SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES
(Autonomous)

DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING-DATA SCIENCE

I B.Tech I Sem

23BSC114T	LINEAR ALGEBRA & CALCULUS (Common to All Branches of Engineering)	L 2	T 1	P -	C 3

COURSE EDUCATIONAL OBJECTIVES:

1. To familiarize the concepts of matrices and mean value theorems and their applications in engineering.
2. To equip the students to solve various application problems in engineering through evaluation of multiple integrals etc.,
3. To equip the students with standard concepts and tools of mathematics to handle various real-world problems and their applications.

UNIT-I: MATRICES (9)

Rank of a matrix by echelon form, normal form. Cauchy–Binet formulae (without proof). Inverse of Non- singular matrices by Gauss-Jordan method, System of linear equations: Solving system of Homogeneous and Non-Homogeneous equations by Gauss elimination method.Iterative methods: Jacobi and Gauss Seidel Methods.

UNIT-II:EIGEN VALUES, EIGEN VECTORS AND ORTHOGONAL TRANSFORMATION (9)

Eigen values, Eigenvectors and their properties, Diagonalization of a matrix, Cayley- Hamilton Theorem (without proof), finding inverse and power of a matrix by Cayley- Hamilton Theorem, Quadratic forms and Nature of the Quadratic Forms, Reduction of Quadratic form to canonical forms by Orthogonal Transformation.

UNIT-III: CALCULUS (9)

Mean Value Theorems: Rolle's Theorem, Lagrange's mean value theorem with their geometrical interpretation, Cauchy's mean value theorem, Taylor's and Maclaurin's theorems with remainders (without proof), Problems and applications on the above theorems.

UNIT-IV:PARTIAL DIFFERENTIATION AND APPLICATIONS (MULTIVARIABLECALCULUS) (9)

Functions of several variables: Continuity and Differentiability, Partial derivatives, total derivatives, chain rule, Taylor's and Maclaurin's series expansion of functions of two variables. Jacobians, Functional dependence, maxima and minima of functions of two variables, method of Lagrange multipliers.

UNIT-V:MULTIPLE INTEGRALS(MULTIPLE VARIABLE CALCULUS) (9)

Double integrals, Triple integrals, change of order of integration, change of variables to polar, cylindrical and spherical coordinates. Finding areas(by double integrals) and volumes (by double integrals and triple integrals).

TOTAL HOURS: 45

COURSE OUTCOMES:

On successful completion of the course, students will be able to		Pos
CO1	To solve a system of homogenous and non-homogeneous linear equations	**PO1, PO2,PO3**
CO2	Develop and use of matrix algebra techniques that are needed by engineers for practical applications	**PO1, PO2,PO3**
CO3	Learn important tools of calculus in higher dimensions. Utilize mean value theorems to real life problems.	**PO1, PO2,PO3**
CO4	Familiarize with functions of several variables which is useful in optimization	**PO1, PO2,PO3**
CO5	Familiarize with double and triple integrals of functions of several variables in two dimensions using Cartesian and polar coordinates and in three dimensions using cylindrical and spherical coordinates.	**PO1, PO2,PO3**

TEXT BOOKS:

1. Higher Engineering Mathematics, B. S. Grewal, Khanna Publishers, 2017, 44th Edition
2. Advanced Engineering Mathematics, Erwin Kreyszig, John Wiley & Sons, 2018, 10th Edition.

REFERENCE BOOKS:

1. Thomas Calculus, George B. Thomas, Maurice D. Weir and Joel Hass, Pearson Publishers, 2018, 14th Edition.
2. Advanced Engineering Mathematics, R. K. Jain and S. R. K. Iyengar, Alpha ScienceInternational Ltd., 2021 5th Edition(9th reprint).
3. Advanced Modern Engineering Mathematics, Glyn James, Pearson publishers, 2018, 5th Edition.
4. Advanced Engineering Mathematics, Micheael Greenberg, Pearson publishers, 9th edition
5. Higher Engineering Mathematics, H.K Das,Er.RajnishVerma,S.ChandPublications,2014, Third Edition (Reprint 2021)

REFERENCE WEBSITE:

1. https://nptel.ac.in/courses/110/105/111105111/
2. https://www.youtube.com/watch?v=8D3WViAyJvc
3. https://www.youtube.com/watch?v=fKzDtjq0ks4
4. https://www.youtube.com/watch?v=wMd4YRyBmjA
5. https://www.youtube.com/watch?v=ArkDa6d5h9I
6. https://www.youtube.com/watch?v=KgItZSst2sU
7. https://www.youtube.com/watch?v=-I3HUeHi1Ys

CO-PO MAPPING:

CO\PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO.1	3	3	2	-	-	-	-	-	-	-	-	-
CO.2	3	3	2	-	-	-	-	-	-	-	-	-
CO.3	3	3	2	-	-	-	-	-	-	-	-	-
CO.4	3	3	2	-	-	-	-	-	-	-	-	-
CO.5	3	3	2	-	-	-	-	-	-	-	-	-
CO*	3	3	2	-	-	-	-	-	-	-	-	-

LINEAR ALGEBRA AND CALCULUS

UNIT - I : MATRICES

Matrix: An arrangement of numbers (real or complex) in rows or columns is called a matrix.

- The numbers present in the matrix are called elements.
- In general matrix are denoted by capital letters i.e., A, B, C ----
- Elements are denoted by small letters a, b, c, ----
- The elements are written between brackets say square brackets [ ] (or) small brackets ( ) (or) branches || ||.
- If a matrix A has m rows and n columns then order of A is m×n i.e., $A_{m\times n}$ is given by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{i1} & a_{i2} & a_{ij} \cdots & a_{in} \\ \vdots & & & \\ a_{mn} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$A = [a_{ij}]_{m\times n}$$

Types of matrices:

→ **Square matrix:** A matrix is said to be square matrix if no. of rows is equal to no. of columns

Ex: $\begin{bmatrix} 1 & 4 \\ 2 & -9 \end{bmatrix}_{2\times 2}$ $\begin{bmatrix} 0 & 1 & 2 \\ -4 & 10 & 11 \\ 2 & 4 & 6 \end{bmatrix}_{3\times 3}$

→ **Rectangular matrix:** A matrix is said to be a rectangular matrix if no. of rows is not equal to no. of columns

Ex: $\begin{bmatrix} 1 & 4 & 5 \\ -2 & 10 & 2 \end{bmatrix}_{2\times 3}$, $\begin{bmatrix} 1 & 2 \end{bmatrix}_{1\times 2}$

→ **Null or Zero matrix:** A matrix is said to be null or zero matrix if all its elements are zero.

Ex: $O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

→ **Unit or Identity matrix:** A square matrix is said to be a unit or identity matrix if all the elements in the principal diagonal are equal to one and remaining elements are zero.

Ex: $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

→ **Diagonal matrix**: A square matrix is said to be diagonal matrix if except principal diagonal elements, remaining elements are zero.

Ex: $\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 5 & 0 \\ 0 & 9 \end{bmatrix}_{2\times 2}$

→ **Scalar matrix**: A square matrix is said to be scalar matrix, if all the principal diagonal elements are equal to same constant except zero and remaining elements are zero.

Ex: $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}_{3\times 3}$, $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}_{4\times 4}$

→ **Row matrix**: A matrix is said to be a row matrix if it has only one row.

Ex: $\begin{bmatrix} 1 & -1 \end{bmatrix}_{1\times 2}$, $\begin{bmatrix} 2 & 3 & 4 & 5 & 6 \end{bmatrix}_{1\times 5}$

→ **Column matrix**: A matrix is said to be a ~~to~~ column matrix if it has only one column.

Ex: $\begin{bmatrix} 2 \\ -3 \end{bmatrix}_{2\times 1}$, $\begin{bmatrix} 9 \\ 2 \\ 4 \\ -1 \end{bmatrix}_{4\times 1}$

→ **Upper triangular matrix**: A square matrix is said to be upper triangular matrix if all the elements below the principal diagonal are zero.

Ex: $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & 2 & 9 \\ 0 & 0 & 10 & 11 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

→ **Lower triangular matrix**: A square matrix is said to be lower triangular matrix if all the elements above the principal diagonal are zero.

Ex: $\begin{bmatrix} 2 & 0 & 0 \\ 3 & -2 & 0 \\ 1 & 5 & 10 \end{bmatrix}$

→ **Trace of a matrix:** The sum of all principal diagonal elements of a matrix is called the trace of matrix.

Ex: $A = \begin{bmatrix} 1 & 5 & 4 \\ 2 & -4 & 10 \\ 3 & 1 & 7 \end{bmatrix}$, $Tr(A) = 1+(-4)+7 = 4$.

→ **Transpose of a matrix:** If A is a square matrix then its transpose is denoted by A^T, is also a matrix obtained by interchanging rows and columns of A.

Ex: If $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 9 \\ 10 & 11 & 20 \end{bmatrix}$ then $A^T = \begin{bmatrix} 2 & 4 & 10 \\ 1 & 2 & 11 \\ 3 & 9 & 20 \end{bmatrix}$.

→ **Symmetric matrix:** A square matrix A is said to be symmetric matrix if A^T is equal to A, i.e., $A^T = A$.

Ex: If $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, then $A^T = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$

$\therefore A^T = A$.

→ **Skew-symmetric matrix:** A square matrix is said to be a skew-symmetric matrix if $A^T = -A$.

Ex: If $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix}$, then $A^T = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$

$A^T = -\begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix} \Rightarrow A^T = -A$.

→ **Addition of matrices:** Two matrices A and B are added if both have same order. It is denoted by '$A+B$'.

→ **Multiplication of matrices:** Two matrices A and B are multiplied if no. of columns in A is equal to no. of rows in B. It is denoted by '$A \cdot B$'.

→ **Determinant of 2x2 matrix:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix, then its determinant denoted by det-A or $|A|$ is given by

$|A| = ad - bc$.

Ex: If $A = \begin{bmatrix} 4 & -5 \\ 10 & 2 \end{bmatrix}$, $|A| = 8-(-50) = 8+50 = 58$.

→ **Singular matrix**: A matrix A is said to be a singular matrix if $|A| = 0$.

→ **Non-singular matrix**: A matrix A is said to be a non-singular matrix if $|A| \neq 0$.

→ **Inverse of 2x2 matrix**: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be any non-singular matrix, then the inverse is defined by

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$, denoted by A^{-1}.

→ **Determinant of 3x3 matrix**:

- **Minor**: Minor of a element is the determinant of the matrix which is obtained by removing row and column in which the element is present.

Ex: Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then, Minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

Minor of $a_{21} = M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$.

- **Co-factor**: Co-factor of a element is the product of sign and minor, i.e., co-factor of $a_{ij} = (-1)^{i+j} M_{ij}$.

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

then co-factor of $a_{11} = (-1)^{1+1} M_{11}$

$= (-1)^2 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

$= a_{22}a_{33} - a_{23}a_{32}$

co-factor of $a_{21} = (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$

$= (-1)^3 [a_{12}a_{33} - a_{13}a_{32}]$

$[= a_{13}a_{32} - a_{12}a_{33}]$

- **Determinant** of matrix A is sum of the products of element of any row or any column with their corresponding cofactor.

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then $|A| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$

$$= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

also, $|A| = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$

$$= -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

→ Adjoint matrix: The adjoint of A is the transposed matrix of cofactors of A.

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Then $\operatorname{adj}A = \begin{bmatrix} M_{11} & -M_{12} & M_{13} \\ -M_{21} & M_{22} & -M_{23} \\ M_{31} & -M_{32} & M_{33} \end{bmatrix}^T = \begin{bmatrix} M_{11} & -M_{21} & M_{31} \\ -M_{12} & M_{22} & -M_{32} \\ M_{13} & -M_{23} & M_{33} \end{bmatrix}$.

→ Inverse of a matrix: If A be any non-singular matrix then its inverse is given by $A^{-1} = \frac{1}{|A|}\operatorname{adj}A$.

→ Sub matrix: Submatrix are also a matrix obtained by removing any row or column or both of a given matrix.

Ex: If $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 4 & 2 \\ 5 & -7 & 9 \end{bmatrix}$,

then submatrices are $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & -7 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & -3 \\ 5 & -7 & 9 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

→ Minor of a matrix: The determinate of a square ~~matrix~~ sub matrix of a given matrix is called minor of a matrix.

→ Rank of a matrix: A matrix is said to be of rank r, when
(i) Every minor of order higher than r vanishes (is zero).
(ii) There exists atleast one non-zero minor of order r.
Rank of matrix A shall be denoted by $\rho(A)$ [read as row of A]

Note: • Rank of a matrix is unique
• Rank of a zero or null matrix is zero.
• Rank of a non-singular square matrix of order n is n.
• Rank of a unit matrix I_n is n.
• Rank of a matrix of order $m \times n$, then $\rho(A) \leq \min(m, n)$.
• If A is a matrix of order

Q① Find the rank of the matrix $\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

Sol: let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = 2(6-3) - 2(2-1) + 1(2-3)$$
$$= 2(3) - 2(1) + 1(-1)$$
$$= 6 - 2 - 1$$
$$|A| = 3 \neq 0.$$

$\therefore$ Rank of $A = \rho(A) = 3.$

② Find the rank of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

Sol: let the given matrix be $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{vmatrix} = 1(36-36) - 2(18-18) + 3(12-12)$$
$$= 1(0) - 2(0) + 3(0)$$
$$|A| = 0.$$

consider any square submatrix of order 2×2

$$\begin{vmatrix} 4 & 6 \\ 6 & 9 \end{vmatrix} = 36 - 36 = 0$$

Similarly ~~determinant~~ minor of every 2×2 is zero.

Hence rank of $A = \rho(A) = 1.$

③ Find the rank of the matrix $\begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

Sol: let the given matrix be $A = \begin{bmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} -1 & 0 & 6 \\ 3 & 6 & 1 \\ -5 & 1 & 3 \end{vmatrix} = (-1)(18-1) - 0(9+5) + 6(3+30)$$
$$= (-1)(17) - 0 + 6(33)$$
$$= -17 + 198$$
$$|A| = 181 \neq 0.$$

$\therefore$ Rank of $A = \rho(A) = 3.$

(4) Find the rank of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$.

$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{vmatrix} = 1(6-6) - 1(6-6) + 1(6-6)$
$= 0.$

we observe that minor of any 2×2 matrix is zero.
Hence rank of $A = \rho(A) = 1$.

(5) Find the rank of a matrix $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 8 & 7 & 0 & 5 \end{bmatrix}$

since ~~rank of~~ $\rho(A) \leq \min(m,n)$, if A is $m\times n$.

$\rho(A) \leq 3.$

Let $B = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{bmatrix}$, $|B| = \begin{vmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 8 & 7 & 0 \end{vmatrix}$

$= 1(0-49) - 2(0-56) + 3(35-48)$
$= 1(-49) - 2(-56) + 3(-13)$
$= -49 + 112 - 39$
$= -88 + 112$
$|B| = 27 \neq 0.$

$\therefore \rho(A) = \text{order of } B = 3.$

(6) Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{vmatrix} = 1(24-25) - 2(18-20) + 3(15-16)$
$= 1(-1) - 2(-2) + 3(-1)$
$= -1 + 4 - 3$
$|A| = 0$

Minor of 2×2 matrix $\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} = 24 - 25$

$= -1 \neq 0.$

$\therefore$ Rank of $A = \rho(A) = 2.$

→ Zero row: A row in which all element are zero are called zero row.

→ Non-zero row: A row which is having atleast 1 non-zero element is called non-zero row.

Ex: $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{array}{l} \rightarrow \text{non-zero row} \\ \rightarrow \text{non-zero row} \\ \rightarrow \text{zero row} \\ \rightarrow \text{non-zero row.} \end{array}$

→ Elementary transformations or operations of a matrix:

1. Elementary row operations:
   a) Interchange of two rows i.e., $R_i \leftrightarrow R_j$, i^{th} row is interchanged with j^{th} row.
   b) Multiplication of a row by a non-zero scalar k, i.e., $R_i \rightarrow kR_i$, i^{th} row is multiplied by non-zero element k
   c) Addition of a non-zero multiple of one row to another row i.e., $R_i \rightarrow R_i + kR_j$, k times j^{th} row is added to i^{th} row.
2. Elementary column transformations:
   a) Interchange of two columns. i.e., $C_i \leftrightarrow C_j$, i^{th} column is interchanged with j^{th} column
   b) Multiplication of a column by a non-zero scalar k. i.e., $C_i \rightarrow kC_i$, i^{th} column is multiplied by non-zero scalar k.
   c) Addition of a non-zero scalar multiple of one column to another column i.e., $C_i \rightarrow C_i + kC_j$, k times j^{th} colum is added to i^{th} column

→ Equivalent matrices: Two matrices A and B are said to be equivalent if one can be obtained from the other by a sequence of elementary transformations.

Two equivalent matrices have the same order and same rank.

The symbol ~ is used for equivalence.

RANK OF A MATRIX BY ECHELON FORM.

→ Echelon form of a matrix:

A matrix is said to be in row echelon form if it satisfies the following conditions.

① All non-zero rows are above all zero rows, if any.

② In each non-zero row, the first non-zero entry called the leading entry (or pivot) is to the right of the leading entry of the row above it.

③ All entries below the leading entry are zeroes.

- we should apply only row-operations
- The no. of non-zero rows in the echelon-form of a matrix gives the rank of the matrix.

Problems:

① Find the rank of the given matrix by reducing to echelon form

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}.$$

Sol: Let $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & -5 & 6 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$.

$$\sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

The above matrix is in echelon form.

$\therefore$ Rank of $A = \rho(A) =$ No. of non-zero rows in echelon form

$\therefore \rho(A) = 2$.

② Find the rank of the following matrix, by reducing into the echelon form.

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$$

Sol: Let $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 + 2R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix}$$

Applying $R_3 \to 2R_3 - 11R_2$

$R_4 \to R_4 + 2R_2$.

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $R_3 \to R_3 + 6R_4$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Applying $R_3 \leftrightarrow R_4$

$$\sim \begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is echelon form.

$\therefore$ Rank of $A = \rho(A) = 4$.

(3) Find the rank of the given matrix by reducing into echelon form $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Sol: Given let $A = \begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 4R_1$
$R_4 \to R_3 - 4R_1$

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & -15 & -21 \end{bmatrix}$$

$R_3 \to R_3 - R_2$; $R_4 \to R_4 - 3R_2$.

$$\sim \begin{bmatrix} 2 & 1 & 3 & 5 \\ 0 & 0 & -5 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is the echelon form.

$\therefore \rho(A) = 2$.

(4) Find the rank of the given matrix by reducing into echelon form of $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 2 & 1 & -1 & 2 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 2 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 2 & 1 & -1 & 2 \end{bmatrix}$

Applying $R_2 \to R_2 - 2R_1$
$R_3 \to R_3 + R_1$
$R_4 \to R_4 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -2 & 1 & -3 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 1 & -2 \end{bmatrix}$$

$R_3 \to 2R_3 + R_2$
$R_4 \to 2R_4 - 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 5 \end{bmatrix}$$

$R_4 \to R_4 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 2 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

which is the echelon matrix,

$\therefore \rho(A) = 4$.

(5) $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

Sol: Let the given matrix be

$$A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$$

$R_3 \to 5R_3 - R_1$

$$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & -8 & 4 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 8R_1$

$$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 20 & 4 \end{bmatrix}$$

$\therefore$ The above matrix is in echelon form. Hence Rank of $A = 3$.

(6) For what value of k, the matrix

$$\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

has Rank 3 by reducing to echelon form.

Sol: Let $A = \begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$

$R_1 \leftrightarrow R_2$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 4 & -3 & 1 \\ k & 2 & 2 & 2 \\ 9 & 9 & k & 3 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - kR_1$, $R_4 \to R_4 - 9R_1$

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 2-k & 2+k & 2 \\ 0 & 0 & k+9 & 3 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$.

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-k & 2+k & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & k+9 & 3 \end{bmatrix}$$

$R_4 \to R_4 - (k+9)R_3$.

$$\sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 2-k & 2+k & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -(k+6) \end{bmatrix}$$

Hence, if the given matrix have rank 3, then

$-(k+6) = 0$

$k+6 = 0$

$\therefore \boxed{k = -6.}$

RANK OF THE MATRIX BY NORMAL FORM

Normal (or) Canonical form:

If the given matrix is reduced to one of the following form $I_r, [I_r . 0], \begin{bmatrix} I_r \\ 0 \end{bmatrix}, \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$. by a finite no. of elementary row & column operations, where I_r is a unit matrix of order r, then the matrix is said to be in normal form.

* Here 'r' is the rank of the matrix.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Q: Find the rank of the following matrix by reducing it to normal form.

(1) $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$

Applying $C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$C_3 \to C_3 - 2C_1$, $C_4 \to C_4 + 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 2 & 1 & 3 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 2 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to 2C_3 - C_2$, $C_4 \to 4C_4 - 6C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to \frac{R_2}{4}$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = \rho(A) = 2$.

(2) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ (1)

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1,\ R_4 \to R_4 - 6R_1$

$$\sim\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1,\ C_3 \to C_3 - 3C_1$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -11 & 5 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-4}$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & -11 & 5 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -8 & 3 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$C_3 \to C_3 + 8C_2,\ C_4 \to C_4 - 3C_2$.

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_3$.

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to \frac{C_3}{-3}$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_4 \to C_4 - 2C_3$

$$\sim\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ Rank of $A = \rho(A) = 3$.

(3) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 6 & 5 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 2 & 6 & 5 \end{bmatrix}$

$R_2 \to R_2 - 3R_1,\ R_3 \to R_3 - 2R_1$

$$\sim\begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -4 \\ 0 & 2 & -1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1,\ C_3 \to C_3 - 3C_1$

$$\sim\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -4 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_2 \to \frac{R_2}{-2}$

$$\sim\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$.

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & -5 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$.

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

$C_3 \to \frac{C_3}{-5}$ (or) $R_3 \to \frac{R_3}{-5}$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\sim [I_3]$

$\therefore$ Rank of $A = \rho(A) = 3$.

④ $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & -4 \\ 2 & 3 & 5 & -5 \\ 3 & -4 & -5 & 8 \end{bmatrix}$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$$

$C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, $C_4 \to C_4 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 3 & -7 \\ 0 & -7 & -8 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_2$, $R_4 \to R_4 + 7R_2$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$$

$C_3 \to C_3 - 2C_2$, $C_4 \to C_4 + 5C_2$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 6 & -30 \end{bmatrix}$$

$R_4 \to R_4 - 6R_3$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

$C_4 \to C_4 + 2C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

$R_4 \to \frac{R_4}{-18}$.

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\sim [I_4]$

$\therefore \rho(A) = 4$.

⑤ $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

Sol: Let $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 + 3R_1$

$$\sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

$C_2 \to C_2 - 4C_1$, $C_3 \to C_3 - 3C_1$, $C_4 + 2C_1$,
$C_5 \to C_5 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

$R_2 \to \frac{R_2}{5}$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

$R_3 \to R_3 - 10R_1$
$R_4 \to R_4 - 15R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$C_5 \to C_5 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \rho(A) = 2.$

CAUCHY-BINET FORMULA.

let A be a $m \times n$ matrix and B be a $n \times m$ matrix.

The cauchy-binet formula for the matrices A and B is given by:

1. If $m \leq n$, $\det(AB)$ = sum of the products of the determinants of all posible $m \times m$ submatrices of A and B.
2. If $m = n$, $|AB| = |A|\,|B|$
3. If $m > n$, then $|AB| = 0$.

Def: If A and B be two square matrices, then the cauchy-binet formula is given by

$|AB| = |A|\,|B|$

(or)

$\det(AB) = \det A . \det B$

Ex ① let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} -1+4 & 1-6 \\ -3+8 & 3-12 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & -5 \\ 5 & -9 \end{bmatrix}$$

$$|AB| = \begin{vmatrix} 3 & -5 \\ 5 & -9 \end{vmatrix} = -27 + 25 = -2.$$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2.$$

$$|B| = \begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix} = 3 - 2 = 1$$

Now, $|A|\,|B| = (-2)(1) = -2$

$\therefore |AB| = |A|\,|B|.$

② Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$

Here A is 2×3, B is 3×2 matrices,

Hence AB = 2×2.

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix} = \begin{bmatrix} 7+18+33 & 8+20+36 \\ 28+45+66 & 32+50+72 \end{bmatrix}$$

$$AB = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

$$|AB| = 8932 - 8896$$

$$LHS = |AB| = 36.$$

$$RHS = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} \begin{vmatrix} 7 & 8 \\ 9 & 10 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \begin{vmatrix} 7 & 8 \\ 11 & 12 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \begin{vmatrix} 9 & 10 \\ 11 & 12 \end{vmatrix}$$

$$= (-3)(-2) + (-6)(-4) + (-3)(-2)$$

$$= 6 + 24 + 6$$

$$= 36$$

$\therefore$ LHS = RHS.

③ Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

then $AB = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{bmatrix}$

$\therefore |AB| = 0.$

INVERSE OF NON-SINGULAR MATRICES BY GUASS - JORDAN METHOD.

Guass - Jordan method: The elementary row operations which are used to convert non-singular matrix A to identity matrix, when performed on a Identity matrix I gives the inverse of A. This method is called Guass Jordan method of find the inverse of A.

Working rule: ① Write two matrices A and I side by side.

② Then perform the same elementary row operations on both.

③ As soon as A is reduced to I, the other matrix represents A^{-1}.

(or)

<u>Gauss - Jordan method</u>: Gauss - Jordan method is the method of finding the inverse of a non-singular ~~[illegible]~~ square matrix by elementary row operations only.

<u>Working rule</u>:

① Let A be a square matrix of order n.

② We can write $A = I_n A$.

③ Now we apply elementary row operations only to the matrix A in LHS and the pre-factor I of the RHS, We will do this till we get an equation of the form

$$I_n = B \cdot A.$$

Here B is the inverse of A., i.e., $\boxed{A^{-1} = B.}$

Q: Find the inverse of the following matrices by Gauss-Jordan method.

① $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

<u>Sol</u>: Let $A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

We write $A = I_3 A$.

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

Applying $R_2 \to 2R_2 - R_1$

$R_3 \to 2R_3 - R_1$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix} A.$$

$R_1 \to 3R_1 + R_2$

$R_3 \to 3R_3 + R_2$

$$\begin{bmatrix} 6 & 0 & 8 \\ 0 & 3 & -1 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ -1 & 2 & 0 \\ -4 & 2 & 6 \end{bmatrix} A.$$

$R_1 \to R_1 + 2R_3$

$R_2 \to 4R_2 - R_3$.

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} -6 & 6 & 12 \\ 0 & 6 & -6 \\ -4 & 2 & 6 \end{bmatrix} A.$$

$R_1 \to \frac{R_1}{6}$, $R_2 \to \frac{R_2}{12}$, $R_3 \to \frac{R_3}{-4}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix} A.$$

$$I_3 = B \cdot A.$$

$$\therefore A^{-1} = B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 1/2 & -1/2 \\ 1 & -1/2 & -3/2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -2 & 2 & 1 \\ 0 & 1 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

② $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

<u>Sol</u>: Let $A = \begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

We write $A = I_4 A$.

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} A.$$

$R_1 \to R_2 + R_1,\ R_3 \to 2R_3 + 2R_1,\ R_4 \to R_4 - R_1$

$$\begin{bmatrix} -1 & -3 & 3 & -1 \\ 0 & -2 & 2 & -1 \\ 0 & -11 & 8 & -5 \\ 0 & 4 & -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A.$$

$R_1 \to 2R_1 - 3R_2,\ R_3 \to 2R_3 - 11R_2,\ R_4 \to R_4 + 2R_2$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -7 & -11 & 2 & 0 \\ 1 & [illegible] & 0 & 1 \end{bmatrix} A$$

$R_2 \to 3R_2 + R_3,\ R_4 \to 6R_4 + R_3.$

$$\begin{bmatrix} -2 & 0 & 0 & 1 \\ 0 & -6 & 0 & -2 \\ 0 & 0 & -6 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -3 & 0 & 0 \\ -4 & -8 & 2 & 0 \\ -7 & -11 & 2 & 0 \\ -1 & 1 & 2 & 6 \end{bmatrix} A$$

$R_1 \to R_1 - R_4,\ R_2 \to R_2 + 2R_4,\ R_3 \to R_3 - R_4.$

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4 & -2 & -6 \\ -6 & -6 & 6 & 12 \\ -6 & -12 & 0 & -6 \\ -1 & 1 & 2 & 6 \end{bmatrix} A.$$

$R_1 \to \frac{R_1}{-2},\ R_2 \to \frac{R_2}{-6},\ R_3 \to \frac{R_3}{-6}.$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix} A.$$

$\Rightarrow \quad I_4 = B.A.$

$$\therefore A^{-1} = B = \begin{bmatrix} 0 & 2 & 1 & 3 \\ 1 & 1 & -1 & -2 \\ 1 & 2 & 0 & 1 \\ -1 & 1 & 2 & 6 \end{bmatrix}$$

(3) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Sol: let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

We write $A = I_3 A$.

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -4 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 2R_1 + R_2,\ R_3 \to 4R_3 - R_2.$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ -3 & 1 & 0 \\ -1 & -1 & 4 \end{bmatrix} A.$$

$R_1 \to 2R_1 - R_3$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & -4 \\ -3 & 1 & 0 \\ -1 & -1 & 4 \end{bmatrix} A.$$

$R_1 \to \frac{R_1}{4},\ R_2 \to \frac{R_2}{-4},\ R_3 \to \frac{R_3}{4}.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/4 & 3/4 & -4/4 \\ +3/4 & -1/4 & 0 \\ -1/4 & -1/4 & 4/4 \end{bmatrix} A.$$

$$I_3 = \frac{1}{4}\begin{bmatrix} -1 & 3 & -4 \\ 3 & -1 & 0 \\ -1 & -1 & 4 \end{bmatrix} A$$

$I_3 = B.A.$

$$\therefore A^{-1} = B = \frac{1}{4}\begin{bmatrix} -1 & 3 & -4 \\ 3 & -1 & 0 \\ -1 & -1 & 4 \end{bmatrix}$$

(4) $A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

Sol: We write $A = I_3 A$.

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} A.$$

$R_1 \to 3R_1 - R_3,\ R_2 \to 3R_2 - 2R_3.$

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix} A.$$

$R_1 \to R_1/3,\ R_2 \to R_2/3,\ R_3 \to R_3/3.$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/3 & -1/3 & -1/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & 1/3 & 1/3 \end{bmatrix} A$$

$I_3 = BA$

$$\therefore A^{-1} = B = \frac{1}{3}\begin{bmatrix} 1 & -1 & -1 \\ 2 & 1 & -2 \\ 2 & 1 & 1 \end{bmatrix}.$$

SYSTEM OF LINEAR EQUATIONS: SOLVING SYSTEM OF HOMOGENEOUS AND NON-HOMOGENEOUS EQUATIONS.

* An equation of the form $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$, where $x_1, x_2, \ldots, x_n$ are unknown variables and $a_1, a_2, \ldots, a_n, b$ are constants is called linear equation in n variables.
* ~~Consider~~ A system of m linear equations in n variables $x_1, x_2, \ldots, x_n$ ~~is~~ given below.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \longrightarrow (1).$$

where a_{ij}'s and b_i's are constants.

* Matrix representation of system (1) is given by.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & & & \\ \cdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\text{i.e., } AX = B \longrightarrow (2).$$

* The augmented matrix is written as $[A : B]$ which will be reduced to echelon form to find rank.
* An ordered n-tuple $(x_1, x_2, \ldots, x_n)$ satisfying all the equations in (1) simultaneously is called the solution of system (1).
* If $B = 0$ in (2), then the system is said to be <u>homogeneous</u>, otherwise the system is said to be <u>non-homogeneous system of linear equations</u>
* For a non-homogeneous system of linear equations
  (1) If rank (A) = rank $(A : B)$ = no. of variables, then the system is consistent and has unique solution.
  (2) If rank (A) = rank $(A : B)$ $\neq$ no. of variables, then the system is consistent and have infinite no. of solutions.
  (3) If rank$(A) \neq$ rank $(A : B)$, then the system is inconsistent i.e., it has no solution.

* For a homogeneous system of linear equations
① If rank(A) = no. of variables, then the system has trivial & zero solution.
② If rank(A) < no. of variables, then the system have infinite no. of solution.

Q: Find the nature of solution of the equations:
① $x_1+x_2+x_3+x_4=0$, $x_1+x_2+x_3-x_4=4$, $x_1+x_2-x_3+x_4=-4$ and $x_1-x_2+x_3+x_4=2$.

Sol: Given system of linear equations are

$$x_1+x_2+x_3+x_4=0$$
$$x_1+x_2+x_3-x_4=4$$
$$x_1+x_2-x_3+x_4=-4$$
$$x_1-x_2+x_3+x_4=2$$

~~The augmented matrix is~~
Matrix representation of above system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix}$$

i.e., $A\,X = B$

$\therefore$ Augmented matrix $= [A:B]$

$$= \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & -1 & 4 \\ 1 & 1 & -1 & 1 & -4 \\ 1 & -1 & 1 & 1 & 2 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 4 \\ 0 & 0 & -2 & 0 & -4 \\ 0 & -2 & 0 & 0 & 2 \end{array}\right]$$

Applying $R_2 \leftrightarrow R_4$

$$\sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 & -4 \\ 0 & 0 & 0 & -2 & 4 \end{array}\right]$$

The above matrix is echelon form.

* $\rho(A) = 4 = \rho(A:B) =$ no. of variables

$\therefore$ The given system is consistent and it has unique solution

(2) Solve the system of linear equations: $x+2y+3z=0$, $3x+4y+4z=0$ and $7x+10y+12z=0$.

Sol: Given eqns are $x+2y+3z=0$, $3x+4y+4z=0$, $7x+10y+12z$

Matrix form is

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e., $AX = 0$

Consider, $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{bmatrix}$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 7R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & -4 & -9 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \rho(A) = 3 =$ no. of variables.

Hence, the system has trivial solution

i.e., $x=0$, $y=0$, $z=0$ is the only solution.

(3) Find the values of a, b in which the equations $x+y+z=3$, $x+2y+2z=6$, $x+ay+3z=b$ have (i) No solution (ii) a unique solution (iii) infinite no. of solutions.

Sol: Given system of eqns are

$x+y+z=3$

$x+2y+2z=6$

$x+ay+3z=b$.

The matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 6 \end{bmatrix}$$

i.e., $A \; X = B$

Consider augmented matrix $[A : B]$

$$= \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 6 \\ 1 & a & 3 & b \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & a-1 & 2 & b-3 \end{array}\right]$$

Applying $R_3 \to R_3 - (a-1)R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 3-a & b-3a \end{array}\right]$$

The above matrix is in echelon form.

Rank of the matrix depends on a&b values.

Case (i) : If $a = 3$ and $b \neq 9$.

Then $\rho(A) = 2$, $\rho(A:B) = 3$.

$\rho(A) \neq \rho(A:B)$

$\therefore$ The system has no solution.

Case (ii) : If $a \neq 3$ ~~and~~ ~~b~~

$\rho(A) = 3 = \rho(A:B) =$ no. of variables.

$\therefore$ The system has a unique solution.

Case (iii) : If $a = 3$ and $b = 9$, then

$\rho(A) = 2 = \rho(A:B) <$ No. of variables $= 3$.

$\therefore$ The system have infinite no. of solutions.

GAUSS ELIMINATION METHOD.

The method of solving a system of linear equations in n unknown variables consists of eliminating the co-efficients in such a way that the system reduces to echelon form which may be solved by backward substitution is called Gauss elimination method.

Q) Solve the system of equations $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$ by Gauss elimination method. $(x=1, y=2, z=-1)$

① Solve the linear equations by gauss elimination method.

$$2x_1+x_2+2x_3+x_4=6$$
$$6x_1-6x_2+6x_3+12x_4=36$$
$$4x_1+3x_2+3x_3-3x_4=-1$$
$$2x_1+2x_2-x_3+x_4=10.$$

Sol: The matrix representation of given eqns is

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 6 & -6 & 6 & 12 \\ 4 & 3 & 3 & -3 \\ 2 & 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 36 \\ -1 \\ 10 \end{bmatrix}$$

i.e. $AX = B$.

Consider the augmented matrix,

$$[A:B] = \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 6 & -6 & 6 & 12 & 36 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_2 \to \frac{R_2}{6}$

$$\sim \left[\begin{array}{cccc|c} 2 & 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 2 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_1 \leftrightarrow R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 6 \\ 2 & 1 & 2 & 1 & 6 \\ 4 & 3 & 3 & -3 & -1 \\ 2 & 2 & -1 & 1 & 10 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$
$R_4 \to R_4 - 2R_1$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 7 & -1 & -11 & -25 \\ 0 & 4 & -3 & -3 & -2 \end{array}\right]$$

$R_3 \to 3R_3 - 7R_2$, $R_4 \to 3R_4 - 4R_2$

$$\sim \left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & -9 & 3 & 18 \end{array}\right]$$

$R_4 \to R_4 - 3R_3$.

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & 6 \\ 0 & 3 & 0 & -3 & -6 \\ 0 & 0 & -3 & -12 & -33 \\ 0 & 0 & 0 & 39 & 117 \end{array}\right]$$

The above matrix is in echelon form.

We have,

$$x_1 - x_2 + x_3 + 2x_4 = 6 \quad \text{--- } (1)$$

$$3x_2 - 3x_4 = -6 \quad \text{--- } (2)$$

$$-3x_3 - 12x_4 = -33 \quad \text{--- } (3)$$

$$39x_4 = 117 \quad \text{--- } (4)$$

By back substitution.

(4) $\Rightarrow$ $\boxed{x_4 = \frac{117}{39} = 3}$

(3) $\Rightarrow$ $-3x_3 - 12(3) = -33$.

$-3x_3 - 36 = -33$

$-3x_3 = -33+36$

$-3x_3 = 3$

$x_3 = \frac{3}{-3}$

$\boxed{x_3 = -1}$

(2) $\Rightarrow$ $3x_2 - 3(3) = -6$.

$3x_2 - 9 = -6$

$3x_2 = -6+9$

$3x_2 = 3$

$x_2 = 3/3$

$\boxed{x_2 = 1}$

(1) $\Rightarrow$ $x_1 - (1) + (-1) + 2(3) = 6$

$x_1 - 1 - 1 + 6 = 6$

$x_1 + 4 = 6$

$x_1 = 6 - 4$

$\boxed{x_1 = 2}$

$\therefore$ The solution of given system is

$x_1 = 2,\ x_2 = 1,\ x_3 = -1,\ x_4 = 3$.

(2) Solve $x+2y+z=4,\ 2x-y+3z=9,\ 3x-y-z=2$

Sol: Given eqns are $x+2y+z=4,\ 2x-y+3z=9,\ 3x-y-z=2$

The matrix form is

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 3 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 2 \end{bmatrix}$$

i.e., $AX = B$.

Consider the augmented matrix,

$$[A:B] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 9 \\ 3 & -1 & -1 & 2 \end{array}\right]$$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -5 & 1 & 1 \\ 0 & -7 & -4 & -10 \end{array}\right]$$

$R_3 \to 5R_3 - 7R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -5 & 1 & 1 \\ 0 & 0 & -27 & -57 \end{array}\right]$$

The above matrix is echelon form.

$\rho(A) = 3 = \rho(A:B) = 3 =$ no. of variables.

$\therefore$ The system has a unique solution.

~~By backward substitution~~

$x + 2y + z = 4 \quad \text{— (1)}$

$-5y + z = 1 \quad \text{— (2)}$

$-27z = -57 \quad \text{— (3)}$

By backward substitution,

(3) $\Rightarrow$ $z = \dfrac{-\cancel{57}^{19}}{-\cancel{27}_{9}} = \dfrac{19}{9} = 2.111\ldots$

$$\boxed{\therefore z = \frac{19}{9}}$$

(2) $\Rightarrow$ $-5y + \dfrac{19}{9} = 1$

$-5y = 1 - \dfrac{19}{9}$

$-5y = \dfrac{9-19}{9} = \dfrac{-10}{9}$

$y = \dfrac{-\cancel{10}^{2}}{9 \times -\cancel{5}} = \dfrac{2}{9} = 0.22\ldots$

$$\boxed{y = \frac{2}{9}}$$

① $\Rightarrow x + 2\left(\frac{2}{9}\right) + \left(\frac{19}{9}\right) = 4.$

$$x + \frac{4}{9} + \frac{19}{9} = 4$$

$$x + \frac{23}{9} = 4$$

$$x = 4 - \frac{23}{9}$$

$$x = \frac{36 - 23}{9} = \frac{13}{9} = 1.444\cdots$$

$$\boxed{x = \frac{13}{9}}$$

$\therefore$ The solution of given system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 13/9 \\ 2/9 \\ 19/9 \end{bmatrix}.$$

③ Show that the system of equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ is consistent and solve them.

Sol: Given eqns are $x+y+z=6$

$x+2y+3z=14,$

$x+4y+7z=30.$

The matrix form is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix}$$

i.e., $AX = B$.

Consider the augmented matrix

$$[A:B] = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$$

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right]$$

Applying $R_3 \to R_3 - 3R_2$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The above matrix is in echelon form.

We have $\rho(A) = \rho(A : B) = 2 < 3 =$ no. of variables.

$\therefore$ The given system is consistent and have infinite no. of solutions.

We get $x + y + z = 6$ —①

$y + 2z = 8$ —②

Let $z = k$.

By backward substitution,

② $\Rightarrow$ $y + 2k = 8$

$y = 8 - 2k$.

① $\Rightarrow$ $x + (8 - 2k) + k = 6$.

$x + 8 - k = 6$

$x = 6 - 8 + k$

$x = -2 + k$.

$\therefore$ the solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2+k \\ 8-2k \\ k \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \\ 0 \end{bmatrix} + k\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

where

$k = 0, 1, 2, 3, \ldots$

For different values of k we get infinite no. of solutions

④ Solve $4x + 2y + z + 3w = 0$, $6x + 3y + 4z + 7w = 0$, $2x + y + w = 0$.

Sol: Given eqns are $4x + 2y + z + 3w = 0$, $6x + 3y + 4z + 7w = 0$ and $2x + y + w = 0$.

The matrix form is

$$\begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

i.e, $AX = 0$.

consider $A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{bmatrix}$

Applying $R_2 \to 2R_2 - 3R_1$, $R_3 \to 2R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & -1 & -1 \end{array}\right]$$

$R_3 \to 5R_3 + R_2$.

$$\sim \left[\begin{array}{ccc|c} 4 & 2 & 1 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The above matrix is in echelon form.

$\rho(A) = 2 <$ no. of variables $= 4$.

$\therefore$ The system have infinite no. of solutions

we have

$$4x + 2y + z + 3w = 0 \quad \text{--- (1)}$$

$$5z + 5w = 0$$

$$\Rightarrow z + w = 0 \quad \text{--- (2)}.$$

By backward substitution,

let $w = k_1$ and $y = k_2$.

$$(2) \Rightarrow z + k_1 = 0$$

$$\Rightarrow z = -k_1$$

$$(1) \Rightarrow 4x + 2k_2 + (-k_1) + 3(k_1) = 0.$$

$$\Rightarrow 4x = -2k_1 - 2k_2$$

$$\Rightarrow x = -\frac{k_1}{2} - \frac{k_2}{2}.$$

The solution is
$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -\frac{k_1}{2} - \frac{k_2}{2} \\ k_2 \\ -k_1 \\ k_1 \end{bmatrix}$$

$$= k_1 \begin{bmatrix} -1/2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

where $k_1, k_2 = 0, 1, 2, 3, \dots$

For different values of k, we get infinite no. of solutions.

(5) Solve $x-y+z=2$, $3x-y+2z=-6$, $3x+y+z=-18$.

Sol: Given eqns are $x-y+z=2$, $3x-y+2z=-6$,

$3x+y+z=-18$.

The matrix form is

$$\begin{bmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 2 \\ -6 \\ -18 \end{bmatrix}$$

i.e., $AX = B$.

Consider the augmented matrix,

$$[A:B] = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 3 & -1 & 2 & -6 \\ 3 & 1 & 1 & -18 \end{array}\right]$$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 4 & -2 & -24 \end{array}\right]$$

$R_3 \to R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -1 & -12 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The above matrix is in echelon form

$\rho(A) = 2 = \rho(A:B) < 3 =$ no. of variables.

$\therefore$ The given system is consistent and have infinite no. of solutions.

We get $\quad x-y+z=2 \quad —(1)$

$2y-z=-12. \quad —(2)$

By backward substitution,

let $z=k$

(2) $\Rightarrow \quad 2y-k=-12$

$2y=-12+k$

$y=-6+k/2$

(1) $\Rightarrow x-(-6+k/2)+k=2$

$x+6-k/2+k=2$.

$x+6+k/2=2$

$x=2-6-k/2=-4-k/2$.

$\therefore$ The solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -4-k/2 \\ -6+k/2 \\ k \end{bmatrix}$

$= \begin{bmatrix} -4 \\ -6 \\ 0 \end{bmatrix} + k \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix}$, where $k = 0, 1, 2, 3, \ldots$

For different values of k, we get infinite no. of solutions.

(6) Solve $3x+y+2z=3$, $2x-3y-z=3$, $x+2y+z=4$ by Gauss elimination method.

Given eqns are $3x+y+2z=3$, $2x-3y-z=3$, $x+2y+z=4$.

Sol: The matrix form is given by.

$$\begin{bmatrix} 3 & 1 & 2 \\ 2 & -3 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

i.e., $AX = B$.

The augmented matrix, $[A:B] =$

$$\left[\begin{array}{ccc|c} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & 3 \\ 1 & 2 & 1 & 4 \end{array}\right]$$

$R_2 \leftrightarrow R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & 3 \\ 3 & 1 & 2 & 3 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -5 \\ 0 & -5 & -1 & -9 \end{array}\right]$$

$R_3 \to 7R_3 - 5R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -5 \\ 0 & 0 & 8 & -38 \end{array}\right]$$

The above matrix is in echelon form.

$\rho(A) = 3 = \rho(A:B) =$ no. of variables.

$\therefore$ The given system is consistent and have a unique solution.

We have, $x+2y+z=4$ — (1)

$-7y-3z=-5$ — (2)

$8z=-38$ — (3).

By backward substitution,

(3) $\Rightarrow z = -\frac{38}{8} = \frac{-19}{4}$.

(2) $\Rightarrow -7y - 3\left(-\frac{19}{4}\right) = -5$

$-7y + \frac{57}{4} = -5$

$-7y = -5 - \frac{57}{4} = \frac{-20-57}{4}$

$-7y = \frac{-77}{4}$

$y = \frac{\cancel{77}^{11}}{4\times\cancel{7}} = \frac{11}{4}$.

(3) $\Rightarrow x + 2\left(\frac{11}{4}\right) + \left(\frac{-19}{4}\right) = 4$

$x + \frac{22}{4} - \frac{19}{4} = 4$

$x + \frac{3}{4} = 4$

$x = 4 - \frac{3}{4} = \frac{16-3}{4}$

$x = \frac{13}{4}$.

$\therefore$ The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 13/4 \\ 11/4 \\ -19/4 \end{bmatrix}$$

$[A:B] =$

$$\begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & 3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

$R_2 \rightarrow 3R_2 - 2R_1$

$R_3 \rightarrow 3R_3 - R_1$

$$\begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -11 & -7 & 3 \\ 0 & 5 & 1 & 9 \end{bmatrix}$$

$R_3 \rightarrow 11R_3 + 5R_2$

$$\begin{bmatrix} 3 & 1 & 2 & 3 \\ 0 & -11 & -7 & 3 \\ 0 & 0 & -24 & 114 \end{bmatrix}$$

(or)

$3x + y + 2z = 3$ — (1)

$-11y - 7z = 3$ — (2)

$-24z = 114$ — (3)

(3) $\Rightarrow z = \frac{114}{-24} = \boxed{-\frac{19}{4}}$

(2) $\Rightarrow -11y - 7\left(-\frac{19}{4}\right) = 3$

$-11y + \frac{133}{4} = 3$

$-11y = 3 - \frac{133}{4}$

$-11y = \frac{12 - 133}{4}$

$y = \frac{-121}{-11 \times 4} = \boxed{\frac{11}{4}}$

(1) $\Rightarrow 3x + \frac{11}{4} - \frac{38}{4} = 3$

$3x - \frac{27}{4} = 3$

$3x = 3 + \frac{27}{4} = \frac{12+27}{4}$

$3x = \frac{39}{4}$

$x = \frac{39}{4 \times 3}$

$\boxed{x = \frac{13}{4}}$

JACOBI ITERATIVE METHOD:

(use only previous values)

Consider the equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \text{— (1)}$$

If the diagonal elements are dominant, i.e., a_1, b_2, c_3 are large compared to other co-efficients, then the system can be solved using Jacobi method.

The system (1) can be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}[d_1 - b_1y - c_1z] \\ y &= \frac{1}{b_2}[d_2 - a_2x - c_2z] \\ z &= \frac{1}{c_3}[d_3 - a_3x - b_3y] \end{aligned}\right\} \text{— (2)}$$

Let us start with initial approximations $x^{(0)} = 0, y^{(0)} = 0, z^{(0)} = 0$ of x, y, z in RHS of (2). Substituting these values we get first approximations ~~x_1, y_1, z_1~~. $x^{(1)}, y^{(1)}, z^{(1)}$

Substituting the 1^{st} approximations in RHS of (2), we get 2nd approximation., where the k^{th} approximation is given by

$$x^{(k)} = \frac{1}{a_1}[d_1 - b_1y^{(k-1)} - c_1z^{(k-1)}]$$

$$y^{(k)} = \frac{1}{b_2}[d_2 - a_2x^{(k-1)} - c_2z^{(k-1)}]$$

$$z^{(k)} = \frac{1}{c_3}[d_3 - a_3x^{(k-1)} - b_3y^{(k-1)}]$$

This process is repeated till the difference between two consecutive approximations is negligible.

① Solve $6x+2y-z=4$, $x+5y+z=3$, $2x+y+4z=27$ by Gauss-Jacobi method.

Sol: Given eqns are $6x+2y-z=4$,

$x+5y+z=3$

and $2x+y+4z=27$

Then we have.

$x=\frac{1}{6}[4-2y+z]$

$y=\frac{1}{5}[3-x-z]$

$z=\frac{1}{4}[27-2x-y]$

First iteration: Put $x^{(0)}=0,\ y^{(0)}=0,\ z^{(0)}=0$.

$x^{(1)}=\frac{1}{6}[4-2(0)+(0)]=\frac{1}{6}(4)=\frac{2}{3}=0.6667$

$y^{(1)}=\frac{1}{5}[3-0-0]=\frac{3}{5}=0.6$

$z^{(1)}=\frac{1}{4}[27-2(0)-0]=\frac{27}{4}=6.75.$

Second iteration:

$x^{(2)}=\frac{1}{6}[4-2y^{(1)}+z^{(1)}]=\frac{1}{6}[4-2(0.6)+6.75]=1.5916$

$y^{(2)}=\frac{1}{5}[3-x^{(1)}-z^{(1)}]=\frac{1}{5}[3-0.6667-6.75]=-0.8833$

$z^{(2)}=\frac{1}{4}[27-2x^{(1)}-y^{(1)}]=\frac{1}{4}[27-2(0.6667)-0.6]=6.2666.$

Third iteration:

$x^{(3)}=\frac{1}{6}[4-2(-0.8833)+6.2666]=2.0055$

$y^{(3)}=\frac{1}{5}[3-1.5916-6.2666]=-0.9716.$

$z^{(3)}=\frac{1}{4}[27-2(1.5916)-(-0.8833)]=6.1750$

Fourth iteration:

$x^{(4)}=\frac{1}{6}[4-2(-0.9716)+6.1750]=2.0197.$

$y^{(4)}=\frac{1}{5}[3-2.0055-6.1750]=-1.0361$

$z^{(4)}=\frac{1}{4}[27-2(2.0055)-(-0.9716)]=5.9901$

Fifth iteration:

$x^{(5)}=\frac{1}{6}[4-2(-1.0361)+5.9901]=2.0103$

$y^{(5)}=\frac{1}{5}[3-2.0197-5.9901]=-1.00196$

$z^{(5)}=\frac{1}{4}[27-2(2.0197)-(-1.0361)]=\cancel{2.0103}\ 5.9991$

Sixth iteration:

$x^{(6)} = \frac{1}{6}[4-2(-1.00196)+5.9991] = 2.0005$

$y^{(6)} = \frac{1}{5}[3-2.0103-5.9991] = -1.0018$

$z^{(6)} = \frac{1}{4}[27-2(2.0103)-(-1.00196)] = 5.9953$

Seventh iteration:

$x^{(7)} = \frac{1}{6}[4-2(-1.0018)+5.9953] = 1.9998$

$y^{(7)} = \frac{1}{5}[3-2.0005-5.9953] = -0.99916$

$z^{(7)} = \frac{1}{4}[27-2(2.0005)-(-1.0018)] = 6.0002.$

Eighth iteration:

$x^{(8)} = \frac{1}{6}[4-2(-0.99916)+6.0002] = 1.8332$

$y^{(8)} = \frac{1}{5}[3-1.9998-6.0002] = -1$

$z^{(8)} = \frac{1}{4}[27-2(1.9998)-(-0.99916)] = 5.9998$

$\therefore$ The solution is $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1.9998 \\ -0.99916 \\ 6.0002 \end{bmatrix} \simeq \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

② Solve $20x+y-2z=17$, $3x+20y-z=-18$ and $2x-3y+20z=25$ by Jacobi method.

Sol: Given eqns are $20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=25$.

We have

$x = \frac{1}{20}[17-y+2z]$

$y = \frac{1}{20}[-18-3x+z]$

$z = \frac{1}{20}[25-2x+3y]$.

First iteration: Put $x^{(0)}=0, y^{(0)}=0, z^{(0)}=0.$

$x^{(1)} = 0.85, y^{(1)} = -0.9, z^{(1)} = 1.25.$

2nd iteration: $x^{(2)} = 1.02, y^{(2)} = -0.965, z^{(2)} = 1.03$

3rd iteration: $x^{(3)} = 1.00125; y^{(3)} = -1.0015, z^{(3)} = 1.00325.$

4th iteration: $x^{(4)} = 1.0004, y^{(4)} = -1.000025, z^{(4)} = 0.9993875.$

5th iteration: $x^{(5)} = 0.99994, y^{(5)} = -1.00090625, z^{(5)} = 1.00000225.$

6th iteration: $x^{(6)} = 1.000004756, y^{(6)} = -0.999990887, z^{(6)} = 0.999992406.$

7th iteration: $x^{(7)} = 0.999998785, y^{(7)} = -1.000000751, z^{(7)} = 1.000000891.$

8th iteration: $x^{(8)} = 1.000000127, y^{(8)} = -0.999999773$
$z^{(8)} = 1.000000009.$

$\therefore x \simeq 1, y \simeq -1, z \simeq 1.$ is the solution.

GAUSS-SEIDEL METHOD.

This is the modified Jacobi's iteration method. (we updated values immediately)

Consider,
$$\left.\begin{aligned} a_1x+b_1y+c_1z&=d_1\\ a_2x+b_2y+c_2z&=d_2\\ a_3x+b_3y+c_3z&=d_3 \end{aligned}\right\} \text{--- (1)}$$

The system can be written as
$$\left.\begin{aligned} x&=\frac{1}{a_1}[d_1-b_1y-c_1z]\\ y&=\frac{1}{b_2}[d_2-a_2x-c_2z]\\ z&=\frac{1}{c_3}[d_3-a_3x-b_3y] \end{aligned}\right\} \text{--- (2)}$$

We start with initial approximations ~~x_0, y_0, z~~.
$x^{(0)}=0, y^{(0)}=0, z^{(0)}=0$.

Substituting $y^{(0)}=0$, $z^{(0)}=0$ in 1st equation of system (2) we get $x^{(1)}$.

Substituting $x^{(1)}, z^{(0)}=0$ in 2nd equation of system (2) we get $y^{(1)}$.

Substituting $x^{(1)}, y^{(1)}$ in 3rd equation of system (2) we get $z^{(1)}$.

This process of iteration is continued till the difference between two consecutive approximations is negligible.

(1) Solve $3x-y+z=1$, $3x+6y+2z=0$, $3x+3y+7z=4$ by Gauss-Seidel iterative method.

Sol: Given eqns are
$$3x-y+z=1$$
$$3x+6y+2z=0$$
$$3x+3y+7z=4$$

which can be written as
$$x=\frac{1}{3}[1+y-z]$$
$$y=\frac{1}{6}[-3x-2z]$$
$$z=\frac{1}{7}[4-3x-3y]$$

First iteration:

Put $y^{(0)}=0$, $z^{(0)}=0$, we get
$$x^{(1)}=\frac{1}{3}[1+y^{(0)}-z^{(0)}]=\frac{1}{3}[1+0-0]=\frac{1}{3}=0.3333$$

Put $x^{(1)}$ and $z^{(0)}$, we get
$$y^{(1)}=\frac{1}{6}[-3x^{(1)}-2z^{(0)}]=\frac{1}{6}[-3(0.3333)-2(0)]=-0.1666$$

Put $x^{(1)}$ and $y^{(1)}$, we get
$$z^{(1)}=\frac{1}{7}[4-3x^{(1)}-3y^{(1)}]=\frac{1}{7}[4-3(0.3333)-3(-0.1666)]=0.4999$$

2nd iteration:

$$x^{(2)} = \frac{1}{3}[1+(-0.1666)-0.4999] = 0.1111$$

$$y^{(2)} = \frac{1}{6}[-3(0.1111)-2(0.4999)] = -0.2221$$

$$z^{(2)} = \frac{1}{7}[4-3(0.1111)-3(-0.2221)] = 0.619.$$

3rd iteration:

$$x^{(3)} = \frac{1}{3}[1+(-0.2221)-0.619] = 0.0529.$$

$$y^{(3)} = \frac{1}{6}[-3(0.0529)-2(0.619)] = -0.2327.$$

$$z^{(3)} = \frac{1}{7}[4-3(0.0529)-3(-0.2327)] = 1.13485$$

4th iteration:

$$x^{(4)} = \frac{1}{3}[1+(-0.2327)-1.13485] = -0.1225$$

$$y^{(4)} = \frac{1}{6}[-3(-0.1225)-2(1.13485)] = -0.3170.$$

$$z^{(4)} = \frac{1}{7}[4-3(-0.1225)-3(-0.3170)] = 0.7597$$

5th iteration:

$$x^{(5)} = \frac{1}{3}[1+(-0.3170)-0.7597] = -0.0255$$

$$y^{(5)} = \frac{1}{6}[-3(-0.0255)-2(0.7597)] = -0.2404.$$

$$z^{(5)} = \frac{1}{7}[4-3(-0.0255)-3(-0.2404)] = 0.6853.$$

6th iteration:

$$x^{(6)} = \frac{1}{3}[1+(-0.2404)-0.6853] = 0.0247$$

$$y^{(6)} = \frac{1}{6}[-3(0.0247)-2(0.6853)] = -0.2407.$$

$$z^{(6)} = \frac{1}{7}[4-3(0.0247)-3(-0.2407)] = 0.664.$$

7th iteration:

$$x^{(7)} = \frac{1}{3}[1+(-0.2407)-0.664] = 0.0317$$

$$y^{(7)} = \frac{1}{6}[-3(0.0317)-2(0.6664)] = -0.2371$$

$$z^{(7)} = \frac{1}{7}[4-3(0.0317)-3(\underset{-0.2371}{\cancel{-0.2317}})] = 0.6594$$

∴ The solution is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0.0317 \\ -0.2371 \\ 0.6594 \end{bmatrix}$$

(2) Solve the following system of equations by Gauss ~~Seidel~~ Seidel method.

$4x+2y+z=14,\ x+5y-z=10,\ x+y+8z=20$

Sol: Given eqns are

$$4x+2y+z=14$$
$$x+5y-z=10$$
$$x+y+8z=20$$

which can be written as

$$x=\frac{1}{4}[14-2y-z]$$
$$y=\frac{1}{5}[10-x+z]$$
$$z=\frac{1}{8}[20-x-y]$$

First iteration: let $x^{(0)}=0,\ y^{(0)}=0.$

$$x^{(1)}=\frac{1}{4}[14-0-0]=3.5$$
$$y^{(1)}=\frac{1}{5}[10-3.5-0]=1.3$$
$$z^{(1)}=\frac{1}{8}[20-3.5-1.5]=1.875$$

2nd iteration:

$$x^{(2)}=\frac{1}{4}[14-2(1.3)-1.875]=2.3812.$$
$$y^{(2)}=\frac{1}{5}[10-2.3812+1.875]=1.898.$$
$$z^{(2)}=\frac{1}{8}[20-2.3812-1.898]=1.9651.$$

3rd iteration:

$$x^{(3)}=\frac{1}{4}[14-2(1.898)-1.9651]=2.0597$$
$$y^{(3)}=\frac{1}{5}[10-2.0597+1.9651]=1.9812.$$
$$z^{(3)}=\frac{1}{8}[20-2.0597-1.9812]=1.9948.$$

4th iteration:

$$x^{(4)}=\frac{1}{4}[14-2(1.9812)-1.9948]=2.0107.$$
$$y^{(4)}=\frac{1}{5}[10-2.0107+1.9948]=1.9968$$
$$z^{(4)}=\frac{1}{8}[20-2.0107-1.9968]=1.999$$

5th iteration:

$$x^{(5)}=\frac{1}{4}[14-2(1.9968)-1.999]=2.0018$$
$$y^{(5)}=\frac{1}{5}[10-2.0018+1.999]=1.999.$$
$$z^{(5)}=\frac{1}{8}[20-2.0018-1.999]=1.999.$$

6th iteration:

$x^{(6)} = \frac{1}{4}[14 - 2(1.999) - 1.999] = 2.0007$

$y^{(6)} = \frac{1}{5}[10 - 2.0007 + 1.999] = 1.9996$

$z^{(6)} = \frac{1}{8}[20 - 2.0007 - 1.9996] = 1.9999.$

Hence, $x = 2.0007 \simeq 2$

$y = 1.9996 \simeq 2$

$z = 1.9999 \simeq 2.$

Q: Solve by Gauss-Seidel method

$x - 2y = -3$, $2x + 25y = 15.$

Sol: Given, $x - 2y = -3$, $2x + 25y = 15.$

$x = -3 + 2y.$

$y = \frac{1}{25}[15 - 2x]$

We start from $x^{(0)} = 0$, $y^{(0)} = 0.$

1st iteration:

$x^{(1)} = -3 + 2(0) = -3.$

$y^{(1)} = \frac{1}{25}[15 - 2(-3)] = 0.84$

2nd iteration:

$x^{(2)} = -3 + 2(0.84) = -1.32.$

$y^{(2)} = \frac{1}{25}[15 - 2(-1.32)] = 0.7056$

3rd iteration:

$x^{(3)} = -3 + 2(0.7056) = -1.5888$

$y^{(3)} = \frac{1}{25}[15 - 2(-1.5888)] = 0.727104.$

4rth iteration:

$x^{(4)} = -3 + 2(0.727104) = -1.545792.$

$y^{(4)} = \frac{1}{25}[15 - 2(-1.545792)] = 0.72366336.$

Hence, $x \simeq -1.5$

$y \simeq 0.72.$