UNIT-V

State space Analysis of continuous systems

Introduction:

→ The analyse MIMO systems, state variable technique used

state space technique

↳ which reduces complexity.

→ This technique, determines internal behavior of the system.

Conventional Methods

↳ Bode plot and Nyquist plot are frequency domain requires laplace Transform for continuous time systems and z-transform for discrete time systems.

→ But for both continuous and discrete-time systems vector matrix form of state-space representation ~~reduces~~ greatly ~~simply~~ simplifies system representation and gives accurately.

concept of state, state variables and state model:

→ In state space analysis time variant systems can be defined.

→ Individual variables can be defined in the MIMO systems.

Velocity $\frac{dx}{dt}$

State:

→ It represents every smallest part information of the system in order to predict response.

→ Basically, a state separates the future i.e., the response of the system from past.

(or)

→ It gives condition of system at any instant of time.

State variable:

The state variable are a set of variables that completely describe the state or condition of system at any instant of time.

State vector:

→ This is a vector consisting of n number of state variables that completely determine the behavior of the system.

State space:

→ The state space is an n-dimensional space whose coordinate axis are n number of state variables that completely determine the behavior of system.

Ex:

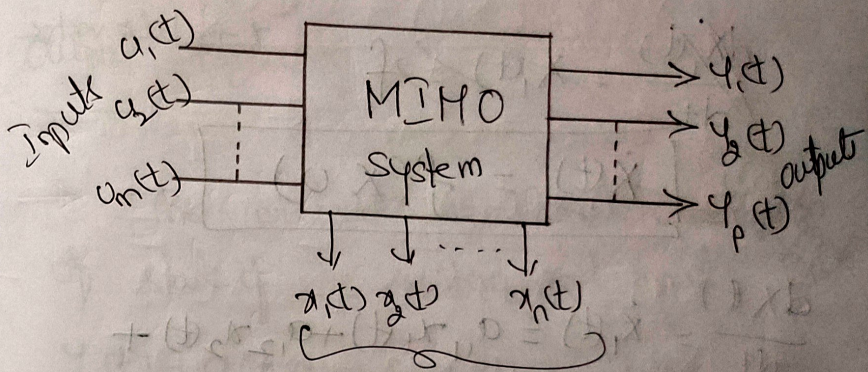
If $x_1, x_2 \dots x_n$ are n-state variables, then the state vector (X) is written by an $n \times 1$ matrix as follows

$$X(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

where $x_1, x_2 \dots x_n$ all are functions of t

State Model:

Consider Multiple Input Multiple output n^{th} order system as shown in fig.



$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$$Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$$

where m — no. of inputs

p — no of output

→ All are column vectors having orders $m \times 1$, $n \times 1$ and $p \times 1$.

→ For such system, the state variable representation can be arranged in the form of (n) first order differential equations

$\cancel{\frac{dx_1(t)}{dt} = x_1(t) = f}$ $x \rightarrow$ state vector

derivation of state vector

$$\boxed{\dot{x}(t) = f(x, u)}$$

$$\frac{dx_1(t)}{dt} = \dot{x}_1(t) = a_{11}x_1(t) + a_{12}x_2(t) + b_{11}u_1(t) + b_{12}u_2(t) \quad \rightarrow (1)$$

$$\frac{dx_2(t)}{dt} = \dot{x}_2(t) = a_{21}x_1(t) + a_{22}x_2(t) + b_{21}u_1(t) + b_{22}u_2(t) \quad \rightarrow (2)$$

Equation (1) & (2) can be represented in matrix form

2×2, 2×1, 2×2

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

∴ Generalized form

$$\boxed{\dot{x}(t) = Ax(t) + Bu(t)}$$

→ The above equation is a state equation.

Output Equation:

→ The output equation combination of state of the system and input.

$$Y = g(x, u)$$

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) + d_{11}u_1(t) + d_{12}u_2(t) \quad (3)$$

$$y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + d_{21}u_1(t) + d_{22}u_2(t) \quad (4)$$

The c and d are coefficients and are constants.

→ Write equations (3) & (4) in matrix form.

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

→ Write in general form

$$\boxed{Y(t) = Cx(t) + Du(t)}$$

The two equations combinedly form the state model of the linear system

$$\therefore \boxed{\begin{aligned} \dot{X}(t) &= A\,X(t) + B\,U(t) \\ Y(t) &= C\,X(t) + D\,U(t) \end{aligned}}$$

→ for linear time invariant system, where A, B, C, D are constant matrices

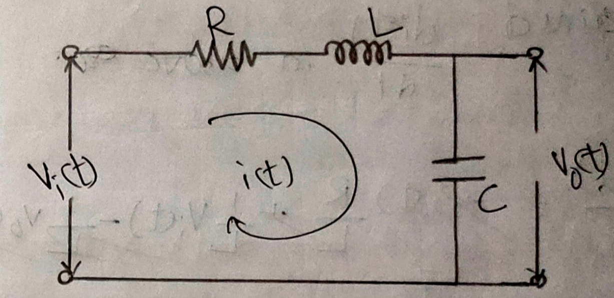
$$\dot{X}(t) = A(t)\,X(t) + B(t)\,U(t)$$
$$Y(t) = C(t)\,X(t) + D(t)\,U(t)$$

For linear time variant system, where A, B, C, D are time dependent matrices

Order of matrices:

A → $n \times n$ called Evolution matrix
B → $n \times m$ called control matrix
C → $p \times n$ called Observation matrix
D → $p \times m$ called Direct Transmission matrix

problem: obtain the state model of the given electrical network in the standard form. Given at $t = t_0$, $i(t) = i(t_0)$ and $V_0(t) = V_0(t_0)$.



sol: There are two energy storage elements (L) and (C) so two two state variables are required.

$X_1(t) = i(t)$ and

$X_2(t) = V_0(t)$

$$X(t_0) = \begin{bmatrix} i(t_0) \\ V_0(t_0) \end{bmatrix}$$

Now, write ~~three~~ differential equations for the given variables by using ~~KCV~~ KCL and KVL.

$$V_i(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

↳ 1 $\qquad \frac{1}{C}\int i(t)\,dt \to V_0(t)$

state equation & output equation.

obtained $\frac{di(t)}{dt}$ in above eq:

$$\frac{di(t)}{dt} = -i(t)\frac{R}{L} + \frac{1}{L}V_i(t) - \frac{1}{L}V_0(t)$$

$$\frac{di(t)}{dt} = -i(t)\frac{R}{L} + \frac{1}{L}V_i(t) - \frac{1}{LC}\int i(t)\,dt$$

$$\boxed{\frac{dq}{dt} = i(t)}$$

$$\frac{di(t)}{dt} = -i(t)\frac{R}{L} + \frac{1}{L}V_i(t) - \frac{q}{LC} \qquad \boxed{\frac{dq}{dt} = i}$$

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{LC} \\ 1 & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ q \end{bmatrix} + \begin{bmatrix} 1/L \\ 0 \end{bmatrix} V(t)$$

$$\dot{X} = AX + BU$$

output $\quad V_0(t) = X_2(t)$

system can be represented
- ↳ 1 T.F model
- ↳ 2 Differential equation model
- ↳ 3 Electrical n/w
- ↳ 4 SFG & Block diagram

State-space representation:

→ The state-space model of a system can be obtained in three different ways when differential equations or transfer function is given.

→ These different ways of representing a system in state space model.

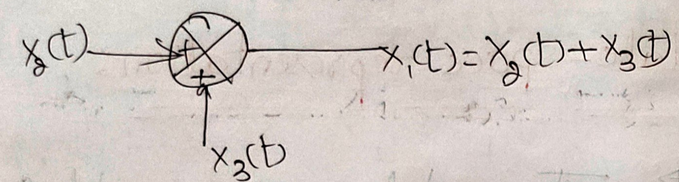
Types of representation
- physical variable form
- phase variable form
- canonical form

→ state diagram consists ③ elements

↳ scalars → it is like amplifiers having gain

$x_2(t) \longrightarrow \boxed{a} \longrightarrow x_1(t) = a x_2(t)$

↳ Adders → are summing points



↳ Integrators:

$\dot{x}_1(t) \longrightarrow \boxed{\int} \longrightarrow x_1(t) = \int \dot{x}_1(t)\, dt$

↳ which integrates, the differentiate of state variable to obtained required state variable.

Standard state model representation

consider standard state model

$$\dot{X}(t) = A X(t) + B U(t) \quad \text{and}$$

$$Y(t) = C X(t) + D U(t)$$

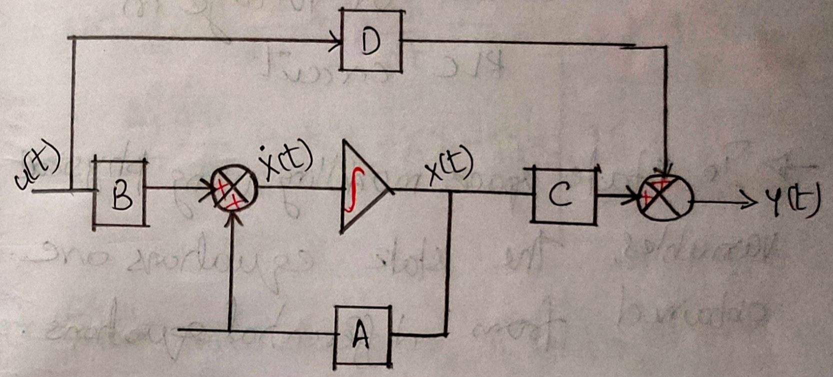


fig: Block diagram of state model

Derivation of state models from schematic models:

In state space analysis, the choice of state variables are random.

→ One of the choice of state variables is physical variables.

→ Ex:

① Mechanical systems → displacement, velocity, acceleration → physical variables are

② Electrical systems → current or voltage in RLC circuit.

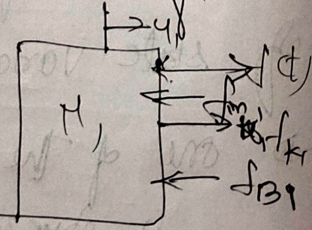
→ In state space modelling using physical variables, the state equations are obtained from differential equations.

↳ it is the basic model.

problem

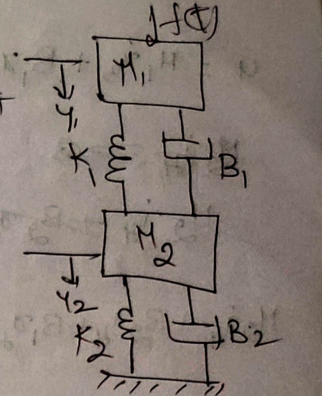
Construct the state model of mechanical model as shown.

Sol: free body diagram for M_1



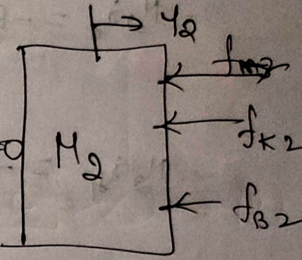
$$M_1 \frac{d^2 y_1}{dt} + B_1 \frac{dy_1}{dt} - B_1 \frac{dy_2}{dt} + K_1 y_1 - K_1 y_2 = f(t)$$

↳ ①



$$M_2 \frac{d^2 y_2}{dt} + B_2 \frac{dy_2}{dt} + B_1 \frac{dy_1}{dt} - B_1 \frac{dy_1}{dt} + K_2 y_2 + K y_2 - K y_1 = 0$$

↳ ②



choose state variables x_1, x_2, x_3 and x_4.

input $f(t) = u$.

relate state variables to physical variables

$\boxed{x_1 = y_1}$; $\boxed{x_2 = y_2}$; $\boxed{x_3 = \frac{dy_1}{dt}}$; $\boxed{x_4 = \frac{d^2 y_2}{dt}}$

$\boxed{\dot{x}_3 = \frac{d^2 y_1}{dt^2}}$ and $f(t) = u$

$u = M_1\dot{x}_3 + B_1 x_3 - B_1 x_3 + K_1 x_2 + K_2 x_2 - K_1 x_1 = 0$

$\cancel{M_2\dot{x}_4 = 0}$

$\cancel{M_2\dot{x}_4 + B_2 x_4 + B_1 x_4 + B_1 x_3 - K_2 x_2 - K_1 x_2 + K_1 x_1} = 0$

$M_2\dot{x}_4 + B_2 x_4 + B_1 x_4 - B_1 x_3 + K_1 x_2 + K_2 x_2 - K_1 x_1 = 0$

$\therefore M_2\dot{x}_4 = -B_2 x_4 - B_1 x_4 + B_1 x_3 - K_2 x_2 - K_1 x_2 + K_1 x_1$

$M_2\dot{x}_4 = -(B_2 - B_1)x_4 + B_1 x_3 - (K_2 + K_1)x_2 + K_1 x_1$

$\therefore \dot{x}_4 = \frac{K_1}{M_2}x - \frac{(K_1+K_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{(B_1+B_2)}{M_2}x_4$

state variable $x_1 = y_1$

differentiating $x_1 = y_1$ w.r. (t)

$\frac{dx_1}{dt} = \dot{x}_1$ and $\frac{dy_1}{dt} = x_3$;

$\dot{x}_1 = x_3$

State variable

$x_2 = y_2$

$\frac{dx_2}{dt} = \frac{dy_2}{dt}$

$\frac{dx_2}{dt} = \dot{x}_2$ and $\frac{dy_2}{dt} = x_4$;

$\boxed{\therefore \dot{x}_2 = x_4}$

$\dot{x}_1 = x_3$ and $\dot{x}_2 = x_4$

$\dot{x}_3 = -\frac{K_1}{M_1}x_1 + \frac{K_1}{M_1}x_2 - \frac{B_1}{M_1}x_3 + \frac{1}{M_1}u$

$\dot{x}_4 = \frac{K_1}{M_2}x_1 - \frac{(K_1+K_2)}{M_2}x_2 + \frac{B_1}{M_2}x_3 - \frac{B_1+B_2}{M_2}x_4$

On arranging state equations in matrix form

$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = [$

$$T(s) = \frac{s^2 + 3s + 3}{s^3 + 2s^2 + 3s + 1}$$

$$T(s) = \frac{Y(s)}{U(s)} = \frac{X(s)}{U(s)} \times \frac{Y(s)}{X(s)}$$

$$= \left[\frac{1}{s^3 + 2s^2 + 3s + 1}\right] \times \left[s^2 + 3s + 3\right]$$

$$\frac{X(s)}{U(s)} = \frac{1}{s^3 + 2s^2 + 3s + 1} \rightarrow \text{state equation} \quad (1)$$

$$\frac{Y(s)}{X(s)} = s^2 + 3s + 3 \rightarrow \text{output eq.} \quad (2)$$

from eq (1)

$$U(s) = s^3 X(s) + 2s^2 X(s) + 3s X(s) + X \quad (2)$$

Taking inverse L.T. → order 3

$$u(t) = \dddot{x} + 2\ddot{x} + 3\dot{x} + x$$

$x_1 = x$

$x_2 = \dot{x} = \dot{x}_1$

$x_3 = \ddot{x} = \dot{x}_2$

$\dot{x}_3 = \dddot{x}$

state equation

$$\dot{X} = AX + BU$$

$$u(t) = \dot{x}_3 + 2x_3 + 3x_2 + x_1$$

$$\boxed{\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + u(t)}$$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$\dot{x}_3 = -x_1 - 3x_2 - 2x_3 + u(t)$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

output Equation

$$Y(s) = s^2 X(s) + 3sX(s) + 3X(s)$$

By taking inverse laplace transform

$$Y(t) = \dddot{x} + 3\dot{x} + 3x$$

$$Y(t) = x_3 + 3x_2 + 3x_1$$

Re arrange

$$Y(t) = 3x_1 + 3x_2 + x_3$$

output eq.

$$Y = CX + DU$$

$$y(t) = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [0][u]$$

(2)

Transfer function

$$U(t) \longrightarrow \boxed{\frac{K}{(S+1)(S+2)(S^2+1)}} \longrightarrow y(t)$$

$$U(S) \qquad\qquad\qquad\qquad Y(S)$$

$$G(S) = \frac{Y(S)}{U(S)} = \frac{K}{(S+1)(S+2)(S^2+1)}$$

$$Y(S)(S+1)(S+2)(S^2+1) = KU(S).$$

$$(S^4 + 3S^3 + 3S^2 + 3S + 2)\,Y(S) = KU(S)$$

Taking Inverse Laplace Transform

$$\frac{d^4y(t)}{dt^4} + \frac{3d^3y(t)}{dt^3} + 3\frac{d^2y(t)}{dt^2} + \frac{3dy(t)}{dt} + 2y(t) = Ku(t)$$

$$\ddddot{y}(t) + 3\dddot{y}(t) + 3\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = Ku(t) \quad (1)$$

Choose state variables

let $y(t) = x_1$

$$\dot{y}(t) = \dot{x}_1 = x_2$$

$$\ddot{y}(t) = \ddot{x}_1 = \dot{x}_2 = x_3$$

$$\dddot{y}(t) = \dddot{x}_1 = \ddot{x}_2 = \dot{x}_3 = x_4$$

$$\ddddot{y}(t) = \ddddot{x}_1 = \dddot{x}_2 = \ddot{x}_3 = \dot{x}_4$$

substitute state variables in eq (1)

$$\boxed{\dot{x}_4 + 3x_4 + 3x_3 + 3x_2 + 2x_1 = KU} \quad (2)$$

$$\left.\begin{aligned} \dot{x}_3 &= x_4 \\ \dot{x}_2 &= x_3 \\ \dot{x}_1 &= x_2 \end{aligned}\right\} \rightarrow \text{state equations.}$$

eq (2) can be written as

$$\dot{x}_4 = -2x_1 - 3x_2 - 3x_3 - 3x_4 + KU$$

State model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \overset{A}{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -3 & -3 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ K \end{bmatrix} u$$

→ If a matrix is of the form

output equation $y(t) = x_1$ by matrix A

then it is called bush or companion form

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Canonical forms:

→ canonical forms are the standard forms of state model

→ There are several canonical forms

companion forms
- ① phase variable canonical form
- ② controllable canonical form
- ③ observable canonical form

modal forms
- ④ diagonal canonical form
- ⑤ Jordan canonical form.

Controllable canonical form

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2}$$

Rewrite above equation in equal order in Nr & Dr

$$\frac{Y(s)}{U(s)} = \frac{0s^2+s+3}{s^2+3s+2}$$

$|||^{ly}$ compare with this y

$$\boxed{\frac{Y(s)}{U(s)} = \frac{b_1 s^{n-1} + \dots + b_{n-1} s b_n}{s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}}$$

get a & b values

$b_0 = 0$, $b_1 = 1$ & $b_2 = 3$ → in output

$a_1 = 3$; $a_2 = 2$ → represented in state matrix

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \cancel{\begin{bmatrix} b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + b_0 u$$

$[b_3 \ \ b_2 \ \ b_1]$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ → is important in pole placement approach.

Diagonal canonical form

Determine the state representation of continuous-time LTI system with system function $G(s) = \frac{(s+5)}{(s+1)(s+3)}$ in diagonal canonical form.

Sol:

$$G(s) = \frac{s+5}{(s+1)(s+3)}$$

Using partial-fraction expansion

$$\frac{s+5}{(s+1)(s+3)} = \frac{A_1}{(s+1)} + \frac{A_2}{(s+3)}$$

$$s+5 = A_1(s+3) + A_2(s+1)$$

Equating coefficients (s) on both

$$1 = A_1 + A_2$$

Comparing constant coefficients

$$5 = 3A_1 + A_2$$

$$\boxed{A_1 = 2} \text{ and } \boxed{A_2 = -1}$$

$$\frac{s+5}{(s+1)(s+3)} = \frac{2}{(s+1)} - \frac{1}{s+3}$$

$$P_1 = -1 \; ; \; P_2 = -3$$

$$C_0 = 0 \; ; \; C_1 = 2 \text{ and } C_2 = -1$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} U$$

pmation

$$y(t) = \begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Jordan canonical form;

$$G(s) = \frac{1}{(s+3)^2(s+2)}$$

Partial fraction

$$\frac{1}{(s+3)^2(s+2)} = \frac{A_1}{(s+3)^2} + \frac{A_2}{(s+3)} + \frac{A_3}{(s+2)}$$

$$1 = A_1(s+2) + A_2(s+3) + A_3(s+3)^2$$

$$A_1 = -1, \; A_2 = -1 \text{ and } A_3 = 1$$

$$T(s) = - \frac{1}{(s+3)^2} - \frac{1}{(s+3)} + \frac{1}{s+2}$$

$$C_1 = 0 \; ; \; C_2 = -1, \; C_3 = -1$$

$$P_1 = -3 \text{ and } P_2 = -2$$

comparing the above eq, with standard Jordan canonical form, the coefficient values for diagonal canonical form sa

observable canonical form: [ocf]

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -a_2\\ 1 & -a_1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}b_2-a_2b_0\\ b_1-a_1b_0\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + b_0u(t)$$

---

problem Determine the state representation of a continuous-time LTI system with system function $G(s) = \dfrac{s+7}{(s+2)(s+3)}$ in observable canonical form.

sol:

$$G(s) = \frac{s+7}{(s+2)(s+3)}$$

$$\boxed{s^2+3s+2s+6}$$

$$\Downarrow$$

$$s^2+5s+6$$

Order $n=2$

$b_0=0;\ b_1=1;\ b_2=7$

$$\boxed{a_1=5;\ a_2=6}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -6\\ 1 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}3\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Diagonalization of state matrix:

The process of converting non-diagonal system matrix into diagonal system matrix.

$$\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

non diag — Non-diagonal matrix

State equation of the state model

$$\dot{X}(t) = A\,X(t) + B\,U(t)$$

Substitute $X(t) = M Z(t)$

$$M\dot{Z}(t) = A M Z(t) + B U(t)$$

$$\dot{Z}(t) = \frac{A M Z(t)}{M} + \frac{B U(t)}{M}$$

$$\boxed{\dot{Z}(t) = M^{-1} A M Z(t) + M^{-1} B U(t)}$$

M = Model matrix

$$M = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \text{ \& } C_2 = \begin{bmatrix} C_{12} \\ C_{22} \end{bmatrix}$$

Ex: obtain the diagonalization system matrix of a system, which is described by the state equation.

$$\dot{x} = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Diagonal system matrix $M^{-1}AM$

Finding Eigen values:

characteristics of equation of state model is

$$|S\underset{\text{Identity matrix}}{I} - \overset{\text{system matrix } 2\times 2}{A}| = 0$$

$$|SI - A| = \left| S \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \right|$$

$$= \left| \begin{bmatrix} S-3 & -4 \\ -2 & S-1 \end{bmatrix} \right|$$

$$= (S-3)(S-1) - 8$$

$$= S^2 - 4S - 5$$

$$= (S+1)(S-5)$$

$$= -1, -5$$

Finding eigen vectors:

For eigen values $S = -1$

$$[S_1 I - A] C_1 = 0$$

$$\left[-1 \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - \begin{vmatrix} 3 & 4 \\ 2 & 1 \end{vmatrix} \right] \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = 0$$

$$\begin{bmatrix} 4 & -4 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} = 0$$

$$-4c_{11} - 4c_{21} = 0$$
$$-2c_{11} - 2c_{21} = 0$$

Let $c_{11} = 1, c_{21} = -1$

$$c_1 = \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(1\times1)-(-1\times2)} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$$M^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

$M^{-1}AM =$

State Transition Matrix

→ For obtaining the solution of homogeneous and non-homogeneous state equations, it is necessary to determine State Transition Matrix (STM).

→ STM represented as $\boxed{\phi(t)}$ for homogeneous-type state equation

$$\boxed{\dot{x}(t) = Ax(t)}$$

Taking Laplace Transform:

$$sX(s) - x(0) = AX(s)$$

$$[sI - A]X(s) = x(0)$$

$$\boxed{X(s) = [sI - A]^{-1} x(0)}$$

Taking Inverse laplace Transform

$$x(t) = L^{-1}\{[sI - A]^{-1} x(0)\}$$

$$\boxed{x(t) = L^{-1}\{[sI - A]^{-1}\} x(0)}$$

$$\boxed{\phi(t) = e^{At} = L^{-1}[sI - A]^{-1}}$$

$$\therefore L\{\phi(t)\} = L\{e^{At}\} = \phi(s) = [sI - A]^{-1}$$

$\phi(s) \rightarrow$ resolvent matrix.

$$\boxed{\phi(t) = e^{At}}$$

$\hookrightarrow$ is a state transition matrix.

properties:

① $\phi(0) = e^{A(0)} = I$

② $\phi(t) = e^{At} = e^{-(-At)^{-1}} = [\phi(-t)]^{-1}$

③ $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \phi(-t)$

④ $\phi(t_1 + t_2) = e^{At_1} \cdot e^{At_2}$

$= \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

⑤ $[\phi(t)]^k = (e^{At})^k = \phi(kt)$

⑥ $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0) =$

$\phi(t_1 - t_0)\phi(t_2 - t_1)$

controllable canonical form (CCF)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \underbrace{b_0 u(t)}_{0}$$

$$y(t) = \begin{bmatrix} b_3 & b_2 & b_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Observable canonical forms

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & -a_3\\ 1 & 0 & -a_2\\ 0 & 0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix} + \begin{bmatrix}b_3\\ b_2\\ b_1\end{bmatrix}u(t)$$

output equation

$b_0 = 0$

$$y(t) = \begin{bmatrix}0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\end{bmatrix}$$

Diagonal canonical form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-P_1 & 0 & 0\\ 0 & -P_2 & 0\\ 0 & 0 & -P_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}u(t)$$

Output

$$y(t) = \begin{bmatrix}c_1 & c_2 & c_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + b_0 u(t)$$

Jordan canonical form

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-P_1 & 1 & 0\\ 0 & -P_2 & 1\\ 0 & 0 & -P_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u(t)$$

$$y(t) = \begin{bmatrix}c_1 & c_2 & \cdots c_3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + b_0 u(t)$$

$$\frac{Y(S)}{U(S)} = \frac{S+3}{S^2+3S+2}$$

$$\cancel{Y}(S)\,S^2 + 3S\,Y(S) + 2Y(S) = U(S)\,S + U(S)\,3$$

$$\frac{Y(S)}{U(S)} \quad \cancel{\frac{X(S)}{U(S)}}$$

$$\frac{X(S)}{U(S)} \times \frac{Y(S)}{\cancel{X}(S)} = \left[\frac{1}{S^2+3S+2}\right](S+3)$$

$$\frac{X(S)}{U(S)} = \frac{1}{S^2+3S+2} \quad \text{and} \quad \frac{Y(S)}{X(S)} = S+3$$

$$Y(S) = S\,X(S) + 3X(S)$$

$U(S) = [S^2 + 3S + 2]\, X(S)$

$U(S) = S^2 X(S) + 3S X(S) + 2X(S)$

Taking inverse laplace Transform

$u(t) = \ddot{x} + 3\dot{x} + 2x$

choose state variables

$$\boxed{\begin{aligned} x_1 &= x \\ \dot{x}_1 &= \dot{x} = x_2 \\ \dot{x}_2 &= \ddot{x} \end{aligned}}$$

$\dot{X} = AX + BU$

$\cancel{u(t) = \dot{x}_2 + 3x_1 +}$

$u(t) = \dot{x}_2 + 3x_2 + 2x_1$

$\dot{x}_2 = -2x_1 - 3x_2 + u(t)$

$\boxed{\dot{x}_1 = x_2}$

$y(t) = x_2 + 3x_1$

$y(t) = 3x_1 + x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Observable canonical form (OCF)

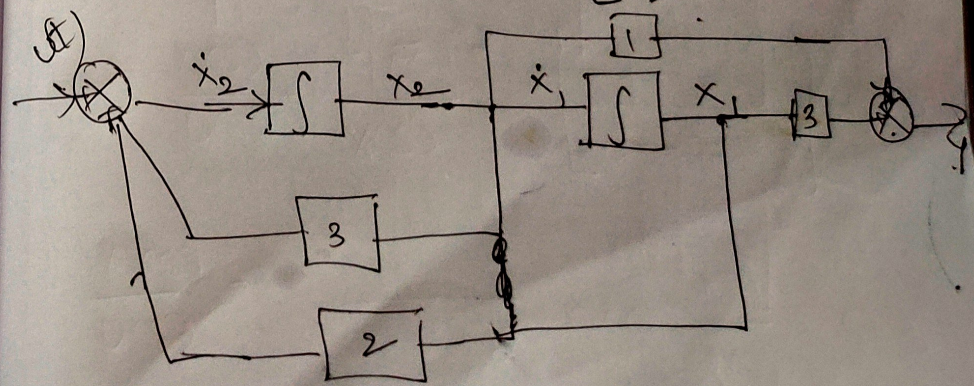
$A \rightarrow$ Replace by A^T

$B \rightarrow C^T$

$C \rightarrow B^T$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} U$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



Input is force

$f(t) = u_1$

$$\boxed{u_1 = kx_1 + Bx_2 + M\dot{x}_2}$$

for state space matrix

$\dot{x}_1 = ?$

$\dot{x}_2 = ?$

$M\dot{x}_2 = u_1 - kx_1 - Bx_2$

$$\boxed{\dot{x}_2 = \frac{u_1}{M} - \frac{K}{M}x_1 - \frac{B}{M}x_2}$$

Re arrange

$$\boxed{\dot{x}_2 = -\frac{K}{M}x_1 - \frac{B}{M}x_2 + \frac{u_1}{M_1}}$$

$$\boxed{f(t) = Kx(t) + B\frac{dx(t)}{dt} + M\frac{d^2x(t)}{dt}}$$

Spring — resistive force — accelerative force

$x(t) \rightarrow$ distance

$V(t) \rightarrow$ differentials of $x(t)$

$u(t) \rightarrow \frac{dV(t)}{dt}$

Consider state space variables

(1) $x(t) = x_1$

(2) $V(t) = \frac{dx(t)}{dt} = \dot{x}_1 = x_2$

(3) $a(t) = \frac{d^2x(t)}{dt} = \frac{dV(t)}{dt} = \dot{x}_2 =$