UNIT – V

STATE SPACE ANALYSIS

The root locus and frequency response methods require the physical system in the form of a transfer function. Even though, the transfer function model provides us with simple and powerful analysis and design techniques, it suffers from certain drawbacks such as

(1) The transfer function is only defined under zero initial conditions

(2) The transfer function model is applicable to linear time-invariant systems

(3) The transfer function model is restricted to single-input-single output systems.

(4) The transfer function does not provide any information regarding internal state of the system.

(5) The classical design methods such as root locus and frequency domain methods are essentially trial and error procedures.

To overcome all these drawbacks, the state variable approach is introduced. It is a direct time-domain approach which provides a basis for modern control theory and system optimization. It is a very powerful technique for the analysis and design of linear and non-linear, time-invariant or time-varying multi-input-multi-output systems. The organization of the state variable approach is such that it is easily amenable to solution through digital computers.

Concepts of State, State Variables & State Model

The state of a dynamical system is a minimal set of variables (known as state variables) such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$ completely determines the behaviour of the system for $t > 0$.

In state variable formation of a system, the state variables are usually represented by $x_1(t)$, $x_2(t)$ ---; the inputs by $u_1(t)$, $u_2(t)$ ---; and the outputs by $y_1(t)$, $y_2(t)$ ----. Let us assume that there are 'm' inputs, 'p' outputs and 'n' state variables.

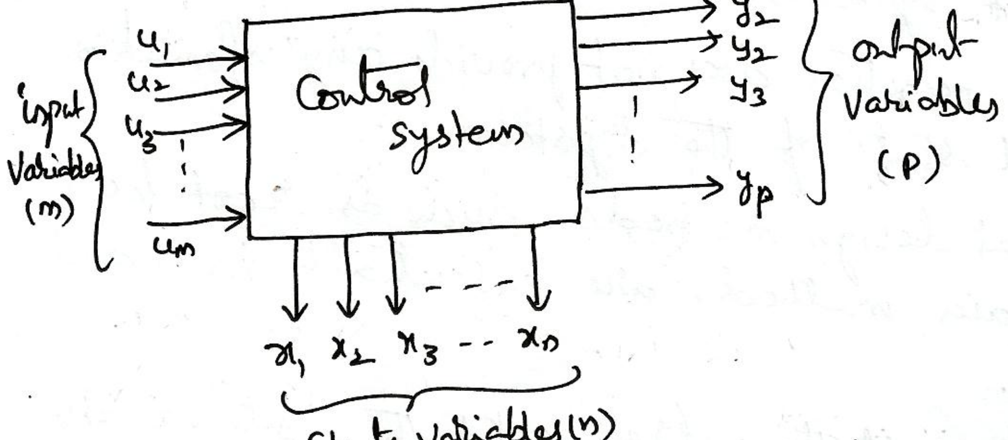


Figure: State Model of a Control System

The input, output and state variables in matrix form are represented as

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}_{m\times 1} ; \quad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}_{p\times 1} ; \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

For a linear system, the state model is given by

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m$$

Thus for linear system, the derivative of each state variable is a linear combination of system states and inputs. where a_{ij} and b_{ij} are constants. In vector form, the state equations can be represented as

$$\dot{x}(t) = Ax(t) + Bu(t) \longrightarrow (1)$$

where $x(t)$ is $n\times1$ state vector, $u(t)$ is $m\times1$ input vector
A is $n\times n$ system matrix defined by
B is $n\times m$ output matrix defined as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n} ; \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}_{n\times m}$$

Similarly, the output variables at time 't' are linear combinations of the values of the input and state variables at time 't', ie

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t) \cdots + c_{1n}x_n(t) + d_{11}u_1(t) + d_{12}u_2(t) \cdots + d_{1m}u_m(t)$$

$$y_2(t) = c_{21}x_1(t) + c_{22}x_2(t) + \cdots + c_{2n}x_n(t) + d_{21}u_1(t) + d_{22}u_2(t) + \cdots + d_{2m}u_m(t)$$

$$\vdots$$

$$y_p(t) = c_{p1}x_1(t) + c_{p2}x_2(t) \cdots + c_{pn}x_n(t) + d_{p1}u_1(t) + \cdots + d_{pm}u_m(t)$$

where the coefficients c_{ij} and d_{ij} are constants. This set of equations may be put in the vector matrix form as

$$y(t) = Cx(t) + Du(t) \longrightarrow (2)$$

where $y(t)$ is $p\times 1$ output vector, C is $p\times n$ output matrix defined by

$$\vec{C} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}_{p\times n}$$

$\vec{D}$ is $p\times m$ transmission matrix defined by

$$\vec{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1m} \\ d_{21} & d_{22} & \cdots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pm} \end{bmatrix}_{p\times m}$$

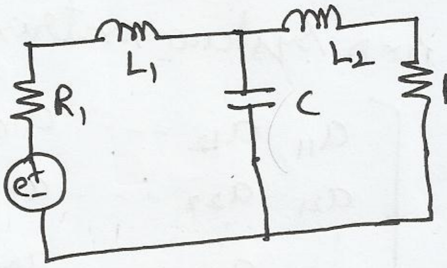
Thus, the state model of a linear time invariant system is given by

$$\dot{x}(t) = Ax(t) + B\,u(t) \; ; \text{ State Equation}$$

$$y(t) = C\,x(t) + D\,u(t) \; ; \text{ output Equation.}$$

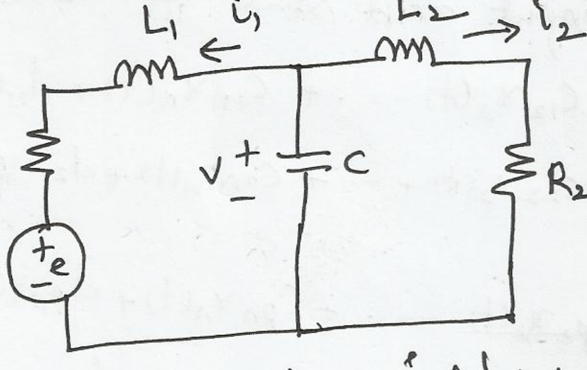
① <u>State space Representation Using physical variables:</u>

① obtain the state model of an electrical network shown in figure.



(Sol) let the current i_1 in inductor L_1, current i_2 in inductor L_2 and voltage drop across capacitor 'c' are the state variables

<u>Note</u>: Number of state variables = Number of storage elements



Let $x_1(t) = v(t)$; $x_2(t) = i_1(t)$; $x_3(t) = i_2(t)$

The differential equations governing the behaviour of the RLC network are

$$i_1 + i_2 + C\frac{dv}{dt} = 0 \longrightarrow ①$$

$$L_1 \frac{di_1}{dt} + R_1 i_1 + e - v = 0 \longrightarrow ②$$

$$L_2 \frac{di_2}{dt} + R_2 i_2 - v = 0 \longrightarrow ③$$

we are interested in expressing the variables $\frac{dv}{dt}$, $\frac{di_1}{dt}$ and $\frac{di_2}{dt}$ as linear Combination of the variables v, i_1 and i_2 and e

From eq ①,
$$\frac{dv}{dt} = -\frac{1}{c} i_1 - \frac{1}{c} i_2 \longrightarrow \text{I}$$

$$\frac{di_1}{dt} = \frac{1}{L_1} v - \frac{R_1}{L_1} i_1 - \frac{1}{L_1} e \longrightarrow \text{II}$$

$$\frac{di_2}{dt} = \frac{1}{L_2} v - \frac{R_2}{L_2} i_2 \longrightarrow \text{III}$$

where input $u(t) = e(t)$; Now the state equations are

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{c} & -\frac{1}{c} \\ \frac{1}{L_1} & -R_1/L_1 & -1/L_1 \\ 1/L_2 & 0 & -R_2/L_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ -1/L_1 \\ 0 \end{bmatrix} u$$

Assume that the voltage across R_2 and current through R_2 are the output variables y_1 and y_2 respectively

$$y_1 = V_2 = R_2 i_2 \; ; \quad y_2 = I = i_2$$

∴ The output equations can be represented as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & R_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

② obtain the state model of armature Controlled DC Motor

(Sol)

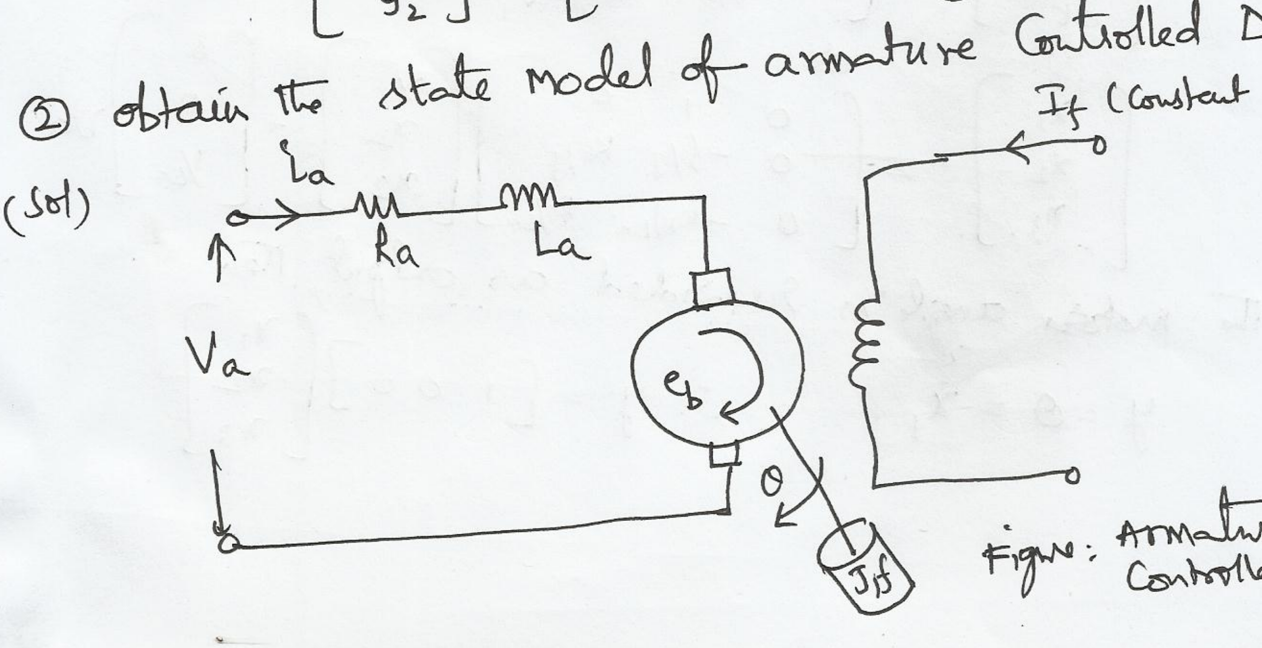


Figure: Armature Controlled DC Motor

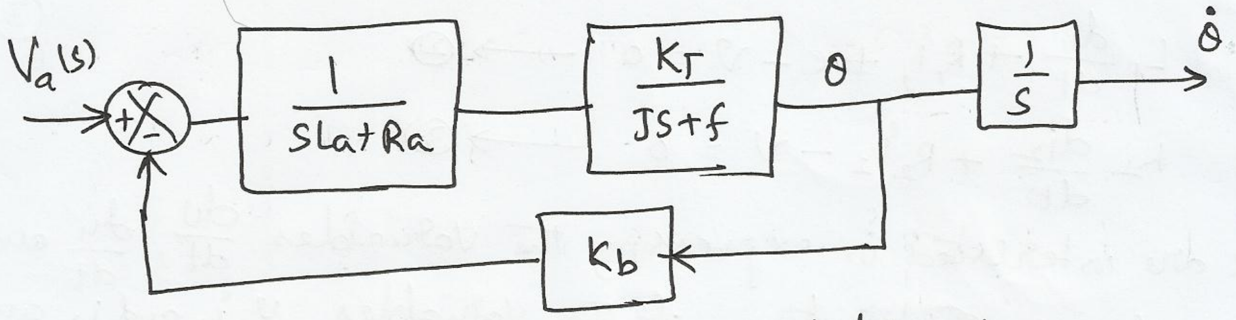


Figure: Block Diagram representation of Armature Controlled DC Motor

The state variables are $x_1 = \theta$; $x_2 = \dot{\theta} = \omega$ and $x_3 = i_a$

Now, we can write the following set of three first-order differential equations relating the inputs and outputs of the first-order factors $\frac{1}{s}$, $\frac{K_T}{JS+f}$ and $\frac{1}{R_a + sL_a}$

$$\dot{x}_1 = x_2$$
$$J\dot{x}_2 + f x_2 = K_T x_3$$
$$V_a - K_b x_2 = R_a x_3 + L_a \dot{x}_3$$

$$\left[\begin{array}{l} \frac{d\theta}{dt} = \omega \\ J\frac{d\omega}{dt} + f\omega = K_T i_a \\ V_a - K_b\frac{d\theta}{dt} = R_a i_a + L_a \frac{d}{dt} i_a \end{array}\right]$$

These three first order differential equations can be represented in vector form as

$$\begin{bmatrix} \frac{d\theta}{dt} \\ \frac{d\omega}{dt} \\ \frac{d}{dt} i_a \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & \frac{K_T}{J} \\ 0 & -\frac{K_b}{L_a} & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \\ i_a \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} V_a$$

or

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & K_T/J \\ 0 & -K_b/L_a & -R_a/L_a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} V_a$$

If the motor angle is regarded as output, then

$$y = \theta = x_1 \qquad \therefore\ y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

② State space representation using phase variables:

The phase variable state model is easily determined if the system model is already known in differential equation or transfer function form.

The phase variables are defined as those particular state variables which are obtained from one of the system variables and its derivatives. Often the variable used is system output and the remaining state variables are then derivatives of the output.

Case (1): when the transfer function does not have zeros, such a transfer function has the form → ①

$$T(s) = \frac{Y(s)}{U(s)} = \frac{b}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s + a_n}$$

The corresponding differential equation is

$$y^{(n)} + a y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = bu \rightarrow ②$$

where $y^{(n)} = \frac{d^n}{dt^n} y$; $\dot{y} = \frac{dy}{dt}$

By letting
$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$
$$\vdots$$
$$x_n = y^{(n-1)}$$; then

$$\dot{x}_1 = \dot{y} = x_2$$
$$\dot{x}_2 = \ddot{y} = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = y^n = -[a_n y + a_{n-1}\dot{y} + \cdots + a_1 y^{n-1}] + bu$$

$$\therefore \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - \cdots - a_1 x_n + bu$$

The above equations result in the following state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix} u$$

or

$$\dot{X} = AX + BU$$

The output equation is $y = CX$
where $C = [1\ 0\ 0\ 0 \cdots 0]$

It is to be observed that the matrix 'A' has very special form. It has all 1's in the upper off-diagonal, its last row is comprised of the negative of the coefficients of the original differential equation and all other elements are zero. This form of matrix 'A' is known as the Bush form or Companion form.

Also note that the matrix B has the speciality that all its elements except that the last are zero. In fact the matrices A and B can be written directly by inspection of the linear differential equation.

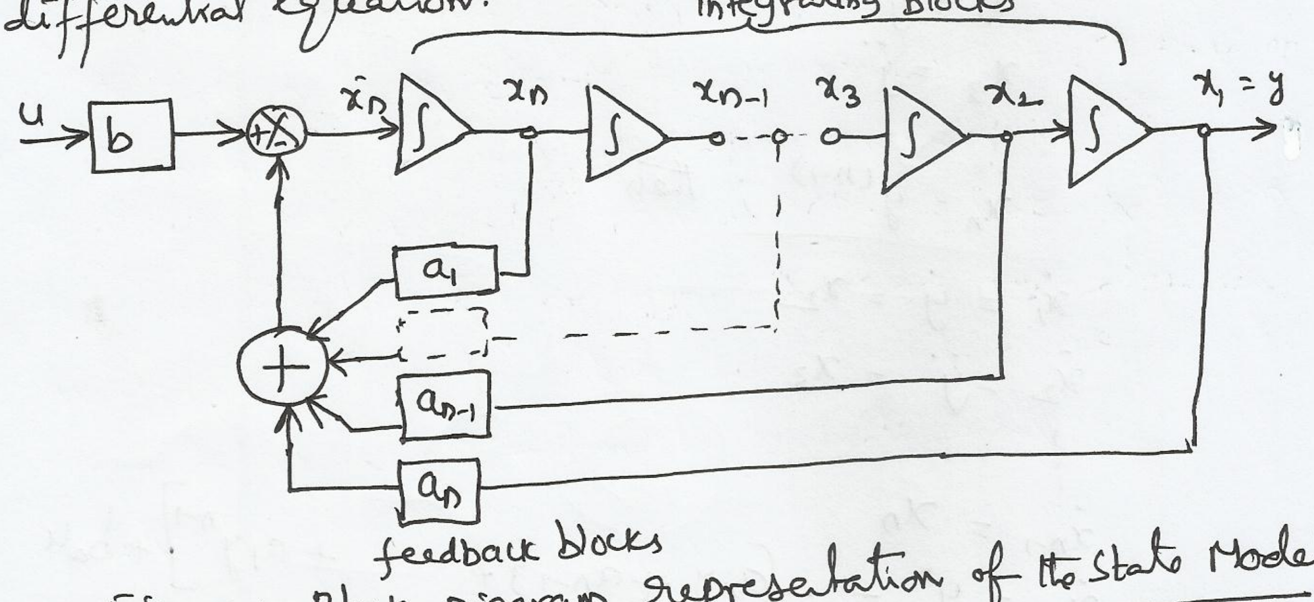


Figure : Block Diagram representation of the State Model

Case (2): phase variable formulation for transfer function with poles and zeros:

Let us consider a third order transfer function

$$\frac{Y(s)}{U(s)} = T(s) = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3} \longrightarrow ①$$

For this third order transfer function let the state variables are x_1, x_2 and x_3. The above equation may be rearranged as

$$T(s) = \frac{b_0 + b_1/s + b_2/s^2 + b_3/s^3}{1-(-a_1/s - a_2/s^2 - a_3/s^3)} \longrightarrow ②$$

We have the Mason's gain formula

$$T(s) = \frac{1}{\Delta} \sum_K P_K \Delta_K \longrightarrow ③$$

From equations ② & ③, we observe that a signal flow graph of equation ② may consists of

(i) three feedback loops (touching each other) with gains $-a_1/s$, $-a_2/s^2$ and a_3/s^3;

(ii) Four forward paths which touch the loops and have gains b_0, b_1/s, b_2/s^2 and b_3/s^3

A signal flow graph configuration which satisfies the above requirements is shown in figure.

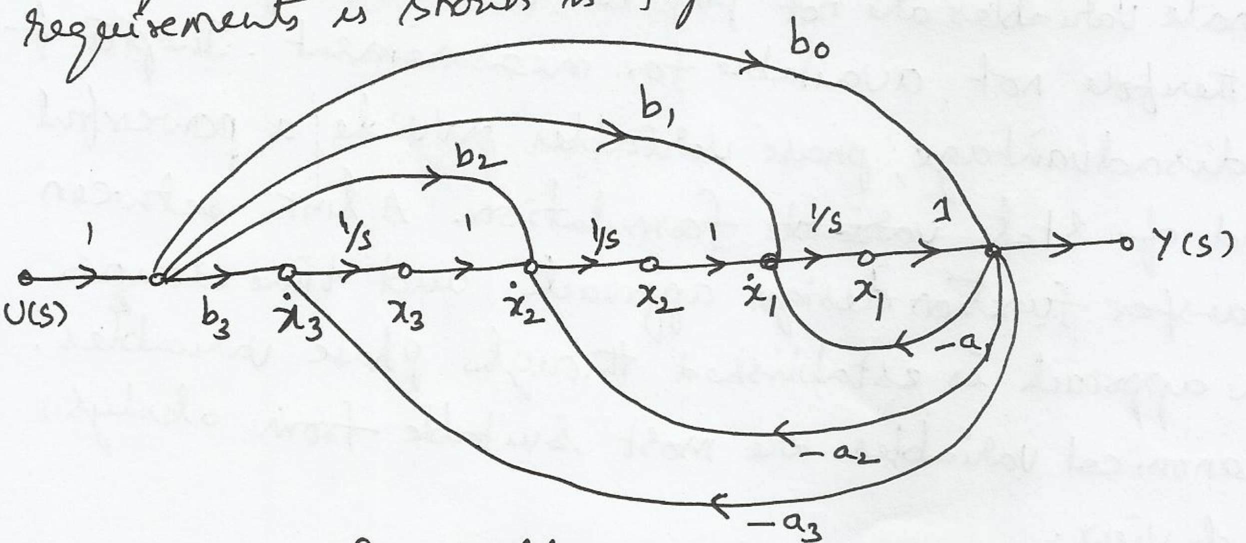


Figure: Signal flow graph

from the Signal flow graph

$$y = x_1 + b_0 u$$

$$\dot{x}_1 = -a_1(x_1 + b_0 u) + x_2 + b_1 u$$

$$= -a_1 x_1 + x_2 + (b_1 - a_1 b_0 u)$$

$$\dot{x}_2 = -a_2 y + x_3 + b_2 u$$

$$= -a_2(x_1 + b_0 u) + b_2 u + x_3 = -a_2 x_1 + x_3 + (b_2 - a_2 b_0)u$$

$$\dot{x}_3 = -a_3 y + b_3 u.$$

$$= -a_3(x_1 + b_0 u) + b_3 u$$

$$= -a_3 x_1 + (b_3 - a_3 b_0)u$$

The above equations can be represented in state model as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 - a_1 b_0 \\ b_2 - a_2 b_0 \\ b_3 - a_3 b_0 \end{bmatrix} u \quad \text{and}$$

$$y = [1\ 0\ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0 u$$

The disadvantage of phase variable representation is that the phase variables are not physical variables of the system and therefore not available for measurement. Inspite of this disadvantage, phase variables provide a powerful method of state variable formulation. A link between the transfer function design approach and time domain design approach is established through phase variables. The canonical variables are most suitable from analysis point of view.

(3) State space representation using Canonical variables

In canonical-variable or normal-form representation of a system, the system matrix A turns out to be a diagonal matrix. This form of state model plays an important role in control theory. The disadvantage of the canonical form is equally important. The canonical variables, like phase variables are not real physical variables of the system.

Let us consider a transfer function shown below

$$\frac{Y(s)}{U(s)} = T(s) = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_{n-1} s + b_n}{s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n} \longrightarrow (1)$$

Assume that the denominator is known in factored form and that the poles of the transfer function located at $\lambda_1, \lambda_2 \ldots \lambda_n$ are all distinct. Then the transfer function can then be expanded into partial fractions as

$$\frac{Y(s)}{U(s)} = T(s) = b_0 + \sum_{i=1}^{n} \frac{c_i}{s-\lambda_i} \longrightarrow (2)$$

where c_i are the residues of the poles at $s = \lambda_i$.

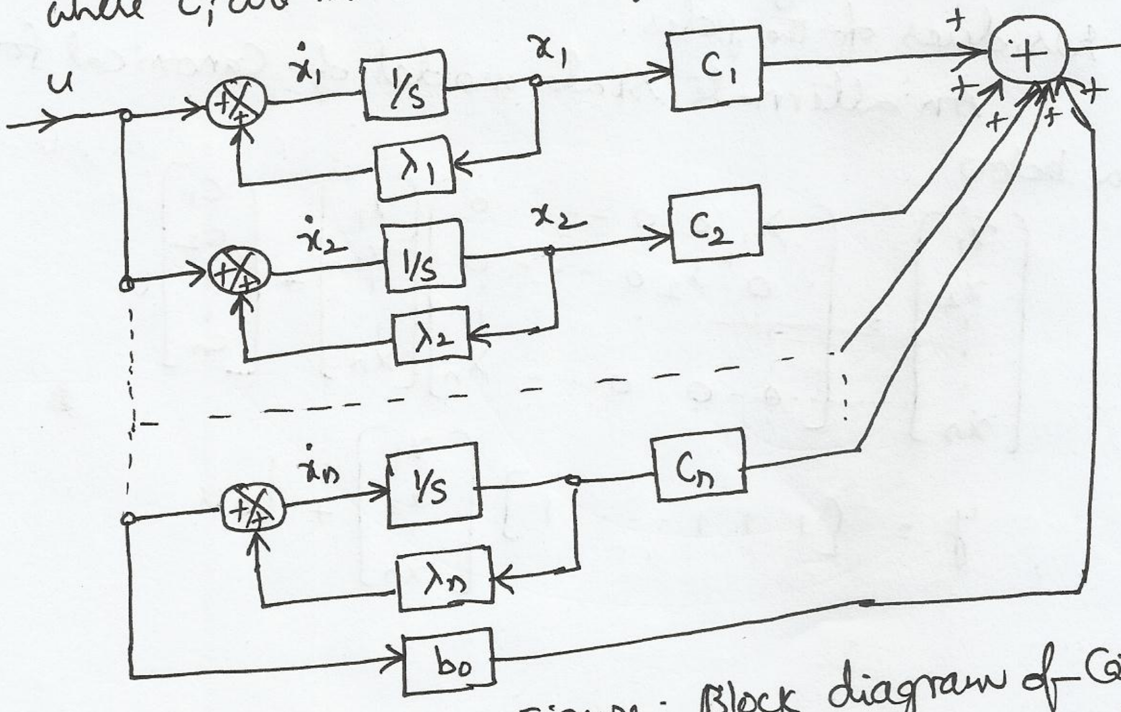


Figure: Block diagram of Canonical model

Defining the output of each integrator to be a state variable, we can write the state equations as

$$\dot{x}_i = \lambda_i x_i + u; \quad i = 1, 2, \dots n$$

The output $y(t)$ is given by

$$y = c_1 x_1 + c_2 x_2 + \dots + c_n x_n + b_0 u$$

This state model can be expressed in the vector-matrix form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

In the canonical model, the system matrix 'A' is a diagonal matrix with the poles of $T(s)$ as its diagonal elements. It is also observed that elements of column vector B are all unity and the elements of the row vector C are residues of the poles.

An alternate state model of canonical form is given below

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + b_0 u$$

① obtain the Canonical form & state model of a system described by a differential equation

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = \dddot{u} + 8\ddot{u} + 17\dot{u} + 8u$$

where y is output and u is input.

(Sol) Taking the laplace transform on both sides with zero initial conditions,

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = s^3U(s) + 8s^2U(s) + 17sU(s) + 8U(s)$$

$$Y(s)[s^3 + 6s^2 + 11s + 6] = U(s)\{s^3 + 8s^2 + 17s + 8]$$

Therefore, the TF $\frac{Y(s)}{U(s)} = \frac{s^3 + 8s^2 + 17s + 8}{s^3 + 6s^2 + 11s + 6}$

$$= \frac{\{s^3 + 6s^2 + 11s + 6] + [2s^2 + 6s + 2]}{s^3 + 6s^2 + 11s + 6}$$

$$= 1 + \frac{2s^2 + 6s + 2}{(s+1)(s+2)(s+3)}$$

$$= 1 + \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

where $A = \lim_{s \to -1} \frac{2s^2 + 6s + 2}{(s+2)(s+3)} = \frac{2-6+2}{2} = -1$

$B = \lim_{s \to -2} \frac{2s^2 + 6s + 2}{(s+1)(s+3)} = \frac{8-12+2}{-1(1)} = 2$

$C = \lim_{s \to -3} \frac{2s^2 + 6s + 2}{(s+1)(s+2)} = \frac{18-18+2}{-2(-1)} = 1$

$$\therefore \frac{Y(s)}{U(s)} = 1 - \frac{1}{s+1} + \frac{2}{s+2} + \frac{3}{s+3}$$

The canonical model representation is shown in the block diagram

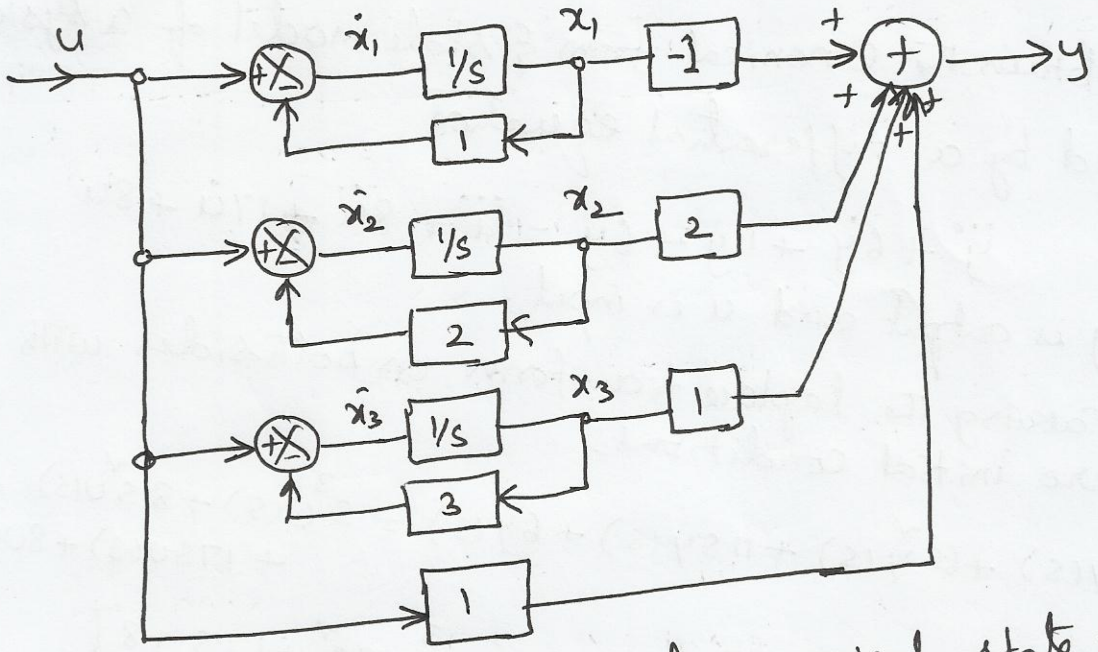


Figure: Block diagram of canonical state model

Therefore, the state equations are

$$\dot{x}_i = \lambda_i x_i + u \; ; \quad i = 1, 2, 3$$

$$\therefore \left.\begin{aligned} \dot{x}_1 &= -x_1 + u \\ \dot{x}_2 &= -2x_2 + u \\ \dot{x}_3 &= -3x_3 + u \end{aligned}\right\}$$

The state model is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u$$

The output equation is

$$y = -x_1 + 2x_2 + x_3 + u$$

$$= \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u$$

② obtain the canonical state model of a system, whose transfer function is given by

$$\frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 7}{(s+1)^2 (s+2)}$$

(Sol) Decomposing the above transfer function by the method of partial fractions yields

$$\frac{Y(s)}{U(s)} = \frac{A}{(s+1)^2} + \frac{B}{(s+1)} + \frac{C}{(s+2)}$$

where $A = \underset{s \to -1}{Lt} \dfrac{2s^2+6s+7}{(s+2)} = \dfrac{2-6+7}{1} = 3$

$B = \underset{s \to -1}{Lt} \dfrac{d}{ds}\left[\dfrac{2s^2+6s+7}{s+2}\right] = \underset{s \to -1}{Lt} \dfrac{(s+2)(4s+6)-(2s^2+6s+7)}{(s+2)^2}$

$= \dfrac{1(2)-(2-6+7)}{1} = \dfrac{2-3}{1} = -1$

$C = \underset{s \to -2}{Lt} \dfrac{2s^2+6s+7}{(s+1)^2} = 8-12+7 = 3$

$\therefore \dfrac{Y(s)}{U(s)} = \dfrac{3}{(s+1)^2} - \dfrac{1}{(s+1)} + \dfrac{3}{(s+2)}$

The block diagram representation of the canonical state model is shown in block diagram

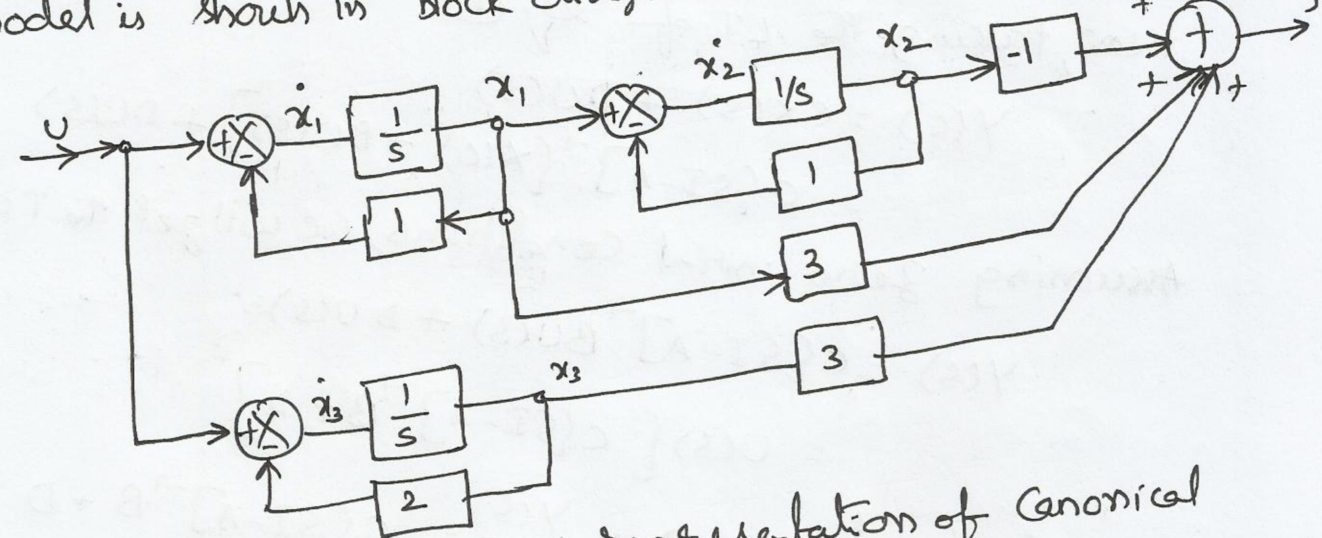


Figure : Block Diagram representation of Canonical state model

The state equations are

$\dot{x}_1 = -x_1 + u$

$\dot{x}_2 = x_1 - x_2$

$\dot{x}_3 = -2x_3 + u$

The output $y = -x_2 + 3x_1 + x_3$

Jordan block

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Because of repeated poles x_1 & x_2 are not in decoupled form. The dotted block is known as Jordan block.

Derivation of Transfer function from State Model:

Let us consider the state Model of a system given by

$$\dot{X} = AX + BU \longrightarrow (1)$$

$$Y = CX + DU \longrightarrow (2)$$

Taking the LT of eq (1), we will get

$$SX(S) - X(0) = AX(S) + BU(S)$$

$$X(S)[SI-A] = X(0) + BU(S)$$

$$\therefore X(S) = [SI-A]^{-1}[X(0) + BU(S)]$$

Now, taking the LT of eq (2)

$$Y(S) = CX(S) + DU(S)$$

$$= C[SI-A]^{-1}[X(0) + BU(S)] + DU(S)$$

Assuming zero initial conditions, we will get the TF

$$Y(S) = C[SI-A]^{-1}BU(S) + DU(S)$$

$$= U(S)[C[SI-A]^{-1}B + D]$$

$$\therefore \text{Transfer function } T(S) = \frac{Y(S)}{U(S)} = C[SI-A]^{-1}B + D$$

$$= C\frac{\text{Adj}[SI-A]}{\text{Det}[SI-A]}B + D$$

Solving the denominator, we will get the characteristic equation $|SI-A| = 0$

An important observation is that, the state model is not unique, but the transfer function is unique.

Diagonalization: The state model of a system is not unique, since the state model employ physical variable, phase variables and Canonical variables.

From the application point of view, the physical variable representation of state model is more useful because they can be easily measured and used for control purposes. However this state model of physical variables is not convenient for investigation of system properties and evaluation of time response.

The Canonical state model in which the system matrix 'A' is in diagonal form is most suitable for investigation of system properties and evaluation of time response. Therefore it is useful to study techniques for transforming a general state model into Canonical form. These techniques are often referred to as diagonalization techniques.

Let us consider an n^{th}-order multi-input-multi-output state model

$$\left.\begin{aligned} \dot{X} &= AX + BU \\ Y &= CX + DU \end{aligned}\right\} \longrightarrow (1)$$

Assume that the matrix 'A' in this model is non diagonal.

Let us define a new state vector V such that

$$X = MV \longrightarrow (2)$$

where M is $n \times n$ nonsingular constant matrix.

Under this transformation, the state model in equation (1) modifies to

$$M\dot{V} = AMV + BU$$

or

$$\dot{V} = M^{-1}AMV + M^{-1}BU$$
$$= \tilde{A}V + \tilde{B}U \longrightarrow (3)$$

$$Y = CMV + DU \text{ or}$$
$$Y = \tilde{C}V + DU \longrightarrow (4)$$

If the matrix M can be selected such that $M^{-1}AM$ is a diagonized matrix Λ, then the model given by eqs ③ & ④ is canonical model. Under this condition, the matrix M is called the diagonalizing matrix or modal matrix.

where $\Lambda = M^{-1}AM$ = diagonal matrix

$\tilde{B} = M^{-1}B$

$\tilde{C} = CM$

The determination of the diagonalizing matrix is facilitated by use of eigen-vectors.

Eigenvalues and Eigenvectors:

The eigenvalues corresponding to system matrix 'A' are the solutions of $|\lambda I - A| = 0. \rightarrow ①$

The above equation may be expressed in expanded form as $q(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n = 0 \rightarrow ②$

The values of λ which satisfy the above equation are called eigenvalues. Equation ② is called the characteristic equation corresponding corresponding to matrix 'A'.

Therefore, it is concluded that the eigen values of the state model and the poles of the system transfer function are the same. Thus a state model is stable if all the eigenvalues have negative real parts.

If all the eigenvalues of matrix 'A' are all distinct, then the rank 'r' of the matrix $(\lambda I - A)$ is $(n-1)$.

The eigen vector m_i associated with the eigenvalue λ_i may be obtained by taking cofactors of matrix $(\lambda_i I - A)$ along any row. ie

$$m_i = \begin{bmatrix} C_{K_1} \\ C_{K_2} \\ \vdots \\ C_{K_n} \end{bmatrix} ; \quad K = 1, 2, \ldots, \text{or } n$$

where C_{K_i} are the co-factors of matrix $(\lambda_i I - A)$

Let $m_1, m_2 \ldots m_n$ be the eigenvectors corresponding to the eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$ respectively. Then the modal matrix or diagonalizing matrix M is given by

$$M = [m_1 \vdots m_2 \vdots m_3 \vdots \ldots m_n]$$

Therefore, the diagonalizing matrix is given by

$$\Lambda = M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & - & - & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & - & \lambda_n \end{bmatrix}$$

when 'A' is expressed in the form given below.

$$A = \begin{bmatrix} 0 & 1 & 0 & - & - & - & 0 \\ 0 & 0 & 1 & \cdots & & & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & - & - & - & 1 \\ -a_n & -a_{n-1} & - & - & - & \cdot & -a_1 \end{bmatrix}$$

then the modal matrix 'M' can be shown to be a special matrix called Vander Monde Matrix

$$M = V = \begin{bmatrix} 1 & 1 & - & - & - & 1 \\ \lambda_1 & \lambda_2 & - & - & - & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & - & - & - & \lambda_n^2 \\ \vdots & & & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & - & - & - & \lambda_n^{n-1} \end{bmatrix}$$

① Consider a system matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$.
Find the eigenvalues, eigen-vectors and diagonalizing matrix.

(Sol) The characteristic equation is $|\lambda I - A| = 0$

$$\Rightarrow \quad \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix} = \begin{bmatrix} \lambda & -1 & 0 \\ -3 & \lambda & -2 \\ 12 & 7 & \lambda+6 \end{bmatrix} = \lambda I - A$$

$$|\lambda I - A| = \lambda(\lambda^2 + 6\lambda + 14) - (-1)[-3\lambda - 18 + 24] + 0$$
$$= \lambda^3 + 6\lambda^2 + 14\lambda - 3\lambda - 18 + 24$$
$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6$$

The roots of $|\lambda I - A| = 0$ are $\lambda = -1, -2, -3$

$\therefore$ The eigenvalues are $\lambda_1 = -1; \ \lambda_2 = -2; \ \lambda_3 = -3$

The eigen vector m_i associated with eigenvector λ_i is obtained from the Co-factors of the matrix $[\lambda_i I - A]$

For m_1, the matrix $(\lambda_1 I - A)$ is

$$(\lambda_1 I - A) = (-1)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 3 & 0 & 2 \\ -12 & -7 & -6 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & 1 & 0 \\ -3 & -1 & -2 \\ 12 & 7 & 5 \end{bmatrix}$$

The Cofactors of 1st row are given by

$$m_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} +[-5+14] \\ -[-15+24] \\ +[-21+12] \end{bmatrix} = \begin{bmatrix} 9 \\ -9 \\ -9 \end{bmatrix}$$

or $m_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ since the eigenvector has unique direction.

Similarly, the eigen vectors associated with eigen values $\lambda_1 = -2$ and $\lambda_3 = -3$ are given by

$$m_2 = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}; \quad m_3 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$$

Therefore, the modal matrix or diagonalizing matrix is given by $M = [m_1 \vdots m_2 \vdots m_3] = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -4 & -3 \\ -1 & 1 & 3 \end{bmatrix}$

② Find the eigenvalues, eigenvectors and modal matrix for a system matrix $A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$

(Sol) The eigenvalues are the solutions of characteristic equation $|\lambda I - A| = 0$.

$$\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda-4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda-3 \end{bmatrix} = \lambda I - A$$

$$\therefore |\lambda I - A| = (\lambda-4)(\lambda^2 - 3\lambda + 2) - (-1)[-\lambda + 3 - 2] + 2[-1 + \lambda]$$

$$= [\lambda^3 - 3\lambda^2 + 2\lambda - 4\lambda^2 + 12\lambda - 8] - \lambda + 1 - 2 + 2\lambda$$

$$= \lambda^3 - 7\lambda^2 + 15\lambda - 9$$

$\therefore$ The eigenvalues are the solutions of $|\lambda I - A| = \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$

$\Rightarrow (\lambda - 1)(\lambda - 3)^2 = 0$; Therefore, the eigenvalues of the system matrix are $\lambda_1 = 1$; $\lambda_2 = 3$ and $\lambda_3 = 3$

The eigenvector associated with eigenvalue $\lambda = 1$ is obtained from the co-factors of any row of $(\lambda I - A)$, where $\lambda = 1$.

where $(\lambda_1 I - A) = (I - A) = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$;

$\therefore$ The co-factors of 1st row are given by

$$m_1 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

co-factors along first row give a null solution. Let us take co-factors along the second row.

$$m_1 = \begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix}$$

To obtain eigenvectors associated with the repeated eigenvalue at $\lambda = 3$, we construct the matrix

$$[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_2 - 4 & -1 & 2 \\ -1 & \lambda_2 & -2 \\ -1 & 1 & \lambda_2 - 3 \end{bmatrix}$$

For $\lambda_2 = 3$, the rank of 3×3 matrix $(\lambda_2 I - A)$ is 2. Therefore one independent eigenvector associated with $\lambda = 3$ can be obtained from the co-factors of 1st row of $(\lambda_2 I - A)$.

$$m_2 = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} \lambda_2(\lambda_2 - 3) + 2 \\ (\lambda_2 - 3) + 2 \\ -1 + \lambda_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

The eigenvector m_3 may be generated from the independent eigenvector m_2 as follows

$$m_3 = \begin{bmatrix} \frac{d}{d\lambda_2} c_{11} \\ \frac{d}{d\lambda_2} c_{12} \\ \frac{d}{d\lambda_2} c_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda_2 - 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

The eigenvector m_3 is known as generalized eigenvector.
Therefore, the modal matrix M is given by

$$M = [m_1 \,\vdots\, m_2 \,\vdots\, m_3] = \begin{bmatrix} 0 & 2 & 3 \\ 8 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

Now, the modal matrix 'M' transform 'A' to the Jordan matrix as follows.

$$M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} = J = \text{Jordan Matrix}$$

↑ Jordan block

SOLUTION OF STATE EQUATIONS: There are three methods for solution of the state equations from which the system transient response can then be obtained.
(1) classical method (2) Laplace transform method
(3) Cayley - Hamilton method.

(1) Computation of state transition matrix by classical method: Let us consider the non-homogeneous state model given by $\dot{x}(t) = Ax(t) + BU(t)$; $x(0) = x_0$
The above state equation can be rewritten as

$$\dot{x}(t) - Ax(t) = BU(t)$$

multiplying both sides by e^{-At}, we can write

$$e^{-At}[\dot{x}(t) - Ax(t)] = \frac{d}{dt}[e^{-At}x(t)] = e^{-At}BU(t)$$

Integrating both sides with respect to 't' between the limits 0 and 't', we get

$$e^{-At}x(t)\Big|_{t=0}^{t} = \int_0^t e^{-A\tau} BU(\tau)d\tau$$

$$e^{-At}x(t) - X(0) = \int_0^t e^{-A\tau} BU(\tau)d\tau$$

Now, pre-multiplying both sides by e^{At}, we have

$$x(t) = e^{At}x(0) + \int e^{A(t-\tau)} BU(\tau)d\tau \longrightarrow \text{I}$$

The above equation represents the solution of non-homo-geneous system.

For homogeneous system $B=0$

$\therefore$ The solution is $x(t) = e^{At}x(0) \longrightarrow \text{II}$

From the above equation, it is observed that the initial state $x(0)$ or x_0 at $t=0$ is driven to a state $x(t)$ at time 't'. This transition in state is carried out by the matrix exponential e^{At}. Because of this property, e^{At} is known as state transition matrix and is denoted by $\phi(t)$.

(2) Computation of State transition Matrix (STM) by Laplace transform method.

Let us consider an unforced system whose state equation is $\dot{x} = AX$, where 'A' is a constant matrix. Taking the Laplace transform of this equation, we obtain

$$SX(S) - X(0) = AX(S)$$

where $X(S)$ is the Laplace transform of the unforced response and $X(0)$ is the initial condition vector. The above equation may be rearranged as

$$[SI - A]\, X(S) = X(0)$$

or $X(S) = [SI-A]^{-1} X(0)$

Taking the inverse laplace transform, we get

$$x(t) = L^{-1}[SI-A]^{-1} X(0) \longrightarrow \text{(I)}$$

where $x(t)$ is the unforced response of the system.

Also we have $x(t) = e^{At} X(0) \longrightarrow \text{(II)}$

from eqs (I) & (II)

The STM $\phi(t) = e^{At} = L^{-1}[SI-A]^{-1} = L^{-1}[\phi(s)]$

where $\phi(s) = (SI-A)^{-1}$ is called the resolvent matrix

Let us now consider the response when the control force U is applied. The state equation for this case is

$$\dot{x} = AX + BU$$

Taking Laplace transform on both sides,

$$SX(S) - X(0) = AX(S) + BU(S); \quad \text{Let } X(0) = X_0$$

$$\therefore (SI-A) X(S) = X_0 + BU(S)$$

Therefore $X(S) = (SI-A)^{-1} X_0 + [SI-A]^{-1} BU(S)$

Taking inverse Laplace transform

$$X(t) = L^{-1}\left[(SI-A)^{-1} X_0\right] + L^{-1}\left[[SI-A]^{-1} BU(s)\right]$$

$$= \phi(t) X_0 + L^{-1}[\phi(s) BU(s)]$$

This is the response of forced system model or non-homogeneous system.

Properties of State Transition Matrix (STM):

We have $\phi(t) = e^{At}$ is the STM. Certain useful properties of STM are given by

(1) $\phi(0) = e^{A0} = I$ = Identity matrix

2) $\phi^{-1}(t) = (e^{At})^{-1} = e^{-At} = \phi(-t)$

3) $\phi(t_1+t_1) = e^{A(t_1+t_2)} = e^{At_1} \cdot e^{At_2}$

$= \phi(t_1)\,\phi(t_2) = \phi(t_2)\,\phi(t_1)$

① Let us consider a system with matrix $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
Find the state transition matrix.

(Sol) The state transition matrix $e^{At} = L^{-1}[SI-A]^{-1}$

where $SI - A = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} S-1 & 0 \\ -1 & S-1 \end{bmatrix}$

$\therefore [SI-A]^{-1} = \dfrac{1}{\det[SI-A]} \operatorname{adj}[A]$

$= \dfrac{1}{(S-1)^2}\begin{bmatrix} S-1 & 0 \\ 1 & S-1 \end{bmatrix}$

The resolvent matrix is given by

$\phi(S) = [SI-A]^{-1} = \begin{bmatrix} \dfrac{1}{S-1} & 0 \\ \dfrac{1}{(S-1)^2} & \dfrac{1}{S-1} \end{bmatrix}$

$\therefore$ The state transition matrix $\phi(t) = L^{-1}(\phi(s))$

$\Rightarrow \phi(t) = \begin{bmatrix} L^{-1}\dfrac{1}{(S-1)} & 0 \\ L^{-1}\left(\dfrac{1}{S-1}\right)^2 & L^{-1}\dfrac{1}{(S-1)} \end{bmatrix} = \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix} = e^{At}$

(2) obtain the time response of the following system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

where $U(t)$ is a unit step occuring at $t=0$ and $x^T(0) = [1 \; 0]$

(Sol) The given system is in the form $\dot{x}(t) = Ax(t) + BU(t)$

Therefore, the response of non-homogeneous system is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} BU(\tau)\,d\tau$$

$$= e^{At}\left[x(0) + \int_0^t e^{-A\tau} BU(\tau)\,d\tau\right]$$

where $e^{At} = \phi(t) = L^{-1}[SI-A]^{-1} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$

From the given data $x(0) = x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Also $e^{-A\tau}B = \begin{bmatrix} e^{-\tau} & 0 \\ -\tau e^{-\tau} & e^{-\tau} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^{-\tau} \\ e^{-\tau}(1-\tau) \end{bmatrix}$

$$\therefore \int_0^t \phi(-\tau)BU(\tau)\,d\tau = \begin{bmatrix} \int_0^t e^{-\tau}\,d\tau \\ \int_0^t e^{-\tau}(1-\tau)\,d\tau \end{bmatrix} = \begin{bmatrix} 1-e^{-t} \\ te^{-t} \end{bmatrix}$$

Therefore, the response of the system is given by

$$x(t) = e^{At} * \left[x(0) + \begin{bmatrix} 1-e^{-t} \\ te^{-t} \end{bmatrix}\right]$$

$$= e^{At}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1-e^{-t} \\ te^{-t} \end{bmatrix}\right\} = e^{At}\begin{bmatrix} 2-e^{-t} \\ te^{-t} \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}\begin{bmatrix} 2-e^{-t} \\ te^{-t} \end{bmatrix} = \begin{bmatrix} 2e^t - 1 \\ 2te^t \end{bmatrix}$$

(3) Consider a Control systems with state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}[u];$$

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad u = \text{unit step.}$$ Compute the STM and state response $x(t)$

(Sol) The given system is in the form $\dot{X} = AX + BU$. The response of this non-homogeneous system is given by

$$x(t) = \phi(t)\left[X(0) + \int_0^t \phi(-\tau) BU(\tau)\, d\tau\right]$$

where $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$; $B = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$; $X(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

The resolvent matrix $\phi(s) = [SI - A]^{-1}$

where $SI - A = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$

$$\therefore \phi(s) = (SI - A)^{-1} = \frac{1}{S^2 + 3S + 2}\begin{bmatrix} S+3 & +1 \\ -2 & S \end{bmatrix}$$

$$\therefore \text{The STM } \phi(t) = L^{-1}[SI - A]^{-1} = L^{-1}\begin{Bmatrix} \frac{S+3}{(S+1)(S+2)} & \frac{+1}{(S+1)(S+2)} \\ \frac{-2}{(S+1)(S+2)} & \frac{S}{(S+1)(S+2)} \end{Bmatrix}$$

where $L^{-1}\dfrac{S+3}{(S+1)(S+2)} = L^{-1}\dfrac{2}{(S+1)} - L^{-1}\dfrac{1}{S+2} = 2e^{-t} - e^{-2t}$

$L^{-1}\dfrac{+1}{(S+1)(S+2)} = L^{-1}\left\{\dfrac{1}{S+1} - \dfrac{1}{S+2}\right\} = e^{-t} - e^{-2t}$

$L^{-1}\dfrac{-2}{(S+1)(S+2)} = L^{-1}\left\{\dfrac{-2}{S+1} + \dfrac{2}{S+2}\right\} = -2e^{-t} + 2e^{-2t}$

$L^{-1}\dfrac{S}{(S+1)(S+2)} = L^{-1}\left\{\dfrac{-1}{S+1} + \dfrac{2}{S+2}\right\} = -e^{-t} + 2e^{-2t}$

$$\therefore \phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\phi(-\tau)BU(\tau) = \begin{bmatrix} 2e^{\tau} - 2e^{2\tau} \\ -2e^{\tau} + 4e^{2\tau} \end{bmatrix}$$

$$\therefore \int_0^t \phi(-\tau)\,BU(\tau)\,d\tau = \begin{cases} \int_0^t (2e^{\tau} - 2e^{2\tau})\,d\tau \\ \int_0^t (-2e^{\tau} + 4e^{2\tau})\,d\tau \end{cases}$$

$$= \begin{bmatrix} 2e^{t} - e^{2t} - 1 \\ -2e^{t} + 2e^{2t} \end{bmatrix}$$

$$\therefore \text{The state response } x(t) = \phi(t)\left[x(0) + \int_0^t \phi(-\tau)BU(\tau)d\tau\right]$$

$$\Rightarrow x(t) = \begin{bmatrix} e^{-t} - e^{-2t} \\ -e^{-t} + 2e^{-2t} \end{bmatrix} + \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 2e^{t} - e^{2t} - 1 \\ -2e^{t} + 2e^{2t} \end{bmatrix}$$

$$= \begin{bmatrix} 1 - e^{-t} \\ e^{-t} \end{bmatrix}$$

The Concepts of Controllability and observability:

The Concepts of Controllability and observability play an important role in Control engineering. These Concepts were introduced by Kalman.

Controllability: A System is Said to be Completely state Controllable if it is possible to transfer the system state from any initial state $x(t_0)$ to any desired state $x(t)$ in specified finite time by a Control vector $U(t)$.

A general nth order multi-input linear time-invariant System with an m-dimensional Control vector is $\dot{x} = AX + BU$ is Completely controllable if and only if the rank of the Composite matrix

$Q_c = [B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ is 'n'.

Since only the matrices A and B are involved, we may say that the pair (A,B) is controlled if rank of the matrix Q_c is n.

<u>Observability</u>: A system is said to be completely observable, if every state $x(t_0)$ can be completely identified by measurements of the outputs $y(t)$ over a finite time interval.

A system which is not completely observable, implies that some of its state variables are shielded from observation.

A general nth order multi-input multi-output linear-time invariant systems

$$\dot{X} = AX + BU$$
$$Y = CX$$

is completely observal if and only if the rank of the composite matrix $Q_0 = [C^T \vdots A^T C^T \vdots \cdots (A^T)^{n-1} C^T]$ is n.

This condition is also referred as the pair (A,C) being observable.

<u>Duality property</u>: (1) The pair (AB) is controllable implies that the pair $(A^T B^T)$ is observable

(2) The pair (AC) is observable implies that the pair $(A^T C^T)$ is controllable.

Thus the concepts of controllability and observability are dual concepts.

① Consider a system with state equations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

check whether the system is completely state controllable or not.

(Sol) The system matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$; $\dot{X} = AX + BU$

The output matrix $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

The Kalman's test states that a system is completely state controllable if the rank of the matrix $Q_c = [B \,\vdots\, AB \,\vdots\, \cdots \,\vdots\, A^{n-1}B]$ is 'n', where 'n' is number of state variables. Therefore for the given system $n=3$

$$\therefore Q_c = [B \,\vdots\, AB \,\vdots\, A^2B]$$

where $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -6 \end{bmatrix}$

$$A^2B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

$$\therefore Q_c = [B \,\vdots\, AB \,\vdots\, A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -6 \\ 1 & -6 & 25 \end{bmatrix}$$

The determinant of Q_c ie $|Q_c| = -1 \neq 0$;

Therefore the rank of $Q_c = n = 3$;

Therefore the system is completely controllable.

(2) Let us examine the observability of the System given below.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{and}$$

$$y = \begin{bmatrix} 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(Sol) The given homogeneous system is in the form $\dot{x} = Ax$ and $y = Cx$; Therefore $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}$; $C^T = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$

The Kalman's test states that a system is Completely observable if the rank of the matrix

$Q_o = \left[C^T \,\vdots\, A^T C^T \,\vdots\, \cdots \,(A^T)^{n-1} C^T \right]$ is n; where n is number of state variables.

In this case $n = 3$

$$\therefore Q_o = \left[C^T \,\vdots\, A^T C^T \,\vdots\, (A^T)^2 C^T \right]$$

$$A^T C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$$

$$\therefore Q_o = \left[C^T \,\vdots\, A^T C^T \,\vdots\, (A^T)^2 C^T \right] = \begin{bmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

The determinant of the matrix Q_o is given by

$$|Q_o| = \begin{vmatrix} 3 & 0 & 0 \\ 4 & 1 & -2 \\ 1 & 1 & -2 \end{vmatrix} = 3(-2+2) = 0$$

$\therefore$ The rank of the matrix is less than 3

i.e. the rank of the matrix Q_o is $r = 2$,

Hence one of the stable variable is unobservable