Z-Transform :-

→ Fundamental differences between continous and discrete time Signal

→ Discre time Signal Representation using complex Exponential & Sinusoidal components.

(Both refer to unit-I)

→ periodicity Properties of Discrete Time Complex functions :-

A discrete time complex Exponential is periodic if it repeats after certain 'N' number of Samples

Consider $x(n) = e^{j\omega_0 n}$

$$x(n+N) = e^{j\omega_0 (n+N)}$$

$$= e^{j\omega_0 n} \cdot e^{j\omega_0 N}$$

for periodicity $x(n) = x(n+N)$

i.e: $\cancel{e^{j\omega_0 n}} = \cancel{e^{j\omega_0 n}} \cdot e^{j\omega_0 N}$

$$\Rightarrow e^{j\omega_0 N} = 1$$

i.e $= e^{j\omega_0 N} = \cos\omega_0 N + j\sin\omega_0 N = 1$

i.e: $\omega_0 N = 2\pi, 4\pi, 6\pi, \ldots$

$\omega_0 N = 2\pi k, \; k \in Z$

$$\frac{\omega_0}{2\pi} = \frac{K}{N}$$

& we also know that

$$\omega_0 = 2\pi f \Rightarrow \frac{\omega_0}{2\pi} = f$$

then $f = \frac{K}{N}$.

for complex exponential to be periodic $\frac{\omega_0}{2\pi} = \frac{K}{N}$.

$\frac{K}{N}$ should be a rational number (i,e: $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \ldots$)

Z-Transform & Fourier Transform Relation:-

Z-transform is given by

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}, \quad Roc: r_2 < |Z| < r_1$$

where 'Z' is defined as $Z = re^{j\Omega}$ in which r is magnitude of Z i,e $|Z|$ & Ω = angle of Z:

i.e $\angle Z$.

Putting for $Z = re^{j\Omega}$ in $X(Z)$, we get

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) [re^{j\Omega}]^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] \cdot e^{-j\Omega n} \rightarrow (1)$$

we also know that fourier transform of a discrete time Signal is given by

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n} \rightarrow (2)$$

by comparing Eq ① & ②, we can say that

$$F[x(n) r^{-n}] = Z\text{-transform} = X(Z)$$

$$\sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\Omega n} = X(Z).$$

Z-Transform Concept

The Z-transform of x(n) is denoted by X(Z) and defined as

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

where Z is complex variable.

x(n) & X(Z) is called Z-transform pair and it is represented as

$$x(n) \xleftrightarrow{Z} X(Z).$$

Z-transform is used to analyze the DT signals and systems, digital filter design & for synthesis of digital filter/systems.

For any input sequence, the Z-transform is complex. It has both real & imaginary parts.

$$x(n) \longrightarrow \boxed{Z\text{-transform}} \longrightarrow X(Z).$$

Region of convergence:

Since Z-transform is an infinite power series, it exists only for those values of Z for which the series converges.

The ROC of X(Z) is the set of all values of Z, for which X(Z) attains finite value.

Constraints on ROC for various classes of signals (or).

Properties of ROCs with proofs:

→ The ROC is a ring or disk in the Z-plane centered at the origin.

→ The ROC cannot contain any poles.

→ If x(n) is a finite duration, causal sequence then the ROC is the entire Z-plane except at $Z=0$.

→ If x(n) is a finite ~~sequence~~, duration, anti-causal sequence, then the ROC is the entire Z-plane except at $Z=\infty$.

→ If x(n) is a finite duration, two sided sequence then the ROC is a entire Z-plane except at $Z=0$ & $Z=\infty$.

→ If x(n) is an infinite duration, two sided

Sequence, then ROC will consist of a ring in Z-plane, bounded on interior and Exterior by a Pole not containing any Poles

→ The ROC of an LTI Stable System Contain unit circle

→ The ROC must be Connected region

Case (i) :- finite duration, right sided (causal) Signal:-

Let $x(n)$ be finite duration Signal with N-Samples, defined as $0 \leq n \leq N-1$

$$\therefore x(n) = \{x(0), x(1), x(2), \cdots x(N-1)\}$$

$$\therefore X(Z) = \sum_{n=0}^{N-1} x(n) Z^{-n}$$

$$= x(0) + x(1) Z^{-1} + x(2) Z^{-2} + \cdots + x(N-1) Z^{-(N-1)}$$

$$= x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^2} + \cdots + \frac{x(N-1)}{Z^{N-1}}$$

when $Z = 0$, all the terms except the first term become infinite.

Hence, $X(Z)$ exists for all values of Z, except at $Z = 0$

Case (ii) :- Infinite duration, left sided (anticausal Signal) :-

Let $x(n)$ be finite duration Signal with N-Samples, defined in the range $-(N-1) \leq n \leq 0$

$$\therefore x(n) = \{x(-(N-1)), \cdots, x(-2), x(-1), x(0)\}.$$

Z-transform of $x(n)$ is given by

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$$

$$X(Z) = \sum_{n=-(N-1)}^{0} x(n) \cdot Z^{-n}$$

$$X(Z) = x(-(N-1)) Z^{+(N-1)} + \cdots + x(-2) Z^{2} + x(-1) + x(0)$$

when $Z = \infty$, all terms except the last term become infinite, Hence $X(Z)$ exists for all values of Z, except $Z = \infty$.

$\therefore$ ROC of $X(Z)$ is entire Z-plane except $Z = \infty$

case - (iii) Finite duration, two Sided (non causal) Signal :-

Let $x(n)$ be a finite duration Signal with N-Samples, defined in the range $-M \leq n \leq M$, where $M = \frac{N-1}{2}$

$$\therefore x(n) = \{x(-M), \cdots x(-2), x(-1), x(0), x(1), x(2), \cdots x(M)\}.$$

Z-transform of $x(n)$ is

$$X(Z) = \sum_{n=-M}^{M} x(n) \cdot Z^{-n}$$

$$X(Z) = x(-M) Z^{M} + \dots + x(-2) Z^{2} + x(-1) Z + x(0) + x(1) Z^{-1} + x(2) Z^{-2} + \dots + x(M) Z^{-M}.$$

$$= x(-M) Z^{M} + \dots + x(-2) Z^{2} + x(-1) Z + x(0) + \frac{x(1)}{Z} + \frac{x(2)}{Z^{2}} + \dots + \frac{x(M)}{Z^{M}}$$

when $Z=0$, the terms with negative power of Z attain infinity

when $Z=\infty$, the terms with positive powers of Z attain infinity.

Hence, $X(Z)$ converges for all values of Z, except at $Z=0$ & $Z=\infty$.

except at $Z=0$ & $Z=\infty$

Roc is entire Z-plane except at $Z=0$ & $Z=\infty$

Case-(iv) :- Infinite duration, right sided (causal) Signal:

Let $x(n) = r_1^{n}, \quad n \geq 0$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} = \sum_{n=0}^{\infty} r_1^{n} Z^{-n} = \sum_{n=0}^{\infty} (r_1 Z^{-1})^{n}$$

if $|r_1 Z^{-1}| < 1$, then $\sum_{n=0}^{\infty} [r_1 Z^{-1}]^{n} = \frac{1}{1 - r_1 Z^{-1}}$

$$= X(Z) = \frac{1}{1 - \frac{r_1}{Z}} = \frac{Z}{Z - r_1}.$$

Condition to be satisfied for convergence of $X(Z)$ is

$$0 < |r_1 Z^{-1}| < 1$$

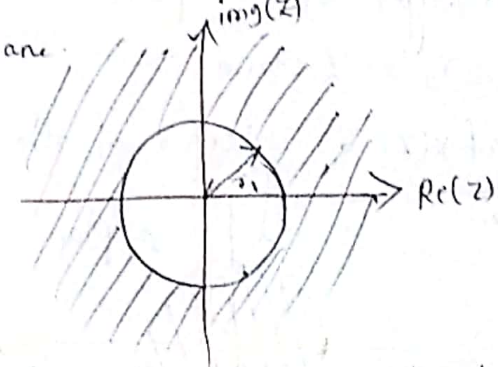
$$|r_1 Z^{-1}| < 1$$

$$\frac{|r_1|}{|Z_1|} < 1 \Rightarrow |Z| > |r_1|.$$

$r_1 \rightarrow$ represent a circle of radius r_1 in Z-plane

$X(Z) \rightarrow$ converges for all points external to the circle of radius r_1 in Z-plane.

$\therefore$ Roc of $X(Z)$ is exterior of the circle of radius r_1 in Z-plane



$\rightarrow X(Z)$ converges for all points external to the circle of radius r_1 in Z-plane.

$\therefore$ Roc of $X(Z)$ is exterior of the circle of radius r_1 in Z-plane

(V) Infinite duration, left sided (anticausal) Signal:

Let $X(n) = r_2^{n}; n \leq 0$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} = \sum_{n=-\infty}^{0} r_2^{n} Z^{-n} = \sum_{n=0}^{\infty} r_2^{-n} Z^{n}$$

$$= \sum_{n=0}^{\infty} \left(r_2^{-1} z\right)^n$$

we know that $\dfrac{1}{1-r_2^{-1} z}$ Roc: $|r_2^{-1} z| < 1$

$\left|\dfrac{z}{r_2}\right| < 1$

$$= \frac{1}{1-\frac{z}{r_2}}$$

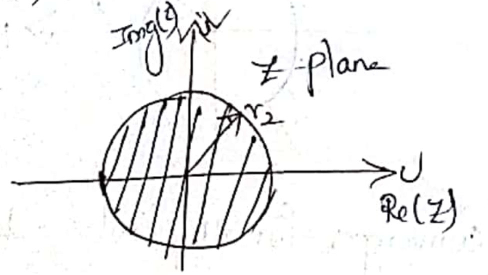
$|z| < |r_2|$.

$$= \frac{r_2}{r_2 - z} = \frac{-r_2}{z - r_2}$$

the term r_2 represents radius of circle in z-plane

X(z) converges for all points internal to circle of radius r_2 in z-plane.

∴ Roc of X(z) is interior of the circle of radius r_2



Z-transform of Some common Sequences :-

I) The unit Sample Sequence (The unit Impulse sequence):-

we know that

$$\delta(n) = 1, \text{ for } n = 0$$
$$\delta(n) = 0, \text{ for } n \neq 0$$

$$X(Z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$Z\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) Z^{-n}$$

by using the impulse property,

$$\sum_{n=-\infty}^{\infty} \delta(n)\, x(n) = x(0)$$

then
$$Z\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) \cdot Z^{-n} = x(0)$$
$$= Z^{0} = 1$$

$$\boxed{Z\{\delta(n)\} \xleftrightarrow{Z.T} 1}$$

II) unit- step Sequence :-

we know that

$$u(n) = 1, \text{ for } n \geq 0$$
$$u(n) = 0, \text{ for } n < 0$$

$$X(Z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$
$$= \sum_{n=-\infty}^{\infty} u(n) Z^{-n} \Rightarrow \sum_{n=0}^{\infty} u(n) Z^{-n}$$

$$= \sum_{n=0}^{\infty} (1) z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

$$= \frac{1}{1-z^{-1}} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}$$

ROC: $|z^{-1}| < 1$.

$\frac{1}{|z|} < 1$ $\quad Z\{u(n)\} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$

$1 < |z|$

$$\boxed{|z| > 1 \quad Z\{u(n)\} = \frac{z}{z-1}}$$

III) unit - Ramp Sequence :-

$x(n) = r(n) = n u(n)$

$r(n) = n$, for $n \geq 0$

$\quad = 0$, for $n < 0$

$$X(z) = Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z\{r(n)\} = \sum_{n=0}^{\infty} n (z^{-1})^{n} \quad \text{ROC}: |z^{-1}| < 1$$

$\left|\frac{1}{z}\right| < 1$

$1 < |z|$

$$= 0 z^{0} + 1 z^{-1} + 2 z^{-2} + 3 z^{-3} + \dots \quad \underline{|z| > 1}$$

$$= z^{-1} + 2z^{-2} + 3z^{-3} + \dots$$

$$= (1 + z^{-1} + 2z^{-2} + 3z^{-3} + \dots) - 1$$

we know that $\cancel{1 + 2x^{2} + 3x^{3} + 4x^{3} + \dots} = \frac{1}{(1-x)^{2}}$

$$= \cancel{\left[1 + (z^{-1}) + 2(z^{-1})^{2} + 3(z^{-1})^{3} + \dots\right]} - 1$$

$$= \cancel{\frac{1}{(1-z^{-1})^{2}}} - 1$$

$$= z^{-1}\left[1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + \dots\right]$$

we know that $1 + 2x + 3x^{2} + 4x^{3} + \dots = \frac{1}{(1-x)^{2}}$

$$= z^{-1}\left[\frac{1}{(1-z^{-1})^{2}}\right] = \frac{z^{-1}}{(1-z^{-1})^{2}} = \frac{\frac{1}{z}}{\frac{(z-1)^{2}}{z^{2}}}$$

$$= \frac{z}{(z-1)^{2}}$$

$$\boxed{Z\{r(n)\} = \frac{z}{(z-1)^{2}}}$$

IV) The Exponential Sequence :- $(x(n) = e^{-j\omega n} u(n))$

$x(n) = e^{-j\omega n} u(n)$, $n \geq 0$

$\quad = 0$, $n < 0$

Fourier transform of $X(z)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n} = \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^{n}$$

ROC : $|e^{-j\omega} z^{-1}| < 1 \Rightarrow \left|\frac{e^{-j\omega}}{z}\right| < 1 \Rightarrow |e^{-j\omega}| < |z|$

$|z| > |e^{-j\omega}|$

we know that $|e^{j\omega}| = |e^{-j\omega}| = 1$

then Roc : $|Z| > 1$.

$$= \sum_{n=0}^{\infty} (e^{-j\omega} Z^{-1})^n$$

$$= (e^{-j\omega} Z^{-1})^0 + (e^{-j\omega} Z^{-1}) + (e^{-j\omega} Z^{-1})^2 + \cdots$$

$$= 1 + (e^{-j\omega} Z^{-1}) + (e^{-j\omega} Z^{-1})^2 + \cdots$$

$$= \frac{1}{1 - e^{-j\omega} Z^{-1}} = \frac{1}{1 - \frac{e^{-j\omega}}{Z}}$$

$$= \frac{Z}{Z - e^{-j\omega}}$$

$$\boxed{Z\{e^{-j\omega n} u(n)\} \longrightarrow \frac{Z}{Z - e^{-j\omega}} ;\ \text{Roc } |Z| > 1}$$

The Sinusoidal Sequence $[x(n) = \sin\omega n\, u(n)]$:

$$x(n) = \sin\omega n\, u(n), \quad n \geq 0$$
$$= 0, \quad n < 0$$

The Z-transform is given by

$$X[Z] = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \sin\omega n \cdot u(n)\, Z^{-n}$$

$$= \sum_{n=0}^{\infty} \sin\omega n \cdot Z^{-n} \Rightarrow \sum_{n=0}^{\infty} \sin\omega n (Z^{-1})^n$$

$$= \cancel{Z_1}$$

Roc : $|Z^{-1}| < 1 \Rightarrow \left|\frac{1}{Z}\right| < 1 \Rightarrow 1 < |Z|$
$\Rightarrow |Z| > 1$.

$$\Rightarrow \sum_{n=0}^{\infty} \left[\frac{e^{j\omega n} - e^{-j\omega n}}{2j}\right] (Z^{-1})^n$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} e^{j\omega n} \cdot (Z^{-1})^n - \sum_{n=0}^{\infty} e^{-j\omega n} (Z^{-1})^n\right]$$

$$= \frac{1}{2j}\left[\sum_{n=0}^{\infty} (e^{j\omega} Z^{-1})^n - \sum_{n=0}^{\infty} (e^{-j\omega} Z^{-1})^n\right]$$

we know that $1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$.

then

$$= \frac{1}{2j}\left[\frac{1}{1 - e^{j\omega} Z^{-1}} - \frac{1}{1 - e^{-j\omega} Z^{-1}}\right]$$

$$= \frac{1}{2j}\left[\frac{Z}{Z - e^{j\omega}} - \frac{Z}{Z - e^{-j\omega}}\right]$$

$$\frac{Z}{2j}\left[\frac{\cancel{Z} - e^{-j\omega} - \cancel{Z} + e^{j\omega}}{(Z - e^{j\omega})(Z - e^{-j\omega})}\right]$$

$$= \frac{Z}{2j}\left[\frac{e^{j\omega} - e^{-j\omega}}{Z^2 - Ze^{-j\omega} - Ze^{j\omega} + e^{0}}\right] = \frac{Z}{\cancel{2j}}\left[\frac{\cancel{2j}\sin\omega}{Z^2 - Z[e^{j\omega} + e^{-j\omega}] + 1}\right]$$

$$= \frac{z \sin\omega}{z^2 - z2\cos\omega + 1}$$

$$\boxed{\therefore Z\{\sin\omega n\, u(n)\} = \frac{z \sin\omega}{z^2 - 2z\cos\omega + 1} \; ; \; ROC: |z| > 1}$$

The cosinusoidal sequence: $x(n) = \cos\omega n\, u(n)$.

Similar to sinusoidal sequence, the Z-transform of cosinusoidal sequence and its ROC is given by.

$$\boxed{Z\{\cos\omega n\, u(n)\} = \frac{z(z - \cos\omega)}{z^2 - 2z\cos\omega + 1} \; ; \; ROC: |z| > 1}$$

Find the Z-transform of

1) $n\,\delta(n)$
2) $\delta(n+3)$
3) $\delta(n+3)$
4) $\delta(n-4)$
5) $n\cdot\delta(n-2)$
6) $y(n) = x(n+1)\,u(n)$
7) $y(n) = x(n-1)\,u(n)$

1) $n\cdot\delta(n)$

we know that $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{n\cdot\delta(n)\} = \sum_{n=-\infty}^{\infty} n\cdot\delta(n) z^{-n}$$

$$= \sum_{n=0} \sum_{n=-\infty}^{\infty} \delta(n)\cdot n z^{-n}$$

by using the impulse property, we get

$$\sum_{n=-\infty}^{\infty} \delta(n) x(n) = x(0)$$

then

$$\sum_{n=-\infty}^{\infty} \delta(n)\, n z^{-n} = 0 z^{-0}$$
$$= 0.$$

$$Z\{n\delta(n)\} = 0$$

5) $n\cdot\delta(n-2)$.

we know that $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$Z\{n\delta(n-2)\} = \sum_{n=-\infty}^{\infty} n\cdot\delta(n-2) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-2)\cdot n z^{-n}$$

by impulse property, $\sum_{n=-\infty}^{\infty} \delta(n-n_0) x(n) = x(n_0)$

$$= \sum_{n=-\infty}^{\infty} \delta(n-2)\, n z^{-n} = 2z^{-2}$$

ROC :- $X(z)$ converges for all values, Except at $z = 0$.

Because, if $z=0$, then $X(z)$ becomes $\frac{2}{0} = \infty$.

$$Z\{n\cdot\delta(n-2)\} = 2z^{-2}.$$

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6) $y(n) = x(n+1)u(n)$.

we know that Z-transform is given by

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} x(n+1)\cdot u(n)\cdot Z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n+1) Z^{-n}$$

Let $n+1 = k$

$n = k-1$.

if $n=0 \Rightarrow k=1$

$n=\infty \Rightarrow k=\infty$

then $$Z\{y(n)\} = \sum_{k=1}^{\infty} x(k)\cdot Z^{-(k-1)}$$

$$= \sum_{k=1}^{\infty} x(k) Z^{-k}\cdot Z.$$

$$= Z \sum_{k=1}^{\infty} x(k)\cdot Z^{-k} \quad Roc: |Z^{-1}|<1$$

$\left|\frac{1}{z}\right|<1$

$1<|z|$

$|Z|>1$

$= \cancel{ZX}$

$$= \cancel{Z/K}\; Z\sum_{k=0}^{\infty} x(k) Z^{-k} - x(0)$$

$$\boxed{Z\{x(n+1)u(n)\} = ZX(Z) - x(0). \; ; Roc: |Z|>1}$$

7) $y(n) = x(n-1)\cdot u(n)$

we know that Z-transform is given by

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$Z\{y(n)\} = \sum_{n=-\infty}^{\infty} x(n-1) u(n) Z^{-n}$$

$$= \sum_{n=0}^{\infty} x(n-1) Z^{-n}$$

Let $n-1=k$

$n=k+1$.

if $n=0$ then $k=-1$

$n=\infty$ then $k=\infty$

then $$Z\{y(n)\} = \sum_{k=-1}^{\infty} x(k) Z^{-(k+1)}$$

$$= \sum_{k=-1}^{\infty} x(k) Z^{-k}\cdot Z^{-1}$$

$$= Z^{-1} \sum_{k=-1}^{\infty} x(k) Z^{-k}$$

$$= Z^{-1} \sum_{k=0}^{\infty} x(k) Z^{-k} + Z^{-1} x(-1) Z^{1}$$

$$= Z^{-1}\left[\sum_{k=0}^{\infty} x(k) Z^{-k} + x(-1) Z\right]$$

$$= Z^{-1}\left[X(Z) + Z x(-1)\right] = Z^{-1}X(Z) + x(-1)$$

Prove that the Sequences a) $x(n) = a^n u(n)$ & b) $x(n) = -a^n u(-n-1)$. have Same $X(z)$ & differ only in Roc. Also plot their Roc's.

a) $x(n) = a^n u(n)$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

Roc: $|az^{-1}| < 1$ we know that $1+x+x^2+x^3+\dots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

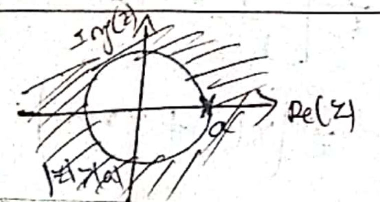
$\left|\frac{a}{z}\right| < 1$

$|a| < |z|$ $\qquad = \frac{1}{1-az^{-1}}$

$|z| > |a|$ $\qquad = \frac{1}{1-\frac{a}{z}}$

$$= \frac{z}{z-a}$$

$$\boxed{Z\{a^n u(n)\} = \frac{z}{z-a}\ ;\ Roc: |z| > |a|}$$



b) $x(n) = -a^n u(-n-1)$ $\qquad u(-n-1) = 1,\ (-n-1) \geq 0$; $-n \geq 1$; $n \leq -1$.

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$= \sum_{n=1}^{-\infty} -a^n z^{-n}$$

$$= -\sum_{n=0}^{-\infty} a^n z^{-n} + 1$$

$$= -\sum_{n=0}^{\infty} (a^{-1} z)^n + 1.$$

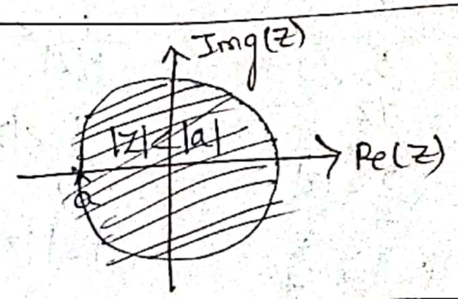
Roc: $|a^{-1} z| < 1$ $\qquad = -\left[\frac{1}{1-a^{-1}z}\right] + 1$

$\left|\frac{z}{a}\right| < 1$ $\qquad = 1 - \frac{1}{1-\frac{z}{a}}$

$|z| < |a|$ $\qquad = 1 - \frac{a}{a-z}$

$$= \frac{a-z-a}{a-z} = \frac{z}{z-a}.$$

$$\boxed{Z\{-a^n u(-n-1)\} = \frac{z}{z-a}\ ;\ Roc: |z| < |a|}$$



1. Find the z-transform & Roc of $X(z)$ for

$$x(n) = 3\left[\frac{5}{7}\right]^n u(n) + 2\left[\frac{-1}{3}\right]^n u(n).$$

and also find pole-zero locations.

Given $x(n) = 3\left[\frac{5}{7}\right]^n u(n) + 2\left(\frac{-1}{3}\right)^n u(n)$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[3\left(\frac{5}{7}\right)^n u(n) + 2\left(\frac{-1}{3}\right)^n u(n)\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[3\left(\frac{5}{7}\right)^n + 2\left(\frac{-1}{3}\right)^n\right] z^{-n}$$

$$= \sum_{n=0}^{\infty} 3\left(\frac{5}{7}\right)^n z^{-n} + \sum_{n=0}^{\infty} 2\left(\frac{-1}{3}\right)^n z^{-n}$$

$$= 3\sum_{n=0}^{\infty}\left(\frac{5}{7}z^{-1}\right)^n + 2\sum_{n=0}^{\infty}\left(\frac{-1}{3}z^{-1}\right)^n$$

Roc: $\left|\frac{5}{7}z^{-1}\right| < 1$ $\qquad$ $\left|\frac{-1}{3}z^{-1}\right| < 1$

$\frac{5}{7z} < 1$ $\qquad$ $\frac{1}{3|z|} < 1$

$\frac{5}{7} < |z|$ $\qquad$ $\frac{1}{3} < |z|$

$|z| > \frac{5}{7}$ $\qquad$ $|z| > \frac{1}{3}$

If we get both Roc in same symbol, i.e. ~~no~~ greater symbol (or) lesser symbol then if both Roc's in greater symbol then we have to take max value in both ~~max~~ if $|z| > a$ & $|z| > b$

~~If both~~ then Roc: $|z| > \max(a, b)$

if both Roc's in lesser than symbol then we have to take min value in both i.e. if $|z| < |a|$ & $|z| < |b|$ then

Roc: ~~min~~ $|z| < \min(|a|, |b|)$.

Similarly Here, ~~if~~ $|z| > \frac{5}{7}$ & $|z| > \frac{1}{3}$.

Roc: $|z| > \max\left(\frac{5}{7}, \frac{1}{3}\right)$

$|z| > \frac{5}{7}$ as ($\frac{5}{7}$ is greater than $1/3$)

$$= 3\left[\frac{1}{1 - \frac{5}{7}z^{-1}}\right] + 2\left[\frac{1}{1 - \left(\frac{-1}{3}z^{-1}\right)}\right]$$

$$3\left[\frac{1}{1 - \frac{5}{7z}}\right] + 2\left[\frac{1}{1 + \frac{1}{3z}}\right]$$

$$= 3\left[\frac{7z}{7z - 5}\right] + 2\left[\frac{3z}{3z + 1}\right]$$

$$= 3\left[\frac{z}{z - \frac{5}{7}}\right] + 2\left[\frac{z}{z + \frac{1}{3}}\right]$$

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$$= \frac{3z\left(z+\frac{1}{3}\right)+2z\left(z-\frac{5}{7}\right)}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)}$$

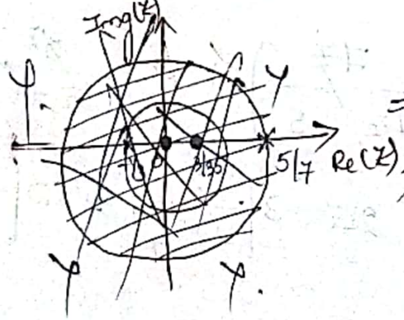
$$= \frac{3z^2+z+2z^2-\frac{10}{7}z}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)}$$

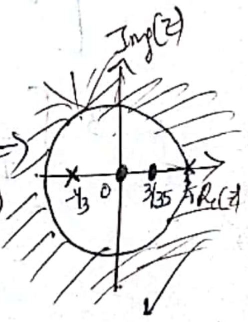
$$= \frac{5z^2-\frac{3}{7}z}{\left(z-\frac{5}{7}\right)\left(z+\frac{1}{3}\right)}$$

$$= \frac{\cancel{z}\cancel{\left[5z\right.}\ 5z\left(z-\frac{3}{35}\right)}{\left[z-\frac{5}{7}\right]\left[z+\frac{1}{3}\right]}.$$

Here holes are seen in Numerator terms i.e. they are zero and $3/35$

Poles are seen in denominator, they are $5/7, -1/3$.





Properties of Z-transform :-

1) Linearity Property: $Z\{a x_1(n)+b x_2(n)\} \xleftrightarrow{Z.T} a X_1(z)+b X_2(z)$

2) Time shifting property: $Z\{x(n-m)\} \xleftrightarrow{Z.T} z^{-m}X(z)$

$Z\{x(n+m)\} \xleftrightarrow{Z.T} z^{m}X(z)$.

3) Multiplication by an Exponential Sequence property: $Z\{a^n x(n)\} \longleftrightarrow X\left(\frac{z}{a}\right)$.

Similarly $Z\{e^{j\omega n}x(n)\} \longleftrightarrow X\left(\frac{z}{e^{j\omega}}\right) = X(e^{-j\omega}z)$

$Z\{e^{-j\omega n}x(n)\} \longleftrightarrow X\left(\frac{z}{e^{-j\omega}}\right) = X(e^{j\omega}z)$

4) Time Expansion property :-

$x(n) \xleftrightarrow{Z.T} X(z)$ with $ROC = R$

$x_K(n) \longleftrightarrow X(z^K)$ with $ROC = R^{1/K}$.

where $x_K(n) = x\left(\frac{n}{K}\right)$, if n is an integer Multiple of K.

Proof :- $Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$.

$Z\{x_K(n)\} = \sum_{n=-\infty}^{\infty} x_K(n)z^{-n} \Rightarrow \sum_{n=-\infty}^{\infty} x\left(\frac{n}{K}\right)z^{-n}$

Let $\frac{n}{k} = p$

$n = pk$

$$Z\{x_k(n)\} = \sum_{p=-\infty}^{\infty} x(p) \cdot Z^{-pk}$$

$$= \sum_{p=-\infty}^{\infty} x(p) (Z^{-k})^{p}$$

$$= X(Z^{k}).$$

$$\therefore x\left(\frac{n}{k}\right) \xleftrightarrow{Z.T} X(Z^{k})$$

Multiplication by n or differentiation in Z-domain

Property :-

if $x(n) \xleftrightarrow{Z.T} X(Z)$, with $ROC = R$

$$n x(n) \xleftrightarrow{Z.T} -Z \frac{d}{dZ} X(Z), \text{ with } ROC = R$$

Proof :-

We know that

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} = X(Z)$$

differentiating on both sides w.r.t to 'Z',

$$\frac{d}{dZ} X(Z) = \sum_{n=-\infty}^{\infty} x(n) \frac{d(Z^{-n})}{dZ}$$

$$\frac{d}{dZ}\{X(Z)\} = \sum_{n=-\infty}^{\infty} x(n) \left[-n Z^{-n-1}\right]$$

$$\frac{d}{dZ}\{X(Z)\} = -\sum_{n=-\infty}^{\infty} n x(n) Z^{-n} \cdot Z^{-1}$$

$$\frac{d}{dZ}\{X(Z)\} = -\sum_{n=-\infty}^{\infty} n \cdot x(n) \cdot Z^{-n} \cdot \frac{1}{Z}$$

$$-Z \frac{d}{dZ}\{X(Z)\} = \sum_{n=-\infty}^{\infty} n x(n) \cdot Z^{-n}$$

$$\boxed{-Z \frac{d}{dZ}\{X(Z)\} = Z\{n x(n)\}.}$$

Conjugation property :-

The conjugation property states that

if $x(n) \xleftrightarrow{Z.T} X(Z)$ with $ROC = R$

$$x^*(n) \xleftrightarrow{Z.T} X^*(Z^*) \text{ with } ROC = R.$$

Proof :-

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot Z^{-n}$$

$$Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) \cdot Z^{-n}$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (Z^*)^{-n}\right]^*$$

$$= [X(Z^*)]^* = X^*(Z^*).$$

$$x^*(n) \xleftrightarrow{Z.T} X^*(z^*)$$

Convolution property :-

If
$$x_1(n) \longleftrightarrow X_1(z) \text{ with } ROC = R_1$$
$$x_2(n) \longleftrightarrow X_2(z) \text{ with } ROC = R_2$$

then
$$x_1(n) * x_2(n) \xleftrightarrow{Z.T} X_1(z)\,X_2(z).$$

Proof :-

we know that
$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

$$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} [x_1(n) * x_2(n)] \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k)\right] z^{-(n-k)} \cdot z^{-k}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) \cdot z^{-k} \cdot \sum_{k=-\infty}^{\infty} x_2(n-k) \cdot z^{-(n-k)}$$

Let $n-k=p$,

then
$$= \left(\sum_{k=-\infty}^{\infty} x_1(k) \cdot z^{-k}\right)\left(\sum_{p=-\infty}^{\infty} x_2(p) \cdot z^{-p}\right)$$

$$= X_1(z) \cdot X_2(z)$$

$$\boxed{\therefore\; x_1(n) * x_2(n) \xrightarrow{Z.T} X_1(z) \cdot X_2(z)}$$

Multiplication property or complex convolution Property :-

The Multiplication property or complex convolution Property of z-transform states that

If
$$x_1(n) \xrightarrow{Z.T} X_1(z).$$
$$x_2(n) \xrightarrow{Z.T} X_2(z).$$

then
$$x_1(n) \cdot x_2(n) \xrightarrow{Z.T} \frac{1}{2\pi j} \oint_c X_1(v) \cdot X_2\left(\frac{z}{v}\right) \cdot v^{-1}\, dv$$

Proof :-
$$Z\{x_1(n) \cdot x_2(n)\} = \sum_{n=-\infty}^{\infty} (x_1(n) \cdot x_2(n)] \cdot z^{-n}$$

from the inverse Laplace transform, we know that
$$x(n) = \frac{1}{2\pi j} \oint_c X_1(z) \cdot z^{n-1}\, dz.$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2\pi j} \oint_c X_1(z) \cdot z^{n-1}\, dz\right] x_2(n)\, z^{-n}$$

$$= \cancel{\sum_{n=-\infty}^{\infty} \frac{1}{2\pi j} \oint_c X_1(z)\, z^{-1}\, dz \sum_{n=-\infty}^{\infty} x_2(n)}$$

Changing the complex variable z to 'v', and then we get

$$Z\{x_1(n).x_2(n)\} = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi j}\oint X_1(v).v^{n-1}.dv\right]x_2(n)z^{-n}$$

$$= \frac{1}{2\pi j}\oint_c X_1(v).\left[\sum_{n=-\infty}^{\infty} x_2(n)(v^{-1}z)^{-n}\right]v^{-1}dv$$

$$= \frac{1}{2\pi j}\oint_c X_1(v)\, X_2(v^{-1}z).\, v^{-1}dv$$

$$x_1(n).x_2(n) \xrightarrow{Z.T} \frac{1}{2\pi j}\oint_c X_1(v)\, X_2\left(\frac{z}{v}\right).v^{-1}dv$$

Correlation Property :-

The correlation property of z-transform states that

if $x_1(n) \xrightarrow{Z.T} X_1(z)$

$x_2(n) \xrightarrow{Z.T} X_2(z)$

then $x_1(n) \otimes x_2(n) \xleftrightarrow{Z.T} X_1(z).X_2(z^{-1})$

Proof :- $Z\{x_1(n)\otimes x_2(n)\} = \sum_{n=-\infty}^{\infty}[x_1(n)\otimes x_2(n)]z^{-n}$

from the correlation property

$$x_1(n)\otimes x_2(n) = \sum_{l=-\infty}^{\infty} x_1(l).x_2(l-n) \text{ (or) } \sum_{l=-\infty}^{\infty} x_1(l-n)x_2(l)$$

then

$$Z\{x_1(n)\otimes x_2(n)\} = \sum_{n=-\infty}^{\infty}\left[\sum_{l=-\infty}^{\infty} x_1(l).x_2(l-n)\right]z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x_1(l).\left[\sum_{n=-\infty}^{\infty} x_2(l-n).z^{-n}\right].$$

let $l-n=m$

$l-m=n.$

$$= \sum_{l=-\infty}^{\infty} x_1(l).\left[\sum_{n=-\infty}^{\infty} x_2(m).z^{-(l-m)}\right]$$

$$= \sum_{l=-\infty}^{\infty} x_1(l).\sum_{n=-\infty}^{\infty} x_2(m)\, z^{-l}\, z^{m}$$

$$= \sum_{l=-\infty}^{\infty} x_1(l)\, z^{-l}.\sum_{n=-\infty}^{\infty} x_2(m).(z^{-1})^{-m}$$

$$= X_1(z).X_2(z^{-1}).$$

$$\boxed{\therefore Z\{x_1(n)\otimes x_2(n)\} = X_1(z)\,X_2(z^{-1})}$$

Accumulation property :-

The accumulation property of z-transform states that $x(n) \xrightarrow{Z.T} X(z)$

$$\sum_{k=-\infty}^{n} x(k) \xrightarrow{Z.T} \frac{1}{1-z^{-1}}X(z)$$

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Proof:- from the defination of Z-transform

$$Z\left\{\sum_{k=-\infty}^{n} x(k)\right\} = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{n} x(k) z^{-n}$$

by Summation Values, we can Say that n is greater than 'k'

~~K~~ $n = k + r$.

$r = n - k$.

$k = n - r$

$$\Rightarrow \sum_{k=-\infty-r}^{\infty-r} \cdot \sum_{r=n+\infty}^{n-n} x(k) z^{-(k+r)}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{r=\infty}^{0} x(k) z^{-k} \cdot z^{-r}$$

Note :- Formula :-

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \; ; \text{ if } a \neq 1,$$

$$= 1 \quad , \quad \text{if } a = 1$$

Parseval's Theorum :-

The Parseval's theorum of Z-transform States that

$$x_1(n) \xleftrightarrow{Z.T} X_1(z)$$

$$x_2(n) \xleftrightarrow{Z.T} X_2(z)$$

then $\sum_{n=-\infty}^{\infty} x_1(n) \cdot x_2^*(n) = \frac{1}{2\pi j}\oint X_1(z) \cdot X_2^*[(z^*)^{-1}] z^{-1} dz$

for complex $x_1(n)$ & $x_2(n)$

Proof:- $\cancel{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$

$$= \sum_{n=-\infty}^{\infty} \left[x_1(n) \cdot x_2^*(n)\right] \cancel{z^{-n}}$$

we know that $x_1(n)$, by Inverse Z-transform,

$$X(z) = \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi j}\oint X_1(z) \cdot z^{n-1} dz\right] x_2^*(n)$$

$$= \sum_{n=-\infty}^{\infty}\left[\frac{1}{2\pi j}\oint X_1(z) \cdot x_2^*(n) (z^{-1})^{-n}\right] \cdot z^{-1} dz$$

$$= \frac{1}{2\pi j}\oint X_1(z) \cdot z^{-1}\left[\sum_{n=-\infty}^{\infty} x_2^*(n) \cdot (z^{-1})^{-n}\right] z^{-1} dz$$

$$= \cancel{\frac{1}{2\pi j}\oint X_1(z) z^{-1} X_2^*(z^{-1}) dz}$$

$$= \cancel{\frac{1}{2\pi j}\oint X_1(z) X_2^* z}$$

$$= \frac{1}{2\pi j}\oint X_1(z) z^{-1}\left[\sum_{n=-\infty}^{\infty} x_2(n) [(z^*)^{-1}]^{-n}\right]^* z^{-1} dz$$

$$= \frac{1}{2\pi j}\oint X_1(z) X_2^*[(z^*)^{-1}] z^{-1} dz$$

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Initial Value Theorum :-

The initial value theorum of z-transform States that, for a causal Signal $x(n)$

if

$$x(n) \xleftrightarrow{z.T} X(z)$$

then $\displaystyle \underset{n \to 0}{Lt}\, x(n) = x(0) = \underset{z \to \infty}{Lt}\, X(z)$

Proof :- we know that for a causal Signal

$$z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$= x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + \dots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \dots$$

Taking the limit $z \to \infty$ on both Sides

$$\underset{z \to \infty}{Lt}\, X(z) = \underset{z \to \infty}{Lt} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \dots\right]$$

$$= x(0) + \frac{x(1)}{\infty} + \frac{x(2)}{\infty^2} + \dots$$

$$= x(0) + 0 + 0$$

$$= x(0) = \underset{n \to 0}{Lt}\, x(n)$$

$$\boxed{\therefore \underset{n \to 0}{Lt}\, x(n) = x(0) = \underset{z \to \infty}{Lt}\, X(z)}$$

Final value Theorum :-

The final value Theorum of z-transform States that for a causal Signal

$$x(n) \xrightarrow{z.T} X(z)$$

and if $X(z)$ has no poles outside the unit circle and it has no double or higher order poles on the unit circle centered at the origin of the z-plane then

$$\underset{n \to \infty}{Lt}\, x(n) = x(\infty) = \underset{z \to 1}{Lt}\, (z-1)X(z)$$

Proof :- we know that for a causal Signal

$$z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$z\{x(n+1)\} = zX(z) - x(0) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n}$$

$$z\{x(n+1)\} - z\{x(n)\} = zX(z) - z\,x(0) - X(z)$$

$$= \sum_{n=0}^{\infty} x(n+1)\, z^{-n} - \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$X(z)(z-1) - x(0) = \sum_{n=0}^{\infty} \left[x(n+1) - x(n)\right] z^{-n}$$

$$(z-1)X(z) - x(0) = [x(1) - x(0)]z^{-0} + [x(2) - x(1)]z^{-1} + [x(3) - x(2)]z^{-2} + \dots$$

Taking $z \xrightarrow{Lt} 1$ on both Sides, we get

$$\underset{\to 1}{Lt}\, (z-1)X(z) - x(0) = x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \dots$$

$$\lim_{z \to 1} (z-1)X(z) - \cancel{x(0)} = -\cancel{x(0)} + x(\infty)$$

$$\lim_{z \to 1} (z-1)X(z) = x(\infty) = \lim_{n \to \infty} x(n)$$

using the final value theorum, find $x(\infty)$, if $X(z)$ is given by

a) $\frac{z+1}{(z-0.6)^2}$ b) $\frac{z+2}{4(z-1)(z+0.7)}$ c) $\frac{2z+3}{(z+1)(z+3)(z-1)}$

a) $\frac{z+1}{(z-0.6)^2}$.

by final value theorum,

$$x(\infty) = \lim_{z \to 1} (z-1)X(z)$$

$$= \lim_{z \to 1} (z-1)\left(\frac{z+1}{(z-0.6)^2}\right)$$

$$= \frac{(1-1)(1+1)}{(1-0.6)^2} = 0$$

$\therefore x(\infty) = 0$

b) $\frac{z+2}{4(z-1)(z+0.7)} = X(z)$

$$(z-1)X(z) = \frac{z+2}{4(z+0.7)}$$

$(z-1)X(z)$ has no poles on or outside the unit circle

then we can find $x(\infty)$ by using final value theorum

$$x(\infty) = \lim_{z \to 1} (z-1)X(z)$$

$$x(\infty) = \lim_{z \to 1} \frac{z+2}{4(z+0.7)}$$

$$= \frac{1+2}{4(1+0.7)} = \frac{3}{4 \times 1.7} = \frac{3}{6.8} = \frac{30}{68} = \frac{15}{34} = 0.44$$

$x(\infty) = 0.44$

c) $\frac{2z+3}{(z+1)(z+3)(z-1)}$

$$(z-1)X(z) = \frac{2z+3}{(z+1)(z+3)}.$$

here poles are $-1, -3$.

Therefore $(z-1)X(z)$ has one pole on the unit circle and one pole (-3) outside the unit circle. So, $x(\infty)$ tends to infinity as $n \to \infty$

Find the intial value $x(0)$ for the following qus by using intial value theorum?

By using intial value theorum,

$$\lim_{n \to 0} x(n) = x(0) = \lim_{z \to \infty} X(z).$$

a) $\frac{z^2+2z+2}{(z+1)(z+0.5)}$.

$$x(z) = \frac{z^2+2z+2}{(z+1)(z+0.5)}$$

by intial value theorum

$$x(0) = \lim_{z\to\infty} X(z)$$

$$= \lim_{z\to\infty} \frac{z^2+2z+2}{(z+1)(z+0.5)}$$

$$= \lim_{z\to\infty} \frac{z^2+2z+2}{z^2+1.5z+0.5}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^2}\left[1+\frac{2}{z}+\frac{2}{z^2}\right]}{\cancel{z^2}\left[1+\frac{1.5}{z}+\frac{0.5}{z^2}\right]}$$

$$= \frac{1+\frac{2}{\infty}+\frac{2}{\infty^2}}{1+\frac{1.5}{\infty}+\frac{0.5}{\infty^2}} = \frac{1+0+0}{1+0+0} = \frac{1}{1}$$

$$x(0) = 1$$

b) $\frac{z+3}{(z+1)(z+2)}$

by using ~~final~~ Intial value theorum,

$$x(0) = \lim_{z\to\infty} X(z)$$

$$= \lim_{z\to\infty} \frac{z+3}{(z+1)(z+2)}$$

$$= \lim_{z\to\infty} \frac{\cancel{z}\left[1+\frac{3}{z}\right]}{\cancel{z}\left[1+\frac{1}{z}\right][z+2]}$$

$$= \frac{1+\frac{3}{\infty}}{\left(1+\frac{1}{\infty}\right)(\infty+2)} = \frac{1}{\infty} = 0$$

$$x(0) = 0$$

c) $X(z) = \frac{z^2}{(z-1)(z-0.2)}$

by using intial value theorum,

$$x(0) = \lim_{z\to\infty} X(z)$$

$$x(0) = \lim_{z\to\infty} \frac{z^2}{(z-1)(z-0.2)}$$

$$= \lim_{z\to\infty} \frac{\cancel{z^2}\ 1}{\cancel{z^2}\left[1-\frac{1}{z}\right]\left[1-\frac{0.2}{z}\right]}$$

$$= \frac{1}{\left(1-\frac{1}{\infty}\right)\left(1-\frac{0.2}{\infty}\right)} = \frac{1}{(1-0)(1-0)}$$

$$= \frac{1}{1} = 1.$$

$$\therefore x(0) = 1$$

Inverse Z-transform:-

The process of finding the time domain s[ignal] from its Z-transform $X(Z)$ is called inverse Z-trans[form] which is denoted by

$$x(n) = Z^{-1}\{X(Z)\}.$$

$$X(Z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty}[x(n)r^{-n}]e^{-j\omega n}$$

This is the DTFT of a signal $x(n)r^{-n}$. Hence th[e] inverse discrete time fourier transform of $X(re^{j\omega})$ must be $x(n)r^{-n}$. therefore, we can write

$$x(n)r^{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega}) \cdot e^{j\omega n}\, d\omega.$$

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(re^{j\omega})(re^{j\omega})^n\, d\omega.$$

$$Z = re^{j\omega}$$

$$\frac{dZ}{d\omega} = rje^{j\omega}$$

$$\frac{dZ}{rje^{j\omega}} = d\omega$$

$$x(n) = \frac{1}{2\pi}\oint_{-\pi}^{\pi} X(Z) \cdot (re^{j\omega})^n \cdot \frac{dZ}{rje^{j\omega}}$$

$$x(n) = \frac{1}{2\pi j}\oint_{-\pi}^{\pi} X(Z) \cdot Z^n \cdot \frac{dZ}{re^{j\omega}}$$

$$= \frac{1}{2\pi j}\oint_{-\pi}^{\pi} X(Z)\, Z^n \frac{dZ}{Z}$$

$$= \frac{1}{2\pi j}\oint_{-\pi}^{\pi} X(Z)\, Z^{n-1} \cdot dZ.$$

$$=$$

~~the~~

1) Determine the inverse Z-transform of function $X(Z) = \frac{3+2Z^{-1}+Z^{-2}}{1-3Z^{-1}+2Z^{-2}}$ by following three methods and prove that it is unique?

Method : 1

Given $X(Z) = \frac{3+2Z^{-1}+Z^{-2}}{1-3Z^{-1}+2Z^{-2}}$.

$$X(Z) = \frac{3Z^2+2Z+1}{Z^2-3Z+2} \quad (\because \times \,\&\, \div \text{ by } Z^2)$$

$$X(Z) = \frac{3Z^2+2Z+1}{(Z-1)(Z-2)}$$

$$\frac{X(Z)}{Z} = \frac{3Z^2+2Z+1}{Z(Z-1)(Z-2)}.$$

$$\frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}$$

$3z^2$ → what ever denominator A has, it should be placed

$$A = z \cdot \frac{3z^2+2z+1}{z(z-1)(z-2)}\Big|_{z=0} = \frac{1}{(0-1)(0-2)} = \frac{1}{2} = 0.5$$

$$B = (z-1)\frac{3z^2+2z+1}{z(z-1)(z-2)}\Big|_{z=1} = \frac{3+2+1}{1(1-2)} = -6.$$

$$C = (z-2)\frac{3z^2+2z+1}{z(z-1)(z-2)}\Big|_{z=2} = \frac{3(2)^2+2(2)+1}{2(2-1)}$$

$$= \frac{17}{2} = 8.5$$

$$\frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}$$

$$X(z) = 0.5 - 6\left[\frac{z}{z-1}\right] + 8.5\left[\frac{z}{z-2}\right]$$

Apply inverse z-transform on both sides

$$x(n) = 0.5\,\delta(n) - 6\,u(n) + 8.5(2)^n u(n)$$

$$x(n) = 0.5\,\delta(n) + \left[-6 + 8.5(2)^n\right]u(n)$$

when $n=0$

$$x(0) = 0.5\,\delta(0) + \left[-6 + 8.5(2)^0\right]u(0)$$

$$= 0.5 + \left[-6 + 8.5\right]$$

$$= 0.5 - 6 + 8.5 = 9 - 6 = 3$$

$n=1$

$$x(1) = 0.5\,\delta(1) + \left[-6 + 8.5(2)^1\right]u(1)$$

$$= 0 - 6 + 17$$

$$= 11.$$

$n=2$

$$x(2) = 0.5\,\delta(2) + \left[-6 + 8.5(2)^2\right]u(2)$$

$$= 0 - 6 + 34$$

$$= 28$$

$$\therefore x(n) = \{\underset{\uparrow}{3}, 11, 28, \ldots\}$$

Method-(2) :- power Series Expansion Method:-

Given $X(z) = \dfrac{3 + 2z^{-1} + z^{-2}}{1 - 3z^{-1} + 2z^{-2}} \times \cancel{3z^2}$

$$
\begin{array}{r|l}
1-3z^{-1}+2z^{-2} & 3+2z^{-1}+z^{-2} \quad \big(3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4} \\
 & \underline{3-9z^{-1}+6z^{-2}} \\
 & 11z^{-1}-5z^{-2} \\
 & \underline{11z^{-1}-33z^{-2}+22z^{-3}} \\
 & 28z^{-2}-22z^{-3} \\
 & \underline{28z^{-2}-84z^{-3}+56z^{-4}} \\
 & 62z^{-3}-56z^{-4} \\
 & \underline{62z^{-3}-186z^{-4}+124z^{-5}} \\
 & 130z^{-4}-124z^{-5} \\
 & 130z^{-4} \\
 & \vdots
\end{array}
$$

$X(z) = 3+11z^{-1}+28z^{-2}+62z^{-3}+130z^{-4}+\dots \rightarrow (1)$

we know that

$X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n}$

$\therefore \dots x(-1)z^{1}+x(0)+x(1).z^{-1}+x(2)z^{-2}+x(3)z^{-3}+\dots \rightarrow (2)$

by comparing eq 1,2, we get

$x(n) = \{ \underset{\uparrow}{3}, 11, 28, 62, 130, \dots \}$

Note :-

→ If the given signal Roc is non-causal signal, then express both Numerator and denominator in ascending (increasing) powers of z or in descending (decreasing) powers of z^{-1} and make long divison

→ If the given signal Roc is causal signal, then express both Numerator and denominator in descending (decrease) powers of z or in ascending (increase) powers of z^{-1} and make long divison

Method-③ :- Residue Method :-

The inverse z-transform of $X(z)$ can be obtained using the equation.

$$x(n) = \frac{1}{2\pi j} \oint_c X(z).z^{n-1}\, dz.$$

where c is a circle in the z-plane in the Roc of $X(z)$. The above equation can be evaluated by finding the sum of all residues of the poles that are inside the circle 'c'.

$$\therefore x(n) = \sum \text{Residues of } X(z).z^{n-1} \text{ at poles inside 'c'}$$

$$= \sum_i (z-z_i).X(z).z^{n-1}\Big|_{z=z_i}$$

If $X(z)z^{n-1}$ has no poles inside the contour 'c' for one or more values of n, then $x(n) = 0$ for these values

Q) using Residue Method, find the inverse Z-transform of

$$X(z) = \frac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}}, \quad ROC: |z| > 3.$$

Given $X(z) = \dfrac{1+2z^{-1}}{1+4z^{-1}+3z^{-2}}$

Here the highest power is z^{-2}, so we have to Multiply both Nr & Dr with z^2, to eliminate inverse z-form, then

$$X(z) = \frac{z^2[1+2z^{-1}]}{z^2[1+4z^{-1}+3z^{-2}]}$$

$$= \frac{z^2+2z}{z^2+4z+3} = \frac{z[z+2]}{z^2+4z+3}$$

$$X(z) = \frac{z[z+2]}{(z+1)(z+3)} \rightarrow \text{poles} = -1, -3$$

by Residue Method,

$x(n) = \sum$ Residues of $X(z) \cdot z^{n-1}$ at poles $X(z) \cdot z^{n-1}$ with

$$= \sum z^{n-1}\left[\frac{z(z+2)}{(z+1)(z+3)}\right] \text{ at pole } -1, -3$$

as $X(z) z^{n-1} = \dfrac{z(z+2)}{(z+1)(z+3)} (z^{n-1}) = \dfrac{z^n(z+2)}{(z+1)(z+3)}$

$$= \underset{z \to -1}{\text{Res}} \frac{z^n[z+2]}{(z+3)\cancel{(z+1)}}\cancel{(z+1)} + \underset{z \to -3}{\text{Res}} \frac{z^n(z+2)}{(z+1)\cancel{(z+3)}} \cdot \cancel{(z+3)}$$

$$= \frac{(-1)^n(-1+2)}{(-1+3)} + \frac{(-3)^n(-3+2)}{(-3+1)}$$

$$= \left[\frac{1}{2}(-1)^n + \frac{1}{2}(-3)^n\right] u(n).$$

Method-(4):- complex integral (or) contour integration Method:

we know that by Residue Method,

$$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} \, dz$$

&

$x(n) = \sum$ Residues of $X(z) \cdot z^{n-1}$ at poles inside 'c'

$$= \sum_i (z-z_i) \cdot X(z) \cdot z^{n-1} \Big|_{z=z_i}$$

if $X(z) \cdot z^{n-1}$ has Multiple poles of order 'k', the residue at pole $z = z_i$ is given by.

$$\frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} \left[(z-z_i)^k \cdot X(z) z^{n-1}\right]_{z=z_i}$$

① Determine the inverse z-transform using the complex integral

$$X(z) = \frac{3z^{-1}}{\left[1 - (1/2)z^{-1}\right]^2} \; ; \; ROC : |z| > $$

by contour integral method ;

$$x(n) = \sum_{\text{'c'}} \text{Residues of } X(z) \cdot z^{n-1} \text{ at poles ins}$$

$$= \sum_i (z - z_i)^k X(z) \cdot z^{n-1} \Big|_{z = z_i}$$

~~If there is a~~

$$X(z) = \frac{3z^{-1}}{(1 - 1/2\, z^{-1})^2}$$

$$= \frac{\frac{3}{z}}{\left(1 - \frac{1}{2z}\right)^2} = \frac{\frac{3}{\cancel{z}}}{\frac{\left(z - \frac{1}{2}\right)^2}{z^{\cancel{2}}}}$$

$$= \frac{3z}{(z - 1/2)^2}$$

Now, if

$$X(z) \cdot z^{n-1} = \frac{3z \cdot z^{n-1}}{(z - 1/2)^2} = \frac{3z^n}{(z - 1/2)^2}$$

If $X(z) \cdot z^{n-1}$ has multiple poles of order k, then the residue at the pole $z = z_i$ is given by

$$= \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} (z - z_i)^k \cdot X(z) z^{n-1} \Big|_{z = z_i}$$

∅ from the qus, $X(z) z^{n-1} = \frac{3z^n}{(z - 1/2)^2}$, Here $k = 2$

poles $= 1/2 = z_i$

$$x(n) = \frac{1}{(2-1)!} \frac{d^{(2-1)}}{dz^{(2-1)}} \cancel{(z - 1/2)^2} \cdot \frac{3z^n}{\cancel{(z - 1/2)^2}} \Big|_{z = 1/2}$$

$$= \frac{1}{1!} \frac{d}{dz} (3z^n) \Big|_{z = 1/2}$$

$$= n 3z^{n-1} \Big|_{z = 1/2} = 3n (1/2)^{n-1}$$

$$\therefore \; x(n) = 3n \left(\frac{1}{2}\right)^{n-1} \cdot u(n)$$