Unit - I

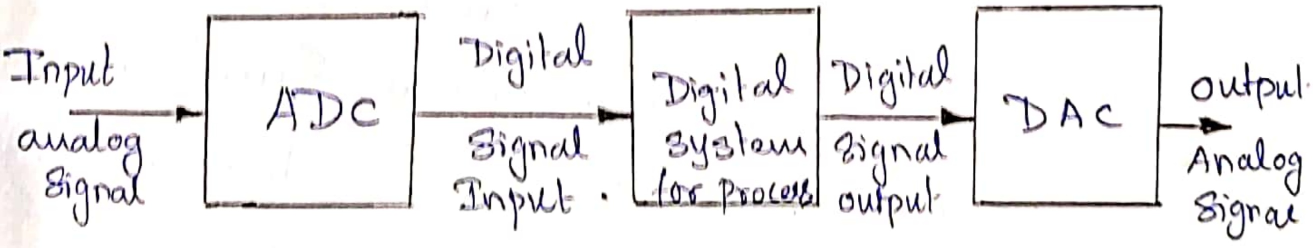
Discrete Fourier Transform

Review of Signals and Systems, Concept of frequency is discrete - time Signals, Summary of analysis & Synthesis equations for FT & DTFT, Frequency domain Sampling, Discrete Fourier Transform (DFT) - deriving DFT from DTFT, Properties of DFT - Periodicity, Symmetry, Circular Convolution. Linear filtering using DFT. Filtering long data sequences - overlap save and overlap add method. Fast Computation of DFT - Radix-2 DIT-FFT, DIF-FFT, Linear filtering using FFT.

Introduction :

Digital Signal Processing (DSP) refers to Processing of Digital Signal using Processor like microprocessor.

The basic Components of a DSP System are Shown in figure. The DSP System involves Conversion of analog Signal to digital Signal, then Processing of the digital Signal by a digital System and then Conversion of the Processed digital Signal back to analog Signal.



Advantage of Digital Signal Processing :

☆ Very Compact
☆ Less expensive
☆ flexible. (Programmable)
☆ Reliable.

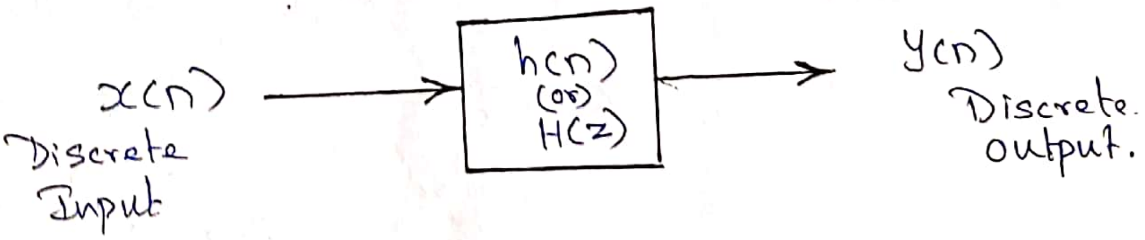
Signal :

Any time Varying quantity that Conveys (or) Carries Some information can be called as Signal. They are two types

1) Continuous -Time Signal [Analog] $x(t)$
2) Discrete - Time Signal [Digital Signal] $x(n)$
[Quantized version of DT signal is Digital signal]

Discrete Time System :

It is interconnection of Some Discrete Components to Perform a desired task. A System which Can Process a discrete time Signal is Called a discrete time system, and so the input and output Signals of a discrete time system are discrete time Signals.



Sampling of Continuous Time Signals:-

The Sampling is the Process of Conversion of a Continuous Time signal into Discrete Time Signal.

The time interval between Successive Sample is called Sampling time (or) Sampling Period (or) Sampling Interval and it is denoted by "T".

Concept of frequency in DT Signals:-

Let $x_a(t) = A\cos(\Omega_0 t + \theta)$

$= A\cos(2\pi F_0 t + \theta)$

where $\Omega_0 =$ Frequency of analog Signal in rad/s

$F_0 = \frac{\Omega_0}{2\pi} =$ Frequency of analog Signal in Hz

Let $x_a(t)$ be Sampled at intervals of T Seconds to get $x(n)$, where $T = \frac{1}{F_s}$

$$\therefore x(n) = x_a(t)\Big|_{t=nT} = A\cos(\Omega_0 t + \theta)\Big|_{t=nT}$$

$$= A\cos\Omega_0 nT + \theta = A\cos\left(\frac{2\pi F_0}{F_s} n + \theta\right)$$

$$= A\cos(2\pi f_0 n + \theta) = A\cos(\omega_0 n + \theta)$$

where $f_0 = \frac{F_0}{F_s}$ = frequency of discrete Sinusoid in Cycles/sample.

$\omega_0 = 2\pi f_0$ = frequency of discrete Sinusoid in radians/Sample.

Discrete Fourier Transform (DFT)

It is defined as

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n K}{N}}$$

where $X(K)$ is frequency domain signal

$x(n)$ is time domain signal.

The Fourier Transform Analysis equation is given by

$$\hat{X}(K) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi K n}{N}}$$

And the Fourier Transform Synthesis equation is given by

$$x(n) = \sum_{K=0}^{N-1} \hat{X}(K)\, e^{j\frac{2\pi K n}{N}}$$

Frequency domain Sampling :-

Sampling in frequency domain is usually used in DFT (Discreate Fourier Transform), where Continuous Signal of Spectrum in Sampled toget discrete Values of Spectrum, which results in Periodicity in time domain.

Discrete Fourier Transform [DFT] -

Deriving DFT from DTFT :-

The Fourier Series describes Periodic Signals by discrete Spectra, where as the DTFT describes discrete Signals by Periodic Spectra. These results are a Consequence of the fact that Sampling on domain induces Periodic extension in the other.

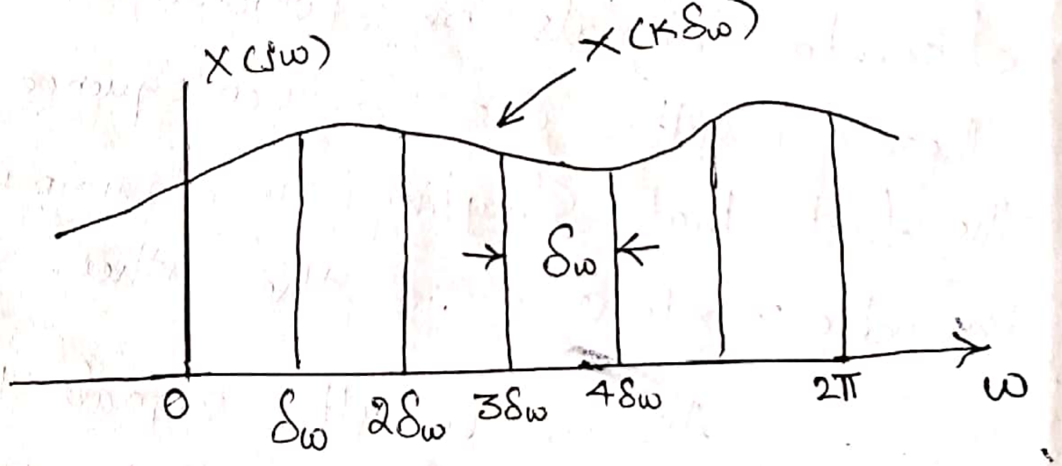
As a result, Signals that are both discrete and Periodic in one domain are also Periodic and discrete in the other.

This is the basis for the formulation of the DFT.

Consider Aperiodic discrete time Signal $x(n)$ with DTFT

$$X(j\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}.$$

Since $X(j\omega)$ is Periodic with Period 2π, Sample $X(j\omega)$ Periodically with N equidistance Samples with Spacing $\delta_\omega = \frac{2\pi}{N}$



$K = 0, 1, 2, \text{-----} N-1$

$$X\left(\frac{2\pi K}{N}\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

The Summation Can be Subdivided into an infinite no. of Summations, where each Sum Contains

$$X\left(\frac{2\pi K}{N}\right) = \cdots\cdots + \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi}{N}kn} +$$

$$\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi}{N}kn} + \cdots\cdots$$

$$= \sum_{\ell=-\infty}^{\infty} \sum_{n=\ell n}^{\ell N+N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

Put $n = n + \ell N$

$$= \sum_{\ell=-\infty}^{\infty} \sum_{n=0}^{N-1} \underset{\cancel{\ell n}}{\overset{x(n+\ell N)}{\cancel{x(n)}}}\, e^{-j\frac{2\pi}{N}k(n+\ell N)}$$

$$= \sum_{n=0}^{N-1} \sum_{\ell=-\infty}^{\infty} x(n+\ell N)\, e^{-j\frac{2\pi}{N}k(n+\ell N)}$$

$$\boxed{X(K) = \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn}.}$$

we know that $x_p(n) = \sum_{K=0}^{N-1} C_K e^{j\frac{2\pi}{N}Kn}$ $n = 0$ to $N-1$

$$C_K = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}Kn} \quad K = 0 \text{ to } N-1$$

$\therefore$ $C_K = \frac{1}{N} X(K)$ $K = 0$ to $N-1$

$\therefore$ DFT is of $x(n)$ is given by

$$\Rightarrow X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi nK}{N}}$$

and Inverse DFT (IDFT) is given by

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi Kn}{N}}$$

Properties of DFT :-

Periodicity :-

if $x_1(n) \xleftrightarrow{DFT} X(K)$

then

$$\boxed{\begin{aligned} x(n+N) &= x(n) \text{ for all values } n \\ x(K+N) &= X(K) \text{ for all values } K \end{aligned}}$$

Proof ÷

$$X(K+N) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n(K+N)}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n K}{N}}\, e^{-j\frac{2\pi n \cancel{N}}{\cancel{N}}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n K}{N}}$$

$$\boxed{\begin{array}{c}\text{note} \\ e^{-j2\pi n} = 1\end{array}}$$

$$X(K+N) = X(K)$$

Circular Convolution ÷

$$\text{if } x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m)\, x_2(n-m)_N$$

then

$$\boxed{\text{DFT}\,[x_1(n) \circledast x_2(n)] = X_1(K)\, X_2(K)}$$

Proof ÷

$$\text{IDFT}\,[X_1(K)\, X_2(K)] = \frac{1}{N} \sum_{K=0}^{N-1} X_1(K)\, X_2(K)\, e^{+j\frac{2\pi n K}{N}}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} \left[\sum_{m=0}^{N-1} x_1(m)\, e^{-j\frac{2\pi m K}{N}}\right]\left[\sum_{p=0}^{N-1} x_2(p)\, e^{-j\frac{2\pi p K}{N}}\right] e^{j\frac{2\pi n K}{N}}$$

$$= \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{K=0}^{N-1} e^{\frac{j2\pi K(n-m-p)}{N}} \quad \text{—}\circledast$$

Since $n-m-P = qN$, $P = n-m\, qN$

$$\sum_{P=0}^{N-1} x_2(P) = \sum_{m=0}^{N-1} x_2(n-m-qN)$$

$$= \sum_{m=0}^{N-1} x_2(n-m, \bmod N)$$

$$= \sum_{m=0}^{N-1} x_2(n-m)_N$$

Sub $\left[\sum_{P=0}^{N-1} x_2(P) \right.$ in equation ⊛$\left.\right]$

$$\text{IDFT}\left[X_1(K)\, X_2(K)\right] = \frac{1}{N} \sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2(n-m)_N N$$

$$= \sum_{m=0}^{N-1} x_1(n)\, x_2(n-m)$$

note
$$\sum_{K=0}^{N-1} e^{\frac{j2\pi K(n-m-P)}{N}} = N$$

$$\text{IDFT}\left[X_1(K) X_2(K)\right] = x_1(n) \circledast x_2(n)$$

$$\therefore \text{DFT}\left[x_1(n) \circledast x_2(n)\right] = X_1(K)\, X_2(K)$$

Symmetry :

Symmetry of real signal.

if $x(n)$ is real signal

then.

$$\boxed{DFT\ [x(n)] = X(K) = X^*(N-K)}$$

and

$$X_r(K) = X_r(N-K)$$

$$X_i(K) = -X_i(N-K)$$

$$|X(K)| = |X(N-K)|$$

$$\angle X(K) = -\angle X(N-K)$$

Symmetry of real and even signal

if $x(n)$ is real and even signal

ie $x(n) = x(N-n)$

then

$$\boxed{DFT\ [x(n)] = X(K) = X_r(K)}$$

and

$$X_i(K) = 0.$$

Symmetry of real and odd signal.

if $x(n)$ is real and odd signal

ie $x(n) = -x(N-n)$

then

$$\boxed{DFT\ [x(n)] = X(K) = jX_i(K)}$$

and $X_r(K) = 0$

Filtering methods based on DFT :-

[Filtering long data Sequences]

In Practical applications involving linear filtering of Signals, the input Sequence x(n) is often a very long Sequence with impulse response of filter h(n) to find the response of System (output) y(n).

The two methods for linear FIR filtering a long Sequence on a block-by-block basics using the DFT

- * over lap Save method
- * over lap add method.

Over lap Add method :-

In this method, the longer Sequence is divided into Smaller Sequences.

Each Section of the longer Sequence and the Smaller Sequence are Converted to the Size of the output Sequence of the Sectioned Convolution and linear Convolution is Performed.

The Overall output Sequence is obtained by Combining the output of the Sectioned Convolution.

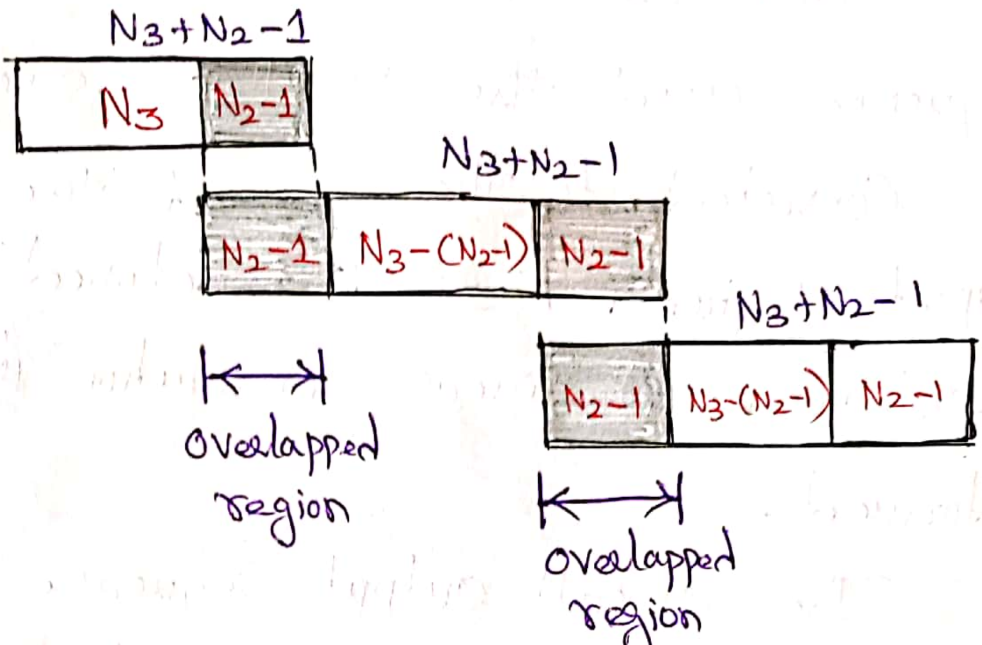
Let N_1 = Length of longer Sequence

N_2 = Lengthe of Small Sequence

Let the longer Sequence be divided into Sections of Size N_3 Samples.

Note : Normally the longer Sequence is divided into Sections of Size Same as that of Smaller Sequence

ie $N_3 = N_2$



Note : Samples in the shaded region are added.

Overlap Save method :

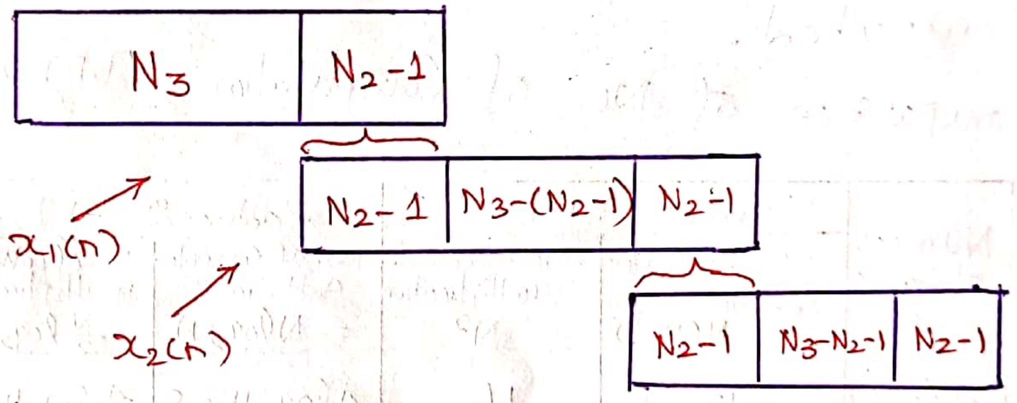
In the Overlap Save method, the result of linear Convolution of the Various Sections are obtained using Circular Convolution.

Let N_1 = Length of Longer Sequence

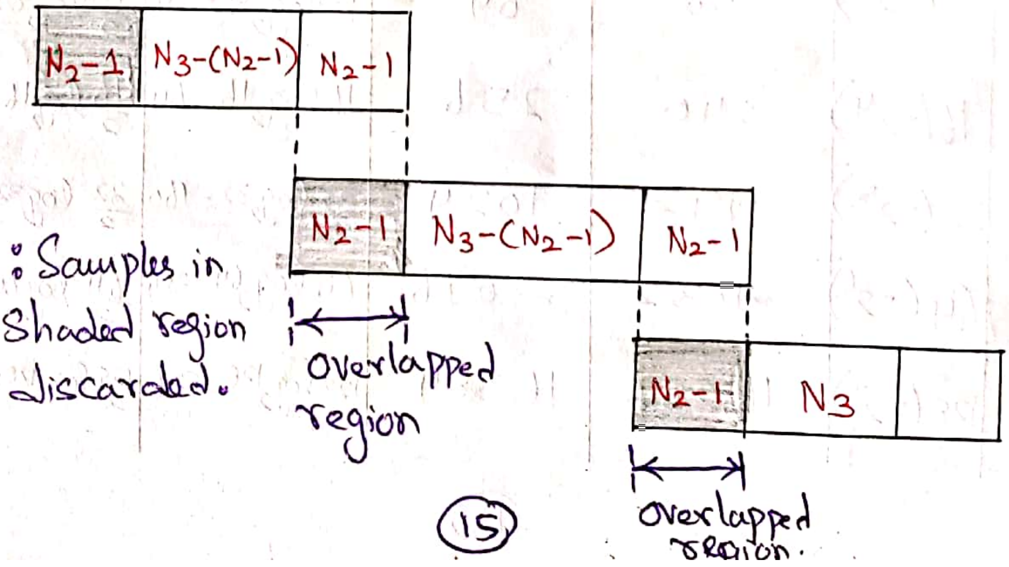
N_2 = Length of Smaller Sequence

Each Section of longer Sequence and the Smaller Sequence are Converted to the Size of output Sequence of Size N_3+N_2-1.

The First N_2-1 Samples of a Section is appended as last N_2-1 Samples of the Previous Section. (ie., the Overlapping Sample are placed at the beginning of the section). The Circular Convolution of each section will Produce an output Sequence of size N_3+N_2-1 Samples. In this output the first N_2-1 Samples are discarded and the remaining Samples of the ouput of the Sectioned Convolutions are Saved as the overall output Sequence.



input Sequence Appending of Sections.



Note: Samples in the Shaded region are discarded.

DFT - Fast Fourier Transform :- [FFT]

The Fast Fourier Transform is a method (or Algorithm) for Computing the discrete Fourier transform [DFT] with reduced number of Calculation.

Radix 2 FFT :-

For radix-2 FFT, the Value of N Should be Such that, $N = 2^m$, So that the N-Point Sequence is Decimated into 2-Point Sequence and the 2-Point DFT for each decimated Sequence is Computed.

Comparison of No. of Computation in DFT & FFT.

Number of Points N	Direct DFT		Radix 2 FFT	
	No. of Complex Addition $= N(N-1)$	No. of Complex multiplication $= N^2$	No. of Complex Addition $= N\log_2 N$	No. of complex multiplication $= \frac{N}{2}\log_2 N$
$4 (=2^2)$	12	16	$4\log_2 4 = 8$	$\frac{4}{2}\log_2 4 = 4$
$8 (=2^3)$	56	64	$8\log_2 8 = 24$	$\frac{8}{2}\log_2 8 = 12$
$16 (=2^4)$	240	256	$16\log_2 16 = 64$	$\frac{16}{2}\log_2 16 = 32$
$32 (=2^5)$	992	1024	$32\log_2 32 = 160$	$\frac{32}{2}\log_2 32 = 80$
$64 (=2^6)$	4032	4096	$64\log_2 64 = 384$	$\frac{64}{2}\log_2 64 = 192$
$128 (=2^7)$	16,256	16,384	$128\log_2 128 = 896$	$\frac{128}{2}\log_2 128 = 448$

Phase Factor (or) Twiddle Factor :-

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nK}{N}}$$

$$W_N^K = e^{-j2\pi K.}$$

Types of FFT Algorithm :-

- * Decimation in Time (DIT)
- * Decimation in Frequency (DIF)

8-Point DFT Using Radix-2 Decimation In Time - FFT Algorithm :-

Step 1 :- Bit reversal

Step 2 :- 2-Point DFT

Step 3 :- 4-Point DFT

Step 4 :- 8-Point DFT.

Step 1 :- Bit reversal

$x(0) \Rightarrow x(000) \Rightarrow x(000) \Rightarrow x(0)$

$x(1) \Rightarrow x(001) \Rightarrow x(100) \Rightarrow x(4)$

$x(2) \Rightarrow x(010) \Rightarrow x(010) \Rightarrow x(2)$

$x(3) \Rightarrow x(011) \Rightarrow x(110) \Rightarrow x(6)$

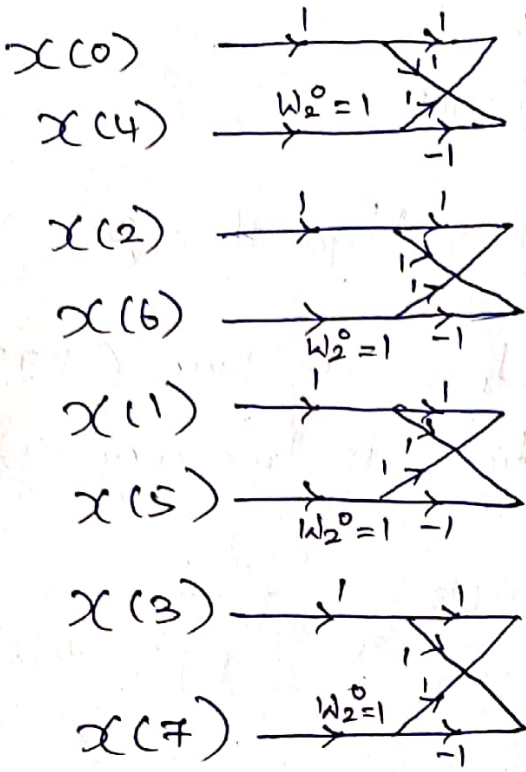
$x(4) \Rightarrow x(100) \Rightarrow x(001) \Rightarrow x(1)$

$x(5) \Rightarrow x(101) \Rightarrow x(101) \Rightarrow x(5)$

$x(6) \Rightarrow x(110) \Rightarrow x(011) \Rightarrow x(3)$

$x(7) \Rightarrow x(111) \Rightarrow x(111) \Rightarrow x(7)$

Step 2 :- 2-Point DFT



$[x(0)] + [x(4)] [W_2^0] = A_0$

$[x(0)] + [x(4)] [-W_2^0] = A_1$

$[x(2)] + [x(6)] [W_2^0] = A_2$

$[x(2)] + [x(6)] [-W_2^0] = A_3$

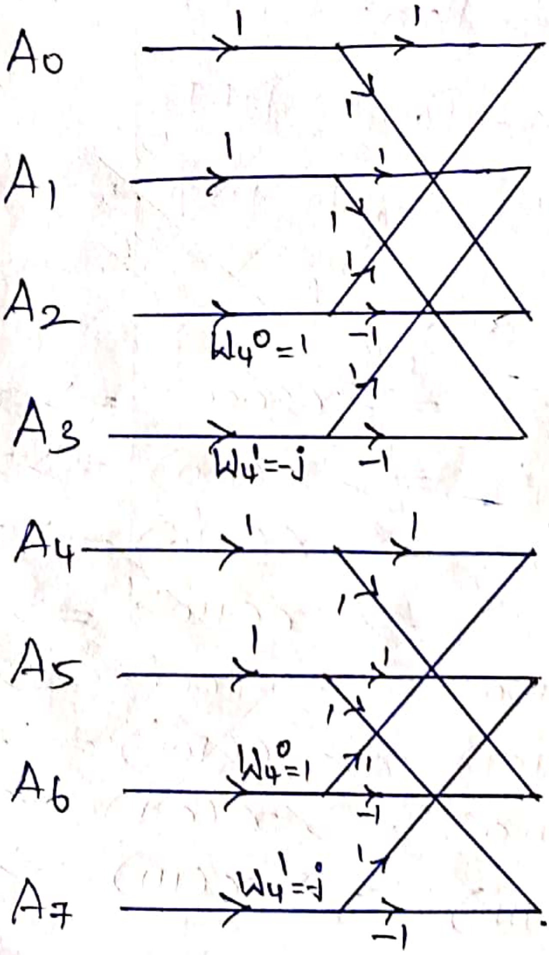
$[x(1)] + [x(5)] [W_2^0] = A_4$

$[x(1)] + [x(5)] [-W_2^0] = A_5$

$[x(3)] + [x(7)] [W_2^0] = A_6$

$[x(3)] + [x(7)] [-W_2^0] = A_7$

Step 3 :- 4-Point DFT



$[A_0] + [A_2] [W_4^0] = B_0$

$[A_1] + [A_3] [W_4^1] = B_1$

$[A_0] + [A_2] [-W_4^0] = B_2$

$[A_1] + [A_3] [-W_4^1] = B_3$

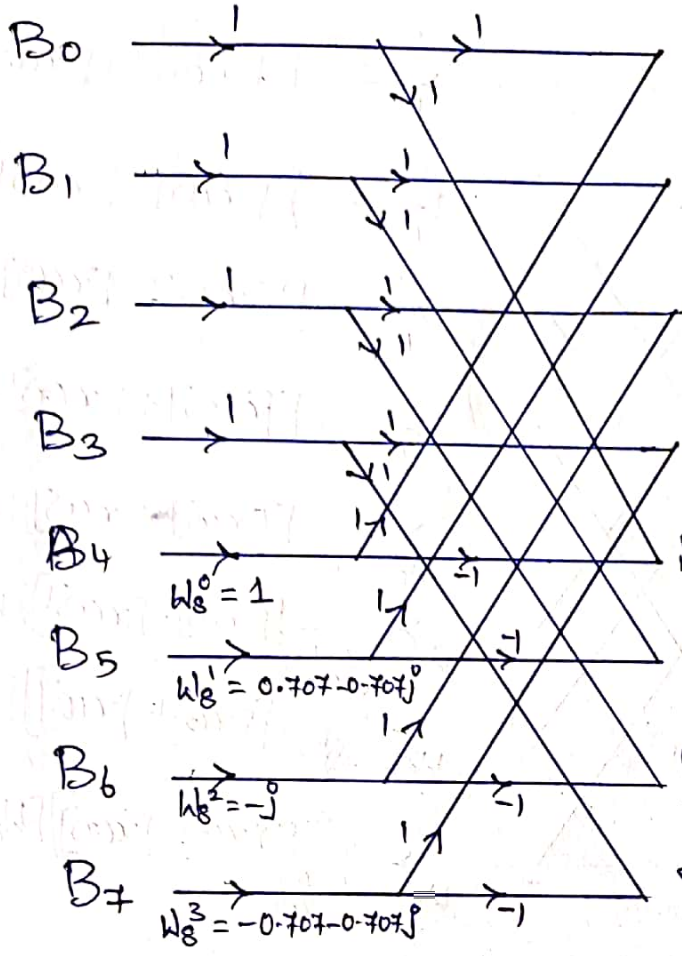
$[A_4] + [A_6] [W_4^0] = B_4$

$[A_5] + [A_7] [W_4^1] = B_5$

$[A_4] + [A_6] [-W_4^0] = B_6$

$[A_5] + [A_7] [-W_4^1] = B_7$

Step A : 8 Point DFT :



$[B_0] + [B_4]\,[W_8^0] = C_0$

$[B_1] + [B_5]\,[W_8^1] = C_1$

$[B_2] + [B_6]\,[W_8^2] = C_2$

$[B_3] + [B_7]\,[W_8^3] = C_3$

$[B_0] + [B_4]\,[-W_8^0] = C_4$

$[B_1] + [B_5]\,[-W_8^1] = C_5$

$[B_2] + [B_6]\,[-W_8^2] = C_6$

$[B_3] + [B_7]\,[-W_8^3] = C_7$

$$X(K) = \{ \underset{\substack{\uparrow \\ K=0}}{C_0}, C_1, C_2, C_3, C_4, C_5, C_6, C_7 \}$$

8 - Point DFT using Radix-2 Decimation In Frequency FFT :

DIF - FFT Algorithm :

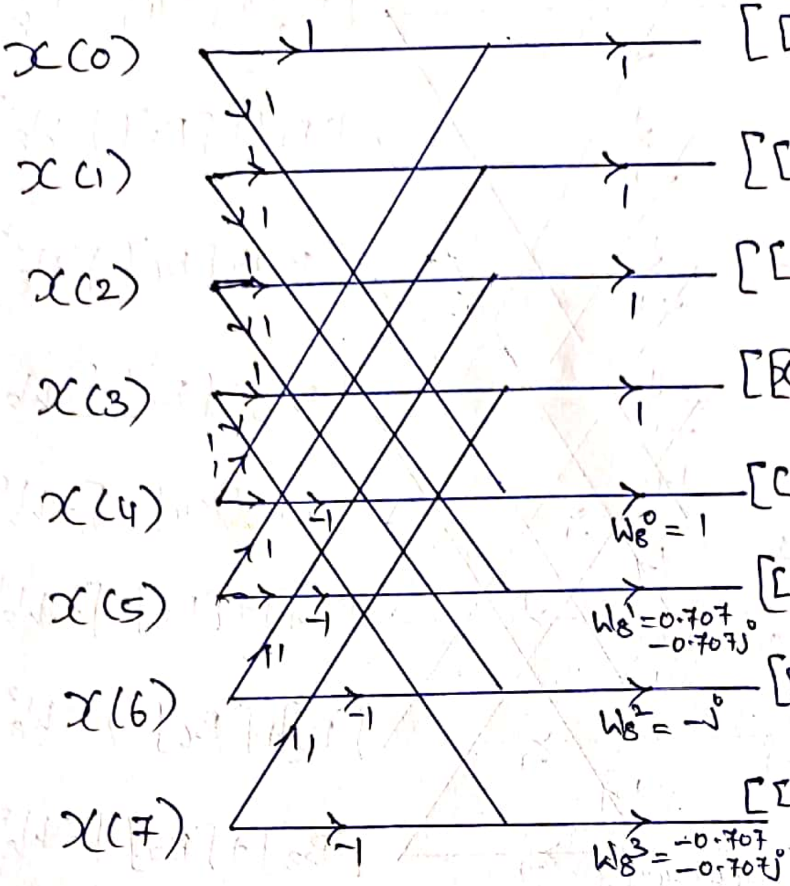
Step 1 : 8-Point DFT

Step 2 : 4-Point DFT

Step 3 : 2-Point DFT

Step 4 : Bit reversal.

Step 1 :- 8-Point DFT



$[[x(0)]+[x(4)]]=A_0$

$[[x(1)]+[x(5)]]=A_1$

$[[x(2)]+[x(6)]]=A_2$

$[[x(3)]+[x(7)]]=A_3$

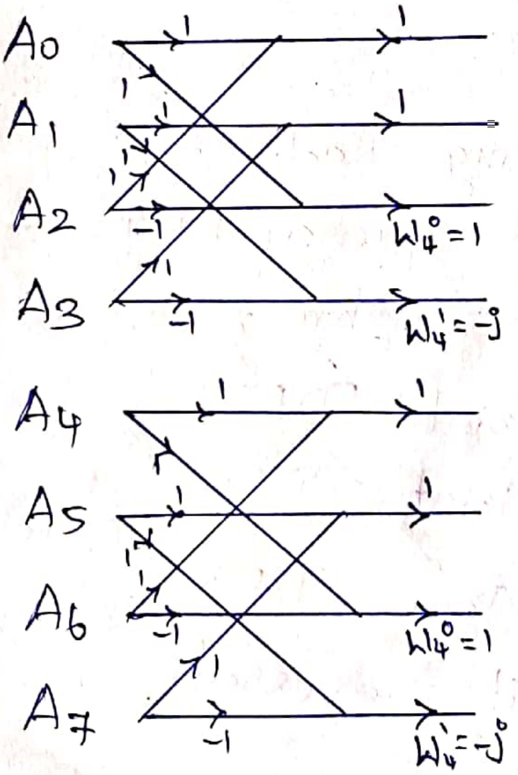
$[[x(0)]-[x(4)]][W_8^0]=A_4$

$[[x(1)]-[x(5)]][W_8^1]=A_5$

$[[x(2)]-[x(6)]][W_8^2]=A_6$

$[[x(3)]-[x(7)]][W_8^3]=A_7$

Step 2 :- 4-Point DFT



$[[A_0]+[A_2]] = B_0$

$[[A_1]+[A_3]] = B_1$

$[[A_0]-[A_2]][W_4^0] = B_2$

$[[A_1]-[A_3]][W_4^1] = B_3$

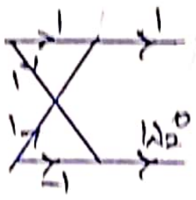
$[[A_4]+[A_6]] = B_4$

$[[A_5]+[A_7]] = B_5$

$[[A_4]-[A_6]][W_4^0] = B_6$

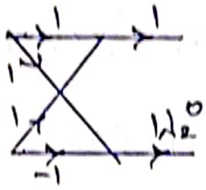
$[[A_5]-[A_7]][W_4^1] = B_7$

Step 3: 2 Point DFT



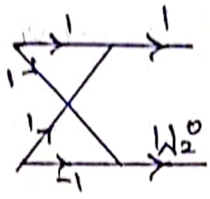
$B_0 \quad [[B_0]+[B_1]] = C_0$

$B_1 \quad [[B_0]-[B_1]][W_2^0] = C_1$



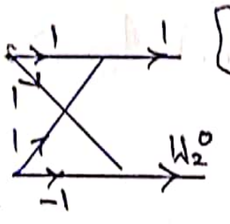
$B_2 \quad [[B_2]+[B_3]] = C_2$

$B_3 \quad [[B_2]-[B_3]][W_2^0] = C_3$



$B_4 \quad [[B_4]+[B_5]] = C_4$

$B_5 \quad [[B_4]-[B_5]][W_2^0] = C_5$



$B_6 \quad [[B_6]+[B_7]] = C_6$

$B_7 \quad [[B_6]-[B_7]][W_2^0] = C_7$

Step 4: Bit reversal

$C_0 \Rightarrow X(0) \Rightarrow X(000) \Rightarrow X(000) \Rightarrow X(0) = C_0$

$C_1 \Rightarrow X(1) \Rightarrow X(001) \Rightarrow X(100) \Rightarrow X(4) = C_4$

$C_2 \Rightarrow X(2) \Rightarrow X(010) \Rightarrow X(010) \Rightarrow X(2) = C_2$

$C_3 \Rightarrow X(3) \Rightarrow X(011) \Rightarrow X(110) \Rightarrow X(6) = C_6$

$C_4 \Rightarrow X(4) \Rightarrow X(100) \Rightarrow X(001) \Rightarrow X(1) = C_1$

$C_5 \Rightarrow X(5) \Rightarrow X(101) \Rightarrow X(101) \Rightarrow X(5) = C_5$

$C_6 \Rightarrow X(6) \Rightarrow X(110) \Rightarrow X(011) \Rightarrow X(3) = C_3$

$C_7 \Rightarrow X(7) \Rightarrow X(111) \Rightarrow X(111) \Rightarrow X(7) = C_7$

$$X(K) = \{\underset{\substack{\uparrow \\ K=0}}{C_0}, C_4, C_2, C_6, C_1, C_5, C_3, C_7\}$$

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Linear Filtering Using FFT :-

The response of LTI System is obtain using FFT

$$y(n) = x(n) * h(n)$$

$$Y(K) = X(K) \cdot H(K)$$

$$y(n) = IDFT\,[X(K) \cdot H(K)]$$

Step 1 :-

x(n) is Converted to X(K) using FFT

Step 2 :-

h(n) is Converted to H(K) using FFT

Step 3 :-

Y(K) can be obtained by

$$Y(K) = X(K) \cdot H(K)$$

Step 4 :-

y(n) can be determined by taking inverse FFT.

Inverse FFT

$$Y^*(K) \xleftrightarrow{FFT} \frac{1}{N} y(n) \Rightarrow y(n)$$

Problem 1 :

Compute 4-Point DFT ~~and~~ of Causal Sample Sequence given by

$$x(n) = \frac{1}{3} \quad ; \quad 0 \leq n \leq 2$$
$$0 \quad ; \quad \text{else}$$

also plot magnitude and Phase spectrum.

Solution

$$x(n) = \{ \underset{\uparrow}{\frac{1}{3}}, \frac{1}{3}, \frac{1}{3}, 0 \}$$

DFT

$$X(K) = \sum_{n=0}^{N-1} x(n) \, e^{-j\frac{2\pi n K}{N}}$$

$$X(K) = \sum_{n=0}^{3} x(n) \, e^{-j\frac{\cancel{2}\pi n K}{\cancel{4}2}}$$

$$X(K) = x(0) \, e^{0} + x(1) \, e^{-j\frac{\pi K}{2}} + x(2) \, e^{-j\frac{\cancel{2}\pi K}{\cancel{2}}}$$
$$+ x(3) \, e^{-j\frac{3\pi K}{2}}$$

$$X(K) = \frac{1}{3} + \frac{1}{3} e^{-j\frac{\pi K}{2}} + \frac{1}{3} e^{-j\pi K} \quad \text{———} \; ⊛$$

where $K = 0$ to $N-1$

$K = 0$

$$X(0) = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = 1$$

$K = 1$

$$X(1) = \frac{1}{3} + \frac{1}{3} e^{-j\frac{\pi}{2}} + \frac{1}{3} e^{-j\pi}$$

$$= \frac{1}{3} + \frac{1}{3}\left[\overset{0}{\cancel{\cos\frac{\pi}{2}}} - j\,\overset{1}{\cancel{\sin\frac{\pi}{2}}}\right] + \frac{1}{3}\left[\overset{-1}{\cancel{\cos\pi}} - j\,\overset{0}{\cancel{\sin\pi}}\right]$$

$$= \cancel{\frac{1}{3}} + \frac{1}{3}[-j] - \cancel{\frac{1}{3}}$$

$$X(1) = -\frac{1}{3} j$$

$K = 2$

$$X(2) = \frac{1}{3} + \frac{1}{3} e^{-j\frac{\pi \cancel{2}}{\cancel{2}}} + \frac{1}{3} e^{-j2\pi}$$

$$= \frac{1}{3} + \frac{1}{3}\left[\overset{-1}{\cancel{\cos}\,\pi} - j\,\overset{0}{\cancel{\sin}\,\pi}\right] + \frac{1}{3}\left[\overset{1}{\cancel{\cos}\,2\pi} - j\,\overset{0}{\cancel{\sin}\,2\pi}\right]$$

$$= \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} + \frac{1}{3} = \frac{1}{3}$$

$K = 3$

$$X(3) = \frac{1}{3} + \frac{1}{3} e^{-j\frac{3\pi}{2}} + \frac{1}{3} e^{-j3\pi}$$

$$X(3) = \frac{1}{3} + \frac{1}{3}\left[\overset{0}{\cancel{\cos\frac{3\pi}{2}}} - j\overset{-1}{\cancel{\sin\frac{3\pi}{2}}}\right] + \frac{1}{3}\left[\overset{-1}{\cancel{\cos 3\pi}} - j\overset{0}{\cancel{\sin 3\pi}}\right]$$

$$= \cancel{\frac{1}{3}} + \frac{1}{3}j - \cancel{\frac{1}{3}}$$

$$X(3) = \frac{1}{3}j$$

$$\therefore\ X(K) = \left\{\underset{\substack{\uparrow \\ K=0}}{1},\ -\frac{1}{3}j,\ \frac{1}{3},\ +\frac{1}{3}j\right\}$$

	$X(K)$	$\lvert X(K)\rvert$ $\sqrt{a^2+b^2}$	$\angle X(K)$ $\tan^{-1}\frac{b}{a}$
$K=0$	1	1	$\angle 0$
$K=1$	$-\frac{1}{3}j$	$\frac{1}{3} = 0.333$	$\angle -0.5\pi$
$K=2$	$\frac{1}{3}$	$\frac{1}{3} = 0.333$	$\angle 0$
$K=4$	$+\frac{1}{3}j$	$\frac{1}{3} = 0.333$	$\angle 0.5\pi$.

magnitud function $|X(K)| = \{1, 0.333, 0.333, 0.333\}$

phase function $\angle X(K) = \{0, -0.5\pi, 0, 0.5\pi\}$

Method 2 :-

$$x(n) = \left\{ \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0 \right\}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + 0 \\ \cancel{\frac{1}{3}} - j\frac{1}{3} - \cancel{\frac{1}{3}} + 0 \\ \frac{1}{3} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - 0 \\ \cancel{\frac{1}{3}} + j\frac{1}{3} - \cancel{\frac{1}{3}} + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -j\frac{1}{3} \\ \frac{1}{3} \\ j\frac{1}{3} \end{bmatrix}$$

$$X(K) = \left\{ 1, -\frac{1}{3}j, +\frac{1}{3}, \frac{1}{3}j \right\}$$

$\sqrt{a^2+b^2}$

$$|X(K)| = \{ 1, 0.333, 0.333, 0.333 \}$$

$\tan^{-1}\frac{b}{a}$

$$\angle X(K) = \{ 0, -0.5\pi, 0, 0.5\pi \}$$

An 8-Point Sequence is given by $x(n) = \{2,1,2,1,1,2,1,2\}$. Compute 8-Point DFT of $x(n)$ by radix 2 Decimation in Time - Fast Fourier Transform. Also sketch the magnitude and phase Spectrum.

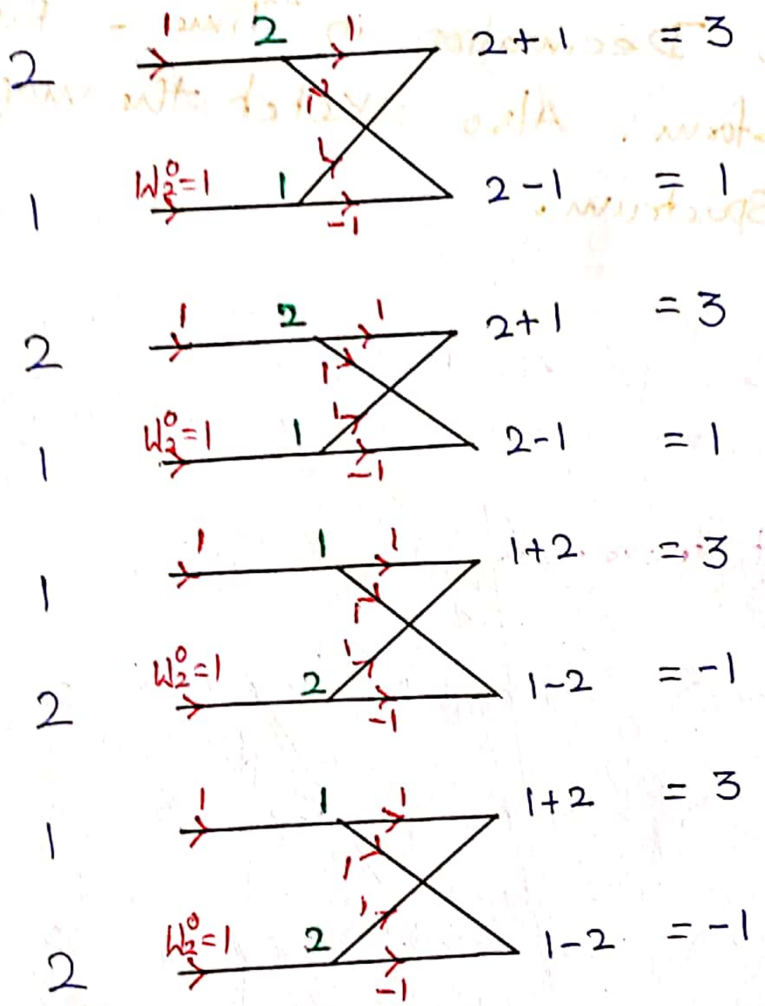
Solution :-

$$x(n) = \{\underset{x(0)}{2}, \underset{x(1)}{1}, \underset{x(2)}{2}, \underset{x(3)}{1}, \underset{x(4)}{1}, \underset{x(5)}{2}, \underset{x(6)}{1}, \underset{x(7)}{2}\}$$

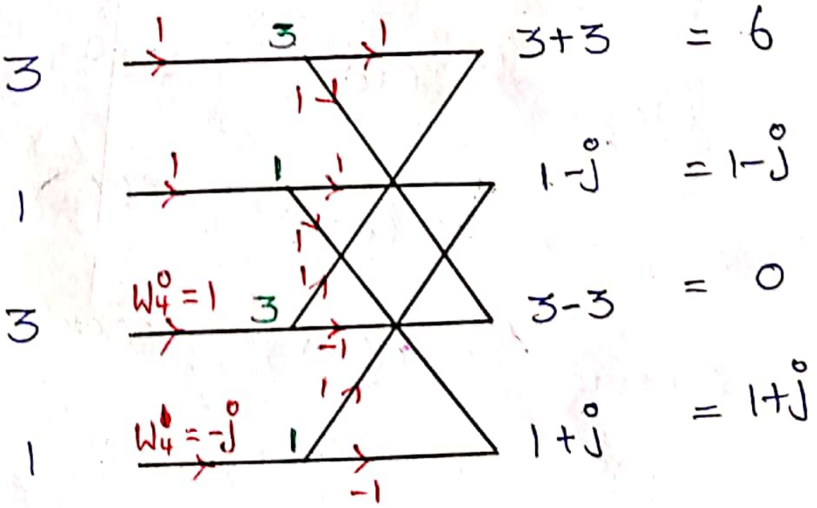
Step 1: Bit reversal

$x(0)$	$x(000)$	$x(000)$	$x(0)$	2
$x(1)$	$x(001)$	$x(100)$	$x(4)$	1
$x(2)$	$x(010)$	$x(010)$	$x(2)$	2
$x(3)$	$x(011)$	$x(110)$	$x(6)$	1
$x(4)$	$x(100)$	$x(001)$	$x(1)$	1
$x(5)$	$x(101)$	$x(101)$	$x(5)$	2
$x(6)$	$x(110)$	$x(011)$	$x(3)$	1
$x(7)$	$x(111)$	$x(111)$	$x(7)$	2

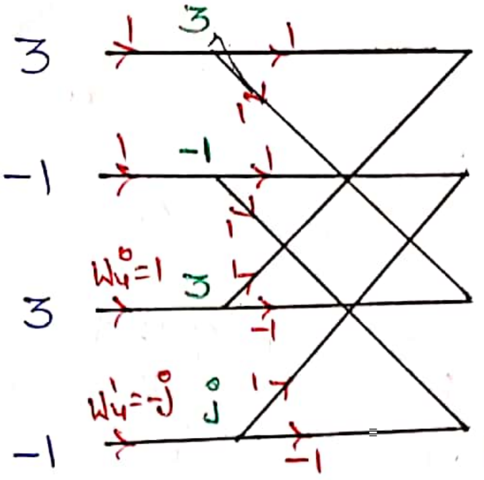
Step 2 : 2 Point Stage



Step 3 : 4 Point Stage



2



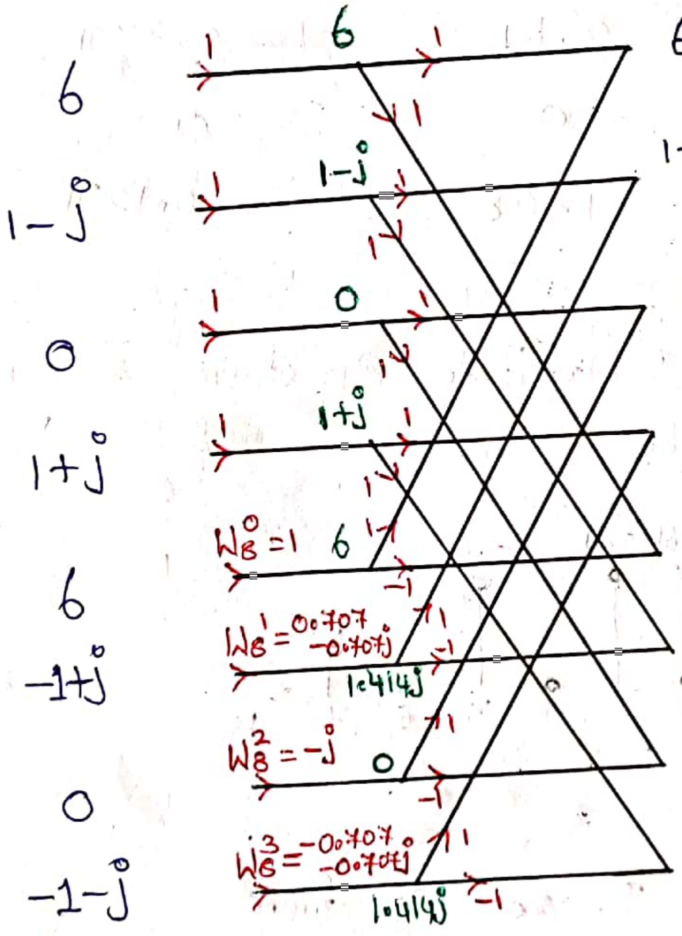
$3+3 = 6$

$-1+j = -1+j$

$3-3 = 0$

$-1-j = -1-j$

Step 4: 8 Point Stage



$6+6 = 12$

$1-j+1.414j = 1+0.414j$

$0+0 = 0$

$1+j+1.414j = 1+2.414j$

$6-6 = 0$

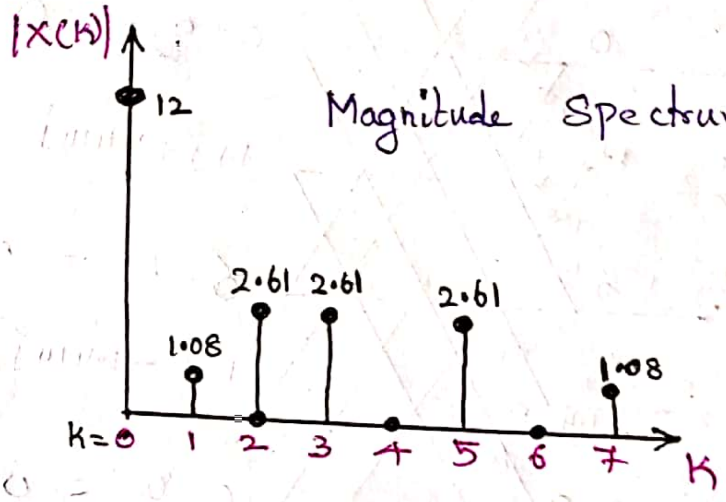
$1-j-1.414j = 1-2.414j$

$0-0 = 0$

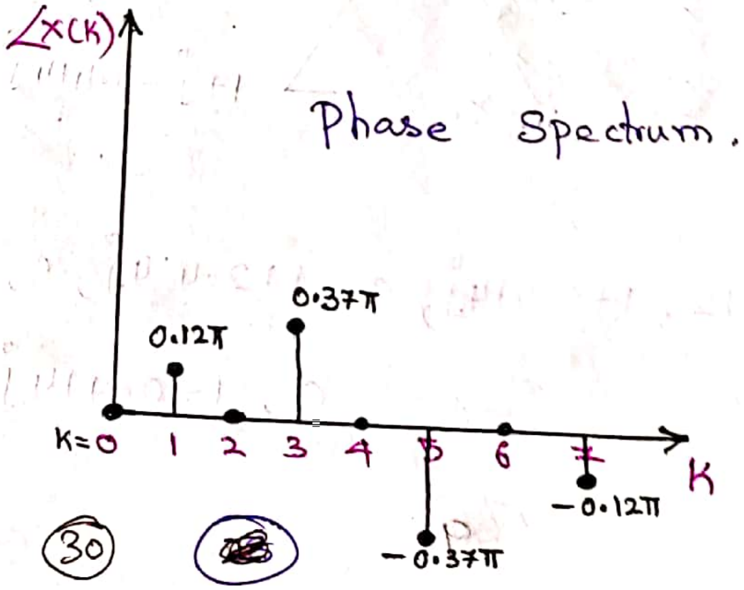
$1+j-1.414j = 1-0.414j$

$X(K) = \{12, 1+0.414j, 0, 1+2.414j, 0, 1-2.414j,$
$0, 1-0.414j\}$

K	X(K)	|X(K)| $\sqrt{a^2+b^2}$	∠X(K) $\tan^{-1}\frac{b}{a}$
0	12	12	0
1	1+0.414j	1.08	0.12π
2	0	2.61	0
3	1+2.414j	2.61	0.37π
4	0	0	0
5	1−2.414j	2.61	−0.37π
6	0	0	0
7	1−0.414j	1.08	−0.12π



Magnitude Spectrum



Phase Spectrum.

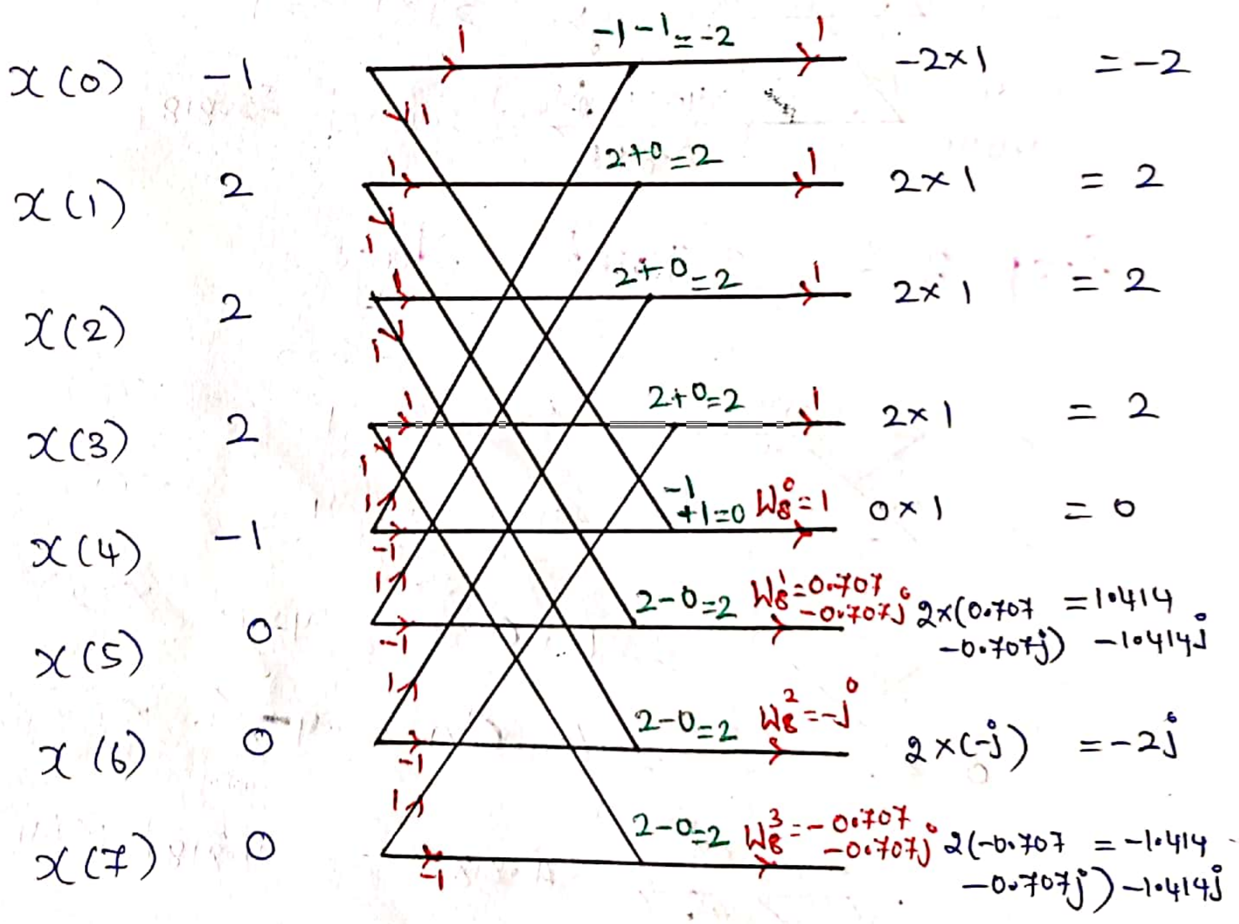
30

An 8-Point Sequence is given by $x(n) = \{-1, 2, 2, 2, -1\}$. by Decimation in Frequency - Fast Fourier Transform Compute DFT

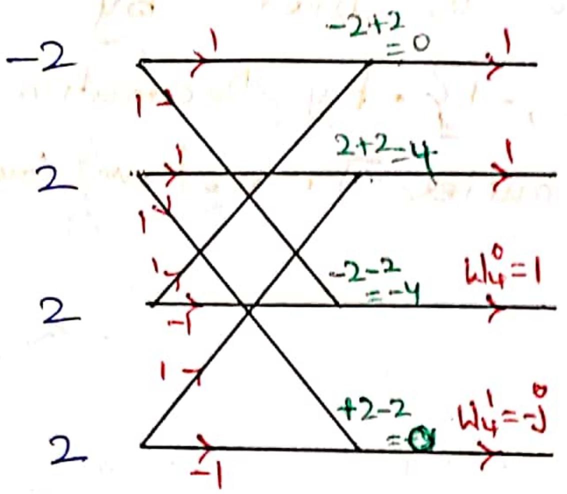
Solution

$$x(n) = \{-1, 2, 2, 2, -1, 0, 0, 0\}$$
$x(0)$ $x(1)$ $x(2)$ $x(3)$ $x(4)$ $x(5)$ $x(6)$ $x(7)$

Step 1: 8 Point Stage



Step 2 : 4 Point Stage

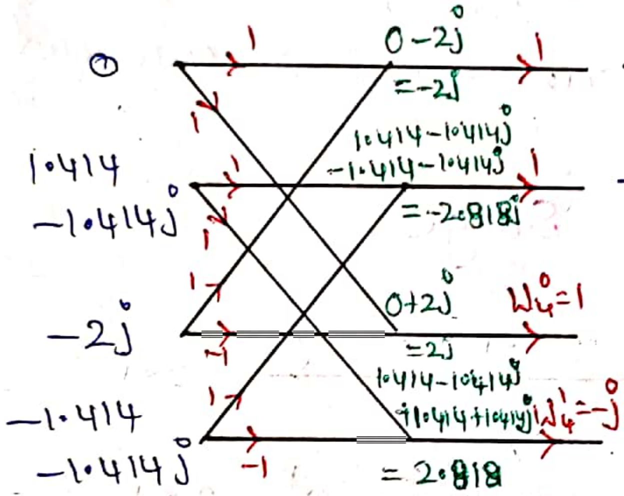


$0\times1 = 0$

$4\times1 = 4$

$-4\times1 = -4$

$-0\times-j = 0$



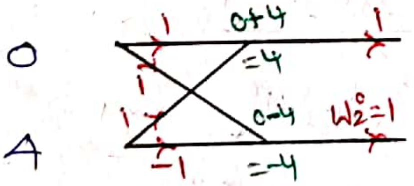
$-2j\times1 = -2j$

$-2.414j\times1 = -2.818j$

$2j\times1 = 2j$

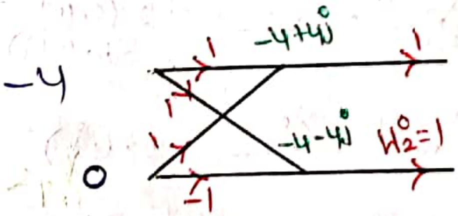
$2.414\times-j = -2.818j$

Step 3 : 2 Point Stage



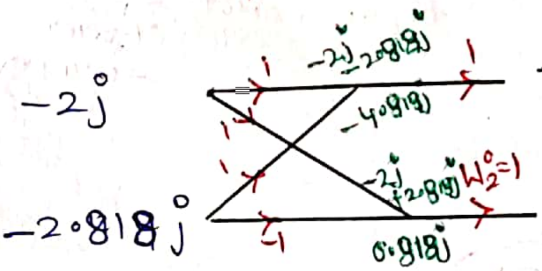
$4\times1 = 4$ X(0)

$-4\times1 = -4$ X(1)



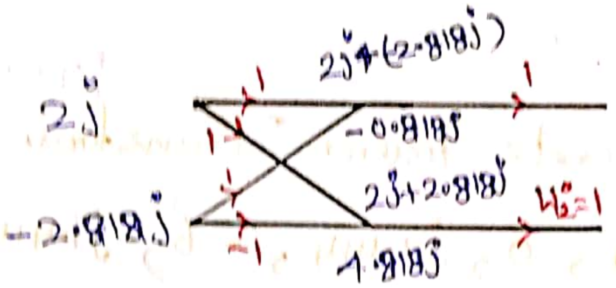
$(-4+0)\times1 = -4$ X(2)

$(-4-0)\times1 = -4$ X(3)



$-4.818j\times1 = -4.818j$ X(4)

$0.818j\times1 = 0.818j$ X(5)



$-0.818j \times 1 = -0.818j$ $X(6)$

$1.818j \times 1 = 4.818j$ $X(7)$.

Step 4 : Bit reversal

$X(0)$	$X(000)$	$X(000)$	$X(0) = 4$
$X(1)$	$X(001)$	$X(100)$	$X(4) = -4.818j$
$X(2)$	$X(010)$	$X(010)$	$X(2) = -4$
$X(3)$	$X(011)$	$X(110)$	$X(6) = -0.818j$
$X(4)$	$X(100)$	$X(001)$	$X(1) = -4$
$X(5)$	$X(101)$	$X(101)$	$X(5) = 0.818j$
$X(6)$	$X(110)$	$X(011)$	$X(3) = -4$
$X(7)$	$X(111)$	$X(111)$	$X(7) = +4.818j$

$$X(K) = \{4, -4.818j, -4, -0.818j, -4, +0.818j, -4, 4.818j\}$$

$$X(K) = \{4, -4.818j, -4, -0.818j, -4, +0.818j, -4, +4.818j\}$$

[IDFT]

Find Inverse Discrete Fourier Transform of $X(K) = \{14, j1.414, 0, j1.414, -6, -j1.414, 0, -j1.414\}$ using FFT-DIF Algorithm?

Solution :

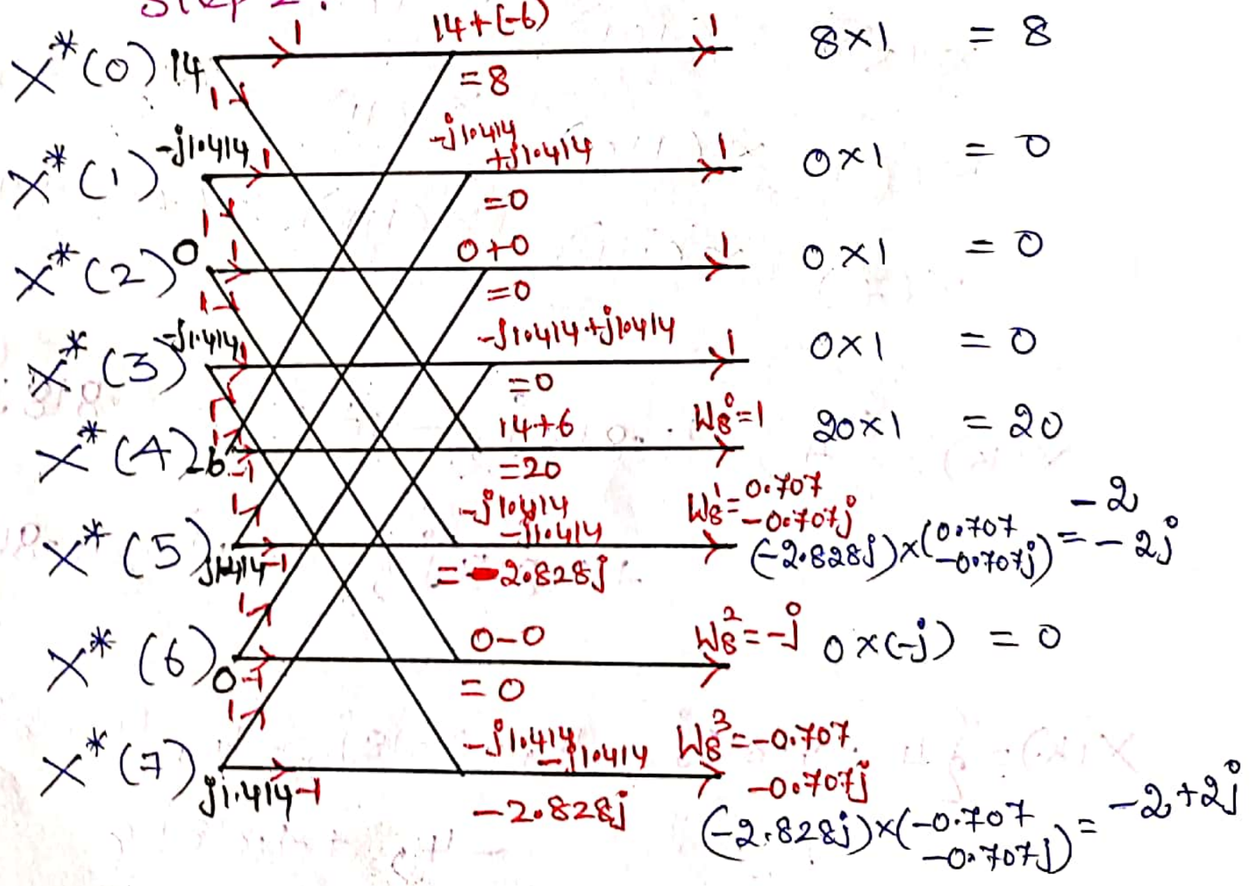
$X(K) = \{14, j1.414, 0, j1.414, -6, -j1.414, 0, -j1.414\}$

Step 1 : Conjucate of X(K)

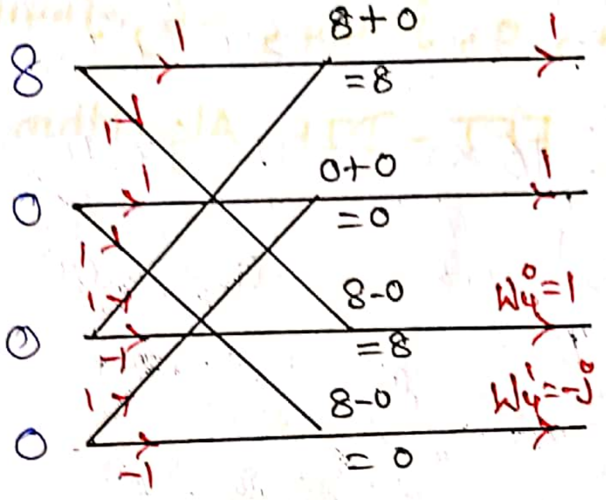
$X^*(K) = \{14, -j1.414, 0, -j1.414, -6, +j1.414, 0, +j1.414\}$

$X^*(0)\ X^*(1)\ X^*(2)\ X^*(3)\ X^*(4)\ X^*(5)\ X^*(6)\ X^*(7)$

Step 2 : 8 Point Stage.



Step 3 : 4 Point Stage

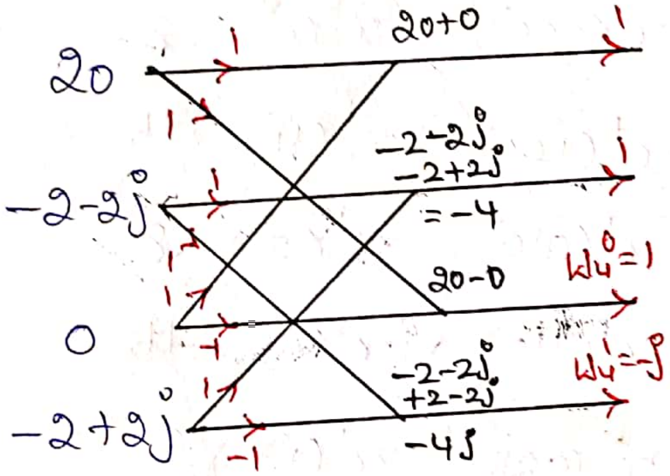


$8 \times 1 = 8$

$0 \times 1 = 0$

$8 \times 1 = 8$

$0 \times -j = 0$



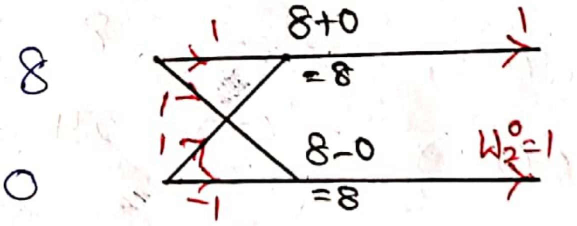
$20 \times 1 = 20$

$-4 \times 1 = -4$

$20 \times 1 = 20$

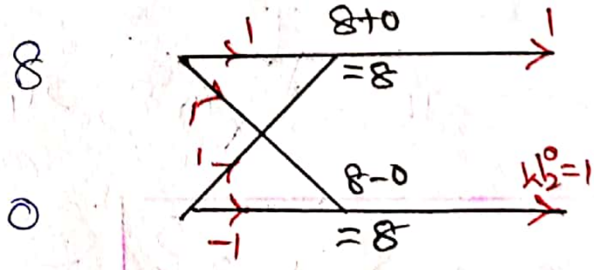
$-4j \times -j = -4$

Step 4 : 2 Point Stage



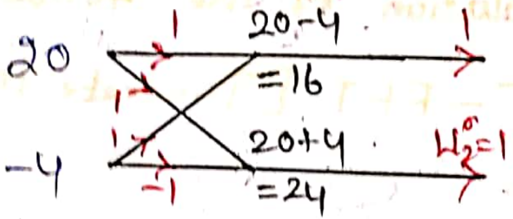
$8 \times 1 = 8$ $x'(0)$

$8 \times 1 = 8$ $x'(1)$



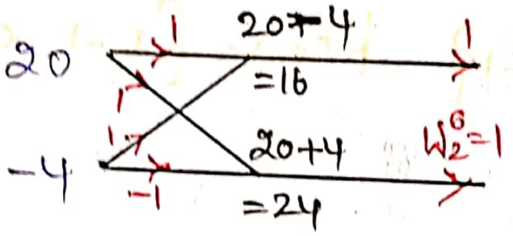
$8 \times 1 = 8$ $x'(2)$

$8 \times 1 = 8$ $x'(3)$



$16\times1 = 16$ $x'(4)$

$24\times1 = 24$ $x'(5)$



$16\times1 = 16$ $x'(6)$

$24\times1 = 24$ $x'(7)$

Step 5 : Bit reversal

$x'(0)$	$x'(000)$	$x'(000)$	$x'(0)$	8
$x'(1)$	$x'(001)$	$x'(100)$	$x'(4)$	16
$x'(2)$	$x'(010)$	$x'(010)$	$x'(2)$	8
$x'(3)$	$x'(011)$	$x'(110)$	$x'(6)$	16
$x'(4)$	$x'(100)$	$x'(001)$	$x'(1)$	8
$x'(5)$	$x'(101)$	$x'(101)$	$x'(5)$	24
$x'(6)$	$x'(110)$	$x'(011)$	$x'(3)$	8
$x'(7)$	$x'(111)$	$x'(111)$	$x'(7)$	24

Step 6 : ÷ N

$$x(n) = \frac{1}{8}\{8, 16, 8, 16, 8, 24, 8, 24\}$$

$$\boxed{x(n) = \{1, 2, 1, 2, 1, 3, 1, 3\}}$$

Compute linear Convolution of the following Sequences using DIT-FFT [Discrete Fourier Transform]

$$x(n) = \{1, 0.2, -1\} \quad h(n) = \{1, -1, -0.2\}$$

Solution :

output sequence Length $= M + L - 1$

$= 3 + 3 - 1$

$= 5$

$\therefore \quad N = 8$

$$x(n) = \{\underset{x(0)}{1}, \underset{x(1)}{0.2}, \underset{x(2)}{-1}, \underset{x(3)}{0}, \underset{x(4)}{0}, \underset{x(5)}{0}, \underset{x(6)}{0}, \underset{x(7)}{0}\}$$

$$\& \; h(n) = \{1, -1, -0.2, 0, 0, 0, 0, 0\}$$

$\rightarrow Y(K) = X(K)\,H(K)$

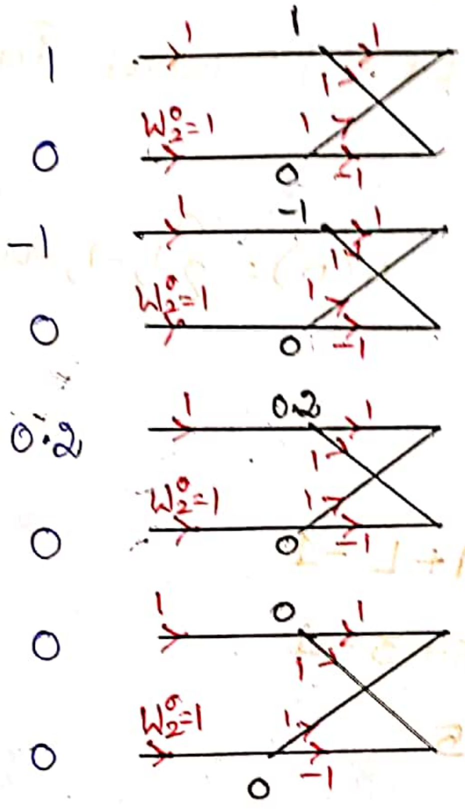
$y(n) = \text{IDFT}\{Y(K)\}$

To Find X(K)

Step 1 : Bit reversal.

$x(0)$	$x(000)$	$x(000)$	$x(0)$	1
$x(1)$	$x(001)$	$x(100)$	$x(4)$	0
$x(2)$	$x(010)$	$x(010)$	$x(2)$	−1
$x(3)$	$x(011)$	$x(110)$	$x(6)$	0
$x(4)$	$x(100)$	$x(001)$	$x(1)$	0.2
$x(5)$	$x(101)$	$x(101)$	$x(5)$	0
$x(6)$	$x(110)$	$x(011)$	$x(3)$	0
$x(7)$	$x(111)$	$x(111)$	$x(7)$	0

Step 2 : 2 Point Stage



1+0 = 1

1-0 = 1

-1+0 = -1

-1-0 = -1

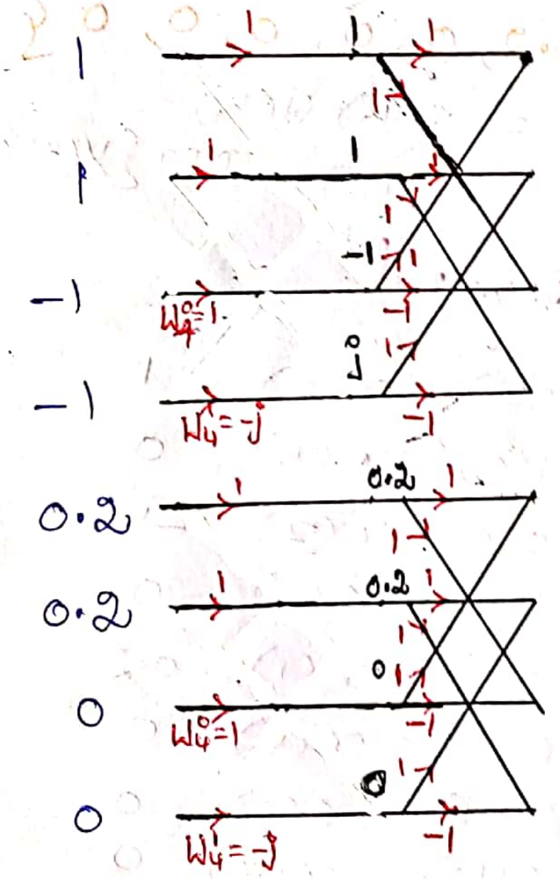
0·2+0 = 0·2

0·2-0 = 0·2

0+0 = 0

0-0 = 0

Step 3 : 4 Point Stage



1-1 = 0

1+j = 1+j

1+1 = 2

1-j = 1-j

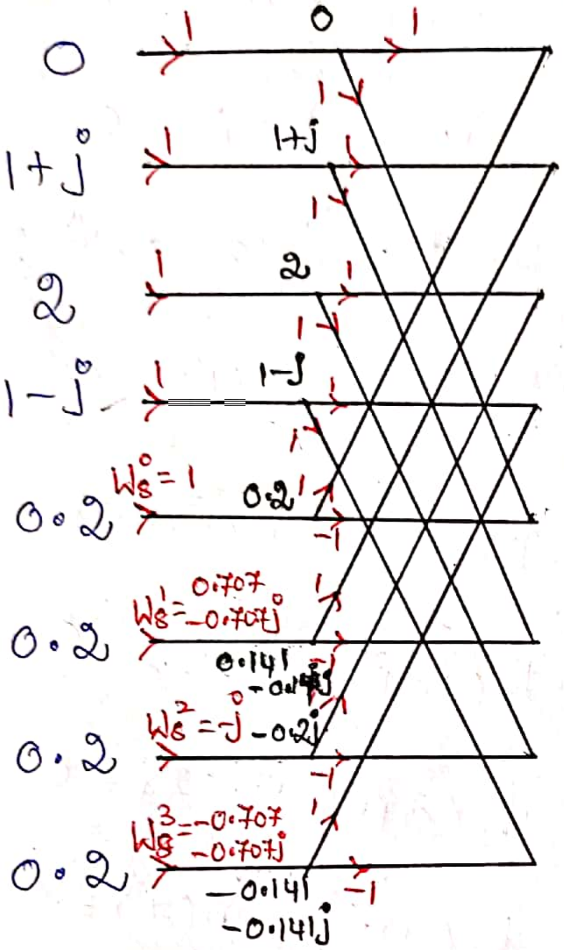
0·2+0 = 0·2

0·2+0 = 0·2

0·2-0 = 0·2

0·2-0 = 0·2

Step 4 : 8 Point Stage



$0 + 0.2 = 0.2$

$1 + j + 0.141 - 0.141j = 1.141 + 0.858j$

$2 - 0.2j = 2 - 0.2j$

$1 - j + (-0.141 - 0.141j) = 0.858 - 1.141j$

$0 - 0.2 = -0.2$

$1 + j - (0.141 - 0.141j) = 0.8581 + 1.141j$

$2 + 0.2j = 2 + 0.2j$

$1 - j - (-0.141 - 0.141j) = 1.414 - 0.858j$

$X(k) = \{0.2, 1.414 + 0.858j, 2 - 0.2j, 0.858 - 1.141j,$
$-0.2, 0.858 + 1.141j, 2 + 0.2j, 1.414 - 0.858j\}$

To Find H(k)

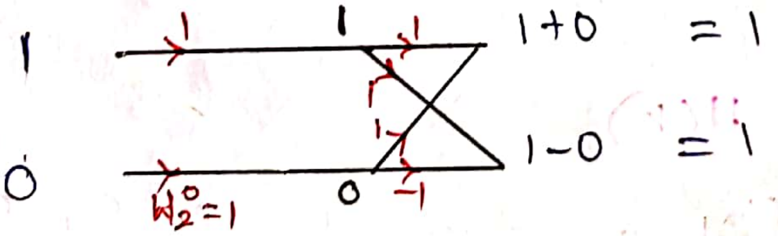
$h(n) = \{1, -1, -0.2, 0, 0, 0, 0, 0\}$

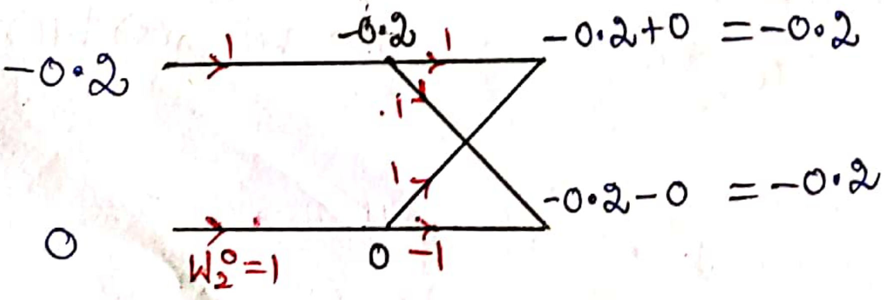
$h(0)\ h(1)\ h(2)\ h(3)\ h(4)\ h(5)\ h(6)\ h(7)$

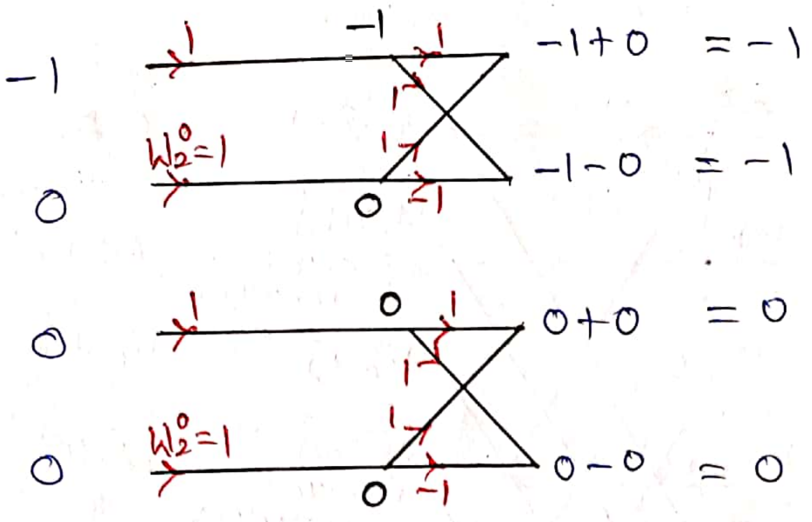
Step 1 : Bit reversal

h(0)	h(000)	h(000)	h(0) = 1
h(1)	h(001)	h(100)	h(4) = 0
h(2)	h(010)	h(010)	h(2) = −0.2
h(3)	h(011)	h(110)	h(6) = 0
h(4)	h(100)	h(001)	h(1) = −1
h(5)	h(101)	h(101)	h(5) = 0
h(6)	h(110)	h(011)	h(3) = 0
h(7)	h(111)	h(111)	h(7) = 0

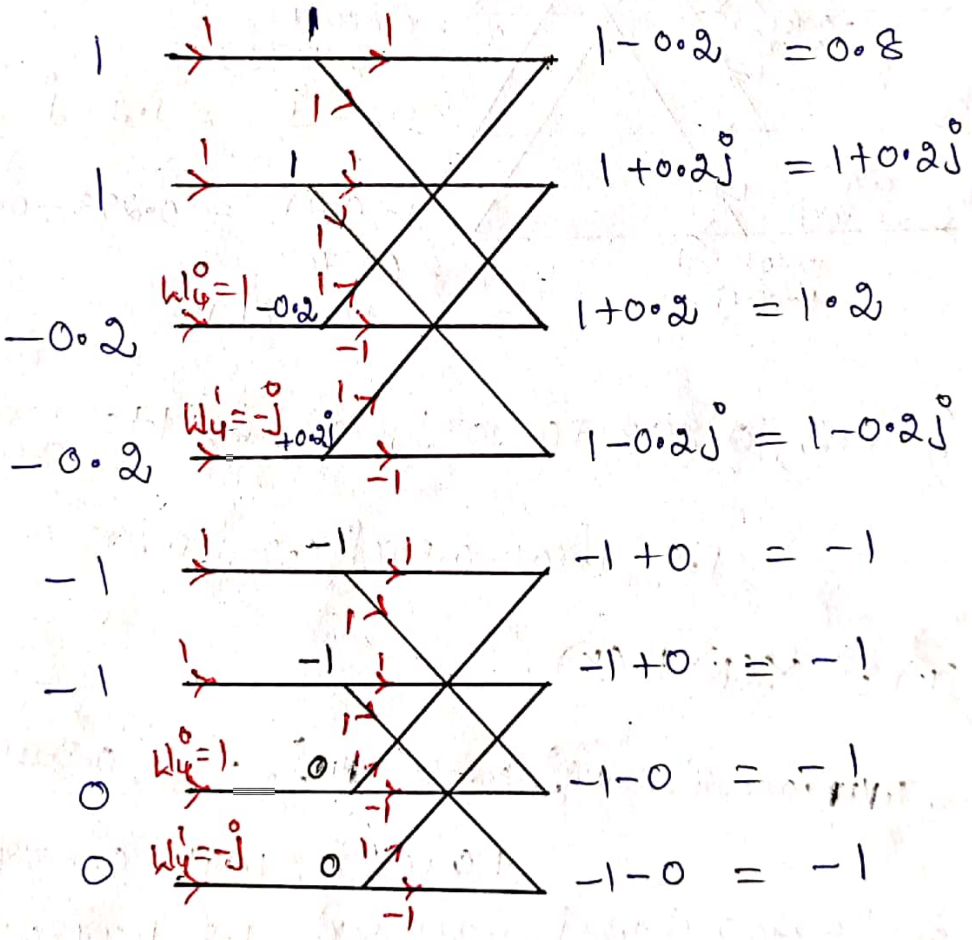
Step 2 : 2 Point Stage

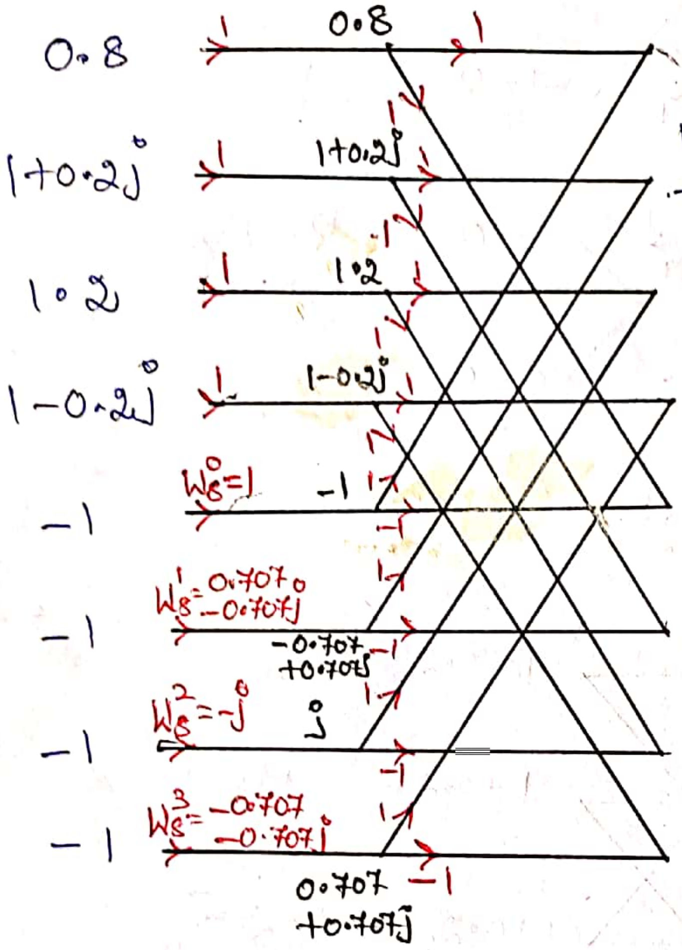






Step 3 :- 4 Point Stage





$0.8-1 = -0.2$

$1+0.2j+(-0.707+0.707j) = 0.293+0.907j$

$1.2+j = 1.2+j$

$1-0.2j+(0.707+0.707j) = 1.707+0.507j$

$0.8+1 = 1.8$

$1+0.2j-(-0.707+0.707j) = 1.707-0.507j$

$1.2-j = 1.2-j$

$1-0.2j-(0.707+0.707j) = 0.293-0.907j$

$$H(K)=\{-0.2,\ 0.293+0.907j,\ 1.2+j,\ 1.707+0.507j,\ 1.8,\ 1.707-0.507j,\ 1.2-j,\ 0.293-0.907j\}$$

$$Y(K) = X(K)\,H(K)$$

$$X(K)=\{0.2,\ 1.1414+0.858j,\ 2-0.2j,\ 0.8586-1.1414j,\ -0.2,\ 0.8586+1.1414j,\ +2+0.2j,\ 1.1414-0.8586j\}$$

$$H(K)=\{-0.2,\ 0.293+0.907j,\ 1.2+j,\ 1.707+0.507j,\ 1.8,\ 1.707-0.507j,\ 1.2-j,\ 0.293-0.907j\}$$

$$Y(K)=\{-0.04,\ -0.44432+1.2868j,\ 2.6+1.76j,\ 2.044-1.513j,\ -0.36$$
$$2.044+1.513j,\ 2.6-1.76j,\ -0.44432-1.2868j\}$$

Step 1:- Conjugate of y(K)

$y^*(K) = \{-0.04, -0.44432 - 1.2868j, 2.6 - 1.076j,$
$2.044 + 1.513j, -0.36, 2.044 - 1.513j,$
$2.6 + 1.076j, -0.44432 + 1.2868j\}$

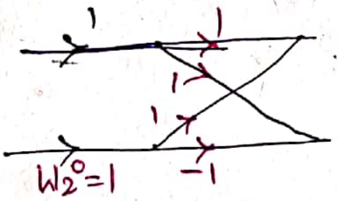
Step 2:- Bit Reversal.

$y^*(0)$	$y^*(000)$	$y^*(000)$	$y^*(0)$	-0.04
$y^*(1)$	$y^*(001)$	$y^*(100)$	$y^*(4)$	-0.36
$y^*(2)$	$y^*(010)$	$y^*(010)$	$y^*(2)$	$2.6-1.076j$
$y^*(3)$	$y^*(011)$	$y^*(110)$	$y^*(6)$	$2.6+1.076j$
$y^*(4)$	$y^*(100)$	$y^*(001)$	$y^*(1)$	$-0.44432-1.02868j$
$y^*(5)$	$y^*(101)$	$y^*(101)$	$y^*(5)$	$2.044-1.513j$
$y^*(6)$	$y^*(110)$	$y^*(011)$	$y^*(3)$	$2.044+1.513j$
$y^*(7)$	$y^*(111)$	$y^*(111)$	$y^*(7)$	$-0.44432+1.02868j$

Step 3:- 2 Point Stage.

-0.04

-0.36

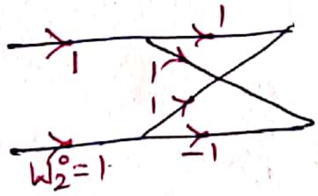


$-0.04 - 0.36 = -0.4$

$-0.04 + 0.36 = 0.32$

$2.6 - 1.076j$

$2.6 + 1.076j$

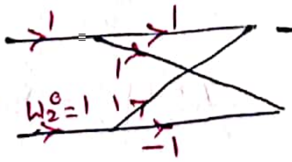


$2.6 - 1.076j + 2.6 + 1.076j = 5.2$

$2.6 - 1.076j - 2.6 - 1.076j = -3.52j$

$-0.44432 - 1.2868j$

$2.044 - 1.513j$

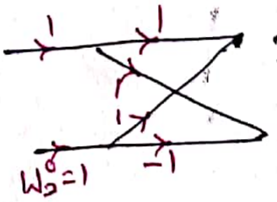


$-0.44432 - 1.2868j + 2.044 - 1.513j$ 2.07998j
$= 1.6$ -~~2.0798j~~

$-0.44432 - 1.2868j - 2.044 + 1.513j = -2.4883 + 0.2262j$

$2.044 + 1.513j$

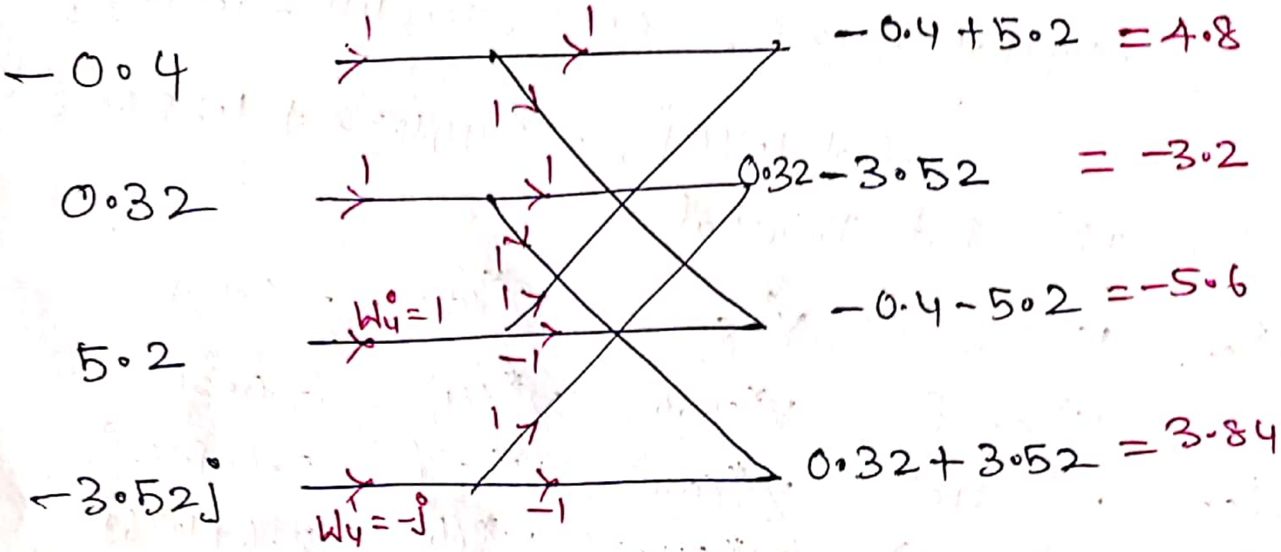
$-0.44432 + 1.2868j$

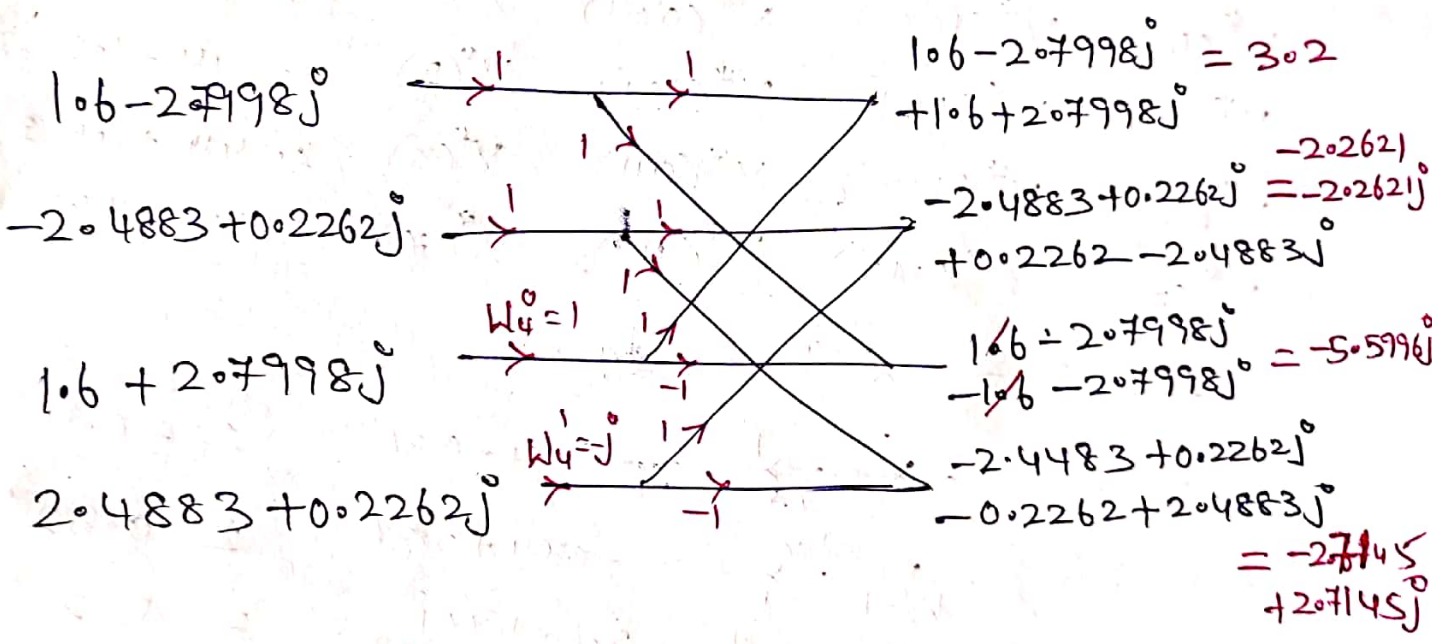


$2.044 + 1.513j - 0.44432 + 1.2868j = 1.6 + 2.07998j$

$2.044 + 1.513j + 0.44432 - 1.2868j = 2.4883 + 0.2262j$

Step 4:- A Point Stage



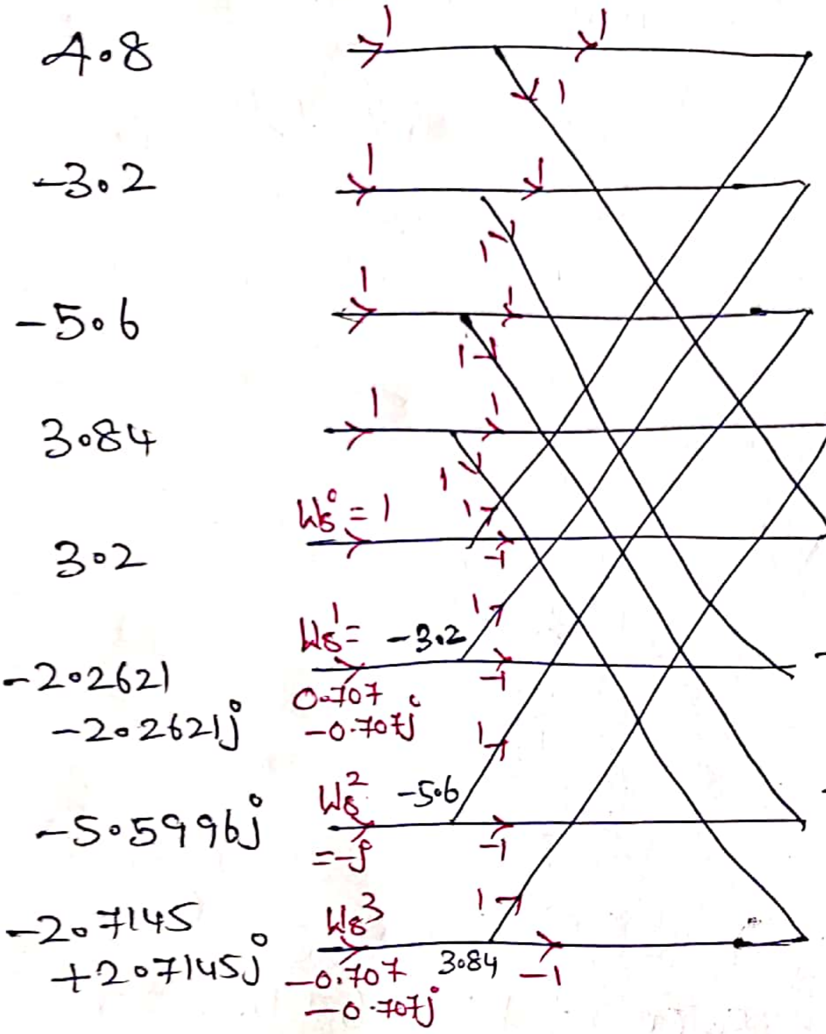


A Point Stage

Result :- {4.8, −3.2, −5.6, 3.84, 3.2

−2.2621 − 2.2621j, −5.5996j, −2.7145 + 2.7145j}

Step 5 : 8 point Stage



4.8 + 3.2 = 8

-3.2 ~~+~~3.2 = -6.4

-5.6 -5.6 = -11.2

3.84 + 3.84 = 7.68

4.8 - 3.2 = 1.6

-3.2 + 3.2 = 0

-5.6 + 5.6 = 0

3.84 - 3.84 = 0

Step 6 : % N

$$y(n) = \frac{1}{8}\{8, -6.4 -11.2, 7.68, 1.6, 0, 0, 0\}$$

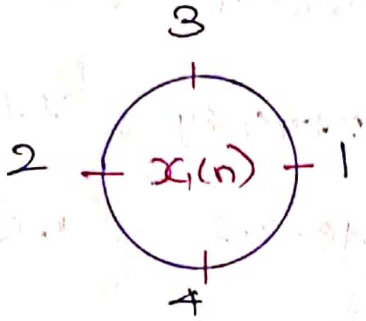
$$\boxed{y(n) = \{1, -0.8 -1.4, 0.96, 0.2, 0, 0, 0\}}$$

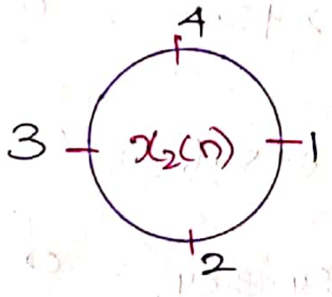
x(n)

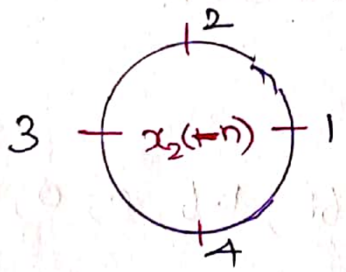
		1	0.2	-1
	1	1	0.2	-1
h(n)	-1	-1	-0.2	+1
	-0.2	-0.2	-0.04	+0.2

y(n) = {1, -0.8, -1.4,
0.96, 0.2, 0, 0, 0}.

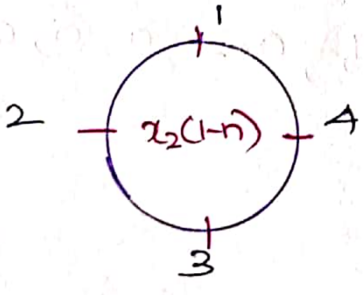
Perform Circular Convolution of the two sequences $x_1(n) = \{\underset{\uparrow}{1}, 3, 2, 4\}$ $x_2(n) = \{\underset{\uparrow}{1}, 4, 3, 2\}$



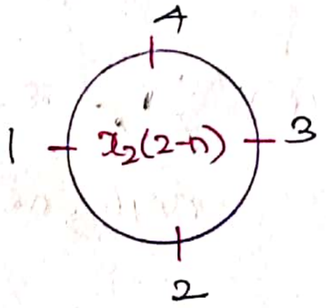




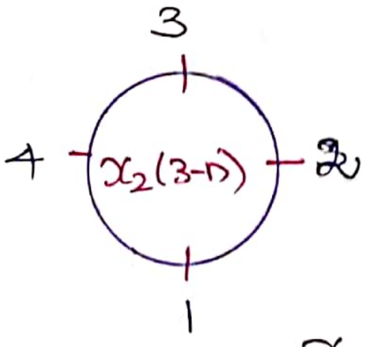
$x_3(0) = 1 + 6 + 6 + 16 = 29$



$x_3(1) = 4 + 3 + 4 + 12 = 23$



$x_3(2) = 3 + 12 + 2 + 8 = 25$



$$x_3(3) = 2+9+8+4 = 23$$

$$x_3(n) = \{\underset{\uparrow}{29}, 23, 25, 23\}$$

Matrix method

$$\begin{bmatrix} 1 & 4 & 2 & 3 \\ 3 & 1 & 4 & 2 \\ 2 & 3 & 1 & 4 \\ 4 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+16+6+6 \\ 3+4+12+4 \\ 2+12+3+8 \\ 4+8+9+2 \end{bmatrix} = \begin{bmatrix} 29 \\ 23 \\ 25 \\ 23 \end{bmatrix}$$

$$x_3(n) = \{\underset{\uparrow}{29}, 23, 25, 23\}$$

Perfrom the linear Convolution of the following Sequences by (a) overlap add method (b) overlap Save method.

$$x(n) = \{1,2,3,-1,-2,-3,4,5,6\} \text{ and}$$

$$h(n) = \{2,1,-1\}.$$

Solution :-

overlap add method

$$x_1(n) = \{1,2,3\}$$

$x_2(n) = \{-1,-2,-3\}$

$x_3(n) = \{4,5,6\}$

$y_1(n) =$

	2	1	-1
1	2	1	-1
2	4	2	-2
3	6	3	-3

$y_1(n) = \{2,5,7,1,-3\}$

$y_2(n)$

	2	1	-1
-1	-2	-1	1
-2	-4	-2	2
-3	-6	-3	3

$y_2(n) = \{-2,-5,-7,-1,3\}$

$y_3(n)$

	2	1	-1
4	8	4	-4
5	10	5	-5
6	12	6	-6

$y_3(n) = \{8,14,13,1,-6\}$

overlap length $= 3-1=2$

$y_1(n)$	2	5	7	1	-3						
$y_2(n)$				-2	-5	-7	-1	3			
$y_3(n)$							8	14	13	1	-6
$y(n)$	2	5	7	-1	-8	-7	7	17	13	1	-6

$y(n) = \{2,5,7,-1,-8,-7,7,17,13,1,-6\}$

Overlap Save method :

$h(n) = \{2, 1, -1\}$

$x_1(n) = \{1, 2, 3\}$

$x_2(n) = \{2, 3, -1\}$

$x_3(n) = \{3, -1, -2\}$

$x_4(n) = \{-1, -2, -3\}$

$x_5(n) = \{-2, -3, 4\}$

$x_6(n) = \{-3, 4, 5\}$

$x_7(n) = \{4, 5, 6\}$

$y_1(n)$

	2	1	-1
1	2	1	-1
2	4	2	-2
3	6	3	-3

$y_1(n) = \{2, 5, 7, 1, -3\}$

$y_2(n)$

	2	1	-1
2	4	2	-2
3	6	3	-3
-1	-2	-1	1

$y_2(n) = \{4, 8, -1, -4, 1\}$

$y_3(n)$

	2	1	-1
3	6	3	-3
-1	-2	-1	1
-2	-4	-2	2

$y_3(n) = \{6, 1, -8, -1, 2\}$

$y_4(n)$

	2	1	-1
-1	-2	-1	1
-2	-4	-2	+2
-3	-6	-3	3

$y_4(n) = \{-2, -5, -7, -1, 3\}$

$y_5(n)$

	2	1	-1
-2	-4	-2	2
-3	-6	-3	3
4	8	4	-4

$y_5(n) = \{-4, -8, 7, 7, -4\}$

$y_6(n)$

	2	1	-1
-3	-6	-3	3
4	8	4	-4
5	10	5	-5

$y_6(n) = \{-6, 5, 17, 1, -5\}$

$y_7(n)$

	2	1	-1
4	8	4	-4
5	10	5	-5
6	12	6	-6

$y_7(n) = \{8, 14, 13, 1, -6\}$

$y_1(n)$	2	5	7	1	-3									
$y_2(n)$				4	8	-1	-4	-1						
$y_3(n)$							6	1	-8	-1	2			
$y_4(n)$										-2	-5	7	-1	3

㊿

$y_1(n)$	2	5	7	~~1~~	~~-3~~																		
$y_2(n)$				~~4~~	~~8~~	-1	~~-4~~	~~-1~~															
$y_3(n)$							~~6~~	~~1~~	-8	~~-1~~	~~2~~												
$y_4(n)$										~~-2~~	~~-5~~	-7	~~-1~~	~~3~~									
$y_5(n)$													~~-4~~	~~-8~~	7	~~7~~	~~-4~~						
$y_6(n)$																~~-6~~	~~5~~	17	~~1~~	~~-5~~			
$y_7(n)$																			~~8~~	~~14~~	13	1	-6
$y(n)$	2	5	7			-1			-8			-7			7			17			13	1	-6

$$y(n) = \{2, 5, 7, -1, -8, -7, 7, 17, 13, 1, -6\}$$