^{-ss}}{(s-2)^4}$

Sol:: *here* $a=5,\ f(s)=\frac{1}{(s-2)^4}$

Therefore $F(t)=L^{-1}\{f(s)\}=L^{-1}\left\{\frac{1}{(s-2)^4}\right\}=e^{2t}L^{-1}\left\{\frac{1}{s^4}\right\}\quad=\frac{e^{2t}t^3}{6}$

Thus $L^{-1}\dfrac{e^{-5s}}{(s-2)^4} = f(t-a)\,H(t-a)$

$$= \frac{e^{2(t-5)}(t-5)^3}{6}H(t-5)$$

(2) $Evaluate: L^{-1}\left\{\dfrac{e^{-\pi s}}{s^2+1}+\dfrac{se^{-2\pi s}}{s^2+4}\right\}$

Sol::

$$Given = F_1(t-\pi)H(t-\pi)+F_2(t-2\pi)H(t-2\pi) \qquad (1)$$

$$Here\;\; F_1(t) = L^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin\, t$$

$$F_2(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos\, 2t$$

Now relation (1) *reads as*

$$Given \;=\; \sin(t-\pi)\,H(t-\pi)+\cos 2(t-2\pi)\,H(t-2\pi)$$

$$= -\cos t\; H(t-\pi)+\cos\,(2t)\,H(t-2\pi)$$

EXERCISE

I. Find the Inverse Laplace Transform of the following

1. $\frac{1}{(s-3)^5}$ 2 $\frac{s}{(s+2)^3}$ 3. $\frac{s+2}{s^2+2s+2}$ 4. $\frac{s+2}{s^2-4s+13}$ 5. $\frac{s-1}{s^2-6s+25}$

6. $\frac{s+1}{s^2-s+1}$ 7. $\frac{s}{s^4+s^2+1}$ 8. $\frac{s}{s^4+64}$

ANSWER

1. $\frac{1}{24}t^4e^{3t}$ 2. $(t-t^2)e^{-2t}$ 3. $e^{-t}[cost+sint]$

4. $e^{2t}\left(cos3t+\frac{4}{3}sin3t\right)$

5. $e^{3t}\left(cos4t+\frac{1}{2}sin4t\right)$ 6. $e^{\frac{t}{2}}\left[cos\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}sin\left(\frac{\sqrt{3}}{2}t\right)\right]$

7. $\frac{1}{8}sin2t\;\; sinh2t$

4.14 INVERSE TRANSFORM OF DERIVATIVE

If $\boldsymbol{L^{-1}\{f(s)\} = F(t)}$ ***then*** $\boldsymbol{L^{-1}\{f^n(s)\} = (-1)^n t^n L^{-1}\{F(t)\}}$

Proof:

We have $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}[f(s)]$

There fore $(-1)^n f^n[s] = L\{t^n F(t)\}$

$L^{-1}\{(-1)^n f^n(s)\} \quad = \{t^n F(t)\}$

$(-1)^n L^{-1}\{f^n(s)\} \quad = \{t^n F(t)\}$

$L^{-1}\{f^n(s)\} \quad = (-1)^n t^n F(t)$

PROBLEMS

(1) *Evaluate*: $L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}$

Sol:: $\quad Let\ f(s) = \log\left(\frac{s+a}{s+b}\right) = \log(s+a) - \log(s+b)$

$Then\ -\frac{d}{ds} f(s) = -\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$

$So\ that\ \ L^{-1}\left\{-\frac{d}{ds} f(s)\right\} = -\left[e^{-at} - e^{-bt}\right]$

$or \quad t\,F(t) = e^{-bt} - e^{-at}$

$Thus \qquad F(t) = \frac{e^{-bt} - e^{-at}}{b}$

(2) Evaluate $\quad L^{-1}\left\{\tan^{-1}\left(\frac{a}{s}\right)\right\}$

Sol:: $\quad Let\ f(s) = \tan^{-1}\left(\frac{a}{s}\right)$

$Then\ -\frac{d}{ds} f(s) = \left[\frac{a}{s^2 + a^2}\right]$

$or \quad L^{-1}\left\{-\frac{d}{ds} f(s)\right\} = \sin\ at \quad so\ that$

$or \quad t\ f(t) = \sin\ at$

$f(t) = \frac{\sin\ at}{t}$

4.15 INVERSE LAPLACE TRANSFORM OF $\left[\frac{F(S)}{S}\right]$

Proof::

$$Since L\left\{\int_0^t f(t)dt\right\} = \frac{F(s)}{s}, we\ have,$$

$$L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t f(t)dt$$

PROBLEMS

(1) *Evaluate*: $L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$

Sol::

$$Let\ us\ denote\ F(s) = \frac{1}{s^2+a^2}\quad so\ that$$

$$f(t) = L^{-1}\{f(s)\} = \frac{\sin at}{a}$$

$$Then\ L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t \frac{\sin at}{a} dt$$

$$= \frac{(1-\cos at)}{a^2}$$

(2) *Evaluate*: $L^{-1}\left\{\frac{1}{s^2(s+a)^2}\right\}$

Sol::

$$we\ have\ L^{-1}\left\{\frac{1}{(s+a)^2}\right\} = e^{-at}t$$

$$Hence\ L^{-1}\left\{\frac{1}{s(s+a)^2}\right\} = \int_0^t e^{-at}t\ dt$$

$$= \frac{1}{a^2}\left[1-e^{-at}(1+at)\right],\ \text{on integratio n by parts.}$$

Using this, we get

$$L^{-1}\left\{\frac{1}{s^2(s+a)^2}\right\} = \frac{1}{a^2}\int_0^t \left[1-e^{-at}(1+at)\right]dt$$

$$= \frac{1}{a^3}\left[at(1+e^{-at})+2(e^{-at}-1)\right]$$

EXERCISE

I. Find the Inverse Laplace Transform of the following

1. $log\left(\frac{s-a}{s-b}\right)$ 2. $log\left(\frac{s+a}{s+b}\right)$ 3. $log\left(1-\frac{a}{s}\right)$ 4. $log\left(\frac{s^2+1}{s(s-1)}\right)$

5. $log\left(\frac{(s+1)(s+3)}{(s+2)(s-3)}\right)$ 6. $log\left(\frac{s+1}{s}\right)$

II. Find the Inverse Laplace Transform of the following

1. $e^{-\pi s}\left(\frac{s}{s^2+9}\right)$ 2. $\frac{e^s}{s+1}$ 3. $e^{2\pi s}\left(\frac{s+2}{s^2+4}\right)$

ANSWERS

I.

1. $\frac{e^{bt}-e^{at}}{t}$ 2. $\frac{e^{-bt}-e^{-at}}{t}$ 3. $\frac{1-e^{at}}{t}$ 4. $\frac{e^{t}+1-2cost}{t}$

5.$-\frac{1}{t}[e^{-t}+e^{-3t}-e^{-2t}-e^{3t}]$ 6. $\frac{1-e^{-t}}{t}$

II 1. $-\boldsymbol{cos3t\, u(t-\pi)}$ 2. $\boldsymbol{e^{-(t+1)}u(t+1)}$ 3. $\boldsymbol{(cos2t+sin2t)u(t+2\pi)}$

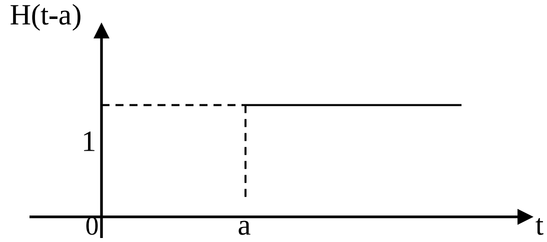
5.1 UNIT STEP FUNCTION:

In many Engineering applications, we deal with an important discontinuous function H (t-a) defined as follows :

$$H(t-a)=\begin{cases}0, & t\le a\\ 1, & t>a\end{cases}$$

where a is a non-negative constant.

This function is known as the **unit step function** or the **Heaviside function**. The function is named after the British electrical engineer Oliver Heaviside. The function is also denoted by U(t-a). The graph of the function is shown below:



Note that the value of the function suddenly jumps from value zero to the value 1 as $t \to a$ from the left and retains the value 1 for all t>a. Hence the function H(t-a) is called the unit step function.

In particular, when a=0, the function H(t-a) become H(t), where

$$H(t) = \begin{cases} 0, & t \le 0 \\ 1, & t > 0 \end{cases}$$

Transform of Unit step function:

By definition, we have

$$L\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)dt$$

$$= \int_0^a e^{-st} 0 dt + \int_a^\infty e^{-st}(1)dt$$

$$= \frac{e^{-as}}{s}$$

In particular, we have $L\{H(t)\} = \dfrac{1}{s}$

Also, $L^{-1}\left\{\dfrac{e^{-as}}{s}\right\} = H(t-a)$ and $L^{-1}\left\{\dfrac{1}{s}\right\} = H(t)$

HEAVISIDE SHIFT THEOREM

Statement :-

L {F(t-a) H(t-a)} = e^{-as} L{F(t)}

Proof :- We have

$$L\{F(t-a)\,H(t-a)\} = \int_0^\infty F(t-a)H(t-a)e^{-st}dt$$

$$= \int_a^\infty e^{-st} F(t-a)dt$$

Setting t-a = u, we get

$$L\{F(t-a)\,H(t-a)\} = \int_0^\infty e^{-s(a+u)} F(u)du$$

$$= e^{-as}\, L\{F(t)\}$$

This is the desired shift theorem.

Problems

1 . Find L{[e^{t-2} + sin(t-2)] H(t-2)}

Sol :

Let

F(t-2) = [e^{t-2} + sin(t-2)]

Then

F(t) = [e^t + sint]

so that

$$L\{F(t)\} = \frac{1}{s-1} + \frac{1}{s^2+1}$$

By Heaviside shift theorem, we have

L{F(t-2) H(t-2)} = e^{-2s} L{F(t)}

Thus,

$$L\{(e^{(t-2)} + \sin(t-2))H(t-2)\} = e^{-2s}\left[\frac{1}{s-1} + \frac{1}{s^2+1}\right]$$

2. Find L{(3t^2 +2t +3) H(t-1)}

Sol :

Let

F(t-1) = 3t^2 +2t +3

so that

F(t) = 3$(t+1)^2$ +2(t+1) +3 = 3t^2 +8t +8

Hence

$$L\{F(t)\} = \frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s}$$

Thus

L{(3t^2 +2t +3) H(t-1)} = L{F(t-1) H(t-1)}

= e^{-s} L {F(t)}

$$= e^{-s}\left[\frac{6}{s^3} + \frac{8}{s^2} + \frac{8}{s}\right]$$

3. Find L{e^{-t} H(t-2)}

Sol :

Let F(t-2) = e^{-t} , so that, F(t) = $e^{-(t+2)}$

Thus,

$$L\{F(t)\} = \frac{e^{-2}}{s+1}$$

By shift theorem, we have

$$L\{F(t-2)H(t-2)\} = e^{-2s}L\{F(t)\} = \frac{e^{-2(s+1)}}{s+1}$$

Thus

$$L\{e^{-t}H(t-2\} = \frac{e^{-2(s+1)}}{s+1}$$

4. Let $F(t) = \begin{cases} f_1(t), & t \le a \\ f_2(t), & t > a \end{cases}$

Verify that

f(t) = $f_1(t) + [f_2(t) - f_1(t)]H(t-a)$

Sol :

Consider

$$f_1(t) + [f_2(t) - f_1(t)]H(t-a) = f_1(t) + \begin{cases} f_2(t) - f_1(t), & t > a \\ 0, & t \le a \end{cases}$$

$$= \begin{cases} f_2(t), & t > a \\ f_1(t), & t \le a \end{cases} = f(t),$$

Given . Thus the required result is verified.

5. Express the functions $f(t) = \begin{cases} t^2, 1 < t \le 2 \\ 4t,\ t > 2 \end{cases}$ in terms of unit step function and hence find their Laplace transforms.

Sol:

Here,

$f(t) = t^2 + (4t-t^2)\,H(t-2)$

Hence,

$$L\{f(t)\} = \frac{2}{s^3} + L(4t-t^2)H(t-2)$$

Let

$\phi(t-2) = 4t - t^2$

so that

$\phi(t) = 4(t+2) - (t+2)^2 = -t^2 + 4$

Now,

$$L\{\phi(t)\} = -\frac{2}{s^3} + \frac{4}{s}$$

Expression (i) reads as

$$L\{f(t)\} = \frac{2}{s^3} + L[\phi(t-2)H(t-2)]$$

$$= \frac{2}{s^3} + e^{-2s}L\phi(t)$$

$$= \frac{2}{s^3} + e^{-2s}\left(\frac{4}{s} - \frac{2}{s^3}\right)$$

This is the desired result.

6 .Express the functions $f(t) = \begin{cases} \cos t, 0 < t \le \pi \\ \sin t, \ t > \pi \end{cases}$ in terms of unit step function and hence find their Laplace transforms.

Sol:

. Here $f(t) = \cos t + (\sin t - \cos t)H(t-\pi)$

Hence,

$$L\{f(t)\} = \frac{s}{s^2+1} + L(\sin t - \cos t)H(t-\pi)$$

Let

$\phi(t-\pi) = \sin t - \cos t$

Then

$\phi(t) = \sin(t+\pi) - \cos(t+\pi) = -\sin t + \cos t$

so that

$$L\{\phi(t)\} = -\frac{1}{s^2+1} + \frac{s}{s^2+1}$$

Expression (ii) reads as

$$L\{f(t)\} = \frac{s}{s^2+1} + L\{\phi(t-\pi)H(t-\pi)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}L\{\phi(t)\}$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s-1}{s^2+1}\right]$$

EXERCISE

Express the following functions in terms of unit step function and hence find their Laplace transforms.

1. $(t-2)^3 H(t-2)$ 2. $e^{-3t}H(t-2)$ 3. $t^2 H(t-2)$ 4. $e^{-2t}(t-5)H(t-5)$

5. $(t-1)^3 H(t-1)$ 6. $e^{a-t}H(t-a)$ 7. $e^{t-3}H(t-3)$

8). $f(t)=\begin{cases} 0, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$ 9). $f(t)=\begin{cases} \cos t, & 0 < t \le \pi \\ 1, & \pi < t \le 2\pi \\ \sin t, & t \ge 2\pi \end{cases}$ 10). $f(t)=\begin{cases} \sin t, & 0 < t \le \pi \\ \sin 2t, & \pi < t \le 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$

ANSWERS

1. $\dfrac{6e^{-2s}}{s^4}$ 2. $\dfrac{e^{-2(s+3)}}{s+3}$ 3. $e^{-2s}\left(\dfrac{2}{s^3}+\dfrac{4}{s^2}+\dfrac{4}{s}\right)$ 4. $\dfrac{e^{10-5s}}{(s-2)^2}$ 5. $\dfrac{6e^{-s}}{s^4}$ 6. $\dfrac{e^{-as}}{s+1}$

7. $\dfrac{e^{-3s}}{s-1}$ 8. $\dfrac{2}{s}\left(e^{-s}-e^{-2s}\right)$ 9. $\dfrac{s}{s^2+1}+e^{-\pi s}\left\{\dfrac{1}{s}+\dfrac{s}{s^2+1}\right\}+e^{-2\pi s}\left\{\dfrac{s}{s^2+1}-\dfrac{1}{s}\right\}$

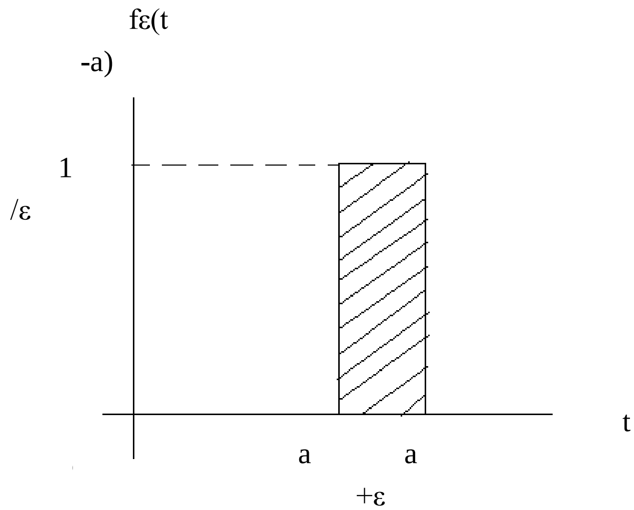
10. $\dfrac{1}{s^2+1}+e^{-\pi s}\left\{\dfrac{2}{s^2+4}+\dfrac{1}{s^2+1}\right\}+e^{-2\pi s}\left\{\dfrac{3}{s^2+9}-\dfrac{2}{s^2+4}\right\}$

5.2 UNIT IMPULSE FUNCTION OR DIRAC'S DELTA FUNCTION

The idea of a large force (like earthquake) acting for a very short duration of time is of practical importance as, for instance, in the collision of two bodies. To deal with such situation, we introduce a function called the Impulse function which is a discontinuous force function.

Thus unit impulse function is considered as the limiting form of the function

$$f_\varepsilon(t-a)=\begin{cases} 0, & for\ t < a \\ \dfrac{1}{\varepsilon}, & for\ a \le t \le a+\varepsilon \\ 0, & for\ t > a \end{cases} \quad (or) \quad \begin{cases} \dfrac{1}{\varepsilon}, & for\ a \le t \le a+\varepsilon \\ 0, & otherwise \end{cases}$$



As $\varepsilon > 0$ it is clear from figure that as $\varepsilon \to 0$, the height of shaded strip increases indefinitely and the width decreases in such a way that its area is always unity.

$$L\{f_\varepsilon(t-a)\} = \int_0^\infty e^{-st} f_\varepsilon(t-a)dt$$

$$= \int_a^{a+\varepsilon} e^{-st} f_\varepsilon(t-a)dt + \int_{a+\varepsilon}^\infty e^{-st} f_\varepsilon(t-a)\,dt$$

$$= \int_a^{a+\varepsilon} e^{-st}\,\frac{1}{\varepsilon}dt + \int_{a+\varepsilon}^\infty e^{-st}(0)\,dt$$

$$= \frac{1}{\varepsilon}\left(\frac{e^{-st}}{-s}\right)_a^{a+\varepsilon} = \frac{e^{-as}\left(1-e^{-\varepsilon s}\right)}{\varepsilon s}$$

The limit of $f_\varepsilon(t-a)$ as $\varepsilon \to 0$ is denoted by $\delta(t-a)$ and is called the Dirac delta function.

Laplace transform of Dirac delta function:

$$L\{\delta(t-a)\} = \underset{\varepsilon\to 0}{Lim}\ L\{f_\varepsilon(t)\}$$

$$= \underset{\varepsilon\to 0}{Lim}\ \frac{e^{-as}\left(1-e^{-\varepsilon s}\right)}{\varepsilon s}\left(\frac{0}{0}\right)$$

$$= e^{-as}\ \underset{\varepsilon\to 0}{Lim}\ \frac{e^{-\varepsilon s}}{s}, usigLHospitalrule$$

$$= e^{-as}1 = e^{-as}$$

Thus unit impulse function $\delta(t-a)$ is defined by

$$\delta(t-a) = \begin{cases} \infty & for\ t = a \\ 0 & for\ t \neq a \end{cases}$$

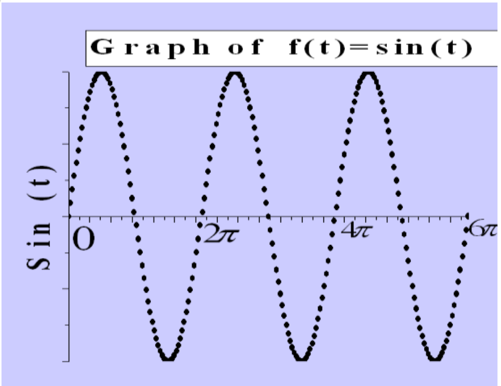
Such that $\int_0^\infty \delta(t-a)dt\ = 1\ \ (a \geq 0)$

5.3 PERIODIC *FUNCTION*

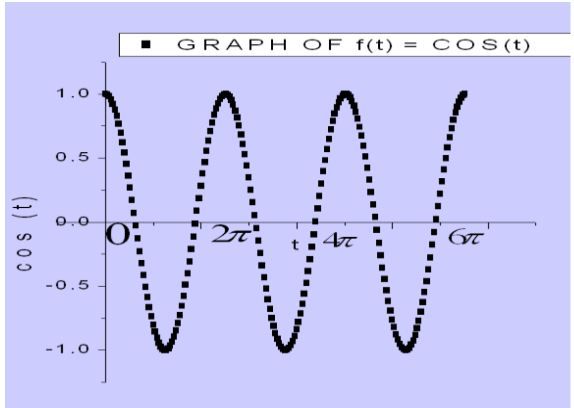
A function f(t) is said to be a periodic function of period T > 0 if f(t) = f(t + nT) where n=1,2,3,….. The graph of the periodic function repeats itself in equal intervals.

For example, sint, cost are periodic functions of period 2π since sin(t + 2nπ) = sin t, cos(t + 2nπ) = cos t.

The graph of f(t) = sin t is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on. The graph of f(t) = cos t is shown below :



Note that the graph of the function between 0 and 2π is the same as that between 2π and 4π and so on.

Theorem:

Let f (t) be a periodic function of period T. Then

$$L\{f(t)\} = \frac{1}{1-e^{-ST}}\int_0^T e^{-st} f(t)dt$$

Proof :

By definition, we have

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-su} f(u)du$$

$$= \int_0^T e^{-su} f(u)du + \int_T^{2T} e^{-su} f(u)du + ... + \int_{nT}^{(n+1)T} e^{-su} f(u)du + ... + \infty$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-su} f(u)du$$

Let us set u = t + nT, then

$$L\{ f(t)\} = \sum_{n=0}^{\infty} \int_{t=0}^{T} e^{-s(t+nT)} f(t+nT)dt$$

Here

f(t+nT) = f(t), by periodic property

Hence

$$L\{f(t)\} = \sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-st} f(t)dt$$

$$= \left[\frac{1}{1-e^{-ST}}\right]\int_0^T e^{-st} f(t)dt,$$

identifying the above series as a geometric series.
Thus

$$L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt$$

This is the desired result.

***1.** For the periodic function f(t) of period 4, defined*

by $f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}$, Find L{ f(t)}.

Sol: Here, period of f(t) = T = 4

We have,

$$L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt$$

$$= \left[\frac{1}{1-e^{-4s}}\right]\int_0^4 e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-4s}}\left[\int_0^2 3te^{-st}dt + \int_2^4 6e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-4s}}\left[3\left\{\left[t\left(\frac{e^{-st}}{-s}\right)\right]_0^2 - \int_0^2 1.\frac{e^{-st}}{-s}dt\right\} + 6\left(\frac{e^{-st}}{-s}\right)_2^4\right]$$

$$= \frac{1}{1-e^{-4s}}\left[\frac{3\left(1-e^{-2s}-2se^{-4s}\right)}{s^2}\right]$$

Thus,

$$L\{f(t)\} = \frac{3(1-e^{-2s}-2se^{-4s})}{s^2(1-e^{-4s})}$$

2. *A periodic function of period* $\frac{2\pi}{\omega}$ *is defined by*

$$f(t) = \begin{cases} E\sin\omega t, & 0 \le t < \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} \le t \le \frac{2\pi}{\omega} \end{cases}$$

where E and ω are positive constants.

Show that

$$L\{f(t)\} = \frac{E\omega}{(s^2 + w^2)(1 - e^{-\pi s/w})}$$

Sol: Here $T = \frac{2\pi}{\omega}$. Therefore

$$L\{f(t)\} = \frac{1}{1 - e^{-s(2\pi/\omega)}} \int_0^{2\pi/\omega} e^{-st} f(t)dt$$

$$= \frac{1}{1 - e^{-s(2\pi/\omega)}} \int_0^{\pi/\omega} Ee^{-st} \sin \omega t dt$$

$$= \frac{E}{1 - e^{-s(2\pi/\omega)}} \left[\frac{e^{-st}}{s^2 + \omega^2} \{-s\sin \omega t - \omega \cos \omega t\} \right]_0^{\pi/\omega}$$

$$= \frac{E}{1 - e^{-s(2\pi/\omega)}} \frac{\omega(e^{-s\pi/\omega} + 1)}{s^2 + \omega^2}$$

$$= \frac{E\omega(1 + e^{-s\pi/\omega})}{(1 - e^{-s\pi/\omega})(1 + e^{-s\pi/\omega})(s^2 + \omega^2)}$$

$$= \frac{E\omega}{(1 - e^{-s\pi/\omega})(s^2 + \omega^2)}$$

This is the desired result.

3. *A periodic function f(t) of period 2a, a>0 is defined by*

$$f(t) = \begin{cases} E, & 0 \le t \le a \\ -E, & a < t \le 2a \end{cases}$$

show that

L {f(t) }= $\frac{E}{s}\tanh\left(\frac{as}{2}\right)$

Sol: Here T = 2a. Therefore

$$L\{f(t)\} = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a Ee^{-st}dt + \int_a^{2a} - Ee^{-st}dt\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[\left(1-e^{-sa}\right)+(e^{-2as}-e^{-as})\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[\left(1-e^{-as}\right)^2\right] = \frac{E(1-e^{-as})^2}{s(1-e^{-as})(1+e^{-as})}$$

$$= \frac{E}{s}\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}\right] = \frac{E}{s}\tanh\left(\frac{as}{2}\right)$$

This is the result as desired.

4. Find the Laplace Transform of the periodic function

f(t) = t for 0 < t < 4 and f (t+4) = f(t) .

Sol: Here, period of f(t) = T = 4

We have,

$$L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt$$

$$= \left[\frac{1}{1-e^{-4s}}\right]\int_0^4 e^{-st} t dt$$

$$= \frac{1}{1-e^{-4s}}\left[t\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right]_0^4$$

$$= \frac{1}{1-e^{-4s}}\left[\left(\frac{4e^{-4s}}{-s} - \frac{e^{-4s}}{s^2}\right) - \left(0 - \frac{1}{s^2}\right)\right]$$

Thus,

$$L\{f(t)\} = \frac{1-4se^{-4s}-e^{-4s})}{s^2(1-e^{-4s})}$$

5. Find the Laplace Transform of the function

$$f(t)=\begin{cases}1, & 0<t<\dfrac{a}{2}\\ -1, & \dfrac{a}{2}<t<a\end{cases}\quad \text{and} \quad f(t+a)=f(t).$$

Sol:

Here, period of f(t) = T = a

We have,

$$L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt$$

$$= \left[\frac{1}{1-e^{-as}}\right]\int_0^a e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{\frac{a}{2}} 1e^{-st}dt + \int_{\frac{a}{2}}^{a} -1e^{-st}dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left(\frac{e^{-st}}{-s}\right)_0^{\frac{a}{2}} - \left(\frac{e^{-st}}{-s}\right)_{\frac{a}{2}}^{a}\right]$$

$$= \frac{1}{1-e^{-as}}\left[\left(\frac{e^{\frac{-as}{2}}}{-s}+\frac{1}{s}\right)+\left(\frac{e^{-as}}{s}-\frac{e^{\frac{-as}{2}}}{s}\right)\right]$$

Thus,

$$L\{f(t)\} = \frac{1}{1-e^{-as}}\left[\frac{1+e^{-as}-2e^{\frac{-as}{2}}}{s}\right]$$

6. *Find the Laplace transform of the square wave function of period 2a*

Defined as $f(t) = \begin{cases} K, & 0 < t < a \\ -K, & a < t < 2a \end{cases}$

Sol: Here, period of f(t) = T = 2a

We have,

$$L\{f(t)\} = \left[\frac{1}{1-e^{-sT}}\right]\int_0^T e^{-st} f(t)dt$$

$$= \left[\frac{1}{1-e^{-2as}}\right]\int_0^{2a} e^{-st} f(t)dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a ke^{-st}dt + \int_a^{2a} -ke^{-st}dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[k\left(\frac{e^{-st}}{-s}\right)_0^a - k\left(\frac{e^{-st}}{-s}\right)_a^{2a}\right]$$

$$= \frac{k}{1-e^{-2as}}\left[\frac{e^{-2as} - 2e^{-as} + 1}{s}\right]$$

$$= \frac{k}{1-e^{-2as}}\left[\frac{(1-e^{-as})^2}{s}\right]$$

$$= \frac{k}{(1-e^{-as})(1+e^{-as})}\left[\frac{(1-e^{-as})^2}{s}\right]$$

Thus,

$$L\{f(t)\} = \frac{k(1-e^{-as})}{s(1+e^{-as})}$$

EXERCISE

1. Find $L\{\,f(t)\,\}$, where f(t) is a periodic function of period 2π and it is given by

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ 0, & \pi < t < 2\pi \end{cases}$$

2. Find $L\{\,f(t)\,\}$, where f(t) is a periodic function of period T and it is given by

$$f(t) = \frac{k}{T}, \; when \;\; 0 < t < T$$

3. Find $L\{|\sin t|\}$

4. Find $L\{\,f(t)\,\}$, where f(t) is a periodic function of period $2b$ and it is given by $f(t) = t\,,\;\; 0 < t < b$ and $f(t) = 2b - t\,,\;\; b < t < 2b$

5. Find the Laplace Transform of the function

$$f(t)=\begin{cases}1, & 0\le t<1\\ -1, & 1\le t<2\end{cases}\quad , f(t+2)=f(t).$$

6. Find $L\{\,f(t)\,\}$, where f(t) is a periodic function of period T and it is given by

$$f(t)=\begin{cases}\dfrac{4Et}{T}-E, & 0\le t<\dfrac{T}{2}\\[2mm] 3E-\dfrac{4Et}{T}, & \dfrac{T}{2}<t<T\end{cases}$$

7. Compute $L\{\,f(t)\,\}$, if $f(t)=t^2$, 0 < t < 2 and f (t+2)= f (t)

ANSWERS

1. $\dfrac{1}{(s^2+1)(1-e^{-\pi s})}$ 2. $\dfrac{k}{s^2T}-\dfrac{ke^{-sT}}{s(1-e^{-sT})}$ 3. $\dfrac{1+e^{-\pi s}}{(s^2+1)(1-e^{-\pi s})}$ 4. $\dfrac{1}{s^2}\tanh\left(\dfrac{bs}{2}\right)$

5. $\dfrac{1-e^{-s}}{s(1+e^{-s})}$ 6. $\dfrac{E}{s(1-e^{-sT})}\left[\dfrac{4}{Ts}\left(1-e^{-\frac{Ts}{2}}\right)-2e^{-\frac{Ts}{2}}-3e^{-Ts}-1\right]$

7. $\dfrac{2}{s^3(1-e^{-2s})}\left[1-e-2s(1+2s+2s^2)\right]$

5.4 CONVOLUTION

The convolution of two functions f(t) and g(t) denoted by f(t) $*$ g(t) is defined as

$$f(t)*g(t)=\int_0^t f(t-u)g(u)\,du$$

Property : f(t) $*$ g(t) = g(t) $*$ f(t)

Proof :- By definition, we have

$$f(t)*g(t)=\int_0^t f(t-u)g(u)\,du$$

Setting t-u = x, we get

$$f(t)*g(t)=\int_t^0 f(x)g(t-x)(-dx)$$

$$= \int_0^t g(t-x) f(x) dx = g(t) * f(t)$$

This is the desired property. Note that the operation $*$ is commutative.

CONVOLUTION THEOREM

Statement: If f(t) and g(t) are piece-wise continuous functions for $t \geq 0$. And are of exponential order then

$$L\{f(t) * g(t)\} = L\{f(t)\}. L\{g(t)\} = \bar{f}(s)\bar{g}(s) \text{ or } \left(\bar{f}.\bar{g}\right)(s)$$

Proof :- Let us denote

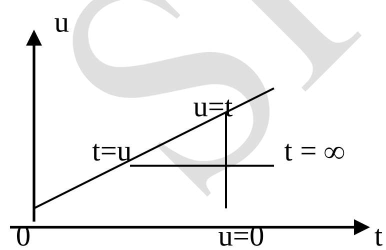
$$f(t) * g(t) = \phi(t) = \int_0^t f(t-u) g(u) du$$

Consider

$$L\{\phi(t)\} = \int_0^\infty e^{-st} [\int_0^t f(t-u) g(u)] dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(t-u) g(u) du \qquad (1)$$

We note that the region for this double integral is the entire area lying between the lines u =0 and u = t. On changing the order of integration, we find that t varies from u to ∞ and u varies from 0 to ∞.



Hence (1) becomes

$$L\{\phi(t)\} = \int_{u=0}^\infty \int_{t=u}^\infty e^{-st} f(t-u) g(u) dt du$$

$$= \int_0^\infty e^{-su} g(u) \left\{ \int_u^\infty e^{-s(t-u)} f(t-u) dt \right\} du$$

$$= \int_0^\infty e^{-su} g(u) \left\{ \int_0^\infty e^{-sv} f(v) dv \right\} du \quad , \quad \text{where } v = t-u$$

$$= \int_0^\infty e^{-su} g(u)du \int_0^\infty e^{-sv} f(v)dv$$

= L{ g(t)} . L{ f(t)}

Thus

L {f(t)} . L{ g(t) }= L[f(t) $*$ g(t)]

This is desired property.

Inverse transform of F(s) by using convolution theorem

We have, if L{f(t)} = F(s) and L{g(t)} = G(s), then

$$L[f(t) * g(t)] = L\{f(t)\} \cdot L\{g(t)\} = F(s)\, G(s) \ \ and\ so$$

$$L^{-1}[F(s)\, G(s)] = f(t) * g(t) = \int_0^t f(t-u) g(u) du$$

This expression is called the convolution theorem for inverse Laplace transform

PROBLEMS

Employ convolution theorem to evaluate the following**:**

(1) $L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\}$

Sol. Let us denote $F(s) = \frac{1}{s+a}, G(s) = \frac{1}{s+b}$

Taking the inverse, we get

$$f(t) = e^{-at}, g(t) = e^{-bt}$$

Therefore , by convolution theorem ,

$$L^{-1}\left\{\frac{1}{(s+a)(s+b)}\right\} = \int_0^t e^{-a(t-u)} e^{-bu} du = e^{-at} \int_0^t e^{(a-b)u} du$$

$$= e^{-at}\left[\frac{e^{(a-b)t} - 1}{a-b}\right] = \frac{e^{-bt} - e^{-at}}{a-b}$$

(2) $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\}$

Sol. Let us denote

$$F(s) = \frac{1}{s^2+a^2}, G(s) = \frac{s}{s^2+a^2} \ \text{ then}$$

$$f(t) = \frac{\sin at}{a},\ g(t) = \cos at$$

Hence by convolution theorem

$$L^{-1}\left\{\frac{s}{\left(s^2+a^2\right)^2}\right\} = \int_0^t \frac{1}{a}\sin a(t-u)\cos au\, du$$

$$= \frac{1}{a}\int_0^t \frac{\sin at + \sin(at-2au)}{2} du,$$

By using compound angle formula

$$= \frac{1}{2a}\left[u \sin at - \frac{\cos(at-2au)}{-2a}\right]_0^t = \frac{t \sin at}{2a}$$

(3) $L^{-1}\left\{\frac{s}{(s-1)(s^2+1)}\right\}$

Sol.

Here $F(s) = \frac{1}{s-1}, \quad G(s) = \frac{s}{s^2+1}$

therefore

$f(t) = e^t,\ g(t) = \sin t$

By convolution theorem, we have

$$L^{-1}\left\{\frac{1}{(s-1)(s^2+1)}\right\} = \int e^{t-u} \sin u\, du \quad = e^t\left[\frac{e^{-u}}{2}(-\sin u - \cos u)\right]_0^t$$

$$= \frac{e^t}{2}\left[e^{-t}(-\sin t - \cos t) - (-1)\right] = \frac{1}{2}\left[e^t - \sin t - \cos t\right]$$

4. Using Convolution theorem, evaluate $L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}$.

Sol. Since $f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$ and

$$g(t) = L^{-1}\left\{\frac{1}{s^2+2s+2}\right\} = L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\} = e^{-t}\sin t$$

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s^2+2s+2)}\right\}=L^{-1}\left\{\frac{1}{s}.\frac{1}{s^2+2s+2}\right\}=f(t)*g(t)=g(t)*f(t)$$

$$=\int_0^t g(u)f(t-u)du=\int_0^t e^{-u}\sin u.1du=\int_0^t e^{-u}\sin udu$$

$$=\left[\frac{e^{-u}}{1+1}(-\sin u-\cos u)\right]_0^t=-\frac{1}{2}[e^{-u}(\sin u+\cos u)]_0^t$$

$$=-\frac{1}{2}[e^{-t}(\sin t+\cos t)-1.(0+1)]=\frac{1}{2}[1-e^{t}(\sin t+\cos t)$$

5. Using Convolution theorem, find $L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}$.

Sol. Let $\overline{f}(s)=\frac{1}{s^2+a^2}$ and $\overline{g}(s)=\frac{1}{s^2+a^2}$. Then

$f(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}$ sin at and $g(t)=L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin at$

$$\therefore L^{1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\frac{1}{s^2+a^2}.\frac{1}{s^2+a^2}\right\}=L^{-1}\{\overline{f}(s).\overline{g}(s)\}=f(t)*g(t)$$

$$\int_0^t f(u)g(t-u)du=\int_0^t \frac{1}{a}\sin au.\frac{1}{a}\sin a(t-u)du$$

$$=\frac{1}{2a^2}\int_0^t 2\sin au\sin(at-au)du$$

$$=\frac{1}{2a^2}\int_0^t[\cos(2au-at)-\cos at]du$$

$$\therefore 2\sin A\sin B=\cos(A-B)-\cos(A+B)$$

$$=\frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a}-\cos atu\right]$$

$$=\frac{1}{2a^2}\left[\frac{1}{2a}\sin at-t\cos at+\frac{1}{2a}\sin at\right]$$

$$==\frac{1}{2a^3}(\sin at-at\cos at)$$

<u>Note:</u> Find inverse Laplace transform of $\frac{1}{(s^2+5^2)^2}$

Sol. Put a=5 in the above problem.

6. Apply Convolution theorem to evaluate $L^{-1}\left\{\frac{1}{(s-2)(s+2)^2}\right\}$.

Sol. Let $\bar{f}(s) = \frac{1}{s-2}$ and $\bar{g}(s) = \frac{1}{(s-2)^2}$ so that

$$f(t) = L^{-1}\left\{\frac{1}{s-2}\right\} = e^{2t} \text{ and}$$

$$g(t) = L^{-1}\left\{\frac{1}{(s+2)^2}\right\} = e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\} = te^{-2t}$$

∴ By Convolution theorem,

$$L^{1}\left\{\frac{1}{(s-2)(s+2)^2}\right\} = L^{-1}\left\{\frac{1}{s-2}.\frac{1}{(s+2)^2}\right\} = L^{-1}\{\bar{f}(s).\bar{g}(s)\} = f(t)*g(t)$$

$$= \int_0^t f(u)g(t-u)du = \int_0^t e^{2u}(t-u)e^{-2(t-u)}du$$

$$= e^{-2t}\int_0^t e^{4u}(t-u)du = e^{-2t}\left[t\int_0^t e^{4u}du - \int_0^t ue^{4u}du\right]$$

$$= e^{-2t}\left[t\left(\frac{e^{4u}}{4}\right)_0^t - \left\{u.\frac{e^{4u}}{4} - 1.\frac{e^{4u}}{16}\right\}_0^t\right] \text{[Integration by parts]}$$

$$e^{-2t}\left[\frac{t}{4}(e^{4t}-1) - \left\{\frac{t}{4}e^{4t} - \frac{1}{16}e^{4t} - 0 + \frac{1}{16}\right\}\right]$$

$$= e^{-2t}\left[-\frac{t}{4} + \frac{1}{16}e^{4t} - \frac{1}{16}\right] = \frac{e^{-2t}}{16}(-4t + e^{4t} - 1)$$

$$= \frac{1}{16}\left[e^{2t} - (4t+1)e^{-2t}\right]$$

Alternate Method: Let $\bar{f}(s) = \frac{1}{(s+2)^2}$ and $\bar{g}(s) = \frac{1}{s-2}$.

Then f(t)= e^{-2t}.t and g(t)=e^{2t}

By Convolution theorem,

$$L^{-1}\left\{\frac{1}{(s-2)+(s+2)^2}\right\} = \int_0^t e^{-2u}.u.e^{2(t-u)}du$$

$$= e^{2t}\int_0^t ue^{-4u}du = e^{2t}\left[u\left(\frac{e^{-4x}}{-4}\right) - 1.\left(\frac{e^{-4u}}{16}\right)\right]_0^t \text{, by parts}$$

$$= \frac{1}{16}[e^{2t} - (4t+1)e^{-2t}]$$

7. Using the Convolution theorem , find

(i) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}$ (ii) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$

(iii) $L^{-1}\left\{\dfrac{s}{s^2(s+1)^2}\right\}$ (iv) $L^{-1}\left\{\dfrac{1}{s(s+1)(s+2)}\right\}$

Sol.

(i) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)^2}\right\}=L^{-1}\left\{\dfrac{s}{s^2+a^2}\cdot\dfrac{1}{s^2+a^2}\right\}$

Let $\overline{f}(s)=\dfrac{s}{s^2+a^2}$ and $\overline{g}(s)=\dfrac{1}{s^2+a^2}$. Then

And $L^{-1}\{\overline{g}(s)\}=L^{-1}\left\{\dfrac{1}{s^2+a^2}\right\}=\dfrac{1}{a}\sin at=g(t),$ say

$\therefore$ By the Convolution theorem, we have

$$L^{-1}\left\{\frac{s}{(s^2+a)^2}\right\}=(\cos at)*\left(\frac{1}{a}\sin at\right)=\frac{1}{a}\int_0^t \cos au\sin a(t-u)du$$

$$=\frac{1}{2a}\int_0^t [\sin(au+at-au)-\sin(au-at+au)]du$$

$$=\frac{1}{2a}\int_0^t [\sin at-\sin(2au-at)]du=\frac{1}{2a}\left[\sin at.u+\frac{1}{2a}\cos(2au-at)\right]_0^t$$

$$=\frac{1}{2a}\left[t\sin at+\frac{1}{2a}\cos at-\frac{1}{2a}\cos at\right]=\frac{t}{2a}\sin at$$

Note: Taking a=1, the above problem becomes $L^{-1}\left[\dfrac{s}{(s^2+1)^2}\right]=\dfrac{t}{2}\sin t$

(ii) $L^{-1}\left\{\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=L^{-1}\left\{\dfrac{s}{s^2+a^2}\cdot\dfrac{s}{s^2+b^2}\right\}$

Let $\overline{f}(s)=\dfrac{s}{s^2+a^2}$ and $\overline{g}(s)=\dfrac{s}{s^2+b^2}$

Then f(t) =cos at and g(t)=cos bt

$$\therefore L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\cos at*\cos bt$$

$$=\int_0^t \cos au.\cos b(t-u)du=\frac{1}{2}\int_0^t 2\cos au.\cos b(t-u)du$$

$$= \frac{1}{2}\int_0^t [\cos(au + bt - bu) + \cos(au - bt + bu)]du$$

$$= \frac{1}{2}\int_0^t \{\cos[(a-b)u + bt] + \cos[(a+b)u - bt]\}du$$

$$= \frac{1}{2}\left[\frac{\sin\{(a-b)u + bt\}}{a-b} + \frac{\sin\{(a+b)u - bt\}}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{a-b}(\sin at - \sin bt) + \frac{1}{a+b}(\sin at + \sin bt)\right]$$

$$= \frac{1}{2}\left[\sin at\left(\frac{1}{a-b} - \frac{1}{a+b}\right) + \sin bt\left(\frac{1}{a+b} - \frac{1}{a-b}\right)\right]$$

$$= \frac{a\sin at - b\sin bt}{a^2 - b^2}$$

Note:1. Putting a=2 and b=3 in the above problem, we obtain

$$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+9)}\right\} = -\frac{1}{5}(2\sin 2t - 3\sin 3t)$$

2. Putting a=2 and b=5 in the above problem , we obtain.

$$L^{-1}\left\{\frac{s^2}{(s^2+4)(s^2+25)}\right\} = \frac{2\sin 2t - 5\sin 5t}{2^2 - 5^2} = \frac{1}{21}(5\sin 5t - 2\sin 2t)$$

(iii) Since $L^{-1}\left\{\frac{1}{s^2}\right\} = t$ and $L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\} = te^{-1}$,

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{(s+1)^2}\cdot\frac{1}{s^2}\right\} = \int_0^t ue^{-u}(t-u)du = t\int_0^t ue^{-u}du - \int_0^t u^2e^{-u}du$$

(IV)Using convolution theorem, we get

$$L^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{s(s+1)}\cdot\frac{1}{s+2}\right\} = \int_0^t e^{-2(t-u)}.(1-e^{-u})du$$

$$e^{-2t}\int_0^t (e^{2u} - e^u)du = e^{-2t}\left(\frac{e^{2u}}{2} - e^u\right)_0^t$$

$$= e^{-2t}\left(\frac{e^{2t}}{2} - e^t - \frac{1}{2} + 1\right) = e^{-2t}\left(\frac{e^{2t}}{2} - e^t + \frac{1}{2}\right) = \frac{1}{2} + \frac{1}{2}e^{-2t} - e^{-1}$$

8. Using convolution theorem, find the Inverse Laplace Transform of $\frac{1}{(s^2+4)(s+1)^2}$.

Sol. $$L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}=L^{-1}\left\{\frac{1}{(s^2+4)(s+1)}.\frac{1}{s+1}\right\}$$

Consider $L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}$

Since $L^{-1}\{\overline{f}(s)\}=L^{1}\left\{\frac{1}{s^2+4}\right\}=\frac{1}{2}\sin 2t = f(t)$

And $L^{-1}\{\overline{g}(s)\}=L^{-1}\left\{\frac{1}{s+1}\right\}=e^{-t} = g(t),$

$\therefore$ By Convolution theorem, we get

$$L^{-1}\left\{\frac{1}{(s^2+4)(s+1)}\right\}= f(t)*f(t)$$

$$\int_0^t f(u)g(t-u)du=\int_0^t \frac{1}{2}\sin 2ue^{-(t-yu)}du$$

$$=\frac{1}{2}e^{-t}\int_0^t e^u \sin 2udu$$

$$=\frac{1}{2}e^{-t}\left[\frac{e^u}{1^2+2^2}(\sin 2u-2\cos 2u\right]_0^t$$

$$=\frac{1}{2}e^{-t}\left[\frac{e^5}{5}(\sin 2t-2\cos 2t)-\frac{1}{5}(0-2)\right]$$

$$=\frac{1}{10}[\sin 2t-2\cos 2y+2e^{-t}]$$

Applying Convolution theorem again , we get

$$L^{-1}\left\{\frac{1}{(s^2+4)(s+1)^2}\right\}=L^{-1}\left\{\frac{1}{(s^2+4)(s+1)}.\frac{1}{s+1}\right\}$$

$$=\int_0^t \frac{1}{10}(\sin 2u-2\cos 2u+2e^{-u})e^{-(t-u)}du$$

$$= \frac{e^{-t}}{10}\left[\int_0^t e^u \sin 2u du - 2\int_0^t e^u \cos 2u du + 2\int_0^t du\right]$$

$$= \frac{e^{-t}}{10}\left[\left\{\frac{e^u}{1^2+2^2}(\sin 2u - 2\cos 2u)\right\}_0^t - 2\left\{\frac{e^u}{1^2+2^2}(\cos 2u + 2\sin 2u)\right\}_0^t + 2(u)_0^t\right]$$

=

$$\frac{e^{-t}}{10}\left[\left\{\frac{e^t}{5}(\sin 2t - \cos 2t - \frac{1}{5}(0-2)\right\} - 2\left\{\frac{e^t}{5}(\cos 2t + 2\sin 2t) - \frac{1}{5}(1+0)\right\} + 2(t-0)\right]$$

$$= \frac{e^{-t}}{50}[e^t(\sin 2t - 2\cos 2t) + 2 - 2e^t(\cos 2t + 2\sin 2t) + 2 + 10t]$$

$$= \frac{e^{-t}}{50}[e^t(\sin 2t - 2\cos 2t - 2\cos 2t - 4\sin 2t) + 4 + 10t]$$

$$= \frac{e^{-t}}{50}[e^t(-3\sin 2t - 4\cos 2t) + 4 + 10t]$$

$$= \frac{e^{-t}}{50}[4 + 10t - e^t(3\sin 2t + 4\cos 2t)]$$

EXERCISE

By employing convolution theorem, evaluate the following :

(1) $L^{-1}\left\{\frac{1}{(s+1)(s+9)^2}\right\}$

(2) $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$

(3) $L^{-1}\left\{\frac{1}{s(s^2+4)}\right\}$

(4) $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$

(5) $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$

6) $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$

(7) $L^{-1}\left\{\frac{1}{s^2(s^2+a^2)}\right\}$

(8) $L^{-1}\left\{\frac{s}{(s^2-a^2)^2}\right\}$

(9) $L^{-1}\left\{\frac{s^2}{s^4-a^4}\right\}$

(10) $L^{-1}\left\{\frac{1}{s^3(s+1)}\right\}$

ANSWER

(1) $\frac{e^{-t}(1-e^{-8t}(1+8t))}{64}$

(2) $\frac{1}{3}(\cos t-\cos 2t)$

(3) $\frac{1}{4}(1-\cos 2t)$

(4) $\frac{1}{3}(\cos t-\cos 2t)$

(5) $\frac{t^2}{2}+\cos t+1$

(6) $\frac{1}{3}\left(e^{t}-e^{-2t}\right)$

(7) $\frac{1}{2a}\left(\sin at+at\cos at\right)$

(8) $\frac{t}{2a}\sinh at$

(9) $\frac{1}{2a}\left(\sin at+\sinh at\right)$

(10) $1-t+\frac{t^2}{2}-e^{-t}$

5.5 APPLICATIONS OF LAPLACE TRANSFORMATION

SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Ordinary linear differential equations with constant coefficients can be easily solved by the Laplace transform method, without the necessity of first finding the general solution and then evaluating the arbitrary constants. This method is , in general , shorter than our earlier methods and is especially suitable to obtain the solution of linear non-homogeneous ordinary differential equation with constant coefficients.

Let us consider a linear differential equation with constant coefficients

$$\frac{d^n y}{dt^n}+a_1\frac{d^{n-1}y}{dt^{n-1}}+a_2\frac{d^{n-2}y}{dt^{n-2}}+...+a_{n-1}\frac{dy}{dt}+a_n y=F(t) \qquad (1)$$

Where F(t) is a function of independent variable t.

Let $y(0)=c_0, y'(0)=c_1....., y^{n-1}(0)=c_{n-1}$ (2)

Be the given initial or boundary conditions , where $c_0,c_1,c_2.....c_{n-1}$ are constants.

If $a_1,a_2,........a_n$ are constants ,then we use

$$L\left\{f^{(n)}(t)\right\}=s^n L\left\{f(t)\right\}-s^{n-1}f(0)-s^{n-2}f'(0)......-f^{n-1}(0) \qquad (3)$$

Taking Laplace Transform of both sides of(1) and applying (3) and using conditions (2) , we obtain an algebraic equation known as "subsidiary equation ", from which $\overline{y}(s) = L\{y(t)\}$ is obtained .The required solution y(t) is obtained by taking the Inverse Laplace Transform of $\overline{y}(s)$.

Working rule to solve differential equation by Laplace Transform Method.

- Take the Laplace Transform of both sides of the given differential equation
- Use the formula
  (i) $L\{y'(t)\} = s\overline{y}(s) - y(0)$
  (ii) $L\{y''(t)\} = s^2\overline{y}(s) - s.y(0) - y'(0)$
  (iii) $L\{y'''(t)\} = s^3\overline{y}(s) - s^2.y(0) - s.y'(0) - y''(0)$
- Replace $y(0), y'(0), y''(0)$ with the given initial conditions.
- Transpose the terms with minus signs to the right.
- Divide by the coefficient of $\overline{y}$, getting $\overline{y}$ as a known function of s.
- Resolve this functions of s(obtained in step5) into partial functions.
- Take the Inverse L.T of $\overline{y}$ obtained in step 5. This gives y as a function of t which is the required solution of the given equation satisfying the given initial conditions.

PROBLEMS

1. Using Laplace transform, solve $(D^2 + 4D + 5)y = 5$ give that $y(0) = 0, y''(0) = 0$ **.**

Sol. Given differential equation can be written as

$$y'' + 4y' + 5y = 5$$

Taking Laplace transform of both sides , we get

$$L\{y''\} + 4.L\{y'\} + 5.L\{y\} = L\{5\}$$

i.e., $[s^2L\{y\} - s.y(0) - y'(0)] + 4[s.L\{y\} - y(0)] + 5.L\{y\} = 5.L\{1\}$

Using the given conditions, it reduces to $(s2 + 4s + 5)L\{y\} = \frac{5}{8}$

$$\Rightarrow L\{y\} = \frac{5}{s(s^2 + 4s + 5)} = \frac{1}{s} - \frac{s+4}{s^2 + 4s + 5} \text{(resolving into partial fractions)}$$

$$\therefore\ y = L^{-1}\left[\frac{1}{s} - \frac{(s+4)}{(s^2 + 4s + 5}\right] = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{(s+2)+2}{(s+2)^2 + 1}\right\}$$

Inverse Laplace transform

$$= 1 - e^{-2t}.L^{-1}\left\{\frac{s+2}{s^2+1}\right\} \text{(Using First shifting theorem)}$$

$$= 1 - e^{2t}.L^{-1}\left\{\frac{s}{s^2+1}+\frac{2}{s^2+1}\right\} = 1 - e^{-2t}(\cos t + 2\sin t)$$

The special advantage of this method in solving differential equations us that the initial conditions are satisfied automatically. It us unnecessary to find the general solution and then determine the constants using the initial conditions.

2. Using Laplace transform method, solve. $(D^2+1)y = 6\cos 2t, t > 0 \ \ if \ \ y = 3, Dy = 1 \ when \ t = 0.$

Sol. Given equation is $y'' + y = 6\cos 2t$

Taking Laplace Transform on both sides of the equation, we get

$L\{y''\} + L\{y\} = 6L\{\cos 2t\}$

Using the given conditions y=3,Dy=1 at t=0, we get

$$(s^2+1)L\{y\} - 3s - 1 = \frac{6s}{s^2+4}$$

Or $L\{y\} = \dfrac{6s}{(s^2+4)(s^2+1)} + \dfrac{3s}{s^2+1} + \dfrac{1}{s^2+1}$

$$= 2\left[\frac{s}{s^2+1} - \frac{s}{s^2+4}\right] + \frac{3s}{s^2+1} + \frac{1}{s^2+1} = \frac{5s}{s^2+1} - \frac{2s}{s^2+4} + \frac{1}{s^2+1}$$

Taking Inverse Laplace transform, we get

$$y = 5L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2L^{-1}\left\{\frac{s}{s^2+4}\right\} + L^{-1}\left\{\frac{1}{s^2+1}\right\} = 5\cos t - 2\cos 2t + \sin t$$

3. Using Laplace transforms solve $(D^2+5D+6)X = 5e^t.given that\ X(0) = 2 \ and \ X'(0) = 1.$

Sol. Given equation is $x'' + 5x' + 6x = 5e^t$ (1)

Taking Laplace Transform of both sides of (1), we get

$L\{x''\} + 5.L\{x'\} + 6L\{x\} = 5.L\{e^t\}$

i.e., $\left[s^2L\{x\} - sx(0) - x'(0)\right] + 5[sL[x\} - x(0)] + 6L\{x\} = \dfrac{5}{s-1}$

Putting the values $x(0) = 2, and x'(0) = 1, we get$

$$\left[s^2L\{x\} - 2x - 1\right] + 5[\{sL\{x\} - 2 + 6L\{x\} = \frac{5}{s-1}$$

i.e., $(s^2+5s+6)L\{x\} - 2s - 11 = \dfrac{5}{s-1}$

i.e., $(s^2+5s+6)L\{x\} = \dfrac{5}{s-1} + 2s + 11$

or $L\{x\} = \dfrac{5}{(s-1)(s^2+5s+6)} + \dfrac{2s+11}{s^2+5s+6}$

$\therefore x = 5.L^{-1}\left\{\dfrac{1}{(s-1)(s+3)(s+3)}\right\} + L^{-1}\left\{\dfrac{2s+11}{(s+2)(s+3)}\right\}$

$= 5.L^{-1}\left\{\dfrac{1}{12}.\dfrac{1}{s-1} - \dfrac{1}{3}.\dfrac{1}{s+2} + \dfrac{1}{4}.\dfrac{1}{s+3}\right\} + L^{-1}\left\{\dfrac{7}{s+2} - \dfrac{5}{s+3}\right\}$

(Resolving into partial fractions)

$= \left[\dfrac{1}{12}L^{-1}\left\{\dfrac{1}{s-1}\right\} - \dfrac{1}{3}.L^{-1}\left\{\dfrac{1}{s+2}\right\} + \dfrac{1}{4}.L^{-1}\left\{\dfrac{1}{s+3}\right\}\right] + 7.L^{-1}\left\{\dfrac{1}{s+2}\right\} - 5.L^{-1}\left\{\dfrac{1}{s+3}\right\}$

$= 5\left[\dfrac{1}{12}e^{t} - \dfrac{1}{3}e^{2t} + \dfrac{1}{4}e^{-3t}\right] + 7e^{-2t} - 5e^{-3t}$

$= \dfrac{5}{12}e^{t} + e^{-12}\left(7 - \dfrac{5}{3}\right) + e^{-3t}\left(\dfrac{5}{4} - 5\right)$

$\therefore x = \dfrac{5}{12}e^{t} + \dfrac{16}{3}e^{-2t} - \dfrac{15}{4}e^{-3t}$

4. By using Laplace transform .solve the initial value problem $y'' = t\cos 2t, y(0) = 0 and y'(0) = 0$

Sol. Given initial value problem is $y'' = t\cos 2t, y(0) = 0$ *and* $y'(0) = 0$

Taking Laplace Transform of both sides of the equation, we get

$L\{y''\} = L\{t\cos 2t\}$

$i.e., s^2L\{y\} - s.y(0) - y'(0) = (-1)\dfrac{d}{ds}L\{\cos 2t\}$

$= (-1)\dfrac{d}{ds}\left(\dfrac{s}{s^2+4}\right) = \dfrac{s^2-4}{(s^2+4)^2}$

Using the given conditions, it reduces to

$s^2L\{y\} = \dfrac{s^2-4}{(s^2+4)^2}$

i.e., $L\{y\} = \dfrac{s^2-4}{s^2(s^2+4)^2}$

$\therefore y = L^{-1}\left\{\dfrac{s^2-4}{s^2(s^2+4)^2}\right\}$ (1)

Consider $\dfrac{s^2-4}{s^2(s^2+4)^2}$

Let $\dfrac{s^2-4}{s^2(s^2+4)^2}=\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{Cs+4}{s^2+4}+\dfrac{Es+F}{(s^2+4)^2}$ (2)

$\Rightarrow s^2-4=As(s^2+4)^2+B(s^2+4)^2+(Cs+D)s^2(s^2+4)+(Es+F)s^2$

$=As(s^4+8s^2+16)+B(s^4+8s^2+16)+(Cs^3+Ds^2)(s^2+4)+(Es^3+Fs^2)$

$=A(s^5+8s^3+16s)+B(s^4+8s^2+16)+(Cs^5+Ds^4+4Cs^3+4Ds^2)+(Es^3+Fs^2)$

$=s^5(A+C)+s^4(B+D)+s^3(8A+4C+E)+s^2(8B+4D+F)+s(16A)+16B$

Comparing like coefficients, we get

A+C=0 .... (3)

B+D=0 .... (4)

8A+4C+E=0 .... (5)

8B+4D+F=1 .....(6)

16A=0⇒A=0 .... (7)

$16B=-4\Rightarrow B=-\dfrac{1}{4}$ .....(8)

From (3) and (7), we get

C=0

From (4) and (8), we get

$D=-B=\dfrac{1}{4}$

From (5), E=-(8A+4C)=0

From (6), F=1-(8B+4D)=1-(-2+1)=1+1=2

Substituting all these values in(2), we get

$$\frac{s^2-4}{s^2(s^2+4)^2}=\frac{-1}{4s^2}+\frac{1}{4(s^2+4)}+\frac{2}{(s^2+4)^2} \quad ....(9)$$

From (1) and (9), we get

$$y=\frac{-1}{4}L^{-1}\left\{\frac{1}{s^2}\right\}+\frac{1}{4}L^{-1}\left\{\frac{1}{s^2+2^2}\right\}+2.L^{-1}\left\{\frac{1}{(s^2+2^2)^2}\right\}$$

$$=\frac{-1}{4}t+\frac{1}{4}.\frac{1}{2}\sin 2t+2.\frac{1}{2(2)^3}[\sin 2t-2t\cos t]$$

$$\left[\therefore L^{-1}\left\{\frac{1}{(s^2+a^2)^2}\right\}=\frac{1}{2a^3}(\sin at-at\cos at)\right]$$

$$\frac{-t}{4}+\frac{1}{8}\sin 2t+\frac{1}{8}(\sin 2t-2t\cos 2t)=\frac{-t}{4}+\frac{1}{4}\sin 2t-\frac{1}{4}t\cos 2t$$

$$= \frac{1}{4}(\sin 2t - t - t\cos 2t)$$

5. Solve $\frac{d^2x}{dt^2} + 9x = \sin t,\ given\ \ that\ x(0) = 1,\ \ x\left(\frac{\pi}{2}\right) = 1$.

Sol: Given equation can be written as

$$x'' + x = \sin t$$

Take Laplace transform on both sides, we get

$$L\{x''\} + L\{x\} = L\{\sin t\}$$

i.e, $\left[s^2 L\{x\} - s\,x(0) - x'(0)\right] + L\{x\} = \frac{1}{s^2+1}$

Using the given condition, it reduces to

$$s^2 L\{x\} - s - b + 9L\{x\} = \frac{1}{s^2+1} \qquad \{taking\ \ x'(0) = b\}$$

i.e, $(s^2+9)L\{x\} = \frac{1}{s^2+1} + s + b$

or $L\{x\} = \frac{1}{(s^2+1)(s^2+9)} + \frac{s+b}{(s^2+9)}$

Take Inverse Laplace transform on both sides, we get

$$x = \frac{1}{8}L^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2+9}\right\} + L^{-1}\left\{\frac{s}{s^2+9}\right\} + bL^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$x = \frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right) + \cos 3t + \frac{b}{3}\sin 3t$$

$$x = \frac{1}{8}\sin t + \cos 3t + \frac{1}{3}\left(b - \frac{1}{8}\right)\sin 3t \qquad \text{........(1)}$$

Given $x\left(\frac{\pi}{2}\right) = 1$

$$\Rightarrow\ 1=\frac{1}{8}\sin\left(\frac{\pi}{2}\right)+\cos\left(\frac{3\pi}{2}\right)+\frac{1}{3}\left(b-\frac{1}{8}\right)\sin\left(\frac{3\pi}{2}\right)$$

$$\Rightarrow 1=\frac{1}{6}-\frac{b}{3}\Rightarrow\frac{b}{3}=\frac{-5}{6} \qquad \text{........(2)}$$

Hence the solution is

$$x=\frac{1}{8}\left(\sin t-\frac{1}{3}\sin 3t\right)+\cos 3t-\frac{5}{6}\sin 3t \quad \left(\textit{from } (1)\ \textit{and } (2)\right)$$

6. Solve by using Laplace transform method

$$y'+y=t\,e^{-t},\ y(o)=2$$

Sol: Taking the Laplace transform of the given equation, we get

$$\left[sL\ \{y(t)\}-y(o)\right]+L\ \{y(t)\}=\frac{1}{(s+1)^2}$$

Using the given condition , we get

$$(s+1)L\ \{y(t)\}-2=\frac{1}{(s+1)^2}$$

So that

$$L\ \{y(t)\}=\frac{2s^2+4s+3}{(s+1)^3}$$

Taking inverse Laplace transforms, we have

$$Y(t)=L^{-1}\left\{\frac{2s^2+4s+3}{(s+1)^3}\right\}$$

$$=L^{-1}\left\{\frac{2(s+1-1)^2+4(s+1-1)+3}{(s+1)^3}\right\}$$

$$=L^{-1}\left\{\frac{2}{s+1}+\frac{1}{(s+1)^3}\right\}$$

$$=\frac{1}{2}e^{-t}\left(t^2+4\right)$$

This is the solution of the given equation.

Solve by using Laplace transform method :

7. $y'' + 2y' - 3y = \sin\, t,\;\; y(o) = y'(o) = 0$

Sol:

Taking the Laplace transform of the given equation, we get

$$\left[s^2 L\{y(t)\} - sy(o) - y'(o)\right] + 2\left[s\, L\{y(t)\} - y(o)\right] - 3\, L\,\{y(t)\} = \frac{1}{s^2+1}$$

$$L\,\{y(t)\}\left[s^2 + 2s - 3\right] = \frac{1}{s^2+1} \text{ (or) } L\,\{y(t)\} = \frac{1}{(s-1)(s+3)\left(s^2+1\right)}$$

$$\text{(or) } y(t) = L^{-1}\left[\frac{1}{(s-1)(s+3)\left(s^2+1\right)}\right]$$

$$= L^{-1}\left[\frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{1}{8}\;\frac{1}{s-1} - \frac{1}{40}\;\frac{1}{s+3} + \frac{-\frac{s}{10} - \frac{1}{5}}{s^2+1}\right]$$

By using method of partial sums and using the given condition , we get

$$= \frac{1}{8}e^t - \frac{1}{40}e^{-3t} - \frac{1}{10}\left(\cos t + 2\sin\, t\right)$$

(8) Solve $\dfrac{d^2x}{dt^2} + 2\dfrac{dx}{dt} + x = 3te^{-t}$, *given that* $x = 4,\; \dfrac{dx}{dt} = 0$ *at* $t = 0$.

Sol: Given equation can be written as

$$x'' + 2x' + x = 3te^{-t}$$

Take Laplace transform on both sides, we get

$$L\{x''\} + 2L\{x'\} + L\{x\} = 3te^{-t}$$

$$\left[s^2L\{x\}-s\,x(0)-x^{'}(0)\right]+2\left[sL\{x\}-x(0)\right]+L\{x\}=\frac{3}{(s+1)^2}$$

Using the given condition , it reduces to

$$s^2L\{x\}-4s+2sL\{x\}-8+L\{x\}=\frac{3}{(s+1)^2}$$

$$\left(s^2+2s+1\right)L\{x\}=\frac{3}{(s+1)^2}+4s+8$$

$$L\{x\}=\frac{3}{(s+1)^4}+\frac{4s+8}{(s+1)^2}$$

Take Inverse Laplace transform on both sides, we get

$$x=3L^{-1}\left\{\frac{1}{(s+1)^4}\right\}+4L^{-1}\left\{\frac{s}{(s+1)^2}\right\}+8L^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

$$x=3e^{-t}L^{-1}\left\{\frac{1}{s^4}\right\}+4e^{-t}L^{-1}\left\{\frac{s-1}{s^2}\right\}+8e^{-t}L^{-1}\left\{\frac{1}{s^2}\right\}$$

$$x=e^{-t}\left(\frac{t^3}{2}+4t+4\right)$$

9. Solve the differential equation

$$\frac{d^2x}{dt^2}-4\frac{dx}{dt}-12x=e^{3t},\ given\ \ that\ x(0)=1,\ \ x^1(0)=-2\ \ .$$

Sol: Given equation can be written as

$$x''-4x^{'}-12x=e^{3t}$$

Take Laplace transform on both sides, we get

$$L\{x''\}-4L\{x^{'}\}-12L\{x\}=L\{e^{3t}\}$$

i.e, $\left[s^2L\{x\}-s\,x(0)-x^{'}(0)\right]-4\left[sL\{x\}-s\,x(0)\right]-12\,L\{x\}=\dfrac{1}{s-3}$

Using the given condition, it reduces to

$$s^2L\{x\}-s+2-4sL\{x\}+4-12L\{x\}=\frac{1}{s-3}$$

i.e. $(s+2)(s-6)L\{x\}=\dfrac{1}{s-31}+s-6$

or $L\{x\}=\dfrac{1}{(s+2)(s-3)(s-6)}+\dfrac{1}{s+2}$

Take Inverse Laplace transform on both sides, we get

$$x=L^{-1}\left\{\frac{1}{(s+2)(s-3)(s-6)}\right\}+L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$Let\ \frac{1}{(s+2)(s-3)(s-6)}=\frac{A}{(s+2)}+\frac{B}{(s-3)}+\frac{C}{(s-6)} \qquad ....(1)$$

$$1=A(s-3)(s-6)+B(s+2)(s-6)+C(s+2)(s-3) \quad ....(2)$$

$$Put\ s=-2\ in\ (2), 1=40A \Rightarrow A=\frac{1}{40}$$

$$Put\ s=3\ in\ (2), 1=-15B \Rightarrow B=-\frac{1}{15}$$

$$Put\ s=6\ in\ (2), 1=24C \Rightarrow C=\frac{1}{24}$$

Substituting the values of A,B and C in (2), we get

$$\frac{1}{(s+2)(s-3)(s-6)}=\frac{1}{40(s+2)}-\frac{1}{15(s-3)}+\frac{1}{24(s-6)}$$

Hence

$$x=\frac{1}{40}L^{-1}\left(\frac{1}{s+2}\right)-\frac{1}{15}L^{-1}\left(\frac{1}{s-3}\right)+\frac{1}{24}L^{-1}\left(\frac{1}{s-6}\right)$$

Hence the solution is

$$x=\frac{1}{40}e^{-2t}-\frac{1}{15}e^{3t}+\frac{1}{24}e^{-3t}$$

10. A particle is moving along a path satisfying , the equation

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 25x = 0$$

where

x denotes the displaceme nt of the particle at time t. If the initial position of the particle is at x = 20 and the initial speed is 10, find the displaceme nt of the particle at any time t using Laplace transform

Sol: Given equation may be rewritten as

$$x''(t) + 6x'(t) + 25x(t) = 0$$

Here the initial conditions are $x(o) = 20, x'(o) = 10.$

Taking the Laplace Transform of the equation, we get

$$L\{x(t)\}[s^2 + 6s + 25] - 20s - 130 = 0 \quad or \; L\{x(t)\} = \frac{20s + 130}{s^2 + 6s + 25}$$

So that

$$x(t) = L^{-1}\left[\frac{20s + 130}{(s+3)^2 + 16}\right] = L^{-1}\left[\frac{20(s+3) + 70}{(s+3)^2 + 16}\right]$$

$$= 20\, L^{-1}\left[\frac{s+3}{(s+3)^2 + 16}\right] + 70\, L^{-1}\left[\frac{1}{(s+3)^2 + 16}\right]$$

$$= 20\, e^{-3t}\cos 4t + 35\,\frac{e^{-3t}\sin 4t}{2}$$

This is the desired solution of the given problem.

(11) A voltage Ee^{-at} is applied at t = 0 to a circuit of inductance L and resistance R.

Show that the current at any time t is $\frac{E}{R - aL}\left[e^{-at} - e^{-\frac{Rt}{L}}\right]$

Sol.

The circuit is an LR circuit. The differential equation with respect to the circuit is

$$L\frac{di}{dt} + Ri = E(t)$$

Here L denotes the inductance, i denotes current at any time t and E(t) denotes the E.M.F.

It is given that $E(t) = E\, e^{-at}$. With this, we have

Thus, we have

$$L\frac{di}{dt} + Ri = Ee^{-at} \quad or \quad Li'(t) + R\, i(t) = Ee^{-at}$$

$$L[L_T\ i'(t)] + R[L_T\ i'(t)] = E\ L_T\left(e^{-at}\right) \qquad or$$

Taking Laplace transform (L_T) on both sides, we get

$$L[s\ L_T\ i(t) - i(o)] + R[L_T\ i(t)] = E\frac{1}{s+a}$$

Since $i(o) = o$, we get $\quad L_T\ i(t)[sL + R] = \dfrac{E}{s+a} \qquad or$

$$L_T\ i(t) = \frac{E}{(s+a)+(sL+R)}$$

Taking inverse transform L_T^{-1}, we get $\quad i(t) = L_T^{-1}\dfrac{E}{(s+a)(sL+R)}$

$$= \frac{E}{R-aL}\left[L_T^{-1}\frac{1}{s+a} - L\ \ L_T^{-1}\frac{1}{sL+R}\right]$$

Thus

$$i(t) = \frac{E}{R-aL}\left[e^{-at} - e^{-\frac{Rt}{L}}\right]$$

This is the result as desired.

EXERCISE

Solve the following Differential equations using Laplace Transforms:

1) $y''' + 2y'' - y' - 2y = 0 \quad given\ \ y(0) = y'(0) = 0,\ \ y''(0) = 6$

2) $y''' - 2y'' + 5y' = 0,\ \ y(0) = y'(0) = 0,\ y''(0) = 1$

3) $y'' + y = t,\ \ y(0) = 1\ ,\ \ y'(0) = -2$

4) $(D^3 + 1)y = 1,\ \ y = Dy = D^2y = D^3y = 0\ at\ t = 0$

5) $y'' + 9y = \cos 2t,\ \ y(0) = 1,\ \ y\left(\dfrac{\pi}{2}\right) = -1$

6) $y'' - 3y' + 2y = 4t + e^{3t},\ \ y(0) = 1,\ y'(0) = -1$

7) $(D^4 - 1)y = 1, \quad y = Dy = D^2 y = D^3 y = 0 \ at \ t = 0$

8) $\frac{d^3 y}{dt^3} - 3\frac{d^2 y}{dt^2} + 3\frac{dy}{dt} - y = t^2 e^t$, *given that* $y = 1, \ \frac{dy}{dt} = 0, \ \frac{d^2 y}{dt^2} = -2 \ at \ t = 0$

9) $(D^4 + 2D^2 + 1)y = 0, \quad y = 0, Dy = 1, D^2 y = 2, D^3 y = -3 \ at \ t = 0$

10) $(D^2 + n^2)x = a\sin(nt + 2), \quad x(0) = x^I(0) = 0$

ANSWERS

1). $y = e^t - 3e^{-t} + 2e^{-2t}$

2) $y = \frac{1}{5} - \frac{1}{5}e^t \cos 2t + \frac{1}{10}e^t \sin 2t$

3) $y = t + \cos t - 3\sin t$

4) $y = 1 + \frac{e^{-t}}{3} - \frac{2}{3}e^{\frac{t}{2}\cos\left(\frac{\sqrt{3}}{2}t\right)}$

5) $y = \frac{1}{5}(\cos 2t + 4\sin 3t + 4\cos 3t)$

6) $y = 2t + 3 + \frac{1}{2}(e^{3t} - e^t) - 2e^{2t}$