

FOURIER SERIES

It converts time domain signal to frequency domain signal for representation. It is applicable to periodic signal.

- 1) Trigonometric fourier series
- 2) Exponential " "
- 3) polar fourier series.

Trigonometric fourier series :-

let us consider the complete set

$$\left\{ \begin{array}{l} 1, \cos \omega_0 t, \cos 2\omega_0 t, \dots \\ 0, \sin \omega_0 t, \sin 2\omega_0 t, \dots \end{array} \right\} \quad f(t) = a_0 g_0(t) + a_1 g_1(t) + \dots + b_1 g_1(t) + \dots$$

$$f(t) = a_0(t) + a_1 \cos \omega_0 t + a_2 \cos 2\omega_0 t + \dots + a_n \cos n\omega_0 t + \dots + b_1 \sin \omega_0 t + b_2 \sin 2\omega_0 t + \dots + b_n \sin n\omega_0 t \rightarrow (1)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \rightarrow (2) \quad t_0 < t < t_0 + T$$

Evaluation of fourier coefficients :-

" a_0 is called a D.C component"

a_0, a_n, b_n are called fourier coefficients

To find a_0 :-

To evaluate a_0 we shall integrate the both

Sides of Eqⁿ (2) over one period ($t_0, t_0 + T$)

$$\int_{t_0}^{t_0+T} f(t) dt = \int_{t_0}^{t_0+T} a_0 dt + \int_{t_0}^{t_0+T} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t) dt$$

$$\int_{t_0}^{t_0+T} f(t) dt = a_0 \int_{t_0}^{t_0+T} dt + \sum_{n=1}^{\infty} \left[a_n \int_{t_0}^{t_0+T} \cos n\omega_0 t dt + b_n \int_{t_0}^{t_0+T} \sin n\omega_0 t dt \right]$$

$$\int_{t_0}^{t_0+T} f(t) dt = a_0 T + \sum_{n=1}^{\infty} a_n \left[\frac{\sin n \omega_0 t}{n \omega_0} \right]_{t_0}^{t_0+T} + \sum_{n=1}^{\infty} b_n \left[\frac{-\cos n \omega_0 t}{n \omega_0} \right]_{t_0}^{t_0+T}$$

$$\int_{t_0}^{t_0+T} f(t) dt = a_0 T + \sum_{n=1}^{\infty} \frac{a_n}{n \omega_0} [\sin n(\omega_0 t_0 + \omega_0 T) - \sin n \omega_0 t_0] \\ - \sum_{n=1}^{\infty} \frac{b_n}{n \omega_0} [\cos n(\omega_0 t_0 + \omega_0 T) - \cos n \omega_0 t_0]$$

$$\int_{t_0}^{t_0+T} f(t) dt = a_0 T \quad \left[\because T = \frac{2\pi}{\omega_0} \right]$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) dt$$

To find a_n :-

$$f(t) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n \omega_0 t + b_n \sin n \omega_0 t)$$

To evaluate the a_n & b_n we use the following results

$$1) \int_{t_0}^{t_0+T} \cos m \omega_0 t \cos n \omega_0 t dt = 0 \text{ if } m \neq n \\ = T/2 \text{ if } m = n$$

$$2) \int_{t_0}^{t_0+T} \sin m \omega_0 t \sin n \omega_0 t dt = 0 \text{ if } m \neq n \\ = T/2 \text{ if } m = n$$

$$3) \int_{t_0}^{t_0+T} \sin m \omega_0 t \cos n \omega_0 t dt = 0 \quad \forall m, n$$

$$f(t) \cos m \omega_0 t = a_0 \cos m \omega_0 t + \sum_{n=1}^{\infty} a_n \cos n \omega_0 t \cos m \omega_0 t \\ + \sum_{n=1}^{\infty} b_n \sin n \omega_0 t \cos m \omega_0 t$$

$$\int_{t_0}^{t_0+T} = \int_{t_0}^{t_0+T} a_0 \cos m \omega_0 t + \sum_{n=1}^{\infty} a_n \cos n \omega_0 t \cos m \omega_0 t + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n \omega_0 t \cos m \omega_0 t dt$$

$$\int_{t_0}^{t_0+T} f(t) \cos m \omega_0 t dt = 0 + a_n(T/2) + 0 \quad (n=m)$$

$$a_m = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos m \omega_0 t dt$$

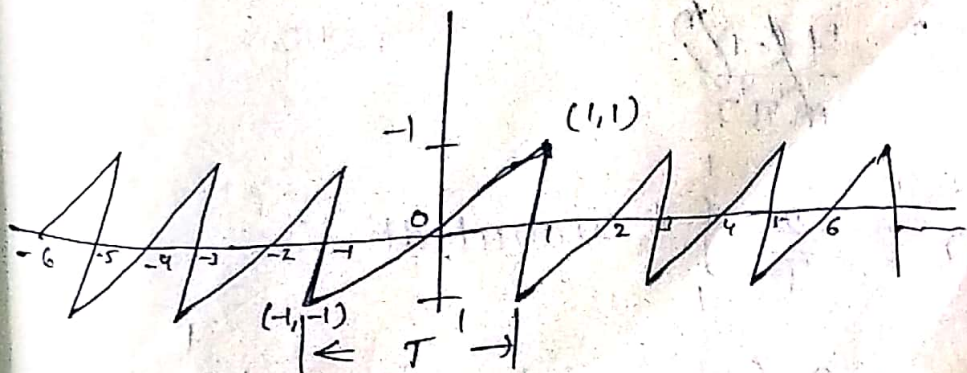
TO find b_n :

$$\int_{t_0}^{t_0+T} f(t) \sin m \omega_0 t = \int_{t_0}^{t_0+T} a_0 \sin m \omega_0 t + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos n \omega_0 t \sin m \omega_0 t dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin n \omega_0 t \sin m \omega_0 t dt$$

$$\int_{t_0}^{t_0+T} f(t) \sin m \omega_0 t dt = 0 + 0 + b_n(T/2) \quad \text{for } n=m$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n \omega_0 t dt$$

1) Find the trigonometric Fourier series for the periodic signal $x(t)$ shown in below fig



$$T = 2, \quad \omega_0 = \frac{2\pi}{T} = \pi$$

$$(-1, -1) \quad (1, 1)$$

$$y+1 = \frac{2}{2}(x+1)$$

$$x-y=0$$

$$x=y$$

$$y=x$$

$$x(t) = t$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

$$a_0 = \frac{1}{2} \int_{-1}^1 t dt = \frac{1}{2} \left(\frac{t^2}{2} \right)_{-1}^1$$

$$= \frac{1}{4} (1^2 - (-1)^2) = \frac{1}{4} \cdot 0$$

cosm not
no.

$$a_n = \frac{2}{T} \int_{-1}^1 t \cdot \cos m\pi t dt$$

$$= \frac{2}{T} \left[t \frac{\sin m\pi t}{m\pi} + \frac{\cos m\pi t}{(m\pi)^2} \right]_{-1}^1$$

$$= \frac{2}{T} \left[\left(0 + \frac{(-1)^m}{m\pi^2} \right) - \left(0 + \frac{(-1)^m}{m\pi^2} \right) \right]$$

$$= \frac{2}{T} \left[\frac{2(-1)^m}{m\pi^2} \right]$$

$$= \frac{4(-1)^m}{m\pi^2 T}$$

$$b_n = \frac{2}{T} \int_{-1}^1 t \sin m\pi t dt$$

$$= \frac{2}{T} \left[-t \frac{\cos m\pi t}{m\pi} + \frac{\sin m\pi t}{m^2\pi^2} \right]_{-1}^1$$

$$= \frac{2}{T} \left[\left(-\frac{(-1)^m}{m\pi} + 0 \right) - \left(\frac{(-1)^m}{m\pi} - 0 \right) \right]$$

$$2 \frac{-4(-1)^n}{n\pi T} = \frac{-2(-1)^n}{n\pi}$$

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

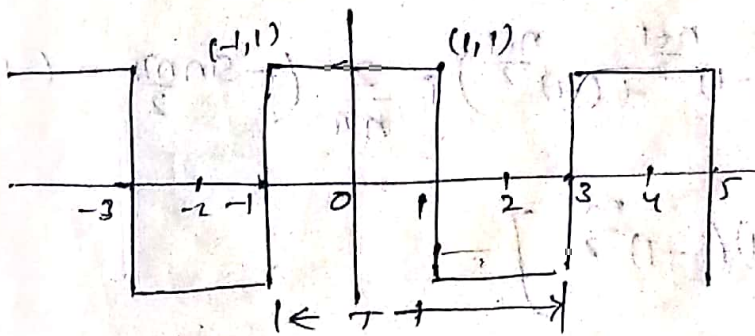
$$= 0 + 0 + \sum_{n=1}^{\infty} \frac{-4(-1)^n}{n\pi T} \sin n\omega_0 t$$

$$= \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n\pi} \sin n\omega_0 t$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\omega_0 t$$

$$\Rightarrow \frac{2}{\pi} \left[\sin \pi t - \frac{\sin 2\pi t}{2} + \frac{\sin 3\pi t}{3} - \dots \right]$$

2) Find the trigonometric fourier series for $x(t)$ show in below fig



$$T = 4, \omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$x(t) = 1 \text{ for } -1 < t < 1$$

$$-1 \text{ for } 1 < t < 3$$

$$a_0 = 0$$

$$a_n = \frac{2}{T} \int_{-1}^3 x(t) \cos n\omega_0 t \, dt$$

$$= \frac{2}{T} \frac{1}{2} \left[\int_{-1}^1 \cos n\omega_0 t \, dt + \int_1^3 -\cos n\omega_0 t \, dt \right]$$

$$= \frac{1}{2} \left[\left(\frac{\sin n\omega_0 t}{n\omega_0} \right)_{-1}^1 - \left(\frac{\sin n\omega_0 t}{n\omega_0} \right)_1^3 \right]$$

$$= \frac{1}{2} \left[\left(\frac{\sin n\pi/2 t}{n\pi/2} \right)_{-1}^1 - \left(\frac{\sin n\pi/2 t}{n\pi/2} \right)_1^3 \right]$$

$$= \frac{1}{2} \left[\frac{2}{n\pi} \left((-1)^{\frac{n+1}{2}} + (-1)^{\frac{n+1}{2}} \right) \right] = \frac{2}{n\pi} \left[\sin^3 \frac{n\pi}{2} - (-1)^{\frac{n+1}{2}} \right]$$

$$= \frac{1}{2} \left[\frac{2}{n\pi} \left((-1)^{\frac{n+1}{2}} + (-1)^{\frac{n+1}{2}} \right) \right] = \frac{2}{n\pi} \left(-\sin \frac{n\pi}{2} - (-1)^{\frac{n+1}{2}} \right)$$

$$= \frac{1}{2} \left[\frac{2}{n\pi} \left((-1)^{\frac{n+1}{2}} \right) \right]$$

$$= \frac{4}{n\pi} \left(\sin n\pi/2 \right)$$

$$b_n = \frac{2}{T} \int_{-1}^3 x(t) \sin n\omega_0 t \, dt$$

$$= \frac{1}{2} \left[\int_{-1}^1 \sin n\omega_0 t \, dt + \int_1^3 -\sin n\omega_0 t \, dt \right]$$

$$= \frac{1}{2} \left[\left(-\frac{\cos n\omega_0 t}{n\omega_0} \right)_{-1}^1 + \left(\frac{\cos n\omega_0 t}{n\omega_0} \right)_1^3 \right]$$

$$= \frac{1}{2} \left[\left(-\frac{\cos n\pi/2 t}{n\pi/2} \right)_{-1}^1 + \left(\frac{\cos n\pi/2 t}{n\pi/2} \right)_1^3 \right]$$

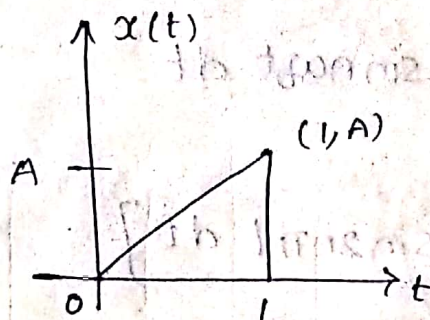
$$\frac{1}{2} \left[\frac{-1}{n\pi} (-1)^n - (-1)^n \right] + \frac{1}{n\pi} \left[(-1)^n - (-1)^n \right]$$

$$= 0$$

$$x(t) = 0 + \sum_{n=1}^{\infty} \frac{4}{n\pi} (\sin n\pi/2) \cos n\pi t$$

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3)



$$T = 1$$

$$(0,0), (1,A) \quad \omega_0 = 2\pi$$

$$y - 0 = A(x)$$

$$x(t) = y = Ax$$

$$x(t) = \underline{At}$$

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} At \, dt$$

$$= \frac{1}{T} \left[A \frac{t^2}{2} \right]_0^1$$

$$= \frac{A}{T} \left[\frac{1}{2} \right] = \frac{A}{2T} = 2\pi \frac{A}{2}$$

$$a_n = \frac{2}{T} \int_0^1 A t \cos n \omega_0 t \, dt$$

$$= 2A \int_0^1 t \cos 2n\pi t \, dt$$

$$= 2A \left[t \frac{\sin 2n\pi t}{2n\pi} + \frac{\cos 2n\pi t}{(2n\pi)^2} \right]_0^1$$

$$= 2A \left[\left(0 + \frac{1}{(2n\pi)^2} \right) - \left(0 + \frac{1}{(2n\pi)^2} \right) \right]$$

$$= \frac{4A}{(2n\pi)^2} - \frac{4A}{(2n\pi)^2} = 0$$

$$b_n = \frac{2}{T} \int_0^1 A t \sin n \omega_0 t \, dt$$

$$= \frac{2A}{T} \left[\int_0^1 t \sin 2n\pi t \, dt \right]$$

$$= 2A \left[-t \frac{\cos 2n\pi t}{2n\pi} + \frac{\sin 2n\pi t}{(2n\pi)^2} \right]_0^1$$

$$= 2A \left[\left(\frac{-1}{2n\pi} + 0 \right) - \left(0 - 0 \right) \right]$$

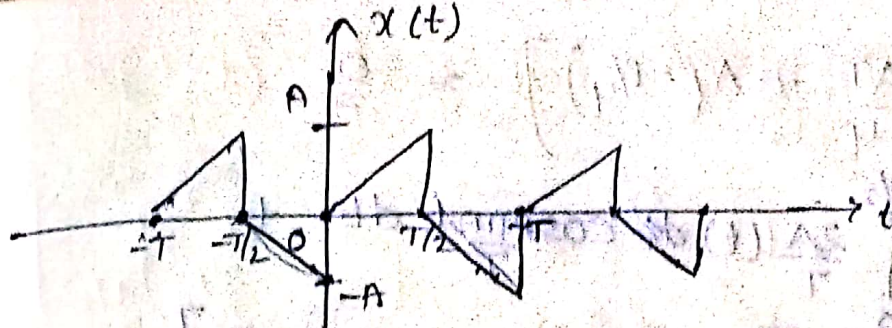
$$= \frac{-A}{n\pi}$$

$$x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{-A}{n\pi} \sin 2n\pi t$$

$$x(t) = \frac{A}{2} - \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin 2n\pi t$$

$$x(t) = \frac{A}{2} - \frac{A}{\pi} \left[\sin 2\pi t + \frac{1}{2} \sin 4\pi t + \dots \right] = \frac{A}{2}$$

4)



$$T = T, \quad \omega_0 = \frac{2\pi}{T}$$

$$(-T/2, 0), (0, A)$$

$$(0, 0), (T/2, A)$$

$$y = \frac{2A}{T} x$$

$$x(t) \cdot T = \frac{2A}{T} t, \quad 0 < t < T/2$$

$$(T/2, 0), (T, -A)$$

$$y = -\frac{A}{T/2} (x - T/2)$$

$$y = -\frac{2A}{T} (x - T/2)$$

$$y = -\frac{2A}{T} x + \frac{2A}{T} (T/2)$$

$$y = -\frac{2A}{T} x + A$$

$$x(t) = -\frac{A}{2} \left[1 - \frac{2t}{T} \right], \quad T/2 < t < T$$

$$a_0 = \frac{1}{T} \int_0^T x(t) dt$$

$$= \frac{1}{T} \left[\int_0^{T/2} \frac{2A}{T} t dt + \int_{T/2}^T \left(-\frac{A}{2} \left(1 - \frac{2t}{T} \right) \right) dt \right]$$

$$= \frac{1}{T} \left[\frac{2A}{T} \left(\frac{t^2}{2} \right)_0^{T/2} + \frac{A}{2} \left(t - \frac{t^2}{T} \right)_{T/2}^T \right]$$

$$= \frac{1}{T} \left[\frac{2A}{T} \left(\frac{T^2}{8} \right) + \frac{A}{2} \left(\left(T - \frac{T^2}{T} \right) - \left(\frac{T}{2} - \frac{T^2}{4T} \right) \right) \right]$$

$$\frac{1}{T} \left[\frac{AT}{4} + A(-T/4) \right] \approx 0$$

$$a_n = \frac{2}{T} \left[\int_0^{T/2} \frac{2A}{T} t \cos \frac{2\pi n t}{T} dt + \int_{T/2}^T A \left(1 - \frac{2}{T}t\right) \cos \frac{2\pi n t}{T} dt \right]$$

$$a_n = \frac{2}{T} \left[\frac{2A}{T} \left(\frac{t \sin \frac{2\pi n t}{T}}{\frac{2\pi n}{T}} \right) \Big|_0^{T/2} + \frac{\cos \frac{2\pi n t}{T}}{\left(\frac{2\pi n}{T}\right)^2} \Big|_0^{T/2} \right. \\ \left. + \left(\frac{A \sin \frac{2\pi n t}{T}}{\frac{2\pi n}{T}} - \frac{2A}{T} \left(\frac{t \sin \frac{2\pi n t}{T}}{\frac{2\pi n}{T}} + \frac{\cos \frac{2\pi n t}{T}}{\left(\frac{2\pi n}{T}\right)^2} \right) \right) \Big|_{T/2}^T \right]$$

$$a_n = \frac{2}{T} \left[\frac{4A}{T^2} \left(\frac{tT}{2\pi n} \right) \Big|_0^{T/2} + \frac{(-1)^n - 1}{\left(\frac{2\pi n}{T}\right)^2} \right] +$$

$$\frac{2}{T} \left[\left(\frac{2A}{T} \cdot \frac{1}{\left(\frac{2\pi n}{T}\right)^2} \right) + \frac{2A}{T} \left(0 - 0 + \frac{A(-1)^n}{\left(\frac{2\pi n}{T}\right)^2} \right) \right]$$

$$a_n = \frac{4A}{T^2} \left[\frac{(-1)^n - 1}{\left(\frac{2\pi n}{T}\right)^2} \right] + \frac{2}{T} \left[\frac{2A}{T} \cdot \frac{T}{\left(\frac{2\pi n}{T}\right)^2} + \frac{2A(-1)^n T}{\left(\frac{2\pi n}{T}\right)^2} \right]$$

$$a_n = 4A \left[\frac{(-1)^n - 1}{\left(\frac{2\pi n}{T}\right)^2} \right] + \frac{2A}{T} \left[\frac{-2T}{\left(\frac{2\pi n}{T}\right)^2} + \frac{2(-1)^n T}{\left(\frac{2\pi n}{T}\right)^2} \right]$$

$$= \frac{4A((-1)^n - 1)}{\left(\frac{2\pi n}{T}\right)^2} + \frac{4A(-1 + (-1)^n)}{\left(\frac{2\pi n}{T}\right)^2} = \frac{2A((-1)^n - 1)}{\pi^2 n^2}$$

$$b_n = \frac{2}{T} \left[\int_0^{T/2} \frac{2A}{T} t \sin \frac{2\pi n t}{T} dt + \int_{T/2}^T A \sin \frac{2\pi n t}{T} dt - \frac{2A}{T} \int_{T/2}^T t \sin \frac{2\pi n t}{T} dt \right]$$

$$b_n = \frac{2}{T} \left[\frac{2A}{T} \left(\frac{t \cos \frac{2\pi n t}{T}}{\frac{2\pi n}{T}} + \frac{\sin \frac{2\pi n t}{T}}{\left(\frac{2\pi n}{T}\right)^2} \right) \Big|_0^{T/2} + \left(\frac{A \cos \frac{2\pi n t}{T}}{\frac{2\pi n}{T}} \right) \Big|_{T/2}^T \right]$$

$$\frac{2A}{T} \left(-\frac{t \cos 2\pi n t}{2\pi n} + \frac{\sin 2\pi n t}{2\pi n} \right) \Big|_0^{T/2}$$

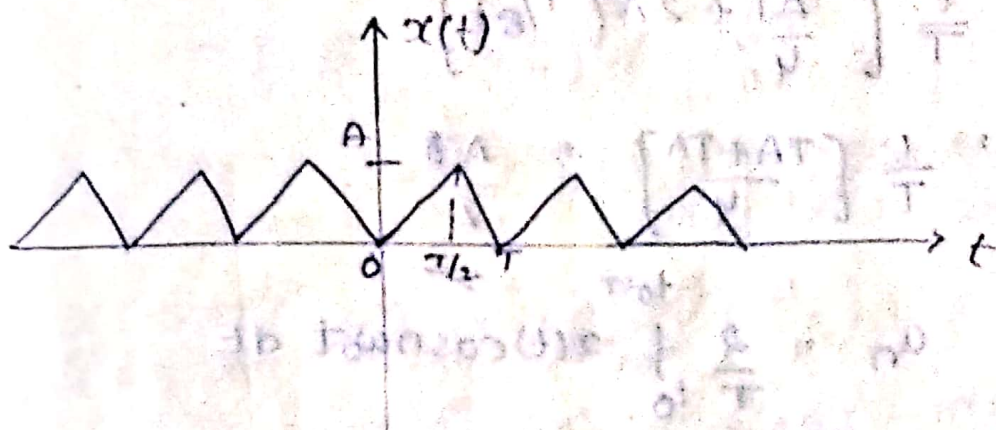
$$\frac{2}{T} \left[\frac{2A}{T} \left(-\frac{T}{2} (-1)^n + 0 \right) - \left(\frac{AT}{2\pi n} - \frac{A(-1)^n T}{2\pi n} \right) \right] +$$

$$\frac{2A}{T} \left(-\frac{T}{2} (1) + 0 + \frac{T}{4\pi n} (-1)^n - 0 \right)$$

$$\frac{2}{T} \left[-\frac{AT}{\pi n} - \frac{AT}{2\pi n} + \frac{(-1)^n AT}{2\pi n} + \frac{AT}{2\pi n} - \frac{AT(-1)^n}{2\pi n} \right]$$

$$\frac{2}{T} \left[-\frac{2AT}{\pi n} \right] = -\frac{2A}{T} \frac{1}{\pi n} - \frac{2A(-1)^n}{\pi n}$$

$$x(t) = \sum_{n=1}^{\infty} -\frac{2A}{\pi n} \sin 2\pi n \frac{t}{T}$$



$$(0,0) \quad (T/2, A)$$

$$y = \frac{2A}{T} x$$

$$x(t) = \frac{2A}{T} t \quad 0 < t < T/2$$

$$(T/2, A), (T, 0)$$

$$y - A = \frac{-A}{T/2} (x - T/2)$$

$$x(t) - A = -\frac{2A}{T} (x) + \frac{2A}{T} (T/2)$$

$$x(t) = 2A - \frac{2A}{T} (t)$$

$$= 2A \left(1 - \frac{t}{T} \right), T/2 < t < T$$

$$a_0 = \frac{1}{T} \left[\int_0^{T/2} \frac{2A}{T} (t) dt + \int_{T/2}^T 2A \left(1 - \frac{t}{T}\right) dt \right]$$

$$= \frac{1}{T} \left[\frac{2A}{T} \left(\frac{t^2}{2} \right)_0^{T/2} + 2A \left(t - \frac{t^2}{2T} \right)_{T/2}^T \right]$$

$$= \frac{1}{T} \left[\frac{2A}{T} \left(\frac{T^2}{8} \right) + 2A \left(\left(T - \frac{T}{2} \right) - \left(\frac{T}{2} - \frac{T}{8T} \right) \right) \right]$$

$$= \frac{1}{T} \left[\frac{2AT}{4} + 2A \left(T - \frac{T}{2} - \frac{T}{2} + \frac{T}{8T} \right) \right]$$

$$= \frac{1}{T} \left[\frac{AT}{4} + 2A \left(\frac{T}{8} \right) \right]$$

$$= \frac{1}{T} \left[\frac{TA + TA}{4} \right] = \frac{A}{2}$$

$t_0 + T$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos n\omega t dt$$

$$a_n = \frac{2}{T} \int_0^T x(t) \cos \frac{2\pi n}{T} t dt$$

$$a_n = \frac{2}{T} \left[\int_0^{T/2} \frac{2A}{T} t \cos \frac{2\pi n}{T} t dt + \int_{T/2}^T 2A \left(1 - \frac{t}{T}\right) \cos \frac{2\pi n}{T} t dt \right]$$

$$= \frac{2}{T} \left[\frac{2A}{T} \left(\frac{t \sin \frac{2\pi n}{T} t}{\frac{2\pi n}{T}} + \frac{\cos \frac{2\pi n}{T} t}{\left(\frac{2\pi n}{T} \right)^2} \right)_{0}^{T/2} + 2A \left(\frac{t \sin \frac{2\pi n}{T} t}{\frac{2\pi n}{T}} \right)_{T/2}^T \right]$$

$$- \frac{2A}{T} \left[\frac{t \sin \frac{2\pi n}{T} t}{\frac{2\pi n}{T}} + \frac{\cos \frac{2\pi n}{T} t}{\left(\frac{2\pi n}{T} \right)^2} \right]_{T/2}^T$$

$$= \frac{2}{T} \left[\frac{2A}{T} \left(\frac{(-1)^n T^2}{(2\pi n)^2} - \frac{T^2}{(2\pi n)^2} \right) + 2A(0) - \frac{2A}{T} \left(\frac{(+1)^n T^2}{(2\pi n)^2} - \frac{(-1)^n T^2}{(2\pi n)^2} \right) \right]$$

$$= \frac{2}{T} \left[\frac{2A(-1)^n T}{(2\pi n)^2} - \frac{2AT}{(2\pi n)^2} - \frac{2AT}{(2\pi n)^2} + \frac{2A(-1)^n T}{(2\pi n)^2} \right]$$

$$a_n = \frac{2}{T} \left[\frac{4A(-1)^n T}{4n\pi} - \frac{4AT}{4n\pi} \right] = \frac{2A(-1)^n - 1}{n\pi}$$

$$b_n = \frac{2}{T} \left[\int_0^{T/2} \frac{2A}{T} \cdot t \sin \frac{2n\pi t}{T} dt + \int_{T/2}^T 2A \left(1 - \frac{t}{T}\right) \sin \frac{2n\pi t}{T} dt \right]$$

$$= \frac{2}{T} \left[\frac{2A}{T} \left[\frac{t \cos \frac{2n\pi t}{T}}{\frac{2n\pi}{T}} + \frac{\sin \frac{2n\pi t}{T}}{\left(\frac{2n\pi}{T}\right)^2} \right]_0^{T/2} + 2A \left[\frac{-\cos \frac{2n\pi t}{T}}{\frac{2n\pi}{T}} \right]_{T/2}^T \right]$$

$$= \frac{2A}{T} \left[\left(\frac{-T \cos \frac{2n\pi T}{2}}{2n\pi} + \frac{\sin \frac{2n\pi T}{2}}{\left(\frac{2n\pi}{T}\right)^2} \right) - \left(\frac{0 \cos 0}{2n\pi} + \frac{\sin 0}{\left(\frac{2n\pi}{T}\right)^2} \right) \right] + 2A \left[\frac{-\cos 2n\pi}{2n\pi} - \frac{-\cos n\pi}{2n\pi} \right]$$

$$= \frac{2}{T} \left[\frac{2A}{T} \left(\left(\frac{-T(-1)^n}{4n\pi} + 0 \right) - (0) \right) + 2A \left(\frac{-T(-1)^n}{2n\pi} + \frac{T(-1)^n}{2n\pi} \right) \right]$$

$$= \frac{2A}{T} \left(\left(\frac{-T}{2n\pi} + 0 \right) - \left(\frac{-T(-1)^n}{4n\pi} - 0 \right) \right)$$

$$= \frac{2}{T} \left[\frac{-2AT(-1)^n}{4n\pi} + \frac{AT}{n\pi} + \frac{AT(-1)^n}{n\pi} \right] + \frac{AT}{n\pi} - \frac{A(-1)^n T}{2n\pi}$$

$$= \frac{2}{T} \left[\frac{AT(-1)^n}{2n\pi} + \frac{AT(-1)^n}{2n\pi} - \frac{AT(-1)^n}{n\pi} \right]$$

$$= \frac{2}{T} \left[\frac{-AT(-1)^n}{2n\pi} - \frac{AT}{n\pi} + \frac{AT(-1)^n}{n\pi} + \frac{AT}{n\pi} - \frac{AT(-1)^n}{2n\pi} \right]$$

$$\frac{A}{T} = \frac{A}{T}$$

$$\frac{A}{T} = \frac{A}{T}$$

$$\frac{A}{T} = \frac{A}{T}$$

$$\frac{A}{T} = \frac{A}{T}$$

$$\frac{A}{T} = \frac{A}{T}$$

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conditions
Symmetry signals :-

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt$$

$$= \frac{1}{T} \int_{t_0}^{t_0+T} (x_e(t) + x_o(t)) dt$$

$$= \frac{1}{T} \left[\int_{-T/2}^{T/2} x_e(t) dt + \int_{-T/2}^{T/2} x_o(t) dt \right]$$

$$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) = \frac{1}{2} [x(t) + x(-t)]$$

$$x_o(t) = \frac{1}{2} [x(t) - x(-t)]$$

$$= \frac{1}{T} \left[\int_{-T/2}^{T/2} \frac{1}{2} \int_{-t_0}^{t_0} x(t) dt dt + \int_{-T/2}^{T/2} \int_{-t_0}^{t_0} x(t) dt dt \right]$$

if $x(t)$ is even then

$$\int_{-t_0}^{t_0} x(t) dt = 2 \int_0^{t_0} x(t) dt$$

Here because,

$$\int_{-t_0}^{t_0} x(t) dt = \int_{-t_0}^0 x(t) dt + \int_0^{t_0} x(t) dt$$

$$= \int_0^{t_0} x(t) dt + \int_0^{t_0} x(t) dt = 2 \int_0^{t_0} x(t) dt$$

(by even cond)

Case (i) :-

If $x(t)$ is odd then

$$\int_{-t_0}^{t_0} x(t) dt = 0$$

If $x(t)$ is even, then odd component

$$x_o(t) = 0$$

$$a_0 = \frac{2}{T} \int_0^{T/2} x_e(t) dt$$

If $x(t)$ is odd then

$$a_0 = \frac{1}{T} \left[\int_{-T/2}^{T/2} x_0(t) dt \right] = 0$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos n\omega_0 t dt$$

$$= \frac{2}{T} \left[\int_{-T/2}^{T/2} x_e(t) \cos n\omega_0 t dt + \int_{-T/2}^{T/2} x_o(t) \cos n\omega_0 t dt \right]$$

case (i), If $x(t)$ is even then

$$a_n = \frac{4}{T} \int_0^{T/2} x(t) \cos n\omega_0 t dt$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin n\omega_0 t dt \quad x_e(t)$$

$$b_n = \frac{2}{T} \left[\int_{-T/2}^{T/2} x_e(t) \sin n\omega_0 t dt + \int_{-T/2}^{T/2} x_o(t) \sin n\omega_0 t dt \right]$$

$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega_0 t dt \quad \boxed{0 = b_n}$$

Case (ii) :- If $x(t)$ is odd then the even component is $x_e(t) = 0$

$$a_0 = \frac{1}{T} \int_{-T/2}^{T/2} x_0(t) dt = 0$$

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} x_0(t) \cos n\omega_0 t dt$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} x_0(t) \sin n\omega_0 t dt$$

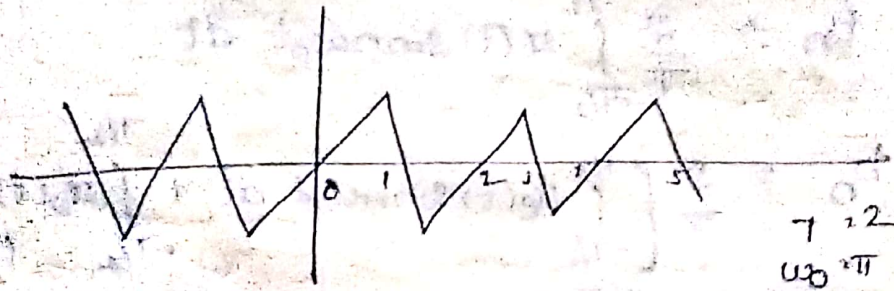
$$\boxed{b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega_0 t dt}$$

* Note that Fourier series expansion of ^{odd} periodic function contains only sine terms

* The Fourier series expansion of even periodic function contains only cosine terms

Half wave symmetric:

A periodic function satisfying the condition $x(t) = -x(t + T/2)$ is said to have a half wave symmetric. The Fourier series expansion of such type of periodic signal contains odd harmonics only.



The given figure is having antisymmetry

$$a_0 = a_n = 0$$

$b_n =$ ∴ The Fourier series expansion of the given curve consists b_n only.

$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega t dt$$

$$(0,0), (1,1)$$

$$y = x$$

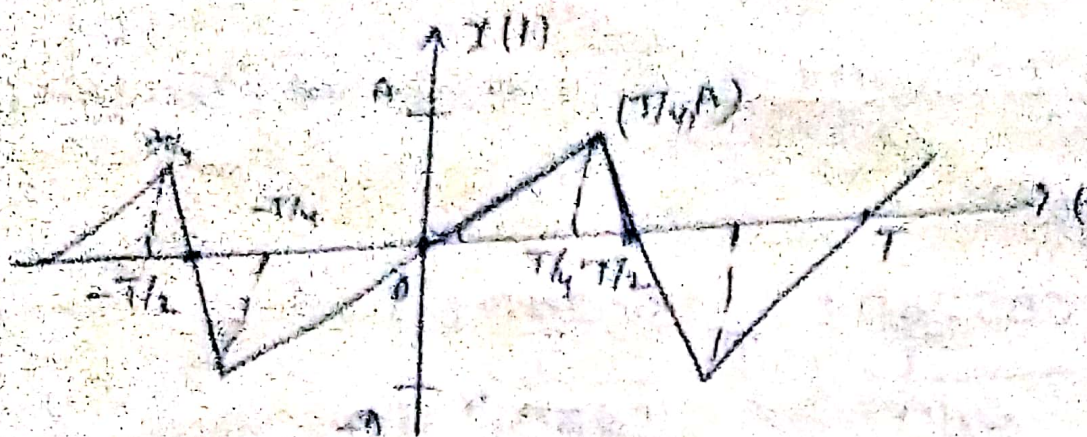
$$x(t) = t$$

$$\frac{4}{T} \int_0^1 x(t) \sin n\pi t dt$$

$$2 \int_0^1 t \sin n\pi t dt$$

$$= 2 \left[-\frac{t \cos n\pi t}{n\pi} + \frac{\sin n\pi t}{(n\pi)^2} \right]_0^1$$

$$= 2 \left[-\frac{(1)^n}{n\pi} \right] = -\frac{2(-1)^n}{n\pi}$$



$$T = T$$

$$\omega_0 = \frac{2\pi}{T}$$

$$(-T/2, 0), (0, 0), (T/2, 0), (-T/2, 0), (T/2, 0)$$

$$y = 0 \quad \text{at } T/2$$

$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin n\omega_0 t \, dt$$

$$\frac{4A}{T} (T/4)$$

$$(0, 0), (T/4, A), (T/2, 0)$$

$$y = 0$$

$$(0, 0), (T/4, A)$$

$$(T/4, A), (T/2, 0)$$

$$y = \frac{A}{T/4} (x)$$

$$(y - A) = \frac{-A}{T/4} (x - T/4)$$

$$y = \frac{4A}{T} x$$

$$y = A - \frac{4A}{T} (t - T/4)$$

$$x(t) = \frac{4A}{T} x(t)$$

$$y = A \left[1 - \frac{4}{T} t + 1 \right]$$

$$y = A \left[2 - \frac{4}{T} t \right]$$

$$b_n = \frac{4}{T} \int_0^{T/4} \frac{4A}{T} (t) \, dt + \int_{T/4}^{T/2} A \left(2 - \frac{4}{T} t \right) \, dt$$

$$= \frac{4}{T} \left[\int_0^{T/4} \frac{4A}{T} t \sin \frac{2\pi n}{T} t dt + \int_{T/4}^{T/2} A \left(2 - \frac{4t}{T} \right) \sin \frac{2\pi n}{T} t dt \right]$$

$$= \frac{4}{T} \left[\frac{4A}{T} \left[-t \cos \frac{2\pi n}{T} t + \frac{\sin \frac{2\pi n}{T} t}{\left(\frac{2\pi n}{T} \right)} \right]_0^{T/4} + \right.$$

$$\left. A \left[-\cos \frac{2\pi n}{T} t \right]_{T/4}^{T/2} - \frac{4}{T} \left[-t \cos \frac{2\pi n}{T} t + \frac{\sin \frac{2\pi n}{T} t}{\left(\frac{2\pi n}{T} \right)} \right]_{T/4}^{T/2} \right]$$

$$= \frac{4}{T} \left[\frac{4A}{T} \left(\frac{-T^2 \cos \frac{n\pi}{2}}{4(2\pi n)} + \frac{T^2 \sin \frac{n\pi}{2}}{(2\pi n)^2} \right) + \right.$$

$$\left. A \left(\frac{-T^2 \cos \frac{n\pi}{2}}{2\pi n} + \frac{T^2 \sin \frac{n\pi}{2}}{2\pi n} \right) - \right.$$

$$\left. \frac{4A}{T} \left(\frac{-T^2 (-1)^n}{2(2\pi n)} + 0 \right) - \left(\frac{-T^2 \cos \frac{n\pi}{2}}{4(2\pi n)} + \frac{T^2 \sin \frac{n\pi}{2}}{(2\pi n)^2} \right) \right]$$

$$= \frac{4}{T} \left[\frac{-AT \cos n\pi/2}{2\pi n} + \frac{AT \sin n\pi/2}{(2\pi n)^2} - \frac{AT (-1)^n}{2\pi n} + \frac{AT \cos n\pi/2}{2\pi n} \right.$$

$$\left. - \frac{AT (-1)^n}{\pi n} - \frac{AT \cos n\pi/2}{2\pi n} + \frac{AT \sin n\pi/2}{(\pi n)^2} \right]$$

$$= \frac{8A (-1)^n}{\pi n}$$

cosine representation (polar representation) ^{fourier series}

The trigonometric fourier series of $x(t)$ contains sine and cosine terms. By using trigonometric identity we can write

$$a_n \cos n\omega_0 t + b_n \sin n\omega_0 t = A_n \cos(n\omega_0 t + \theta_n)$$

Thus we obtain cosine representation of $x(t)$ which contains sinusoids of frequencies $\omega_0, 2\omega_0, 3\omega_0, \dots$

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n (\cos(n\omega_0 t + \theta_n))$$

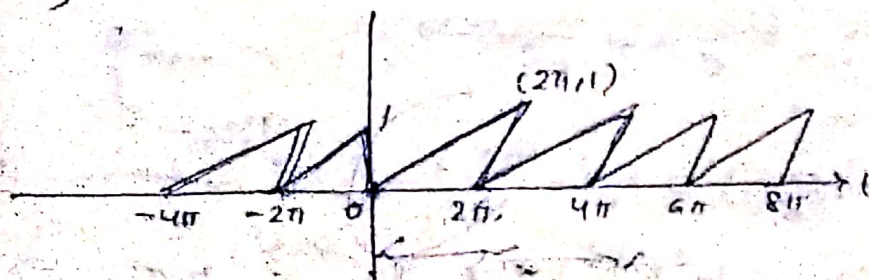
$$\text{where } A_0 = a_0$$

$$A_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

find the cosine representation fourier for half wave rectifier.

1)



$$T = 2\pi$$

$$\omega_0 = 1$$

$$(0,0), (2\pi,1)$$

$$y-0 = \frac{1}{2\pi} x$$

$$\therefore x(t) = \frac{t}{2\pi}$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2\pi} (t) dt$$

$$\Rightarrow \frac{1}{4\pi^2} \left(\frac{t^2}{2} \right)_0^{2\pi} = \frac{1}{4\pi^2} \left(\frac{4\pi^2}{2} \right) = \frac{1}{2}$$

Exponential Fourier series: (complex Fourier Series)

$$\text{let } f_1(t) = e^{jn\omega_0 t}, f_2(t) = e^{jm\omega_0 t}$$

$$\int_{t_0}^{t_0+T} e^{jn\omega_0 t} (e^{jm\omega_0 t})^* dt$$

$$\int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jm\omega_0 t} dt$$

$$\int_{t_0}^{t_0+T} e^{j(n-m)\omega_0 t} dt$$

$$\frac{e^{j(n-m)\omega_0 t}}{j(n-m)\omega_0} \Big|_{t_0}^{t_0+T}$$

$$\frac{1}{j(n-m)\omega_0} \cdot e^{j(n-m)\omega_0 t_0} \left[e^{j(n-m)2\pi} - 1 \right]$$

$$= 0$$

$\therefore f_1(t)$ & $f_2(t)$ are orthogonal

$$\text{Complete set } \left\{ 1, e^{j\omega_0 t}, e^{2j\omega_0 t}, e^{3j\omega_0 t}, \dots, e^{n_1 j\omega_0 t}, e^{-j\omega_0 t}, e^{-2j\omega_0 t}, \dots, e^{-n_2 j\omega_0 t} \right\}$$

$$f(t) = F_0(1) + F_1(e^{j\omega_0 t}) + F_2(e^{2j\omega_0 t}) + \dots + F_{n_1}(e^{n_1 j\omega_0 t}) + F_{-1}(e^{-j\omega_0 t}) + F_{-2}(e^{-2j\omega_0 t}) + \dots + F_{-n_2}(e^{-n_2 j\omega_0 t})$$

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

F_n = Fourier coefficients

F_n for complex functions

$$C_{12} = \int_{T_1}^{T_2} f_1(t) f_2^*(t) dt$$

$$\frac{\int_{T_1}^{T_2} f_1^*(t) f_2(t) dt}{\int_{T_1}^{T_2} f_2^*(t) f_2(t) dt}$$

$$f(t) = \int_{t_0}^{t_0+T} f(t) e^{-jn\omega t} dt$$

$$f(t) = \int_{t_0}^{t_0+T} f(t) e^{-jn\omega t} dt$$

$$f(t) = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega t} dt$$

The relationship b/w Trigonometric Fourier series and exponential Fourier series.

FS representation of $x(t)$ is

$$x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega t + b_n \sin n\omega t]$$

ES representation of $x(t)$ is

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega t}$$

$$x(t) = F_0 + F_1 e^{j\omega t} + F_2 e^{j2\omega t} + \dots + e^{-j\omega t} + F_{-1} e^{-j\omega t} + F_{-2} e^{-j2\omega t} + \dots - \infty$$

$$= F_0 + F_1 (\cos \omega t + j \sin \omega t) + F_2 (\cos 2\omega t + j \sin 2\omega t) + \dots + e^{-j\omega t} + F_{-1} (\cos \omega t - j \sin \omega t) + F_{-2} (\cos 2\omega t - j \sin 2\omega t) + \dots - \infty$$

$$x(t) = F_0 + \cos \omega t (F_1 + F_{-1}) + \cos 2\omega t (F_2 + F_{-2}) + \dots + (-F_1 + F_{-1}) j \sin \omega t + (F_2 - F_{-2}) j \sin 2\omega t + \dots$$

$$x(t) = F_0 + \sum_{n=1}^{\infty} (F_n + F_{-n}) \cos n\omega t + j \sum_{n=1}^{\infty} (F_n - F_{-n}) \sin n\omega t$$

$$a_0 = F_0, a_n = F_n + F_{-n}, b_n = j(F_n - F_{-n})$$

$$\text{Conversely } F_n = \frac{1}{2} [a_n - j b_n], F_{-n} = \frac{1}{2} [a_n + j b_n]$$

* Derive the polar series from exponential fourier series representation and hence

P.V that $|D_n| = 2|C_n|$

E.P.S

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \rightarrow (1)$$

$$= \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t - cn}$$

$$\text{where } F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) e^{-jn\omega_0 t} dt$$

$$f(t) = F_0 + \sum_{n=1}^{\infty} F_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t}$$

$$= F_0 + \sum_{n=1}^{\infty} F_n e^{jn\omega_0 t} + \sum_{n=1}^{\infty} F_{-n} e^{-jn\omega_0 t} \rightarrow (2)$$

$$F_0 = a_0$$

$$F_n = \frac{1}{2} (a_n - jb_n)$$

$$F_{-n} = \frac{1}{2} (a_n + jb_n) \rightarrow (3)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n - jb_n}{2} e^{jn\omega_0 t} + \sum_{n=1}^{\infty} \frac{a_n + jb_n}{2} e^{-jn\omega_0 t}$$

$$f(t) = a_0 + \frac{1}{2} \sum_{n=1}^{\infty} a_n e^{jn\omega_0 t} - \frac{j}{2} \sum_{n=1}^{\infty} b_n e^{jn\omega_0 t} + \frac{1}{2} \sum_{n=1}^{\infty} a_n e^{-jn\omega_0 t} + \frac{j}{2} \sum_{n=1}^{\infty} b_n e^{-jn\omega_0 t}$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{jn\omega_0 t} + e^{-jn\omega_0 t}}{2} \right) - j \sum_{n=1}^{\infty} b_n \left(\frac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2} \right)$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Polar representation of above eqⁿ

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$

$$f(t) = D_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n)$$

where $D_0 = a_0$, $D_n = \sqrt{a_n^2 + b_n^2}$, $\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$

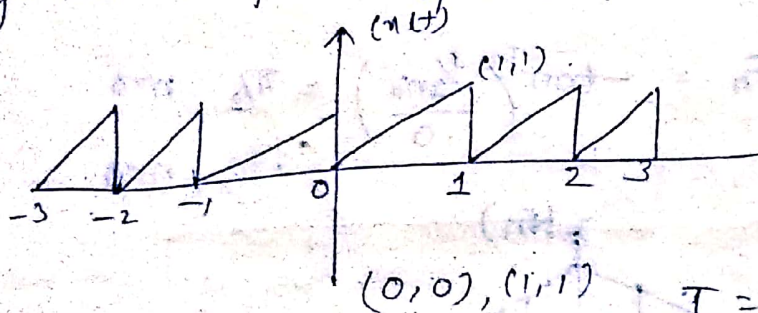
$$|c_n| = \frac{1}{2} |a_n - jb_n|$$

$$= \frac{1}{2} [\sqrt{a_n^2 + b_n^2}]$$

$$= \frac{1}{2} [D_n]$$

$$D_n = 2|c_n|$$

1) find the exponential fourier series of sawtooth waveform as shown in fig. plot the magnitude & phase spectrum.



$$y = x$$

$$x(t) = t$$

$$T = 1$$

$$\omega_0 = 2\pi$$

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$

$$F_n = \frac{1}{T} \int_0^1 t e^{-jn\omega_0 t} dt$$

$$F_n = \frac{1}{T} \left[\frac{te^{-jn\omega_0 t}}{(-jn\omega_0)} - \frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2} \right]_0^1$$

$$F_n = \left[\left(\frac{e^{-jn\omega_0}}{-jn\omega_0} - \frac{e^{-jn\omega_0}}{(-jn\omega_0)^2} \right) - \left(0 - \frac{1}{(-jn\omega_0)^2} \right) \right]$$

$$= \left[\frac{e^{-jn2\pi}}{-jn\omega_0} + \frac{e^{-jn2\pi}}{n^2\omega_0^2} - \frac{1}{n^2\omega_0^2} \right]$$

$$\left[\frac{1}{-jn\omega_0} + \frac{1}{jn\omega_0} - \frac{1}{jn\omega_0} \right]$$

$$\Rightarrow \left[\frac{-1}{j2n\pi} \right] = \frac{j}{2n\pi}$$

$$F_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$

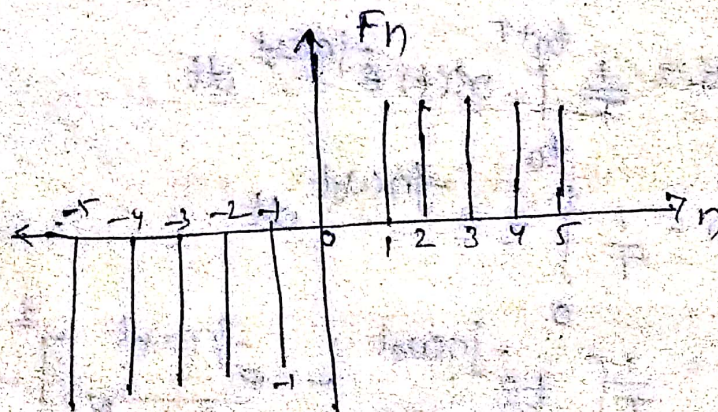
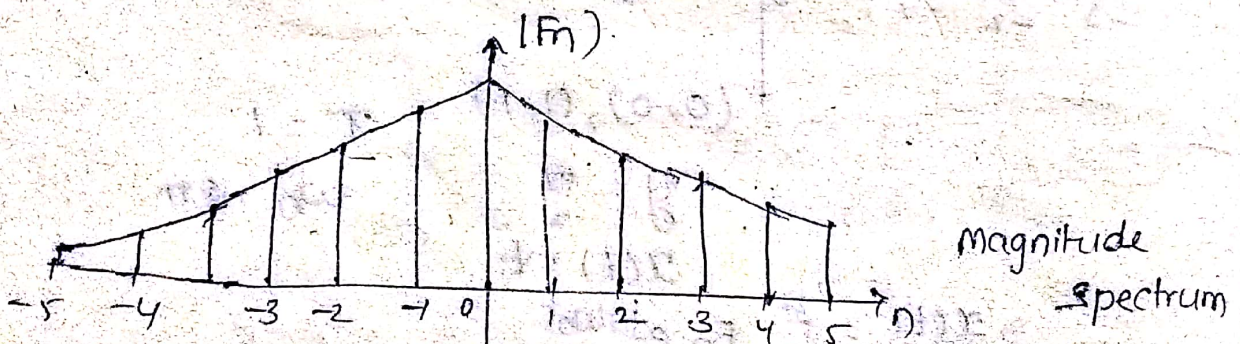
$$\int_0^1 t dt = \left. \frac{t^2}{2} \right|_0^1 = \frac{1}{2}$$

$$x(t) = \frac{1}{2} + \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \left(\frac{j}{2n\pi} \right) e^{jn2\pi t}$$

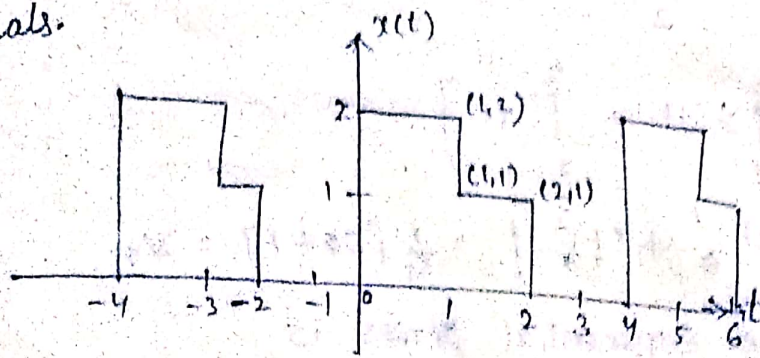
$$|F_n| = \left| \frac{j}{2n\pi} \right| = \sqrt{0 + \left(\frac{1}{2n\pi} \right)^2} = \frac{1}{2n\pi}$$

$$\angle F_n = -\tan^{-1} \left(\frac{1/2n\pi}{0} \right) = \pi/2 \quad n > 0$$

$$= -\pi/2 \quad n < 0$$



2) compute the exponential series of the following signals.



$$T = 4, \quad \omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$(0,2), (1,2)$$

$$(1,1), (2,1)$$

$$y-2 = 0(x-0)$$

$$y-1 = 0$$

$$y = 2$$

$$y = 1$$

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega t}$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega t} dt$$

$$= \frac{1}{4} \int_0^4 x(t) e^{-jn\omega t} dt$$

$$= \frac{1}{4} \left[\int_0^1 2e^{-jn\omega t} dt + \int_1^2 e^{-jn\omega t} dt \right]$$

$$= \frac{1}{4} \left[2 \left(\frac{e^{-jn\omega t}}{-jn\omega} \right)_0^1 + \left(\frac{e^{-jn\omega t}}{-jn\omega} \right)_1^2 \right]$$

$$= \frac{1}{4} \left[2 \left(\frac{e^{-jn\omega}}{-jn\omega} + \frac{1}{jn\omega} \right) + \left(\frac{e^{-j2n\omega}}{-jn\omega} + \frac{e^{-jn\omega}}{jn\omega} \right) \right]$$

$$= \frac{1}{4} \left[-\frac{e^{-jn\omega}}{jn\omega} + \frac{2}{jn\omega} - \frac{e^{-j2n\omega}}{jn\omega} \right]$$

$$= \frac{1}{4} \left[-\frac{e^{-jn\pi/2}}{jn\pi/2} + \frac{2}{jn\pi/2} - \frac{e^{-j2n\pi/2}}{jn\pi/2} \right]$$

$$= \frac{1}{4} \left[-\frac{2e^{-jn\pi/2}}{jn\pi} + \frac{4}{jn\pi} - \frac{2e^{-jn\pi}}{jn\pi} \right]$$

$$= \frac{1}{jn\pi} \left[1 - \frac{1}{2} e^{-jn\pi/2} - \frac{1}{2} e^{-jn\pi} \right]$$

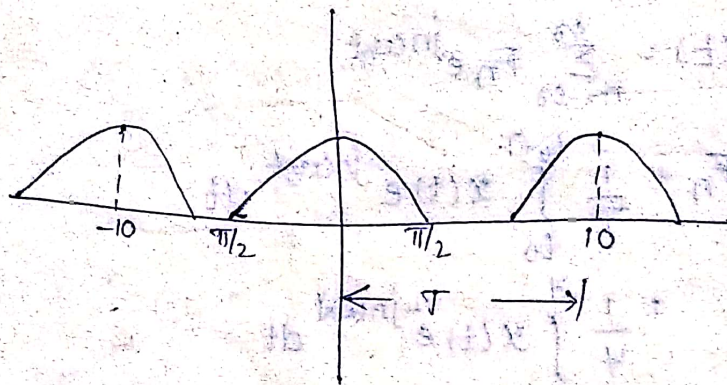
$$F_0 = \frac{1}{4} \left[\int_0^1 2 dt + \int_1^2 dt \right]$$

$$= \frac{1}{4} \left[(2t)_0^1 + (t)_1^2 \right], \quad \frac{1}{4} [2 + 1] = 3/4$$

$\therefore$ The exponential series is

$$x(t) = \frac{3}{4} + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{jn\pi} \left[1 - \frac{1}{2} e^{-jn\pi/2} - \frac{1}{2} e^{-jn\pi} \right]$$

3) Find the exponential fourier series for the fig shown in the below.



$$T = 10, \quad \omega = 2\pi/10$$

$$x(t) = \cos t \quad -\pi/2 < t < \pi/2$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} \cos t e^{-jn\pi/5} dt$$

$$F_n = \frac{1}{T} \int_{-\pi/2}^{\pi/2} \left(\frac{e^{jt} + e^{-jt}}{2} \right) e^{-jn\pi/5} dt$$

$$F_n = \frac{1}{T} \left[\int_{-\pi/2}^{\pi/2} \frac{e^{jt} \cdot e^{-jn\pi/5}}{2} dt + \int_{-\pi/2}^{\pi/2} \frac{e^{-jt} \cdot e^{-jn\pi/5}}{2} dt \right]$$

$$= \frac{1}{2T} \left[\int_{-\pi/2}^{\pi/2} e^{jt} \cdot e^{-jn\pi/5} dt + \int_{-\pi/2}^{\pi/2} e^{-jt} \cdot e^{-jn\pi/5} dt \right]$$

$$= \frac{1}{2T} \left[\int_{-\pi/2}^{\pi/2} e^{jt} \cdot e^{-jn\omega t} dt + \int_{-\pi/2}^{\pi/2} e^{-jt} \cdot e^{-jn\omega t} dt \right]$$

$$= \frac{1}{2T} \left[\int_{-\pi/2}^{\pi/2} e^{jt(1-n\omega_0)} dt + \int_{-\pi/2}^{\pi/2} e^{-jt(1+n\omega_0)} dt \right]$$

$$= \frac{1}{2T} \left[\frac{e^{jt(1-n\omega_0)}}{j(1-n\omega_0)} \Big|_{-\pi/2}^{\pi/2} + \frac{e^{-jt(1+n\omega_0)}}{-j(1+n\omega_0)} \Big|_{-\pi/2}^{\pi/2} \right]$$

$$= \frac{1}{2T} \left[\frac{1}{j(1-n\omega_0)} \left(e^{j\pi/2(1-n\omega_0)} - e^{-j\pi/2(1-n\omega_0)} \right) - \frac{1}{j(1+n\omega_0)} \left(e^{-j\pi/2(1+n\omega_0)} - e^{j\pi/2(1+n\omega_0)} \right) \right]$$

$$= \frac{1}{2T} \left[\frac{1}{j(1-n\omega_0)} \left[e^{j\pi/2} \cdot e^{-j\pi/2 n\omega_0} - e^{-j\pi/2} \cdot e^{j\pi/2 n\omega_0} \right] - \frac{1}{j(1+n\omega_0)} \left[e^{-j\pi/2} \cdot e^{-j\pi/2 n\omega_0} - e^{j\pi/2} \cdot e^{j\pi/2 n\omega_0} \right] \right]$$

$$= \frac{1}{2T} \left[\frac{1}{j(1-n\omega_0)} \left[j e^{-j\pi/2 n\omega_0} + j \cdot e^{j\pi/2 n\omega_0} \right] - \frac{1}{j(1+n\omega_0)} \left(-j e^{-j\pi/2 n\omega_0} - j e^{j\pi/2 n\omega_0} \right) \right]$$

$$= \frac{1}{2T} \left[\frac{j}{j(1-n\omega_0)} \left(e^{-j\pi/2 n\omega_0} + e^{j\pi/2 n\omega_0} \right) + \frac{j}{j(1+n\omega_0)} \left(e^{-j\pi/2 n\omega_0} + e^{j\pi/2 n\omega_0} \right) \right]$$

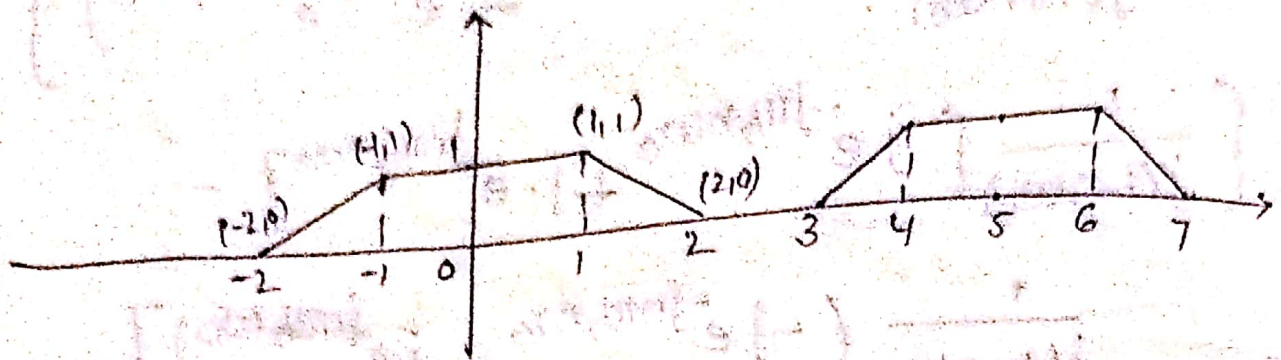
$$= \frac{1}{2T} \left[\frac{j}{(1-n\omega_0)} \left(\sin n\pi/2 \omega_0 \right) + \frac{j}{(1+n\omega_0)} \left(\sin n\pi/2 \omega_0 \right) \right]$$

$$= \frac{1}{T} \left[j \sin n\pi/2 \omega_0 \left(\frac{1}{1-n\omega_0} + \frac{1}{1+n\omega_0} \right) \right]$$

$$= \frac{j}{T} \sin n\pi/2 \omega_0 \left(\frac{2}{1-n^2\omega_0^2} \right)$$

$$= \frac{2j}{T} \sin n\pi/2 \omega_0$$

4) Find the exponential series for the given signal.



$$T = 5, \omega_0 = 2\pi/5$$

$$(-2, 0), (-1, 1)$$

$$y - 0 = \frac{1}{1} (x + 2)$$

$$y = x + 2$$

$$x(t) = t + 2 \quad -2 < t < -1$$

$$(1, 1), (2, 0)$$

$$y - 1 = \frac{-1}{1} (x - 1)$$

$$y = -x + 2$$

$$x(t) = -t + 2 \quad 1 < t < 2$$

$$x(t) = 1, -1 < t < 1, \quad x(t) = 0 \quad 2 < t < 3.$$

$$x(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt$$

$$F_n = \frac{1}{T} \left[\int_{-2}^3 x(t) e^{-jn\omega_0 t} dt \right]$$

$$F_n = \frac{1}{T} \left[\int_{-2}^{-1} (t+2) e^{-jn\omega_0 t} dt + \int_{-1}^1 t e^{-jn\omega_0 t} dt + \int_1^2 (-t+2) e^{-jn\omega_0 t} dt \right]$$

$$\begin{aligned}
 f_n &= \frac{1}{T} \left[\int_{-2}^{-1} t e^{-jn\omega_0 t} dt + 2 \int_{-2}^{-1} e^{-jn\omega_0 t} dt + \int_{-1}^1 e^{-jn\omega_0 t} dt \right. \\
 &\quad \left. + \int_1^2 t e^{-jn\omega_0 t} dt + 2 \int_1^2 e^{-jn\omega_0 t} dt \right] \\
 &= \frac{1}{T} \left[\left(\frac{t e^{-jn\omega_0 t}}{-jn\omega_0} - \frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2} \right)_{-2}^{-1} + 2 \left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right)_{-2}^{-1} \right. \\
 &\quad \left(\frac{t e^{-jn\omega_0 t}}{-jn\omega_0} - \frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2} \right)_{-1}^1 - \left(\frac{t e^{-jn\omega_0 t}}{-jn\omega_0} - \frac{e^{-jn\omega_0 t}}{(-jn\omega_0)^2} \right)_{-1}^1 \\
 &\quad \left. + 2 \left(\frac{e^{-jn\omega_0 t}}{-jn\omega_0} \right)_{-1}^1 \right] \\
 &= \frac{1}{T} \left[\left(\frac{e^{jn\omega_0}}{jn\omega_0} + \frac{e^{jn\omega_0}}{n^2\omega_0^2} \right) - \left(\frac{2e^{j2n\omega_0}}{jn\omega_0} + \frac{e^{j2n\omega_0}}{n^2\omega_0^2} \right) + \right. \\
 &\quad \left. - 2 \left(\frac{e^{jn\omega_0}}{jn\omega_0} - \frac{e^{j2n\omega_0}}{jn\omega_0} \right) + \left(\frac{e^{-jn\omega_0}}{-jn\omega_0} + \frac{e^{-jn\omega_0}}{n^2\omega_0^2} \right) - \left(\right. \right. \\
 &\quad \left. \left. \frac{e^{jn\omega_0}}{jn\omega_0} + \frac{e^{jn\omega_0}}{n^2\omega_0^2} \right) - \left(\left(\frac{-2e^{-j2n\omega_0}}{jn\omega_0} + \frac{e^{-j2n\omega_0}}{n^2\omega_0^2} \right) - \left(\frac{-e^{-jn\omega_0}}{jn\omega_0} + \frac{e^{-jn\omega_0}}{n^2\omega_0^2} \right) \right) + \right. \\
 &\quad \left. \left. 2 \left(\frac{-e^{-j2n\omega_0}}{jn\omega_0} + \frac{e^{-jn\omega_0}}{jn\omega_0} \right) \right] \right] \\
 &= \frac{1}{T} \left[\frac{e^{j2n\omega_0}}{n^2\omega_0^2} + \frac{e^{-j2n\omega_0}}{n^2\omega_0^2} + \frac{e^{jn\omega_0}}{n^2\omega_0^2} + \frac{e^{-jn\omega_0}}{n^2\omega_0^2} \right] \\
 &= \frac{1}{T} \left[\frac{1}{n^2\omega_0^2} \left[(e^{jn\omega_0} - e^{-jn\omega_0}) - (e^{j2n\omega_0} + e^{-j2n\omega_0}) \right] \right] \\
 &= \frac{1}{T n^2 \omega_0^2} [2j \sin n\omega_0 - 2 \cos 2n\omega_0] \\
 &= \frac{2}{T n^2 \omega_0^2} \left[j \sin \frac{2\pi}{5} - \cos \frac{4\pi}{5} \right] \\
 &= \frac{5}{2n^2 \omega_0^2} \left[j \sin \left(\frac{2\pi}{5} + \frac{\pi}{5} \right) - \cos \left(\frac{\pi}{5} - \frac{\pi}{5} \right) \right]
 \end{aligned}$$

$$F_n = \frac{5}{2n\pi} [j \sin n\pi/5 + \cos n\pi/5]$$

$$= \frac{5}{2n\pi} [e^{jn\pi/5}]$$

$$F_0 = \frac{1}{T} \left[\int_{-2}^{-1} (t+2) dt + \int_{-1}^1 dt + \int_1^2 (-t+2) dt \right]$$

$$= \frac{1}{T} \left[\left(\frac{t^2}{2} + 2t \right)_{-2}^{-1} + (t)_{-1}^1 + \left(-\frac{t^2}{2} + 2t \right)_1^2 \right]$$

$$= \frac{1}{T} \left[\left(\frac{1}{2} - 2 \right) - \left(2 - 4 \right) + \left(1 - (-1) \right) + \left((-2 + 4) - \left(-\frac{1}{2} + 2 \right) \right) \right]$$

$$= \frac{1}{T} \left[\frac{1}{2} - 2 + 2 + 2 - 2 + \frac{1}{2} - 2 \right] = 3/5$$

$$x(t) = \frac{3}{5} + \sum_{\substack{n=-\infty \\ (n \neq 0)}}^{\infty} \frac{5}{2n\pi} [e^{jn\pi/5}] \cdot e^{jn2\pi t/5}$$

Fourier Series & Fourier Transform

Fourier Series is used to convert time domain signal to frequency domain signal for representation. It is applicable to periodic signal.

- 1) Trigonometric Fourier Series
- 2) Exponential Fourier Series
- 3) Polar Fourier Series.

Trigonometric Fourier Series :-

It is represented by the function

$$f(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos n\omega_0 t + b_n \sin n\omega_0 t].$$

Here a_0, a_n, b_n are Fourier co-eff's given by

$$a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \cdot dt$$

$$a_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \cos n\omega_0 t \cdot dt$$

$$b_n = \frac{2}{T} \int_{t_0}^{t_0+T} f(t) \sin n\omega_0 t \cdot dt.$$

Symmetry Conditions :- of Trigonometric Fourier Series :-

If $f(t)$ is even then

$$a_0 = \frac{2}{T} \int_0^{T/2} f(t) \cdot dt, \quad a_n = \frac{4}{T} \int_0^{T/2} x(t) \cdot \cos n\omega_0 t \cdot dt.$$

$$b_n = 0$$

If $f(t)$ is odd, then
 $a_0 = 0, a_n = 0, b_n = \frac{4}{T} \int_0^{T/2} x(t) \cdot \sin n\omega_0 t \cdot dt$.

Cosine Representation of Trigonometric Fourier Series:
 (b) Polar
 $x(t) = A_0 + \sum_{n=1}^{\infty} A_n (\cos(n\omega_0 t + \theta_n))$.

where $A_0 = a_0$

$$A_n = \sqrt{a_n^2 + b_n^2}$$

$$\theta_n = -\tan^{-1}\left(\frac{b_n}{a_n}\right)$$

Exponential Fourier Series :-

It is represented by the equation $f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$

$$F_n = \frac{1}{T} \int_{t_0}^{t_0+T} f(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

Relationship between Trigonometric Fourier Series and Exponential Fourier Series :-

$$x(t) = F_0 + \sum_{n=1}^{\infty} (F_n + F_{-n}) \cos n\omega_0 t + j \sum_{n=1}^{\infty} (F_n - F_{-n}) \sin n\omega_0 t$$

This is Eq for trigonometric Fourier Series in terms of exponential Fourier Series.

Here $a_0 = F_0$

$$a_n = F_n + F_{-n}$$

$$b_n = F_n - F_{-n}$$

and conversely

$$F_n = \frac{1}{2} [a_n - j b_n] \quad F_{-n} = \frac{1}{2} [a_n + j b_n]$$

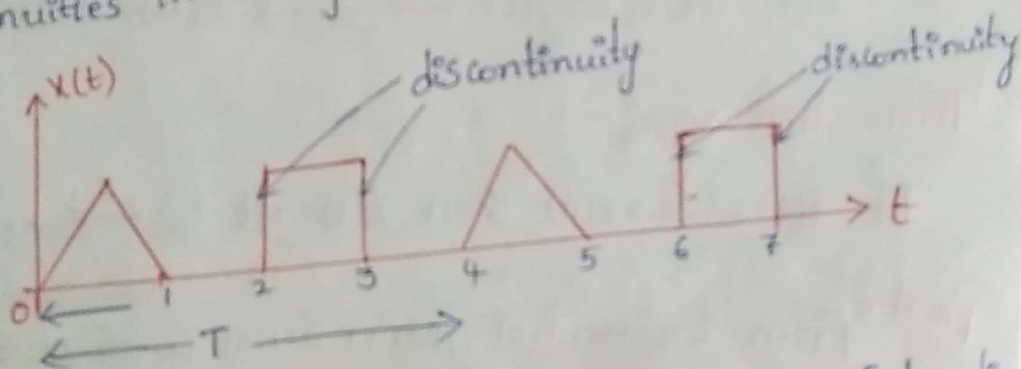
Dirichlet conditions :-

The conditions under which a periodic signal can be represented by a Fourier Series are known as Dirichlet conditions; named after the mathematician Dirichlet who first found them.

1. The function $x(t)$ have only a finite number of Maxima and minima.
2. The function $x(t)$ have a finite number of discontinuities.
3. The function $x(t)$ is absolutely integrable over one period, that is

$$\int_0^T |x(t)| dt < \infty$$

The Dirichlet conditions permit a finite number of discontinuities in each period of $x(t)$ as shown in below.



Let there is a discontinuity at time t_d . An interesting aspect of Fourier Series is that the Series at the point of discontinuity t_d converges to an average of the left hand and right hand limits of $x(t_d)$ is given by

$$x(t_d) = \frac{x(t_d^+) + x(t_d^-)}{2}$$

Properties of Fourier Series :-

Linearity :-

If $x(t) \xleftrightarrow{\text{F.S.}} X(n)$ & $y(t) \xleftrightarrow{\text{F.S.}} Y(n)$, then
 $Z(t) = ax(t) + by(t) \xleftrightarrow{\text{F.S.}} Z(n) = aX(n) + bY(n)$.

Proof :- From exponential Fourier Series co-efficient of $Z(t)$ is

$$Z(n) = \frac{1}{T} \int_0^T Z(t) \cdot e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T [ax(t) + by(t)] \cdot e^{-jn\omega_0 t} dt$$

$$= \frac{1}{T} \int_0^T a \cdot x(t) \cdot e^{-jn\omega_0 t} dt + \frac{b}{T} \int_0^T y(t) \cdot e^{-jn\omega_0 t} dt$$

$$= a \cdot \frac{1}{T} \int_0^T x(t) \cdot e^{-jn\omega_0 t} dt + b \cdot \frac{1}{T} \int_0^T y(t) \cdot e^{-jn\omega_0 t} dt$$

$$Z(n) = aX(n) + bY(n)$$

$$\therefore Z(t) = ax(t) + by(t) \xleftrightarrow{\text{F.S.}} Z(n) = aX(n) + bY(n)$$

Time shifting :-

If $x(t) \xleftrightarrow{\text{F.S.}} X(n)$ then $Z(t) = x(t - t_0) \xleftrightarrow{\text{F.S.}} e^{-jn\omega_0 t_0} X(n) = Z(n)$

Proof :- From Exponential Fourier Series co-eff of $Z(t)$ is

$$Z(n) = \frac{1}{T} \int_0^T Z(t) \cdot e^{-jn\omega_0 t} dt$$

Similarly, Exponential Fourier Series co-eff of $x(t - t_0)$ is

$$Z(n) = \frac{1}{T} \int_0^T x(t - t_0) \cdot e^{-jn\omega_0 t} dt$$

$$\text{Put } t - t_0 = m. \Rightarrow t = m + t_0$$

$$dt = dm$$

$$\text{Here upper limits } t = T \Rightarrow m = T - t_0$$

$$\text{lower limits } t = 0 \Rightarrow m = 0 - t_0 = -t_0, \text{ then}$$

$$Z(n) = \frac{1}{T} \int_{-t_0}^{T-t_0} x(m) e^{-jn\omega_0(m+t_0)} dm$$

$$= \frac{1}{T} \int_{-t_0}^{T-t_0} x(m) \cdot e^{-jn\omega_0 m} \cdot e^{-jn\omega_0 t_0} dm$$

$$= \frac{1}{T} \cdot e^{-jn\omega_0 t_0} \cdot \frac{1}{T} \int_{-t_0}^{T-t_0} x(m) \cdot e^{-jn\omega_0 m} dm$$

$$= e^{-jn\omega_0 t_0} \cdot x(m)$$

$$= e^{-jn\omega_0 t_0} \cdot x(t - t_0)$$

$$Z(t) = x(t - t_0) \xleftrightarrow{\text{F.S}} Z(n) = e^{-jn\omega_0 t_0} \cdot x(t - t_0)$$

$$= e^{-jn\omega_0 t_0} \cdot \frac{1}{T} \int_0^{T-t_0} x(m) \cdot e^{-jn\omega_0 m} dt$$

this is formula for $X(n)$

$$= e^{-jn\omega_0 t_0} \cdot X(n)$$

$$Z(t) = x(t - t_0) \xleftrightarrow{\text{F.S}} Z(n) = e^{-jn\omega_0 t_0} \cdot X(n)$$

Frequency Shift :-

If $x(t) \xleftrightarrow{F.S} X(n)$ then $Z(t) = e^{jm\omega_0 t} \cdot x(t) \xleftrightarrow{F.S} Z(n) = X(n-m)$

Proof :- $Z(n) = \frac{1}{T} \int_0^T x(t) \cdot e^{-jn\omega_0 t} \cdot dt$

$$= \frac{1}{T} \int_0^T e^{jm\omega_0 t} \cdot x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

$$= \frac{1}{T} \int_0^T x(t) \cdot e^{j(m-n)\omega_0 t} \cdot dt$$

$$= \frac{1}{T} \int_0^T x(t) \cdot e^{-j(n-m)\omega_0 t} \cdot dt$$

$$= X(n-m)$$

$Z(t) = e^{jm\omega_0 t} \cdot x(t) \xleftrightarrow{F.S} Z(n) = X(n-m)$

Time Scaling :-

If $x(t) \xleftrightarrow{F.S} X(n)$ then $Z(t) = x(at) \xleftrightarrow{F.S} Z(n) = X(n)$

Proof :- $X(n) = \frac{1}{T} \int_0^T x(t) \cdot e^{-jn\omega_0 t} \cdot dt$

→ Since $x(t)$ is periodic, then $Z(t) = x(at)$ is also periodic

and if 'T' is the period of $x(t)$, then the period of $Z(t)$ will be $\frac{T}{a}$. (eg prob for expln).

→ Similarly, if frequency of $x(t)$ is ω_0 the frequency of $Z(t) = x(at)$ will be $a\omega_0$ ($\because T = \frac{T}{a}$ $\omega_0 = \frac{2\pi}{T}$ $\frac{2\pi}{\omega_0} = \frac{2\pi}{a\omega_0}$ $\Rightarrow \frac{2\pi}{T/a} = 2\pi \times \frac{a}{T}$ $\omega_0 = a\omega_0$)

$$Z(n) = \frac{1}{(T/a)} \int_0^{(T/a)} x(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

$$= \frac{1}{(T/a)} \int_0^{T/a} x(at) \cdot e^{-jn\omega_0 t} \cdot dt$$

Let $\tau = at \Rightarrow \tau = \frac{T}{a} \cdot \frac{t}{T/a} \Rightarrow t = \frac{\tau}{a}$

$$d\tau = a dt$$

$$dt = \frac{1}{a} \cdot d\tau$$

upper limits $\Rightarrow t = \frac{T}{a}$
 $\tau = a\left(\frac{T}{a}\right) = T$

lower limits $\Rightarrow t = 0$

$$\tau = a(0) = 0$$

$$= \frac{1}{T/a} \cdot \int_0^T x(\tau) \cdot e^{-jn\omega_0 \left(\frac{\tau}{a}\right)} \cdot \frac{1}{a} \cdot d\tau$$

$$= \frac{1}{a} \cdot \frac{1}{T/a} \cdot \int_0^T x(\tau) \cdot e^{-jn\omega_0 \tau} \cdot d\tau$$

$$= \frac{1}{T} \int_0^T x(\tau) \cdot e^{-jn\omega_0 \tau} \cdot d\tau$$

$$= X(n)$$

$\therefore x(t) = x(at) \xleftrightarrow{F.S} Z(n) = X(n)$

Time differentiation :-

If $x(t) \xleftrightarrow{F.S} X(k)$, then $\frac{dx(t)}{dt} \xleftrightarrow{F.S} jk\omega_0 X(k) = Z(n)$

Proof:-

By definition of exponential Fourier Series,

$$x(t) = \sum_{n=-\infty}^{\infty} X(n) \cdot e^{+jn\omega_0 t}$$

differentiating on both sides with respect to 't'

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(n) \cdot \frac{d}{dt} e^{jk\omega_0 t}$$

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} (x(n) \cdot jk\omega_0) e^{jk\omega_0 t}$$

$$\frac{dx(t)}{dt} = jk\omega_0 \sum_{n=-\infty}^{\infty} x(n) \cdot e^{jk\omega_0 t}$$

$$\boxed{\begin{array}{l} x(t) \xleftrightarrow{F.S} X(k) \\ \frac{dx(t)}{dt} \xleftrightarrow{F.S} jk\omega_0 X(k) \end{array}}$$

Convolution in time :-

if $x(t) \xleftrightarrow{F.S} X(k)$ & $y(t) \xleftrightarrow{F.S} Y(k)$ then

$$Z(t) = x(t) * y(t) \xleftrightarrow{F.S} Z(k) = T X(k) \cdot Y(k)$$

Proof :-

we know that

$$Z(k) = \frac{1}{T} \int_0^T z(t) \cdot e^{-jk\omega_0 t} \cdot dt$$

$$= \frac{1}{T} \int_0^T [x(t) * y(t)] \cdot e^{-jk\omega_0 t} \cdot dt$$

we know that convolution $x(t) * y(t) = \int x(\tau) \cdot y(t-\tau) d\tau$

$$= \frac{1}{T} \left[\int_0^T \int_0^T x(\tau) \cdot y(t-\tau) \cdot d\tau \cdot e^{-jk\omega_0 t} \cdot dt \right]$$

$$\text{Let } t - \tau = m \Rightarrow t = m + \tau \quad \left| \begin{array}{l} \text{upper limit } \Rightarrow t = T \\ m = t - \tau = T - \tau \\ \text{lower limit } \Rightarrow t = 0 \Rightarrow m = 0 - \tau \end{array} \right.$$

$$dt = dm$$

$$= \frac{1}{T} \left[\int_0^T \int_{-T}^0 x(T) \cdot y(m) \cdot dm \cdot e^{-jk\omega_0(m+T)} \cdot dT \right]$$

$$= \frac{1}{T} \int_0^T x(T) \cdot e^{-jk\omega_0 T} dT \int_{-T}^0 y(m) \cdot e^{-jk\omega_0 m} dm$$

by
As, here we take periodic signal, the upper limit & lower limit we get total fundamental time period is 'T' for y(m), then the period limits can be changed from 'T' to '0', as it is periodic signal the fundamental time period is same, so for y(m).
Therefore

$$= \frac{1}{T} \int_0^T x(T) \cdot e^{-jk\omega_0 T} \cdot dT \cdot \int_0^T y(m) \cdot e^{-jk\omega_0 m} \cdot dm$$

for above eqn, multiply by T & divide by T

$$= T \cdot \frac{1}{T} \int_0^T x(T) \cdot e^{-jk\omega_0 T} \cdot dT \cdot \frac{1}{T} \int_0^T y(m) \cdot e^{-jk\omega_0 m} \cdot dm$$

$$= T \cdot X(k) \cdot Y(k).$$

$$\therefore Z(t) = x(t) * y(t) \xleftrightarrow{F-S} Z(n) = T \cdot X(k) \cdot Y(k).$$

Multiplication or Modulation Theorem:-

If $x(t) \xleftrightarrow{F.S} X(k)$ & $y(t) \xleftrightarrow{F.S} Y(k)$, then

$$z(t) = x(t) \cdot y(t) \xleftrightarrow{F.S} z(k) = X(k) * Y(k).$$

Proof:-

$$Z(k) = \frac{1}{T} \int_0^T z(t) \cdot e^{-jk\omega_0 t} dt.$$

$$= \frac{1}{T} \int_0^T [x(t) \cdot y(t)] \cdot e^{-jk\omega_0 t} dt.$$

from formula of exponential Fourier Series.

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) \cdot e^{+jk\omega_0 t}$$

$$Z(k) = \frac{1}{T} \int_0^T \sum_{k=-\infty}^{\infty} X(k) \cdot e^{+jk\omega_0 t} \cdot y(t) \cdot e^{-jk\omega_0 t} dt.$$

$$= \sum_{k=-\infty}^{\infty} X(k) \left[\frac{1}{T} \int_0^T y(t) \cdot e^{-j(k-m)\omega_0 t} dt \right].$$

$$= \sum_{k=-\infty}^{\infty} X(k) \cdot Y(k-m).$$

by convolution theorem in discrete time

$$= x(k) * y(k) = \sum_{k=-\infty}^{\infty} X(k) \cdot Y(k-m)$$

$$= x(k) * y(k)$$

$$\therefore z(t) = x(t) \cdot y(t) \xleftrightarrow{F.S} z(k) = x(k) * y(k).$$

Parseval's Theorem :-

If $x(t)$ is periodic power signal with fourier co-eff $X(k)$, then average power in the signal is given by.

$$P = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

Proof :- power of the signal is given by

$$P = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2 dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot x^*(t) dt$$

we know that,

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t}$$

$$(x(t))^* = \left[\sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t} \right]^*$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} X^*(k) e^{-jk\omega_0 t}$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t} \cdot \sum_{k=-\infty}^{\infty} X^*(k) e^{-jk\omega_0 t} dt$$

$$= \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} \sum_{k=-\infty}^{\infty} |X(k)|^2 dt$$

Here there are no t terms, so we can eliminate t terms.

$$= \lim_{T \rightarrow \infty} \sum_{k=-\infty}^{\infty} |X(k)|^2 \frac{1}{T} \int_{-T/2}^{T/2} dt$$

$$T \rightarrow \sum_{k=-\infty}^{\infty} |X(k)|^2 \frac{1}{T} \left[\frac{T}{2} - \left(-\frac{T}{2} \right) \right]$$

$$= T \rightarrow \sum_{k=-\infty}^{\infty} |X(k)|^2 \frac{1}{T} [T]$$

$$= \sum_{k=-\infty}^{\infty} |X(k)|^2$$

$$P = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

$$\begin{aligned}
 T &\xrightarrow{L \rightarrow \infty} \sum_{k=-\infty}^{\infty} |X(k)|^2 \frac{1}{T} \left[\frac{T}{2} - \left(-\frac{T}{2}\right) \right] \\
 &= T \xrightarrow{L \rightarrow \infty} \sum_{k=-\infty}^{\infty} |X(k)|^2 \frac{1}{T} [T] \\
 &= \sum_{k=-\infty}^{\infty} |X(k)|^2
 \end{aligned}$$

$$P = \sum_{k=-\infty}^{\infty} |X(k)|^2$$

Fourier Transforms :-

In Fourier series any continuous time periodic signal $f(t)$ can be represented as linear combination of complex exponentials and Fourier co-eff are discrete.

The Fourier series can be applied only to periodic signals, but Fourier transform can also be applied to Non-periodic signals like rectangular function, pulse, step function, Ramp function... etc.

The Fourier transform can be developed by finding Fourier series of periodic function and then tending 'T' to infinity. If 'T' tends to infinity then $f(t)$ becomes aperiodic signal.

Deriving Fourier transform from Fourier Series:-
 Exponential form of Fourier Series is

$$f(t) = \sum_{n=-\infty}^{\infty} F_n \cdot e^{jn\omega_0 t}$$

$$F_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \cdot e^{-jn\omega_0 t} \cdot dt$$

for aperiodic

$$TF_n = \int_{-\infty}^{\infty} f(t) \cdot e^{-jn\omega_0 t} \cdot dt \rightarrow (1)$$

Here limits changes to ∞ , as we are taking aperiodic signal, there will not be any per time period specifically
 Here, replacing $n\omega_0$ with ω .

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} \cdot dt \rightarrow (2)$$

by comparing eq (1) & (2)

$$TF_n = F(\omega)$$

$$F_n = \frac{1}{T} \cdot F(\omega)$$

then

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} \cdot F(\omega) \cdot e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{T} \left[F(n\omega_0) \cdot e^{jn\omega_0 t} \right]$$

$$\omega_0 = \frac{2\pi}{T}; T = \frac{2\pi}{\omega_0}$$

$$f(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t}$$

but $\omega = n\omega_0 \Rightarrow \omega_0 = \frac{\omega}{n}$
 $d\omega = n d\omega_0$

if $n \rightarrow \infty$ then $n\omega_0 = \omega$. ($\because n$ is large value, Inverse of large value is negligible)

~~$$f(t) = \frac{\omega_0}{2\pi} \sum_{n=-\infty}^{\infty} F(n\omega_0) e^{jn\omega_0 t}$$~~

Here we are as $n\omega_0$ approaches ω , that is from discrete form to continuous form. So the 'Σ' changes to integration in continuous form.

$$= \frac{\omega_0}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} d\omega \quad \left(\begin{array}{l} \because \text{Here } n\omega_0 \rightarrow \omega \\ \omega = n\omega_0 \\ d\omega = n d\omega_0 \\ 1 \end{array} \right)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} d\omega \Rightarrow \text{IFT}$$

Note:-

For periodic signals discrete waveform is formed and for aperiodic signal continuous waveform is formed.

Fourier transform of a signal $f(t)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \cdot e^{-j\omega t} dt$$

Inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \cdot e^{j\omega t} d\omega$$

$$F(\omega) = |F(\omega)| \cdot e^{j\theta(\omega)}$$

$|F(\omega)|$ = magnitude
 $\theta(\omega)$ = phase

Fourier Transform of any arbitrary Signal & Standard Signal:-

i) Fourier Transform of Single Sided Exponential Signal

The Single Sided exponential is defined as

$$x(t) = A \cdot e^{-at}; \text{ for } t \geq 0$$

by definition of Fourier transform.

$$X(\omega) = X(j\Omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\Omega t} \cdot dt \quad (\because \text{as we know that})$$

$$= \int_0^{\infty} A \cdot e^{-at} \cdot e^{-j\Omega t} \cdot dt$$

$$= A \int_0^{\infty} e^{-(a+j\Omega)t} \cdot dt$$

$$= A \left[\frac{e^{-(a+j\Omega)t}}{-(a+j\Omega)} \right]_0^{\infty} = \frac{-A}{a+j\Omega} \left[e^{-(a+j\Omega)t} \right]_0^{\infty}$$

$$= \frac{-A}{a+j\Omega} \left[e^{-\infty} - e^0 \right]$$

$$= \frac{-A}{a+j\Omega} (0 - 1) = \frac{A}{a+j\Omega}$$

$$\boxed{F\{A \cdot e^{-at}\} = \frac{A}{a+j\Omega} = X(\omega)}$$

$$|X(\omega)| = \frac{A}{\sqrt{a^2 + \Omega^2}}$$

($\because$ mod is applicable only for complex terms).

ii) Fourier Transform of double Sided Exponential Signal:-

$$x(t) = A \cdot e^{-a|t|}; \text{ for all } t$$

$$x(t) = A \cdot e^{at}; \text{ for } t = -\infty \text{ to } 0$$

$$= A \cdot e^{-at}; \text{ for } t = 0 \text{ to } \infty.$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \quad (\because \text{by formula of Fourier transform})$$

$$= \int_{-\infty}^0 A e^{at} \cdot e^{-j\omega t} dt + \int_0^{\infty} A e^{-at} \cdot e^{-j\omega t} dt$$

$$= A \int_{-\infty}^0 e^{(a-j\omega)t} dt + A \int_0^{\infty} e^{-(a+j\omega)t} dt$$

$$= A \left[\frac{e^{(a-j\omega)t}}{a-j\omega} \right]_{-\infty}^0 + A \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty}$$

$$= \frac{A}{a-j\omega} [e^0 - e^{-\infty}] - \frac{A}{a+j\omega} [e^{-\infty} - e^0]$$

$$= \frac{A}{a-j\omega} [1-0] - \frac{A}{a+j\omega} [0-1]$$

$$= \frac{A}{a-j\omega} + \frac{A}{a+j\omega} = A \left[\frac{1}{a-j\omega} + \frac{1}{a+j\omega} \right]$$

$$= A \left[\frac{a+j\omega + a-j\omega}{(a-j\omega)(a+j\omega)} \right] = \frac{2Aa}{a^2 + \omega^2}$$

$$F \{ A e^{-|a|t} \} = \frac{2Aa}{a^2 + \omega^2}$$

c) Fourier Transform of a constant :-

Let $x(t) = A$ (constant).

By definition of Fourier transform, the constant will not satisfy the condition $\int_{-\infty}^{\infty} |x(t)| dt < \infty$.

Let $x_1(t) =$ double sided exponential signal

Let $x_1(t) = A e^{-|a|t}$.

then $x(t) = A e^{-at} u(t)$

on taking Fourier transform on both sides

$$F\{x(t)\} = F\{A e^{-at} u(t)\}$$

$$= A e^{-a \cdot 0} F\{u(t)\}$$

$$= A e^{-a \cdot 0} X(\omega) \quad (\because F\{u(t)\} = F\{A e^{-at}\})$$

$$= A e^{-a \cdot 0} \frac{2aA}{\omega^2 + a^2}$$

$= 0$, for all values of ω , except at $\omega = 0$

At $\omega = 0$, let $X(\omega) = K \cdot \delta(\omega)$; at $\omega = 0$, $X(\omega) = K$, at $\omega \neq 0$, $X(\omega) = 0$

by Property of impulse function, $\int_{-\infty}^{\infty} \delta(t) dt = 1$

here formula to find magnitude,

$$|X(\omega)| = K = \int_{-\infty}^{\infty} \frac{2aA}{\omega^2 + a^2} \cdot d\omega = 2aA \int_{-\infty}^{\infty} \frac{1}{\omega^2 + a^2} \cdot d\omega$$

$$= 2aA \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 + a^2}$$

$$= 2aA \left[\frac{1}{a} \tan^{-1}\left(\frac{\omega}{a}\right) \right]_{-\infty}^{\infty}$$

$$= 2A \left[\tan^{-1}(\infty) - \tan^{-1}(-\infty) \right]$$

$$= 2A \left[\tan^{-1}(\infty) + \tan^{-1}(\infty) \right]$$

$$= 2A \left[2 \tan^{-1}(\infty) \right]$$

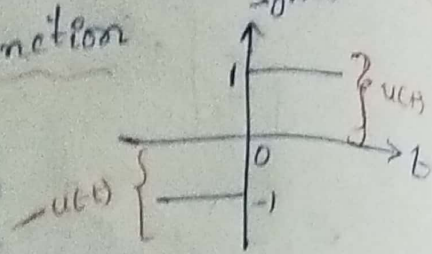
$$= 2A \left[2 \tan^{-1}\left(\tan\left(\frac{\pi}{2}\right)\right) \right]$$

$$= 2A \left[2 \left(\frac{\pi}{2}\right) \right] = 2A\pi$$

$$F\{A\} = 2\pi A \delta(\omega)$$

Fourier transform of Signum-function

$$\text{sgn}(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \\ 0, & t = 0. \end{cases}$$



$$\text{sgn}(t) = \lim_{\alpha \rightarrow 0} [e^{-\alpha t} u(t) - e^{\alpha t} u(-t)]$$

$$F\{\text{sgn}(t)\} = \lim_{\alpha \rightarrow 0} F\{e^{-\alpha t} u(t) - e^{\alpha t} u(-t)\}$$

$$= \lim_{\alpha \rightarrow 0} F\{e^{-\alpha t} u(t)\} - \lim_{\alpha \rightarrow 0} F\{e^{\alpha t} u(-t)\}$$

$$= \lim_{\alpha \rightarrow 0} \int_0^{\infty} e^{-\alpha t} e^{-j\omega t} dt - \lim_{\alpha \rightarrow 0} \int_{-\infty}^0 e^{\alpha t} e^{-j\omega t} dt$$

$$= \lim_{\alpha \rightarrow 0} \int_0^{\infty} e^{-(\alpha + j\omega)t} dt - \lim_{\alpha \rightarrow 0} \int_{-\infty}^0 e^{(\alpha - j\omega)t} dt$$

$$= \lim_{\alpha \rightarrow 0} \left[\frac{e^{-(\alpha + j\omega)t}}{-(\alpha + j\omega)} \right]_0^{\infty} - \lim_{\alpha \rightarrow 0} \left[\frac{e^{(\alpha - j\omega)t}}{(\alpha - j\omega)} \right]_{-\infty}^0$$

$$= \lim_{\alpha \rightarrow 0} \left[\frac{-1}{\alpha + j\omega} [e^{-\infty} - e^0] - \frac{1}{\alpha - j\omega} [e^0 - e^{-\infty}] \right]$$

$$= \lim_{\alpha \rightarrow 0} \left[\frac{-1}{\alpha + j\omega} [0 - 1] - \frac{1}{\alpha - j\omega} [1 - 0] \right]$$

$$= \lim_{\alpha \rightarrow 0} \left[\frac{1}{\alpha + j\omega} - \frac{1}{\alpha - j\omega} \right]$$

$$= \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega} = \frac{2}{j\omega} = \frac{1}{j\omega}$$

Fourier transform of unit step function:-

$$u(t) = 1; t \geq 0 \\ = 0, t < 0.$$

$$\text{sgn}(t) = 2u(t) - 1. \\ 2u(t) = \text{sgn}(t) + 1$$

$$\Rightarrow u(t) = \frac{1}{2} [\text{sgn}(t) + 1]$$

$$F\{u(t)\} = \frac{1}{2} F\{\text{sgn}(t) + 1\} = \frac{1}{2} F\{\text{sgn}(t)\} + \frac{1}{2} F\{1\}$$

$$= \frac{1}{2} \left[\frac{1}{j\pi f} \right] + \frac{1}{2} 2\pi(1) \cdot \delta(\omega).$$

$$\hookrightarrow (\because F\{A\} \Leftrightarrow 2\pi A \delta(\omega))$$

$$= \frac{1}{j 2\pi f} + \pi \delta(\omega)$$

$$F\{u(t)\} = \frac{1}{j\omega} + \pi \delta(\omega)$$

~~XXXXXX~~

Fourier transform of Sinusoidal Signal :-

$$x(t) = A \sin \omega t = A \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right]$$

by taking Fourier transform on both sides.

$$F\{x(t)\} = F\left\{A \left[\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right]\right\}$$

$$= \frac{1}{2j} F\{e^{j\omega t}\} - \frac{1}{2j} F\{e^{-j\omega t}\}$$

~~(we know that, Fourier transform of exponential signal is)~~

$$= \frac{A}{2j} \left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0) \right]$$

~~(we know that the Fourier transform of exponential signal)~~

= we know that the Fourier transform of a constant

$$F\{A\} = 2\pi A \delta(\omega)$$

→ $A \cdot e^{j\Omega_0 t} \Rightarrow$ that is here we are having freq shifting,
 that is because, we are multiplying constant 'A' with $e^{j\Omega_0 t}$.
 So, it is having freq shifting, then

$$F\{A\} = 2\pi A \delta(\Omega)$$

$$F\{A \cdot e^{j\Omega_0 t}\} = 2\pi A \delta(\Omega - \Omega_0)$$

$$F\{A \cdot e^{-j\Omega_0 t}\} = 2\pi A \delta(\Omega + \Omega_0)$$

$$= \frac{1}{2j} [2\pi A \delta(\Omega - \Omega_0)] - \frac{1}{2j} [2\pi A \delta(\Omega + \Omega_0)]$$

$$= \frac{\pi A}{j} [\delta(\Omega - \Omega_0)] - \frac{\pi A}{j} [\delta(\Omega + \Omega_0)]$$

$$= \frac{\pi A}{j} [\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]$$

$$\boxed{F\{A \sin \Omega_0 t\} = \frac{\pi A}{j} [\delta(\Omega - \Omega_0) - \delta(\Omega + \Omega_0)]}$$

Fourier Transform of cosinusoidal signal :-

$$x(t) = A \cos \Omega_0 t = A \left[\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} \right]$$

$$F\{A \cos \Omega_0 t\} = F\left\{A \left[\frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} \right]\right\}$$

$$F\{A e^{j\Omega_0 t}\} = 2\pi A \delta(\Omega - \Omega_0)$$

$$F\left\{\frac{A e^{j\Omega_0 t}}{2}\right\} \Rightarrow \pi A \delta(\Omega - \Omega_0)$$

$$F\left\{\frac{A e^{-j\Omega_0 t}}{2}\right\} \leftarrow \pi A \delta(\Omega + \Omega_0)$$

$$= F\left\{\frac{A e^{j\Omega_0 t}}{2}\right\} + F\left\{\frac{A \cdot e^{-j\Omega_0 t}}{2}\right\}.$$

$$= \pi A \cdot \delta(\Omega - \Omega_0) + \pi A \cdot \delta(\Omega + \Omega_0)$$

$$A \cos \Omega_0 t \longleftrightarrow \pi A [\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$$

Properties of Fourier Transform:-

1) Linearity :- If $x(t) \xrightarrow{F.T} X(\omega)$ & $y(t) \xrightarrow{F.T} Y(\omega)$, then
 $z(t) = ax(t) + by(t) \xrightarrow{F.T} aX(\omega) + bY(\omega).$

Proof :-

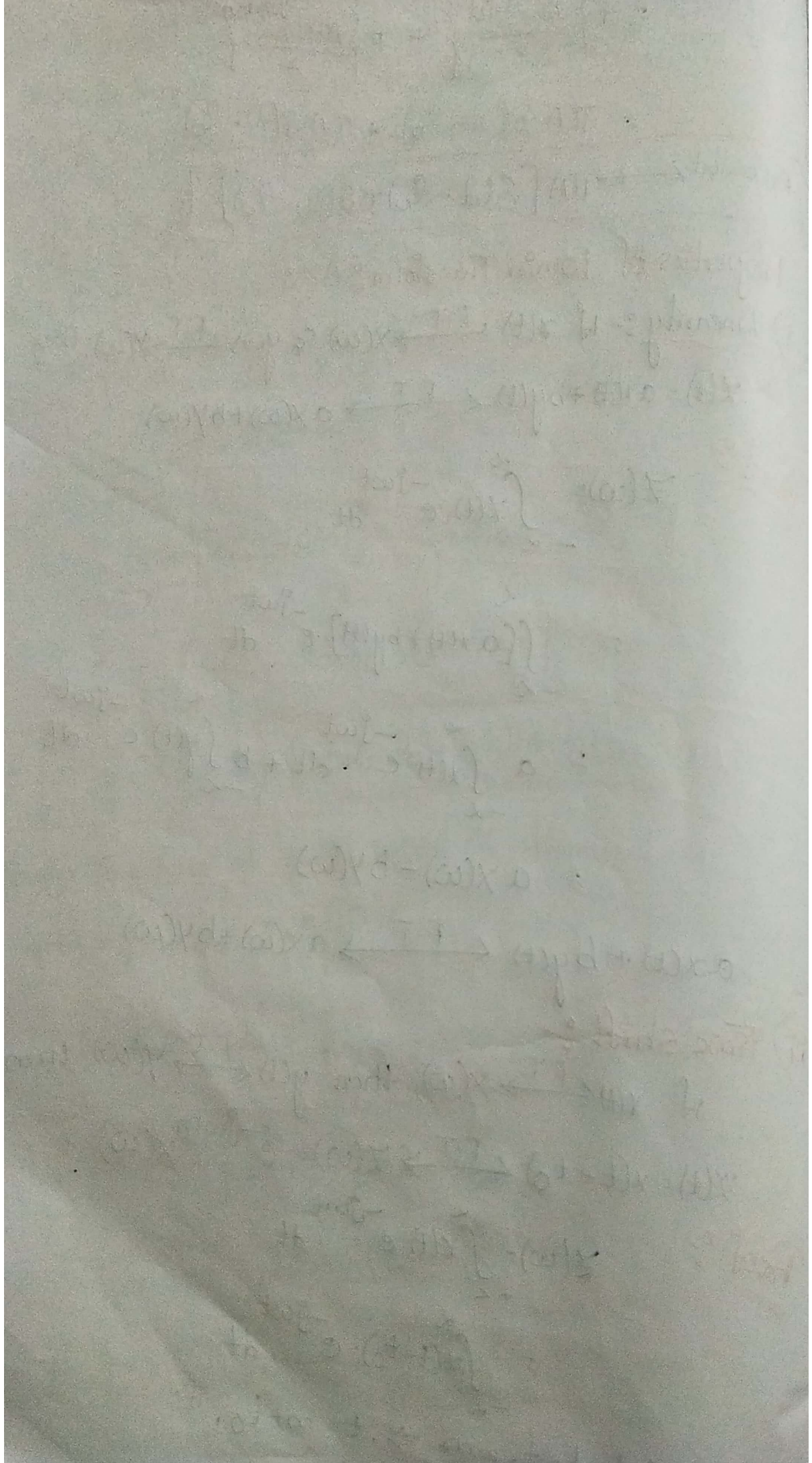
$$Z(\omega) = \int_{-\infty}^{\infty} z(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} [ax(t) + by(t)] \cdot e^{-j\omega t} \cdot dt$$

$$= a \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt + b \int_{-\infty}^{\infty} y(t) \cdot e^{-j\omega t} \cdot dt$$

$$= aX(\omega) + bY(\omega).$$

$$ax(t) + by(t) \xrightarrow{F.T} aX(\omega) + bY(\omega).$$



Time shifting of Fourier Transform^o
 $F\{x(t)\} \rightarrow X(\omega)$, then $F\{x(t-t_0)\} = e^{-j\omega t_0} \cdot X(\omega)$.

Proof:-

$$Z(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$Z(\omega) = \int_{-\infty}^{\infty} x(t-t_0) \cdot e^{-j\omega t} dt$$

let $t-t_0 = m$

$t = m+t_0$

$dt = dm$

$$Z(\omega) = \int_{-\infty}^{\infty} x(m) \cdot e^{-j\omega(m+t_0)} dm$$

$$= \int_{-\infty}^{\infty} x(m) \cdot e^{-j\omega m} \cdot e^{-j\omega t_0} dm$$

$$= e^{-j\omega t_0} \cdot \int_{-\infty}^{\infty} x(m) \cdot e^{-j\omega m} dm$$

$$= e^{-j\omega t_0} \cdot X(\omega)$$

$$\boxed{F\{x(t-t_0)\} = e^{-j\omega t_0} \cdot X(\omega)}$$

Time reversal^o
 $F\{x(t)\} = X(\omega)$, then $F\{x(-t)\} = X(-\omega)$.

Proof:-

$$Z(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} x(-t) \cdot e^{-j\omega t} dt$$

$t = m$

Frequency Shift :-

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then

$$z(t) = e^{j\omega_0 t} \cdot x(t) \xleftrightarrow{F.T} Z(\omega) = X(\omega - \omega_0)$$

Proof :-

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} e^{j\omega_0 t} \cdot x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-j(\omega - \omega_0)t} \cdot dt$$

$$= X(\omega - \omega_0)$$

$$\therefore F\{e^{j\omega_0 t} \cdot x(t)\} = X(\omega - \omega_0)$$

Time Scaling :-

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then $z(t) = x(at)$

$$z(t) = x(at) \xleftrightarrow{F.T} Z(\omega) = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Proof :-

$$Z(\omega) = \int_{-\infty}^{\infty} z(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(at) \cdot e^{-j\omega t} \cdot dt$$

Let $at = m \Rightarrow dt = \frac{1}{a} \cdot dm$
 $t = \frac{m}{a}$

$$= \int_{-\infty}^{\infty} x(m) \cdot e^{-j\omega(\frac{m}{a})} \cdot \frac{1}{a} \cdot dm$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} x(m) \cdot e^{-j\frac{\omega}{a}m} \cdot dm$$

$$= \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

$$\therefore F\{x(at)\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Frequency differentiation:

$$\text{If } x(t) \xleftrightarrow{F.T} X(\omega), \text{ then } -jt x(t) = Z(t) \xleftrightarrow{F.T} Z(\omega) = \frac{dX(\omega)}{d\omega}$$

Proof:

$$Z(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} -jt x(t) \cdot e^{-j\omega t} \cdot dt$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

differentiating on both sides w.r.t ' ω '

$$\frac{d}{d\omega}[X(\omega)] = \int_{-\infty}^{\infty} x(t) \cdot (-jt) \cdot e^{-j\omega t} \cdot dt$$

$$= -jt \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt$$

$$= -jt X(\omega)$$

$$= -j \int_{-\infty}^{\infty} t x(t) \cdot e^{-j\omega t} dt$$

we know that

$$F\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt$$

Similarly

$$\int_{-\infty}^{\infty} [t x(t)] \cdot e^{-j\omega t} dt = F\{t x(t)\}$$

$$= -j F\{t x(t)\}$$

$$= F\{-j t x(t)\} = \frac{d}{d\omega} X(\omega)$$

Time differentiation :-

If $x(t) \xleftrightarrow{FT} X(\omega)$, then

$$\frac{d}{dt} x(t) \xleftrightarrow{FT} j\omega X(\omega)$$

Proof :-

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$$

differentiation on both sides

with respect to time 't'

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot \frac{d}{dt} e^{j\omega t} d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) j\omega \cdot e^{j\omega t} d\omega$$

$$\frac{d}{dt}[x(t)] = j\omega \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

we know that

$$\text{I.F.T}\{X(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} \cdot d\omega$$

$$\frac{d}{dt}[x(t)] = j\omega \cdot \text{I.F.T}[X(\omega)]$$

$$\boxed{\text{F.T}\left\{\frac{d}{dt}[x(t)]\right\} = j\omega X(\omega)}$$

Time integration:-

If $\text{F}\{x(t)\} \longrightarrow X(\omega)$, then

$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\text{F.T}} \frac{1}{j\omega} X(\omega)$$

Proof:-

Let $x(t)$ is expressed as

$$x(t) = \frac{d}{dt} \left(\int_{-\infty}^t x(\tau) \cdot d\tau \right)$$

$$\text{F}\{x(t)\} = \text{F}\left\{\frac{d}{dt} \int_{-\infty}^t x(\tau) \cdot d\tau\right\}$$

$$= \text{F}\left\{\frac{d}{dt} \int_{-\infty}^t x(\tau) \cdot d\tau\right\}$$

By using differentiation property on right hand side, we get

$$F[x(t)] = j\omega \cdot F\left[\int_{-\infty}^t x(\tau) \cdot d\tau\right]$$

$$X(\omega) = j\omega \cdot F\left[\int_{-\infty}^t x(\tau) \cdot d\tau\right]$$

$$\boxed{\frac{1}{j\omega} X(\omega) = F\left[\int_{-\infty}^t x(\tau) \cdot d\tau\right]}$$

Multiplication or Modulation:-

If $x(t) \xleftrightarrow{F.T} X(\omega)$ & $y(t) \xleftrightarrow{F.T} Y(\omega)$, then

$$z(t) = x(t) \cdot y(t) \xleftrightarrow{F.T} z(\omega) = \frac{1}{2\pi} [X(\omega) * Y(\omega)]$$

Proof:-

$$z(\omega) = \int_{-\infty}^{\infty} z(t) \cdot e^{-j\omega t} \cdot dt$$

$$z(\omega) = \int_{-\infty}^{\infty} x(t) \cdot y(t) \cdot e^{-j\omega t} \cdot dt$$

by Inverse Fourier transform states that

~~$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \cdot d\omega$$~~

$$IFT[x(t)] = IFT[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \cdot d\omega$$

Sub $x(t)$ in $Z(\omega)$

$$Z(\omega) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\lambda) \cdot e^{j\lambda t} \cdot d\lambda \cdot y(t) \cdot e^{-j\omega t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\lambda) \cdot d\lambda \int_{-\infty}^{\infty} y(t) \cdot e^{-(\omega - \lambda)t} dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\lambda) \cdot d\lambda \cdot y(\omega - \lambda)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} x(\lambda) \cdot y(\omega - \lambda) \cdot d\lambda$$

$$= \frac{1}{2\pi} [x(\omega) * y]$$

So, we know that ^{by} convolution theorem

$$\int_{-\infty}^{\infty} x(t) \cdot y(T-t) \cdot dt = x(T) * y(T)$$

$$= \frac{1}{2\pi} [x(\omega) * y(\omega)]$$

Time convolution (or) freq multiplication theorem:-

If $x_1(t) \longleftrightarrow X_1(\omega)$ & $x_2(t) \longleftrightarrow X_2(\omega)$, then

$$x_1(t) * x_2(t) \xrightarrow{FT} X_1(\omega) \cdot X_2(\omega)$$

Proof:- we know that Fourier transform of $x(t)$ is

$$F[x(t)] = X(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt.$$

then

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] \cdot e^{-j\omega t} \cdot dt \quad \text{--- (1)}$$

but we know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \cdot d\tau \quad \text{--- (2)}$$

Sub this eqn, then

$$F[x_1(t) * x_2(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) \cdot e^{-j\omega t} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot \int_{-\infty}^{\infty} x_2(t-\tau) \cdot e^{-j\omega t} \cdot d\tau \rightarrow (3)$$

by using time Shifting Property,

$$\int_{-\infty}^{\infty} x(t-t_0) \cdot e^{-j\omega t} dt = e^{-j\omega t_0} \cdot X(\omega).$$

Similarly

$$\int_{-\infty}^{\infty} x_2(t-\tau) \cdot e^{-j\omega t} dt = e^{-j\omega\tau} \cdot X_2(\omega) \rightarrow \text{④}$$

For substitute $-\alpha$ eq(4) in eq(3), then

$$= \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(\omega) \cdot e^{-j\omega\tau} d\tau$$

$$= x_2(\omega) \cdot \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-j\omega\tau} d\tau$$

$$= X_2(\omega) \cdot X_1(\omega)$$

$$\therefore F\{x_1(t) * x_2(t)\} = X_1(\omega) \cdot X_2(\omega).$$

Parseval's Theorem or Rayleigh's theorem:-

If $x(t) \xleftrightarrow{F.T} X(\omega)$, then Energy of the Signal

$$\text{Should be } E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \int_{-\infty}^{\infty} |x(f)|^2 df$$

Proof:-

Here we have to prove $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |x(f)|^2 df$

Given $E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$

$$E = \int_{-\infty}^{\infty} x(t) \cdot x^*(t) \cdot dt.$$

Taking conjugate on both sides for $x(t)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot e^{-j\omega t} d\omega$$

$$E = \int_{-\infty}^{\infty} x(t) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot e^{-j\omega t} d\omega \cdot dt$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt \cdot d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) X(\omega) \cdot d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \cdot d\omega$$

but we know that $\omega = 2\pi f$
 $d\omega = 2\pi df$

then

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(f)|^2 \cdot 2\pi \cdot df$$

$$= \frac{1}{\cancel{2\pi}} \cdot \cancel{2\pi} \int_{-\infty}^{\infty} |X(f)|^2 \cdot df$$

$$E = \int_{-\infty}^{\infty} |X(f)|^2 \cdot df$$

Fourier Transform of a constant :-

Fourier Transform of a constant is equal to undefined form or infinite or zero by directly applying the formula.

So, in order to eliminate these values, we are finding the ^{inverse} Fourier transform of a constant & f

Let $X(\omega) = A \delta(\omega)$, Here $\delta(\omega) = 1$, as it is impulse sequence

$$\text{I.F.T} \{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} A \cdot \delta(\omega) \cdot e^{j\omega t} d\omega$$

$$= \frac{A}{2\pi} \int_{-\infty}^{\infty} \delta(\omega) \cdot e^{j\omega t} d\omega$$

Here, by using impulse property - (1), i.e.:

$$\int_{-\infty}^{\infty} x(t) \cdot \delta(t - t_0) \cdot dt = x(t_0)$$

$$\text{Similarly } \int_{-\infty}^{\infty} e^{j\omega t} \delta(\omega - \omega_0) d\omega = x(0)$$

$$= x(0) = e^{j\omega(0)} = e^0 = 1$$

$$= \frac{A}{2\pi} [1] = \frac{A}{2\pi}$$

$$\text{I.F.T} \{A \cdot \delta(\omega)\} = \frac{A}{2\pi}$$

$$A\delta(\omega) = \text{F.T} \left\{ \frac{A}{2\pi} \right\} \Rightarrow A\delta(\omega) = \frac{1}{2\pi} \text{F.T} \{A\}.$$

$$\boxed{\text{F.T} \{A\} = 2\pi A\delta(\omega).}$$

$$= \lim_{a \rightarrow 0} \left[\frac{e^{-\infty}}{-a+j\omega} - \frac{e^0}{-a+j\omega} - \frac{e^0}{a-j\omega} + \frac{e^{-\infty}}{a-j\omega} \right]$$

$$= \lim_{a \rightarrow 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right] = \frac{1}{j\omega} + \frac{1}{j\omega} = \frac{2}{j\omega}$$

INTRODUCTION TO HILBERT TRANSFORM:

The hilbert transform of $h(t)$ is defined by

$$\hat{h}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{h(\tau)}{t-\tau} d\tau$$

The inverse hilbert transform by means of which the original signal $h(t)$ is recovered from $\hat{h}(t)$ is defined by

$$h(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\hat{h}(\tau)}{t-\tau} d\tau$$

→ The functions $h(t)$ and $\hat{h}(t)$ are said to constitute a hilbert transform pair.

APPLICATIONS:

- Areas of signal processing, analysis and synthesis of signals, design of filters etc.
- To represent band pass signals
- To realize phase selectivity in the generation of SSB systems.
- To relate the gain & phase characteristics of linear communication channels & filters of minimum phase type.