Discrete Mathematics & Graph theory

Unit - I R23 Mathematical logic (23HS0836)

* Introduction :-

Logic is the science dealing with the methods of reasoning. Reasoning plays a very important role in every area of knowledge, particularly mathematics. Logic expressed in such a language has come to be called 'Symbolic logic' or 'Mathematical logic'.

* Statements and Notation :-

A declarative sentance that is either true or false, but not both is called a statement or proposition.

Example: The following sentences are statement.

1. $3+4=5$ (F)
2. 3 is a prime number (T)
3. New Delhi is in India (T)

The following sentences are not statements.

1. $x+y=4$
2. let me go.

Statements are of two types

1. Simple statement
2. Compound statement.

Simple statement :

Simple Statement is a simple sentence and it can not contain any connectives.

Example : ① $3+4=5$.

② Delhi is in India.

Compound statements :

The composition of two or more statements is called compound statement.

Example : ① $2+3=5$ and $4+5=10$.

② If $2+3=5$ then $4+5=10$.

Truth value :

The truthfullness or falsity of a statement is called its truth value denoted by T (True) and F (False) respectively. If statement is true we will indicate its truth value by 1 and if it is false by the symbol 0.

* Connectives :-

Connectives are used to form a compound sentences from two or more statements.

The connectives are negation ($\sim$ or $\neg$), conjunction ($\wedge$), Disjunction ($\vee$), conditional or implication ($\rightarrow$), Biconditional or Bi-implication ($\leftrightarrow$).

Negation :-

A statement is obtained by inserting the word 'not' in an appropriate place, then it is called negation of given statement.

It is denoted by $\sim$ or $\neg$.

If p is a statement, then negation p is the statement 'not p' denoted by $\sim p$ or $\neg p$.

Truth table :-

p	$\sim p$
T	F
F	T

p	$\sim p$
1	0
0	1

Example : p : 3 is a prime number

$\sim p$: 3 is not a prime number.

Conjunction :- If P and Q are any two statements then the conjunction of P and Q is denoted by $P \wedge Q$, which can be read as 'P and Q'. The truth value of $P \wedge Q$ is true only when P is true and Q is true, otherwise it has the truth value F.

Truth table :-

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction :- If P and Q are any two statements then the disjunction of P and Q denoted by $P \vee Q$, which is read as 'P or Q'. The truth value of $P \vee Q$ is false when both P and Q has the truth value false, otherwise it has the truth value true.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

Conditional :- If P and Q are any two statements then the conditional of P and Q is denoted by $P \rightarrow Q$, which is read as P implies Q. The truth value of $P \rightarrow Q$ is F when P has the truth value T and Q has the truth value F, otherwise it has the truth value T.

Truth table:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional :-

If P and Q are any two statements then the biconditional of P and Q is denoted by $P \leftrightarrow Q$ or $P \rightleftarrows Q$ which is read as 'P if and only if Q'. The truth value of $P \leftrightarrow Q$ is T when both P and Q have the same truth value, otherwise it has the truth value F.

Truth table :-

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Problems :

* Construct the truth table for the following statements.

① $P \wedge \sim P$

② $(P \vee Q) \vee \sim P$

①

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

② $(P \vee Q) \vee \sim P$

P	Q	$P \vee Q$	$\sim P$	$(P \vee Q) \vee \sim P$
T	T	T	F	T
T	F	F	F	T
F	T	F	T	T
F	F	F	T	T

* Construct the truth table for the formula $\neg(\neg P \vee \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(\neg P \vee \neg Q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

* Construct the truth table for the following formula

① $P \wedge (\neg Q \wedge Q)$

② $\neg (P \vee Q) \vee (\neg P \wedge \neg Q)$

①

P	Q	$\neg Q$	$\neg Q \wedge Q$	$P \wedge (\neg Q \wedge Q)$
T	T	F	F	F
T	F	T	F	F
F	T	F	F	F
F	F	T	F	F

② $\neg (P \vee Q) \vee (\neg P \wedge \neg Q)$

P	Q	$\neg P$	$\neg Q$	$P \vee Q$	$\neg (P \vee Q)$	$\neg P \wedge \neg Q$	$\neg (P \vee Q) \vee (\neg P \wedge \neg Q)$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

Ex: $\neg P \vee (q \vee \neg r)$

$(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$

$(P \rightleftarrows Q) \rightleftarrows (P \wedge Q) \vee (\neg P \wedge \neg Q)$

$\neg (\neg P \vee \neg Q)$

$\sim (P \vee (Q \wedge R)) \rightleftarrows [(P \vee Q) \wedge (P \vee R)]$

$\neg (\neg P \vee \neg Q) \vee (R \rightarrow Q)$

Converse, inverse and contrapositive :-

Converse: If P and Q are any two statements. If $P \to Q$ is conditional statement then the converse of $P \to Q$ is $Q \to P$.

Inverse: If P and Q are any two statements. If $P \to Q$ is a conditional statement then the inverse of $P \to Q$ is $\neg P \to \neg Q$.

Contrapositive: let P and Q are any two statements. If $P \to Q$ is conditional statement then the Contrapositive of $P \to Q$ is $\neg Q \to \neg P$.

Example: $P \to Q$: If the calculator is working then the battery is good.

let P: The calculator is working
Q: The battery is good.

Converse: If the battery is good then the Calculator is working

Inverse: If the calculator is not working then the battery is not good.

contrapositive: If the battery is not good then the calculator is not working.

Other connectives :-

XOR (or) Exclusive or :-

let P and Q be the two statements.
The XOR of P and Q denoted by $P \oplus Q$, is the statement that is true when exactly one of P and Q is true but not both. and is false otherwise.

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

NAND:

NAND is a combination of not and AND.
It is denoted by symbol $\uparrow$.

let P and Q be any two statements.
The truth values of $P \uparrow Q$ is false when both P and Q has the truth value T, otherwise it has the truth value T.

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

NOR:

NOR is a combination of not and OR.

It is denoted by the symbol $\downarrow$.

let P and Q be any two statements. The truth value of $P \downarrow Q$ is T when both P and Q has the truth value F, otherwise it has the truth value F.

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

* Well formed formulas:

A statement involving connectives in symbolic form and consisting of variables, Paranthesis. A correct expression is called a well formed formula.

A w.f.f can be generated by the following rules.

1. All variables and constants are w.f.f
2. If P is a w.f.f, the $\neg P$ is also w.f.f.
3. If P and Q are w.f.f then $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$, $P \leftrightarrow Q$ are all w.f.f

Example: $(P \leftrightarrow Q) \wedge (Q \rightarrow R)$, $\sim(P \vee Q)$ are w.f.f.

* Tautology:

A compound statement which is true for all possible truth values of its components is called a Tautology. A Tautology is generally denoted by T_0.

Example: $P \vee \sim P$ is a Tautology.

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

Contradiction:

A compound statement which is false for all possible truth values of its components is called a contradiction. A contradiction is generally denoted by F_0.

<u>Example:</u> $P \wedge \neg P$ is a contradiction.

P	$\neg P$	$P \wedge \neg P$
T	F	F
F	T	F

* Duality law:—

Suppose u is a compound statement that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurrence of $\wedge$ and $\vee$ in u by $\vee$ and $\vee$ respectively. Also, if u contains T_0 and F_0 as components, suppose we replace each occurrence of T_0 and F_0 by F_0 and T_0 respectively. Then the resulting compound statement is called the 'dual' of u and is denoted by u^d.

<u>Example:</u>
$$u = P \wedge (q \vee \neg r) \vee (s \wedge T_0)$$
$$u^d = P \vee (q \wedge \neg r) \wedge (s \vee F_0).$$

* Write down the duals of the following propositions

① $\neg(p \vee q) \wedge [p \vee \neg(q \wedge \neg s)]$

$\neg(p \wedge q) \vee [p \wedge \neg(q \vee \neg s)]$

② $[(p \wedge F_0) \vee (q \wedge T_0)] \wedge [(r \vee s) \vee F_0]$

$[[p \vee T_0] \wedge (q \vee F_0)] \vee [(r \wedge s) \wedge T_0]$

③ $p \rightarrow q$.

we have $p \rightarrow q \Leftrightarrow \neg p \vee q$.

$u: p \rightarrow q \Rightarrow u: \neg p \vee q$.

$u^d: \neg p \wedge q$.

Equivalence implications :-

The compound statement P and Q are said to be logically equivalent whenever P and Q have the same truth value and it is denoted by $P \Leftrightarrow Q$ (or) $P \equiv Q$.

Example: $P \rightarrow$ is logically equivalent to $\sim P \vee Q$.

P	Q	$P \rightarrow Q$	$\sim P$	$\sim P \vee Q$.
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

both are same

$\therefore \; P \rightarrow Q \Leftrightarrow \sim P \vee Q$.

laws of logic :-

(1) $P \wedge P \Leftrightarrow P$, $P \vee P \Leftrightarrow P$ — Idempotent law

(2) $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$.
$P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$ } Associative law.

(3) $P \wedge T \Leftrightarrow P$, $P \vee F \Leftrightarrow P$ — Identity law.

(4) $P \wedge \sim P \Leftrightarrow F$, $P \vee \sim P \Leftrightarrow T$ — Inverse law.

(5) $P \wedge Q \Leftrightarrow Q \wedge P$, $P \vee Q \Leftrightarrow Q \vee P$ — commutative law.

(6) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ } Distributive law

(7) $\sim(P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$.

$\sim(P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$ } Demorgan's Laws.

(8) $P \vee (P \wedge Q) \Leftrightarrow P$, $P \wedge (P \vee Q) \Leftrightarrow P$ — Absorption law.

(9) $P \vee T \Leftrightarrow T$, $P \wedge F \Leftrightarrow F$.

(10) $\sim(\sim P) = P$.

Tautological implications:—

A statement A is said to be tautologically imply a statement B if and only if $A \rightarrow B$ is a tautology. It is denoted by $A \Rightarrow B$, which is read as A implies B. Here $\Rightarrow$ is not connective.

(1) $P \wedge Q \Rightarrow P$

(2) $P \wedge Q \Rightarrow Q$.

(3) $P \Rightarrow P \vee Q$

(4) $\neg P \Rightarrow P \rightarrow Q$

(5) $Q \Rightarrow P \rightarrow Q$

(6) $\neg(P \rightarrow Q) \Rightarrow P$.

(7) $\neg(P \rightarrow Q) \Rightarrow \neg Q$.

(8) $P \wedge (P \rightarrow Q) \Rightarrow Q$.

(9) $\neg Q \wedge (P \rightarrow Q) \Rightarrow \neg P$

(10) $\neg P \wedge (P \wedge Q) \Rightarrow Q$.

(11) $(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow P \rightarrow R$.

(12) $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow R) \Rightarrow R$.

* Prove that $[(P \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (P \rightarrow r)$ is a tautology.

Truth table:

P	q	r	$P \rightarrow q$	$q \rightarrow r$	$(P \rightarrow q) \wedge (q \rightarrow r)$	$(P \rightarrow q) \wedge (q \rightarrow r) \rightarrow (P \rightarrow r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	T
T	F	T	F	T	F	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	T	F	T	F	F	T
F	F	T	T	T	T	T
F	F	F	T	T	T	T

$\therefore [(P \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (P \rightarrow r)$ is a tautology.

* Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent by using truth tables.

Truth table:

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	F	F
F	T	F	F	F	T	T	F
F	F	T	F	F	F	F	F
F	F	F	F	F	F		

$\therefore$ $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \wedge r)$ are logically equivalent.

* Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent with out using truth tables.

$$\neg(p \vee (\neg p \wedge q)) = \neg((p \vee \neg p) \wedge (p \vee q)) \quad \text{(By distributive law)}$$

$$= \neg(T \wedge (p \vee q))$$

$$= \neg(p \vee q)$$

$$= \neg p \wedge \neg q \quad \text{(By demorgan law)}$$

* Prove that $(P \to q) \to r \leftrightarrow (\neg P \vee q) \to r$ is a Tautology.

* Prove the following logical equivalences.

① $P \vee [P \wedge (P \vee q)] \Leftrightarrow P$

② $[(P \vee q) \wedge (P \vee \neg q)] \vee q \Leftrightarrow P \vee q$

Normal forms :-

Elementary product :-

A product of the variables and their negations in a formula is called an elementary product.

<u>Example</u> :- P, $P \wedge Q$, $\neg P \wedge P$, $\neg P \wedge Q$, $\neg Q \wedge P$ are some examples of elementary product.

Elementary sum :-

A sum of the variables and their negations is called an elementary sum.

<u>Example</u> :- P, $P \vee Q$, $\neg P \vee Q$ are some examples of elementary sum.

Disjunctive normal form :-

A formula which is equivalent to a given formula and which consists of a sum of elementary products is called the disjunctive normal form of the given formula.

Obtain the disjunctive normal forms of.

(a) $P \wedge (P \rightarrow Q)$ (b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$.

(a) $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$ $[\because P \rightarrow Q \Leftrightarrow \neg P \vee Q]$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

(b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

we know that $P \rightleftarrows Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

$\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

$\Leftrightarrow (\neg(P \vee Q) \wedge (P \wedge Q)) \vee (\neg(\neg(P \vee Q)) \wedge \neg(P \wedge Q))$

$\Leftrightarrow [(\neg P \wedge \neg Q) \wedge (P \wedge Q)] \vee [(P \vee Q) \wedge (\neg P \vee \neg Q)]$

$\Leftrightarrow [\neg P \wedge \neg Q \wedge P \wedge Q] \vee [(P \vee Q) \wedge \neg P] \vee [(P \vee Q) \wedge \neg Q]$

$\Leftrightarrow [\neg P \wedge \neg Q \wedge P \wedge Q] \vee [(P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)]$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q] \vee (P \wedge \neg P) \vee (Q \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg Q)$

Conjunctive normal form:-

A formula which is equivalent to a given formula and which consist of a product of elementary sums is called a conjunctive normal form.

* Obtain a conjunctive normal form of the following formulae.

(a) $P \wedge (P \rightarrow Q)$ (b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$

(a) $P \wedge (P \rightarrow Q) \Leftrightarrow P \wedge (\neg P \vee Q)$

(b) We know that $P \rightleftarrows Q \Leftrightarrow (\neg P \vee Q) \wedge (\neg Q \vee P)$

$\neg(P \vee Q) \rightleftarrows (P \wedge Q) \Leftrightarrow [\neg(\neg(P \vee Q)) \vee (P \wedge Q)]$

$\wedge [\neg(P \wedge Q) \vee \neg(P \vee Q)]$

$\Leftrightarrow [(P \vee Q) \vee (P \wedge Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)]$

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge [(\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)]$

* Obtain the disjunctive normal forms of the following

ⓐ $P \wedge (P \rightarrow Q)$ ⓑ $\neg(P \rightarrow (q \wedge r))$

ⓐ $P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

ⓑ $\neg[\neg P \vee (q \wedge r)]$

$\Leftrightarrow P \wedge \neg(q \wedge r)$.

$\Leftrightarrow P \wedge (\neg q \vee \neg r)$

$\Leftrightarrow (P \wedge \neg q) \vee (P \wedge \neg r)$

* Find the disjunctive normal form of $\neg(P \rightarrow (q \wedge r))$.

$\neg(P \rightarrow (q \wedge r))$

$\Leftrightarrow \neg(\neg P \vee (q \wedge r))$.

$\Leftrightarrow P \wedge \neg(q \wedge r)$.

$\Leftrightarrow P \wedge (\neg q \vee \neg r)$.

$\Leftrightarrow (P \wedge \neg q) \vee (P \wedge \neg r)$

* Find the disjunctive normal form of $P \vee (\neg P \rightarrow (q \vee (q \rightarrow \neg r)))$

$\Leftrightarrow P \vee (\neg P \rightarrow (q \vee (\neg q \vee \neg r)))$

$\Leftrightarrow P \vee (\neg P \rightarrow (q \vee \neg q \vee \neg r))$.

$\Leftrightarrow P \vee (P \vee q \vee \neg q \vee \neg r)$

$\Leftrightarrow P \vee P \vee q \vee \neg q \vee \neg r$.

* Find the disjunctive normal form of

PNF $(\neg P \rightarrow r) \wedge (P \rightarrow q)$

$(\neg P \rightarrow r) \wedge (P \rightarrow q)$

$\Leftrightarrow (P \vee r) \wedge (\neg P \vee q)$

$\Leftrightarrow [(P \vee r) \wedge \neg P] \vee [(P \vee r) \wedge q]$.

$\Leftrightarrow (P \wedge \neg P) \vee (r \wedge \neg P) \vee (P \wedge q) \vee (r \wedge q)$.

* Find the distributive normal form of.

$(P \wedge \neg(q \vee r)) \vee (((P \wedge q) \vee \neg r) \wedge P)$

$(P \wedge \neg(q \vee r)) \vee (((\underline{P \wedge q}) \vee \neg r) \underline{\wedge P})$

$(P \wedge \neg q \wedge \neg r) \vee ((P \wedge q \wedge P) \vee (\neg r \wedge P))$

$(P \wedge \neg q \wedge \neg r) \vee (P \wedge q \wedge P) \vee (P \wedge \neg r)$.

* Find the conjunctive normal form

$(p \wedge q) \vee (\neg p \wedge q \wedge r)$.

$(p \wedge q) \vee (\neg p \wedge q \wedge r)$

$\Leftrightarrow [(p \wedge q) \vee \neg p] \wedge [(p \wedge q) \vee q] \wedge ((p \wedge q) \vee r)$

$\Leftrightarrow [p \vee \neg p] \wedge [q \vee \neg p] \wedge [p \vee q] \wedge [q \vee q] \wedge [p \vee r] \wedge [q \vee r]$

* Find the conjunctive normal form form of

$q \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)$.

$\underline{q \vee (p \wedge \neg q)} \vee (\neg p \wedge \neg q)$.

$\Leftrightarrow [(q \vee p) \wedge (q \vee \neg q)] \vee [\neg p \wedge \neg q]$.

$\Leftrightarrow [(q \vee p) \vee (\neg p \wedge \neg q)] \wedge [(q \vee \neg q) \vee (\neg p \wedge \neg q)]$.

$\Leftrightarrow (q \vee p \vee \neg p) \wedge (q \vee p \vee \neg q) \wedge [(q \vee \neg q) \vee (\neg p \wedge \neg q)]$

$\Leftrightarrow (q \vee p \vee \neg p) \wedge [q \vee p \vee \neg q] \wedge [q \vee \neg q \vee \neg p] \wedge [q \vee \neg q \vee \neg q]$.

* Find the conjunctive normal form of

$$P \to ((P \to q) \wedge \neg(\neg P \vee \neg q)).$$

$P \to ((P \to q) \wedge \neg(\neg P \vee \neg q))$

$\Leftrightarrow P \to ((\neg P \vee q) \wedge (P \wedge q)).$

$\Leftrightarrow \neg P \vee ((\neg P \vee q) \wedge (P \wedge q)).$

$\Leftrightarrow [\neg P \vee (\neg P \vee q)] \wedge [\neg P \vee (P \wedge q)]$

$\Leftrightarrow [\neg P \vee \neg P \vee q] \wedge [\neg P \vee (P \wedge q)]$

$\Leftrightarrow [\neg P \vee \neg P \vee q] \wedge [\neg P \vee P] \wedge [\neg P \wedge q]$

Minterms :—

let P and Q be two statements variables. let us construct all possible formulae which consist of conjunction of P or its negation and conjunction of Q or its negation. None of the formulae should contain both a variable and its negation. Such formulae are called min terms.

For two variables P and Q there are $2^2 = 4$ possible min terms. such formulae given as,

$P \wedge Q$, $P \wedge \neg Q$, $\neg P \wedge Q$, $\neg P \wedge \neg Q$ are called min terms or boolean conjunction of P and Q.

P	Q	$P \wedge Q$	$P \wedge \neg Q$	$\neg P \wedge Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	F

Principal disjunctive normal forms:-

For a given formula, an equivalent formula consisting of disjunctions of minterms only is known as its principal disjunctive normal form.

Such a normal form is also called the sum of products canonical form.

Note: Min terms for 3 variables P, Q and R are $P\wedge Q\wedge R$, $P\wedge Q\wedge \neg R$, $P\wedge \neg Q\wedge R$, $P\wedge\neg Q\wedge\neg R$, $\neg P\wedge Q\wedge R$, $\neg P\wedge Q\wedge\neg R$, $\neg P\wedge\neg Q\wedge R$, $\neg P\wedge\neg Q\wedge\neg R$

* Obtain the principal disjunctive normal form for the following formulae.

(a) $P\to Q$ (b) $P\vee Q$ (c) $\neg(P\wedge Q)$

(a) $P\to Q \Leftrightarrow \neg P\vee Q \Leftrightarrow (\neg P\wedge T)\vee(Q\wedge T)$

$\Leftrightarrow [\neg P\wedge(Q\vee\neg Q)]\vee[Q\wedge(P\vee\neg P)]$

$\Leftrightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(Q\wedge P)\vee(Q\wedge\neg P)$

$\Leftrightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(P\wedge Q)\vee(\neg P\wedge Q)$

$\Leftrightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(P\wedge Q)$

(b) $P\vee Q \Leftrightarrow (P\wedge T)\vee(Q\wedge T)$

$\Leftrightarrow [P\wedge(Q\vee\neg Q)]\vee[Q\wedge(P\vee\neg P)]$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(Q\wedge P)\vee(Q\wedge\neg P)$

$\Leftrightarrow (P\wedge Q)\vee(P\wedge\neg Q)\vee(\neg P\wedge Q)$

* Obtain the principal disjunctive normal form of

(a) $\neg P \vee Q$ (b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

(a) $\neg P \vee Q \Leftrightarrow (\neg P \wedge T) \vee (Q \wedge T)$

$\Leftrightarrow [\neg P \wedge (Q \vee \neg Q)] \vee [Q \wedge (P \vee \neg P)]$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \vee \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$

(b) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow [(P \wedge Q) \wedge (R \vee \neg R)] \vee [(\neg P \wedge R) \wedge (Q \vee \neg Q)] \vee [(Q \wedge R) \wedge (P \vee \neg P)]$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q)$

$\vee [(Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)]$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge \neg Q \wedge R)$

* Obtain the PDNF of $\neg(P \to (q \wedge r))$.

$\neg(P \to (q \wedge r))$

$\Leftrightarrow \neg(\neg P \vee (q \wedge r))$.

$\Leftrightarrow P \wedge \neg(q \wedge r)$

$\Leftrightarrow P \wedge (\neg q \vee \neg r)$

$\Leftrightarrow (P \wedge \neg q) \vee (P \wedge \neg r)$

$\Leftrightarrow (P \wedge \neg q \wedge T) \vee (P \wedge \neg r \wedge T)$

$\Leftrightarrow [(P \wedge \neg q) \wedge (r \vee \neg r)] \vee [(P \wedge \neg r) \wedge (q \vee \neg q)]$

$\Leftrightarrow (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge \neg r \wedge q) \vee (p \wedge \neg r \wedge \neg q)$

$\Leftrightarrow (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg r)$

Max terms :-

let P and Q be two statement variables. let us construct all possible formulae which consist of disjunction of P or its negation and disjunction of Q or its nagation, none of the formulae should contain both variables and its nagation. Such formulae are called max terms.

For two variables P and Q, there are $2^2 = 4$ possible max terms. Such formulae given by $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$ and $\neg P \vee \neg Q$ are called max terms or boolean disjunction of P and Q.

P	Q	$P \vee Q$	$P \vee \neg Q$	$\neg P \vee Q$	$\neg P \vee \neg Q$
T	T	T	T	T	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	T	T	T

Note: Max terms for 3 variables P, Q and R are

$P \vee Q \vee R$ $\quad$ $P \vee Q \vee \neg R$ $\quad$ $P \vee \neg Q \vee R$ $\quad$ $P \vee \neg Q \vee \neg R$

$\neg P \vee Q \vee R$ $\quad$ $\neg P \vee Q \vee \neg R$ $\quad$ $\neg P \vee \neg Q \vee R$ $\quad$ $\neg P \vee \neg Q \vee \neg Q$.

Principal conjunctive normal form :-

For a given formula, an equivalent formula consisting of conjunctions of a max terms only is known as its principal conjunctive normal form. This normal form is also the product of sums canonical form.

* Obtain PCNF for the following formula.

(1) $P \wedge Q$ (2) $q \wedge (P \vee \neg q)$

(1) $P \wedge Q \Leftrightarrow (P \vee F) \wedge (Q \vee F)$

$\Leftrightarrow [P \vee (Q \wedge \neg Q)] \wedge [Q \vee (P \wedge \neg P)]$

$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (Q \vee P) \wedge (Q \vee \neg P)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (Q \vee \neg P)$

(2) $q \wedge (P \vee \neg q)$

$\Leftrightarrow (q \vee F) \wedge (P \vee \neg q)$

$\Leftrightarrow [q \vee (P \wedge \neg P)] \wedge (P \vee \neg q)$.

$\Leftrightarrow (q \vee P) \wedge (q \vee \neg P) \wedge (P \vee \neg q)$

$\Leftrightarrow (P \vee q) \wedge (\neg P \vee q) \wedge (P \vee \neg q)$

* Obtain the PCNF for the formula given by $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$.

let $S \Leftrightarrow (\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$

$\Leftrightarrow (\neg P \rightarrow R) \wedge (Q \rightarrow P) \wedge (P \rightarrow Q)$

$\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \vee R \vee F) \wedge (\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)$

$\Leftrightarrow [(P \vee R) \vee (Q \wedge \neg Q)] \wedge [\neg Q \vee P \vee (R \wedge \neg R)] \wedge [(\neg P \vee Q) \vee (R \wedge \neg R)]$

$\Leftrightarrow [(P \vee R \vee Q) \wedge (P \vee R \vee \neg Q)] \wedge [\underline{(\neg Q \vee P \vee R)} \wedge (\neg Q \vee P \vee \neg R)]$
$\wedge [(\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)]$

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$\wedge (\neg P \vee Q \vee \neg R)$

<u>Note:</u> The conjunctive normal form of $\neg S$ above problem can easily be obtained by writing the conjunction of the remaining max terms, thus $\neg S$ has the principal conjunctive normal form.

$(P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

by considering $\neg\neg S$, we obtain.

$\neg (P \vee Q \vee \neg R) \vee \neg (\neg P \vee \neg Q \vee R) \vee \neg (\neg P \vee \neg Q \vee \neg R)$

$\Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

which is the principal disjunctive normal form of S.

* Obtain PCNF of $A = (P\wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$ by Constructing PDNF.

$$A \Leftrightarrow (P\wedge Q) \vee (\neg P\wedge Q) \vee (Q\wedge R)$$

$$\Leftrightarrow (P\wedge Q\wedge T) \vee (\neg P\wedge Q\wedge T) \vee (Q\wedge R\wedge T)$$

$$\Leftrightarrow ((P\wedge Q)\wedge(R\vee\neg R)) \vee ((\neg P\wedge Q)\wedge(R\vee\neg R)) \vee ((Q\wedge R)\wedge(P\vee\neg P))$$

$$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge\neg R)$$
$$\vee (Q\wedge R\wedge P)\vee(Q\wedge R\wedge\neg P)$$

$$\Leftrightarrow (P\wedge Q\wedge R)\vee(P\wedge Q\wedge\neg R)\vee(\neg P\wedge Q\wedge R)\vee(\neg P\wedge Q\wedge\neg R)$$

The PDNF of $\neg A$ is obtained by using the remaining min terms of P, Q, R.

P	Q	R
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

The PDNF of $\neg A$ is given by

$$\neg A \Leftrightarrow (P\wedge\neg Q\wedge R)\vee(P\wedge\neg Q\wedge\neg R)\vee(\neg P\wedge\neg Q\wedge R)$$
$$\vee(\neg P\wedge\neg Q\wedge\neg R)$$

Take negation on both sides, we get PCNF.

~~$\neg(\neg A) \Leftrightarrow \neg(P\wedge Q\wedge R)\wedge\neg(P\wedge Q\wedge\neg R)\wedge\neg(\neg P\wedge$~~

$$\neg(\neg A) \Leftrightarrow \neg(P\wedge\neg Q\wedge R)\wedge\neg(P\wedge\neg Q\wedge\neg R)\wedge\neg(\neg P\wedge\neg Q\wedge R)$$
$$\wedge\neg(\neg P\wedge\neg Q\wedge\neg R)$$

$$\Leftrightarrow (\neg P\vee Q\vee\neg R)\wedge(\neg P\vee Q\vee R)\wedge(P\vee Q\vee\neg R)\wedge(P\vee Q\vee R)$$

which is the required PCNF.

* Functionally complete set of connectives :-

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from this set is called a functionally complete set of connectives.

* write an equivalent formula for $P \wedge [(Q \rightleftarrows R) \vee (R \rightleftarrows P)]$ which does not contain the biconditional.

$P \wedge [(Q \rightleftarrows R) \vee R \rightleftarrows P]$

$\Leftrightarrow P \wedge [((Q \rightarrow R) \wedge (R \rightarrow Q)) \vee [(R \rightarrow P) \wedge (P \rightarrow R)]]$

* write an equivalent formula for $P \wedge (Q \rightleftarrows R)$ which contains neither the biconditional nor the conditional.

$P \wedge (Q \rightleftarrows R)$

$\Leftrightarrow P \wedge [(Q \rightarrow R) \wedge (R \rightarrow Q)]$

$\Leftrightarrow P \wedge [(\neg Q \vee R) \wedge (\neg R \vee Q)]$.

* Inference theory of ~~pred~~ statement calculus:-

The rules of inference is a criteria for determining the validity of an argument.

An argument is said to be valid iff the conjunction of premesis implies the conclusion C.

For a set of premesis $H_1, H_2, \dots H_n$ follows $H_1 \wedge H_2 \wedge H_3 \dots H_n \rightarrow C$ is a tautology then the given argument is valid otherwise it is invalid.

Rules of inference :-

Any conclusion which is arrived by following these rules is called a valid conclusion and the argument is valid argument.

Rule P :- A premesis may be introduced at any point in the derivation.

Rule T :- A formula may be introduced in a derivation at any point.

Rule CP :- If we can derive S from R and a set of premesis then we can derive $R \rightarrow S$ from the set of premesis alone.

Rules of inference in implications:—

I_1 : $P \wedge Q \Rightarrow P$

I_2 : $P \wedge Q \Rightarrow Q$.

I_3 : $P \Rightarrow P \vee Q$.

I_4 : $Q \Rightarrow P \vee Q$

I_5 : $\neg P \Rightarrow P \rightarrow Q$.

I_6 : $Q \Rightarrow P \rightarrow Q$

I_7 : $\neg(P \rightarrow Q) \Rightarrow P$

I_8 : $\neg(P \rightarrow Q) \Rightarrow \neg Q$

I_9 : $P, Q \Rightarrow P \wedge Q$

I_{10} : $\neg P, P \vee Q \Rightarrow Q$

(or) $\neg P \wedge (P \vee Q) \Rightarrow Q$.

I_{11} : $P, P \rightarrow Q \Rightarrow Q$

I_{12} : $\neg Q, P \rightarrow Q \Rightarrow \neg P$

I_{13} : $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$

I_{14} : $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$.

Equivalences:—

E_1 : $\neg\neg P \Leftrightarrow P$

E_2 : $P \wedge Q \Leftrightarrow Q \wedge P$.

E_3 : $P \vee Q \Leftrightarrow Q \vee P$

E_4 : $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

E_5 : $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

E_6 : $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

E_7 : $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

E_8 : $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$.

E_9 : $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$.

E_{10} : $P \vee P \Leftrightarrow P$

E_{11} : $P \wedge P \Leftrightarrow P$.

E_{12} : $R \vee (P \wedge \neg P) \Leftrightarrow R$.

E_{13} : $R \wedge (P \vee \neg P) \Leftrightarrow R$.

E_{14} : $R \vee (P \vee \neg P) \Leftrightarrow T$

E_{15} : $R \wedge (P \wedge \neg P) \Leftrightarrow F$.

E_{16} : $P \rightarrow Q \Leftrightarrow \neg P \vee Q$

E_{17} : $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$.

E_{18} : $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$.

E_{19} : $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

E_{20} : $\neg(P \rightleftarrows Q) \Leftrightarrow P \rightleftarrows \neg Q$.

E_{21} : $P \rightleftarrows Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

E_{22} : $P \rightleftarrows Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

* Show that $S \vee R$ is a Tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

Step		Derivation	rule/Formula.
$\{1\}$	(1)	$P \vee Q$	Rule P.
$\{1\}$	(2)	$\neg P \to Q$	Rule T, (1), E_1, E_{16}.
$\{3\}$	(3)	$Q \to S$.	Rule P.
$\{1,3\}$	(4)	$\neg P \to S$.	Rule T, (2)(3), I_{13}.
$\{1,3\}$	(5)	$\neg S \to P$.	Rule T, (4) E_{18}, E_1
$\{6\}$	(6)	$P \to R$.	Rule P.
$\{1,3,6\}$	(7)	$\neg S \to R$.	Rule T, (5)(6), I_{13}
$\{1,3,6\}$	(8)	$S \vee R$	Rule T, (7), E_{16}, E_1

* Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premesis $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Step		Derivation	Formula/rule.
$\{1\}$	(1)	$P \vee Q$.	Rule P.
$\{1\}$	(2)	$\neg P \to Q$	Rule T, (1), E_{16}.
$\{3\}$	(3)	$Q \to R$.	Rule P.
$\{1,3\}$	(4)	$\neg P \to R$.	Rule T, (2), (3), I_{13}
$\{1,3\}$	(5)	$\neg R \to P$	Rule T (4), E_1, E_{18}
$\{6\}$	(6)	$P \to M$	Rule P.
$\{1,3,6\}$	(7)	$\neg R \to M$	Rule T, (5), (6) I_{13}.

$\{1,3,6\}$	(8)	$\neg M \rightarrow \neg R$	Rule T, (7), E_1, E_{18}
$\{9\}$	(9)	$\neg M$	Rule P.
$\{1,3,6,9\}$	(10)	R.	Rule T (8),(9), I_{11}.
$\{1,3,6,9\}$	(11)	$P \wedge (P \vee Q)$	Rule T (1), (10), I_9.

* Show that $R \rightarrow S$ can be derived from the premesis $P \rightarrow (Q \rightarrow S)$, $\neg R \vee P$ and Q.

Step		derivation	rule
$\{1\}$	(1)	$\neg R \vee P$	Rule P.
$\{2\}$	(2)	R	Rule P, Assumed premese.
$\{1,2\}$	(3)	P	Rule T, (1) (2), I_{10}.
$\{4\}$	(4).	$P \rightarrow (Q \rightarrow S)$	Rule P.
$\{1,2,4\}$	(5)	$Q \rightarrow S$	Rule T, I_{11}.
$\{6\}$	(6)	Q	Rule P.
$\{1,2,4,6\}$	(7)	S	Rule T, (5),(6), I_{11}.
$\{1,2,4,6\}$	(8)	$R \rightarrow S$	Rule CP.

* Show that S is a valid conclusion from the Premisis $P \to q$, $P \to r$, $\neg(q \wedge r)$ and $(S \vee P)$.

Step		Derivation	Rule
{1}	(1)	$\neg(q \wedge r)$	Rule P.
{1}	(2)	$\neg q \vee \neg r$	Rule T, (1), E_9.
{1}	(3)	$q \to \neg r$	Rule T, (2), E_{16}.
{4}	(4)	$P \to q$	Rule P,
{1,4}.	(5)	$P \to \neg r$	Rule T, (3), (4), I_{13}.
{6}.	(6)	$P \to r$	Rule P.
{6}	(7)	$\neg r \to \neg P$.	Rule T, (6), E_{18}.
{1,4,6}.	(8)	$P \to \neg P$.	Rule T, (5), (7), I_{13}.
{1,4,6}	(9)	$\neg P \vee \neg P$.	Rule T, (8), E_{16}
{1,4,6}	(10)	$\neg P$	Rule T, (9), E_{10}.
{11}.	(11)	$S \vee P$.	Rule P.
{1,4,6,11}.	(12).	S	Rule T, (10), (11), E_{10}

*

* Show that $\sim P$ is a valid conclusion from the Premesis $\sim(P \wedge \sim Q)$, $\sim Q \vee R$, $\sim R$.

Step	Derivation	Rule
{1} (1)	$\neg(P \wedge \neg Q)$	Rule P.
{1} (2)	$\neg P \vee Q$	Rule T, (1), E_8.
{1} (3)	$P \rightarrow Q$	Rule T, (2), E_{16}.
{4} (4)	$\neg Q \vee R$	Rule P.
{4}. (5)	$Q \rightarrow R$	Rule T, (4), E_{16}.
{1, 4} (6)	$P \rightarrow R$	Rule T, (3),(5) I_{13}.
{7} (7)	$\sim R$.	Rule P.
{1,4,7}. (8)	$\sim P$.	Rule T, (6),(7), I_{12}.

* Prove by indirect method $\neg q$, $p \rightarrow q$ and $p \vee t$ then t.

we shall use indirect method of proof by assuming $\neg t$ as an additional premise.

{1} (1)	$\neg q$	Rule P.
{2} (2)	$p \rightarrow q$	Rule P.
{3} (3)	$p \vee t$	Rule P.
{4}. (4).	$\neg t$.	Rule P, Assumed premise.
{3}. (5)	$t \vee p$.	Rule T, (3), E_4.
{3,4} (6).	p.	Rule T, (4), (5), I_{10}.
{2,3,4} (7)	q.	Rule T, (2),(6). I_{11}.
{1,2,3,4}. (8).	$q \wedge \neg q$.	Rule T, (1).(7), I_9.

It is a contradiction.

Pridicate calculus :-

Predicative logic :-

The logic based upon the analysis of predicates in any statement is called predicate logic.

Predicate :-

A Predicate tells the nature of the objects. In general, predicates are represented by capital letters and objects are represented by small letters.

Example: Dog is an animal
cat is an animal
Rat is an animal.

Here 'is an animal' is a predicate, which is denoted by A. Dog, Cat, Rat are objects which are denoted by d, c, r respectively, then the statements can be written as A(d), A(c), A(R).

n-placed predicate :-

A predicate requiring n objects to represent a statement function is called n-placed predicate (n≥1)

Example :- 'is a man' is a predicate that needs only one object to represent a statement function then it is called a one placed predicate.

A predicate is 'taller than' needs two objects then it is called 2-placed predicate.

Simple statement function :-

A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable.

Example: $M(x)$: x is a man, is a statement function.

compound statement function :-

The statement functions which are obtained from combining one or more simple statement functions and the logical connectives is called compound statement function.

Example :- let $M(x)$: x is a man.

$H(x)$: x is a mortal.

$M(x) \wedge H(x)$: x is a man and x is a mortal.

$\neg M(x)$: x is not a mortal.

$M(x) \rightarrow \neg H(x)$: If x is a man then x is not a mortal.

$M(x) \vee \neg H(x)$: x is a man or x is not a mortal.

Free and bound variables:-

Given a formula containing a part of the form $(\forall x)\ P(x)$ (or) $(\exists x)\ P(x)$, such a part is called a x-bound part of the formula.

Any occurance of x in an x-bound part of the formula is called a bound occurance of x. while any occurance of x one of any variable is not a bound occurance is called free occurance.

Quantifiers:-

Quantifiers are words reffer to quantities such as some or all, and ~~~~ indicate how frequently a certain statement is true.

Quantifiers are two types.

Universal quantifiers:-

The universal quantifier of $P(x)$ is the statement, $P(x)$ is true for all values of x in the universe of discourse.

The notation is $(\forall x)\ P(x)$ (or) $(x)\ P(x)$

Here $\forall$ is called universal quantifiers.

Example: ① All men are mortal.

for all x, If x is a man then x is mortal.

$(\forall x)\ [M(x) \rightarrow H(x)]$.

② Every apple is red.

for all x, If x is apple then x is red.

$\forall(x)\ [A(x) \rightarrow R(x)]$

③ Any integer is either positive or negative

for all x, if x is an integer then
x is either positive or negative.

$\forall(x)\ [I(x) \rightarrow (P(x) \vee N(x))]$

Existential quantifiers:-

The existential quantification of $P(x)$ is the statement there exist an element x in the universe of discourse such that $P(x)$ is true we use the notation $(\exists x) P(x)$.

Here $\exists x$ is called existential quantifier.

Example: ① Some men are clever.

for some x, x is a man and x is clever.

$M(x)$: x is a man

$C(x)$: x is clever.

$(\exists x)\ [M(x) \wedge C(x)]$

(2) Some cats are black.

for some x, x is cat and x is black.

$$\exists(x)\,[C(x) \wedge B(x)]$$

(3) Some real numbers are rational.

$$(\exists x)\,[R_1(x) \wedge R_2(x)]$$

Formulas :—

E_{25} : $\neg(\exists x)\, A(x) \Leftrightarrow \forall(x)\, \neg A(x)$

E_{26} : $\neg(\forall x)\, A(x) \Leftrightarrow (\exists x)\, \neg A(x)$

E_{24} : $(x)\,[A(x) \wedge B(x)] \Leftrightarrow (x)\, A(x) \wedge (x)\, B(x)$.

E_{23} : $(\exists x)\,(A(x) \vee B(x)) \Leftrightarrow (\exists x)\, A(x) \vee (\exists x)\, B(x)$.

E_{27} : $(x)\,(A \vee B(x)) \Leftrightarrow A \vee (x)\, B(x)$

E_{28} : $(\exists x)\,(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x)\, B(x)$.

E_{29} : $(x)\, A(x) \rightarrow B \Leftrightarrow (\exists x)\,(A(x) \rightarrow B)$

E_{30} : $(\exists x)\, A(x) \rightarrow B \Leftrightarrow (x)\,(A(x) \rightarrow B)$

E_{31} : $A \rightarrow (x)\, B(x) \Leftrightarrow (x)\,(A \rightarrow B(x))$

E_{32} : $A \rightarrow (\exists x)\, B(x) \Leftrightarrow (\exists x)\,(A \rightarrow B(x))$

I_{15} : $(x)\, A(x) \vee (x)\, B(x) \Rightarrow (x)\,(A(x) \vee B(x))$

I_{16} : $(\exists x)\,(A(x) \wedge B(x)) \Rightarrow (\exists x)\, A(x) \wedge \exists x\, B(x)$.

Theory of Inference for predicate calculus :-

Rule US (universal specification):-

From $(\forall x)\, A(x)$ one can conclude $A(y)$.

Rule ES (Existential specification:-

From $(\exists x)\, A(x)$ one can conclude $A(y)$.

Rule UG (universal generalization):-

From $A(x)$ one can conclude $(\forall y)\, A(y)$

Rule EG (~~universal~~ Existential generalization):-

From $A(x)$ one can conclude $(\exists y)\, A(y)$

Show that all men are mortal
Socrates is a man
therefore socrates is mortal.

let $M(x)$: x is a man.
$H(x)$: x is a mortal.
s : socrates.

Given $\forall(x)\, [M(x) \rightarrow H(x)]$, $M(s) \Rightarrow H(s)$

$\{1\}$	(1)	$(\forall x)\, M(x) \rightarrow H(x)$	Rule P.
$\{1\}$	(2)	$M(s) \rightarrow H(s)$	Rule US, (1)
$\{3\}$	(3)	$M(s)$	Rule P.
$\{1,3\}$	(4)	$H(s)$	Rule T, (2),(3) I_{11}.

* Show that $(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$

Step	Derivation	rule
$\{1\}$ (1)	$(x)(P(x) \rightarrow Q(x))$	Rule P.
$\{1\}$ (2)	$P(y) \rightarrow Q(y)$	Rule US, (1).
$\{3\}$ (3)	$(x)(Q(x) \rightarrow R(x))$	Rule P.
$\{3\}$ (4)	$Q(y) \rightarrow R(y)$	Rule US, (3).
$\{1,3\}$ (5)	$P(y) \rightarrow R(y)$	Rule T, (2),(4), I_{13}.
$\{1,3\}$ (6)	$(x)(P(x) \rightarrow R(x))$	Rule UG, (5).

* Show that $(\exists x)\ M(x)$ follows logically from the premises $(x)\ (H(x) \rightarrow M(x))$ and $(\exists x)\ H(x)$.

Step	Derivation	rule
$\{1\}$ (1)	$(\exists x)\ H(x)$	Rule P.
$\{1\}$ (2)	$H(y)$	Rule ES, (1)
$\{3\}$ (3)	$(x)(H(x) \rightarrow M(x))$	Rule P.
$\{3\}$ (4)	$H(y) \rightarrow M(y)$	Rule US, (3)
$\{1,3\}$ (5)	$M(y)$	Rule T, (2),(4), I_{11}.
$\{1,3\}$. (6)	$(\exists x)\ M(y)$	Rule EG (5)

* Prove that $\forall x\,(P(x) \rightarrow (Q(y) \wedge R(x))),\ \exists P(x)$

$\Rightarrow Q(y) \wedge \exists x\,(P(x) \wedge R(x))$

Step		derivation	rule.
{1}	(1)	$\forall x\,(P(x) \rightarrow (Q(y) \wedge R(x)))$	Rule P.
{1}	(2)	$P(z) \rightarrow (Q(y) \wedge R(z))$	Rule US, (1)
{3}	(3)	$(\exists x)\,P(x)$.	Rule P.
{3}	(4)	$P(z)$	Rule ES, (3)
{1,3}	(5)	$Q(y) \wedge R(z)$	Rule T, (2), (4), I_{11}
{1,3}	(6)	$Q(y)$	Rule T, (5), I_1.
{1,3}	(7)	$R(z)$	Rule T, (5), I_2.
{1,3}	(8)	$P(z) \wedge R(z)$	Rule T, (4)(7), I_9.
{1,3}	(9)	$\exists(x)\,(P(x) \wedge R(x))$	Rule EG, (8)
{1,3}	(10)	$Q(y) \wedge (\exists x)(P(x) \wedge R(x))$	Rule T, (6), (9), I_9.

* Prove that $(\exists x)\,(P(x) \wedge Q(x)) \Rightarrow (\exists x)\ P(x)\ \wedge (\exists x)\ Q(x)$.

{1}	(1)	$(\exists x)\,(P(x) \wedge (Q(x)))$	Rule P.
{1}	(2)	$P(y) \wedge Q(y)$	Rule ES, (1)
{1}	(3)	$P(y)$	Rule T, (2), I_1.
{1}	(4)	$Q(y)$	Rule T, (2), I_2
{1}	(5)	$(\exists x)\ P(x)$	Rule EG, (3)
{1}	(6)	$(\exists x)\ Q(x)$	Rule E(G), (4).
{1}	(7)	$(\exists x)\ P(x) \wedge \exists x\, Q(x)$	Rule T, (5), (6), I_9.

* Verify the validity of the following arguments:
Lions are dangerous animals, there are Lions,
Therefore, there are dangerous animals.

let $L(x)$: x is a Lion.

$D(x)$: x is a dangerous animal.

Then the inference pattern is

$$\forall x\,(L(x) \rightarrow D(x)),\ \exists x\, L(x) \Rightarrow \exists x\, D(x).$$

Step		Derivation	rule
$\{1\}$	(1)	$\exists x\, L(x)$	Rule P.
$\{1\}$	(2)	$L(y)$	Rule ES, (1)
$\{3\}$	(3)	$\forall x\,[L(x) \rightarrow D(x)]$	Rule P
$\{3\}$	(4)	$L(y) \rightarrow D(y)$	Rule US, (3)
$\{1,3\}$	(5)	$D(y)$	Rule T, (2), (4), I_{11}
$\{1,3\}$	(6)	$\exists x\, D(x)$	Rule EG.