Recurrence relations

* Generating functions of sequences :-

let $a_0; a_1, a_2, \ldots$ be a sequence of real numbers. The function $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

is called generating function.

Here $\{a_0, a_1, a_2, \ldots a_k, \ldots\}$ is called numeric function 'a'.

Example:

The generating function for the numeric in $\{3^0, 3, 3^2, \ldots 3^k, \ldots\}$ is $G(x) = \sum_{n=0}^{\infty} 3^n x^n$.

Problems :-

1. Find the generating function for the sequence $1, 1, 0, 1, 1, 1 \ldots$

The given sequence is $1, 1, 0, 1, 1, 1, \ldots$

let $a_0 = 1, a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 1, a_5 = 1, \ldots$

The generating function is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \cdots$$

$$= 1 + x + 0x^2 + 1 \cdot x^3 + \cdots$$

$$= (1 + x + x^2 + x^3 + \cdots) - x^2$$

$$= (1-x)^{-1} - x^2.$$

→ Find the generating function for the sequence 1, 1, 1, 3, 1, 1, ...

Given sequence is 1, 1, 1, 3, 1, 1, ...

let $a_0=1$, $a_1=1$, $a_2=1$, $a_3=3$, $a_4=1$, $a_5=1$, ...

The generating function is

$$G(x)=\sum_{n=0}^{\infty} a_n x^n$$

$$=a_0+a_1x+a_2x^2+\dots$$

$$=1+x+x^2+3x^3+x^4+x^5+\dots$$

$$=(1+x+x^2+x^3+x^4+\dots)+2x^3.$$

$$=\frac{1}{1-x}+2x^3$$

$$=\frac{2x^3-2x^4+1}{1-x}.$$

→ Determine the sequence generated by

(i) $f(x)=2e^x+3x^2$ (ii) $7e^{8x}-4e^{3x}$.

(i) Given $f(x)=2e^x+3x^2$

$$=2\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots\right]+3x^2$$

$$=2+2x+x^2+\frac{2x^3}{3!}+\dots+3x^2$$

$$=2+2x+4x^2+\frac{2x^3}{3!}+\dots$$

(ii) Given $f(x)=7e^{8x}-4e^{3x}$.

$$=7\left[1+8x+\frac{(8x)^2}{2!}+\frac{(8x)^3}{3!}+\dots\right]-4\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+\dots\right]$$

$$=7+56x+224x^2+\dots-4-12x-18x^2-\dots$$

$$=3+44x+206x^2+\dots$$

The sequence is $\{3, 44, 206, \dots\}$

→ Find the sequence generated by the following generating functions ③

① $(2x-3)^3$ ② $\frac{x^4}{1-x}$.

① Given $f(x) = (2x-3)^3$

By binomial theorem,

$(x+y)^n = n_{c_0} x^n y^0 + n_{c_1} x^{n-1} y + n_{c_2} x^{n-2} y^2 + \dots + n_{c_n} y^n$.

$(2x-3)^3 = 3_{c_0}(2x)^3 + 3_{c_1}(2x)^2(-3) + 3_{c_2}(2x)(-3)^2 + 3_{c_3}(-3)^3$.

$= 8x^3 - 36x^2 + 54x - 27$.

$= -27 + 54 - 36x^2 + 8x^3$.

The sequence is $\{-27, 54, -36, 8\}$

② Given $f(x) = \frac{x^4}{1-x}$

$= x^4[1 + x + x^2 + x^3 + \dots]$

$= x^4 + x^5 + x^6 + x^7 + \dots$

The sequence is

$\{0, 0, 0, 0, 1, 1, 1, \dots\}$.

→ Find the sequence for the function $\frac{1}{1-ax}$.

Given $f(x) = \frac{1}{1-ax} = (1-ax)^{-1}$

$= 1 + ax + (ax)^2 + (ax)^3 + (ax)^4 + \dots$

$= 1 + ax + a^2x^2 + a^3x^3 + a^4x^4 + \dots$

The sequence is $\{1, a, a^2, a^3, a^4, \dots\}$

* Calculating coefficients of generating functions:—

① $\binom{n}{r} = {}^{n}C_{r} = \dfrac{n!}{r!\,(n-r)!}$

② $1+x+x^2+\cdots+x^n = \dfrac{1-x^{n+1}}{1-x}$.

③ The coefficient of x^r in $(1+x+x^2+\cdots)^n$ is

$$\binom{n+r-1}{r}$$

(i.e) $[(1-x)^{-1}]^n$, the coefficient of x^r is $C(n+r-1, r)$

Problems:

→ Find the coefficient of x^{20} in $(x^3+x^4+x^5+\cdots)^5$.

Sol: Given $(x^3+x^4+x^5+\cdots)^5$

$$= (x^3)^5(1+x+x^2+\cdots)^5.$$

$$= x^{15}\left[(1-x)^{-1}\right]^5$$

$$= x^{15}(1-x)^{-5}.$$

$$= x^{15}\sum_{r=0}^{n} C(n+r-1, r)\, x^r$$

$$= x^{15}\sum_{r=0}^{5} C(5+r-1, r)\, x^r$$

$$= x^{15}\sum_{r=0}^{5} C(4+r, r)\, x^r.$$

for $r=5$, we get x^{20} term.

$$= x^{15}C(5+5-1, 5)\,x^5 = C(9,5)\,x^{20}.$$

The co-efficient of x^{20} is $= C(9,5) = {}^{9}C_{5}$

$= 126$.

→ Find the co-efficient of x^5 in $(1-2x)^{-7}$.

we have $(1-2x)^{-7} = \sum_{r=0}^{\infty} \binom{7+r-1}{r} (2x)^r$

$$= \sum_{r=0}^{\infty} \binom{6+r}{r} (2x)^r.$$

for $r=5$, we get x^5 term.

$$= \binom{6+5}{5} (2x)^5$$

$$= {}^{11}C_5 (2x)^5 = \frac{11!}{5!\,6!} 2^5 x^5.$$

The coefficient of $x^5 = \frac{11!}{5!\,6!} 2^5$

$$= 14,784.$$

→ Find the coefficient of x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$.

Given $(x^2+x^3+x^4+x^5+x^6)^5$

$$= [x^2(1+x+x^2+x^3+x^4)]^5.$$

$$= x^{10}(1+x+x^2+x^3+x^4)^5.$$

$$= x^{10}\left[\frac{1-x^{4+1}}{1-x}\right]^5 = x^{10}(1-x^5)^5(1-x)^{-5}.$$

$$= x^{10}[{}^5C_0 - {}^5C_1 x^5 + {}^5C_2 x^{10} - {}^5C_3 x^{15} + {}^5C_4 x^{20} - {}^5C_5 x^{25}]$$

$$\left[\sum_{r=0}^{\infty} C(5-1+r, r)\, x^r\right]$$

$$= x^{10}[{}^5C_0 - {}^5C_1 x^5 + {}^5C_2 x^{10} - {}^5C_3 x^{15} + {}^5C_4 x^{20} - {}^5C_5 x^{25}]$$

$$[{}^4C_0 x^0 + {}^5C_1 x^1 + {}^6C_2 x^2 + {}^7C_3 x^3 + \cdots]$$

$$= x^{10}\left[{}^5C_0\,{}^4C_0 + {}^5C_0\,{}^5C_1 x + {}^5C_0\,{}^6C_2 x^2 + \cdots\right.$$

$$\left. - {}^5C_1\,{}^4C_0 x^5 - {}^5C_1\,{}^5C_1 x^6 - \cdots + {}^5C_2 x^{10} + {}^5C_2\,{}^5C_1 x^{11} + \cdots\right]$$

The coefficient of x^{20} is

$${}^5C_0 \times {}^4C_{10} - {}^5C_1\,{}^{9}C_4 + {}^5C_2\,{}^4C_0$$

$$= {}^{14}C_4 - 5 \times {}^9C_4 + {}^5C_2$$

$$= 381.$$

Recurrence relations:—

Recurrence relation is an equation that express for any integer $n \geq 2$, the n^{th} term of the sequence a_n interms of one or more a_i's, $i < n$

one can carryout a setp by step computation to determine a_n from $a_{n-1}, a_{n-2}, \cdots$ provided that the values of the function at one or more points are given. The given values are called initial conditions or boundary Conditions of the recurrence relation.

Example: The numeric function $(5, 8, 11, 14, \cdots)$ is defined by the recurrence $a_n = a_{n-1} + 3, n \geq 1$ with the initial condition $a_0 = 5$.

Order of recurrence relation:— The order of the recurrence relation is the difference between the largest and smallest subscript appearing in the relation.

Example: ① $a_n = -3a_{n-1}$ is a R.R then the order is $n-(n-1)=1$

② $a_{n+2} = -a_{n+1} = 2a_n$, then the order is $n+2-n = 2$.

linear recurrence relation with constant Co-efficient :—

A recurrence relation of the form

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$$

where c_i s are constant, is called a linear recurrence relation of k^{th} order provided both c_0 and c_k are non zero.

* If $f(n)$ is zero (ie $f(n)=0$) then the relation is known as linear homogeneous equation.

* If $f(n)$ is not zero (ie $f(n)\neq 0$) then the relation is known as linear non homogeneous equations.

Solution of linear recurrence relation:—

An explicit formula which satisfy the R.R with initial condition is called a solution of R.R.

1. Iteration (or) substitution method.
2. Characteristic roots.
3. Generating functions.

Example ① The R.R : $a_n = 2a_{n-1}$ is a linear homogeneous relation with constant coefficients of degree 1.

② The R.R $a_n - a_{n-1} = 3$ is a linear non homogeneous relation with constant coefficient of degree 1.

* Solving recurrence relation by substitution and generating functions:—

Solving recurrence relation by substitution method:—

In this method R.R for a_n is used repeatedly to solve for a general expression for a_n interms of n.

Problems:

** Solve the R.R $a_n = a_{n-1} + 2$, $n \geq 2$ subject to initial condition $a_1 = 3$.

Given $a_n = a_{n-1} + 2 \rightarrow ①$

Replace n by n-1 in ① we obtain

$a_{n-1} = a_{n-2} + 2$

substituting a_{n-1} in ① we get

① ⇒ $a_n = a_{n-2} + 2 + 2$

$= a_{n-2} + 4 \rightarrow ②$

Replace n by $n-2$, in ①, again

$a_{n-2} = a_{n-3} + 2$

Sub a_{n-2} in ②, we get.

$a_n = a_{n-3} + 2 + 2 \cdot 2$.

$a_n = a_{n-3} + 4 \cdot 2$

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$a_n = a_{n-k} + k \cdot 2$

Put $k = n-1 \Rightarrow a_n = a_{n-(n-1)} + (n-1)2$.

$= a_1 + 2(n-1)$

$= 3 + 2n - 2$

$a_n = 2n+1$

→ Solve $a_k = k(a_{k-1})^2$, $k \geq 1$, $a_0 = 1$.

Given $a_k = k(a_{k-1})^2$, $k \geq 1$, $a_0 = 1$

by substitution method, we have

Put $k=1 \Rightarrow a_1 = 1 \cdot (a_0^2) = 1$

$k=2 \Rightarrow a_2 = 2(a_1^2) = 2(1)^2 = 2$

$k=3 \Rightarrow a_3 = 3(a_2)^2 = 3 \cdot 2^2$

$k=4 \Rightarrow a_4 = 4(a_3)^2 = 4 \cdot 3^2 \cdot 2^4$

$k=5 \Rightarrow a_5 = 5 \cdot 4^2 \cdot 3^4 \cdot 2^8$

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Put $k=n \Rightarrow a_n = n(n-1)^2 (n-2)^4 (n-3)^8 \cdots$

→ Solve $a_n = a_{n-1} + f(n)$ for $n \geq 1$ by using substitution method.

Given $a_n = a_{n-1} + f(n)$ for $n \geq 1$

By substition method, we have

Put $n=1 \Rightarrow a_1 = a_0 + f(1)$

Put $n=2 \Rightarrow a_2 = a_1 + f(2)$

$$a_2 = a_0 + f(1) + f(2)$$

Put $n=3 \Rightarrow a_3 = a_2 + f(3)$

$$= a_0 + f(1) + f(2) + f(3)$$

$$- - - - - - - - -$$

$$a_n = a_0 + f(1) + f(2) + \cdots + f(n)$$

$$a_n = a_0 + \sum_{k=1}^{n} f(k)$$

Solving recurrence relations by generating functions method: —

Suppose the second order recurrence relation of homogeneous is

$a_n + A a_{n-1} + B a_{n-2} = 0$ for $n \geq 2$, or equivalently

$a_{n+2} + A a_{n+1} + B a_n = 0$ for $n \geq 0 \rightarrow (1)$

let us multiply both sides of the relation (1) by x^{n+2} and take the sum of all the resulting relations that correspond to $n = 0, 1, 2, 3, \ldots$ then we obtain

$$\sum_{n=0}^{\infty} a_{n+2}x^{n+2} + A\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + B\sum_{n=0}^{\infty} a_n x^{n+2} = 0$$

$$\sum_{n=2}^{\infty} a_n x^n + Ax\sum_{n=1}^{\infty} a_n x^n + Bx^2\sum_{n=0}^{\infty} a_n x^n = 0.$$

(or)

$$\sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x + Ax\sum_{n=0}^{\infty} a_n x^n - Axa_0 + Bx^2\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_n x^n(1+Ax+Bx^2) - a_0 - x(a_1+a_0A) = 0$$

$$G(x)(1+Ax+Bx^2) = a_0 + (a_1+a_0A)x.$$

$$G(x) = \frac{a_0 + (a_1+a_0A)x}{1+Ax+Bx^2}$$

where $G(x)$ is a generating function

Next substitute a_0, a_1 values in $G(x)$ and then resolve in partial fractions, the coefficient of x^n in RHS of $G(x)$ determines a_n.

Problems:-

→ Solve the recurrence relation using generating functions $a_n - 9a_{n-1} + 20a_{n-2} = 0$ for $n \geq 2$ and $a_0 = -3$, $a_1 = -10$.

Given $a_n - 9a_{n-1} + 20a_{n-2} = 0$ for $n \geq 2 \rightarrow ①$

Also given $a_0 = -3$, $a_1 = -10$.

let $G(x) = \sum_{n=0}^{\infty} a_n x^n \longrightarrow (2)$

$= a_0 + a_1 x + a_2 x^2 + \dots$

Multiplying the given equation by x^n and taking $\sum_{n=2}^{\infty}$

$$\sum_{n=2}^{\infty} a_n x^n - 9 \sum_{n=2}^{\infty} a_{n-1} x^n + 20 \sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$(a_2x^2 + a_3x^3 + \dots) - 9(a_1x^2 + a_2x^3 + \dots) + 20(a_0x^2 + a_1x^3 + \dots) = 0.$$

$$(a_0 + a_1x + a_2x^2 + \dots) - a_0 - a_1x - 9x(a_0 + a_1x + a_2x^2 \dots - a_0) + 20x^2(a_0 + a_1x + a_2x^2 + \dots) = 0.$$

$$G(x) - a_0 - a_1x - 9x(G(x) - a_0) + 20x^2 G(x) = 0$$

$$G(x)[1 - 9x + 20x^2] - a_0 - a_1x - 9x a_0 = 0.$$

$$G(x)[1 - 9x + 20x^2] + 3 + 10x - 9x(-3) = 0.$$

$$G(x)[1 - 9x + 20x^2] = -3 + 17x.$$

$$G(x) = \frac{-3 + 17x}{1 - 9x + 20x^2} \longrightarrow (3)$$

$$\frac{-3 + 17x}{1 - 9x + 20x^2} = \frac{-3 + 17x}{(1-5x)(1-4x)} = \frac{A}{1-5x} + \frac{B}{1-4x}.$$

$$-3 + 17x = A(1-4x) + B(1-5x) \longrightarrow (4)$$

Put $x = \frac{1}{4}$ in (4) we get

$$-3 + 17\left(\tfrac{1}{4}\right) = A\left(1 - 4\left(\tfrac{1}{4}\right)\right) + B\left(1 - \tfrac{5}{4}\right)$$

$$\boxed{B = -5.}$$

Put $x = \frac{1}{5}$ in ④, we get.

$$-3+17\left(\frac{1}{5}\right) = A\left(1-\frac{4}{5}\right) + B\left(1-5\left(\frac{1}{5}\right)\right)$$

$$-3+\frac{17}{5} = A\left(\frac{1}{5}\right)$$

$$\boxed{A=2}$$

from ③, $G(x) = \frac{2}{1-5x} - \frac{5}{1-4x}$.

$$= 2(1-5x)^{-1} - 5(1-4x)^{-1}$$

$$G(x) = 2\left(1+5x+(5x)^2+\dots\right] - 5\left[1+4x+(4x)^2+\dots\right]$$

$$= 2\sum_{n=0}^{\infty}(5x)^n - 5\sum_{n=0}^{\infty}(4x)^n.$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[2(5^n) - 5(4^n)\right]x^n.$$

$$a_n = 2(5^n) - 5(4^n).$$

→ Solve $a_n - 5a_{n-1} + 6a_{n-2} = 2^n$, $n \geq 2$ with initial conditions $a_0 = 1$, $a_1 = 1$ using generating functions.

Given $a_n - 5a_{n-1} + 6a_{n-2} = 2^n \rightarrow ①$

let $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

Multiplying x^n on both sides in ① and taking summation $\sum_{n=2}^{\infty}$.

$$\sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1}x^n + 6\sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=2}^{\infty} 2^n x^n$$

$$\left[a_2x^2 + a_3x^3 + \cdots\right] + a_0 + a_1x - a_0 - a_1x - 5\left[a_1x^2 + a_2x^2 + \cdots\right]$$

$$+ 6\left[a_0x^2 + a_1x^3 + \cdots\right] = 2^2x^2 \sum_{n=2}^{\infty} 2^{n-2} x^{n-2}$$

$$G(x) - a_0 - a_1x - 5x\left[G(x) - a_0\right] + 6x^2 G(x) = \frac{4x^2}{1-2x}$$

$$G(x)\left[1-5x+6x^2\right] - a_0 - a_1x + 5a_0x = \frac{4x^2}{1-2x}.$$

$$G(x)\left[1-5x+6x^2\right] = \frac{4x^2}{1-2x} + a_0 + a_1x - 5a_0x$$

$$G(x) = \frac{4x^2}{(1-2x)(1-5x+6x^2)} + \frac{1-4x}{(1-2x)(1-3x)} \quad \left[\because \begin{array}{l} a_0=1 \\ a_1=1 \end{array}\right]$$

$$G(x) = \frac{4x^2}{(1-2x)^2(1-3x)} + \frac{1-4x}{(1-2x)(1-3x)}$$

$$= \frac{4x^2(1) + (1-4x)(1-2x)}{(1-2x)^2(1-3x)}$$

$$= \frac{4x^2 + 1 - 2x - 4x + 8x^2}{(1-2x)^2(1-3x)}$$

$$= \frac{12x^2 - 6x + 1}{(1-2x)^2(1-3x)}$$

let, $\dfrac{12x^2-6x+1}{(1-2x)^2(1-3x)} = \dfrac{A}{1-2x} + \dfrac{B}{(1-2x)^2} + \dfrac{C}{1-3x}.$

$$12x^2 - 6x + 1 = A(1-2x)(1-3x) + B(1-3x) + (1-2x)^2$$

Put $x = \frac{1}{2} \Rightarrow \frac{12}{4} - \frac{6}{2} + 1 = B\left(1 - \frac{3}{2}\right)$

$$B\left(\frac{-1}{2}\right) = 1$$

$\boxed{B = -2}$

Put $x = \frac{1}{3} \Rightarrow \frac{12}{9} - \frac{6}{3} + 1 = C\left(1 - \frac{2}{3}\right)^2$

$$C\left(\frac{1}{9}\right) = \frac{4}{3} - 2 + 1$$

$$C = \frac{4 - 6 + 3}{3} \cdot 9$$

$$\boxed{C = 3}.$$

Put $x = 0 \Rightarrow A + B + C = 1$

$$A = 1 - B - C$$

$$A = 1 + 2 - 3$$

$$\boxed{A = 0}$$

$$G(x) = \frac{0}{1-2x} + \frac{-2}{(1-2x)^2} + \frac{3}{1-3x}.$$

$$G(x) = -2\,(1-2x)^{-2} + 3(1-3x)^{-1}$$

$$G(x) = -2\left[1 + 2(2x) + 3(2x)^2 + \cdots\right] + 3\left[1 + 3x + (3x)^2 + \cdots\right]$$

$$\sum_{n=0}^{\infty} a_n x^n = -2 \sum_{n=0}^{\infty} (n+1) 2^n x^n + 3 \sum_{n=0}^{\infty} 3^n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \left[(-2)(n+1) 2^n + 3 \cdot 3^n\right] x^n$$

$$a_n = -(n+1)\, 2^{n+1} + 3^{n+1}.$$

→ Using generating function solve $a_n = 3a_{n+1} + 2$, $a_0 = 1$.

Given $a_n = 3a_{n+1} + 2 \rightarrow ①$

let $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$

Multiplying x^n on bothsides in ① and taking summation $\sum_{n=0}^{\infty}$.

$$\sum_{n=0}^{\infty} a_n x^n - 3\sum_{n=0}^{\infty} a_{n+1} x^n = 2\sum_{n=0}^{\infty} x^n.$$

$$G(x) - 3\left[a_1 + a_2 x + a_3 x^2 + \cdots\right] = 2\left[1 + x + x^2 + \cdots\right]$$

$$G(x) = \frac{3}{x}\left[a_1 x + a_2 x^2 + a_3 x^3 + \cdots\right] = 2(1-x)^{-1}.$$

$$G(x) = \frac{3}{x}\left[G(x) - a_0\right] = \frac{2}{1-x}.$$

$$G(x) - \frac{3}{x}G(x) + \frac{3a_0}{x} = \frac{2}{1-x}.$$

$$G(x)\left[1 - \frac{3}{x}\right] = \frac{2}{1-x} - \frac{3}{x} \qquad [\because a_0 = 1]$$

$$G(x)\left[1 - \frac{3}{x}\right] = \frac{2x - 3 + 3x}{x(1-x)}$$

$$G(x) = \frac{5x-3}{x(1-x)} \cdot \frac{x}{x-3}.$$

$$= \frac{3-5x}{(x-1)(x-3)}.$$

Now $\dfrac{3-5x}{(x-1)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-3}$.

$$3 - 5x = A(x-3) + B(x-1)$$

Put $x=1 \Rightarrow 3-5 = A(1-3)$

$$-2 = -2A$$

$$\boxed{A=1}$$

Put $x=3 \Rightarrow 3-15 = B(3-1)$

$$-12 = 2B$$

$$\boxed{B=-6}.$$

$$G(x) = \frac{1}{x-1} - \frac{6}{x-3}$$

$$G(x) = -(1-x)^{-1} - \frac{6}{-3}\left(1-\frac{x}{3}\right)^{-1}$$

$$= -(1-x)^{-1} + 2\left(1-\frac{x}{3}\right)^{-1}$$

$$= -(1+x+x^2+\cdots) + 2\left(1+\frac{x}{3}+\left(\frac{x}{3}\right)^2+\cdots\right)$$

$$= -\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n.$$

$$= -\sum_{n=0}^{\infty} x^n + 2\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n x^n.$$

$$= \sum_{n=0}^{\infty}\left[-1+\frac{2}{3^n}\right]x^n$$

$$\therefore\ a_n = -1 + 2\left[\frac{1}{3}\right]^n.$$

* The method of Characteristic roots :—

In this method the solution of a recurrence relation is obtained as the sum of two parts.

① The homogeneous solution, which satisfy the recurrence relation when the RHS of the relation is zero (i-e $f(n)=0$)

② The particular solution, which satisfies the relation with $f(n)$ on the right hand side

The general solution is given by

$$\boxed{a_n = a_n^{(h)} + a_n^{(P)}}$$

where $a_n^{(h)}$ = Homogeneous solution

$a_n^{(P)}$ = Particular solution.

Note: If RHS of the given recurrence relation is zero then the general solution is $a_n = a_n^{(h)}$.

Solution of linear homogeneous equations :—

let $a_n = r^n$ is a solution of the recurrence relation $C_0 a^n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_K a_{n-K} = 0$.

$$c_0 r^n + c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k} = 0$$

Dividing both sides by r^{n-k}, we get

$$r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k = 0.$$

The above equation is called as the characteristic equation of the recurrence relation of k^{th} order and it has k roots called characteristic roots.

① Distinct roots :— If the characteristic roots are distinct roots, say $r_1, r_2, \cdots r_k$ then the general solution for homogeneous equation is

$$a_n = a_n^{(h)} = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n + \cdots + b_k r_k^n.$$

where $b_1, b_2, b_3, \cdots b_k$ are constants. which may be choosen to satisfy any initial conditions.

Example: characteristic roots $r = 2, 3, 4$ then the solution is

$$a_n = a_n^{(h)} = b_1 2^n + b_2 3^n + b_3 4^n.$$

② Multiple roots :— If 'r' is a root of the characteristic equation of m^{th} order with multiplicity m (repeated m times) then the general solution is

$$a_n = a_n^{(h)} = (b_1 + b_2 n + b_3 n^2 + \dots + n^{m-1} b_m) r^n.$$

Example: If the characteristic equation is $(r-4)^3 = 0$ then the roots are $r = 4, 4, 4$ (repeated 3 times) then the general solution is $a_n = (b_1 + n b_2 + n^2 b_3) 4^n$.

(3) Mixed roots :— If some roots of a characteristic equation ~~[illegible]~~ ~~[illegible]~~ are distinct and some roots are equal.

Example: If the characteristic equation is

$(r-2)(r-4)(r-3)^3 = 0$

then the roots are $r = 2, 4, 3, 3, 3$.

$$a_n = b_1 2^n + b_2 4^n + (b_3 + n b_4 + n^2 b_5) 3^n.$$

Problems:

Solve $a_n - a_{n-1} - 2a_{n-2} = 0$

Given $a_n - a_{n-1} - 2a_{n-2} = 0 \rightarrow (1)$

let $a_n = r^n$ is a solution of (1)

$$r^n - r^{n-1} - 2r^{n-2} = 0$$

Dividing r^{n-2} on both sides.

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0$$
$$r = -1, 2.$$

The roots are distinct

General solution is $a_n = b_1 2^n + b_2 (-1)^n$

→ Solve $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$. with the initial conditions $a_0 = 0$, $a_1 = 1$

Given $a_n = a_{n-1} + 2a_{n-2}$, $n \geq 2$.

$\Rightarrow a_n - a_{n-1} - 2a_{n-2} = 0 \rightarrow (1)$

This is the 2nd order linear homogeneous recurrence relation with constant coefficients

let $a_n = r^n$ is a solution of (1).

then $r^n - r^{n-1} - 2r^{n-2} = 0$

Dividing r^{n-2} on both sides.

$$r^2 - r - 2 = 0$$
$$(r-2)(r+1) = 0.$$
$$r = 2, -1.$$

∴ The roots are distinct.

General solution is

$$a_n = b_1 2^n + b_2 (-1)^n \rightarrow (2).$$

Also given $a_0 = 0$, $a_1 = 1$.

Put $n=0$ in (2) $a_0 = b_1(2^0) + b_2(-1)^0$.

$$b_1 + b_2 = 0 \rightarrow (3).$$

Put $n=1$ in (2), $a_1 = b_1(2^1) + b_2(-1)^1$

$$2b_1 - b_2 = 1 \rightarrow (4).$$

solving (3) & (4), $b_1 + b_2 = 0$

$$2b_1 - b_2 = 01$$

$$3b_1 \quad = 1$$

$$\boxed{b_1 = \frac{1}{3}}$$

from (3) $b_2 = -b_1$

$$\boxed{b_2 = -\frac{1}{3}.}$$

The general solution is

$$a_n = \left(\frac{1}{3}\right)2^n + \left(\frac{1}{3}\right)(-1)^n.$$

→ Solve $a_n = 2a_{n-1} - a_{n-2}$ with initial conditions $a_1 = 1.5$, $a_2 = 3$.

Given $a_n = 2a_{n-1} - a_{n-2}$

$$a_n - 2a_{n-1} + a_{n-2} = 0 \rightarrow (2)$$

let $a_n = r^n$ is a solution of (1), then

$$r^n - 2r^{n-1} + r^{n-2} = 0$$

Dividing r^{n-2} on both sides, we get

$$r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0$$

$r = 1, 1$.

The roots are equal.

The general solution is

$$a_n = (b_1 + b_2 n)(1)^n \rightarrow ②$$

Also given $a_0 = 1.5$, $a_1 = 3$.

put $n = 1$, in ②, $a_1 = (b_1 + b_2)(1)^1$

$$b_1 + b_2 = 1.5 \rightarrow ③$$

Put $n = 2$ in ②, $a_2 = (b_1 + 2b_2)(1)^2$.

$$b_1 + 2b_2 = 3 \rightarrow ④$$

By solving ③ & ④

$$b_1 + b_2 = 1.5$$
$$b_1 + 2b_2 = 3$$
$$-b_2 = -1.5$$
$$\boxed{b_2 = 1.5}$$

from ③ $b_1 = 1.5 - b_2$

$$b_1 = 0$$

the general solution is

$$a_1 = 1.5n.$$

→ Solve $a_n = 3a_{n-1} - a_{n-2}$ with initial conditions $a_1 = -2$ and $a_2 = 4$.

Given $a_n = 3a_{n-1} - a_{n-2}$

$$a_n - 3a_{n-1} + a_{n-2} = 0 \rightarrow (1)$$

let $a_n = r^n$ is a solution of (1) then

$$r^n - 3r^{n-1} + r^{n-2} = 0$$

Dividing r^{n-2} on both sides, we get

$$r^2 - 3r + 1 = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$r = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}$$

$$r = \frac{3 \pm \sqrt{5}}{2} = \frac{3+\sqrt{5}}{2}, \frac{3-\sqrt{5}}{2}.$$

The roots are distinct.

The general solution is $a_n = b_1\left(\frac{3+\sqrt{5}}{2}\right)^n + b_2\left(\frac{3-\sqrt{5}}{2}\right)^n \rightarrow (2)$

Also given $a_1 = -2$ and $a_2 = 4$.

Put $n=1$ in (2), $a_1 = b_1\left(\frac{3+\sqrt{5}}{2}\right) + b_2\left(\frac{3-\sqrt{5}}{2}\right)$

$$\frac{1}{2}\left[b_1(3+\sqrt{5}) + b_2(3-\sqrt{5})\right] = -2$$

$$3(b_1+b_2) + \sqrt{5}(b_1 - b_2) = -4 \rightarrow (3)$$

Put $n=2$ in (2), $a_2 = b_1\left(\frac{3+\sqrt{5}}{2}\right)^2 + b_2\left(\frac{3-\sqrt{5}}{2}\right)^2$

$$\frac{b_1}{4}(9+5+6\sqrt{5}) + \frac{b_2}{4}(9+5-6\sqrt{5}) = 4.$$

$$14(b_1+b_2) + 6\sqrt{5}(b_1 - b_2) = 16.$$

$$7(b_1+b_2) + 3\sqrt{5}(b_1-b_2) = 8 \rightarrow (4).$$

Solving (3) & (4) we get

$$(3)\times 3 \Rightarrow \quad 9(b_1+b_2)+3\sqrt{5}(b_1-b_2)=-12$$
$$7(b_1+b_2)+3\sqrt{5}(b_1-b_2)=8$$

$$2(b_1+b_2) = -20$$
$$\boxed{b_1+b_2=-10} \rightarrow (5)$$

Again solving (3) & (4) we get

$$(3)\times 7 \Rightarrow \quad 21(b_1+b_2)+7\sqrt{5}(b_1-b_2)=-28$$
$$(4)\times 3 \Rightarrow \quad 21(b_1+b_2)+9\sqrt{5}(b_1-b_2)=24$$

$$-2\sqrt{5}(b_1-b_2)=-52$$
$$\sqrt{5}(b_1-b_2)=26.$$
$$b_1-b_2=\frac{26}{\sqrt{5}} \rightarrow (6)$$

solve (5) & (6) $\quad b_1+b_2=-10$

$$b_1-b_2=\frac{26}{\sqrt{5}}$$

$$2b_1=-10+\frac{26}{\sqrt{5}}$$
$$b_1=\frac{13-5\sqrt{5}}{\sqrt{5}}.$$

from (5), $b_2=-10-b_1=-10-\left(\frac{13-5\sqrt{5}}{\sqrt{5}}\right)$

$$b_2=-\frac{(13+5\sqrt{5})}{\sqrt{5}}.$$

The general solution of recurrence relation is $a_n=\left(\frac{13-5\sqrt{5}}{\sqrt{5}}\right)\left(\frac{3+\sqrt{5}}{2}\right)^n-\left(\frac{13+5\sqrt{5}}{\sqrt{5}}\right)\left(\frac{3-\sqrt{5}}{2}\right)^n.$

$$a_n=\frac{1}{\sqrt{5}}\left[(13-5\sqrt{5})\left(\frac{3+\sqrt{5}}{2}\right)^n-(13+5\sqrt{5})\left(\frac{3-\sqrt{5}}{2}\right)^n\right]$$

→ Solve $a_n = a_{n-1} + 2a_{n-2}$, $n>2$ with initial conditions $a_0 = 2$, $a_1 = 1$

Given $a_n = a_{n-1} + 2a_{n-2}$

$$a_n - a_{n-1} - 2a_{n-2} = 0 \rightarrow (1)$$

let $a_n = r^n$ is a solution of (1) then

$$r^n - r^{n-1} - 2r^{n-2} = 0.$$

Dividing r^{n-2} we get

$$r^2 - r - 2 = 0$$

$$r = 2, -1.$$

The roots are distinct.

General solution is $a_n = b_1 2^n + b_2(-1)^n \rightarrow (2)$

Also given that $a_0 = 2$, $a_1 = 1$.

Put $n=0 \Rightarrow a_0 = b_1(2^0) + b_2(-1)^0$

$$2 = b_1 + b_2(1)$$

$$b_1 + b_2 = 2$$

Put $n=1 \Rightarrow a_1 = b_1(2^1) + b_2(-1)$

$$2b_1 - b_2 = 1 \rightarrow (4).$$

by solving (3) and (4) we get

$$b_1 = 1, \quad b_2 = 1$$

The general solution is

$$a_n = 2^n + (-1)^n.$$

Solutions of inhomogeneous recurrence relation:-

The general solution of a linear non-homogeneous recurrence relation.

$$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = f(n)$$

where $C_0, C_1, C_2, \ldots C_k$ are constants, is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

There is no general method for finding the particular solution of a R.R for every function of $f(n)$. So the solution can be obtained by the method of inspection.

In this method for finding a particular solution we use trail solution in each case are shown in the following table.

$f(n)$	Trail solution
① If $f(n)$ = constant	① $a_n^{(P)} = A_0$
② If $f(n)$ = Polynomial of degree m	② $a_n^{(P)} = A_0 + A_1 n + A_2 n^2 + \ldots A_m n^m$.
③ If $f(n) = b^n$ (b is not a root)	③ $a_n^{(P)} = A_0 b^n$.
④ If $f(n) = b^n$ (b is a root of multiplicity m)	④ $a_n^{(P)} = A_0 n^m b^n$.
⑤ If $f(n) = b^n \times$ polynomial (b is a root of multiplicity m)	⑤ $a_n^{(P)} = n^m (A_0 + A_1 n + \ldots)$

Problems:

→ Solve $a_{n+2} - 5a_{n+1} + 6a_n = 2$ with the initial condition $a_0 = 1, a_1 = -1$

Given $a_{n+2} - 5a_{n+1} + 6a_n = 2 \rightarrow (1)$

Given equation is a linear homogeneous eq'n.

So General solution is

$$a_n = a_n^{(h)} + a_n^{(p)} \rightarrow (2)$$

To find $a_n^{(h)}$:-

Now let $a_{n+2} - 5a_{n+1} + 6a_n = 0 \rightarrow (3)$

let $a_n = r^n$ is a solution of (3) then

$$r^{n+2} - 5r^{n+1} + 6r^n = 0$$

Dividing r^n, we get

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

$$r = 2, 3$$

The roots are distinct.

$$a_n^{(h)} = b_1 2^n + b_2 3^n \rightarrow (4).$$

To find $a_n^{(p)}$:- The RHS of the given relation is constant (i.e) $f(n) =$ constant.

So $a_n^{(p)} = A_0 \rightarrow (5)$.

Putting $a_n^{(p)}$ for a_n in the given relation

$$A_0 - 5A_0 + 6A_0 = 2$$

$$2A_0 = 2$$

$$A_0 = 1$$

Substitute Ao value in (5) we get.

$$a_n^{(P)} = 1 \rightarrow (6)$$

The general solution is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = b_1 2^n + b_2 3^n + 1 \rightarrow (7)$$

Also given that $a_0 = 1$, $a_1 = -1$

Put $n=0$ in (7) $\Rightarrow a_0 = b_1 2^0 + b_2 3^0 + 1$

$$b_1 + b_2 + 1 = 1$$

$$b_1 + b_2 = 1 \rightarrow (8)$$

Put $n=1$ in (7) $\Rightarrow a_1 = b_1 2^1 + b_2 3^1 + 1$

$$2b_1 + 3b_2 + 1 = 0$$

$$2b_1 + 3b_2 = -2 \rightarrow (9)$$

by solving (8) & (9)

(8) $\times 2 \Rightarrow 2b_1 + 2b_2 = 0$

(9) $\Rightarrow 2b_1 + 3b_2 = -2$

$$b_2 = -2$$

From (8) $b_1 = -b_2$

$$b_1 = 2.$$

Substituting b_1 and b_2 values in (7)

we get

$$a_n = 2 \cdot 2^n + (-2) 3^n + 1$$

$$a_n = 2^{n+1} - 2 \cdot (3^n) + 1$$

→ Solve $a_n - 5a_{n-1} + 6a_{n-2} = 1$.

Given $a_n - 5a_{n-1} + 6a_{n-2} = 1 \rightarrow (1)$

Given eq'n is a nonhomogeneous equation.

So that, the general solution is

$$a_n = a_n^{(p)} + a_n^{(h)} \rightarrow (2)$$

To find $a_n^{(h)}$:- Now let $a_n - 5a_{n-1} + 6a_{n-2} = 0 \rightarrow (3)$

let $a_n = r^n$ is a solution of (3), then

$$r^n - 5r^{n-1} + 6r^{n-2} = 0$$

Dividing r^{n-2}, we get

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

$r = 2, 3$. The roots are distinct.

$$a_n^{(h)} = b_1 2^n + b_2 3^n \rightarrow (4)$$

To find $a_n^{(p)}$:- The RHS of the given relation is a constant, (i.e) $f(n)=$ constant.

So $a_n^{(p)} = A_0 \rightarrow (5)$

Substituting a_n is same as A_0 in the given R. R

$$a_n = A_0 \quad a_{n-1} = A_0 \quad a_{n-2} = A_0$$

then $A_0 - 5A_0 + 6A_0 = 1$

$$2A_0 = 1$$

$$A_0 = \frac{1}{2}$$

Putting A_0 value in (5) $a_n^{(p)} = \frac{1}{2}$

The general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$$a_n = b_1 2^n + b_2 3^n + \frac{1}{2}.$$

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(33)

Solve the R.R $a_{n+2} - 2a_{n+1} + a_n = 2^n$ with initial condition $a_0 = 2$, $a_1 = 1$.

Given $a_{n+2} - 2a_{n+1} + a_n = 2^n \rightarrow (1)$

Given equation is a non homogeneous eq'n.

So the general solution is

$$a_n = a_n^{(h)} + a_n^{(P)} \rightarrow (2)$$

<u>To find $a_n^{(h)}$</u> :—

Now let $a_{n+2} - 2a_{n+1} + a_n = 0 \rightarrow (3)$

let $a_n = r^n$ is a solution of (3) then

$$r^{n+2} - 2r^{n+1} + r^n = 0$$

Dividing r^n we get

$$r^2 - 2r + 1 = 0$$

$$(r-1)(r-1) = 0$$

$r = 1, 1$, The roots are equal

$$a_n^{(h)} = (b_1 + b_2 n)(1)^n \rightarrow (4)$$

<u>To find $a_n^{(P)}$</u> : The RHS of the given relation is 2^n (i.e) $f(n) = b^n$)

$$a_n^{(P)} = A_0 2^n \rightarrow (5)$$

substituting a_n is same as $A_0 2^n$ in eqn (1)

$$a_n = A_0 2^n, \quad a_{n+1} = A_0 2^{n+1}, \quad a_{n+2} = A_0 2^{n+2}$$

from (1), $A_0 2^{n+2} - 2A_0 2^{n+1} + A_0 2^n = 2^n$.

$$2^n [4A_0 - 4A_0 + A_0] = 2^n$$

$$A_0 = 1$$

Substituting A_0 value in (5), we get

$$a_n^{(P)} = 2^n$$

The general solution for a_n is

$$a_n = a_n^{(h)} + a_n^{(p)}$$

$$a_n = (b_1 + b_2 n) + 2^n \rightarrow (6)$$

Also given that $a_0 = 2,\ a_1 = 1$.

Put $n=0$ in (6) $\Rightarrow a_0 = b_1 + b_2(0) + 2^0$

$$b_1 + 1 = 2$$

$$b_1 = 1$$

Put $n=1$ in (6) $\Rightarrow a_1 = (b_1 + b_2) + 2^1$

$$b_1 + b_2 + 2 = 1$$

$$b_2 = -1 - b_1$$

$$b_2 = -2.$$

Substituting b_1, b_2 values in (6)

$$a_n = 1 - 2n + 2^n.$$

→ Solve $a_n - 7a_{n-1} + 10a_{n-2} = 4^n$.

Given $a_n - 7a_{n-1} + 10a_{n-2} = 4^n \rightarrow (1)$

Given equation is non-homogeneous eq'n.

So the general solution is $a_n = a_n^{(h)} + a_n^{(p)} \rightarrow (2)$

To find $a_n^{(h)}$:-

Now let $a_n - 7a_{n-1} + 10a_{n-2} = 0 \rightarrow (3)$.

let $a_n = r^n$ is a solution of (3), then

$$r^n - 7r^{n-1} + 10r^{n-2} = 0$$

Dividing r^{n-2}, we get

$$r^2 - 7r + 10 = 0$$

$$(r-2)(r-5) = 0$$

$r = 2, 5$.

The roots are distinct.

$$a_n^{(h)} = b_1 2^n + b_2 5^n \rightarrow (4)$$

<u>To find $a_n^{(P)}$:-</u> The RHS of the given relation is 4^n.

$$a_n^{(P)} = A_0 4^n \rightarrow (5).$$

Putting this $a_n^{(P)}$ for an in the given relation

$a_n = A_0 4^n$, $a_{n-1} = A_0 4^{n-1}$, $a_{n-2} = A_0 4^{n-2}$

From (1) $\Rightarrow A_0 4^n - 7 A_0 4^{n-1} + 10 A_0 4^{n-2} = 4^n$

$$4^n \left[A_0 - \frac{7}{4} A_0 + \frac{10}{16} A_0\right] = 4^n$$

$$\frac{8 A_0 - 14 A_0 + 5 A_0}{8} = 1$$

$$-A_0 = 8$$

$$A_0 = -8$$

Substituting A_0 value in (5), we get

$$a_n^{(P)} = -8 4^n$$

The general solution is

$$a_n = a_n^{(h)} + a_n^{(P)}$$

$$a_n = b_1 2^n + b_2 5^n - 8 (4^n)$$

→ solve $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)^2$ given $a_0 = 0$, $a_1 = 1$.

Given $a_n - 4a_{n-1} + 4a_{n-2} = (n+1)^2$

Given equation is a non homogeneous eq'n.

So the solution is $a_n = a_n^{(h)} + a_n^{(P)} \rightarrow (2)$

To find $a_n^{(h)}$:-

Now let $a_n - 4a_{n-1} + 4a_{n-2} = 0 \rightarrow (3)$

let $a_n = r^n$ is a solution of (3), then

$$r^n - 4r^{n-1} + 4r^{n-2} = 0$$

Dividing r^{n-2}, we get

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

$$r = 2, 2.$$

The roots are equal.

$$a_n^{(h)} = (b_1 + b_2 n) 2^n \rightarrow (4)$$

To find $a_n^{(P)}$ = The R.H.S of the given relation is a polynomial of degree 2.

So $a_n^{(P)} = A_0 + A_1 n + A_2 n^2 \rightarrow (5)$

Substituting a_n is same as $A_0 + A_1 n + A_2 n^2$ in the given relation.

$a_n = A_0 + A_1 n + A_2 n^2$, $a_{n-1} = A_0 + A_1(n-1) + A_2(n-1)^2$

$a_{n-2} = A_0 + A_1(n-2) + A_2(n-2)^2$.

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From ①

$$\Rightarrow A_0 + A_1 n + A_2 n^2 - 4\left(A_0 + A_1(n-1) + A_2(n-1)^2\right.$$
$$\left. + 4\left(A_0 + A_1(n-2) + A_2(n-2)^2\right)\right] = n^2 + 2n + 1$$

$$\Rightarrow A_0 + A_1 n + A_2 n^2 - 4A_0 - 4A_1(n-1) - 4A_2(n^2+1-2n)$$
$$+ 4A_0 + 4A_1(n-2) + 4A_2(n^2-4n+4) = n^2 + 2n + 1$$

$$\Rightarrow A_0 + A_1 n + A_2 n^2 - 4A_0 - 4A_1 n + 4A_1 - 4A_2 n^2 - 4A_2$$
$$+ 8A_2 n + 4A_0 + 4A_1 n - 8A_2 + 4A_2 n^2 - 16A_2 n + 16A_2$$
$$= n^2 + 2n + 1.$$

$$\Rightarrow n^2 A_2 + 4(A_1 - 8A_2) + (A_0 - 4A_1 + 12A_2) = n^2 + 2n + 1$$

equating n^2 coefficient on both sides

$$A_2 - 4A_2 + 4A_2 = 1$$
$$A_2 = 1.$$

equating n coefficients on both sides

$$A_1 - 8A_2 = 2$$
$$A_1 = 2 + 8A_2$$
$$A_1 = 2 + 8(1)$$
$$A_1 = 10$$

Equating constant coefficient on both sides

$$A_0 - 4A_1 + 12A_2 = 1$$
$$A_0 = 1 + 4A_1 - 12A_2.$$
$$A_0 = 1 + 4(10) - 12(1)$$
$$A_0 = 29.$$

Sub A_0, A_1, A_2 values in ⑤, we get

$$a_n^{(P)} = 29 + 10n + n^2$$

The general solution is $a_n = a_n^{(h)} + a_n^{(P)}$

$$a_n = (b_1 + b_2 n)\, 2^n + 29 + 10n + n^2 \rightarrow ⑥$$

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Also given $a_0 = 0$, $a_1 = 1$.

Put $n=0$ in ⑥ $\Rightarrow a_0 = (b_1 + b_2(0))\, 2^0 + 29 + 0 + 0$

$$b_1 + 29 = 0$$

$$b_1 = -29.$$

Put $n=1$ in ⑥ $\Rightarrow a_1 = (b_1 + b_2(1))\, 2^1 + 29 + 10(1) + 1^2$

$$(b_1 + b_2)\, 2 + 40 = 1$$

$$(-29 + b_2)\, 2 = -39$$

$$b_2 = -\frac{39}{2} + 29$$

$$b_2 = \frac{19}{2}.$$

Substituting b_1, b_2 values in ⑥

$$a_n = \left(-29 + \frac{19}{2} n\right) 2^n + 29 + 10n + n^2,$$

→ Solve $y_{n+2} - y_{n+1} - 2y_n = n^2$.

Given $y_{n+2} - y_{n+1} - 2y_n = n^2 \rightarrow$ ①

Given relation is non homogeneous eq'n.

So general solution is $y_n = y_n^{(h)} + y_n^{(P)} \rightarrow$ ②

To find $y_n^{(h)}$:-

Now let $y_{n+2} - y_{n+1} - 2y_n = 0 \rightarrow$ ③

let $y_n = r^n$ is a solution of ③, then

$$r^{n+2} - r^{n+1} - 2r^n = 0$$

Dividing r^n we get

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = -1, 2.$$

The roots are distinct

$$y_n^{(h)} = b_1(-1)^n + b_2(2^n)$$

<u>To find $y_n^{(P)}$</u> :- The R.H.S of the given relation is a polynomial of degree 2.

So $y_n^{(P)} = A_0 + A_1 n + A_2 n^2 \rightarrow (5)$

Putting $y_n^{(P)}$ for y_n in the given relation.

$y_n = A_0 + A_1 n + A_2 n^2$, $y_{n+1} = A_0 + A_1(n+1) + A_2(n+1)^2$

$y_{n+2} = A_0 + A_1(n+2) + A_2(n+2)^2$.

from (1) $\Rightarrow A_0 + A_1(n+2) + A_2(n+2)^2$

$- [A_0 + A_1(n+1) + A_2(n+1)^2]$

$- 2[A_0 + A_1 n + A_2 n^2] = n^2$.

$\Rightarrow A_0 + A_1 n + 2A_1 + A_2(n^2 + 4 + 4n)$

$- A_0 - A_1 n - A_1 - A_2(n^2 + 1 + 2n)$

$- 2A_0 - 2A_1 n - 2A_2 n^2 = n^2$

$\Rightarrow$ equating n^2 coefficient on both sides

$$A_2 - A_2 - 2A_2 = 1.$$

$$A_2 = \frac{-1}{2}.$$

equating n coefficients on both sides

$$A_1 + 4A_2 - A_1 - 2A_2 - 2A_1 = 0$$

$$2A_1 = 2A_2$$

$$A_1 = A_2 = \frac{-1}{2}.$$

Equating constant terms on both sides

$$A_0 + 4A_2 + 2A_1 - A_0 - A_1 - A_2 - 2A_0 = 0$$

$$2A_0 = 3A_2 + A_1$$

$$2A_0 = 3\left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)$$

$$2A_0 = \frac{-3-1}{2} = -2$$

$$A_0 = -1$$

Substituting A_0, A_1, A_2 values in (5)

$$y_n^{(P)} = -1 + \left(\frac{-1}{2}\right)^n + \left(\frac{-1}{2}\right)n^2.$$

The general solution is

$$y_n = b_1(-1)^n + b_2(2)^n + \left(\frac{-1}{2}\right)n + \left(\frac{-1}{2}\right)n^2 + (-1)$$

$$= b_1(-1)^n + b_2 2^n - 1 - \frac{1}{2}n - \frac{1}{2}n^2.$$