Unit - III - Elementary Combinatorics

Basis of counting :-

There are two fundamental principles of counting called

① Sum rule (or) Addition principle (or) principle of disjunctive counting.

② Product rule (or) Multiplication principle (or) The principle of sequential counting.

Sum rule :- If a set X is the union of disjoint non empty subsets $S_1, S_2, S_3 --- S_n$ then ~~the~~ $|X| = |S_1| + |S_2| + |S_3| + --- + |S_n|$ which is also be stated as if $E_1, E_2, E_3 --- E_n$ are mutually exclusive events and E_1 can happen e_1 ways, E_2 can happen e_2 ways, --- E_n can happen in e_n ways then E_1 or E_2 or E_3 --- or E_n can happen $e_1 + e_2 + e_3 + --- + e_n$ ways

Examples

① If there are 14 boys and 12 girls in a class find the number of ways of selecting one student.

Sol :- For a selection of one student either of the following two tasks is to be performed.

① selecting a boy among 14 boys (or)

② selecting a girl among 12 girls.

Selecting a boy among 14 boys is $14C_1$ ways
selecting a girl among 12 girls is $12C_1$ ways

$\therefore$ By Sum rule either of the two tasks can be performed in $14C_1 + 12C_1 = 14 + 12 = 26$ way.

② In how many ways can we draw a heart or a spade card from an ordinary deck of playing cards A heart or an ace? an ace or a king? A card numbered 2 through 10? A numbered card or a king?

Sol:- ① The number of ways of selecting a heart $= 13C_1$
The number of ways of selecting a spade card $= 13C_1$
$\therefore$ The number of ways of selecting a heart card or a spade card is $13C_1 + 13C_1 = 13 + 13 = 26$

② The number of ways of selecting a heart card $= 13C_1$
The number of ways of selecting an ace $= 4C_1 - 1 = 3$
$\therefore$ The number of ways of selecting a heart or an ace card is $13C_1 + 3 = 16$.

③ Total no of Ace $= 4$
Total no of kings $= 4$
$\therefore$ we get in $4 + 4 = 8$ way.

④ Total no of numbered cards 2 through $10 = 9$.

Getting of 2 through 10 cards $= 9 + 9 + 9 + 9 = 36$.

⑤ Total numbered cards $= 36$
Total number of kings $= 4$
we get in $36 + 4 = 40$ ways.

Product rule :- If an event can occur in 'm' ways and a second event in 'n' ways and if the number of ways the second event occurs does not depend upon the occurance of the first event then the two events can occur simultaneously in 'mn' ways.

In general if events $E_1, E_2, E_3 --- E_n$ can happen in $n_1, n_2, n_3 --- n_n$ ways then the sequence of events E_1 first follows by E_2 and so on follows by E_n can happen in $n_1 \times n_2 \times n_3 --- \times n_n$ ways.

Examples :-

① Three persons enter into car there are 5 seats in a car. In how many ways can they take up their seats.

Sol :- The first person has a choice of 5 seats and can sit in any one of those 5 seats.

So there are 5 ways of occupying the first seat.

The second person has a choice of 4 seats.

similarly 3^{rd} person has a choice of 3 seats.

Hence the required number of ways in which all the 3 persons can seat is $5 \times 4 \times 3 = 60$ ways.

② How many 3 digit numbers can be formed using the digit 1, 3, 4, 5, 6, 8 and 9? How many can be formed if no digit can be repeated.

Sol:- Given digits 1, 3, 4, 5, 6, 8, 9.

① Repitition is allowed $7 \times 7 \times 7 = 7^3$ ways $= 343$ ways

② Repitition is not allowed $7 \times 6 \times 5 = 210$ ways.

② How many different licence plates are there that involved 1, 2 or 3 letters followed by 4 digits.

Sol:- ① One letter followed by 4 digits.

$\underline{26}\ \underline{10}\ \underline{10}\ \underline{10}\ \underline{10} = 26 \times 10^4$

② Two letters followed by 4 digits

$26\ 26\ 10\ 10\ 10\ 10 = 26^2 \times 10^4$

③ 3 letters followed by 4 digits

$26\ 26\ 26\ 10\ 10\ 10\ 10 = 26^3 \times 10^4$

The number of ways of form licence plates are $= (26 + 26^2 + 26^3)\, 10^4$ ways.

③ How many numbers can be formed using the digits 1, 3, 4, 5, 6, 8 and 9. If no repetions are allowed.

Sol:- one digit number $7 = 7$ ways

Two digit numbers $\underline{7}\ \underline{6} = 7 \times 6$ ways

Three " " $\underline{7}\ \underline{6}\ \underline{5} = 7 \times 6 \times 5$

4 " " $\underline{7}\ \underline{6}\ \underline{5}\ \underline{4} = 7 \times 6 \times 5 \times 4$

5 " " $\underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3} = 7 \times 6 \times 5 \times 4 \times 3$

6 " " $\underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2} = 7 \times 6 \times 5 \times 4 \times 3 \times 2$

7 " " $7\ 6\ 5\ 4\ 3\ 2\ 1 = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

Total numbers $= 7 + (7\times6) + (7\times6\times5) + (7\times6\times5\times4) + (7\times6\times5\times4\times3) + (7\times6\times5\times4\times3\times2) + (7\times6\times5\times4\times3\times2\times1)$.

Total numbers $= 13,699$ ways.

(4) In how many ways can one select two books from different subjects from among six distinct computer science books, 3 distinct maths books and two distinct chemistry books.

Sol:- Using product rule one can select two books from different subjects as follows.

one C.S and one maths in 6×3 ways $=18$
one C.S and one chem in 6×2 ways $=12$
one maths and one chem in 3×2 ways $=6$.

These sets are pairwise disjoint one can use the sum rule to get the required number of ways which is $18+12+6=36$ ways.

Note:-

(1) Sum rule: If A and B are disjoint sets i.e $A\cap B=\phi$ then $|A+B| = |A|+|B|$ (or) $|A\cup B|$

(2) Product rule: If $A\times B$ is the cartesian product of sets A and B then $|A\times B| = |A|\cdot|B|$ (or) $|A\cap B|$.

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Enumerating combinations and Permutations:-

Factorial:- The Product of first n natural numbers $1, 2, 3, \ldots n$ is denoted by $n!$.

Thus $n! = n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$

$n! = n(n-1)!$

$n! = n(n-1)(n-2)!$.

Permutation:- An arrangement of a given set of objects taken some or all of them at a time is called a permutation of the objects.

A permutation of n objects taken 'r' at a time and is denoted by $P(n,r)$ or ^nP_r

Results:-

1. $P(n,r)$ or $^nP_r = \frac{n!}{(n-r)!}$
2. The no of permutations of n distinct objects taken r at a time without repetition is

$$P(n,r) = n(n-1)(n-2) \ldots (n-r+1).$$

3. $P(n,n) = n!$, i.e $^nP_n = n!$
4. $P(n,n-1) = P(n,n)$
5. $0! = 1$.

Combination :- An ordered selection of a given set of objects taken some or all of them at a time is called a combination. Thus any ordered selection of r objects from a set of n objects is called an r-combinations of n objects and is denoted by $c(n,r)$ or ^nc_r.

Results :-

(1) $^nc_r = \dfrac{n!}{r!(n-r)!}$

(2) $c(n,0) = 1$

(3) $c(n,n) = 1$, i.e $^nc_n = 1$.

(4) $c(n,r) = c(n,n-r)$

(5) $c(n+1,r) = c(n,r-1) + c(n,r)$

(6) $r!\, ^nc_r = {^np_r}$.

Permutation of objects with repetitions :-

The no of permutation of n objects in which p objects are of one type, q objects are of 2^{nd} type, 'r' objects are of 3^{rd} type and rest are all distinct is $\dfrac{n!}{p!q!r!}$

Constrained repetitions :-

* The no. of combinations of r objects among n objects if the repetitions is allowed is

$$c(n+r-1,\ n-1) = \frac{(n+r-1)!}{r!\,(n-1)!}$$

* The no. of ways of distributing n objects to r children so that each child got atleast one is ~~C(n-1,)~~ $C(n-1, n-r)$

* The no. of non negative integer solutions of $x_1+x_2+ \cdots +x_r=n$ such that $x_i>0$ is $C(n-1, n-r)$.

* The no. of non negative integer solutions of $x_1+x_2+ \cdots +x_r=n$ such that $x_i \geq 0$ is $C(n+r-1, r)$.

Problems:-

① In how many ways five students can be selected from 12 student ① without replacement ② If two students not to be included.

Sol:- Total number of students $n=12$.

① 5 students can be selected from 12 students without replacement is ${}^{12}C_5 = 792$ ways

② If two students not to be included means $12-2=10$.

$\therefore {}^{10}C_5 = 252$ ways.

② How many 4 digit number can be formed using the digits 0, 1, 2, 3, 4, 5 if ① Repetition of digit is not allowed ② Repetition of digit is allowed.

Sol:- ① In a 4 digit number '0' cannot appear in the thousands place.

so 1^{st} place can be filled in 5 ways

2^{nd} place is filled in again 5 ways including 0

3^{rd} place is filled in again 4 ways

4^{th} place is filled in again 3 ways.

Hence by product rule

Required no. of ways $= 5 \times 5 \times 4 \times 3 = 300$ ways.

(2) Repetition is allowed.

$\underline{5}\ \underline{6}\ \underline{6}\ \underline{6} = 5 \times 6 \times 6 \times 6 = 1080$ ways

(3) In how many ways can the letters of the word "COMPUTER" be arranged? How many of them begin with C and end with R? How many of them do not begin with C but end with R?

Sol:- The word COMPUTER consists of 8 letters which can be arranged in $P(8,8) = 8! = 40320$ ways.

If C occupies the 1^{st} place and R is the last place then there are 6 letters (O, M, P, U, T, E) left to be arranged in 6 places in between C and R. This can be done in $P(6,6) = 6! = 720$ ways.

If C doesnot occupy the first place but R occupies the last place. The first place can be occupied in 6 ways by one of the letters O, M, P, U, T, E for the 2^{nd} place, again 6 letters are there including C.

The $3^{rd}, 4^{th}, 5^{th}, 6^{th}, 7^{th}$ places can be filled up by 5, 4, 3, 2, 1 ways.

Hence by Product rule,

The required number of arrangements is

$6 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 4320$ ways.

(4) How many different words can be formed with the letters of the word. MISSISSIPPI

Sol:- There are 11 letters in the word of which 4 are S's, 4 are I's, 2 are P's and M's are 1.

∴ The number of arrangements of 11 digits of which 4 are similar of one kind, 4 are of 2^{nd} kind and 2 are of 3^{rd} kind is

$$\frac{n!}{p!\,q!\,r!} = \frac{11!}{4!\,4!\,2!} = 34650$$

(5) out of 9 girls and 15 boys. How many different committees can be formed each consisting of 6 boys and 4 girls.

Sol:- There are 9 girls and 15 boys. then we can form 2 committees such that each consisting of 6 boys and 4 Girls.

(1) To select 6 boys out of 15 boys and 4 Girls out of 9 Girls.

The number of ways to select 6 boys out of 15 Boys is $15_{C_6} = 5005$.

The number of ways to select 4 girls out of

9 Girls is $^9C_4 = 126$.

By using the Product rule, the number of ways of selecting 6 boys and 4 girls to form a committe is $^{15}C_6 \times {^9C_4} = 630630$.

(2) After forming a first committee, there are remai-ning 9 boys and 5 girls in which we can form another 2nd committee also.

i,e we have to select again 6 boys out of 9 boys and 4 girls out of 5 girls.

$\therefore$ The number of ways of selecting 6 boys from 9 boys is $^9C_6$

The number of ways of selecting 4 girls from 5 girls is $^5C_4$

$\therefore$ By Product rule, The number of ways of form 2nd committe is $^9C_6 \times {^5C_4} = 420$.

Hence by sum rule, the number of committees can be formed each consisting of 6 boys and 4 girls is $630630 + 420 = 631050$.

(6) In how many ways a committee of 5 members can be selected from 6 men and 5 women consisting of 3 men and 2 women.

Sol:- The number of ways of selecting 3 men out of 6 men is $^6C_3$.

The number of ways of selecting 2 women out of 5 women is $5C_2$ ways.

By Product rule, The total number of ways of selecting a committe is $6C_3 \times 5C_2 = 200$ ways.

(7) out of 5 men and 2 women a committee of 3 is to be formed. In how many ways can it be formed. If atleast one women is to be included.

Sol:- There are 2 possible ways to form a committee if atleast one women is to be included.

(1) 2 men and 1 women.

(2) 1 men and 2 women

∴ The number of ways of selecting 2 men and one women is $5C_2 \times 2C_1 = 20$.

The number of ways of selecting one men and 2 women is $5C_1 \times 2C_2 = 5$

∴ The no of ways of forming the committee is $20+5=25$

(8) The question paper of mathematics contains two questions divided into two groups of 5 questions each. In how many ways can an examine answer six questions taking atleast two ~~gro~~ questions from each group.

Sol:- The examinee can answers questions from the two groups in the following ways.

(1) 2 from first group and 4 from second group.

(2) 3 from first group and 3 from second group
(3) 4 from first group and 2 from second group.

The number of ways of ~~selecting~~ Answering the questions in case (1) is $5C_2 \times 5C_4 = 50$.

The number of ways of Answering the questions in case (2) is $5C_3 \times 5C_3 = 100$.

The number of ways of Answering the questions in case (3) is $5C_4 \times 5C_2 = 50$.

$\therefore$ The required no of ways $= 50 + 100 + 50 = 200$.

(9) How many permutations can be formed out of the letters of word "SUNDAY". How many of these (1) Begin with S (2) End with Y (3) Begin with S and end with Y (4) S & Y always together.

Sol:- The word "SUNDAY" consist of 6 letters which can be arranged in $6!$ ways.

(1) 's' occupies in first place then there are 5 letters left to be arranged in 5 places

The total number of permutations begin with 's' is $\underline{S}UNDAY - 5!$

(2) 'y' occupies in the last place then there are 5 letters left to be arranged in 5 places.

The total number of permutations end with 'y' is $SUNDA\underline{Y} - 5!$

③ S & Y occupies first and last places then there are 4 letters left to be arranged in 4 places.
The total number of permutations begins with S and end with Y is $\underline{S}$ U N D A $\underline{Y}$ $- 4!$

④ S & Y always together (i,e S and Y (or) Y and S).
The total number of permutations when S and Y always together is $\underline{SY}$ U N D A $- 5! \times 2!$.

⑩ Find the number of arrangements of the letters in the word "ACCOUNTANT".

Sol:- There are 10 letters in the word ACCOUNTANT of which 2 are A's, 2 are C's, 2 are T's, 2 are N's, 1 is 'O' and one is 'U'.

∴ The no of arrangements of 10 digits of which 2 are similar of one king, 2 are 2nd kind, 2 are 3rd kind and 2 are 4th kind is

$$\frac{10!}{2!\,2!\,2!\,2!} =$$

⑪ How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5=20$ where each ① $x_i \geq 2$ ② $x_i > 2$.

Sol:- Given $x_1+x_2+x_3+x_4+x_5=20$.

① Given x_i is non negative integer

i,e $x_i \geq 2 \Rightarrow x_i - 2 \geq 0$.

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Let us consider $y_1=x_1-2$, $y_2=x_2-2$, $y_3=x_3-2$

$y_4=x_4-2$, $y_5=x_5-2$.

$\Rightarrow x_1=y_1+2$, $x_2=y_2+2$, $x_3=y_3+2$, $x_4=y_4+2$, $x_5=y_5+2$

Substituting x_1, x_2, x_3, x_4, x_5 in given equation,

$$y_1+2+y_2+2+y_3+2+y_4+2+y_5+2=20.$$

$$y_1+y_2+y_3+y_4+y_5=10.$$

The number of non negative integer solutions of this equation is the required number and the number is (here $n=10$, $r=5$)

$$C(n+r-1, n)=C(10+5-1, 10)$$
$$=C(14,10)$$
$$={}^{14}C_{10}$$

(2) Given $x_i>2 \Rightarrow x_i-2>0. \Rightarrow y_i=x_i-2>0.$

Let us take $y_1=x_1-2$, $y_2=x_2-2$, $y_3=x_3-2$, $y_4=x_4-2$ and $y_5=x_5-2$.

$\Rightarrow x_1=y_1+2$, $x_2=y_2+2$, $x_3=y_3+2$, $x_4=y_4+2$, $x_5=y_5+2$

Substituting this x values in the given equation

$$y_1+2+y_2+2+y_3+2+y_4+2+y_5+2=20.$$

$$y_1+y_2+y_3+y_4+y_5=10.$$

Now compare the above equation with $x_1+x_2+x_3+\cdots+x_r=n$ with respect to the

~~xxx~~ constraint $x_i > 0$. we got

$n = 10, r = 5$.

The no of non negative integers is

$C(n-1, n-r) = C(10-1, 10-5)$

$= C(9,5)$

$= {}^{9}C_5$

$= 126$.

(12) Enumerate the number of non negative integral solutions to the inequality $x_1+x_2+x_3+x_4+x_5 \leq 19$.

Sol:- If $x_1+x_2+x_3+x_4+x_5 = 0$ then $n=0, r=5$

$C(n+r-1, n) = C(0+5-1, 0)$

$= C(4,0)$

$= {}^{4}C_0$.

If $x_1+x_2+x_3+x_4+x_5 = 1$ then $n=1, r=5$.

$C(n+r-1, n) = C(1+5-1, 1)$

$= C(5,1)$

$= {}^{5}C_1$

If $x_1+x_2+x_3+x_4+x_5 = 2$ then $n=2, r=5$.

$C(n+r-1, n) = C(2+5-1, 2)$

$= C(6,2)$

$= {}^{6}C_2$.

$x_1+x_2+x_3+x_4+x_5 = 3$; $n=3, r=5 \Rightarrow C(5+3-1, 3) = {}^{7}C_3$

$x_1+x_2+x_3+x_4+x_5 = 4$, $n=4, r=5 \Rightarrow C(4+5-1, 4) = {}^{8}C_4$

$x_1+x_2+x_3+x_4+x_5 = 5$, $n=5, r=5 \Rightarrow C(5+5-1, 5) = {}^{9}C_5$

$x_1+x_2+x_3+x_4+x_5=6$, $n=6$, $r=5 \Rightarrow C(6+5-1,6)={}^{10}C_6$

$x_1+x_2+x_3+x_4+x_5=7$, $n=7$, $r=5 \Rightarrow C(7+5-1,7)={}^{11}C_7$

$x_1+x_2+x_3+x_4+x_5=8$, $n=8$, $r=5 \Rightarrow C(8+5-1,8)={}^{12}C_8$

$x_1+x_2+x_3+x_4+x_5=9$, $n=9$, $r=5 \Rightarrow C(9+5-1,9)={}^{13}C_9$

$x_1+x_2+x_3+x_4+x_5=10$, $n=10$, $r=5 \Rightarrow C(10+5-1,10)={}^{14}C_{10}$

Similarlly for $n=11,12,13,14,15,16,17,18,19$ we get

${}^{15}C_{11}$, ${}^{16}C_{12}$, ${}^{17}C_{13}$, ${}^{18}C_{14}$, ${}^{19}C_{15}$, ${}^{20}C_{16}$, ${}^{21}C_{17}$, ${}^{22}C_{18}$, ${}^{23}C_{19}$.

The number of non negative integral solutions is ${}^4C_0+{}^5C_1+{}^6C_2+{}^7C_3+{}^8C_4+{}^9C_5+{}^{10}C_6+$
${}^{11}C_7+{}^{12}C_8+{}^{13}C_9+{}^{14}C_{10}+{}^{15}C_{11}+{}^{16}C_{12}+{}^{17}C_{13}+$
${}^{18}C_{14}+{}^{19}C_{15}+{}^{20}C_{16}+{}^{21}C_{17}+{}^{22}C_{18}+{}^{23}C_{19}$

$\Rightarrow 1+5+15+35+70+126+210+330+495+715+1001$
$+1365+1820+2380+3060+$

$\Rightarrow 42504.$

Binomial and Multinomial theorems:-

Binomial theorem:- Let x and y be any real numbers and n be any non negative integer then

$$(x+y)^n=\sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r}.$$

Here $C(n,r)={}^nC_r$ is the coefficient of $x^r y^{n-r}$, this result is known as the Binomial theorem for a +ve integral index.

The numbers $C(n,r)$ for $r=0,1,2---n$ in the above results are known as Binomial coefficients

Note:-

1. Binomial theorem is also stated as

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r.$$

2. $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^r y^{n-r} = {}^nC_0 x^0 y^n + {}^nC_1 x y^{n-1} + {}^nC_2 x^2 y^{n-2} + ---$

3. $(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r = {}^nC_0 x^n y^0 + {}^nC_1 x^{n-1} y^1 + {}^nC_2 x^{n-2} y^2 + ---$

Multinomial theorem:- For a positive integers n and t, the coefficient of $x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3} --- x_t^{n_t}$ in the expansion of $(x_1+x_2+x_3+---+x_t)^n$ is

$$\frac{n!}{n_1!\, n_2!\, n_3! --- n_t!}$$

where each n_i is a non-negative integer which is $\leq n$ and $n_1+n_2+n_3+---+n_t = n$.

Note:-

1. The expression $\frac{n!}{n_1!\, n_2! --- n_t!}$ is also written as

$\binom{n}{n_1, n_2, n_3 --- n_t}$ and is called a multinomial coefficient.

2. The general term of multinomial theorem is

given by,

$$\boxed{(x_1+x_2+\cdots+x_t)^n = \frac{n!}{n_1!\,n_2!\cdots n_t!}\, x_1^{n_1} x_2^{n_2} x_3^{n_3}\cdots x_t^{n_t}}$$

Problems:-

(1) Find the binomial expansion of $(2a-3b)^4$

Sol:- Here $x=2a$, $y=-3b$ and $n=4$.

By using the Binomial theorem

$$(x+y)^n = nC_0 x^0 y^{n-0} + nC_1 x y^{n-1} + nC_2 x y^{n-2} + \cdots + nC_n x^n$$

$$(2a-3b)^4 = 4C_0(-3b)^4 + 4C_1(2a)(3b)^{4-1} + 4C_2(2a)^2(-3b)^2 + 4C_3(2a)^3(-3b) + 4C_4(2a)^4$$

$$= (-3b)^4 + 4(2a)(-3b)^3 + 6(2a)^2(-3b)^2 + 4(2a)^3(-3b) + (2a)^4$$

$$= 81b^4 - 216ab^3 + 216a^2b^2 - 96a^3b + 16a^4.$$

(2) what is the coefficient of (1) x^3y^7 in $(x+y)^{10}$ (2) x^2y^4 in $(x-2y)^6$.

Sol:- (1) Here $n=10$.

By using Binomial theorem

$$(x+y)^n = \sum_{r=0}^{n} nC_r\, x^r y^{n-r}$$

$$(x+y)^{10} = \sum_{r=0}^{10} 10C_r\, x^r y^{10-r}.$$

Taking $r=3$ in ~~to~~ the above equation

$$(x+y)^{10} = 10C_3\, x^3 y^7$$

$\therefore$ The coefficient of x^3y^7 in $(x+y)^{10}$ is $10C_3 = 120$.

(2) ~~Here~~ $x=x$, ~~and~~ $y=-2y$ and $n=6$.

By using Binomial theorem,

$$(x+y)^n = \sum_{r=0}^{n} nC_r \, x^r \, y^{n-r}$$

$$(x-2y)^6 = \sum_{r=0}^{6} 6C_r \, x^r \, y^{6-r}$$

Taking $r=2$ term in the above equation,

we get $6C_2 \, x^2 (-2y)^4$

$$\Rightarrow 6C_2 (-2)^4 x^2 y^4$$

$\therefore$ The coefficient of x^2y^4 is $6C_2(-2)^4 = 240$.

(3) what is the coefficient of x^6y^3 in $(x-3y)^9$.

Sol:- Here $x=x$, $y=-3y$ and $n=9$.

By using the Binomial ~~coefficient~~ theorem,

$$(x+y)^n = \sum_{r=0}^{n} nC_r \, x^r \, y^{n-r}$$

$$(x-3y)^9 = \sum_{r=0}^{9} 9C_r \, x^r \, y^{9-r} (-3)^{9-r}$$

Taking $r=6$ term in the above equation, we get $9C_6 (-3)^3 x^6 y^3$

$$\Rightarrow 9C_6 (-3)^3 x^6 y^3$$

The coefficient of x^6y^3 is $9C_6(-3)^3 = -2268$.

(4) Find the coefficient of $x^3y^2z^2$ in $(2x-y+z)^7$.

Sol:- Here $x_1=2x$, $x_2=-y$, $x_3=z$ and $n=7$.

By using general term of multinomial theorem,

$$(x_1+x_2+\dots+x_t)^n = \frac{n!}{n_1!\,n_2!\dots n_t!}\, x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}$$

$$(2x-y+z)^7 = \frac{n!}{n_1!\,n_2!\,n_3!} (2x)^{n_1} (-y)^{n_2} z^{n_3}$$

$$= \frac{9!}{n_1!\,n_2!\,n_3!} 2^{n_1} (-1)^{n_2} x^{n_1} y^{n_2} z^{n_3}$$

For the term containing $x^3y^2z^2$, we should have $n_1=3$, $n_2=2$, $n_3=2$.

Also $n_1+n_2+n_3=n$

$3+2+2=n$

$\boxed{n=7}$

Thus the coefficient of $x^3y^2z^2$ is $\frac{9!}{3!\,2!\,2!} 2^3 (-1)^2$

$$\Rightarrow \frac{9!}{3!\,2!\,2!} 2^3$$

The Principle of Inclusion and exclusion :-

Let A and B are subsets of some universal set U then $|A \cup B| = |A| + |B| - |A \cap B|$

where $|A|$ = Number of elements in the set A

$|B|$ = Number of elements in the set B

$|A \cup B|$ = Number " " " $A \cup B$

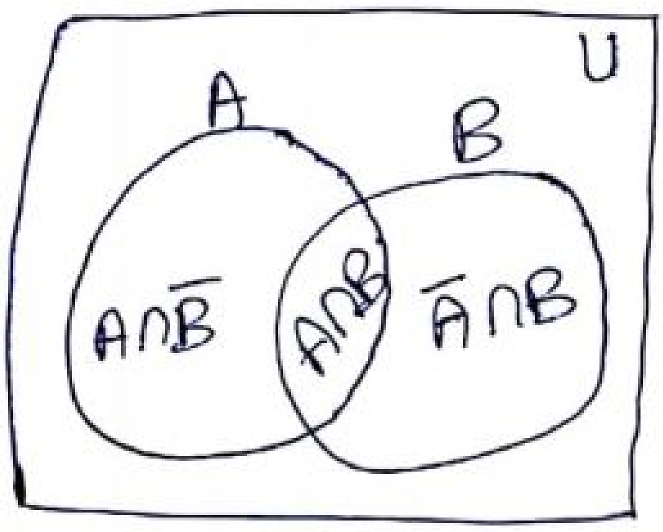
$|A \cap B|$ = " " " $A \cap B$

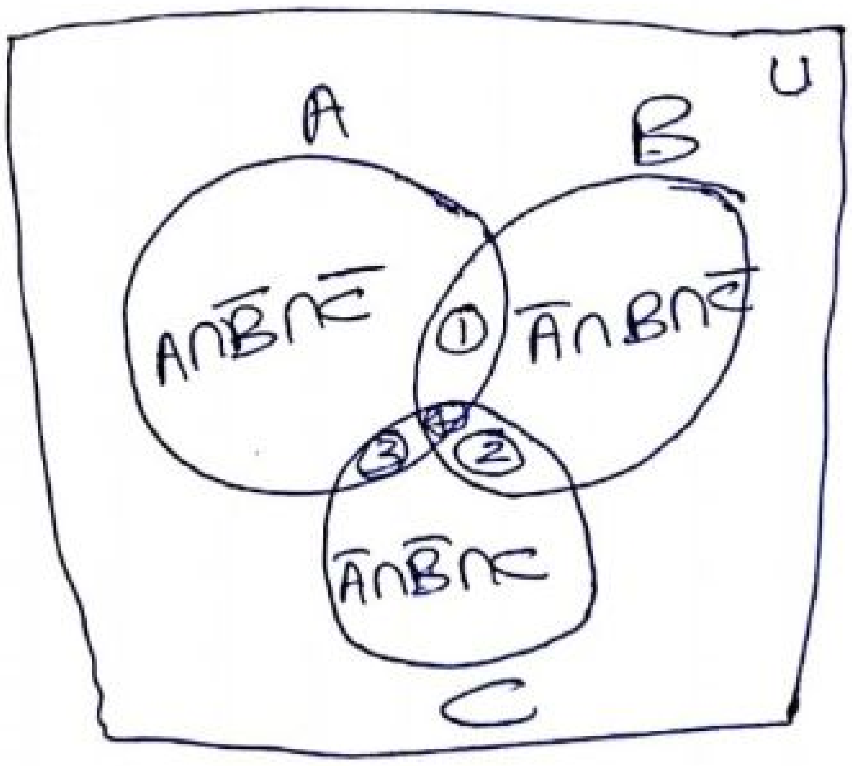
Note:-

① Let A, B and are subset of universal set U,

then, $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

②





① $A \cap B \cap \bar{C}$

② $\bar{A} \cap B \cap C$

③ $A \cap \bar{B} \cap C$

④ $A \cap B \cap C$.

(3) $|\bar{A}| = N - |A|$

where N is the number of elements in the universal set

(4) $\lfloor x \rfloor$ = The least positive integer.

Problems:-

(1) Find how many integers between 1 and 60 that are divisible by 2 nor by 3 and nor by 5. Also determine the number of integers divisible by 5 not by 2, not by 3.

Sol:- Let $S = \{1, 2, 3 - - - 60\}$ then $N = n(S) = 60$.

Let $|A_1|$ denotes the number of integers which are divisible by 2.

i;e $|A_1| = \left\lfloor \frac{60}{2} \right\rfloor = 30$.

Let $|A_2|$ denotes the number of integers which are divisible by 3 is $|A_2| = \left\lfloor \frac{60}{3} \right\rfloor = 20$.

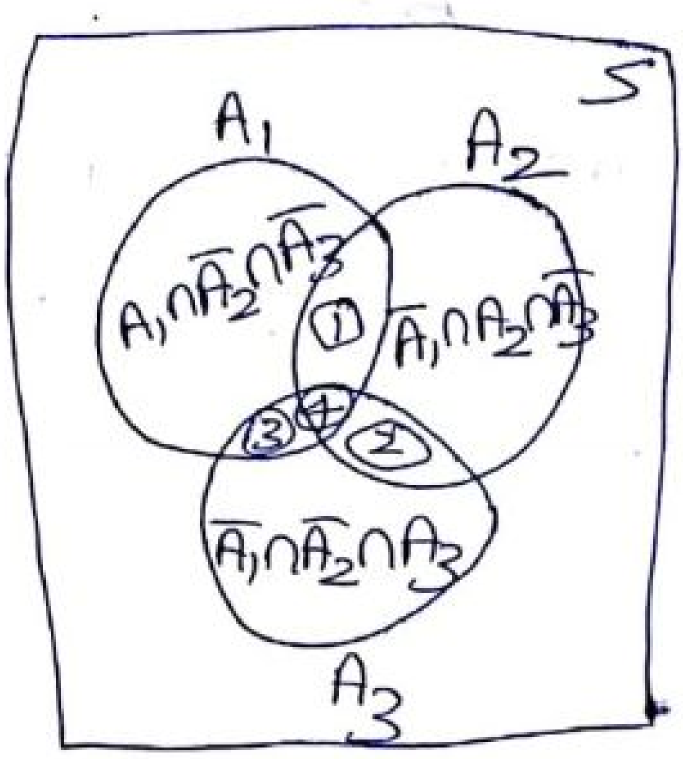
Let $|A_3|$ denotes the number of integers which are divisible by 5 is $|A_3| = \left\lfloor \frac{60}{5} \right\rfloor = 12$

Let $|A_1 \cap A_2|$ is the number of elements which are divisible by 2 and 3 is $|A_1 \cap A_2| = \left\lfloor \frac{60}{LCM\ of\ (2,3)} \right\rfloor = 10$

Let $|A_1 \cap A_3|$ is the number of elements which are divisible by 2 and 5 is $|A_1 \cap A_3| = \left\lfloor \frac{60}{LCM(2,5)} \right\rfloor = \frac{60}{10} = 6$.

Let $|A_2 \cap A_3)$ is the number of elements which are divisible by 3 & 5 is $|A_2 \cap A_3) = \left\lfloor \frac{60}{LCM(3,5)} \right\rfloor = \left\lfloor \frac{60}{15} \right\rfloor = 4.$

||ly $|A_1 \cap A_2 \cap A_3) = \left\lfloor \frac{60}{LCM(2,3,5)} \right\rfloor = \frac{60}{30} = 2.$

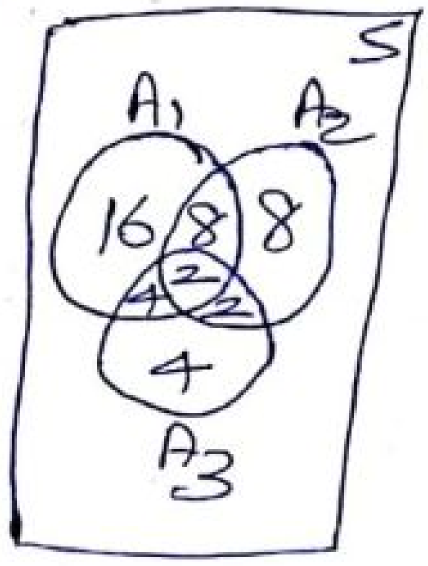


① $A_1 \cap A_2 \cap \bar{A_3}$

② $\bar{A_1} \cap A_2 \cap A_3$

③ $A_1 \cap \bar{A_2} \cap A_3$

④ $A_1 \cap A_2 \cap A_3$



From the above ven diagram,

The number of integers which are divisible by 2 nor 3 and nor 5 is $|A_1 \cap \bar{A_2} \cap \bar{A_3})$

$$|A_1 \cap \bar{A_2} \cap \bar{A_3}| = |A_1| - |A_1 \cap A_2 \cap \bar{A_3}) - |A_1 \cap \bar{A_2} \cap A_3) - |A_1 \cap A_2 \cap A_3)$$

$$= 30 - 8 - 4 - 2$$

$$= 16.$$

The number of integers which are divisible by 5 not by 2, not by 3. is

$$|\bar{A_1} \cap \bar{A_2} \cap A_3| = |A_3) - |\bar{A_1} \cap A_2 \cap A_3) - |A_1 \cap \bar{A_2} \cap A_3) - |A_1 \cap A_2 \cap A_3)$$

$$= 12 - 4 - 2 - 2$$

$$= 4.$$

(2) out of 80 students in a class, 60 play foot ball, 53 play hockey and 35 both the games. How many students (1) do not play of these games
(2) play only hockey but not football.

Sol:- The number of students in a class $N=80$.

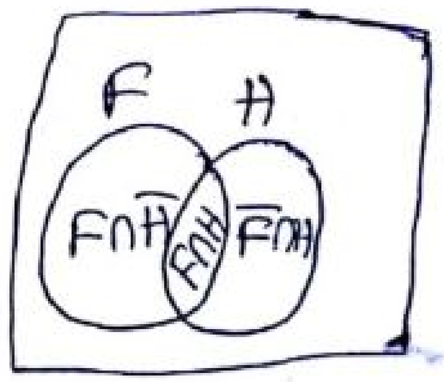
Let F: The students play football
H: The students play hockey

$\therefore |F|=60$
$|H|=53$
$|F\cap H|=35$.

(1) The number of students do not play hockey and football is $|\bar{F}\cap\bar{H}|$

$$|\bar{F}\cap\bar{H}|=|\overline{(F\cup H)}|=N-|F\cup H|$$
$$=N-[|F|+|H|-|F\cap H|]$$
$$=80-[60+53-35]$$
$$=2.$$

(2) The number of students play only hockey but not football is $|\bar{F}\cap H|$



$$\therefore |\bar{F}\cap H|=|H|-|F\cap H|$$
$$=55-35$$
$$=18.$$

③ A survey among 100 students shows that of the three ice cream flavours vanilla, chocolate, strawberry. 50 students like vanilla, 43 likes chochobate, 28 like strawberry and vanilla and 5 like all of them. Find the following ① chocolate but not strawberry ② chokolate and strawberry but not vanilla. ③ vanila or chocolate but not strawberry.

Sol:- Given, $N=100$.

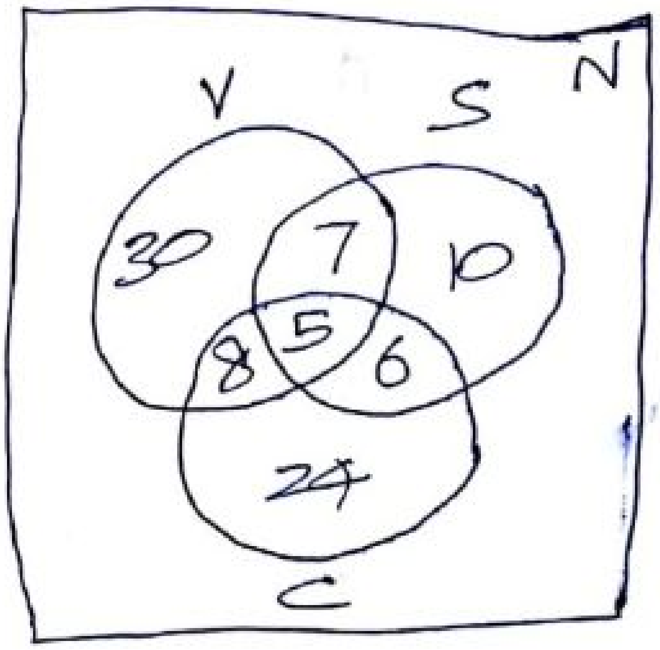
V: The students like vanilla.

C: The students like chocolate

S: The students like strawberry

Also given, $|V|=50$, $|C|=43$, $|S|=28$

$|V\cap C|=13$, $|C\cap S|=11$, $|S\cap V|=12$, $|S\cap V\cap C|=5$.



① From the above ven diagram, the number of students like chocolate but not strawberry is

$$|C\cap\bar{S}|=|C|-|C\cap S\cap\bar{V}|-|C\cap S\cap V|$$

$$=43-6-5$$

$$=32.$$

(2) The number of students like chocolate and strawberry but not vanilla is

$$|C\cap S\cap \bar{V}| = |C\cap S| - |C\cap S\cap V|$$
$$= 11-5$$
$$= 6.$$

(3) The number of students like vanilla or chocolate but not strawberry is

$$|(V\cup C)\cap \bar{S}| = |V\cap \bar{C}\cap \bar{S}| + |C\cap \bar{V}\cap \bar{S}| + |V\cap C\cap \bar{S}|$$
$$= 30+8+24$$
$$= 62.$$

(4) Consider a set of integers from 1 to 250. Find how many of these numbers are divisible by 3 or 5 or 7. Also indicate how many are divisible by 3 or 7 but not by 5 and divisible by 3 or 5.

Sol:- Let $S=\{1,2,3,---250\}$ i,e $N=250$.

Let A_1: The numbers are divisible by 3

A_2: The numbers which are divisible by 5

A_3: The numbers which are divisible by 7.

$$|A_1| = \left\lfloor \frac{250}{3} \right\rfloor = 83$$

$$|A_2| = \left\lfloor \frac{250}{5} \right\rfloor = 50$$

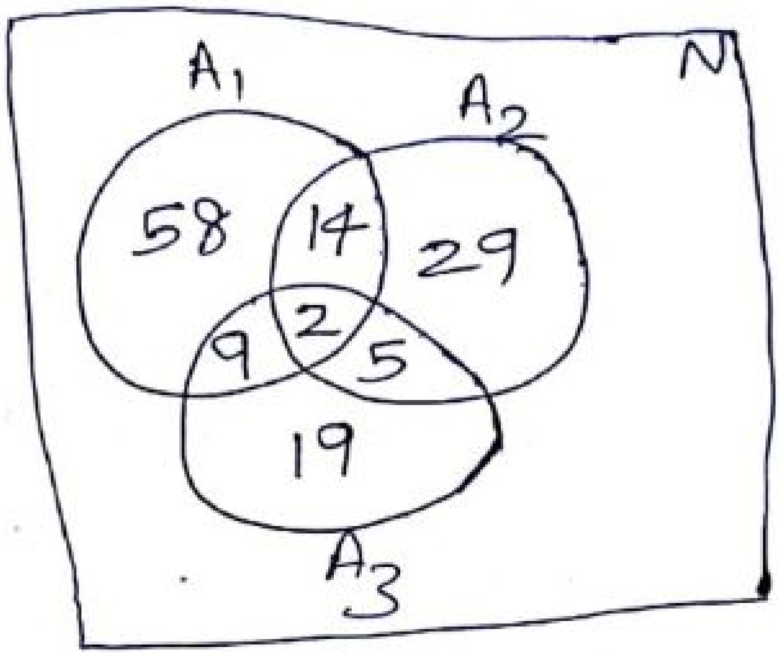
$$|A_3| = \left\lfloor \frac{250}{7} \right\rfloor = 35.$$

$|A_1 \cap A_2| = \left\lfloor \frac{250}{LCM(3,5)} \right\rfloor = 16.$

$|A_1 \cap A_3| = \left\lfloor \frac{250}{LCM(3,7)} \right\rfloor = 11$

$|A_2 \cap A_3| = \left\lfloor \frac{250}{LCM(5,7)} \right\rfloor = 7.$

$|A_1 \cap A_2 \cap A_3| = \left\lfloor \frac{250}{LCM(3,5,7)} \right\rfloor = 2.$



From the above ven diagram, the number of integers which are divisible by 3 or 5 or 7 is

$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3|$
$+ |A_1 \cap A_2 \cap A_3|$

$= 83 + 50 + 35 - 16 - 11 - 7 + 2$

$= 136.$

The number of integers which are divisible by 3 or 7 but not by 5 is $|(A_1 \cup A_3) \cap \overline{A_2}|$

$$|(A_1 \cup A_3) \cap \overline{A_2}| = |A_1 \cap \overline{A_2} \cap \overline{A_3}| + |\overline{A_1} \cap \overline{A_2} \cap A_3| + |A_1 \cap A_3 \cap \overline{A_2}|$$

$$= 58 + 19 + 9$$

$$= 86.$$

The number of integers which are divisible by 3 or 5 is $|A_1 \cup A_3|$

$$|A_1 \cup A_3| = |A_1| + |A_2| - |A_1 \cap A_3|$$

$$= 83 + 35 - 11$$

$$= 107.$$

Pigeon hole Principle and its applications:-

statement:- If m pigeons are assigned to n pigeon holes and $m > n$ then two or more pigeons occupies the same pigeon hole.

(or)

In otherwords, m pigeons occupies n pigeon holes and if $m > n$ then atleast one pigeon hole must contains 2 or more pigeons in it.

The extended pigeon hole Principle:- If m pigeons are assigned to n pigeon holes then one of the pigeon hole must contain $\left[\frac{m-1}{n}\right] + 1$ pigeons

Note:- To apply the Principle we must decide which objects will play the role of pigeon and which objects will play the role of pigeon hole.

Problems :-

① Show that if 8 people are in a room, atleast two of them have birthdays that occur on the same day of the week.

Sol:- Let $m=8$ and $n=7$.

i,e Treat the 8 people as the pigeons and 7 days of the week as the pigeon holes.

Here $8>7$, i,e $m>n$.

Then By using the pigeon hole principle we can say that atleast two people have birthday on the same day of the week.

② Find the minimum number of students in a class to be sure that 4 out of them are born on the same month.

Sol:- Here we have to find the number of pigeons m and using the extended Principle the 4 students were born on the same month, the number of months in a year $n=12$.

$$\left\lfloor\frac{m-1}{n}\right\rfloor+1=4$$

$$\left\lfloor\frac{m-1}{12}\right\rfloor+1=4$$

$$\frac{m-1}{12}=3$$

$$m-1=36$$

$$\boxed{m=37}$$

(3) Applying pigeon hole Principle show that if any 14 integers are selected from the set $S=\{1,2,3---25\}$ there are atleast two whose sum is 26. Also write a statement that generalizes this result.

Sol:- Given $S=\{1,2,3,---25\}$

Let the integer sets having the sum 26 as

$S_1=\{1,25\}$, $S_2=\{2,24\}$, $S_3=\{3,23\}$, $S_4=\{4,22\}$

$S_5=\{5,21\}$, $S_6=\{6,20\}$, $S_7=\{7,19\}$, $S_8=\{8,18\}$

$S_9=\{9,17\}$, $S_{10}=\{10,16\}$, $S_{11}=\{11,15\}$, $S_{12}=\{12,14\}$

$S_{13}=\{13,13\}$

$\therefore$

Treat 14 intege sets as the pigeons and $S_1,S_2,---S_{13}$ as the pigeon holes. Then $m=14, n=13$

$$\therefore \left\lfloor\frac{m-1}{n}\right\rfloor+1=\left\lfloor\frac{14-1}{13}\right\rfloor+1=2.$$

Here $m > n$ (i,e $14 > 13$) then by using pigeon hole Principle at least one integer set must be repeated twice.