Unit II

Infinite Impulse Response Filters

Characteristics of Practical frequency Selective filters. Characteristics of Commonly used analog filters - Butterworth filters, Chebyshev filters. Design of IIR filters from analog filters (LPF, HPF, BPF, BRF) - Approximation of derivatives, Impulse invariance method, Bilinear transformation. Frequency transformation in the analog domain. Structure of IIR filter direct form I, direct form II, Cascade, Parallel realizations.

Discrete time IIR filter from analog filter :-

Impulse Invariant Transformation.

The objective of Impulse Invariant transformation is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter

$$H(s) = \frac{Y(s)}{X(s)}$$

Can be Express in a form

$$H(s) = \sum_{i=1}^{N} \frac{A_i}{s+P_i}$$

$$H(s) = \frac{A_1}{s+P_1} + \frac{A_2}{s+P_2} + \frac{A_3}{s+P_3} + \cdots + \frac{A_N}{s+P_N}$$

$$\frac{A_i}{S+P_i} \longrightarrow \frac{A_i}{1-e^{-P_iT}z^{-1}}$$

$$H(z) = \frac{A_1}{1-e^{-P_1T}z^{-1}} + \frac{A_2}{1-e^{-P_2T}z^{-1}} + \frac{A_3}{1-e^{-P_3T}z^{-1}} + \cdots + \frac{A_N}{1-e^{-P_NT}z^{-1}}$$

where T = Sampling Period.

Bilinear Transformation :-

The bilinear transformation is a Conformal mapping that transforms the imaginary axis of S-Plane into the Unit Circle in the Z-Plane

$$S = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$$

where

T is Sampling time Period.

Relation Between Analog and Digital frequency In Impulse Invariant

$$\omega = \Omega T$$

Analog frequency

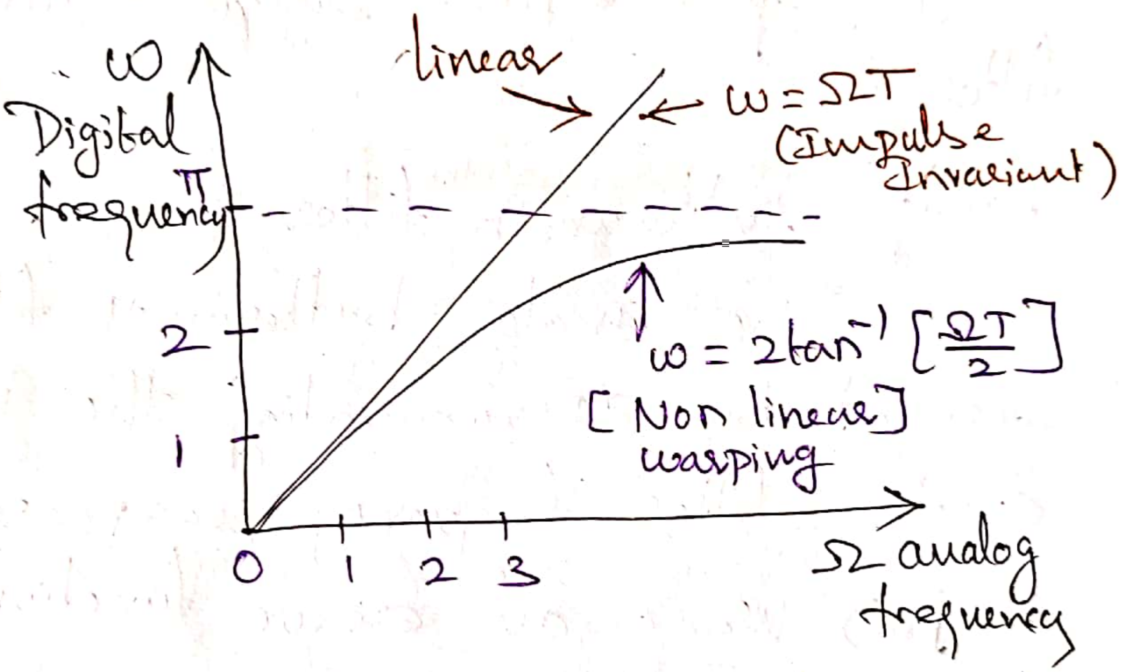
Digital frequency

In Bilinear Transformation

$$\omega = 2\tan^{-1}\frac{\Omega T}{2}$$

Analog frequency

Digital frequency



The non-linear relation between analog frequency and Digital frequency in Bilinear Transform is called "Warping"

To overcome the non-linear relation between analog frequency and digital frequency in Bilinear Transform Pre-warping is done in Analog filter before transform.

Pre-warping is given by

$$\Omega = \frac{2}{T} \tan \frac{\omega}{2}$$

Design of Low pass Digital Butterworth filter :-

Analog Butterworth filter :-

The analog butterworth filter is designed by approximating the ideal analog filter frequency response, $H(j\Omega)$ using an error function

The magnitude is maximally flat in the Pass band and monotoni-cally decreasing in the Stop band

The magnitude response of low Pass filter is given by

$$|H(j\Omega)|^2 = \frac{1}{1+\left[\frac{\Omega}{\Omega_c}\right]^{2N}}$$

where

N - is order of filter

Ω_c - is cut off frequency.

Properties of Butterworth filters:-

1. The Butterworth filter are all pole design
2. The magnitude is maximally flat at the origin
3. The magnitude is a monotonically decreasing ~~f~~ as function of Ω
4. The magnitude response approaches the ideal response as the value of N increases.

Design Procedure for Lowpass Digital Butterworth IIR filter:-

Let

ω_p = Pass band edge digital frequency

ω_s = Stopband edge digital frequency

$T = \frac{1}{F_s}$ = Sampling time in Sec

A_p = Gain at a Passband frequency

A_s = Gain at a Stopband frequency

Step 1 :-

Choose either bilinear (or) Impulse Invariant transformation.

Step 2 :-

Calculate the ratio $\frac{\Omega_s}{\Omega_p}$

For Impulse invariant transformation

$$\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p}$$

For Bilinear transformation

$$\frac{\Omega_s}{\Omega_p} = \frac{\tan \frac{\omega_s}{2}}{\tan \frac{\omega_p}{2}}$$

Step 3 :-

Calculate the order of the filter N

$$N_1 = \frac{1}{2} \frac{\log \left[\left[\frac{1}{A_s^2} - 1 \right] / \left[\frac{1}{A_p^2} - 1 \right] \right]}{\log \left[\frac{\Omega_s}{\Omega_p} \right]}$$

$N \geq N_1$ [Should be a Integer]

Step 4 :

Determine the normalized transfer function $H(S_n)$

when N is even.

$$H(S_n) = \prod_{K=1}^{N/2} \frac{1}{S_n^2 + b_K S_n + 1}$$

when N is odd

$$H(S_n) = \frac{1}{S_n+1} \prod_{K=1}^{\frac{N-1}{2}} \frac{1}{S_n^2 + b_K S_n + 1}$$

where

$$b_K = 2 \sin\left[\frac{(2K-1)\pi}{2N}\right]$$

Step 5 :

Calculate the analog cut off frequency Ω_c

$$\Omega_c = \frac{\frac{\omega_s}{T}}{\left[\left[\frac{1}{A_s^2}\right] - 1\right]^{[1/2N]}}$$ for Impulse Invariant Transformation

$$\Omega_c = \frac{\frac{2}{T}\tan\frac{\omega_s}{2}}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}}$$ for Bilinear Transformation.

Step 6 :-

Determine the unnormalized analog transform function $H(s)$

LPF $\quad H(s) = H(S_n)\Big|_{S_n = \frac{s}{\Omega_c}}$

HPF $\quad H(s) = H(S_n)\Big|_{S_n = \frac{\Omega_c}{s}}$

BPF $\quad H(s) = H(S_n)\Big|_{S_n = \frac{Q(s^2+\Omega_0^2)}{\Omega_0 s}}$

BRF $\quad H(s) = \underline{H(S_n)}\Big|_{S_n = \frac{\Omega_0 s}{Q(s^2+\Omega_0^2)}}$

where

$$\Omega_0 = \sqrt{\Omega_p \Omega_s}$$

and

$$Q = \frac{\Omega_0}{\Omega_s - \Omega_p}$$

Step 7 :-

Convert H(s) to H(z)

for Impulse Invariant transformation

$$\frac{A_i}{S+P_i} \longrightarrow \frac{A_i}{1-e^{-P_i T} z^{-1}}$$

for Bilinear Transformation

$$S = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$$

Step 8 :-

Realize the digital filter H(z) by a suitable structure.

Design of Low Pass Digital Chebyshev filter :-

The analog Chebyshev filter is designed by approximating the ideal frequency response using an error function.

The approximation function is Selected Such that the error is minimized over a Prescribed band of frequencies. There are two types of Chebyshev approximation.

* Chebyshev filter [Type - I]
* Chebyshev filter [Type - II]

Chebyshev filter Type - I :

In Type - I approximation, the error function is Selected Such that, the magnitude response is equiripple in the Pass band and monotonic in the Stop band.

Chebyshev filter Type - II :

In Type - II approximation, the error function is Selected Such that, the magnitude respone is Monotonic in Pass band and equiripple in stopband. It is also called inverse Chebyshev response

The magnitude response of Type-I LPF is

$$|H(j\Omega)^2| = \frac{1}{1+\epsilon^2 C_N^2\left[\frac{\Omega}{\Omega_c}\right]}$$

The attenuation Constant

$$\epsilon = \left[\frac{1}{A_p^2} - 1\right]^{\frac{1}{2}}$$

Properties of Chebyshev filter (Type-1)

1. The magnitude $|H(j\Omega)|$ oscillates between 1 and $1/\sqrt{1+\epsilon^2}$ within the Passband and so the filter is called epuiripple in the passband.

2. The normalized magnitude response has a value of $1/\sqrt{1+\epsilon^2}$ at cut off frequency Ω_c

3. The magnitude is monotonic outside the pass band.

4. The Chebyshev Type-1 filters are all pole design

Design Procedure for Lowpass Digital Chebyshev IIR Filter :-

Step 1 :-

Choose either bilinear (or) Impulse Invariant transformation.

Step 2 :-

Calculate the ratio $\frac{\Omega_s}{\Omega_p}$

For Impulse invariant Transformation

$$\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p}$$

For Bilinear Transformation

$$\frac{\Omega_s}{\Omega_p} = \frac{\tan\frac{\omega_s}{2}}{\tan\frac{\omega_p}{2}}$$

Step 3 :-

Calculate the order of the filter N

$$N_1 = \frac{\cosh^{-1}\left[\left[\frac{1}{A_s^2}-1\right]^{1/2} / \left[\frac{1}{A_p^2}-1\right]^{1/2}\right]}{\cosh^{-1}\left[\frac{\Omega_s}{\Omega_p}\right]}$$

where $N \geq N_1$ [Should be a Integer]

Step 4 :-

Determine the normalized transfer function $H(S_n)$

when N is even

$$H(S_n) = \prod_{K=1}^{N/2} \frac{B_K}{S_n^2 + b_K S_n + C_K}$$

when N is odd

$$H(S_n) = \frac{B_0}{S_n + C_0} \prod_{K=1}^{\frac{N-1}{2}} \frac{B_K}{S_n^2 + b_K S_n + C_K}$$

where

$$b_K = 2\, Y_N \sin\left[\frac{(2K-1)\pi}{2N}\right]$$

$$C_K = Y_N^2 + \cos^2\left[\frac{(2K-1)\pi}{2N}\right]$$

$$C_0 = Y_N = \frac{1}{2}\left\{\left[\left[\frac{1}{\epsilon^2}+1\right]^{1/2}+\frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left[\frac{1}{\epsilon^2}+1\right]^{1/2}+\frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right\}$$

$$\epsilon = \left[\left[\frac{1}{A_p^2}-1\right]^{1/2}\right]$$

For even values of N, find B_K such that

$$H(0) = \frac{1}{(1+\epsilon^2)^{1/2}}$$

For odd values of N, find B_K such that

$$H(0) = 1.$$

Step 5 :

Calculate the analog cut off frequency Ω_c

$$\Omega_c = \frac{\omega_p}{T}$$ for Impulse Invariant Transformation

$$\Omega_c = \frac{2}{T}\tan\frac{\omega_p}{2}$$ for Bilinear Transformation.

Step 6 :

Determine the unnormalized analog transform function H(s)

LPF $H(s) = H(s_n)\Big|_{s_n = \frac{s}{\Omega_c}}$

HPF $H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_c}{s}}$

BPF $H(s) = H(s_n)\Big|_{s_n = \frac{Q(s^2+\Omega_0^2)}{\Omega_0 s}}$

BRF $H(s) = H(s_n)\Big|_{s_n = \frac{\Omega_0 s}{Q(s^2+\Omega_0^2)}}$

where
$$\Omega_0 = \sqrt{\Omega_p \Omega_s}$$

and
$$Q = \frac{\Omega_0}{\Omega_s - \Omega_p}$$

Step 7 :

Convert H(s) to H(z)

$$\frac{A_i}{s+P_i} \longrightarrow \frac{A_i}{1-e^{-P_iT}z^{-1}}$$ for Impulse Invariant Transformation

$$S = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}}$$ for Bilinear transformation.

Step 8 :-

Realize the digital filter $H(z)$ by a suitable structure.

Frequency translation :-

	Filter Type	Transformation
1.	Low pass	$S_n \rightarrow \frac{S}{\Omega_c}$
2.	High pass	$S_n \rightarrow \frac{\Omega_c}{S}$
3.	Band Pass	$S_n \rightarrow \frac{Q(S^2+\Omega_0^2)}{\Omega_0 S}$
4.	Band Stop	$S_n \rightarrow \frac{\Omega_0 S}{Q(S^2+\Omega_0^2)}$

where

$$\Omega_0 = \sqrt{\Omega_p \Omega_s}$$

$$Q = \frac{\Omega_0}{\Omega_s - \Omega_p}$$

Summary of Butterworth Lowpass Filter Normalized Transfer function.

Order, (N)	Normalized transfer function $H(S_n)$
1	$\dfrac{1}{S_n+1}$
2	$\dfrac{1}{S_n^2+1.414S_n+1}$
3	$\dfrac{1}{(S_n+1)(S_n^2+S_n+1)}$
4.	$\dfrac{1}{(S_n^2+0.765S_n+1)(S_n^2+1.848S_n+1)}$
5	$\dfrac{1}{(S_n+1)(S_n^2+0.618S_n+1)(S_n^2+1.618S_n+1)}$
6	$\dfrac{1}{(S_n^2+1.932S_n+1)(S_n^2+1.414S_n+1)(S_n^2+0.518S_n+1)}$

Structures for Realization of IIR Systems.

In general, the time domain representation of an N^{th} order IIR System is

$$y(n) = -\sum_{m=1}^{N} a_m y(n-m) + \sum_{m=0}^{M} b_m x(n-m)$$

and the Z-domain representation of an N^{th} order IIR System is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

The different types of Structure for realizing the IIR Systems are

1. Direct form I Structure
2. Direct form II Structure
3. Cascade form Structure
4. Parallel form Structure.

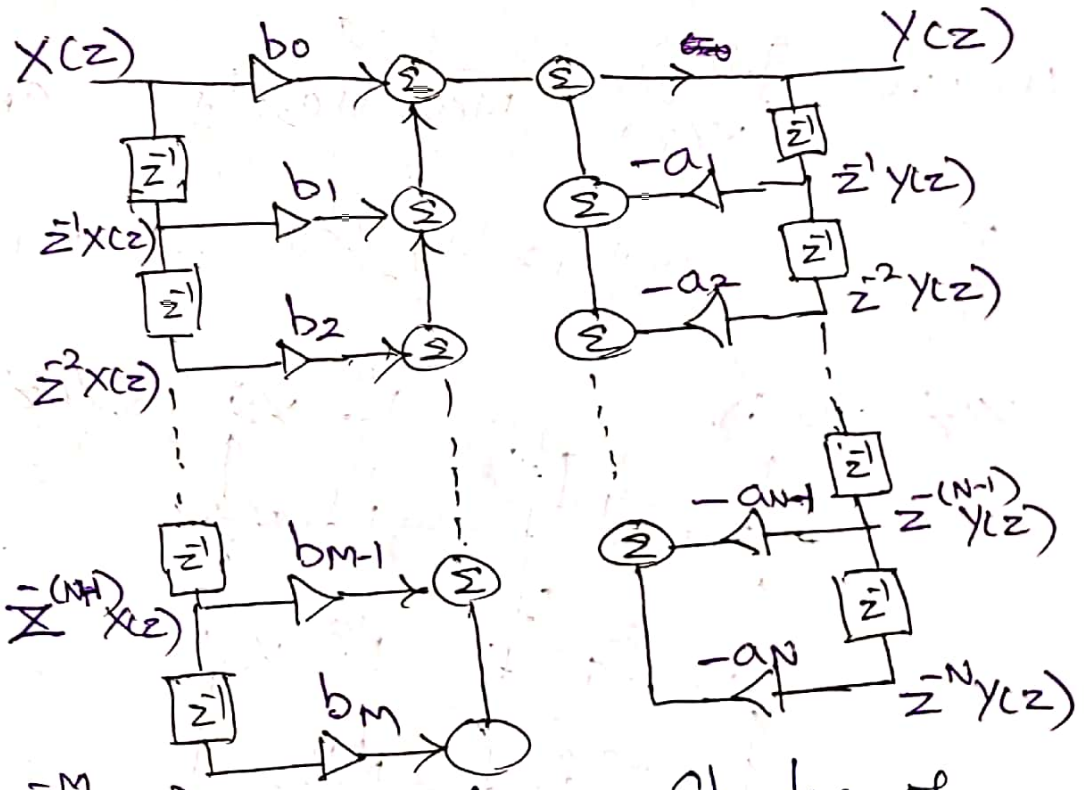
Direct form I Structure of IIR system

Consider the difference equation of an IIR System.

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) \ldots\ldots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + \ldots + b_M x(n-M)$$

on taking z-transform of the above equation we get

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) \ldots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \ldots + b_M z^{-M} X(z)$$



Direct form I Structure of IIR Structure.

Direct form II Structure of IIR System

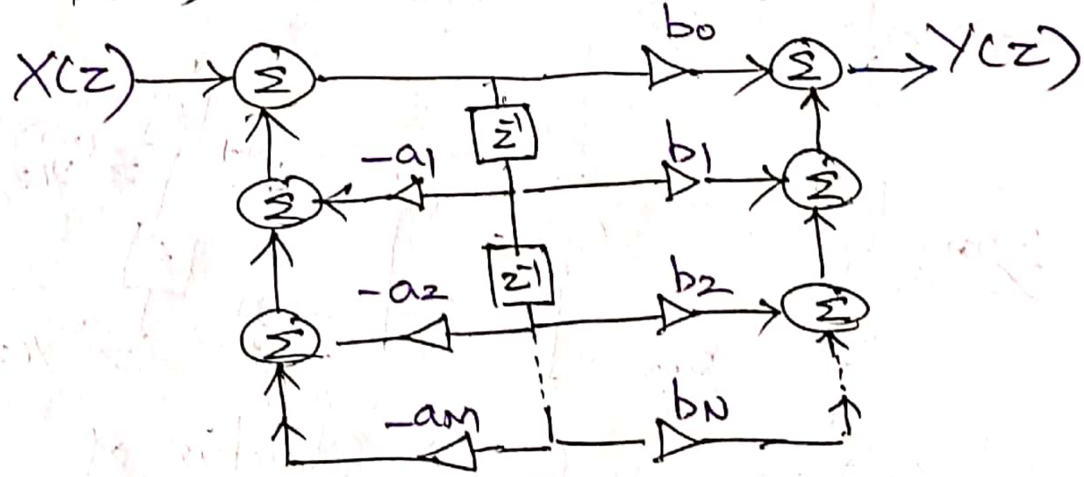
An alternative Structure called direct form II Structure can be realized which uses less number of delay than the direct form I Structure.

$$\frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)}$$

where $$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots a_N z^{-N}}$$

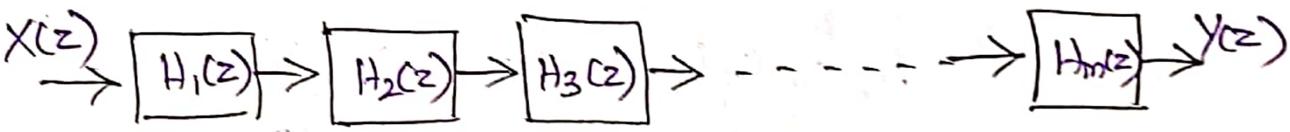
$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}$$



Cascade form Realization of IIR System:-

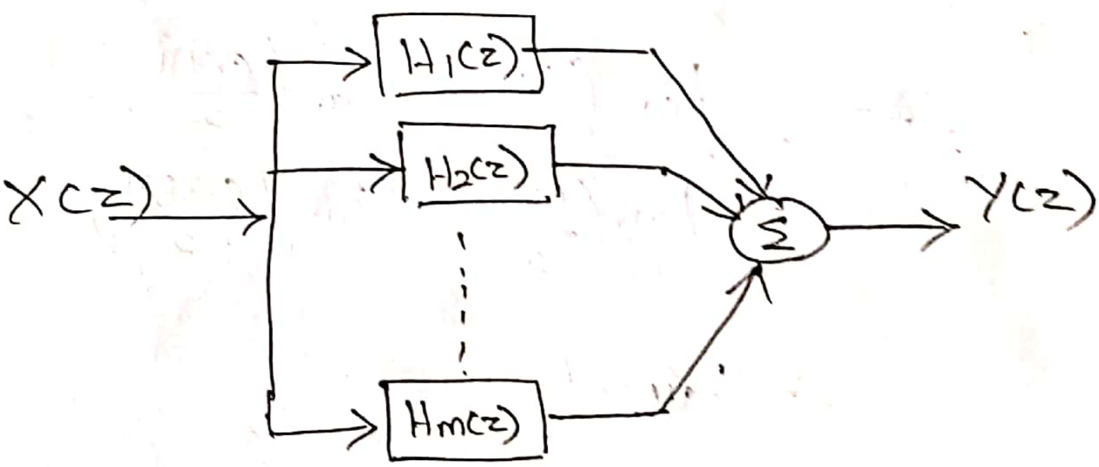
The transfer function H(z) can be expressed as a Product of a number of Second-order (or) first-order Sections

$$H(z) = \frac{Y(z)}{X(z)} = H_1(z) \times H_2(z) \times H_3(z) \cdots \times H_m(z)$$



Parallel form Realization of IIR System:-

The transfer function H(z) can expressed as a Sum of a number of Second order (or) first-order Sections.



Problem 1:

Design a Butterworth digital IIR Low Pass filter using bilinear transform by Taking $T = 0.1$ sec, to Satisfy the following Specifications

$$0.6 \le |H(e^{j\omega})| \le 1.0 \; ; \quad \text{for } 0 \le \omega \le 0.35\pi$$

$$|H(e^{j\omega})| \le 0.1 \; ; \quad \text{for } 0.7\pi \le \omega \le \pi$$

Solution:

$$A_p = 0.6 \qquad \omega_p = 0.35\pi$$

$$A_s = 0.1 \qquad \omega_s = 0.7\pi$$

Step 1:

Bilinear Transform.

Step 2:

Calculate the ratio $\frac{\Omega_s}{\Omega_p}$

$$\frac{\Omega_s}{\Omega_p} = \frac{\tan\left(\frac{\omega_s}{2}\right)}{\tan\left(\frac{\omega_p}{2}\right)} = \frac{\tan\left(\frac{0.7\pi}{2}\right)}{\tan\left(\frac{0.35\pi}{2}\right)}$$

$$= 3.2026$$

$$\frac{\Omega_s}{\Omega_p} = 3.2026$$

Step 3 :

- Calculate the order of the filter N

$$N_1 = \frac{\frac{1}{2}\log\left[\frac{\left[\frac{1}{A_s^2}-1\right]}{\left[\frac{1}{A_p^2}-1\right]}\right]}{\log\left[\frac{\Omega_s}{\Omega_p}\right]} = \frac{\frac{1}{2}\log\left[\frac{\left[\frac{1}{0.1^2}-1\right]}{\left[\frac{1}{0.6^2}-1\right]}\right]}{\log\left[3.2026\right]}$$

$$N_1 = 1.7267 \qquad N \geq N_1$$

$$N = 2$$

Step 4 :

Calculate the Cut-off frequency Ω_c.

$$\Omega_c = \frac{\Omega_s}{\left[\frac{1}{A_s^2}-1\right]^{1/2N}} = \frac{\frac{2}{T}\tan\left(\frac{\omega_s}{2}\right)}{\left[\frac{1}{A_s^2}-1\right]^{1/2N}}$$

$$= \frac{\frac{2}{0.1}\tan\left(\frac{0.7\pi}{2}\right)}{\left[\frac{1}{0.1^2}-1\right]^{(1/2\times 2)}} = \frac{20\tan\left(\frac{0.7\pi}{2}\right)}{(99)^{1/4}}$$

$$\Omega_c = 12.4439 \text{ rad/sec.}$$

Step 5 :

Calculate the analog Transfer function $H_a(s)$

$$H_a(s) = \prod_{K=1}^{\frac{N}{2}} \frac{\Omega_c^2}{s^2 + \Omega_c b_K s + \Omega_c^2}$$

$$= \prod_{K=1}^{\cancel{2}/\cancel{2}\,1} \frac{(12.4439)^2}{s^2 + 12.4439\, b_K s + (12.4439)^2}$$

$$H_a(s) = \frac{154.8506}{s^2 + 12.4439\, b_1 s + 154.8506}$$

$$b_K = 2 \sin \frac{(2K-1)\pi}{2N} \quad b_1 = 2\sin\left(\frac{\pi}{4}\right) = 1.4142$$

$$H_a(s) = \frac{154.8506}{s^2 + 17.5982 s + 154.8506}$$

Step 6 :

Convert $H_a(s)$ to $H(z)$ by using bilinear Transformation.

$$s = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{0.1} \frac{1-z^{-1}}{1+z^{-1}} = \frac{20(1-z^{-1})}{(1+z^{-1})}$$

$$H(z) = \frac{154.8506}{\left(\frac{20(1-z^{-1})}{(1+z^{-1})}\right)^2 + 17.5982\left(\frac{20(1-z^{-1})}{(1+z^{-1})}\right) + 154.8506}$$

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$$= \frac{154.8506}{\dfrac{400(1-z^{-1})^2 + 17.5982(20(1-z^{-1})(1+z^{-1})) + 154.8506(1+z^{-1})^2}{(1+z^{-1})^2}}$$

$$= \frac{154.8506\,(1+z^{-1})^2}{400(1-z^{-1})^2 + 17.5982\,(20(1-z^{-2})) + 154.8506(1+z^{-2}+2z^{-1})}$$

$$\boxed{H(z) = \frac{1.5485 + 3.097z^{-1} + 1.05485z^{-2}}{9.0681 - 4.903z^{-1} + 2.0289z^{-2}}}$$

Problem 2:

Design a Butterworth digital IIR low pass filter using impulse invariant transformation by taking $T = 1$ sec, with Pass band attenation $\alpha_p = 3.01\,dB$ and Stop band attenation $\alpha_s = 13.97\,dB$, Pass band frequency $\omega_p = 0.3\pi$ and Stop band frequency $\omega_s = 0.75\pi$

Solution:

$$A_p = 10^{\frac{-\alpha_p}{20}} = 10^{\frac{-3}{20}} = 0.707$$

$$A_s = 10^{\frac{-\alpha_s}{20}} = 10^{\frac{-13.97}{20}} = 0.2$$

$$\omega_p = 0.3\pi$$

$$\omega_s = 0.75\pi$$

Step 1 :

Impulse Invariant Method

Step 2 :

Calculate the ratio $\frac{\Omega_s}{\Omega_p}$

Is

$$\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p} = \frac{0.75\pi}{0.3\pi} = 2.5$$

Step 3 :

Calculate the order of the filter N

$$N_1 = \frac{\frac{1}{2}\log\frac{\left[\frac{1}{A_s^2}-1\right]}{\left[\frac{1}{A_p^2}-1\right]}}{\log\left[\frac{\Omega_s}{\Omega_p}\right]} = \frac{\frac{1}{2}\log\frac{\left[\frac{1}{0.2^2}-1\right]}{\left[\frac{1}{0.707^2}-1\right]}}{\log(2.5)}$$

$$N_1 = 1.07339$$

$$\boxed{N = 2}$$

Step 4 : Calculate the analog Transfer function

Ω_c

$$\Omega_c = \frac{\frac{\omega_s}{T}}{\left[\frac{1}{A_s^2}-1\right]^{1/2N}} = \frac{\frac{0.75\pi}{1}}{\left[\frac{1}{0.2^2}-1\right]^{1/2\times 2}}$$

$$= \frac{0.75\pi}{(24)^{1/4}}$$

$\Omega_c = 1.0645$ rad/sec

Step 5 :-

Calculate the analog Transfer function Ha

Ha(s)

$$Ha(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + \Omega_c b_k s + \Omega_c^2} = \prod_{k=1}^{N/2} \frac{(1.0645)^2}{s^2 + 1.0645 b_k s + (1.0645)^2}$$

$$Ha(s) = \frac{1.1332}{s^2 + 1.0645 b_1 s + 1.1332}$$

$$b_k = 2 \sin\left(\frac{(2k-1)\pi}{2N}\right)$$

$$b_1 = 2 \sin \frac{\pi}{4} = 1.4142$$

$$Ha(s) = \frac{1.1332}{s^2 + 1.5054 s + 1.1332}$$

Step 6 :-

Convert Ha(s) to H(z) by using Impulse Invariant method

$$\frac{A_i}{S - P_i} \longrightarrow \frac{A_i}{1 - e^{P_i T} z^{-1}}$$

$$\frac{1.1382}{(S-(-0.7527+0.7527j))(S-(-0.7527-0.7527j))} = \frac{A}{(S-(-0.7527+0.7527j))} + \frac{B}{(S-(-0.7527-0.7527j))}$$

$$1.1382 = A(S-(-0.7527-0.7527j)) + B(S-(-0.7527+0.7527j))$$

$$S = -0.7527 - 0.7527j$$

$$1.1382 = A(0) + B(\cancel{-0.7527} - 0.7527j + \cancel{0.7527} - 0.7527j).$$

$$1.1382 = B(-1.5054j).$$

$$B = 0.7527j.$$

$$\therefore A = B^* = -0.7527j$$

$$Ha(s) = \frac{-0.7527j}{S-(-0.7527+0.7527j)} + \frac{0.7527j}{S-(-0.7527-0.7527j)}.$$

$$H(z) = \frac{-0.7527j}{1-e^{(-0.7527+0.7527j)(1)}z^{-1}} + \frac{0.7527j}{1-e^{-0.7527-0.7527j}z^{-1}}$$

Problem 3 :

Design a chebyshev digital IIR lowpass filter using impulse invariant transformation by taking $T = 1$ sec. to Satisfy the following Specifications.

$$0.9 \le |H(e^{j\omega})| \le 1.0 \qquad 0 \le \omega \le 0.25\pi$$

Solution :

$$|H(e^{j\omega})| \le 0.24 \qquad 0.5\pi \le \omega \le \pi$$

$A_p = 0.9 \qquad \omega_p = 0.25\pi$

$A_s = 0.24 \qquad \omega_s = 0.5\pi.$

Step 1 :

Impulse invariant transformation.

Step 2 :

Calculate the ratio $\frac{\Omega_s}{\Omega_p}$

$$\frac{\Omega_s}{\Omega_p} = \frac{\omega_s}{\omega_p} = \frac{0.5\pi}{0.25\pi} = 2$$

$$\frac{\Omega_s}{\Omega_p} = 2$$

Step 2a : Calculate the attenuation Constant ε

$$\varepsilon = \left[\frac{1}{A_p^2} - 1\right]^{1/2} = \left[\frac{1}{0.9^2} - 1\right]^{1/2}$$

$$\boxed{\varepsilon = 0.4843}$$

Step 3 :

Calculate the order of filter N

$$N_1 = \frac{\cosh^{-1}\left[\frac{1}{\epsilon}\left[\frac{1}{A_s^2}-1\right]^{1/2}\right]}{\cosh^{-1}\left[\frac{\Omega_s}{\Omega_p}\right]}$$

$$= \frac{\cosh^{-1}\left[\frac{1}{0.4843}\left[\frac{1}{0.24^2}-1\right]^{1/2}\right]}{\cosh^{-1}[2]}$$

$$N_1 = 2.1077$$

$$\boxed{N = 3}$$

Step 4 :

Calculate the analog Cut-off frequency Ω_c

$$\Omega_c = \frac{\omega_p}{T} = \frac{0.25\pi}{1} = 0.7854$$

$\Omega_c = 0.7854$ rads/sec

Step 5 :

Calculate the analog Transfer function $H_a(s)$

$$\frac{N-1}{2} \leftarrow \frac{3-1}{2} = 1$$

$$H_a(s) = \frac{B_0 \Omega_c}{s + \Omega_c c_0} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k \Omega_c^2}{s^2 + \Omega_c b_k s + \Omega_c^2 c_k}$$

$$H_a(s) = \frac{B_0 \Omega_c}{s + \Omega_c c_0} \times \frac{B_1 \Omega_c^2}{s^2 + \Omega_c b_1 s + \Omega_c^2 c_1}$$

$$H_a(s) = \frac{B_0 B_1 \; 0.7854 \times (0.7854)^2}{(s + 0.7854 c_0)(s^2 + 0.7854 b_1 s + (0.7854)^2 c_1)}$$

$$H_a(s) = \frac{B_0 B_1 \; 0.4844}{(s + 0.7854 c_0)(s^2 + 0.7854 b_1 s + 0.61685 c_1)}$$

$$c_0 = y_N = \frac{1}{2}\left[\left[\left[\frac{1}{\epsilon^2}+1\right]^{1/2} + \frac{1}{\epsilon}\right]^{\frac{1}{N}} - \left[\left[\frac{1}{\epsilon^2}+1\right]^{1/2} + \frac{1}{\epsilon}\right]^{-\frac{1}{N}}\right]$$

$$= \frac{1}{2}\left[\left[\left[\frac{1}{0.4843^2}+1\right]^{1/2} + \frac{1}{0.4843}\right]^{\frac{1}{3}} - \left[\left[\frac{1}{0.4843^2}+1\right]^{1/2} + \frac{1}{0.4843}\right]^{-1/3}\right]$$

$$= \frac{1}{2}\left[(0.22942)^{1/3} - (0.22942)^{-1/3}\right]$$

$$c_0 = \frac{1}{2}\left[1.6335 - 0.6122\right]$$

$$\boxed{c_0 = y_N = 0.5107}$$

$$b_K = 2 y_N \sin\frac{(2K-1)\pi}{2N}$$

$$b_1 = 2 \times 0.5107 \sin\frac{\pi}{6} = 0.5107$$

$$C_K = Y_N^2 + \cos^2\left[\frac{(2K-1)\pi}{2N}\right]$$

$$C_1 = (0.5107)^2 + \cos^2\left(\frac{\pi}{6}\right)$$

$$C_1 = 1.0108$$

$$Ha(s) = \frac{B_0 B_1 \, 0.4844}{(S + 0.7854 \times 0.5107)(S^2 + 0.7854 \times 0.5107 S + 0.61685 \times 1.0108)}$$

$$Ha(s) = \frac{B_0 B_1 \, 0.4844}{(S + 0.4011)(S^2 + 0.4011 S + 0.6235)}.$$

$$Ha(0) = 1$$

$$Ha(0) = 1 = \frac{B_0 B_1 \, 0.4844}{(0.4011)(0.6235)}$$

$$B_0 B = 0.5162$$

$$Ha(s) = \frac{0.2501}{(S + 0.4011)(S^2 + 0.4011 S + 0.6235)}$$

$$H_a(s) = \frac{0.2501}{(s+0.4011)(s-(-0.2+0.7637j))(s-(-0.2-0.7637j))}$$

$$H_a(s) = \frac{A}{s+0.4011} + \frac{B}{s-(-0.2+0.7637j)} + \frac{C}{s-(-0.2-0.7637j)}$$

$$0.2501 = A(s-(-0.2+0.7637j))(s-(-0.2-0.7637j))$$
$$+B(s+0.401)(s-(-0.2-0.7637j))$$
$$+C(s+0.401)(s-(-0.2+0.7637j))$$

$s = -0.401$

$$0.2501 = A(-0.401+0.2-0.7637j)(-0.401+0.2+0.7637j) + B(0) + C(0)$$
$$= A(-0.201-0.7637j)(-0.201+0.7637j)$$
$$= A\ 0.62363$$

$A = 0.4$

$s = -0.2+0.7637j$

$$0.2501 = A(0) + B(-0.2+0.7637j+0.401)(-\cancel{0.2}+0.7637j+\cancel{0.2}+0.7637j)$$

$$\frac{0.2501}{-1.167+0.3j} = B$$

$B = -0.2-0.053j \quad \therefore C = -0.2+0.053j$

$$H_a(s) = \frac{0.4}{s+0.401} + \frac{-0.2-0.053j}{s-(-0.2+0.7637j)} + \frac{-0.2+0.053j}{s-(-0.2-0.7637j)}$$

$$H(z) = \frac{0.4}{1-e^{-0.401}z^{-1}} + \frac{-0.2-0.053j}{1-e^{-0.2+0.7637j}z^{-1}} + \frac{-0.2+0.053j}{1-e^{-0.2-0.763j}z^{-1}}$$

Problem 4 :-

Design a chebyshev digital IIR lowpass filter using bilinear transformation by taking T = 1 sec to satisfy the following Specifications

$$0.8 \le |H(e^{j\omega})| \le 1.0 \quad \text{for } 0 \le \omega \le 0.2\pi$$

$$|H(e^{j\omega})| \le 0.2 \quad \text{for } 0.32\pi \le \omega \le \pi$$

Solution :-

$A_p = 0.8 \qquad \omega_p = 0.2\pi$

$A_s = 0.2 \qquad \omega_s = 0.32\pi$

Step 1 :-

Bilinear Transformation

Step 2 :- calculate the ratio $\frac{\Omega_s}{\Omega_p}$

$$\frac{\Omega_s}{\Omega_p} = \frac{\tan\left(\frac{\omega_s}{2}\right)}{\tan\left(\frac{\omega_p}{2}\right)}$$

$$= \frac{\tan\left(\frac{0.32\pi}{2}\right)}{\tan\left(\frac{0.2\pi}{2}\right)} = 1.6921$$

$$\frac{\Omega_s}{\Omega_p} = 1.6921$$

Step 2a :- Calculate ε

$$\varepsilon = \left[\frac{1}{A_p^2} - 1\right]^{1/2} = \left[\frac{1}{0.8^2} - 1\right]^{1/2}$$

$$\boxed{\varepsilon = 0.75}$$

Step 3 :-

Calculate the order of filter N

$$N_1 = \frac{\cosh^{-1}\left[\frac{1}{\epsilon}\left[\frac{1}{A_s^2}-1\right]^{1/2}\right]}{\cosh^{-1}\left[\frac{\Omega_s}{\Omega_p}\right]}$$

$$= \frac{\cosh^{-1}\left[\frac{1}{0.75}\left[\frac{1}{0.2^2}-1\right]^{1/2}\right]}{\cosh^{-1}\left[1.6921\right]}$$

$$N_1 = 2.2944$$

$$N = 3$$

Step 4 :-

Calculate the analog ~~transfer fun~~

Cut-off frequency Ω_c

$$\Omega_c = \frac{\frac{2}{T}\tan\frac{\omega p}{2}}{\cancel{\frac{1}{A}}}$$

$$= \frac{2}{1}\tan\left(\frac{0.2\pi}{2}\right)$$

$$\Omega_c = 0.6498 \text{ rad/sec}$$

Step 5 :

Calculate the Analog Transfer function $H_a(s)$

$$\frac{N-1}{2} \Leftarrow \frac{3-1}{2} = 1.$$

$$H_a(s) = \frac{B_0 \Omega_c}{S+\Omega_c C_0} \prod_{K=1}^{\frac{N-1}{2}} \frac{B_K \Omega_c^2}{S^2 + b_K \Omega_c S + \Omega_c^2 C_K}.$$

$$H_a(s) = \frac{B_0 \cdot (0.6498)}{S + 0.6498\, C_0} \times \frac{B_1\, (0.6498)^2}{S^2 + b_1\, 0.6498\, S + (0.6498)^2 C_1}$$

$$H_a(s) = \frac{B_0 B\;\; 0.27437}{(S+0.6498 C_0)\;(S^2 + b_1 0.6498 S + 0.4222 C_1)}$$

$$C_0 = Y_N = \frac{1}{2}\left[\left[\left[\frac{1}{\epsilon^2}+1\right]^{1/2} + \frac{1}{\epsilon}\right]^{\frac{1}{3}} - \left[\left[\frac{1}{\epsilon^2}+1\right]^{1/2} + \frac{1}{\epsilon}\right]^{-1/3}\right]$$

$$= \frac{1}{2}\left[\left[\left[\frac{1}{0.75^2}+1\right]^{1/2} + \frac{1}{0.75}\right]^{1/3} - \left[\left[\frac{1}{0.75^2}+1\right]^{1/2} + \frac{1}{0.75}\right]^{-1/3}\right]$$

$$= \frac{1}{2}\left[\left[1.4422 - 0.6934\right]\right]$$

$$C_0 = Y_N = 0.3744$$

$$b_K = 2\, Y_N \sin\left[\frac{(2K-1)\pi}{2N}\right]$$

$$b_1 = 2 \times 0.3744 \sin\left[\frac{\pi}{6}\right]$$

$$b_1 = 0.3744$$

$$C_K = Y_N^2 + \cos^2\left[\frac{(2K-1)\pi}{2N}\right]$$

$$C_1 = (0.3744)^2 + \cos^2\left(\frac{\pi}{6}\right)$$

$$C_1 = 0.8902$$

$$Ha(s) = \frac{B_0 B_1 \; 0.27437}{(S+0.6498\times0.3744)(S^2+0.3744\times0.6498S+0.4222\times0.8902)}$$

$$Ha(s) = \frac{B_0 B_1 \; 0.27437}{(S+0.24328)(S^2+0.24328S+0.3758)}$$

$$Ha(0) = 1$$

$$1 = \frac{B_0 B_1 \; 0.27437}{(0.24328)(0.3758)}$$

$$B_0 B = 0.3333$$

$$Ha(s) = \frac{0.0914}{(S+0.24328)(S^2+0.24328S+0.3758)}$$

Step 6 :- Convert $Ha(s)$ to $H(z)$

$$S = \frac{2}{T} \frac{1-z^{-1}}{1+z^{-1}}$$

$$H_a(z) = \frac{0.0914}{\left(\frac{2(1-z^{-1})}{1+z^{-1}} + 0.24328\right)\left(\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right)^2 + 0.24328\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right) + 0.3758\right)}$$

$$= \frac{0.0914}{(2-2z^{-1}+0.24328+0.24328z^{-1})\;\dfrac{(4(1-z^{-1})^2+0.24328(2-2z^{-1})(1+z^{-1})+0.3758(1+z^{-1})^2}{(1+z^{-1})^3}}$$

$$H_a(z) = \frac{0.0914(1+z^{-1})^3}{1-2.2821z^{-1}+1.9589z^{-2}-0.624z^{-3}}$$

$$H(z) = \frac{0.0083+0.0251z^{-1}+0.0251z^{-2}+0.0083z^{-3}}{1-2.2821z^{-1}+1.9589z^{-2}-0.624z^{-3}}$$

Problem 5 :

For the analog transfer function, $H(s) = \frac{2}{s^2+3s+2}$, determine $H(z)$ using impulse invariant method if (a) $T = 1$ sec and (b) $T = 0.1$ sec

$$H_a(s) = \frac{2}{s^2+3s+2} = \frac{2}{(s+1)(s+2)}$$

$$\frac{2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\frac{2}{\cancel{(s+1)(s+2)}} = \frac{A(s+2)+B(s+1)}{\cancel{(s+1)(s+2)}}.$$

$s = -2$

$2 = B(-1) \qquad B = -2$

$s = -1$

$2 = A(1) + B(0)$

$A = 2$

$$H_a(s) = \frac{2}{s+1} - \frac{2}{s+2}.$$

$$\frac{A_i}{s-P_i} \longrightarrow \frac{A_i}{1-e^{P_iT}z^{-1}}$$

(a) T = 1 sec

$$H(z) = \frac{2}{1-e^{-1(1)}z^{-1}} - \frac{2}{1-e^{-2(1)}z^{-1}}$$

$$H(z) = \frac{2}{1-e^{-1}z^{-1}} - \frac{2}{1-e^{-2}z^{-1}}$$

$$H(z) = \frac{2}{1-0.3678z^{-1}} - \frac{2}{1-0.1353z^{-1}}$$

(b) T = 0.5 se.

$$H(z) = \frac{2}{1-e^{-1(0.5)}z^{-1}} - \frac{2}{1-e^{-2(0.5)}z^{-1}}$$

$$= \frac{2}{1-e^{-0.5}z^{-1}} - \frac{2}{1-e^{-1}z^{-1}}$$

$$H(z) = \frac{2}{1-0.6065z^{-1}} - \frac{2}{1-0.3678z^{-1}}$$

Problem 6 :

Obtain the direct form-I, direct form II, Cascade and Parallel form realization of the LTI System governed by the equation

$$y(n) + \frac{3}{8}y(n-1) - \frac{3}{32}y(n-2) - \frac{1}{64}y(n-3) = x(n) + 3x(n-1) + 2x(n-2)$$

Solution :

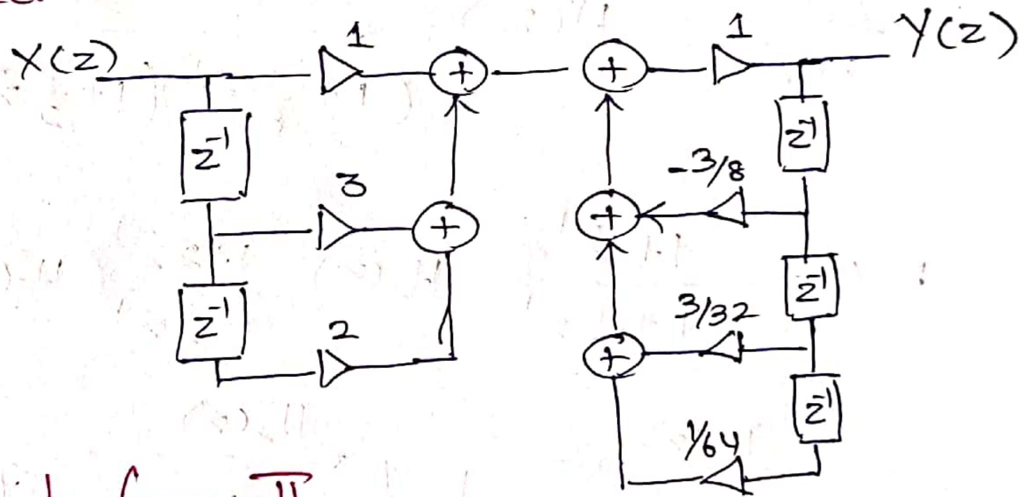
$$Y(z) + \frac{3}{8} Y(z) z^{-1} - \frac{3}{32} Y(z) z^{-2} - \frac{1}{64} Y(z) z^{-3}$$

$$= X(z) + 3X(z) z^{-1} + 2X(z) z^{-2} \quad (1)$$

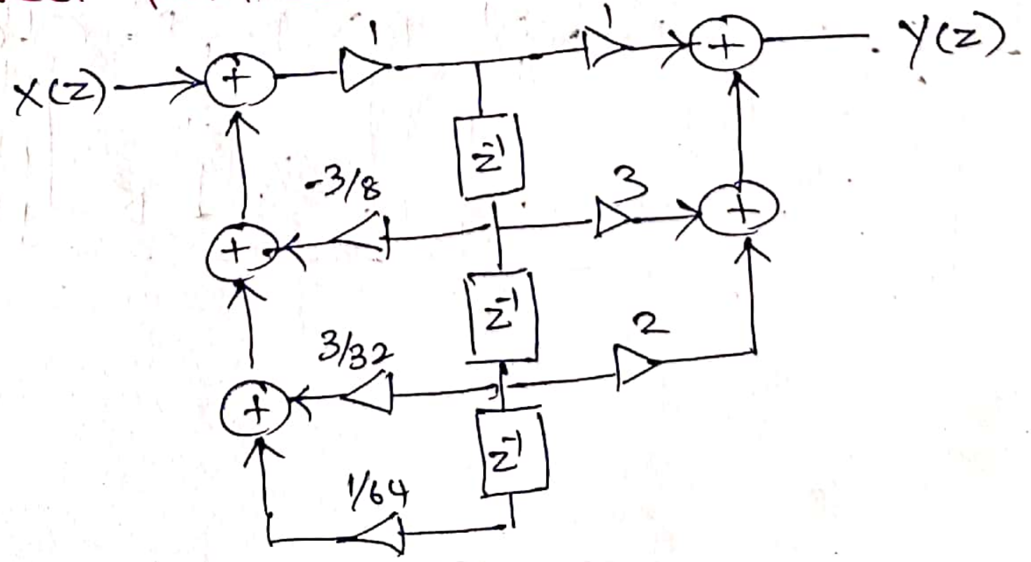
$$Y(z) = X(z) + 3X(z) z^{-1} + 2X(z) z^{-2}$$

$$- \frac{3}{8} Y(z) z^{-1} + \frac{3}{32} Y(z) z^{-2} + \frac{1}{64} Y(z) z^{-3}$$

Direct form I



Direct form II



Cascade Form :

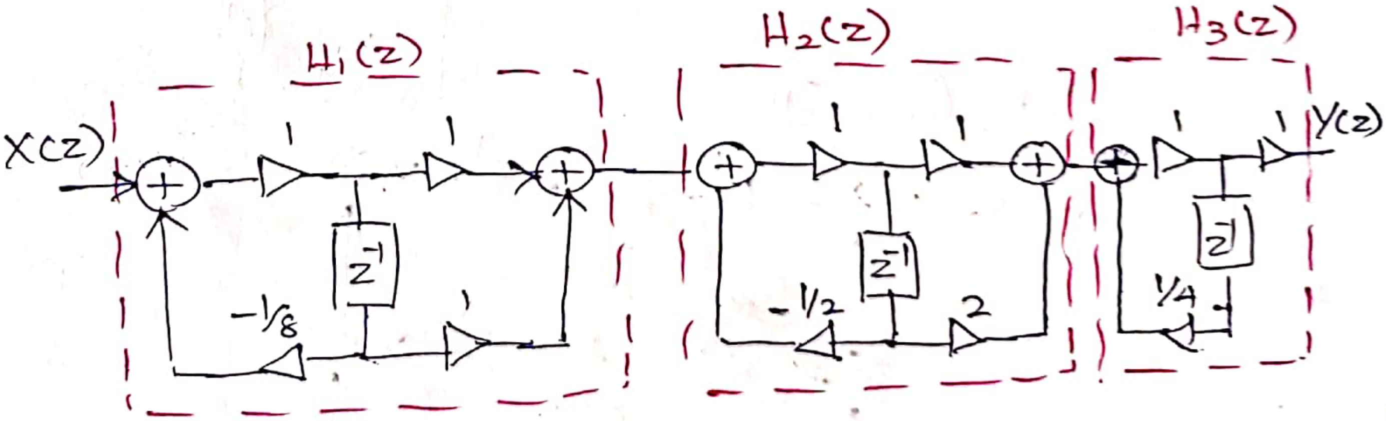
from equ (1)

$$Y(z)\left[1+\frac{3}{8}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}\right]=X(z)\left[1+3z^{-1}+2z^{-2}\right]$$

$$\frac{Y(z)}{X(z)}=\frac{1+3z^{-1}+2z^{-2}}{1+\frac{3}{8}z^{-1}-\frac{3}{32}z^{-2}-\frac{1}{64}z^{-3}}$$

$$\frac{Y(z)}{X(z)}=H(z)=\frac{(1+z^{-1})(1+2z^{-1})}{(1+\frac{1}{8}z^{-1})(1+\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} \quad -(2)$$

$$H_1(z)=\frac{1+z^{-1}}{1+\frac{1}{8}z^{-1}} \qquad H_2(z)=\frac{1+2z^{-1}}{1+\frac{1}{2}z^{-1}} \qquad H_3(z)=\frac{1}{1-\frac{1}{4}z^{-1}}$$



Parallel Form :

Consider the equ (2)

$$H(z) = \frac{(1+z^{-1})(1+2z^{-1})}{\left(1+\frac{1}{8}z^{-1}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

$$H(z) = \frac{A}{1+\frac{1}{8}z^{-1}} + \frac{B}{1+\frac{1}{2}z^{-1}} + \frac{C}{1-\frac{1}{4}z^{-1}}$$

$$A = -\frac{35}{3}$$

$$B = \frac{8}{3}$$

$$C = 10$$

$$H(z) = \underbrace{\frac{-\frac{35}{3}}{1+\frac{1}{8}z^{-1}}}_{H_1(z)} + \underbrace{\frac{\frac{8}{3}}{1+\frac{1}{2}z^{-1}}}_{H_2(z)} + \underbrace{\frac{10}{1-\frac{1}{4}z^{-1}}}_{H_3(z)}$$