Unit-3

Partial Differentiation equations

Introduction: An equation which contains two or more PD are called PDE which have one dependent variable and its derivative w.r.to two or more independent variable is called PDE. Eg: $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Formation of PDE by eliminating of arbitrary constants

1. Form PDE by eliminating the arbitrary constants a, b from $z = ax + by + a^2 + b^2$

Soln:-

Given $z = ax + by + a^2 + b^2 \rightarrow (1)$

$\frac{\partial z}{\partial x} = a$ $\qquad$ $\frac{\partial z}{\partial y} = b$

$p = a$, $\qquad$ $q = b$

put p, q in (1)

$z = px + qy + p^2 + q^2$ //

2. $z = axy + b$

$\frac{\partial z}{\partial x} = ay$ | $\frac{\partial z}{\partial y} = ax$

$p = ay$ | $q = ax$

$\frac{p}{y} = a$ | $\frac{q}{x} = a$

$\therefore \frac{p}{y} = \frac{q}{x}$

$px - qy = 0$ //

3) $z = ax^2 + by^2$

$\frac{\partial z}{\partial x} = 2ax$, $\frac{\partial z}{\partial y} = b2y$

$p = 2ax$, $q = 2by$

$\frac{p}{2x} = a$, $\frac{q}{2y} = q$

$\therefore (1) \Rightarrow z \frac{p^2}{4x}$

$z = \frac{p}{2x} x^2 + \frac{q}{2y} y^2$

4. $z = ax^3 + by^3$

$\frac{\partial z}{\partial x} = 3ax^2$

$p = 3ax^2$

$\frac{p}{3x^2} = a$

$\frac{\partial z}{\partial y} = b3y^2$

$q = 3by^2$

$\frac{q}{3y^2} = b.$

$\therefore \quad z = \frac{p}{3x^2} x^3 + \frac{q}{3y^2} y^3$

$z = \frac{px}{3} + \frac{qy}{3}$ //

5. $z = (x+a)(y+b)$

soln:

$p = y+b$

$q = (x+a)$

① $\Rightarrow z = pq$ //

Order:- The highest derivative Present in the PDE is called order.

Degree: The highest derivative power present in the PDE is called degree.

Eg:- 1) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$

order $= 2$, degree $= 1$.

2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} = 2$

degree $= 1$, order $= 2$

Example for first order and I degree PDE:-

$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

order $= 1$

degree $= 1$

6. $z = (x^2+a)(y^2+b)$

$\frac{\partial z}{\partial x} = p = 2x(y^2+b)$

$\frac{p}{2x} = y^2+b$

$\frac{\partial z}{\partial y} = (x^2+a)\,2y$

$\frac{q}{2y} = (x^2+a)$

$\therefore$ ① $\Rightarrow z = \frac{p}{2x} \cdot \frac{q}{2y} = \frac{pq}{4xy}$ //

7. $2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \rightarrow ①$

Soln:

$\cancel{2}p = \frac{\cancel{2}x}{a^2} + 0.$ $\qquad$ $\cancel{2}q = \frac{\cancel{2}y}{b^2}$

$\frac{p}{x} = \frac{1}{a^2}$ $\qquad$ $\frac{q}{y} = \frac{1}{b^2}$

$\therefore ① \Rightarrow \quad 2z = x^2\left(\frac{p}{x}\right) + y^2\left(\frac{q}{y}\right)$

$2z = px + qy$ //.

8. $z = (x-a)^2 + (y-b)^2 \rightarrow ①$

Soln:

$p = 2(x-a)$ $\qquad$ $q = 2(y-b)$

$\frac{p}{2} = (x-a)$ $\qquad$ $\frac{q}{2}$ $(y-b)$

$① \Rightarrow \quad z = \left(\frac{p}{2}\right)^2 + \left(\frac{q}{2}\right)^2$

$= \frac{p^2}{4} + \frac{q^2}{4} \quad \Rightarrow \quad 4z = p^2 + q^2$ //

9. $x^2 + y^2 + (z-c)^2 = a^2$

Soln:

$2x + 2(z-c)p = 0$ $\qquad$ $2y + 2(z-c)q = 0$

$x + (z-c)p = 0$ $\qquad$ $y + (z-c)q = 0$

$(z-c)p = -x$ $\qquad$ $(z-c)q = -y$

$z - c = \frac{-x}{p} \rightarrow ①$ $\qquad$ $(z-c) = \frac{-y}{q} \rightarrow ②$

$\therefore \frac{①}{②} \Rightarrow \quad 1 = \frac{\cancel{-}x/p}{\cancel{-}y/q} \Rightarrow \frac{y}{q} = \frac{x}{p} \Rightarrow yp = xq$ //

10 $(x-h)^2+(y-k)^2+z^2=a^2$, h,k are consts

soln:

$2(x-h)+2zp=0$

$(x-h)+zp=0$

$x-h=-zp$

$2(y-k)+2zq=0$

$(y-k)+zq=0$

$y-k=-zq$

$$① \Rightarrow (-zp)^2+(-zq)^2+z^2=a^2$$

$$z^2p^2+z^2q^2+z^2=a^2$$

11. $\log(az-1)=x+ay+b$

soln:

$$\frac{1}{az-1}(ap)=1$$

$$\frac{ap}{az-1}=1$$

$ap=az-1$

$ap-az=-1$

$a(p-z)=-1$

$$a=\frac{-1}{p-z}$$

$$\frac{1}{az-1}(aq)=a$$

$$\frac{q}{az-1}=1$$

$q=az-1$

$q+1=az$

$$a=\frac{q+1}{z}$$

$$\therefore \frac{-1}{p-z}=\frac{q+1}{z}$$

$-z=(q+1)(p-z)$

$\cancel{-z}=pq-qz+p\cancel{-z}$

$$\boxed{qz=p(q+1)}$$

Type-2: Elimination of arbitrary functions:-

Form a PDE by eliminating arbitrary functions

1. $z = f(x^2+y^2)$

<u>Soln:-</u>

$p = f'(x^2+y^2) \cdot 2x$

$\frac{p}{2x} = f'(x^2+y^2) \rightarrow (1)$

$q = f'(x^2+y^2)\, 2y$

$\frac{q}{2y} = f'(x^2+y^2) \rightarrow (2)$

$\frac{(1)}{(2)} \Rightarrow \frac{\frac{p}{2x}}{\frac{q}{2y}} = 1 \Rightarrow \frac{p}{\cancel{2}x} \times \frac{\cancel{2}y}{q} = 1$

$\Rightarrow \frac{py}{xq} = 1$ //

$\Rightarrow py = xq$ //

2. $lx + my + nz = \phi(x^2+y^2+z^2)$

$l + np = \phi'(x^2+y^2+z^2)(2x+2zp)$

$\frac{l+np}{2x} = \phi'(x^2+y^2+z^2) \rightarrow (1)$

$m + nq = \phi'(x^2+y^2+z^2)\, 2y$

$\frac{m+nq}{2y+2zq} = \phi'(x^2+y^2+z^2) \rightarrow (2)$

$\frac{(1)}{(2)} \Rightarrow \frac{\frac{l+np}{2x+2zp}}{\frac{l+nq}{2y+2zq}} = 1$

$\frac{l+np}{\cancel{2}x+2zp} = \frac{m+nq}{\cancel{2}y+2zq}$

$\frac{l+np}{x+zp} = \frac{m+nq}{y+zq} \Rightarrow$ //

3) $z = (x+y)\, f(x^2-y^2)$

$P = (x+y) f'(x^2-y^2)\, 2x + f(x^2-y^2)$

$P - f(x^2-y^2) = (x+y)\, f'(x^2-y^2)\, 2x \rightarrow ①$

$q = (x+y)\, f'(x^2-y^2)\,(-2y) + f(x^2-y^2)$

$q - f(x^2-y^2) = (x+y)\, f'(x^2-y^2)\,(-2y) \rightarrow ②$

$\frac{①}{②} \Rightarrow \dfrac{P - f(x^2-y^2)}{q - f(x^2-y^2)} = \dfrac{-x}{y}$

$\dfrac{P - \frac{z}{x+y}}{q - \frac{z}{x+y}} = -\dfrac{x}{y}$

$\dfrac{P(x+y) - z}{q(x+y) - z} = \dfrac{-x}{y}$

$yP + qx = z$ //

4) $z = f(x^2-y^2)$

$P = f'(x^2-y^2)\, 2x$

$\frac{P}{2x} = f'(x^2-y^2) \rightarrow ①$

$\frac{q}{2y} = f'(x^2-y^2) \rightarrow ②$

$\frac{①}{②}$ $\frac{P}{\cancel{2}x} = \frac{q}{\cancel{2}y}$

$Py - qx = 0$ //

5) $z = f(x+y)$

$P = f'(x+y)$

$q = f'(x+y)$

$\frac{P}{q} = 1$ //

6. $z = \phi\left(\frac{y}{x}\right)$

$p = \phi'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$ $\quad q = \phi'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right)$

$\frac{px^2}{-y} = \phi'\left(\frac{y}{x}\right) \rightarrow ①$ $\quad \frac{q}{1/x} = \phi'\left(\frac{y}{x}\right) \rightarrow ②$

$\frac{①}{②} \Rightarrow \frac{px^2}{-y} = \frac{q}{1/x}$

$\frac{px^2}{-y} = qx$

$-\frac{px}{y} = q$

$-px = yq$ //

Type : 3 Eliminating 2 arbitrary functions :

1. $\phi(x^2+y^2+z^2,\ z^2-2xy) = 0$

wkt, $v = \phi(u)$

$z^2 - 2xy = f(x^2+y^2+z^2)$

$2zp - 2y = f'(x^2+y^2+z^2)(2x+2zp)$

$zp - y = f'(x^2+y^2+z^2)(x+zp)$

$\frac{zp-y}{x+zp} = f'(x^2+y^2+z^2)$

$2zq - 2x = f'(x^2+y^2+z^2)(2y+2zq)$

$zq - x = f'(x^2+y^2+z^2)(y+zq)$

$\frac{zq-x}{y+zq} = f'(x^2+y^2+z^2)$

$\Rightarrow \frac{zp-y}{x+zp} = \frac{zq-x}{y+zq}$ //

② $z = f(x+y) + g(x-y)$

$p = f'(x+y) + g'(x-y)$

$q = f'(x+y) - g'(x-y)$

$r = f''(x+y) + g''(x-y)$

$t = f''(x+y) + g''(x-y)$

$t = r$

$\boxed{t = r}$

Lagrange's linear eqn :-

The eq $Pp + Qq = R$

is called Lagrange's LE

Its AE is

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

First order linear PDE:- A linear PDE of order-1 involving a dependent variable z and independent variable x, y of the form $Pp+Qq=R$ is called linear PDE, where P, Q, R are functions of x, y, z

Eg: $px+qy^2=z$

First order non-linear PDE:-

A ~~linear~~ PDE which involves first order partial derivatives p, q with degree higher than 1 is called a non-linear PDE,

Eg: $p^2+q^2=1$

Type:1) Method of grouping

1. Solve $xp+yq=z$

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

consider $\frac{dx}{x}=\frac{dy}{y}$

IOBS, $\log x=\log y+\log c_1$

$x=yc_1$

IIly $y=zc_2$

$\therefore \phi(c_1,c_2)=0$

② $yzp+zxq=xy$

$$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy}$$

$\frac{dx}{y\cancel{z}}=\frac{dy}{\cancel{z}x}$

$x\,dx=y\,dy$

$\frac{x^2}{2}=\frac{y^2}{2}+\frac{c_1}{2}$

$x^2-y^2=c_1$

$\frac{dx}{\cancel{y}z}=\frac{dz}{xy}$

$x\,dx=z\,dz$

$\frac{x^2}{2}=\frac{z^2}{2}+\frac{c}{2}$

$x^2-z^2=c_1$

$\therefore$ The G.S is

③ $\frac{y^2z}{x}p+zxq=y^2$

$$\frac{dx}{\frac{y^2z}{x}}=\frac{dy}{zx}=\frac{dz}{y^2}$$

$\frac{dx}{\frac{y^2\cancel{z}}{x}}=\frac{dy}{\cancel{z}x}$

$x^2dx=y^2dy$

IOBS, $\boxed{x^3-y^3=c_1}$

$\frac{dx}{\frac{y^2z}{x}}=\frac{dz}{y^2}$

$x\,dx=z\,dz$

$\frac{x^2}{2}=\frac{z^2}{2}+\frac{c_2}{2}$

$\boxed{x^2-z^2=c_2}$

Multiplier method: A linear PDE of order one involving a dependent variable z and two independent variables x and y of the form $Pp+Qq=R$ is called Lagrange's linear eqn.

It's AE is $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

① solve $(y-z)p+(z-x)q=x-y$

$$\frac{dx}{y-z}=\frac{dy}{z-x}=\frac{dz}{x-y}$$

Consider, 1, 1, 1 as multipliers

$$EF=\frac{dx+dy+dz}{0}$$

$\Rightarrow dx+dy+dz=0$

$\Rightarrow x+y+z=c_1$

choose, x, y, z as mul

$$E.F=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\Rightarrow x\,dx+y\,dy+z\,dz=0$

$\Rightarrow x^2+y^2+z^2=c_2$

$\therefore \phi(x+y+z,\ x^2+y^2+z^2)=0.$

② $(mz-ny)p+(nx-lz)q=ly-mx$

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

choose l, m, n as multipliers

$$EF=\frac{l\,dx+m\,dy+n\,dz}{0}$$

$\Rightarrow l\,dx+m\,dy+n\,dz=0$

$\Rightarrow lx+my+nz=c_1$

choose x, y, z as multiplie

$$EF=\frac{x\,dx+y\,dy+z\,dz}{0}$$

$\Rightarrow x\,dx+y\,dy+z\,dz=0$

$\Rightarrow x^2+y^2+z^2=c_2$

③ $x^2(y-z)p+y^2(z-x)q=z^2(x-y)$

$$\frac{dx}{x^2(y-z)}=\frac{dy}{y^2(z-x)}=\frac{dz}{z^2(x-y)}$$

choose $\frac{1}{x},\frac{1}{y},\frac{1}{z}$

$xyz=c_1$

choose $\frac{1}{x^2},\frac{1}{y^2},\frac{1}{z^2}$

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c_2$

4) $x(y-z)p+y(z-x)q=z(x-y)$

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$EF = \frac{dx+dy+dz}{xy-xz+yz-xy+xz-zy}$$

$$= \frac{dx+dy+dz}{0}$$

$\Rightarrow$ IOBS, $x+y+z=c_1$

$$EF = \frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y-z+z-x+x-y}$$

$$= \frac{dx/x+\frac{dy}{y}+\frac{dz}{z}}{0}$$

IOBS, $\log xyz = \log c_2$

$xyz = c_2$

Non-linear PDE's: (Standard form)

Type: 1 (only P, q) $f(P,q)=0$, Put $P=a$

1. Solve $p+q-pq=0$

Gn $p+q-pq=0$

$p=a \Rightarrow a+\cancel{b}-ab=0$

$q=b$

$b-ab=-a$

$b(1-a)=-a$

$b=\frac{a}{a-1}$

$\therefore$ The G.S $z=ax+by+c$

$z=ax+\frac{a}{a-1}y+c$ //

2. $p^2+q^2=1$

soln:

Put $p=a$, $q=b$

$a^2+b^2=1$

$b^2=1-a^2$

$b=\sqrt{1-a^2}$

The G.S is

$z=ax+\sqrt{1-a^2}y+c$ //

3. $p+q-2=1$

Put $p=a$

$q=b$

$a+b-2=1$

$b-2=1-a$

$b=1-a+2$

$b=1-a$

$\therefore$ The G.S is

$z = ax+(1-a)y+c$ //

4. $pq=K$

$ab=K$

$b=\frac{K}{a}$

The G.S is

$z=ax+by+c$

$z=ax+\frac{K}{a}y+c$ //

5. $p+q=k$

$a+b=k$

$b=k-a$

the G.S is $z=ax+by+c$

$z=ax+(k-a)+c$

Type : 2 $f(p, q, z) = 0$

Procedure :- Put $p = aq$ in ①

Find p, q in terms of a, z

Sub p, q in $dz = p\,dx + q\,dy$

$\int$ate to get complete solution.

1. solve $z = p^2 + q^2$

put $q = ap$

$z = p^2 + a^2 p^2$

$z = p^2(1+a^2)$

$p^2 = \frac{z}{1+a^2}$

$p = \sqrt{\frac{z}{1+a^2}}$, $q = ap$
$= a\sqrt{\frac{z}{1+a^2}}$

The C.S is $dz = p\,dx + q\,dy$

$dz = \frac{\sqrt{z}}{1+a^2}dx + a\frac{\sqrt{z}}{1+a^2}dy \Rightarrow 2\sqrt{z} = \frac{x}{\sqrt{1+a^2}} + \frac{ay}{\sqrt{1+a^2}} + b$

2) $p^2 z^2 + q^2 = p^2 q$

③ $z = pq$

put $q = ap$

$z = ap^2 \Rightarrow p^2 = \frac{z}{a}$

$\Rightarrow p = \sqrt{\frac{z}{a}}$

$\therefore q = a\frac{\sqrt{z}}{\sqrt{a}} = \sqrt{az}$

$dz = \frac{\sqrt{z}}{a}dx + \sqrt{a}\sqrt{z}\,dy$

3 $zpq = p + q$

4. $z^2(p^2+q^2+1) = c^2$

put $q = ap$

$p = \frac{1}{z}\sqrt{\frac{c^2-z^2}{1+a^2}}$

$q = \frac{a}{z}\sqrt{\frac{c^2-z^2}{1+a^2}}$

$dz = \frac{1}{z}\frac{\sqrt{c^2-z^2}}{\sqrt{1+a^2}}dx + \frac{a}{z}\sqrt{\frac{c^2-z^2}{1+a^2}}\,dy$

$\frac{z\,dz}{\sqrt{c^2-z^2}} = \int\frac{dx}{\sqrt{1+a^2}} + \int\frac{a}{\sqrt{1+a^2}}dy$

$-\sqrt{c^2-z^2} = \frac{x}{\sqrt{1+a^2}} + \frac{ay}{\sqrt{1+a^2}} + b$

5. solve $p^2 + pq = z^2$

Put $q = ap$

$p^2 + p(ap) = z^2$

$p^2 + ap^2 = z^2$

$p^2(1+a) = z^2$

$p^2 = \frac{z^2}{1+a}$

$p = \frac{z}{\sqrt{1+a}}$, $q = \frac{az}{\sqrt{1+a}}$

$dz = \frac{z}{\sqrt{1+a}}dx + \frac{az}{\sqrt{1+a}}dy$

$\log z = \frac{x}{\sqrt{1+a}} + \frac{a}{\sqrt{1+a}}y + b$

Type: 3 $\quad f(x, p) = g(y, q)$

Method :-

Put $\quad f(x, p) = g(y, q) = k$

Find $\quad p, q$ in terms of x, y

Sub $\quad p, q,$ in $dz = p\,dx + q\,dy$

Integrate to get c.s.

Ex.

1. Solve $\quad p^2 + q^2 = x + y$ $\qquad$ 2. $xp - yq = y^2 - x^2$

$$p^2 - x = y - q^2$$

Put $\quad p^2 - x = a, \qquad y - q^2 = a$

$$p^2 = a + x, \qquad q^2 = y - a$$

$$p = \sqrt{a+x}, \qquad q = \sqrt{y-a}$$

$$dz = p\,dx + q\,dy$$

$$dz = \sqrt{a+x}\,dx + \sqrt{y-a}\,dy$$

IOBS, $\quad z = 2\sqrt{a+x} + 2\sqrt{y-a}\,dy$

3. $\left(\frac{p}{2} + x\right)^2 + \left(\frac{q}{2} + y\right)^2 = 1$ $\qquad$ 4. $p - x^2 = q + y^2$

Put $\quad \left(\frac{p}{2} + x\right)^2 = a^2, \quad \left(\frac{q}{2} + y\right)^2 = b^2$

$$\frac{p}{2} = a - x$$

$$p = 2(a - x), \quad q = 2(b - y)$$

$$dz = 2(a - x)\,dx + 2(b - y)\,dy$$

$$z = -(a - x)^2 - (b - y)^2 + c$$

5. $q^2 - p = y - x$

6. $p + q = \sin x + \sin y$

$$p - \sin x = \sin y - q$$

Put $\quad p - \sin x = a$

$$p = a + \sin x, \quad q = \sin y - a$$

$$dz = (a + \sin x)\,dx + (\sin y - a)\,dy$$

$$z = ax - \cos x - \cos y - ay + b$$

Type : 4 clairaut's eqn

The equation of the form $z = px + qy + f(p,q)$ is called clairauts eqn.

For solution, put $p = a$, $q = b$ //.

① Solve $z = px + qy + pq$

put $p = a$, $q = b$

$z = ax + by + ab$ //

2 $z = px + qy + p^2 - q^2$

$z = ax + by + a^2 - b^2$

16×5

3 $z = px + qy + \sqrt{pq}$

$z = ax + by = \sqrt{aq}$

4. $z - px - qy = (p+q)^3$

$z = px + qy + (p+q)^3$

$z = ax + by + (a+b)^3$ //

5. $z = px + qy + pq + q^3$

$z = ax + by + ab + b^3$ //

Method of separation of variables:-

1 Solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$, where $u(x,0) = 6e^{-3x}$

By method of sov.

Soln. Gn $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \rightarrow (1)$

Let $U = XT \rightarrow (2)$

diff p.w.r.t x

$\frac{\partial u}{\partial x} = X'T$

diff p.w.r.t T

$\frac{\partial u}{\partial T} = XT'$

$\therefore (1) \Rightarrow X'T = 2XT' + XT$

$X'T = X(2T' + T)$

$\frac{X'}{X} = \frac{2T' + T}{T} = \lambda$ (say)

Let $\frac{X'}{X} = \lambda$

$\frac{1}{X}\frac{dx}{dx} = \lambda$

$\frac{1}{X} dx = \lambda dx$

IOBS, $\log x = \lambda x + c_1$

$X = e^{\lambda x + c_1}$

$= e^{\lambda x} \cdot e^{c_1}$

$\boxed{X = Ae^{\lambda x}}$

and

$\frac{2T' + T}{T} = \lambda$

$\frac{2T'}{T} + 1 = \lambda$

$\frac{2}{T}\frac{dT}{dt} + 1 = \lambda$

$\frac{2}{T}\frac{dT}{dt} = \lambda - 1$

$\frac{2}{T} dT = (\lambda - 1) dt$

$\log T = \left(\frac{\lambda - 1}{2}\right)t + c_2$

$T = e^{\left(\frac{\lambda-1}{2}\right)t + c_2}$

$= Be^{\left(\frac{\lambda-1}{2}\right)t}$

$\therefore (2)\ U = Ae^{\lambda x} \, Be^{\left(\frac{\lambda-1}{2}\right)t}$

$= Ce^{\lambda x + \left(\frac{\lambda-1}{2}\right)t} \rightarrow (3)$

Given $u(x,0) = 6e^{-3x}$

$\Rightarrow t=0, \quad u = 6e^{-3x}$

$\therefore$ (3) $\Rightarrow$ $6e^{-3x} = c\,e^{\lambda x}$

$\Rightarrow$ $c = 6, \quad \lambda = -3$

$\therefore$ $u = 6e^{-3x+(-2t)}$

$= 6e^{-3x+2t}$ //

2. Solve $4u_x + u_y = 3u$, $u(0,y) = e^{-3y}$ by M.SOV.

Soln:

$4u_x + u_y = 3u \longrightarrow (1)$

Let $u = xy$

diff p.w.r.to x, diff p.w.r.to y.

$\frac{\partial u}{\partial x} = x'y$ $\qquad$ $\frac{\partial u}{\partial y} = xy'$

$\therefore$ (1) $\Rightarrow$ $4x'y + xy' = 3xy$

$4x'y = 3xy - xy'$

$4x'y = x(3y - y')$

$\frac{4x'}{x} = \frac{3y - y'}{y} = \lambda$ (say)

Let $\frac{4x'}{x} = \lambda$ and $\frac{3y-y'}{y} = \lambda$

$4x' = \lambda x$ $\qquad$ $3 - \frac{y'}{y} = \lambda$

$4\frac{dx}{dx} = \lambda x$ $\qquad$ $3 - \frac{1}{y}\frac{dy}{dy} = \lambda$

$4\frac{dx}{x} = \lambda dx$ $\qquad$ $3 - \lambda = \frac{1}{y}\frac{dy}{dy}$

IOBS, $\log x = \frac{\lambda x}{4} + c_1$ $\qquad$ $(3-\lambda)\,dy = \frac{1}{y}dy$

$x = e^{\frac{\lambda x}{4} + c_1}$ $\qquad$ IOBS, $c_2 + (3-\lambda)y = \log y$

$= Ae^{\lambda x/4}$

$$Y = B e^{(3-\lambda)y}$$

$\therefore$ $$U = \overset{c}{AB} \, e^{\frac{\lambda}{4}x + (3-\lambda)y} \rightarrow (3)$$

Gn $$u(0,y) = e^{-3y}$$

$$\Rightarrow \quad x = 0, \quad u = e^{-3y}$$

$\therefore$ (3) $\Rightarrow$ $$U = c e^{(3-\lambda)y}$$

$$e^{-3y} = c e^{(3-\lambda)y}$$

$c = 1$, $$3 - \lambda = -3$$

$$-\lambda = -6$$

$$\boxed{\lambda = 6}$$

$\therefore$ $$U = e^{\frac{6}{4}x + (-3)y}$$

$$U = e^{3/2 x - 3y}$$ //