Vector Integration:

Integral of a vector function over curve c

Consider a vector function $\bar{F}$ and an open curve C. Summation of F values over c is said to be integral of F. It is denoted by $\int_c F \cdot d\bar{r}$.

Let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$
$d\bar{r} = dx\bar{i} + dy\bar{j} + dz\bar{k}$

$$F \cdot d\bar{r} = F_1 dx + F_2 dy + F_3 dz$$

Line-Integral :- Let $r = f(t)$ define a smooth curve c joining A and B. Let ds be differential of arc length at $P \in C$.

~~The~~ Any integral which is evaluated over a curve c is called line integral of F. It is given by $\int_c F \cdot d\bar{r}$

Circulation work done

work done by a force is given by

$$\oint_c \bar{F} \cdot d\bar{r} = \oint_c (F_1 dx + F_2 dy + F_3 dz)$$

Note:

1. If curve c is closed, then find $\int$ al along anti-clock.
2. If $C = C_1 + C_2$ then $\int_c F \cdot d\bar{r} = \int_{c_1} F d\bar{r} + \int_{c_2} F \cdot d\bar{r}$
3. $\dfrac{y-y_1}{y_2-y_1} = \dfrac{x-x_1}{x_2-x_1}$ is the eqn of line joining points (x_1, y_1), (x_2, y_2)

Problems:-

1. Find $\int_c \bar{F}\cdot d\bar{r}$ where $\bar{F} = xy\bar{i} + y\bar{j}$ and c is a straight line from (0,0) to (1,1).

$\bar{F}\cdot d\bar{r} = (xy\bar{i} + y\bar{j})\cdot(dx\bar{i} + dy\bar{j})$

The eqn of line (0,0) to (1,1) is $\frac{y-0}{1} = \frac{x-0}{1}$

$y = x$

$dy = dx$

$= xy\,dx + y\,dy$

$\Rightarrow x^2dx + x\,dx$

$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 (x^2dx + x\,dx)$

$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_0^1 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ //

2. Find $\int_c \bar{F}\cdot d\bar{r}$ where $\bar{F} = y\bar{i} - x\bar{j}$ where c is st line joining (0,0) and (2,2)

$\bar{F}\cdot d\bar{r} = y\,dx - x\,dx$

$= x\,dx - x\,dx$

$= 0$

Eqn (0,0), (2,2)

$\frac{y-0}{2} = \frac{x-0}{2}$

$y = x$

$dy = dx$

$\int_c \bar{F}\cdot d\bar{r} = 0$ //

3. Find $\int_c \bar{F}\cdot d\bar{r}$ where $\bar{F} = (3x^2+6x)\bar{i} - 14yz\bar{j} + 20xz^2\bar{k}$ along a straight line joining (0,0,0) to (1,1,1)

$\bar{F}\cdot d\bar{r} = (3x^2+6x)\,dx - 14yz\,dy + 20xz^2\,dz$

$= (3x^2+6x)\,dx - 14x^2\,dx + 20x^3\,dx$

$\frac{x-0}{1} = \frac{y-0}{1} = \frac{z-0}{1}$

$x = y = z$

$dx = dy = dz$

$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 (3x^2+6x-14x^2+20x^3)\,dx$

$= \int_0^1 (20x^3 - 11x^2 + 6x)\,dx$

$$= \left(\cancel{20}^{5}\frac{x^4}{4} - 11\frac{x^3}{3} + \frac{6x^2}{2}\right)_0^1$$

$$= 5 - \frac{11}{3} + 3$$

$$= 8 - \frac{11}{3} = \frac{24-11}{3} = \frac{13}{3}$$ //

4. If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$ then find $\int_C \bar{F}.d\bar{r}$ where C is the curve $y = 2x^2$ in xy-plane from $(0,0)$ to $(1,2)$

$$\bar{F}.d\bar{r} = 3xy\,dx - y^2 dy$$

Gny $y = 2x^2$

$dy = 4x\,dx$

$$= 3x(2x^2)\,dx - 4x^4(4x\,dx)$$

$$= 6x^3 dx - 16x^5 dx$$

$$\int_C \bar{F}.d\bar{r} = \int(6x^3 - 16x^5)\,dx$$

$$= \frac{6x^4}{4} - \frac{16x^6}{6}$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = \frac{9-16}{6} = \frac{-7}{6}$$ //

5. If $\bar{F} = (5xy - 6x^2)\bar{i} + (2y - 4x)\bar{j}$. Find $\int_C \bar{F}.d\bar{r}$ where C is the curve $y = x^3$ in xy-plane from $(1,1)$ to $(2,8)$

$$\bar{F}.d\bar{r} = (5xy - 6x^2)\,dx + (2y - 4x)\,dy$$

$y = x^3$

$dy = 3x^2 dx$

$$= (5x x^3 - 6x^2)\,dx + (2x^3 - 4x)\,3x^2 dx$$

$$= (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

$$\int_C \bar{F}.d\bar{r} = \int_1^2 (5x^4 - 6x^2 + 6x^5 - 12x^3)\,dx$$

$$= \left(\frac{5x^5}{5} - \frac{6x^3}{3} + \frac{6x^6}{6} - \frac{12x^4}{4}\right)_1^2 = 35$$ //

6. IF $\bar{F} = y\bar{i} - x\bar{j}$. Find $\int_C \bar{F}\cdot d\bar{r}$ where c is the curve $y = x^2$ in xy-plane from (0,0) to (1,1).

$y = x^2$

$dy = 2x\,dx$

$\bar{F}\cdot d\bar{r} = y\,dx - x\,dx$

$= x^2 dx - x\,dx$

$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (x^2 - x)\,dx = \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_0^1 = \frac{1}{3} - \frac{1}{2} = \frac{-1}{6} = \frac{-1}{6}$

7. If $\bar{F} = y\bar{i} + 2x\bar{j}$. Find $\int_C \bar{F}\cdot d\bar{r}$ where c is the straight line joining $(\overset{x_1}{2}, \overset{y_1}{0})$ to $(\overset{x_2}{1}, \overset{y_2}{3})$

$\bar{F}\cdot d\bar{r} = y\,dx + 2x\,dy$

$\frac{x-2}{1-2} = \frac{y-0}{3-1}$

$= 2t(-dt) + 2(-t+2)\,2dt$

$\frac{x-2}{-1} = \frac{y}{2} = t$

$= -2t\,dt - 2t\,dt + 8dt$

$x - 2 = -t, \quad y = 2t$

$= -4t\,dt + 8dt$

$x = -t + 2 \quad dy = 2dt$

$dx = -dt$

$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (-4t + 8)\,dt$

$= \left[-\frac{4t^2}{2} + 8t\right]_0^1$

$= -2 + 8$

$= 6$

8. Find work done in moving a particle in the force field. $\bar{F} = 3x^2\bar{i} + (2xz-y)\bar{j} + 2\bar{k}$ along the straight line from $(0,0,0)$ to $(2,1,3)$?

$(0,0,0)$ $(2,1,3)$

$\Rightarrow \frac{x}{2} = y = \frac{z}{3} = t$

$\Rightarrow x = 2t$, $dx = 2dt$

$y = t$, $dy = dt$

$z = 3t$, $dz = 3dt$

$$\bar{F}\cdot d\bar{r} = 3x^2 dx + (2xz-y)\,dy + 2\,dz$$

$$= 3(2t)^2(2dt) + (2.2t.3t - t)\,dt + 2.3dt$$

$$= 24t^2 dt + (12t^2 - t)\,dt + 6\,dt$$

$$= (24t^2 + 12t^2 - t + 6)\,dt$$

$$= (36t^2 - t + 6)\,dt$$

$$\int \bar{F}\cdot d\bar{r} = \left(\frac{36t^3}{3} - \frac{t^2}{2} - 6t\right)_0^1 = \frac{35}{2}$$

9. If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and c is the curve $x = t^2$, $y = 2t$, $z = t^3$ from $t=0$ to $t=1$. Find $\int_c \bar{F}\cdot d\bar{r}$?

Soln: $\frac{51}{70}$ //

Surface integral :-

Any integral which is to be evaluated over a surface is called surface integral.

It is given by $\int_S F.\bar{N}\, ds$.

where $\bar{N}$ is unit normal to a point P.

i) Surface integral over I octant is

$$\int_S \bar{F}.\bar{N}\, ds = \iint_{R(x,y)} F.\bar{N} \frac{dx\, dy}{|\bar{N}.\bar{k}|}$$, $\bar{k}$ is unit normal along xy-plane

Problems:

1. Evaluate $\int_S \bar{F}.\bar{N}\, ds$ for $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$ in I octant and S is a surface $2x+3y+6z=12$

Soln:

The surface integral in I octant is gn by

$$\int_S \bar{F}.\bar{N}\, ds = \iint_{R(x,y)} \bar{F}.\bar{N} \frac{dx\, dy}{|\bar{N}.\bar{k}|} \longrightarrow (1)$$

Gn $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$

& $\phi = 2x+3y+6z-12$

$\nabla\phi = 2\bar{i}+3\bar{j}+6\bar{k}$

$|\nabla\phi| = \sqrt{4+9+36} = \sqrt{49} = 7$

unit Normal $\bar{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$

Now, $\bar{F}.\bar{N} = (18z\bar{i} - 12\bar{j} + 3y\bar{k}) \cdot \left(\frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}\right)$

$= \frac{1}{7}(36z - 36 + 18y)$

$= \frac{18}{7}(2z-2+y)$

$|\bar{N}\cdot\bar{K}| = \left|\frac{2\bar{i}+3\bar{j}+6\bar{k}}{7}\cdot k\right| = \left|\frac{6}{7}\right| = \frac{6}{7}$

$\therefore \; ① \Rightarrow \int_S \bar{F}\cdot\bar{n}\,ds = \iint \frac{18}{\cancel{7}}(2z-2+y)\cdot\frac{\cancel{6}}{\cancel{7}}\;\frac{dx\,dy}{\cancel{6/7}}$

$= 3\cancel{\iint (2z-2+y)\,dx\,dy}$

On $2x+3y+6z=12$

$6z = 12-2x-3y$

$z = \frac{12-2x-3y}{6}$

$= \cancel{3}\iint \left[\frac{2(12-2x-3y)}{\cancel{6}\,2}-2+y\right] dx\,dy$

$= \iint (12-2x-3y-2+y)\,dx\,dy$

Limits

On $2x+3y+6z=12$

$= \iint 10\cdot$

Put $z=0$, $2x+3y=12$

$3y = 12-2x$

$y = \frac{12-2x}{3}$

$\therefore y: 0 \to \frac{12-2x}{3}$

Put $y=0$, $2x=12$

$x=6$

$\therefore x: 0 \to 6$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} 3\left[\cancel{2}\left(\frac{12-2x-3y}{\cancel{6}\,3}\right)-2+y\right] dx\,dy$$

$$= \iint \cancel{3}\left(\frac{12-2x-3y-6+3y}{\cancel{3}}\right) dx\,dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (-2x+6)\,dx\,dy$$

$$= \int_{x=0}^{6} (-2x+6)\,(y)_0^{\frac{12-2x}{3}}\,dx$$

$$= \int_{x=0}^{6} (-2x+6)\left(\frac{12-2x}{3}\right) dx$$

$$= \int_{x=0}^{6} 2(-x+3)\, 2\, \frac{(6-x)}{3} dx$$

$$= \frac{4}{3} \int_{x=0}^{6} (-6x + x^2 + 18 - 3x)\, dx$$

$$= \frac{4}{3} \int_{x=0}^{6} (x^2 - 9x + 18)\, dx$$

$$= \frac{4}{3} \left(\frac{x^3}{3} - 9 \cdot \frac{x^2}{2} + 18x \right)_0^6$$

$$= \frac{4}{3} \left(\frac{6^3}{3} - \frac{9 \cdot 6^2}{2} + 18.6 \right)$$

$$= \frac{4}{3}\, 6^2 \left(\frac{\cancel{6}^2}{\cancel{3}} - \frac{9}{2} + 3 \right)$$

$$= \frac{4}{3} \times 36 \left(-\frac{9}{2} + 5 \right)$$

$$= 4 \times \cancel{12}^{6} \times \frac{1}{\cancel{2}} = 24 //$$

(2). Evaluate $\int_S \bar{F} \cdot \bar{n}\, ds$, $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and S is the sphere $x^2+y^2+z^2=1$ located in I octant.

The surface integral in I octant is gn by

$$\int_S \bar{F} \cdot \bar{n}\, ds = \iint_R \bar{F} \cdot \bar{n} \frac{dx\, dy}{|\bar{n} \cdot \bar{k}|} \longrightarrow (1)$$

Gn $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$

& $\phi = x^2+y^2+z^2-1$

$\nabla\phi = 2x\bar{i} + 2y\bar{j} + 2z\bar{k}$

$|\nabla\phi| = \sqrt{4x^2+4y^2+4z^2} = 2\sqrt{x^2+y^2+z^2} = 2.1 = 2$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\cancel{2}(x\bar{i} + y\bar{j} + z\bar{k})}{\cancel{2}} = x\bar{i} + y\bar{j} + z\bar{k}$$

$\bar{F}\cdot\bar{N} = (yz\bar{i} + zx\bar{j} + xy\bar{k})\cdot(x\bar{i}+y\bar{j}+z\bar{k})$

$= xyz + xyz + xyz = 3xyz$

$|\bar{N}\cdot\bar{k}| = |x\bar{i}+y\bar{j}+z\bar{k}\cdot\bar{k}| = |z| = z$

Limits:

On $x^2+y^2+z^2=1$

Put $z=0$, $x^2+y^2=1 \Rightarrow y^2 = 1-x^2$

$\Rightarrow y = \sqrt{1-x^2}$ $\therefore y: 0 \to \sqrt{1-x^2}$

Put $y=0$, $x^2=1 \Rightarrow x=1$ $\therefore x: 0\to 1$

$\therefore$ (1) $\Rightarrow \int_S \bar{F}\cdot\bar{N}\,ds = \int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} 3xy\cancel{z}\,\frac{dx\,dy}{\cancel{z}}$

$= 3\int_{x=0}^{1} x\left(\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} dx$

$= \frac{3}{2}\int_{x=0}^{1} x(1-x^2)\,dx$

$= \frac{3}{2}\int_0^1 (x-x^3)\,dx$

$= \frac{3}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)$

$= \frac{3}{2}\left(\frac{1}{2}-\frac{1}{4}\right)$

$= \frac{3}{2}\left(\frac{1}{4}\right) = 3/8$ //

Green's Theorem:- Let $P, Q, \frac{\partial Q}{\partial x}, \frac{\partial P}{\partial y}$ are continuous over a region R bdd by a simple closed curve C then

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Problems:

1. Evaluate $\int_C [(3x^2-8y^2)\,dx + (4y-6xy)\,dy]$ where c is the bdd by $x=0, y=0, x+y=1$ by Green's theorem.

Soln:

Here $P = 3x^2-8y^2$, $Q = 4y-6xy$

$\frac{\partial P}{\partial y} = -16y$ $\quad$ $\frac{\partial Q}{\partial x} = -6\cancel{x}y$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$

$\therefore$ By Green theorem, $\int (P\,dx + Q\,dy) = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Limits

$y=\sqrt{x}, y=x^2$

$\Rightarrow \sqrt{x} = x^2$

$\Rightarrow x^4 - x = 0$

$\Rightarrow x = 0, 1$

$\therefore x: 0 \to 1$

$$\text{RHS} = \iint 10y\,dx\,dy$$

$$= \int_0^1 \int_{\sqrt{x}}^{x^2} 10y\,dx\,dy$$

$$= \int_0^1 5(x^4-x)\,dx$$

$$= 3/2$$

③ verify Green's theorem, $\int_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where C is bounded by $y^2 = x$, $y = x^2$?

Soln: By Green' thm, $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

RHS:

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 10y$$

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \int_0^1 \int_x^{x^2} 10y\,dx\,dy$$

$$= \int_0^1 10\left(\frac{y^2}{2}\right)_x^{x^2} dx$$

$$= 3/2$$

LHS: $\int_C P\,dx + Q\,dy = \int_{C_1} (P\,dx + Q\,dy) + \int_{C_2} (P\,dx + Q\,dy)$

$= I_1 + I_2$

Along I_1:-

let $y = \sqrt{x}$

$\Rightarrow y^2 = x$

$2y\,dy = x\,dx$

$\therefore \int_{C_1} P\,dx + Q\,dy = \int_{C_1} (6y^5 - \cancel{2}2y^3 + 4y)\,dy$

$= -5/2$

$\therefore I = -\frac{5}{2} + 1 = \frac{-3}{2} = 3/2$

Along I_2:-

$y = x^2$

$dy = 2x\,dx$

$\therefore \int_{C_2} P\,dx + Q\,dy$

$= \int_{C_2} (-20x^4 + 8x^3 + 3x^2)\,dx$

$= 1$

LHS = RHS, Green's theorem is verified

Gauss divergence Theorem:-

Let S be a closed surface enclosing by a volume V. If $\bar{F}$ be differentiable vector point function then

$$\iint_S \bar{F} \cdot \bar{N}\, ds = \int_V \operatorname{div} \bar{F}\, dv$$

Note; Another form G D theorem is

$$\int_S (F_1\, dy\, dz + F_2\, dx\, dz + F_3\, dx\, dy) = \int_V \operatorname{div} \bar{F}\, dv$$

1. Evaluate $\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ where $x^2+y^2+z^2=a^2$ using GD theorem.

Soln: By GD theorem $\iint_S \bar{F}\cdot\bar{n}\,ds = \int_V \text{div}\,\bar{F}\,dv \rightarrow (1)$

Here $\bar{F} = x\bar{i} + y\bar{j} + z\bar{k}$

$\text{div}\,\bar{F} = 1+1+1 = 3$

$\therefore (1) \Rightarrow \iint_S \bar{F}\cdot\bar{n}\,ds = \int 3\,dv = 3V$

$= 3$ (volume of sphere)

$= \cancel{3}\,\frac{4}{\cancel{3}}\pi r^3 = 4\pi r^3$

$= 4\pi a^3$

2. using GD theorem, find $\iint_S x\,dy\,dz + y\,dx\,dz + z\,dx\,dy$ where S is the sphere $x^2+y^2+z^2=1$

By GDT, $\iint_S \bar{F}\cdot\bar{n}\,ds = \int_V \text{div}\,\bar{F}\,dv$

$= \int_V 3\,dV$

$= 3V$

$= 3$ volume of the sphere

$= \cancel{3}\,\frac{4}{\cancel{3}}\pi r^3$

$= 4\pi r^3$

$= 4\pi$ $(\because r=1)$

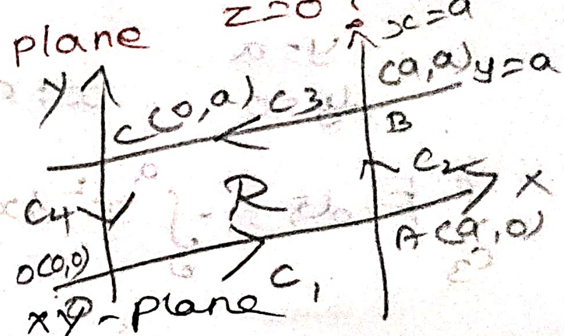
stoke's theorem:-

Let S be a closed surface bounded by a closed curve c. If $\bar{F}$ be any function then

$$\int_C F\cdot d\bar{r} = \iint \text{curl}\,(\bar{F})\,\bar{n}\, ds$$

* problems

1. verify stoke's theorem for $\bar{f} = x^2\bar{i} + xy\bar{j}$ where integrated sides are along $x=0$, $y=0$, $x=a$, $y=a$. around the square in the plane $z=0$?



Soln: Gn $\bar{F} = x^2\bar{i} + xy\,\bar{j}$

Gn $z=0$, so it lies in xy-plane

By stokes theorem,

$$\int_C \bar{F}\cdot d\bar{r} = \iint_S \text{curl}\,\bar{F}\cdot\bar{n}\, ds \longrightarrow (1)$$

where $ds = \dfrac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

To find LHS:

$$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r} + \int_{C_3} \bar{F}\cdot d\bar{r} + \int_{C_4} \bar{F}\cdot d\bar{r}$$

Now, $\bar{F}\cdot d\bar{r} = (x^2\bar{i} + xy\bar{j})\cdot(dx\bar{i} + dy\bar{j})$

$= x^2\,dx + xy\,dy$

Case i: Along OA

$y=0$

$\Rightarrow dy=0$ $\quad x: 0 \to a$

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} x^2 dx + xy\, dy$$

$$= \int_0^a x^2 dx$$

$$= \left(\frac{x^3}{3}\right)_0^a$$

$$= \frac{a^3}{3}$$

Case ii Along AB

$x=a$ $\quad y: 0 \to a$

$dx=0$

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_0^a ay\, dy$$

$$= a\left(\frac{y^2}{2}\right)_0^a$$

$$= \frac{a^3}{2}$$

Case iii : Along BC

$y=a$ $\quad x: a \to 0$

$dy=0$

$$\int_{C_3} \bar{F}\cdot d\bar{s} = -\int_0^a x^2 dx$$

$$= -\frac{a^3}{3}$$

Case iv Along CO

$x=0$ $\quad y:$

$dx=0$

$$\int_{C_4} \bar{F}\cdot d\bar{r} = 0$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \cancel{\frac{a^3}{3}} + \frac{a^3}{2} - \cancel{\frac{a^3}{3}} + 0 = \frac{a^3}{2}$$

To find RHS:

Consider $\iint_S \text{curl } \bar{F}\cdot\bar{n}\, ds = \iint \text{curl } \bar{F}\cdot\bar{n} \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|}$

$$\text{curl } \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

$= y\bar{k}$

$\text{curl } \bar{F} \cdot \bar{N} = y\bar{k} \cdot \bar{k} = y$

$|\bar{N} \cdot \bar{k}| = |\bar{k} \cdot \bar{k}| = 1$

$\therefore \iint_S \text{curl } \bar{F} \cdot \bar{N} \, ds = \iint_S y \frac{dx\,dy}{1}$ $\quad\quad \because x : 0 \to a$, $y : 0 \to a$

$= \int_0^a \int_0^a y \, dx \, dy$

$= \int_0^a (x)_0^a \, y \, dy$

$= \int_0^a a y \, dy$

$= a \left(\frac{y^2}{2}\right)_0^a$

$= a \frac{a^2}{2} = \frac{a^3}{2}$

$\therefore$ LHS = RHS

Hence $\int_C \bar{F} \cdot d\bar{r} = \iint_S \text{curl } \bar{F} \cdot \bar{N} \, ds$

Hence stoke's theorem is verified