Laplace Transform :-

Laplace Transform :-

Laplace Transform represents continous time Signals in terms of complex Exponential Signals i.e:- e^{-st}. It is used to analyze the Signals or func which are not absolutely integrable.

More effectively continous time Signals can be analyzed using laplace transform

Laplace Transform provides broader characterization than fourier Transform.

Definition :-

To transform a time domain Signal $x(t)$ to S-domain, multiply the Signal by e^{-st} & then integrate from $-\infty$ to ∞

The transformed Signal is represented by $X(S)$ & transformation is denoted by 'L'.

Laplace transform of a Continous time Signal $x(t)$ is given by

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt$$

where 'S' is in Complex nature & given as

$$S = \underbrace{\sigma}_{\text{real part}} + \underbrace{j\omega}_{\text{img part}}$$

If $x(t)$ is defined for $t \geq 0$, i.e. $x(t)$ is causal then

$$L\{x(t)\} = X(S) = \int_{0}^{\infty} x(t) \cdot e^{-st} \, dt.$$

Types of Laplace Transform :-

> <u>Bilateral or two Sided laplace Transform</u> :-

If the integration is taken from $-\infty$ to ∞ it is called Bilateral laplace Transform

> <u>unilateral or one Sided Laplace transform</u> :-

If the integration is taken from 0 to ∞ it is called unilateral laplace transform

Inverse Laplace Transform :-

The s-domain Signal $X(S)$ can be obtained transformed to time domain Signal $x(t)$ by using Inverse laplace transform and is defined as

$$L^{-1}\{X(S)\} = x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s) \cdot e^{st} \cdot ds.$$

The Signal $x(t)$ & $X(s)$ are called Laplace Transform Pair

$$x(t) \xrightarrow{L} X(S)$$

$$X(S) \xrightarrow{L^{-1}} x(t).$$

Relation b/w Fourier Transform and Laplace Transform :-

Fourier Transform is given by

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \cdot dt \longrightarrow (1)$$

fourier transform can be calculated if $x(t)$ is absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty \longrightarrow (2)$$

Laplace transform is written as

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt.$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t - j\omega t} \cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} \{x(t) \cdot e^{-\sigma t}\} \cdot e^{-j\omega t}\, dt$$

$$= F\{x(t) \cdot e^{-\sigma t}\}. \longrightarrow (3)$$

by Eq (3),

Laplace transform of $x(t)$ is basically Fourier transform of $x(t) \cdot e^{-\sigma t}$

If $\sigma = 0$ in Eq (3), then $\Rightarrow S = \sigma + j\omega = 0 + j\omega = j\omega$

$$\text{Eq (3)} \Rightarrow X(S) = \int_{-\infty}^{\infty} x(t) \cdot e^{-0} \cdot e^{-j\omega t} \, dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} \, dt$$

$$X(S) = X(j\omega).$$

It is basically fourier transform on imaginary $(j\omega)$ axis in S-plane.

Region of Convergence :-

The Laplace Variable 'S' is a complex variable defined by $S = \sigma + j\Omega$,

where σ is the real part of 'S' & denoted as $Re(s)$.

Ω is the imaginary part of 'S' & denoted as $Img(s)$.

The range of variation of S can be represented schematically as S-plane.

The horizontal axis of this plane is known as real axis, Since it is defined by

$$s = \sigma + j\Omega = \sigma + j(0) = \sigma.$$

The vertical axis of this plane is known as imaginary axis, Since it is defined by

$$s = \sigma + j\Omega = 0 + j\Omega = j\Omega.$$

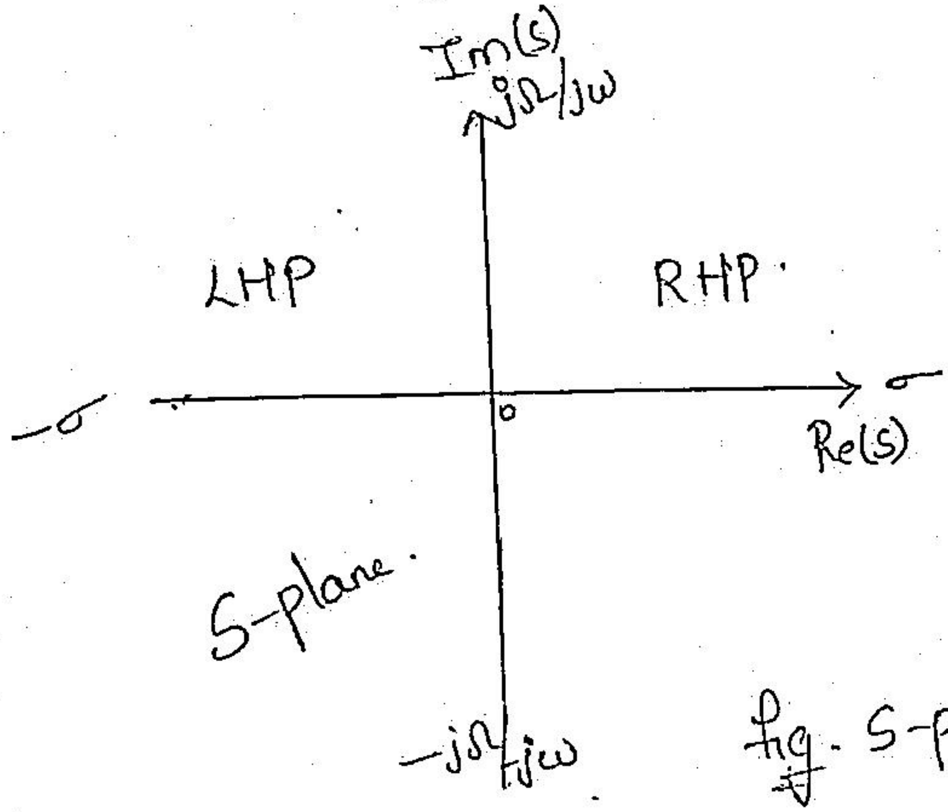


fig. S-plane

LHP → Left Half plane

RHP → Right Half plane.

The range of σ for which the Laplace Transform converges is called Region of Convergence (ROC).

Find the Laplace transform of $e^{-at}u(t)$ & Roc.

Given $x(t) = e^{-at}u(t)$

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} e^{-at}u(t) \cdot e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-(s+a)t}\, dt \quad \left(\because u(t)=1, t\geq 0;\ =0, t<0\right)$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \left[\frac{e^{-\infty} - e^{0}}{-(s+a)}\right]$$

$$= \frac{0-1}{-(s+a)} = \frac{1}{s+a}$$

Here $Re(s) > 0$ for Roc, Here in given prob

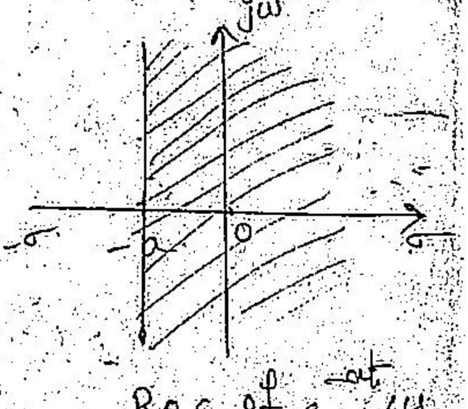
$\Rightarrow Re(s+a) > 0$

$Re(s) > -a$

$\sigma > -a$

Roc : $\sigma > -a$

$X(s) = \dfrac{1}{s+a}$



Roc of $e^{-at}u(t)$

Properties of Laplace Transform :-

i) <u>Amplitude Scaling</u>;

If $L\{x(t)\} = X(s)$ then $L\{Ax(t)\} = AX(s)$.

$$\Rightarrow L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$L\{Ax(t)\} = \int_{-\infty}^{\infty} A\,x(t)\cdot e^{-st}\,dt.$$

$$= A\int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt.$$

$$= AX(s)$$

$$\therefore \boxed{L\{Ax(t)\} \longrightarrow AX(s)}$$

ii) <u>Linearity</u> :

If $L\{x_1(t)\} = X_1(s)$ and $L\{x_2(t)\} = X_2(s)$ then

$$L\{a_1x_1(t) + a_2x_2(t)\} = a_1X_1(s) + b\,a_2X_2(s).$$

$$\text{If} \Rightarrow L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$L\{a_1x_1(t) + a_2x_2(t)\} = \int_{-\infty}^{\infty} [a_1x_1(t) + a_2x_2(t)]e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} a_1x_1(t)\cdot e^{-st}\,dt + \int_{-\infty}^{\infty} a_2x_2(t)e^{-st}\,dt$$

$$= a_1\int_{-\infty}^{\infty} x_1(t)\cdot e^{-st}\,dt + a_2\int_{-\infty}^{\infty} x_2(t)\cdot e^{-st}\,dt$$

$$= a_1X_1(s) + a_2X_2(s)$$

$$L\{a_1x_1(t)+a_2x_2(t)\} \Leftrightarrow a_1X_1(S)+a_2X_2(S)$$

(iii) Time differentiation:

→ If $L\{x(t)\} = X(S)$ then

$$L\left\{\frac{d}{dt}x(t)\right\} = SX(S)-X(0)\text{ ; where } X(0) \text{ is}$$

value of $x(t)$ at $t=0$

$$L\{x(t)\} = \int_0^{\alpha} x(t)\cdot e^{-St}\,dt.$$

$$L\left\{\frac{d}{dt}x(t)\right\} = \int_0^{\alpha}\frac{d\,x(t)}{dt}\cdot e^{-St}\,dt = \int_0^{\alpha} e^{-St}\frac{d\,x(t)}{dt}\,dt$$

we know that

$$\int uv = u\int v - \int u'\int v.$$

Here $u = e^{-St}$ $v = \frac{d}{dt}[x(t)]$

$$L\left\{\frac{d\,x(t)}{dt}\right\} = e^{-St}\int_0^{\alpha}\frac{d}{dt}x(t) - \int_0^{\alpha}\frac{d}{dt}(e^{-St})\int\frac{d\,x(t)}{dt}$$

$$= \left[e^{-St}x(t)\right]_0^{\alpha} - \int_0^{\alpha} e^{-St}(-S)x(t)\,dt$$

$$= \left[e^{-St}x(t)\right]_0^{\alpha} + S\int_0^{\alpha} x(t)\cdot e^{-St}\,dt$$

$$= \left[e^{-\alpha}\cdot x(\alpha) - e^{0}x(0)\right] + SX(S)$$

$$= 0 - x(0) + s\,X(s)$$

$$= sX(s) - x(0).$$

$$L\left\{\frac{d}{dt}x(t)\right\} = sX(s) - x(0).$$

iv) <u>Time</u> <u>integration</u>:-

If $L\{x(t)\} = X(s)$, then

$$L\left\{\int x(t)\cdot dt\right\} = \frac{X(s)}{s} + \frac{\left[\int x(t)\cdot dt\right]_{t=0}}{s}$$

$$X(s) = L\{x(t)\} = \int_0^{\infty} x(t)\cdot e^{-st}\,dt$$

$$L\left\{\int x(t)\cdot dt\right\} = \int_0^{\infty}\left[\int x(t)\cdot dt\right]\cdot e^{-st}\,dt.$$

$$\int UV = U\int V - \int U'\int V.$$

Here $u = \int x(t)\,dt$, $v = e^{-st}$, then

$$L\left\{\int x(t)\cdot dt\right\} = \int x(t)\cdot dt\int_0^{\infty} e^{-st}\,dt - \int\frac{d}{dt}\left[\int x(t)\cdot dt\right]\int e^{-st}\,dt$$

$$= \left[\int x(t)\,dt\left[\frac{e^{-st}}{-s}\right]\right]_0^{\infty} - \int_0^{\infty} x(t)\,dt\left[\frac{e^{-st}}{-s}\right]$$

$$= \frac{-1}{s}\int\frac{-1}{s}\int x(\infty)\,dt\;e^{-\infty} - \left[\frac{-1}{s}\int x(0)\,dt\;e^{-0}\right]$$

$$+ \frac{1}{s}\int_0^{\infty} x(t)\,e^{-st}\,dt$$

$$= 0 + \frac{1}{s}\int x(0)\,dt + \frac{1}{s}\cdot X(s)$$

$$= \frac{X(s)}{s} + \frac{\left[\int x(t)\,dt\right]_{t=0}}{s}$$

Frequency Shifting :-

$\rightarrow$ If $L\{x(t)\} = X(s)$, then

$$L\{e^{-at}\cdot x(t)\} = X(s+a)$$

$$X(s) \Rightarrow L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt \quad \text{---} \; (1)$$

$$L\{e^{-at}x(t)\} = \int_{-\infty}^{\infty} e^{-at}\cdot x(t)\cdot e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot e^{-(s+a)t}\,dt \rightarrow (2)$$

by comparing Eq (1) & (2), we get

$$= X(s+a)$$

$$L\{e^{-at}x(t)\} \longrightarrow X(s+a)$$

Similarly $L\{e^{at}x(t)\} \longrightarrow X(s-a)$.

Time Shifting :-

If $\mathcal{L}\{x(t)\} = X(s)$, then $\mathcal{L}\{x(t \pm a)\} = e^{\pm as} X(s)$.

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \, dt$$

$$\mathcal{L}\{x(t \pm a)\} = \int_{-\infty}^{\infty} x(t \pm a) e^{-st} \, dt$$

Let $t \pm a = \tau$

$t = \tau \mp a$

$dt = d\tau$

$$\mathcal{L}\{x(t \pm a)\} = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s(\tau \mp a)} \cdot d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau \pm sa} \, d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} e^{\pm sa} \, d\tau$$

$$= e^{\pm as} \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\tau} \cdot d\tau$$

$$= e^{\pm as} X(s).$$

$$\mathcal{L}\{x(t \pm a)\} \longrightarrow e^{\pm as} X(s).$$

Frequency differentiation :-

If $L\{x(t)\} = X(s)$, then $L\{t\,x(t)\} = -\frac{d}{ds}X(s)$.

$$X(s) = L\{x(t)\} = \int_{-\alpha}^{\alpha} x(t) \cdot e^{-st}\, dt \quad \text{--- (1)}$$

differentiating on both sides,

with respect to 's'.

$$\frac{d}{ds}[X(s)] = \int_{-\alpha}^{\alpha} x(t) \cdot \frac{d}{ds}[e^{-st}]\, dt$$

$$\frac{d}{ds}[X(s)] = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}[-t]\cdot dt$$

$$= -\int_{-\infty}^{\infty} tx(t)\cdot e^{-st}\, dt$$

$$-\frac{d}{ds}[X(s)] = \int_{-\infty}^{\infty} tx(t)\cdot e^{-st}\, dt$$

by comparing above eqn with eq ①, we get

$$-\frac{d}{ds}[X(s)] = L\{t\,x(t)\}$$

$$\therefore L\{t\,x(t)\} \longrightarrow -\frac{d}{ds}[X(s)].$$

Frequency integration :-

If $L\{x(t)\} = X(s)$ then $L\{\frac{1}{t}x(t)\} = \int_{s}^{\infty} X(s)\,ds.$

we know that :

$$L\{x(t)\} = X(s) = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\, dt \rightarrow ①$$

integrating on both sides with respect to s from s to ∞.

$$\int_{s}^{\infty} X(s)\,ds = \int_{s}^{\infty}\int_{-\infty}^{\infty} x(t)\cdot e^{-st}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\cdot \int_{s}^{\infty} e^{-st}\, ds\cdot dt$$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{e^{-st}}{-t}\right]_{s}^{\infty} \cdot dt$$

$$= -\int_{-\infty}^{\infty} \frac{x(t)}{t} \left[e^{-\infty} - e^{-st}\right] \cdot dt$$

$$= -\int_{-\infty}^{\infty} \frac{x(t)}{t} \left[0 - e^{-st}\right] \cdot dt$$

$$= -\int_{-\infty}^{\infty} \frac{x(t)}{t} \left[-e^{-st}\right] \cdot dt$$

$$= \int_{-\infty}^{\infty} \frac{x(t)}{t} \cdot e^{-st} \, dt .$$

$$\int_{s}^{\infty} X(s)\, ds = L\left\{\frac{x(t)}{t}\right\} .$$ ($\therefore$ by comparing above Eq with Eq ①).

$$L\left\{\frac{x(t)}{t}\right\} \longrightarrow \int_{s}^{\infty} X(s) \cdot ds .$$

Time Scaling :-

If $L\{x(t)\} = X(S)$ then $L\{x(at)\} = \frac{1}{|a|} X\left(\frac{s}{a}\right)$

$$L\{x(t)\} = X(S) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \, dt$$

$$L\{x(at)\} = \int_{-\infty}^{\infty} x(at) e^{-st} \, dt$$

let $at = \tau$

$$t = \frac{\tau}{a}$$

$$dt = \frac{1}{a} \cdot d\tau.$$

$$\Rightarrow \int_{-\infty}^{\infty} x(\tau) \cdot e^{-s\left(\frac{\tau}{a}\right)} \cdot \frac{1}{a} \cdot d\tau.$$

$$\Rightarrow \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) \cdot e^{-\left(\frac{s}{a}\right)\tau} \cdot d\tau.$$

This is in the form of $\int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt = X(s)$

Similarly $= \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-\left(\frac{s}{a}\right)\tau} \cdot d\tau.$

$$= \frac{1}{a} X\left(\frac{s}{a}\right).$$

The above transformation is applicable for positive values as well as negative values of 'a'

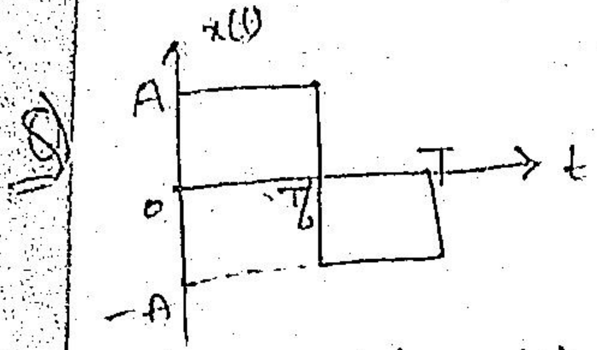
then $L\{x(at)\} \longrightarrow \frac{1}{|a|} X\left(\frac{s}{a}\right).$

Periodicity :-

If $x(t) = x(t + nT)$ and $x_1(t)$ be one period of $x(t)$ and $L\{x_1(t)\} = \int_0^T x_1(t) \cdot e^{-st} \cdot dt$ then

$$L\{x(t+nT)\} = \frac{1}{1 - e^{-st}} \int_0^T x_1(t) \cdot e^{-st} dt$$

Laplace Transform of certain Signals using Waveform Synthesis :-



In given graph, $x(t) = A, \quad 0 < t \leq T/2$

$\quad = -A, \quad T/2 \leq t \leq T.$

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$= \int_{0}^{T/2} A e^{-st} dt + \int_{T/2}^{T} -A e^{-st} dt$$

$$= A \int_{0}^{T/2} e^{-st} dt - A \int_{T/2}^{T} e^{-st} dt$$

$$= A \left[\frac{e^{-st}}{-s}\right]_{0}^{T/2} - A \left[\frac{e^{-st}}{-s}\right]_{T/2}^{T}$$

$$= -\frac{A}{s}\left[e^{-T/2 s} - e^{0}\right] + \frac{A}{s}\left[e^{-sT} - e^{-\frac{sT}{2}}\right]$$

$$= -\frac{A}{s} e^{-\frac{sT}{2}} + \frac{A}{s} + \frac{A}{s} e^{-sT} - \frac{A}{s} e^{-\frac{sT}{2}}$$

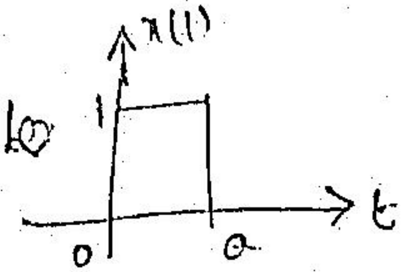
$$= \frac{A}{s} - \frac{2A}{s} \cdot e^{-\frac{sT}{2}} + \frac{A}{s} \cdot e^{-sT}$$

$$= \frac{A}{S}\left[1 - 2e^{-\frac{ST}{2}} + e^{-ST}\right]$$

$$\left[1 - e^{-\frac{ST}{2}}\right]^2 = 1 - e^{-ST} - 2e^{-\frac{ST}{2}}$$

$$= \frac{A}{S}\left[1 - e^{-\frac{ST}{2}}\right]^2$$

b)



$$x(t) = 1, \quad 0 \le t \le a$$
$$= 0, \quad t > a$$

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt$$

$$= \int_0^a 1 \cdot e^{-st}\, dt$$

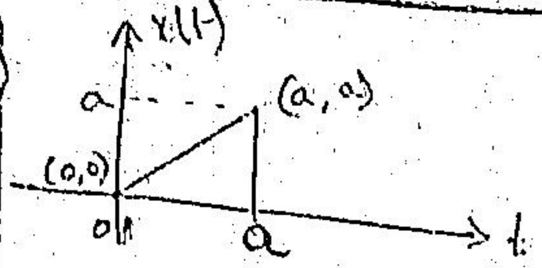
$$= \int_0^a e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_0^a$$

$$= \frac{-1}{S}\left[e^{-as} - e^{0}\right]$$

$$= \frac{-1}{S}\left[e^{-as} - 1\right]$$

$$= \frac{1}{S}\left[1 - e^{-as}\right]$$

c)



$(0, 0)$ $\quad$ (a, a)

$x_1 \; y_1$ $\quad$ $x_2 \; y_2$

Equation of line is $y - y_1 = \frac{x_2 - x_1}{y_2 - y_1}\left[x - x_1\right]$

$$y - 0 = \frac{a-0}{a-0}\left[x - 0\right]$$

$$y = x.$$

$$x(t) = t.$$

$$\mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$= \int_{\cancel{-\infty}}^{a} t \cdot e^{-st}\, dt = \int_{0}^{a} t\, e^{-st}\, dt$$

$$\int uv = u\int v - \int u' \int v$$

Here $u = t$ $\quad$ $v = e^{-st}.$

$$\int_0^a t\, e^{-st}\, dt = t\int_0^a e^{-st}\, dt - \int \frac{dt}{dt} \cdot \int e^{-st}\, dt$$

$$= \left[t\left[\frac{e^{-st}}{-s}\right]\right]_0^a - \int_0^a \int e^{-st}\, dt$$

$$= \left(-\frac{a}{s}e^{-as} - 0\right) - \int_0^a \frac{e^{-st}}{-s}\, dt$$

$$= -\frac{a}{s}\cdot e^{-as} + \frac{1}{s}\int_0^a e^{-st}\,dt$$

$$= -\frac{a}{s}\cdot e^{-as} + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^a$$

$$= -\frac{a}{s}\cdot e^{-as} - \frac{1}{s^2}\left[e^{-st}\right]_0^a$$

$$= -\frac{a}{s}e^{-as} - \frac{1}{s^2}\left[e^{-as} - e^{0}\right]$$

$$= -\frac{a}{s}e^{-as} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}$$

$$= \frac{1}{s^2}\left[1 - e^{-as} - as\,e^{-as}\right]$$

$$= \frac{1}{s^2}\left[1 - e^{-as}(1+as)\right].$$

Find the Laplace Transform and also Region of Convergence:-

i) $x(t) = A\,u(t)$

ii) $x(t) = t\,u(t)$

iii) $x(t) = e^{-3t}u(t)$

iv) $x(t) = e^{-3t}u(-t)$

v) $x(t) = e^{-4|t|}$.

1) $x(t) = A\,u(t)$

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$L\{A u(t)\} = \int_{-\infty}^{\infty} A u(t)\cdot e^{-st}\,dt$$

$u(t) = 1,\ t \geq 0$ ⇒ Here Signal Starts from 0 to ∞

$\quad\ = 0,\ t < 0$

then the limits become.

$$L\{A u(t)\} = \int_{0}^{\infty} A u(t)\cdot e^{-st}\,dt$$

$$= \int_{0}^{\infty} A(1)\cdot e^{-st}\,dt$$

$$= \int_{0}^{\infty} A e^{-st}\,dt = A\int_{0}^{\infty} e^{-st}\,dt$$

$$= A\left[\frac{e^{-st}}{-s}\right]_{0}^{\infty} \Rightarrow -\frac{A}{s}\left[e^{-s\infty} - e^{-s(0)}\right]$$

$$= -\frac{A}{s}\left[e^{-s\infty} - e^{0}\right] = -\frac{A}{s}\left[e^{-s\infty} - 1\right] \rightarrow (1)$$

$$= -\frac{A}{s}\left[0 - 1\right] = \frac{A}{s}.$$

$$L\{A u(t)\} \longrightarrow \frac{A}{s}.$$

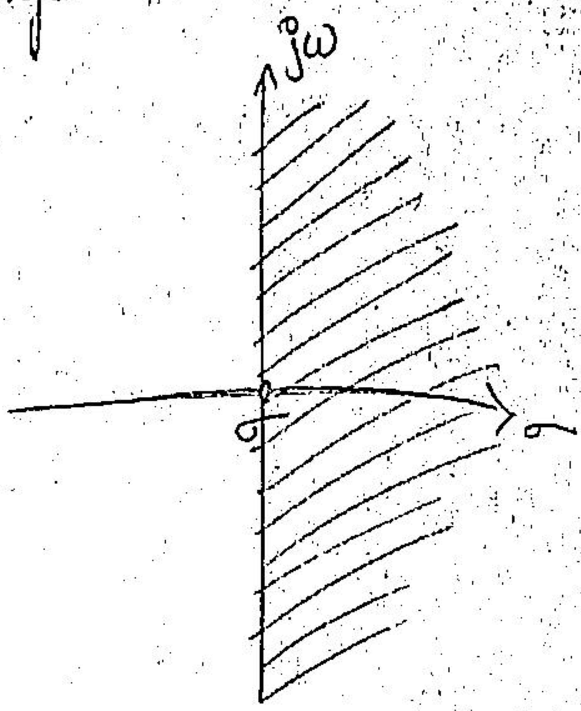
in Eq (1) $s\infty$, Here $Re(s) > 0$

$\sigma > 0.$

for Region of convergence (ROC) :-

$Re(s) > 0$

$\sigma > 0$



(1) $x(t) = t\,u(t)$.

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$= \int_{-\infty}^{\infty} t\,u(t)\cdot e^{-st}\,dt$$

$$= \int_{0}^{\infty} t\,u(t)\cdot e^{-st}\,dt \Rightarrow \int_{0}^{\infty} t(1)\,e^{-st}\,dt$$

$$= \int_{0}^{\infty} t\,e^{-st}\,dt$$

This is in the form of $\int uv = u\int v - \int u' \int v . dr$

Here $u = t$, $v = e^{-st}$

$\cancel{\int t\, e^{-st}}$ $\Rightarrow$

$$\int t\, e^{-st} = t\int e^{-st}\,dt - \int \frac{dt}{dt} \cdot \int e^{-st}\,dt$$

$$= t\left[\frac{e^{-st}}{-s}\right] - \int s.e^{-st}\,dt$$

$$= t\left[\frac{e^{-st}}{-s}\right] - \int\left[\frac{e^{-st}}{-s}\right]$$

$$= t\left[\frac{e^{-st}}{-s}\right] + \frac{1}{s}\int e^{-st}\,dt$$

$$= -\frac{t}{s}\left[e^{-st}\right] + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]$$

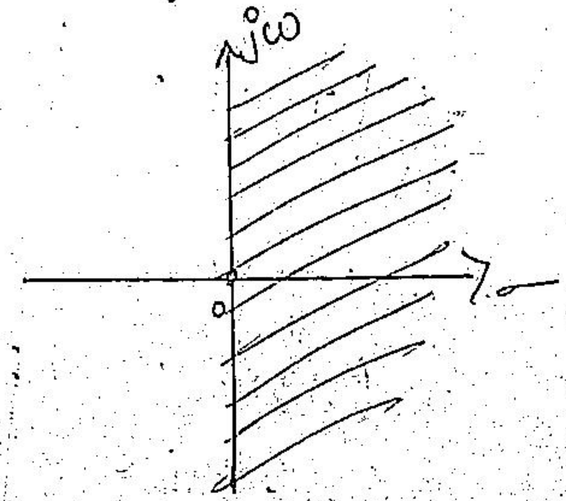
$$= \left(-\frac{t}{s}\left[e^{-st}\right] - \frac{1}{s^2}\left[e^{-st}\right]\right)_0^{t}$$

$$= \left[-\frac{\alpha e^{-\alpha s}}{s} - \frac{1}{s^2}e^{-\alpha s}\right] - \left[0 - \frac{1}{s^2}e^{0}\right]$$

$$0 - 0 - \left[0 - \frac{1}{s^2}\right] = \frac{1}{s^2}.$$

for Region of convergence: $Re(s) > 0$

$\sigma > 0$



$$L\{t\, u(t)\} \longrightarrow \frac{1}{s^2}$$

iii) $x(t) = e^{-3t} u(t)$.

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-3t} u(t) \cdot e^{-st}\, dt \quad (\because \text{as } u(t) \text{ Signal starts from 0 to } \infty) \; \{u(t) = 1\}$$

$$= \int_{0}^{\infty} e^{-3t} \cdot e^{-st}\, dt$$

$$= \int_{0}^{\infty} e^{-t(s+3)} \cdot dt = \left[\frac{e^{-t[s+3]}}{-(s+3)}\right]_0^{\infty}$$

$$\Rightarrow \frac{-1}{(s+3)} \left[e^{-\infty(s+3)} - e^{0(s+3)}\right]$$

$$= \frac{-1}{s+3} \left[e^{-\infty} - e^{0}\right]$$

$$\frac{-1}{s+3} \left[0 - 1\right] = \frac{1}{s+3}$$

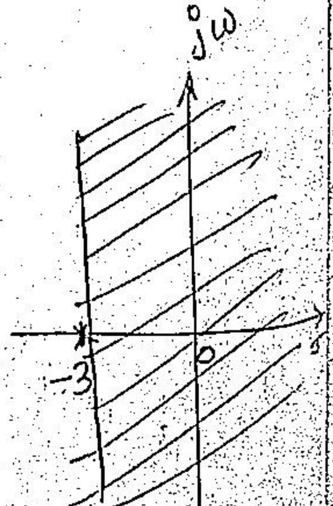
for Region of Convergence :-

ROC :- $Re(s+3) > 0$

$Re(\sigma + j\omega + 3) > 0$

$\sigma + 3 > 0$

$\sigma > -3$



$$L\{e^{-3t} u(t)\} \longrightarrow \frac{1}{s+3}$$

v) $x(t) = e^{-3t} u(-t)$.

$$\mathcal{L}\{x(t)\} = \int_{-\alpha}^{\alpha} x(t) \cdot e^{-st} dt$$

$$\mathcal{L}\{e^{-3t} u(-t)\} = \int_{-\alpha}^{\alpha} e^{-3t} u(-t) \cdot e^{-st} dt.$$

$$u(-t) = 1, \quad -t \geq 0$$
$$t \leq 0$$, Signal starts from $-\alpha$ to 0
$$= 0, \quad t > 0.$$

$$= \int_{-\alpha}^{0} e^{-3t} \cdot e^{-st} dt$$

$$= \int_{-\alpha}^{0} e^{-t(s+3)} \cdot dt \Rightarrow \left[\frac{e^{-t(s+3)}}{-(s+3)}\right]_{-\alpha}^{0}$$

$$= \frac{-1}{s+3}\left[e^{-0} - e^{\alpha(s+3)}\right].$$

$$= \frac{-1}{s+3}\left[1 - e^{\alpha(s+3)}\right].$$

Here ~~Re(s+3) > 0 (or)~~ Re(s+3) < 0, should satisfy this condition, if not the total condition becomes ~~zero~~ infinite

~~$Re(\sigma + j\omega + 3) > 0$~~

~~$\sigma + 3 > 0$~~

~~$\sigma > -3$~~

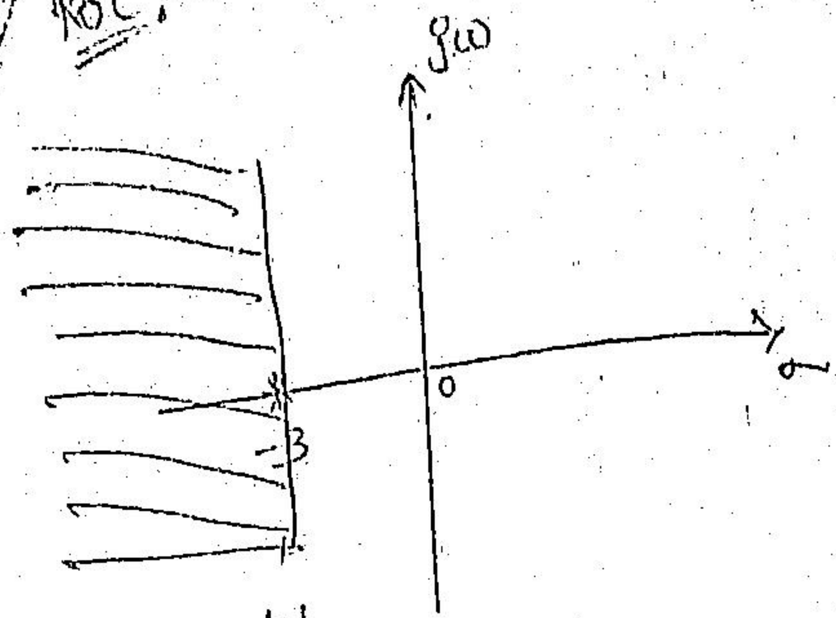
$Re(s+3) < 0$

$Re(\sigma + j\omega + 3) < 0 \Rightarrow \sigma + 3 < 0$

$\sigma < -3.$

ROC :- $\sigma < -3$

ROC: $\sigma < -3$



v) $x(t) = e^{-4|t|}$

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\,dt$$

$$L\{e^{-4|t|}\} = \int_{-\infty}^{\infty} e^{-4|t|}\cdot e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{4t}\cdot e^{-st}\,dt + \int_{0}^{\infty} e^{-4t}\, e^{-st}\,dt$$

$$= \int_{-\infty}^{0} e^{-t(s-4)}\,dt + \int_{0}^{\infty} e^{-t(s+4)}\,dt$$

$$= \left[\frac{e^{-t(s-4)}}{-(s-4)}\right]_{-\infty}^{0} + \left[\frac{e^{-t(s+4)}}{-(s+4)}\right]_{0}^{\infty}$$

$$= \frac{-1}{s-4}\left[e^{-0} - e^{+\infty(s-4)}\right] - \frac{1}{s+4}\left[e^{-0} - e^{-\infty(s+4)}\right]$$

Roc :- $Re(s-4) < 0$

$Re(\sigma + j\omega - 4) < 0$

$\sigma - 4 < 0$ ~~$\sigma - 4 > 0$~~

$\sigma < 4$ ~~$\sigma > 4$~~

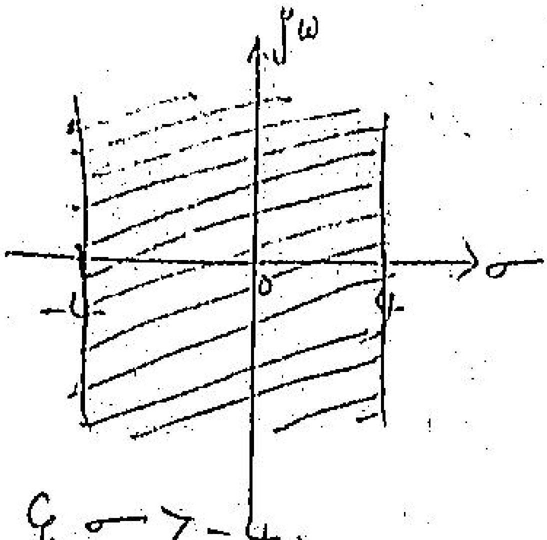
Roc. $Re(s+4) > 0$

$Re(\sigma + j\omega + 4) > 0$

$\sigma + 4 > 0$

$\sigma > -4$

Roc :-



Roc :- $\sigma < 4$ & $\sigma > -4$.

Determine the Laplace Transform of the following Signals

i) $x(t) = \sin\omega_0 t \cdot u(t)$

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} \cdot dt.$$

$$= \int_{-\infty}^{\infty} \sin\omega_0 t \cdot u(t) \cdot e^{-st} dt.$$

as $u(t) = 1, t \geq 0$ $\Rightarrow$ Here Signal starts from 0 to ∞.

$= 0, t < 0$

therefore, $L\{x(t)\} = \int_0^{\infty} \sin\omega_0 t \cdot (1) \cdot e^{-st} dt$

$$= \int_0^{\infty} \sin\omega_0 t \cdot e^{-st} dt$$

$$\int UV = U\int V - \int U'\int V$$

Here $u = \sin n\omega_0 t$, $V = e^{-st}$

$$= \sin n\omega_0 t \int e^{-st} - \int \frac{d}{dt}(\sin n\omega_0 t) \cdot \int e^{-st}$$

$$= \sin n\omega_0 t \left[\frac{e^{-st}}{-s}\right] - \int (n\cos n\omega_0 t)\, \underline{e^{-st}}$$

$$\sin\omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$$= \int_0^{\infty} \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] \cdot e^{-st}\, dt$$

$$= \frac{1}{2j}\int_0^{\infty} e^{j\omega_0 t} \cdot e^{-st} - \frac{1}{2j}\int_0^{\infty} e^{-j\omega_0 t} \cdot e^{-st}\, dt$$

$$= \frac{1}{2j}\int_0^{\infty} e^{-t(s-j\omega_0)}\, dt - \frac{1}{2j}\int_0^{\infty} e^{-t(s+j\omega_0)}\, dt$$

$$= \frac{1}{2j}\left[\frac{e^{-t(s-j\omega_0)}}{-(s-j\omega_0)}\right]_0^{\infty} - \frac{1}{2j}\left[\frac{e^{-t(s+j\omega_0)}}{-(s+j\omega_0)}\right]_0^{\infty}$$

$$= \frac{-1}{2j}\left[\frac{e^{-\infty} - e^{0}}{s-j\omega_0}\right] + \frac{1}{2j}\left[\frac{e^{-\infty} - e^{0}}{s+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{0-1}{s-j\omega_0}\right] + \frac{1}{2j}\left[\frac{0-1}{s+j\omega_0}\right]$$

$$= \frac{1}{2j(s-j\omega_0)} - \frac{1}{2j}\left[\frac{1}{s+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right]$$

$$= \frac{1}{2j}\left[\frac{\cancel{s}+j\omega_0-\cancel{s}+j\omega_0}{s^2+\omega^2}\right]$$

$$= \frac{1}{\cancel{2j}}\left[\frac{\cancel{2j}\omega_0}{s^2+\omega_0^2}\right] = \frac{\omega_0}{s^2+\omega_0^2}$$

$$\boxed{L\left\{\sin\omega_0 t\, u(t)\right\} = \frac{\omega_0}{s^2+\omega_0^2}}$$

ii) $x(t) = \cos\omega_0 t\; u(t)$.

$$~~L\{\sin~~\quad L\{x(t)\} = \int_{-\alpha}^{\alpha} x(t)\cdot e^{-st}\,dt$$

$$L\{\cos\omega_0 t\cdot u(t)\} = \int_{-\alpha}^{\alpha} \cos\omega_0 t\cdot u(t)\cdot e^{-st}\,dt$$

but we know that $u(t) = 1,\ t \geq 0 \Rightarrow$ from 0 to α

$= 0,\ t < 0 \Rightarrow -\alpha$ to 0

$$\therefore L\{\cos\omega_0 t\, u(t)\} = \int_{0}^{\alpha} \cos\omega_0 t\,(1)\,e^{-st}\,dt$$

We know that $\Rightarrow \cos\omega_0 t = \dfrac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

$$= \int_0^{\infty} \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{j\omega_0 t} \cdot e^{-st}\, dt + \frac{1}{2}\int_0^{\infty} e^{-j\omega_0 t} \cdot e^{-st}$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t(s-j\omega_0)}\, dt + \frac{1}{2}\int_0^{\infty} e^{-t(s+j\omega_0)}\, dt$$

$$= \frac{1}{2}\left[\frac{e^{-t(s-j\omega_0)}}{-(s-j\omega_0)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-t(s+j\omega_0)}}{-(s+j\omega_0)}\right]_0^{\infty}$$

$$= \frac{+1}{2}\left[\frac{e^{-\infty} - e^{0}}{-(s-j\omega_0)}\right] + \frac{1}{2}\left[\frac{e^{-\infty} - e^{0}}{-(s+j\omega_0)}\right]$$

$$= -\frac{1}{2}\left[\frac{0-1}{s-j\omega_0}\right] - \frac{1}{2}\left[\frac{0-1}{s+j\omega_0}\right]$$

$$= \frac{+1}{2(s-j\omega_0)} + \frac{1}{2(s+j\omega_0)}$$

$$= \frac{+1}{2}\left[\frac{1}{s+j\omega_0} + \frac{1}{s-j\omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s - \cancel{j\omega_0} + s + \cancel{j\omega_0}}{s^2 + \omega_0^2}\right]$$

$$= \frac{1}{\cancel{2}}\left[\frac{\cancel{2}s}{s^2 + \omega_0^2}\right]$$

$$= \frac{s}{s^2 + \omega_0^2}.$$

$$\boxed{L\{\cos\omega_0 t \cdot u(t)\} = \frac{s}{s^2 + \omega_0^2}}$$

Q) $x(t) = \cosh\omega_0 t \cdot u(t)$.

Q) $x(t) = \sinh\omega_0 t \cdot u(t)$.

Q) $x(t) = e^{-at}\sin\omega_0 t\, u(t)$.

Q) $x(t) = e^{-at}\cos\omega_0 t\, u(t)$.

Q) $x(t) = \cosh\omega_0 t\, u(t)$.

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t)\cdot e^{-st}\, dt$$

$$L\{\cosh\omega_0 t\, u(t)\} = \int_{-\infty}^{\infty} \cosh\omega_0 t \cdot u(t)\cdot e^{-st}\, dt$$

we know that

$u(t) = 1,\ t \geq 0$ Signal starts from 0 to ∞

$\quad\ \ = 0,\ t < 0$

and

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2}$$

Then $L\{\cosh\omega_0 t\cdot u(t)\} = \int_0^{\infty}\left(\frac{e^{\omega_0 t}+e^{-\omega_0 t}}{2}\right)1\cdot e^{-st}\,dt$

$$= \frac{1}{2}\int_0^{\infty} e^{\omega_0 t}\cdot e^{-st}\,dt + \frac{1}{2}\int_0^{\infty} e^{-\omega_0 t}\cdot e^{-st}\,dt$$

$$= \frac{1}{2}\int_0^{\infty} e^{-t(s-\omega_0)}\,dt + \frac{1}{2}\int_0^{\infty} e^{-t(s+\omega_0)}\,dt$$

$$= \frac{1}{2}\left[\frac{e^{-t(s-\omega_0)}}{-(s-\omega_0)}\right]_0^{\infty} + \frac{1}{2}\left[\frac{e^{-t(s+\omega_0)}}{-(s+\omega_0)}\right]_0^{\infty}$$

$$= \frac{1}{2}\left[\frac{e^{-\infty}-e^{0}}{-(s-\omega_0)}\right] + \frac{1}{2}\left[\frac{e^{-\infty}-e^{0}}{-(s+\omega_0)}\right]$$

$$= \frac{1}{2}\left[\frac{0-1}{-(s-\omega_0)}\right] + \frac{1}{2}\left[\frac{0-1}{-(s+\omega_0)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-\omega_0} + \frac{1}{s+\omega_0}\right]$$

$$= \frac{1}{2}\left[\frac{s+\cancel{\omega_0}+s-\cancel{\omega_0}}{s^2-\omega_0^2}\right]$$

$$= \frac{1}{2}\left[\frac{2s}{s^2-\omega_0^2}\right] = \frac{s}{s^2-\omega_0^2}$$

$$L\{\cosh\omega_0 t\; u(t)\} = \frac{s}{s^2-\omega_0^2}$$

Q) $x(t) = \sinh \omega_0 t \cdot u(t)$

$$\sinh \omega_0 t = \frac{e^{\omega_0 t} - e^{-\omega_0 t}}{2}$$

Similarly $L\{\sinh \omega_0 t \, u(t)\} = \frac{\omega_0}{s^2 - \omega_0^2}$

Q) $x(t) = e^{-at} \sin \omega_0 t \cdot u(t)$.

$$L\{x(t)\} = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

$$L\{e^{-at} \sin \omega_0 t \, u(t)\} = \int_{-\infty}^{\infty} e^{-at} \sin \omega_0 t \cdot u(t) \cdot e^{-st} \cdot dt$$

Here $u(t) = 1$, for $t \geq 0 \Rightarrow$ Signal starts from 0 to ∞

$= 0, \quad t < 0$

& $\sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

$$L\{e^{-at} \sin \omega_0 t \, u(t)\} = \int_0^{\infty} \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] e^{-at} (1) e^{-st} dt$$

$$= \frac{1}{2j} \int_0^{\infty} e^{j\omega_0 t} \cdot e^{-at} e^{-st} dt - \frac{1}{2j} \int_0^{\infty} e^{-j\omega_0 t} e^{-at} e^{-st} dt$$

$$= \frac{1}{2j} \int_0^{\infty} e^{-t(s+a-j\omega_0)} dt - \frac{1}{2j} \int_0^{\infty} e^{-t(s+a+j\omega_0)} dt$$

$$= \frac{1}{2j} \left[\frac{e^{-t(s+a-j\omega_0)}}{-(s+a-j\omega_0)}\right]_0^{\infty} - \frac{1}{2j} \left[\frac{e^{-t(s+a+j\omega_0)}}{-(s+a+j\omega_0)}\right]_0^{\infty}$$

$$\frac{1}{2j}\left[\frac{e^{-\infty}-e^{0}}{-(s+a-j\omega_0)}\right]-\frac{1}{2j}\left[\frac{e^{-\infty}-e^{0}}{-(s+a+j\omega_0)}\right]$$

$$=\frac{1}{2j}\left[\frac{0-1}{-(s+a-j\omega_0)}\right]-\frac{1}{2j}\left[\frac{0-1}{-(s+a+j\omega_0)}\right]$$

$$=\frac{1}{2j}\left[\frac{1}{(s+a)-j\omega_0}-\frac{1}{(s+a)+j\omega_0}\right]$$

$$=\frac{1}{2j}\left[\frac{\cancel{s}+\cancel{a}+j\omega_0-\cancel{s}-\cancel{a}+j\omega_0}{(s+a)^2-(j\omega_0)^2}\right]$$

$$=\frac{1}{\cancel{2j}}\left[\frac{\cancel{2j}\omega_0}{(s+a)^2-j^2\omega_0^2}\right]$$

$$=\frac{\omega_0}{(s+a)^2+\omega_0^2}\quad(\because j^2=-1).$$

$$\therefore\Rightarrow\boxed{L\{e^{-at}\sin\omega_0 t\cdot u(t)\}=\frac{\omega_0}{(s+a)^2+\omega_0^2}}$$

Q) $x(t)=e^{-at}\cos\omega_0 t\, u(t)$

$$\cos\omega_0 t=\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}$$

$$\boxed{L\{e^{-at}\cos\omega_0 t\, u(t)\}=\frac{(s+a)}{(s+a)^2+\omega_0^2}}$$

Initial Value Theorum:-

The initial value theorum allows us to calculate initial value of $x(t)$ i.e. $x(0)$ directly from the transform $X(s)$, with out need for finding the inverse laplace transform of $X(s)$ i.e.

$$\lim_{s \to \infty} sX(s) - x(0) = 0$$

$$\Rightarrow \lim_{s \to \infty} sX(s) = x(0)$$

Proof: Here we consider a unilateral laplace transform.

$$L\{x(t)\} = \int_0^{\infty} x(t) \cdot e^{-st} dt$$

$$L\left\{\frac{dx(t)}{dt}\right\} = \int_0^{\infty} \frac{dx(t)}{dt} \cdot e^{-st} dt$$

by time diff property

i.e. $$L\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0)$$

on taking Lt $s \to \infty$ on both sides of eqn

$$\lim_{s \to \infty} L\left\{\frac{dx(t)}{dt}\right\} = \lim_{s \to \infty} sX(s) - x(0)$$

$$\lim_{s \to \infty} \int_0^{\infty} \frac{dx(t)}{dt} \cdot e^{-st} dt = \lim_{s \to \infty} sX(s) - x(0)$$

$$\int_0^{\infty} \frac{dx(t)}{dt} \lim_{s \to \infty} e^{-st} dt = \lim_{s \to \infty} sX(s) - x(0)$$

$$\int_0^\infty \frac{dx(t)}{dt}\cdot e^{-\infty}\, dt = \lim_{s\to\infty} sX(s) - x(0)$$

$$0 = \lim_{s\to\infty} sX(s) - x(0)$$

$$\boxed{\lim_{s\to\infty} sX(s) = x(0).}$$

Final value theorum :-

The final value theorum enables us to determine the final value of a function $x(t)$ i.e $x(\infty)$ directly from its Laplace transform $X(s)$, without need for finding the inverse Laplace Transform of $X(s)$.

It states that if, $x(t) \longleftrightarrow X(s)$

then, $\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} sX(s)$

Proof :-

$$L\left\{\frac{d\,x(t)}{dt}\right\} = sX(s) - x(0)$$

by taking limits $s \to 0$, on both sides

$$\lim_{s\to 0} L\left\{\frac{d\,x(t)}{dt}\right\} = \lim_{s\to 0} sX(s) - x(0).$$

Note :- Final value theorum is applicable only to the values at which Poles lies in left

$$\lim_{s\to 0}\int_0^\infty \frac{d\,x(t)}{dt}\, e^{-st}\, dt = \lim_{s\to 0} sX(s) - x(0)$$

$$\int_0^\infty \frac{d\,x(t)}{dt}\left[\lim_{s\to 0} e^{-st}\right] dt = \lim_{s\to 0} sX(s) - x(0)$$

$$\int_0^\infty \frac{d}{dt}x(t)\cdot e^{-0}\, dt = \lim_{s\to 0} sX(s) - x(0)$$

$$\int_0^\infty \frac{d\,x(t)}{dt}\,dt = \lim_{s\to 0} sX(s) = x(0) \quad (\because e^{-\infty} = 0)$$

$$\lim_{s\to 0} sX(s) = x(0)$$

$$x(0) = \lim_{t\to 0} x(t)$$

$$\therefore \lim_{t\to 0} x(t) = x(0) = \lim_{s\to}$$

$$\int_0^\infty \frac{d}{dt}x(t)\cdot dt = \lim_{s\to 0} sX(s) - x(0)$$

$$\left[x(t)\right]_0^\infty = \lim_{s\to 0} sX(s) - x(0)$$

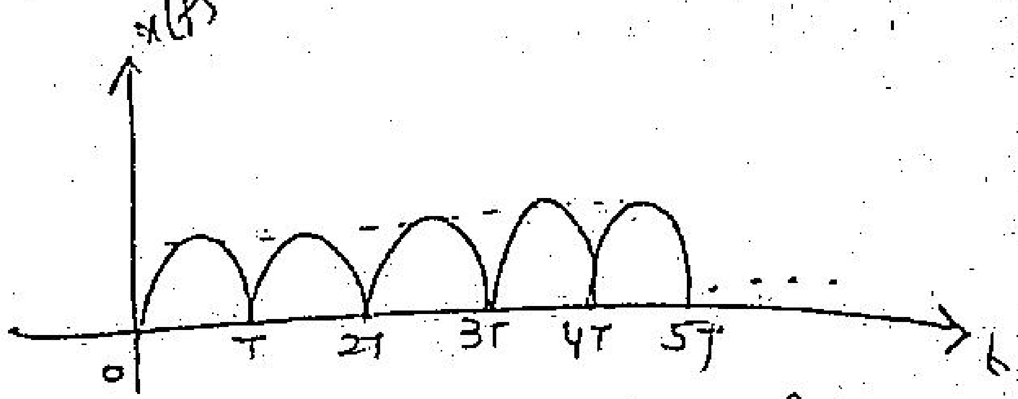
$$x(\infty) - x(0) = \lim_{s\to 0} sX(s) - x(0)$$

$$x(\infty) = \lim_{s\to 0} sX(s)$$

$$\boxed{\lim_{t\to\infty} x(t) = x(\infty) = \lim_{s\to 0} sX(s)}$$

Time periodicity property :-

Let $x(t)$ be a periodic waveform which satisfies the condition $x(t) = x(t+nT)$, for all $t > 0$, where T is the period of the function, & $n = 0, 1, 2, 3, \dots$



By using unilateral laplace transform,

$$L\{x(t)\} = \int_0^{\infty} x(t)\, e^{-st}\, dt$$

The Laplace transform for the above graph, can be written as

$$X(s) = L\{x(t)\} = \int_0^{T} x(t) \cdot e^{-st}\, dt + \int_T^{2T} x(t)\, e^{-st}\, dt + \int_{2T}^{3T} x(t)\, e^{-st}\, dt + \dots$$

we know that if the Signal is periodic, then we can write $x(t) = x(t+T)$.

let us Consider a Exponential Signal $x(t) = e^{j\omega t}$ which is a periodic Signal then $x(t+T) = e^{j\omega(t+T)}$

$$x(t+T) = e^{j\omega t + j\omega T} = e^{j\omega t} \cdot e^{j\omega T}$$

$$x(t+T) = x(t) \cdot e^{j\omega T}$$

~~Similarly~~ $x(t) = x(t+T) \cdot e^{-j\omega T}$

Similarly

$$x(t) = x(t+2T)\, e^{-j\omega 2T}$$

$$x(t) = x(t+3T)\, e^{-j\omega 3T}$$

but in laplace transform, we know that $s = j\omega$

then

$$\left.\begin{aligned} x(t) &= x(t+T) \cdot e^{-sT} \\ x(t) &= x(t+2T) \cdot e^{-2sT} \\ x(t) &= x(t+3T) \cdot e^{-3sT} \end{aligned}\right\} - Eq\ 2$$

then by sub Eq 2 values in Eq 1, we get

$$X(s) = L\{x(t)\} = \int_0^T x(t)\, e^{-st}\, dt + e^{-sT} \int_{0}^{T} x(t+T) \cdot e^{-st}\, dt$$

$$+ e^{-2sT} \int_0^T x(t+2T) \cdot e^{-st}\, dt + \cdots$$

$$+ e^{-nsT} \int_0^T x(t+nT)\, e^{-st}\, dt + \cdots$$

$$X(s) = \int_0^T x(t)\, e^{-st}\, dt + e^{-sT} \int_0^T x(t)\, e^{-st}\, dt + e^{-2sT} \int_0^T x(t)\, e^{-st}\, dt$$

$$+ \cdots e^{-nsT}\int_0^T x(t)e^{-sT}dt + \cdots$$

$$= \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right]\int_0^T x(t)e^{-sT}dt$$

$$= \left[1 + e^{-sT} + e^{-2sT} + e^{-3sT} + \cdots\right]X_1(s)$$

$$= \left[1 + e^{-sT} + (e^{-sT})^2 + (e^{-sT})^3 + \cdots + (e^{-sT})^n + \cdots\right]X_1(s)$$

$$= \frac{1}{1-e^{-sT}} \cdot X_1(s), \text{ where } X_1(s) = \int_0^T x(t)e^{-sT}dt$$

Laplace transform of periodic func $= \frac{1}{1-e^{-sT}}\{X_1(s)\}$.

Determine the initial value $x(0^+)$ for the following Laplace Transform

a) $X(s) = \dfrac{4}{s^2+3s-5}$ b) $X(s) = \dfrac{3s+2}{s^0(s^2+3s+2)}$

a) By using initial value theorem,

$$x(0) = \underset{s\to\alpha}{Lt}\ sX(s)$$

$$= \underset{s\to\alpha}{Lt}\ s\left[\frac{4}{s^2+3s-5}\right]$$

$$= \underset{s\to\alpha}{Lt}\ \frac{4s}{s^2+3s-5}$$

$$= \lim_{s \to \alpha} \frac{4\cancel{s}}{\cancel{s}\left[s + \frac{3}{s} - \frac{5}{s^2}\right]}$$

$$= \lim_{s \to \alpha} \frac{4}{s + 3 - \frac{5}{s}}$$

$$= \frac{4}{\alpha + 3 - \frac{5}{\alpha}} = \frac{4}{\alpha} = 0$$

$X(s) = \frac{4}{s^2 + 3s - 5} \Rightarrow$ initial value $= 0$

b) $X(s) = \frac{3s+2}{s(s^2+3s+2)}$

by using initial value theorum

$$x(0) = \lim_{s \to \alpha} \frac{3s+2}{s(s^2+3s+2)}$$

$$\lim_{s \to \alpha} \frac{\cancel{s}\left[3 + \frac{2}{s}\right]}{\cancel{s}\left[s^2+3s+2\right]}$$

$$= \lim_{s \to \alpha} \left[\frac{3 + \frac{2}{s}}{s^2 + 3s + 2}\right]$$

$$= \frac{3 + \frac{2}{\alpha}}{\alpha^2 + 3(\alpha) + 2} = \frac{3+0}{\alpha} = \frac{3}{\alpha}$$

$= 0$

Find the final value for the following question

a) $x(s) = \frac{1}{s-4}$ b) $x(s) = \frac{s-2}{s(s+4)}$ c) $X(s) = \frac{2}{s^2+1}$

d) $X(s) = \frac{2}{s^2+4s-2}$

a) $X(s) = \frac{1}{s-4}$

Here pole is $s-4=0 \Rightarrow s=4$.

Pole $= 4$, it lies on right half plane of

s-plane

So, final value theorum donot Exist, it

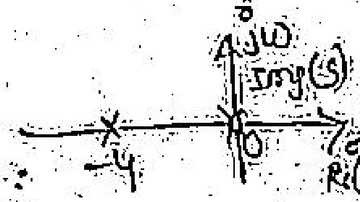
Exist only when pole lies on left half plane

of S-plane.

∴ Final value theorum donot exist, So

Final value also doesnot exist i.e. $x(\infty) = \infty$.

b) $X(s) = \frac{s-2}{s(s+4)}$



Here poles are $0, -4$,

both Poles lies on left half plane of S-plane

Hence final value Exist

So, by final value theorum

$$x(\alpha) = \lim_{s \to 0} s\,x(s)$$

$$x(\alpha) = \lim_{s \to 0} \cancel{s}\frac{(s-2)}{\cancel{s}(s+4)}$$

$$= \lim_{s \to 0} \frac{s-2}{s+4} = \frac{0-2}{0+4} = \frac{-2}{4} = \frac{-1}{2}$$

$$x(\alpha) = \frac{-1}{2}$$

c) $x(s) = \frac{2}{s^2+3}$

$s^2+3=0 \Rightarrow s = \sqrt{-3} = \sqrt{3}\,i$

Here s, poles are imaginary so, the final value theorum donot exist, as it does not lies on left half plane of Re(s).

d) $x(s) = \frac{2}{s^2+4s-2}$

by final value theorum $\Rightarrow$ $x(s)$

$$x(\alpha) = \lim_{s \to 0} s\,x(s) = \lim_{s \to 0} \frac{2s}{s^2+4s-2}$$

$$\cancel{\lim_{s \to 0} \frac{2s}{s\left(s+\frac{4}{s}-\frac{2}{s}\right)}} = \frac{2(0)}{0+4(0)-2}$$

$$= \frac{-0}{2} = 0.$$

$$x(\alpha) = 0$$

$$s = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{-4 \pm \sqrt{16+4}}{2}$$

$$= \frac{-4 \pm \sqrt{20}}{2}$$

$$= \frac{-4+2\sqrt{5}}{2}, \frac{-4-2\sqrt{5}}{2}$$