Unit - III

Finite Impulse Response Filters

Design of FIR filters - Symmetric and Anti symmetric FIR filters - design of Linear Phase FIR filters Using Fourier Series method - FIR filter design using windows (Rectangular, Hamming and Hanning window), Frequency Sampling method. FIR filter Structure - Linear phase structure, direct form Realiz-ations.

FIR Filters :-

The filters designed by using finite samples of impulse response are called FIR (Finite Impulse Response) filters.

Various Steps in Designing FIR Filters :-

(i) Choose an ideal (desired) frequency response, $H_d(e^{jw})$

(ii) Take inverse Fourier transform of $H_d(e^{jw})$ to get $h_d(n)$

(iii) Infinite duration $h_d(n)$ ~~is~~ converted into finite duration $h(n)$.

(iv) Take z-transform of $h(n)$ to get $H(z)$ and realize the filter

Frequency Response of Linear phase FIR filters :-

Depending on the value of N (odd (or) even) and the type of symmetry of the filter impulse response sequence (Symmetric (or) antisymmetric) there are

Six possible types of linear phase FIR filters.

The following are the six cases of impulse response for linear phase FIR filter.

Case (i) : Symmetric impulse response and N is odd with centre of symmetry at $(N-1)/2$

Case (ii) : Symmetric impulse response and N is even with centre of symmetry at $(N-1)/2$

Case (iii) : Antisymmetric impulse response and N is odd with centre of antisymmetry at $(N-1)/2$

Case (iv) : Antisymmetric impulse response and N is even with centre of antisymmetry at $(N-1)/2$

Case (v) : Symmetric impulse response and N is odd with centre of symmetry at $n=0$

Case (vi) : Antisymmetric impulse response and N is odd with centre of antisymmetry at $n=0$

Case(i) Frequency response of linear phase FIR filter when impulse response is Symmetric and N is Odd with Centre of Symmetry at $(N-1)/2$

The frequency response of Linear phase FIR filter when impulse response is Symmetric and N is odd with Centre of Symmetric at $(N-1)/2$ is given by

$$H(e^{j\omega}) = \left[h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \cos \omega n \right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

Let $H(e^{j\omega}) = A(\omega)\, e^{j\theta(\omega)}$

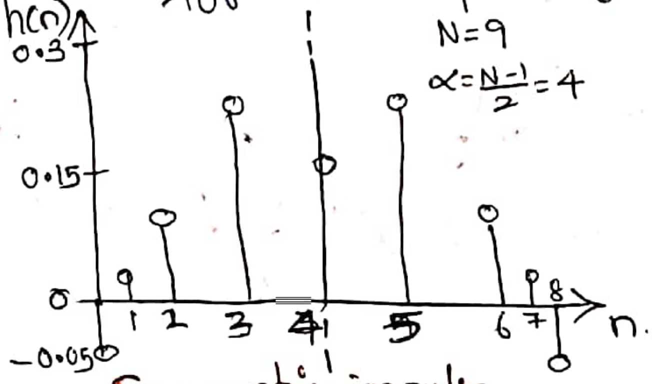
where

$$A(\omega) = h\left(\frac{N-1}{2}\right) + \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \cos \omega n$$

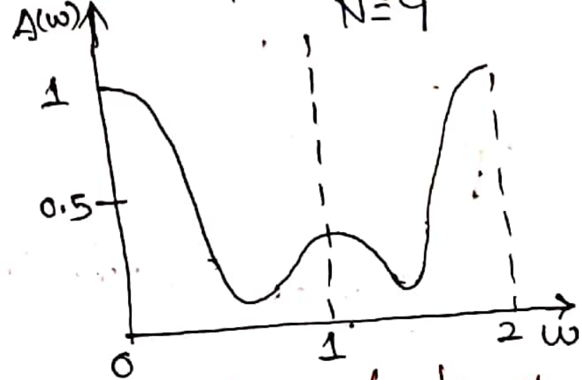
Amplitude function

$$\theta(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \quad \text{where } \alpha = \frac{N-1}{2}$$

for example Symmetric impulse response N=9



Symmetric impulse response N=9



Amplitude function of $H(e^{j\omega})$

Case (ii): Frequency response of linear phase FIR filter when impulse response is Symmetric and N is even with Centre of Symmetry at $(N-1)/2$

The frequency response of linear phase FIR filter when impulse response is Symmetric and N is even with Centre of Symmetry at $(N-1)/2$ is given by

$$H(e^{j\omega}) = \left[\sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\cos\left(\omega\left(n-\frac{1}{2}\right)\right)\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

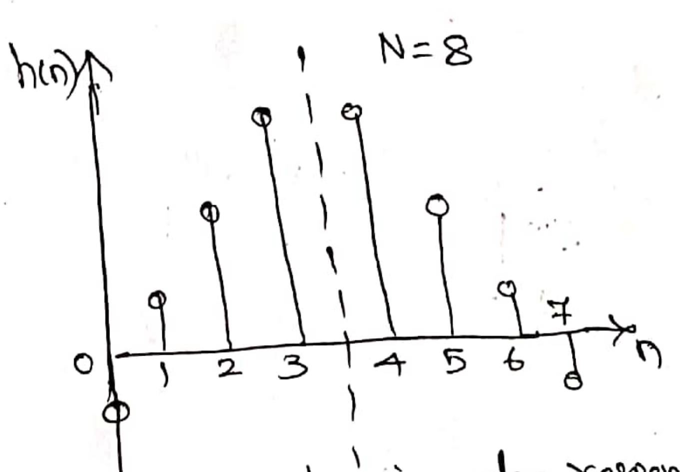
Let $H(e^{j\omega}) = A(\omega)\, e^{j\theta(\omega)}$

Amplitude function

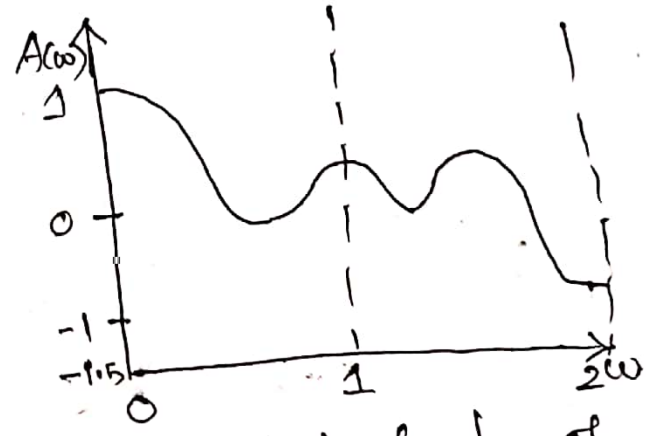
$$A(\omega) = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right)\cos\left(\omega\left(n-\frac{1}{2}\right)\right)$$

Phase function

$$\theta(\omega) = -\omega\left(\frac{N-1}{2}\right) = -\omega\alpha \quad \text{where } \alpha = \frac{N-1}{2}$$



Symmetric impulse response N=8



Amplitude function of $H(e^{j\omega})$

Case (iii): Frequency response of linear phase FIR filter when impulse response is anti symmetric and N is odd with centre of anti symmetry at $(N-1)/2$.

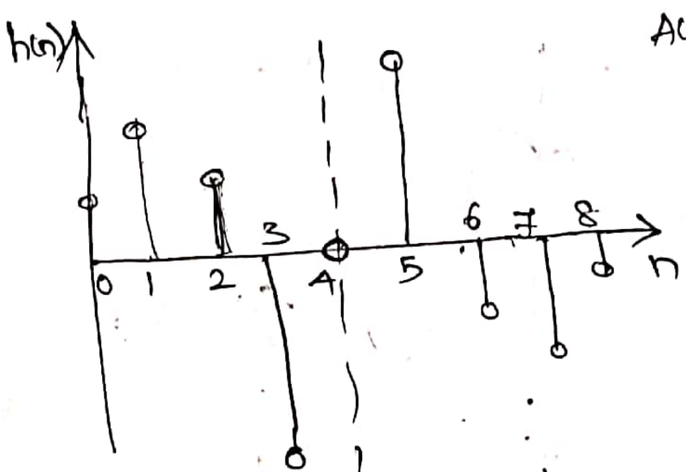
The frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of anti symmetric at $(N-1)/2$ is given by

$$H(e^{j\omega}) = \left[\sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \sin \omega n\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

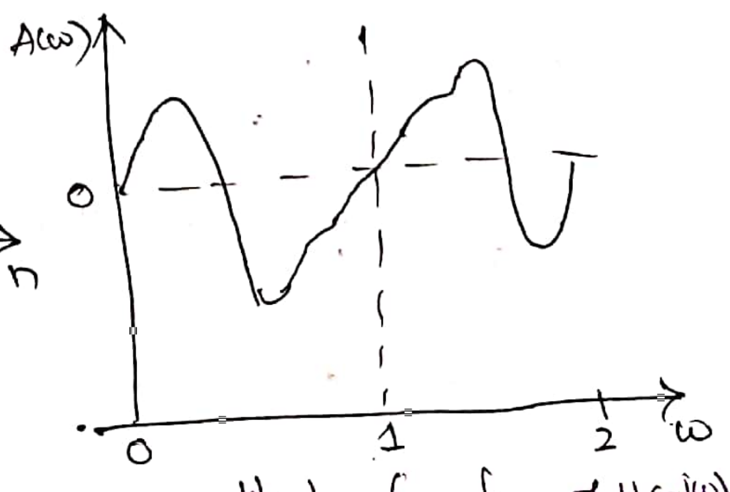
$$H(e^{j\omega}) = A(\omega)\, e^{j\theta\omega}$$

$$A(\omega) = \sum_{n=1}^{\frac{N-1}{2}} 2h\left(\frac{N-1}{2}-n\right) \sin \omega n$$ ← Amplitude function

$$\theta(\omega) = \frac{\pi}{2} - \frac{\omega(N-1)}{2} = \beta - \alpha\omega$$ ← phase function.



Antisymmetric impulse response N=9



Amplitude function of $H(e^{j\omega})$

Case (iv) Frequency response of linear phase FIR filter when impulse response is antisymmetric and N is even with centre of antisymmetry at $(N-1)/2$

The frequency response of linear phase FIR filter when impulse response is antisymmetric and N is even with centre of antisymmetry at $(N-1)/2$ is given by,

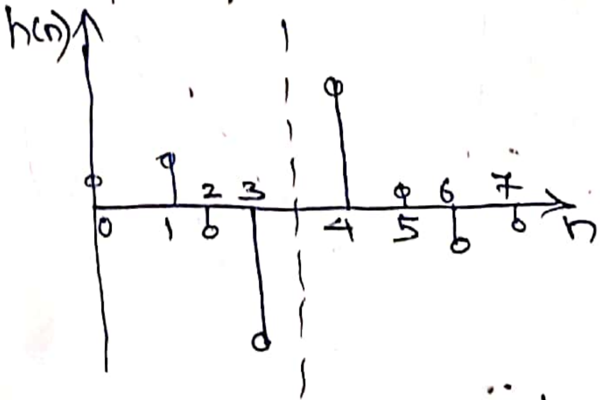
$$H(e^{j\omega}) = \left[\sum_{n=1}^{\frac{N}{2}} 2h\left(\frac{N}{2}-n\right) \sin\left(\omega(n-\tfrac{1}{2})\right)\right] e^{j\left(\frac{\pi}{2}-\frac{\omega(N-1)}{2}\right)}$$

Let $H(e^{j\omega}) = A(\omega)\, e^{j\theta(\omega)}$

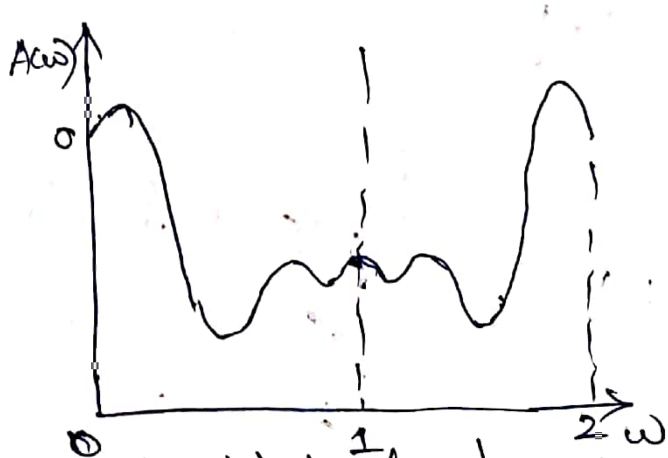
Amplitude function

$$A(\omega) = \sum_{n=1}^{N/2} 2h\left(\frac{N}{2}-n\right) \sin\left[\omega(n-\tfrac{1}{2})\right]$$

Phase function. $\theta(\omega) = \frac{\pi}{2} - \omega\left(\frac{N-1}{2}\right) = \beta - \alpha\omega$



Antisymmetric impulse response N=8



Amplitude function of $H(e^{j\omega})$

Case (v): Frequency response of linear phase FIR filter when impulse response is Symmetric and N is odd with Centre of Symmetry at n=0

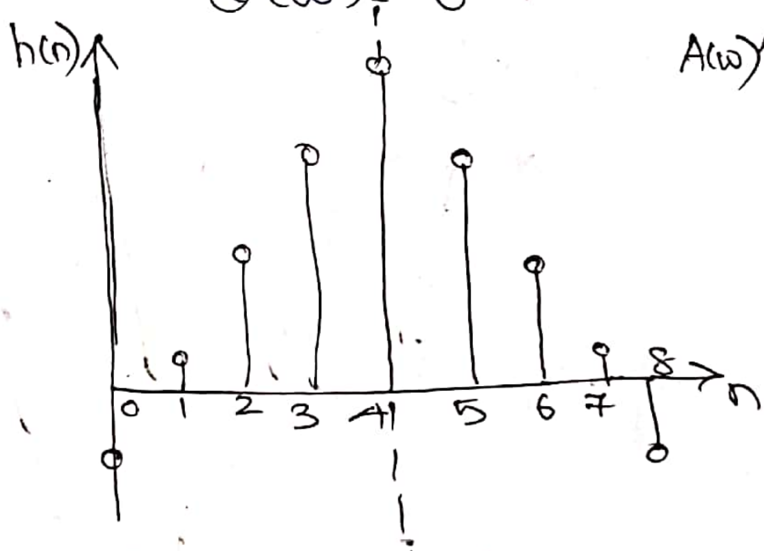
The frequency response of linear phase FIR filter when impulse response is Symmetric and N is Odd with Centre of Symmetry at n=0 is given by,

$$H(e^{j\omega}) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \cos \omega n$$

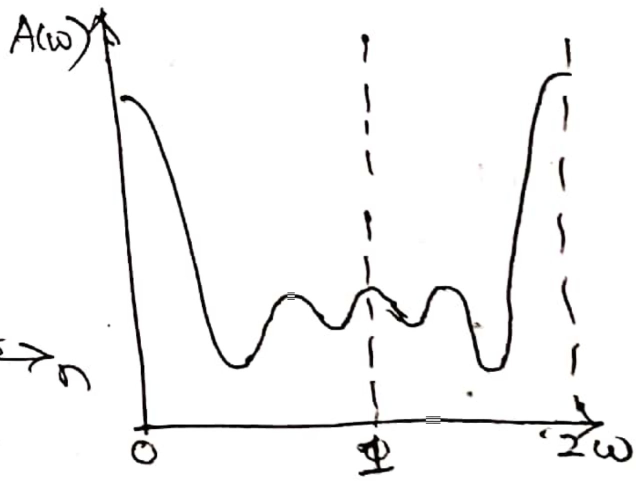
$$H(e^{j\omega}) = A(\omega) e^{j\theta(\omega)}$$

$$A(\omega) = h(0) + \sum_{n=1}^{\frac{N-1}{2}} 2h(n) \cos \omega n$$

$$\theta(\omega) = 0$$



Symmetric impulse response for N=9



Amplitude function of $H(e^{j\omega})$

Case (vi) Frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of anti symmetry at $n=0$

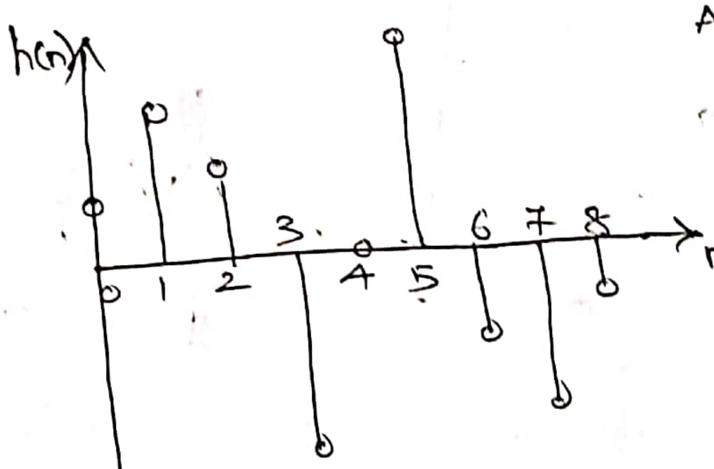
The frequency response of linear phase FIR filter when impulse response is antisymmetric and N is odd with centre of antisymmetry at $n=0$ is given by

$$H(e^{j\omega}) = \left[\sum_{n=1}^{\frac{N-1}{2}} 2h(n)\sin\omega n\right] e^{-j\frac{\pi}{2}}$$

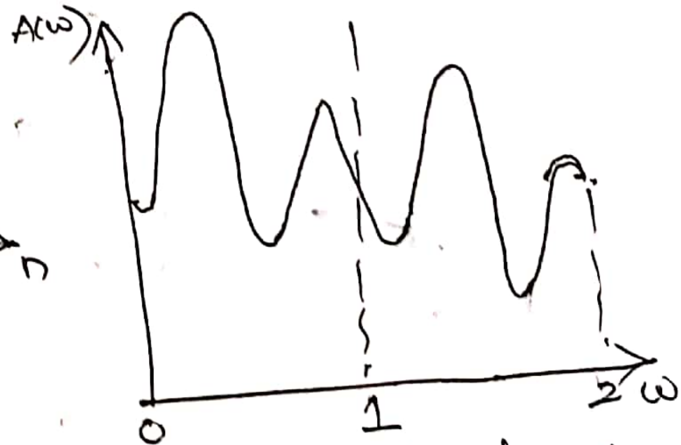
$$H(e^{j\omega}) = A(\omega)\, e^{j\theta(\omega)}$$

Amplitude function $A(\omega) = \sum_{n=1}^{N-1/2} 2h(n)\sin\omega n$

Phase function $\theta(\omega) = -\frac{\pi}{2}$



Antisymmetric impulse response $N=9$



Amplitude function of $H(e^{j\omega})$

Design Procedure for Digital FIR by using Fourier Series method

Step 1:
Choose desired frequency response $H_d(e^{j\omega})$

Step 2:
Determine the desired impulse response $h_d(n)$ by taking inverse Fourier transform.

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

Step 3:
Calculate N Samples of $h_d(n)$ for $n = -\left(\frac{N-1}{2}\right)$ to $\left(\frac{N-1}{2}\right)$ and form the impulse response $h(n)$ of FIR filter

∴ Impulse response

$$h(n) = h_d(n)\Big|_{n=-\left(\frac{N-1}{2}\right) \text{ to } +\left(\frac{N-1}{2}\right)}$$

The impulse response is Symmetric

with $n=0$ and so $h(-n) = h(n)$.

Step 4:

Take Z-transform of the impulse response to get the non causal transfer function of FIR filter $H_N(z)$

$$\therefore H_N(z) = Z\{h(n)\} = \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n)\, z^{-n}$$

Step 5:

Convert the non causal transfer function $H_N(z)$ to causal transfer function, $H(z)$ by multiplying $H_N(z)$ by $z^{-\frac{(N-1)}{2}}$

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n)\, z^{-n}.$$

Alternatively,

Transfer function

$$H(z) = z^{\frac{N-1}{2}} \left[h(0) + \sum_{n=1}^{\frac{N-1}{2}} h(n)\left[z^{n} + z^{-n}\right] \right]$$

Step 6:

Realization of Linear phase Structure of $H(z)$

FIR filter design Using windows :-

Design Procedure for FIR filter using window Technique

Step 1 :-

Choose the desired frequency response $H_d(e^{j\omega})$

LPF

$$H_d(e^{j\omega}) = \begin{cases} 0 & -\pi \le \omega \le -\omega_c \\ e^{-j\alpha\omega} & -\omega_c \le \omega \le \omega_c \\ 0 & \omega_c \le \omega \le \pi \end{cases}$$

HPF

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & -\pi \le \omega \le -\omega_c \\ 0 & -\omega_c \le \omega \le \omega_c \\ e^{-j\alpha\omega} & \omega_c \le \omega \le \pi . \end{cases}$$

BPF

$$H_d(e^{j\omega}) = \begin{cases} 0 & -\pi \le \omega \le -\omega_{c_2} \\ e^{-j\alpha\omega} & -\omega_{c_2} \le \omega \le -\omega_{c_1} \\ 0 & -\omega_{c_1} \le \omega \le \omega_{c_1} \\ e^{-j\alpha\omega} & \omega_{c_1} \le \omega \le \omega_{c_2} \\ 0 & \omega_{c_2} \le \omega \le \pi \end{cases}$$

BRF

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & -\pi \leq \omega \leq -\omega c_2 \\ 0 & -\omega c_2 \leq \omega \leq -\omega c_1 \\ e^{-j\alpha\omega} & -\omega c_1 \leq \omega \leq \omega c_1 \\ 0 & \omega c_1 \leq \omega \leq \omega c_2 \\ e^{-j\alpha\omega} & \omega c_2 \leq \omega \leq \pi. \end{cases}$$

Step 2 :-

Determine the desired impulse response $h_d(n)$ by taking inverse Fourier transform of $H_d(e^{j\omega})$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

Step 3 :-

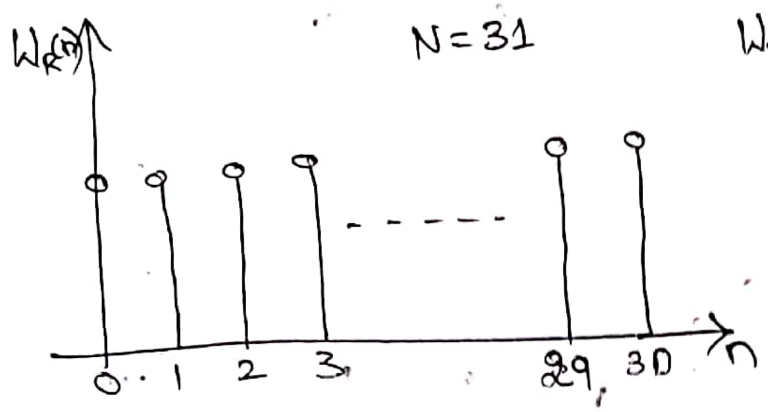
Determine the $h(n)$ finite impulse response by multiply $W(n)$ with $h_d(n)$

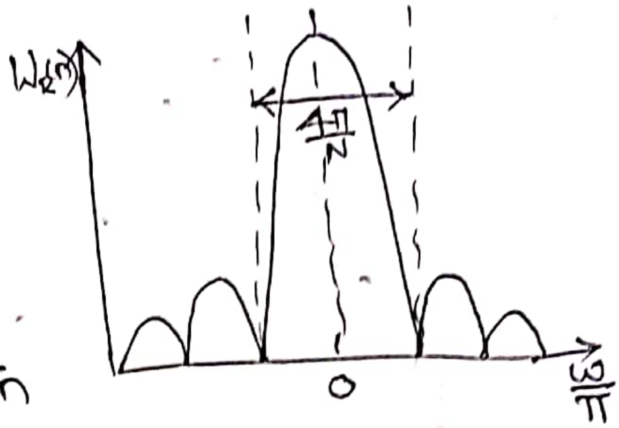
$$h(n) = h_d(n) \times W(n)$$

where $W(n)$ is window function.

Rectangular window

$$W_R(n) = \begin{cases} 1 & \text{for } n = 0 \text{ to } N-1 \\ 0 & \text{otherwise.} \end{cases}$$

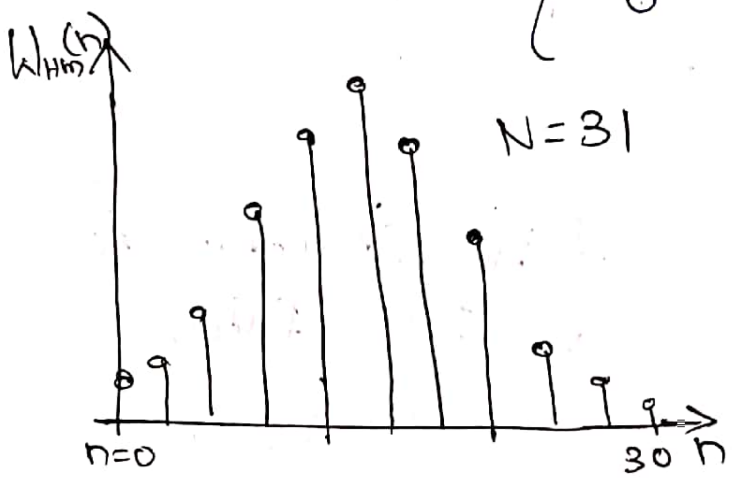


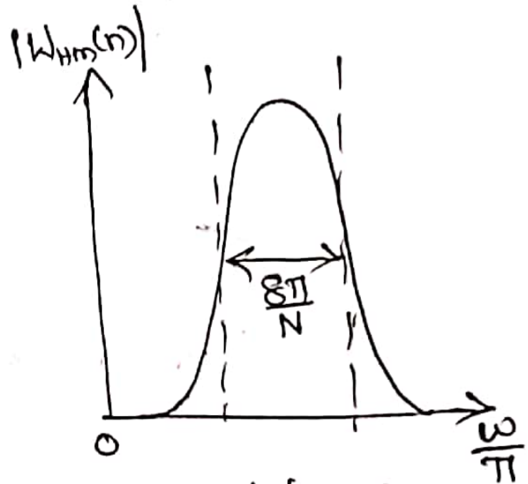


Approximate width of main lobe $\frac{4\pi}{N}$

Hamming Window :

$$W_{Hm}(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right) & \text{for } 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$

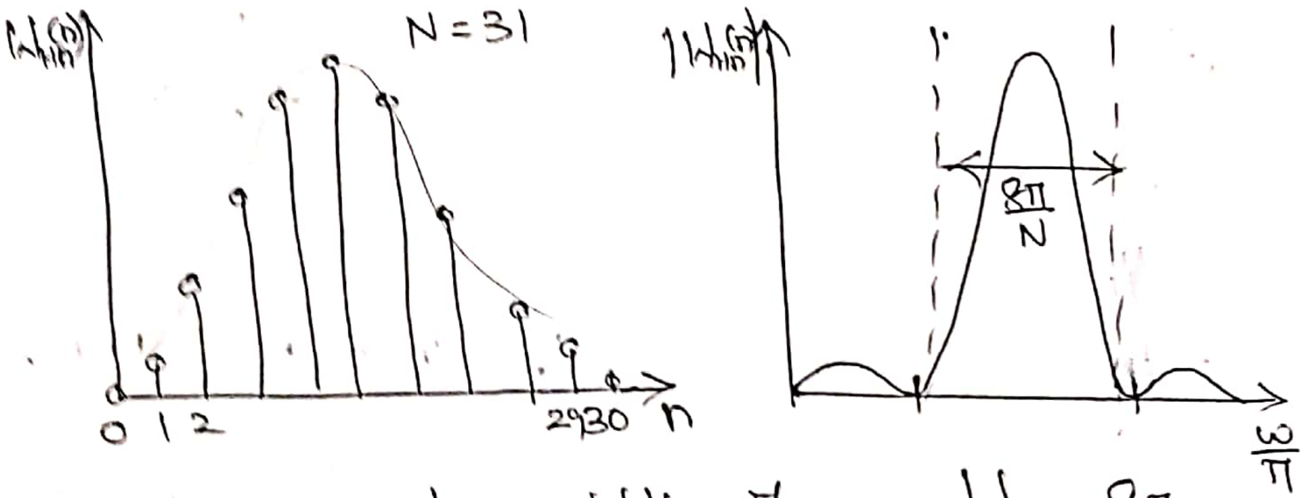




Approximate width of main lobe $\frac{8\pi}{N}$

Hanning Window :

$$W_{Hn}(n) = \begin{cases} 0.5 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) & \text{for } 0 \leq n \leq N \\ 0 & \text{otherwise} \end{cases}$$



Approximate width of main lobe $\frac{8\pi}{N}$

Step 4 :-

Take z-transform of the impulse response h(n) to get the transfer function H(z)

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}.$$

Step 5 :-

Draw a suitable structure for Realization of FIR filter.

Design of FIR filters by Frequency Sampling Technique [Procedure]

Step 1 :-

Choose the ideal (desired) frequency response $H_d(e^{j\omega})$.

Step 2 :-

Sample $H_d(e^{j\omega})$ at N-points by taking $\omega = \omega_K = \frac{2\pi K}{N}$ where $K = 0, 1, 2, 3$ ---- $(N-1)$ to generate the sequence $H(K)$

$$\therefore H(K) = H_d(e^{j\omega})\Big|_{\omega = \frac{2\pi K}{N}}$$

for $K = 0, 1, \text{------} (N-1)$

Step 3 :-

Compute the N samples of impulse response $h(n)$ using the following equation

when N is odd

Impulse response

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} Re\left[H(K)\, e^{\frac{j2\pi nK}{N}}\right]\right]$$

when N is even

Impulse response

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N}{2}-1} Re\left[H(K)\, e^{\frac{j2\pi nK}{N}}\right]\right]$$

Here $H\left(\frac{N}{2}\right) = 0$

Step 4 :-

Take Z-transform of the impulse response h(n) to get the filter transfer function, H(z)

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}.$$

Problem 1 :

Design a FIR lowpass filter with Cutoff frequency of 1 KHz and Sampling frequency of 4 KHz with 11 Samples using Fourier Series method.

Solution :

$f_c = 1KHz \quad f_s = 4KHz$

$$\omega_c = \frac{2\pi f_c}{f_s} = \frac{2\pi \times 1 \times 10^3}{4 \times 10^3} = 0.5\pi \text{ rad/sample}$$

LPF & N = 11 Samples

Step 1 :-

Desired frequency response LPF

$$H_d(e^{j\omega}) = \begin{cases} 0 & -\pi \le \omega \le -\omega_c \\ 1 & -\omega_c \le \omega \le \omega_c \\ 0 & +\omega_c \le \omega \le +\pi. \end{cases}$$

Step 2 :

Find desired impulse response

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega_c} 0 \; e^{j\omega n} d\omega + \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \right.$$

$$\left. + \int_{\omega_c}^{\pi} 0 \; e^{j\omega n} d\omega \right]$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega n}}{jn} \right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn} \right]$$

$$= \frac{1}{\pi n} \left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j} \right]$$

$$h_d(n) = \frac{\sin \omega_c n}{\pi n}.$$

when $n = 0$. Using L' Hospital rule

$$h_d(n) = \frac{\omega_c}{\pi}$$

$$\underset{\theta \to 0}{Lt} \frac{\sin A\theta}{\theta} = A.$$

$$\alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$$

Step 3 :

find $h(n)$ for $n = -5$ to 5

$h(-n) = h(n)$

$$\therefore h(-5) = h(5) = \frac{\sin(0.5\pi \times 5)}{\pi \times 5} = 0.0637$$

$$h(-4) = h(4) = \frac{\sin(0.5\pi \times 4)}{\pi \times 4} = 0$$

$$h(-3) = h(3) = \frac{\sin(0.5\pi \times 3)}{\pi \times 3} = -0.1061$$

$$h(-2) = h(2) = \frac{\sin(0.5\pi \times 2)}{\pi \times 2} = 0$$

$$h(-1) = h(1) = \frac{\sin(0.5\pi \times 1)}{\pi \times 1} = 0.3183$$

$$h(0) = \frac{\sin(0.5\pi \times 0)}{\pi (0)} = \frac{0}{0}$$

$$\therefore \frac{\omega_c}{\pi} = \frac{0.5\cancel{\pi}}{\cancel{\pi}} = 0.5$$

$$h(n) = \{0.0637, 0, -0.1061, 0, 0.3183, \underset{\uparrow}{0.5}, 0.3183, 0, -0.1061, 0, 0.0637\}$$

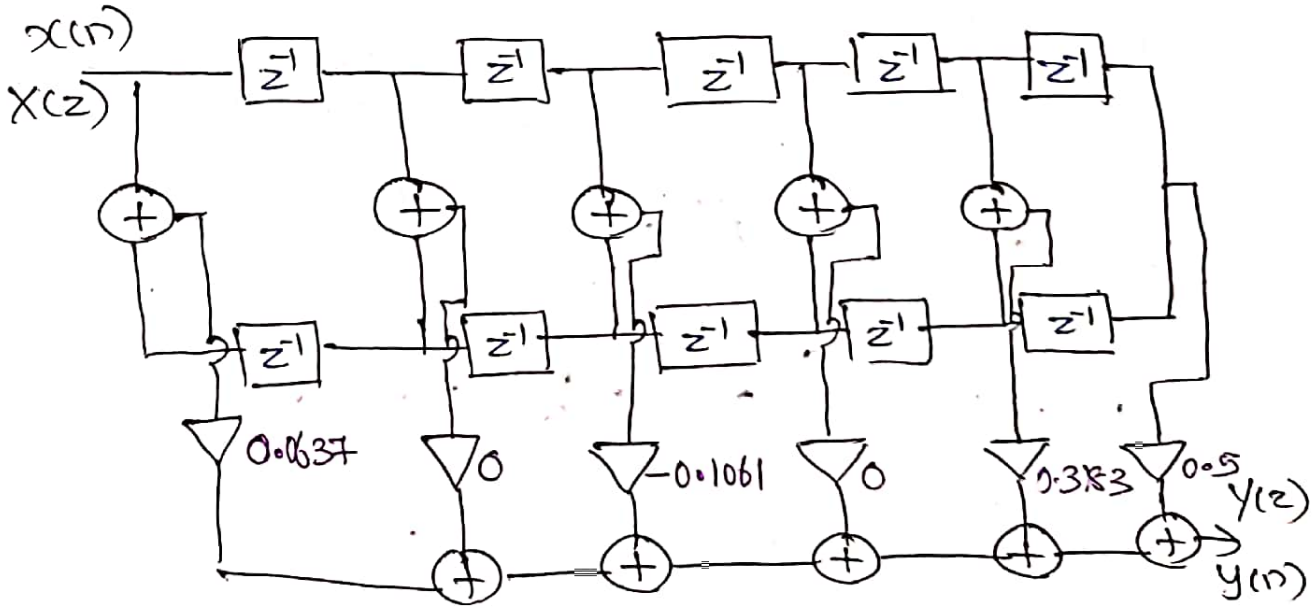
Step 4 :

$$H(z) = z^{-\left(\frac{N-1}{2}\right)} \sum_{n=-\left(\frac{N-1}{2}\right)}^{\frac{N-1}{2}} h(n)\, z^{-n}$$

$$= z^{-5}\Big[h(-5)\, z^{5} + h(-4)\, z^{4} + h(-3)\, z^{3} + h(-2)\, z^{2} + h(-1)\, z^{1} + h(0)\, z^{0} + h(1)\, z^{-1} + h(2)\, z^{-2} + h(3)\, z^{-3} + h(4)\, z^{-4} + h(5)\, z^{-5}\Big]$$

$$= 0.0637 z^{0} + 0 z^{-1} - 0.1061 z^{-2} + 0 z^{-3} + 0.3183 z^{-4} + 0.5 z^{-5} + \cancel{0 z^{-6}} - 0.1061 z^{-8} + 0 z^{-9} + 0.0637 z^{-10}$$

$$+ 0.3183 z^{-6} + 0 z^{-7}$$



Problem 2 :-

Design an FIR filter for the ideal frequency response using Hamming window with $N=7$.

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & -\frac{\pi}{8} \le \omega \le \frac{\pi}{8} \\ 0 & \frac{\pi}{8} \le \omega \le \pi \end{cases}$$

Solution :-

LPF $\qquad \omega_c = \frac{\pi}{8}$

$N=7$ $\qquad$ Hamming window.

Step 1:-

Choose the desired frequency Response of Low Pass filter.

$$H_d(e^{j\omega}) = \begin{cases} 0 & -\pi \le \omega \le -\omega_{c_1} \\ e^{-j\alpha\omega} & -\omega_{c_1} \le \omega \le \omega_c \\ 0 & \omega_c \le \omega \le \pi \end{cases}$$

Step 2 :- Determine the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \, e^{j\omega n} \, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega c} 0 \, \cancel{e^{j\omega n}} \, d\omega + \int_{-\omega c}^{\omega c} e^{-j\alpha\omega} \, e^{j\omega n} \, d\omega \right.$$

$$\left. + \int_{\omega c}^{\pi} 0 \, \cancel{e^{j\omega n}} \, d\omega \right]$$

$$= \frac{1}{2\pi} \int_{-\omega c}^{\omega c} e^{-j\omega(\alpha - n)} \, d\omega$$

$$= \frac{1}{2\pi} \left[\frac{e^{-j\omega(\alpha-n)}}{-j(\alpha-n)} \right] = \frac{1}{\pi(\alpha-n)} \left[\frac{e^{-j\omega_c(\alpha-n)} - e^{j\omega_c(\alpha-n)}}{2j} \right]$$

$$h_d(n) = \frac{\sin(\omega c(\alpha - n))}{[\pi(\alpha - n)]} \qquad n \neq \alpha$$

$$h_d(n) = \frac{\omega c}{\pi} \qquad n = \alpha$$ using L' Hosp rule.

Step 3 ÷ Find the finite Impulse response

$$h(n) = h_d(n)\, W_{Hm}(n)$$

$$h(n) = \left[\frac{\sin[\omega_c(\alpha-n)]}{[\pi(\alpha-n)]}\right] \times \left[0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)\right]$$

$\pi/8$ $\quad \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$

$$h(n) = \frac{\left[\sin[0.3927(3-n)]\right]\left[0.54 - 0.46\cos\left(\frac{2\pi n}{\cancel{6}3}\right)\right]}{[\pi(3-n)]}$$

$$h(n) = \frac{\left[\sin[0.3927(3-n)]\right]\left[0.54 - 0.46\cos\left(\frac{\pi n}{3}\right)\right]}{[\pi(3-n)]}$$

$n = 0$ to $N-1 = 0$ to 6

$h(0) = 0.007842$

$h(1) = 0.03488$

$h(2) = 0.09379$

$h(3) = \frac{\omega_c}{\pi} = \frac{\pi/8}{\cancel{\pi}} = 0.125$

$h(4) = 0.09379$

$h(5) = 0.03488$

$h(6) = 0.007842$ 22

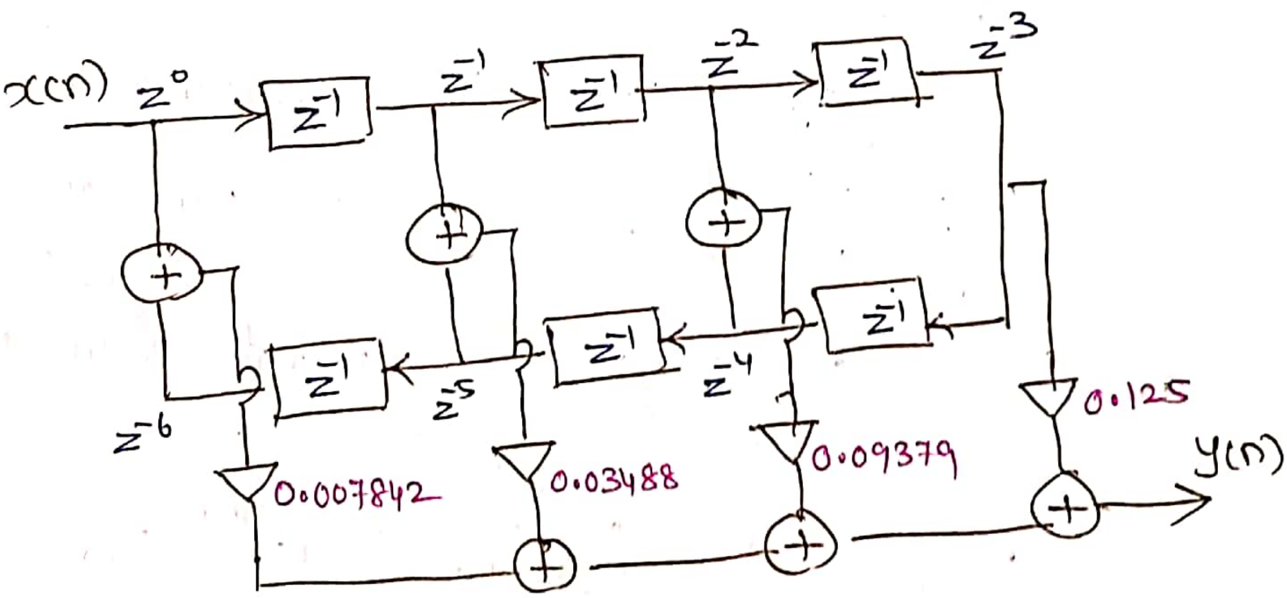
$$h(n) = \{0.007842,\ 0.03488,\ 0.09379,\ 0.125,\ 0.09379,\ 0.03488,\ 0.007842\}$$

Step 4 :- Find Transfer function $H(z)$ by taking Z-transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

$$H(z) = 0.007842\, z^{0} + 0.03488\, z^{-1} + 0.09379\, z^{-2} + 0.125\, z^{-3} + 0.09379\, z^{-4} + 0.03488\, z^{-5} + 0.007842\, z^{-6}$$

Step 5 :- Realize using Linear Phase Structure.



Problem 3 :

Design an ideal high pass filter whose cut off frequency 1000 Hz with sampling frequency 8000 Hz using Hanning window with 11 samples.

Solution :

HPF

N = 11

Hanning Window

$$\omega_c = \frac{2\pi f_{c}}{f_s} = \frac{2\pi \times 1000}{8000}$$

$$= 0.785 \text{ rads/sec}$$

Step 1: Choose the ~~ideal~~ desired frequency response

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & -\pi \le \omega \le -\omega_c \\ 0 & -\omega_c \le \omega \le \omega_c \\ e^{-j\alpha\omega} & \omega_c \le \omega \le \pi \end{cases}$$

Step 2: find desired impulse response

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) \, e^{j\omega n} \, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-w_c} e^{-j\alpha w} e^{jwn}\, dw + \int_{-w_c}^{w_c} \cancel{0\, e^{jwn}}\, dw + \int_{w_c}^{\pi} e^{-j\alpha w} e^{jwn}\, dw\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{-w_c} e^{-jw(\alpha-n)}\, dw + \int_{w_c}^{\pi} e^{-jw(\alpha-n)}\, dw\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{-jw(\alpha-n)}}{-j(\alpha-n)}\right]_{-\pi}^{-w_c} + \left[\frac{e^{-jw(\alpha-n)}}{-j(\alpha-n)}\right]_{+w_c}^{+\pi}\right]$$

$$= \frac{-1}{\pi(\alpha-n)}\left[\frac{e^{+jw_c(\alpha-n)} - e^{j\pi(\alpha-n)} + e^{-jw_c(\alpha-n)} + e^{-j\pi(\alpha-n)}}{2j}\right]$$

$$= \frac{+1}{\pi(\alpha-n)}\left[\left[\frac{e^{j\pi(\alpha-n)} - e^{-j\pi(\alpha-n)}}{2j}\right] - \left[\frac{e^{jw_c(\alpha-n)} - e^{-jw_c(\alpha-n)}}{2j}\right]\right]$$

$$h_d(n) = \frac{[\sin[\pi(\alpha-n)] - \sin[w_c(\alpha-n)]]}{[\pi(\alpha-n)]} \qquad n \neq \alpha$$

$$h_d(n) = \frac{\pi - w_c}{\pi} \qquad n = \alpha. \quad \text{L' Hosp rule.}$$

Step 3 :- Find finite impulse response h(n)

$$h(n) = h_d(n) \times W_{Hn}(n)$$

$$h(n) = \frac{\left[\sin(\pi(\alpha-n)) - \sin(\omega_c(\alpha-n))\right]\left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right]}{\left[\pi(\alpha-n)\right]}$$

$$\alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$$

$$h(n) = \frac{\left[\sin(\pi(5-n)) - \sin(0.7854(5-n))\right]\left[0.5 - 0.5\cos\left(\frac{\cancel{2}\pi n}{\cancel{10}\,5}\right)\right]}{\left[\pi(5-n)\right]}$$

$h(0) = 0$

$h(1) = 0.$

$h(2) = -0.026$

$h(3) = -0.1041$

$h(4) = -0.2035$

$h(5) = \frac{\pi - \omega_c}{\pi} = 0.75$

$h(6) = -0.2035$

$h(7) = -0.1041$

$h(8) = -0.026$

$h(9) = 0$ $\quad h(10) = 0$

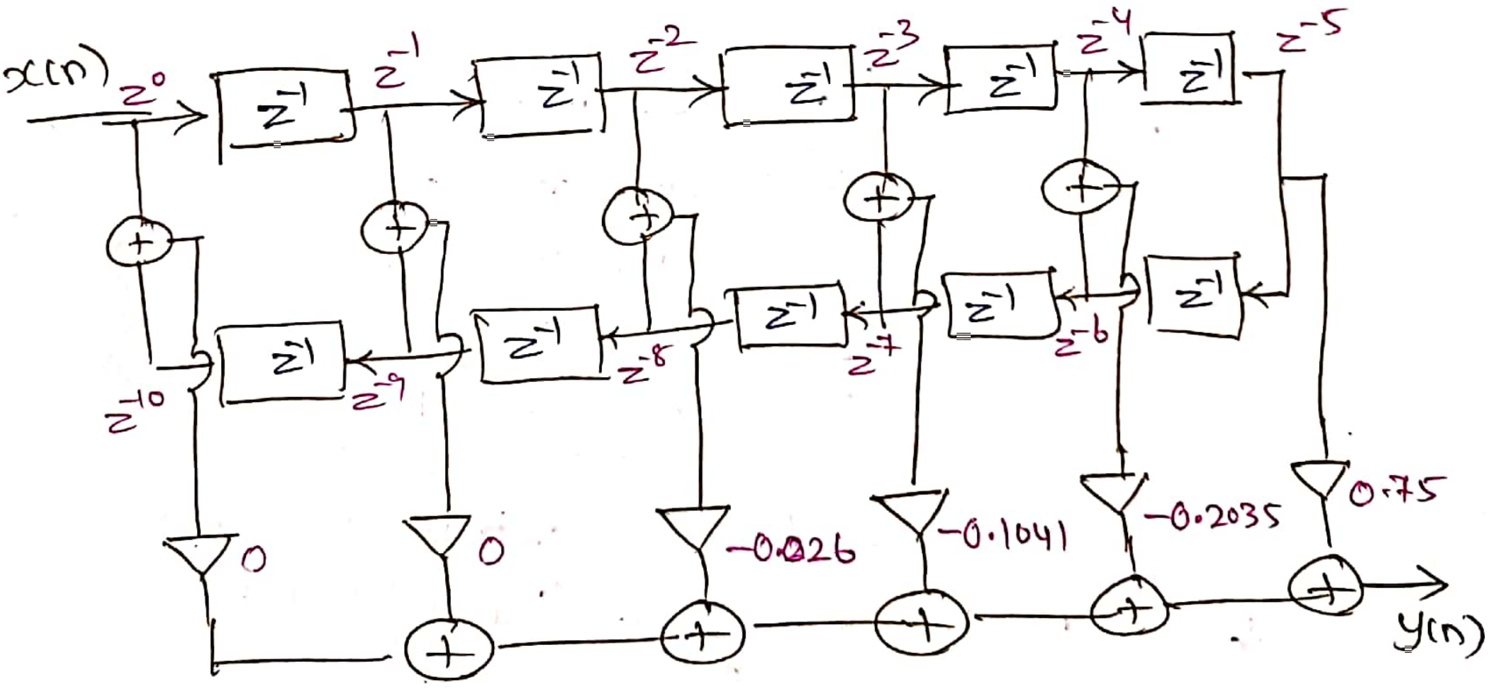
$$h(n) = \{0, 0, -0.026, -0.1041, -0.2035, 0.75, -0.2035, -0.1041, -0.026, 0, 0\}.$$

Step 4 :

Find the Transfer function $H(z)$

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

$$H(z) = 0z^{0} + 0z^{-1} - 0.026\, z^{-2} - 0.1041\, z^{-3} - 0.2035 z^{-4} + 0.75 z^{-5} - 0.2035 z^{-6} - 0.1041\, z^{-7} - 0.026 z^{-8} + 0 z^{-9} + 0 z^{-10}.$$



Problem 4 :

Design a linear phase FIR bandpass filter to Pass frequencies in the range 0.4π to 0.65π rad/sample by taking 7 Samples of hanning window Sequence.

Solution :

BPF

$\omega c_1 = 0.4\pi$

$\omega c_2 = 0.65\pi$

Hanning window

$N = 7$

Step 1 : Choose the desired frequency response $H_d(e^{j\omega})$

$$H_d(e^{j\omega}) = \begin{cases} 0 & -\pi \le \omega \le -\omega c_2 \\ e^{-j\alpha\omega} & -\omega c_2 \le \omega \le -\omega c_1 \\ 0 & -\omega c_1 \le \omega \le \omega c_1 \\ e^{-j\alpha\omega} & \omega c_1 \le \omega \le \omega c_2 \\ 0 & \omega c_2 \le \omega \le \pi \end{cases}$$

Step 2 : Find the desired impulse response $h_d(n)$ by taking inverse Fourier Transform

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega c_2} \cancel{0\, e^{+j\omega n}\, d\omega} + \int_{-\omega c_2}^{-\omega c_1} e^{-j\alpha\omega} e^{j\omega n}\, d\omega\right.$$

$$\left. \int_{-\omega c_1}^{\omega c_1} \cancel{0\, e^{j\omega n}} d\omega + \int_{+\omega c_1}^{\omega c_2} e^{-j\alpha\omega} e^{j\omega n}\, d\omega + \int_{\omega c_2}^{\pi} \cancel{0\, e^{j\omega n}}\, d\omega \right]$$

$$h_d(n) = \frac{1}{2\pi}\left[\int_{-\omega c_2}^{-\omega c_1} e^{-j\omega(\alpha-n)}\, d\omega + \int_{\omega c_1}^{\omega c_2} e^{-j\omega(\alpha-n)}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[\frac{e^{-j\omega(\alpha-n)}}{-j(\alpha-n)}\right]_{-\omega c_2}^{-\omega c_1} + \left[\frac{e^{-j\omega(\alpha-n)}}{-j(\alpha-n)}\right]_{\omega c_1}^{\omega c_2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega c_1(\alpha-n)} - e^{j\omega c_2(\alpha-n)}}{-j(\alpha-n)} + \frac{e^{-j\omega c_2(\alpha-n)} - e^{-j\omega c_1(\alpha-n)}}{-j(\alpha-n)}\right]$$

$$= \frac{1}{\pi(\alpha-n)}\left[\left[\frac{e^{j\omega c_2} - e^{-j\omega c_2}}{2j}\right] - \left[\frac{e^{j\omega c_1} - e^{-j\omega c_1}}{2j}\right]\right]$$

$$h_d(n) = \frac{\sin(\omega c_2(\alpha-n)) - \sin(\omega c_1(\alpha-n))}{(\pi(\alpha-n))} \qquad n \neq \alpha$$

$$h_d(n) = \frac{\omega c_2 - \omega c_1}{\pi} \qquad n = \alpha \qquad \alpha = \frac{N-1}{2}, \quad \alpha = 3$$

Step 3 :- Find the finite impulse response $h(n)$

$$h(n) = h_d(n)\, W_{Hn}(n)$$

$$h(n) = \frac{[\sin(\omega c_2(\alpha-n)) - \sin(\omega c_1(\alpha-n))]}{[\pi(\alpha-n)]} \times \left[0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right)\right]$$

$$h(n) = \frac{[\sin(0.65\pi(3-n)) - \sin(0.4\pi(3-n))]\left[0.5-0.5\cos\left(\frac{2\pi n}{\cancel{6}3}\right)\right]}{[\pi(3-n)]}$$

$n = 0$ to 6

$h(0) = 0$

$h(1) = -0.0556$

$h(2) = -0.0143$

$h(3) = \frac{\omega c_2 - \omega c_1}{\pi} = \frac{0.65\pi - 0.4\pi}{\pi} = 0.25$

$h(4) = -0.0143$

$h(5) = -0.0556$

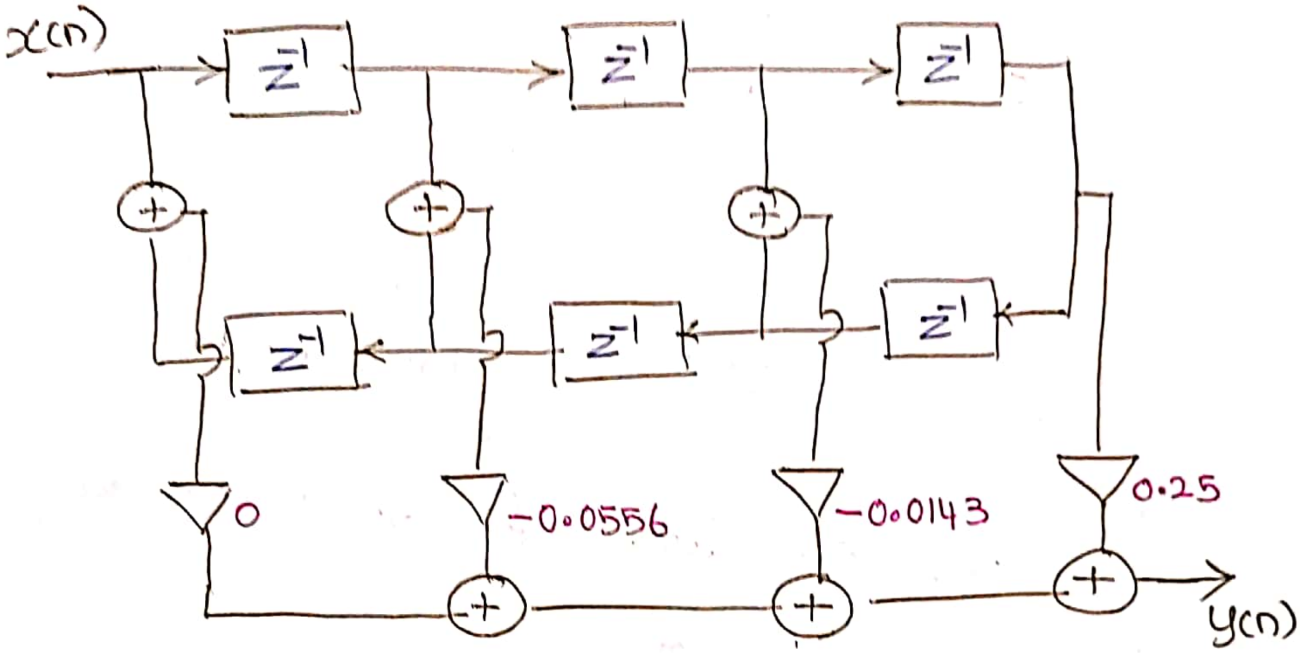
$h(6) = 0$

$h(n) = \{0, -0.0556, -0.0143, 0.25, -0.0143, -0.0556, 0\}$

Step 4 :- Find $H(z)$ by taking $h(n)$ Z-transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

$$H(z) = 0z^0 - 0.0556\,z^{-1} - 0.0143\,z^{-2} + 0.25\,z^{-3}$$
$$- 0.0143\,z^{-4} - 0.0556\,z^{-5} + 0\,z^{-6}$$

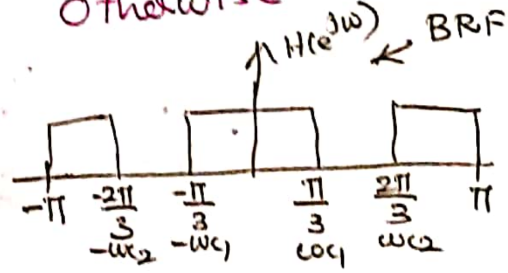


Problem 5 :

Design an FIR filter using Rectangular window for the given frequency response

$$H(e^{j\omega}) = \begin{cases} e^{-j3\omega} & |\omega| \le \frac{\pi}{3} \text{ and } |\omega| \ge \frac{2\pi}{3} \\ 0 & \text{otherwise} \end{cases}$$

Solution :



BRF (or) BSF

$\omega c_1 = \frac{\pi}{3} = 1.047$ $\alpha = 3$

$N = 2\alpha + 1$

$\omega c_2 = \frac{2\pi}{3} = 2.094$ $N = 7$

Rectangular window

Step 1 :-

Choose the desired frequency response

$H_d(e^{j\omega})$

$$H_d(e^{j\omega}) = \begin{cases} e^{-j\alpha\omega} & -\pi \le \omega \le -\omega c_2 \\ 0 & -\omega c_2 \le \omega \le -\omega c_1 \\ e^{-j\alpha\omega} & -\omega c_1 \le \omega \le \omega c_1 \\ 0 & \omega c_1 \le \omega \le \omega c_2 \\ e^{-j\alpha\omega} & \omega c_2 \le \omega \le \pi \end{cases}$$

Step 2 :-

Find the desired ~~freque~~ Impulse response

$h_d(n)$

$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega})\, e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega c_2} e^{-j\alpha\omega} e^{j\omega n}\, d\omega + \int_{-\omega c_2}^{-\omega c_1} \cancel{0\, e^{j\omega n}}\, d\omega \right.$$

$$\left. + \int_{-\omega c_1}^{\omega c_1} e^{-j\alpha\omega} e^{j\omega n}\, d\omega + \int_{\omega c_1}^{\omega c_2} \cancel{0\, e^{j\omega n}}\, d\omega + \int_{\omega c_2}^{\pi} e^{-j\alpha\omega} e^{j\omega n}\, d\omega \right]$$

$$= \frac{1}{2\pi} \left[\int_{-\pi}^{-\omega c_2} e^{-j\omega(\alpha-n)}\, d\omega + \int_{-\omega c_1}^{\omega c_1} e^{-j\omega(\alpha-n)}\, d\omega + \int_{\omega c_2}^{\pi} e^{-j\omega(\alpha-n)}\, d\omega \right]$$

$$= \frac{1}{2\pi} \left[\left[\frac{e^{-j\omega(\alpha-n)}}{-j(\alpha-n)} \right]_{-\pi}^{-\omega c_2} + \left[\frac{e^{-j\omega(\alpha-n)}}{-j(\alpha-n)} \right]_{-\omega c_1}^{\omega c_1} + \left[\frac{e^{-j\omega(\alpha-n)}}{-j(\alpha-n)} \right]_{\omega c_2}^{\pi} \right]$$

$$h_d(n) = \frac{-1}{\pi(\alpha-n)}\left[\left[\frac{e^{j\omega_{c_2}(\alpha-n)} - e^{j\pi(\alpha-n)}}{2j}\right] + \left[\frac{e^{-j\omega_{c_1}(\alpha-n)} - e^{j\omega_{c_1}(\alpha-n)}}{2j}\right] + \left[\frac{e^{-j\pi(\alpha-n)} - e^{-j\omega_{c_2}(\alpha-n)}}{2j}\right]\right]$$

$$h_d(n) = \frac{1}{\pi(\alpha-n)}\left[\left[\frac{e^{j\pi(\alpha-n)} - e^{-j\pi(\alpha-n)}}{2j}\right] - \left[\frac{e^{j\omega_{c_2}(\alpha-n)} - e^{-j\omega_{c_2}(\alpha-n)}}{2j}\right] + \left[\frac{e^{j\omega_{c_1}(\alpha-n)} - e^{-j\omega_{c_1}(\alpha-n)}}{2j}\right]\right]$$

$$h_d(n) = \frac{\sin(\pi(\alpha-n)) - \sin(\omega_{c_2}(\alpha-n)) + \sin(\omega_{c_1}(\alpha-n))}{\pi(\alpha-n)}$$

$n \neq \alpha$

$$h_d(n) = \frac{\pi - \omega_{c_2} + \omega_{c_1}}{\pi}$$

$n = \alpha \quad \alpha = \frac{N-1}{2} = \frac{7-1}{2}$

$\boxed{\alpha = 3}$

Step 3: Find the finite impulse response $h(n)$

$$h(n) = h_d(n) \cdot W_R(n)$$

$$W_R(n) = \begin{cases} 1 & 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$h(n) = \frac{[\sin(\pi(\alpha-n)) - \sin(\omega c_2(\alpha-n)) + \sin(\omega c_1(\alpha-n))]}{[\pi(\alpha-n)]} \times 1$$

$$h(n) = \frac{[\sin(\pi(3-n)) - \sin(2.094(3-n)) + \sin(1.047(3-n))]}{[\pi(3-n)]} \times 1$$

$h(0) = 0.00018$

$h(1) = 0.2756$

$h(2) = 0$

$h(3) = \frac{\pi - \omega c_2 + \omega c_1}{2} = \frac{\pi - 2.094 + 1.047}{2} = 0.666$

$h(4) = 0$

$h(5) = 0.2756$

$h(6) = 0.00018$

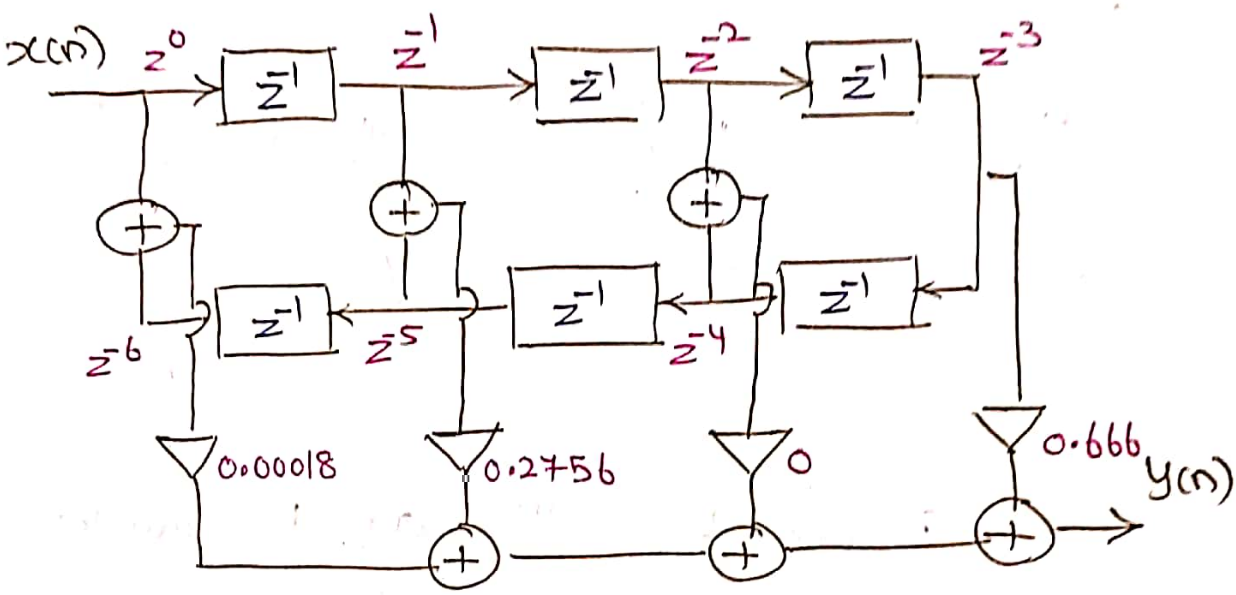
$h(n) = \{0.00018, 0.2756, 0, 0.666, 0, 0.2756, 0.00018\}$

Step 4 : Find $H(z)$ by taking z-transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}.$$

$$H(z) = 0.00018\,z^{0} + 0.2756\,z^{-1} + 0\,z^{-2} + 0.666\,z^{-3}$$
$$+ 0\,z^{-4} + 0.2756\,z^{-5} + 0.00018\,z^{-6}$$

Step 5 : Realize the $H(z)$ using Linear phase Structure.



Problem 6 :

Determine the co-efficients of a linear phase FIR filter of length $N = 11$ which has a symmetric unit sample response and a frequency response that satisfies the conditions

$$H\left(\frac{2\pi k}{11}\right) = \begin{cases} 1 & \text{for } k = 0, 1, 2, 3 \\ 0 & \text{for } k = 4, 5 \end{cases}$$

~~Stov~~ Solution :

$N = 11$ $\qquad \alpha = \frac{N-1}{2} = \frac{11-1}{2} = 5$

Step 1 :

Choose the ideal frequency response $H_d(e^{j\omega})$

$$H_d(e^{j\omega}) = \begin{cases} 1 \cdot e^{-j\alpha\omega} & \text{for } K = 0, 1, 2, 3 \\ 0 \cdot e^{-j\alpha\omega} & \text{for } K = 4, 5 \end{cases}$$

Step 2 :

Sample $H_d(e^{j\omega})$ at N-Samples by taking $\omega = \omega_K = \frac{2\pi K}{N}$ where $K = 0, 1, 2 \cdots N-1$

$$H(K) = H_d(e^{j\omega})\Big|_{\omega = \omega_K = \frac{2\pi K}{N}}$$

$$H(K) = \begin{cases} e^{-j5\left(\frac{2\pi K}{11}\right)} & K = 0, 1, 2, 3 \\ 0 & K = 4, 5 \end{cases}$$

$H(0) = 1$

$$H(1) = e^{-j\frac{10\pi}{11}}$$

$$H(2) = e^{-j\frac{10\pi\times 2}{11}} = e^{-j\frac{20\pi}{11}}$$

$$H(3) = e^{-j\frac{10\pi\times 3}{11}} = e^{-j\frac{30\pi}{11}}$$

$$H(4) = 0$$

$$H(5) = 0$$

Step 4: Compute the N Samples of impulse response h(n)

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} Re\left\{H(K)\, e^{\frac{j2\pi nK}{N}}\right\}\right]$$

$$h(n) = \frac{1}{11}\left[1 + 2\sum_{K=1}^{5} Re\left\{H(K)\, e^{\frac{j2\pi nK}{11}}\right\}\right]$$

$$h(n) = \frac{1}{11}\left[1 + 2\left[Re\left\{e^{-j\frac{10\pi}{11}}\, e^{\frac{j2\pi n}{11}} + e^{-j\frac{20\pi}{11}}\, e^{\frac{j2\pi n\times 2}{11}} + e^{-j\frac{30\pi}{11}}\, e^{\frac{j2\pi n\times 3}{11}}\right\}\right]\right]$$

$$= \frac{1}{11}\left[1 + 2\left[Re\left\{e^{-j\frac{2\pi}{11}(5-n)} + e^{-j\frac{4\pi}{11}(5-n)} + e^{-j\frac{6\pi}{11}(5-n)}\right\}\right]\right]$$

$$h(n) = \frac{1}{11}\left[1 + 2\cos\left(\frac{2\pi(5-n)}{11}\right) + 2\cos\left(\frac{4\pi(5-n)}{11}\right) + 2\cos\left(\frac{6\pi(5-n)}{11}\right)\right]$$

$h(0) = -0.0496$

$h(1) = 0.0989$

$h(2) = -0.0338$

$h(3) = -0.127$

$h(4) = 0.2935$

$h(5) = 0.6363$

$h(6) = 0.2935$

$h(7) = -0.127$

$h(8) = -0.0338$

$h(9) = 0.0989$

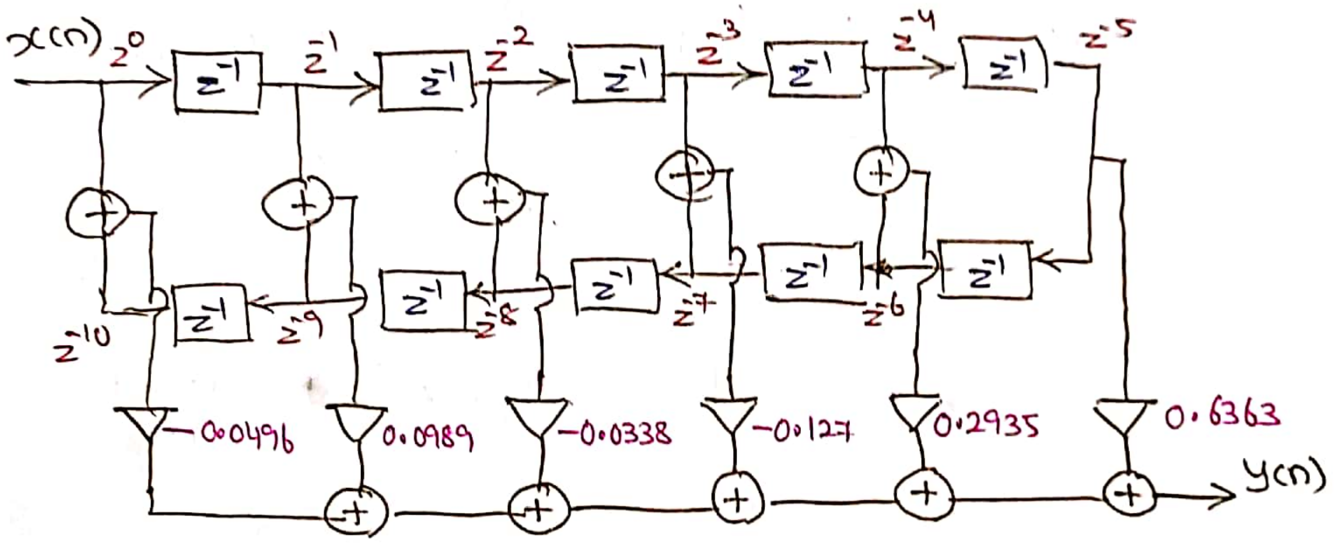
$h(10) = -0.0496$

$h(n) = \{-0.0496, 0.0989, -0.0338, -0.127,$
$0.2935, 0.6363, 0.2935, -0.127, -0.0338,$
$0.0989, -0.0496\}$

Step 5 : Find H(z) by taking Z-transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)\, z^{-n}$$

$$H(z) = -0.0496\, z^{0} + 0.0989\, z^{-1} - 0.0338\, z^{-2} - 0.127\, z^{-3} + 0.2935\, z^{-4} + 0.6363\, z^{-5} + 0.2935\, z^{-6} - 0.127\, z^{-7} - 0.0338\, z^{-8} + 0.0989\, z^{-9} - 0.0496\, z^{-10}$$



Problem 7 :

Design a linear phase FIR lowpass filter for the desired frequency response as given below, by frequency Sampling technique for N=7

$$H_d(e^{j\omega}) = \begin{cases} e^{-j3\omega} & ; 0 \le \omega \le 0.6\pi \text{ and } 1.4\pi \le \omega \le 2\pi \\ 0 & ; 0.6\pi \le \omega \le 1.4\pi. \end{cases}$$

Solution :

$N = 7 \qquad \alpha = \frac{N-1}{2} = \frac{7-1}{2} = 3$

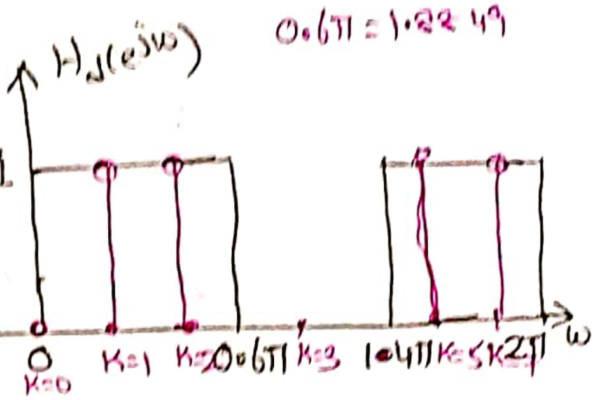
$\frac{2\pi K}{N} = \frac{2\pi K}{7}$

$K=0 \quad \omega_0 = 0$

$K=1 \quad \omega_1 = \frac{2\pi}{7}$

$K=2 \quad \omega_2 = \frac{4\pi}{7}$

$K=3 \quad \omega_3 = \frac{6\pi}{7}$



Step 1 :

Choose the ideal frequency response

$H_d(e^{jw})$

$$H_d(e^{jw}) = \begin{cases} 1 \cdot e^{-j\alpha w} & K = 0, 1, 2 \\ 0 & K = 3 \end{cases}$$

Step 2 :

Sample $H_d(e^{jw})$ at N-Points

$$\omega = \omega_K = \frac{2\pi K}{N}$$

$$H(K) = H_d(e^{jw}) \Big|_{\omega = \omega_K = \frac{2\pi K}{N}}$$

$$H(K) = \begin{cases} 1 \cdot e^{-j3 \times \frac{2\pi K}{7}} & K = 0, 1, 2 \\ 0 & K = 3 \end{cases}$$

$H(0) = 1 \qquad K = 0$

$H(1) = e^{-j\frac{6\pi}{7}} \qquad K = 1.$

$$H(2) = e^{-j\frac{12\pi}{7}} \qquad K=2$$

$$H(3) = 0 \qquad K=3$$

Step 4 :

Compute the N Samples of impulse response h(n)

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N-1}{2}} Re\left\{H(K)\, e^{\frac{j2\pi nK}{N}}\right\}\right]$$

$$= \frac{1}{7}\left[1 + 2\sum_{K=1}^{3} Re\left\{H(K)\, e^{\frac{j2\pi nK}{N}}\right\}\right]$$

$$h(n) = \frac{1}{7}\left[1 + 2\, Re\left\{e^{\frac{-j6\pi}{7}} e^{\frac{j2\pi n}{7}} + e^{-j\frac{12\pi}{7}} e^{\frac{j4\pi n}{7}} + 0\right\}\right]$$

$$= \frac{1}{7}\left[1 + 2\, Re\left\{e^{-j\frac{2\pi}{7}(3-n)} + e^{-j\frac{4\pi}{7}(3-n)}\right\}\right]$$

$$h(n) = \frac{1}{7}\left[1 + 2\cos\left(\frac{2\pi(3-n)}{7}\right) + 2\cos\left(\frac{4\pi(3-n)}{7}\right)\right]$$

$n = 0$ to $N-1 = 0$ to 6

$h(0) = 0.0635$

$h(1) = -0.178$

$h(2) = 0.2574$

$h(3) = 0.7142$

$h(4) = 0.2574$

$h(5) = -0.178$

$h(6) = 0.0635$

$$h(n) = \{0.0635, -0.1781, 0.2574, 0.7142, 0.2574, -0.1781, 0.0635\}$$

Step 5 : Find $H(z)$ by taking z-transform.

$$H(z) = 0.0635 z^{0} - 0.1781 z^{-1} + 0.2574 z^{-2} + 0.7142 z^{-3} + 0.2574 z^{-4} - 0.1781 z^{-5} + 0.0635 z^{-6}$$