UNIT - II

EIGEN VALUES, EIGEN VECTORS AND ORTHOGONAL TRANSFORMATION

EIGEN VALUES AND EIGEN VECTORS:

Let $A = [a_{ij}]$ be an $n \times n$ matrix. A non-zero vector X is said to the characteristic vector of A, if there exists a scalar λ such that $AX = \lambda X$.

If $AX = \lambda X$ $(X \neq 0)$, we say that X is a eigen vector or characteristic vector of A corresponding to the eigen value or characteristic value λ of A.

Ex. ① Take $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1.\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda X.$$

Here $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value $\lambda = 1$ of A.

② $A = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}$, $X = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

$$AX = \begin{bmatrix} 5 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 24 \\ 6 \end{bmatrix} = 6.\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \lambda X.$$

Here $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to the eigen value $\lambda = 6$ of A.

To find the eigen values & eigen vectors of a matrix:

let $A = [a_{ij}]$ be an $n \times n$ matrix. let X be an eigen vector of A corresponding to the eigen value λ.

By definition $AX = \lambda X$

$$AX - \lambda I X = 0$$

$$(A - \lambda I) X = 0.$$

This is a homogeneous system of n equations in n variables. This will have non-zero solutions X, if and only if

$|A - \lambda I| = 0$.

$A - \lambda I$ is called characteristic matrix of A.

$|A - \lambda I| = 0$ is called the characteristic ~~funct~~ equation of A.

This will be polynomial equation in λ of degree n.

By solving this equation we get n roots.

$$\lambda = \lambda_1, \lambda_2, ---, \lambda_n.$$

These are the characteristic roots or eigen values of a matrix.

Consider the homogeneous system,

$(A - \lambda_i I) X_i = 0$ for $i = 1, 2, --, n$.

Solve of X_i, we get eigen vectors.

PROPERTIES OF EIGEN VALUES AND EIGEN VECTORS.

1. The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.
2. If λ is an eigen value of A corresponding to the eigen vector X, then λ^n is eigen value of A^n corresponding to the eigen vector X.
3. A square matrix A and its transpose A^T have the same eigen values
4. If $\lambda_1, \lambda_2, ----, \lambda_n$ are the eigen values of matrix A, then $k\lambda_1, k\lambda_2, ---, k\lambda_n$ are the eigen values of the matrix kA, where k is a non-zero scalar.
5. If λ is an eigen value of the matrix A, then $\lambda + k$ is an eigen value of the matrix $A + kI$.

6. If λ is an eigen value of a non-singular matrix corresponding to the eigen vector X, then $\lambda^{-1} = \frac{1}{\lambda}$ is an eigen value of A^{-1} corresponding to eigen vector X itself.

7. If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value.

8. If λ is an eigen value of A, then the eigen value of $B = a_0A^2 + a_1A + a_2I$ is $a_0\lambda^2 + a_1\lambda + a_2$.

9. The eigen values of a triangular matrix are just the diagonal elements of the matrix.

① Find the eigen values of the matrix $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

We have, $[A - \lambda I]X = 0$.

$$\left[\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—①}$$

To find eigen values.

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 5-\lambda & -2 & 0 \\ -2 & 6-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{vmatrix} = 0$$

$$(5-\lambda)[(6-\lambda)(7-\lambda) - 4] + 2[(-2)(7-\lambda) - 0] + 0[-4 - 0] = 0$$

$$(5-\lambda)[42 - 6\lambda - 7\lambda + \lambda^2 - 4] + 2[-14 + 2\lambda] = 0$$

$$(5-\lambda)[38 - 13\lambda + \lambda^2] + 2[-14 + 2\lambda] = 0$$

$$190 - 65\lambda + 5\lambda^2 - 38\lambda + 13\lambda^2 - \lambda^3 - 28 + 4\lambda = 0$$

$$-\lambda^3 + 18\lambda^2 - 99\lambda + 162 = 0.$$

$$\lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0$$

Put $\lambda = 3$.

$$(3)^3 - 18(3)^2 + 99(3) - 162 = 0$$

$$\Rightarrow 27 - 18(9) + 297 - 162 = 0$$

$$\Rightarrow 27 - 162 + 297 - 162 = 0$$

$$0 = 0.$$

$\therefore \lambda = 3$ is a root.

$$\begin{array}{c|cccc} 3 & 1 & -18 & 99 & -162 \\ & 0 & 3 & -45 & 162 \\ \hline & 1 & -15 & 54 & \boxed{0} \end{array}$$

$\lambda = 3$, $\lambda^2 - 15\lambda + 54 = 0$

$\lambda^2 - 6\lambda - 9\lambda + 54 = 0$

$\lambda(\lambda - 6) - 9(\lambda - 6) = 0$

$\Rightarrow (\lambda - 6)(\lambda - 9) = 0$

$\therefore \lambda = 3, 6, 9.$

$\therefore$ The Eigen values of A are 3, 6, 9.

(or)

Given, $A = \begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$

To find eigen values we write

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where S_1 = sum of the diagonal elements of A.

$= 5 + 6 + 7 = 18$

S_2 = sum of minors of diagonal elements of A.

$= \begin{vmatrix} 6 & 2 \\ 2 & 7 \end{vmatrix} + \begin{vmatrix} 5 & 0 \\ 0 & 7 \end{vmatrix} + \begin{vmatrix} 5 & -2 \\ -2 & 6 \end{vmatrix}$

$= (42 - 4) + (35 - 0) + (30 - 4)$

$= 38 + 35 + 26$

$= 99.$

$S_3 = |A| = \begin{vmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{vmatrix}$

$= 5(42 - 4) + 2(-14 - 0) + 0$

$= 5(38) - 28 = 190 - 28$

$= 162.$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 99\lambda - 162 = 0.$

$\therefore \lambda = 3, 6, 9.$

$\therefore$ Eigen values of A are 3, 6, 9.

(5) Find the eigen values and corresponding eigen vectors of the matrix $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

We have $[A - \lambda I]X = 0$

$$\Rightarrow \begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{——} \; (1)$$

To find eigen values; we write

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

$$S_1 = 6+3+3 = 12.$$

$$S_2 = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 6 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 6 & -2 \\ -2 & 3 \end{vmatrix}$$

$$= (9-1) + (18-4) + (18-4)$$

$$= 8 + 14 + 14 = 36.$$

$$S_3 = |A| = 6(9-1) + 2(-6+2) + 2(2-6)$$

$$= 6(8) + 2(-4) + 2(-4)$$

$$= 48 - 8 - 8$$

$$= 32$$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\Rightarrow \lambda^2 - 12\lambda^2 + 36\lambda - 32 = 0.$$

$\lambda = 2$	1	-12	36	-32
	0	2	-20	32
	1	-10	16	$\boxed{0}$

$$(\lambda-2)(\lambda^2 - 10\lambda + 16) = 0$$

$$\lambda = 2, \quad \lambda^2 - 10\lambda + 16 = 0$$

$$\lambda^2 - 8\lambda - 2\lambda + 16 = 0$$

$$\lambda(\lambda-8) - 2(\lambda-8) = 0$$

$$(\lambda-2)(\lambda-8) = 0$$

$$\therefore \lambda = 2, 2, 8.$$

To find eigen vector corresponding to $\lambda = 8$:

For $\lambda = 8$

$$(1) \Rightarrow \begin{bmatrix} 6-8 & -2 & 2 \\ -2 & 3-8 & -1 \\ 2 & -1 & 3-8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - R_1, \quad R_3 \rightarrow R_3 + R_1$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & -3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} -2 & -2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1-2x_2+2x_3=0 \quad (2)$$
$$-3x_2-3x_3=0 \quad (3)$$

let $x_3=k$

$(3) \Rightarrow -3x_2-3k=0$
$$x_2=\frac{3k}{-3}=-k.$$

$(2) \Rightarrow -2x_1-2(-k)+2k=0$
$$-2x_1+2k+2k=0$$
$$-2x_1=-4k$$
$$x_1=2k.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 2k \\ -k \\ k \end{bmatrix}=k\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore X_1=\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to $\lambda=8$.

Eigen vectors corresponding to $\lambda=2$:

For $\lambda=2$

$$(1) \Rightarrow \begin{bmatrix} 6-2 & -2 & 2 \\ -2 & 3-2 & -1 \\ 2 & -1 & 3-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow 2R_2+R_1, \quad R_3 \rightarrow 2R_3-R_1$

$$\begin{bmatrix} 4 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$4x_1-2x_2+2x_3=0$$

let $x_3=k_1, x_2=k_2$.

then $4x_1-2k_2+2k_1=0$
$$4x_1=-2k_1+2k_2$$
$$x_1=-\frac{k_1}{2}+\frac{k_2}{2}.$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -\frac{k_1}{2}+\frac{k_2}{2} \\ k_2 \\ k_1 \end{bmatrix}=k_1\begin{bmatrix} -1/2 \\ 0 \\ 1 \end{bmatrix}+k_2\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}=k_1\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}+k_2\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore X_2=\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$ and $X_3=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are the eigen vectors of A corresponding to eigen value $\lambda=2$.

(3) Find the eigen values & eigen vectors of the matrix

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Sol: let $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

To find eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

where $S_1 = 8+7+3 = 18.$

$$S_2 = \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$$

$= (21-16) + (24-4) + (56-36)$

$= 5 + 20 + 20$

$= 45$

$$S_3 = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix} = 8(21-16) + 6(-18+8) + 2(24-14)$$

$= 8(5) + 6(-10) + 2(10)$

$= 40 - 60 + 20$

$= 0.$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda = 0$

$\Rightarrow \lambda[\lambda^2 - 18\lambda + 45] = 0$

$\Rightarrow \lambda[\lambda^2 - 15\lambda - 3\lambda + 45] = 0$

$\Rightarrow \lambda[\lambda(\lambda-15) - 3(\lambda-15)] = 0$

$\Rightarrow \lambda(\lambda-15)(\lambda-3) = 0$

$\therefore \lambda = 0, 15, 3.$

~~Eigen vector corresponding to $\lambda \neq 0$~~

To find eigen vectors:

$$(A - \lambda I)X = 0.$$

$$\Rightarrow \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 0$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 - 4x_2 + 3x_3 = 0$ — (1)

$-5x_2 + 5x_3 = 0$ — (2)

let $x_3 = k$

(2) $\Rightarrow -5x_2 + 5k = 0$

$\Rightarrow -5x_2 = -5k$

$x_2 = k$

(1) $\Rightarrow 2x_1 - 4k + 3k = 0$

$2x_1 = k$

$x_1 = k/2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k/2 \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix} = k \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ is the eigen vector of A corresponding to the eigen value $\lambda = 0$.

<u>For $\lambda = 3$:</u>

$$\begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2 + 6R_1$, $R_3 \to 5R_3 - 2R_1$

$$\begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & -8 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to 2R_3 - R_2$

$$\begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 6x_2 + 2x_3 = 0$ — (3)

$-16x_2 - 8x_3 = 0$ — (4)

Let $x_3 = k_1$

(4) $\Rightarrow -16x_2 - 8k_1 = 0$

$x_2 = \frac{8k_1}{-16}$

$x_2 = -k_1/2$.

(3) $\Rightarrow 5x_1 - 6(-k_1/2) + 2k_1 = 0$

$5x_1 + 3k_1 + 2k_1 = 0$

$5x_1 = -5k_1$

$x_1 = -k_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ -k_1/2 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ -1/2 \\ 1 \end{bmatrix} = k_1\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ is the eigen vector of A corresponding to eigen value $\lambda = 3$.

$R_1 \leftrightarrow R_3$.

(or) $\begin{bmatrix} 2 & -4 & 0 \\ -6 & 4 & -4 \\ 5 & -6 & 2 \end{bmatrix}$

$R_2 \to R_2 + 3R_1$

$R_3 \to 2R_3 - 5R_1$

$$\begin{bmatrix} 2 & -4 & 0 \\ 0 & -8 & -4 \\ 0 & 8 & 4 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} 2 & -4 & 0 \\ 0 & -8 & -4 \\ 0 & 0 & 0 \end{bmatrix}$$

$2x_1 - 4x_2 = 0$

$-8x_2 - 4x_3 = 0$

Let $x_3 = k_1$

$-8x_2 - 4k_1 = 0$

$x_2 = \frac{4k_1}{-8} = \frac{-k_1}{2}$

$2x_1 - 4(-k_1/2) = 0$

$2x_1 = -2k$

$x_1 = -k$.

For $\lambda = 15$:

$$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix} 2 & -4 & -12 \\ -6 & -8 & -4 \\ -7 & -6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 3R_1$, $R_3 \to 2R_3 + 7R_1$

$$\begin{bmatrix} 2 & -4 & -12 \\ 0 & -20 & -40 \\ 0 & -40 & -80 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$.

$$\begin{bmatrix} 2 & -4 & -12 \\ 0 & -20 & -40 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 - 4x_2 - 12x_3 = 0$

$\Rightarrow x_1 - 2x_2 - 6x_3 = 0$ —— (5)

$-20x_2 - 40x_3 = 0$

$\Rightarrow x_2 + 2x_3 = 0$ —— (6)

let $x_3 = k_2$

(6) $\Rightarrow x_2 + 2k_2 = 0$

$x_2 = -2k_2$

(5) $\Rightarrow x_1 - 2(-2k_2) - 6k_2 = 0$

$x_1 + 4k_2 - 6k_2 = 0$

$x_1 = 2k_2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2k_2 \\ -2k_2 \\ k_2 \end{bmatrix} = k_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$\therefore X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to the eigen value $\lambda = 15$.

(4) Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & -4 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

Sol: Given, $A = \begin{bmatrix} 3 & -4 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$

To find eigen values:

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where S_1 = sum of the diagonal elements of the matrix.

$= 3 + 0 + (-1) = 2$.

S_2 = Sum of minors of the diagonal elements

$$= \begin{vmatrix} 0 & 1 \\ 2 & -1 \end{vmatrix} + \begin{vmatrix} 3 & 3 \\ 1 & -1 \end{vmatrix} + \begin{vmatrix} 3 & -4 \\ 1 & 0 \end{vmatrix}$$

$= (0-2) + (-3-3) + (0+4)$

$= -2 - 6 + 4$

$= -4$.

$$S_3 = |A| = \begin{vmatrix} 3 & -4 & 3 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{vmatrix} = 3(0-2) + 4(-1-1) + 3(2-0)$$

$= 3(-2) + 4(-2) + 3(2)$

$= -6 - 8 + 6 = -8$.

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$\lambda = 2$, $\lambda^2 - 4 = 0$

$(\lambda+2)(\lambda-2) = 0$

$\therefore \lambda = 2, 2, -2.$

$$\lambda = 2 \; \begin{array}{|cccc} 1 & -2 & -4 & 8 \\ 0 & 2 & 0 & -8 \\ \hline 1 & 0 & -4 & \boxed{0} \end{array}$$

To find eigen vectors:

$[A - \lambda I]X = 0.$

$$\begin{bmatrix} 3-\lambda & -4 & 3 \\ 1 & 0-\lambda & 1 \\ 1 & 2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 2$:

$$\begin{bmatrix} 3-2 & -4 & 3 \\ 1 & 0-2 & 1 \\ 1 & 2 & -1-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 & 3 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & -4 & 3 \\ 0 & 2 & -2 \\ 0 & 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

2 + 4

$R_3 \to R_3 - 3R_2$

$$\begin{bmatrix} 1 & -4 & 3 \\ 0 & 2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 - 4x_2 + 3x_3 = 0$ — (1)

$2x_2 - 2x_3 = 0$ — (2)

let $x_3 = k$.

(2) $\Rightarrow 2x_2 - 2k = 0$

$x_2 = k.$

(1) $\Rightarrow x_1 - 4(k) + 3(k) = 0$

$x_1 - k = 0$

$x_1 = k.$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k \\ k \\ k \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen vector of A corresponding to the eigen value $\lambda = 2$. Similarly, $X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

For $\lambda = -2$:

$$\begin{bmatrix} 3+2 & -4 & 3 \\ 1 & 0+2 & 1 \\ 1 & 2 & -1+2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -4 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 5 & -4 & 3 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 5R_2 - R_1$

$$\begin{bmatrix} 5 & -4 & 3 \\ 0 & 14 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$5x_1 - 4x_2 + 3x_3 = 0$ — (3)

$14x_2 + 2x_3 = 0$ — (4)

Let $x_3 = k$.

(4) $\Rightarrow$ $14x_2 + 2k = 0$

$x_2 = -k/7$

(3) $\Rightarrow$ $5x_1 - 4(-k/7) + 3k = 0$

$5x_1 + \frac{4}{7}k + \frac{21}{7}k = 0$

$5x_1 = -\frac{25}{7}k$.

$x_1 = -\frac{5}{7}k$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{5}{7}k \\ -\frac{1}{7}k \\ k \end{bmatrix} = k \begin{bmatrix} -\frac{5}{7} \\ -\frac{1}{7} \\ 1 \end{bmatrix} = k \begin{bmatrix} -5 \\ -1 \\ 7 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} -5 \\ -1 \\ 7 \end{bmatrix}$ is the eigen ~~value~~ vector of A corresponding to the eigen value $\lambda = -2$.

(3) Find the eigen values and eigen vectors of the matrix

$$A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Sol: Given, $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

To find eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = -2+1+0 = -1$

$$S_2 = \begin{vmatrix}1 & -6\\ -2 & 0\end{vmatrix} + \begin{vmatrix}-2 & -3\\ -1 & 0\end{vmatrix} + \begin{vmatrix}-2 & 2\\ 2 & 1\end{vmatrix} = (0-12)+(0-3)+(-2-4)$$
$$= -12-3-6 = -21$$

$$S_3 = -2(0-12) - 2(0-6) - 3(-4+1)$$
$$= -2(-12) - 2(-6) - 3(-3)$$
$$= +24+12+9$$
$$S_3 = 45$$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\Rightarrow \lambda^3 + 1\lambda^2 - 21\lambda - 45 = 0$$

$$\begin{array}{c|cccc} \lambda=5 & 1 & 1 & -21 & -45 \\ & 0 & 5 & 30 & 45 \\ \hline & 1 & 6 & 9 & \boxed{0} \end{array}$$

$$(\lambda-5)(\lambda^2+6\lambda+9)=0$$
$$(\lambda-5)(\lambda+3)^2=0$$
$$\lambda = 5, -3, -3.$$

To find eigen vectors:

$$(A-\lambda I)X = 0$$

$$\begin{bmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & 0-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \quad \text{— (1)}$$

For $\lambda = 5$

$$\begin{bmatrix}-7 & 2 & -3\\ 2 & -4 & -6\\ -1 & -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1 \leftrightarrow R_3$

$$\begin{bmatrix}-1 & -2 & -5\\ 2 & -4 & -6\\ -7 & 2 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2 \to R_2 + 2R_1$, $R_3 \to R_3 - 7R_1$

$$\begin{bmatrix}-1 & -2 & -5\\ 0 & -8 & -16\\ 0 & 16 & 32\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix}-1 & -2 & -5\\ 0 & -8 & -16\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$-x_1 - 2x_2 - 5x_3 = 0 \quad \text{— (2)}$$
$$-8x_2 - 16x_3 = 0 \quad \text{— (3)}$$

let $x_3 = k$

(3) $\Rightarrow$ $-8x_2 = 16k$

$$x_2 = -2k$$

(2) $\Rightarrow$ $-x_1 - 2(-2k) - 5k = 0$

$$-x_1 = +9k$$
$$x_1 = -9k$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9k \\ -2k \\ k \end{bmatrix} = k\begin{bmatrix} 9 \\ -2 \\ 1 \end{bmatrix}.$$

$\therefore X_1 = \begin{bmatrix} 9 \\ -2 \\ 1 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda = 5$.

For $\lambda = -3$

$$① \Rightarrow \begin{bmatrix} -2-(-3) & 2 & -3 \\ 2 & 1-(-3) & -6 \\ -1 & -2 & 0-(-3) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow R_2 - 2R_1, \; R_3 \rightarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$

Let $x_3 = k_1, \; x_2 = k_2$

$\Rightarrow x_1 + 2k_2 - 3k_1 = 0$

$x_1 = 3k_1 - 2k_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3k_1 - 2k_2 \\ k_2 \\ k_1 \end{bmatrix} = k_1\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ are the eigen vectors of A corresponding to the eigen value $\lambda = -3$.

⑥ Write the characteristic equation of $\begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$.

Sol: Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix}$

The characteristic eqn is

$$|A - \lambda I| = 0$$

$$\left| \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\begin{vmatrix} 1-\lambda & 1 \\ 2 & 5-\lambda \end{vmatrix} = 0$$

$(1-\lambda)(5-\lambda) - 2 = 0$

$5 - \lambda - 5\lambda + \lambda^2 - 2 = 0$

$\therefore \lambda^2 - 6\lambda + 3 = 0$ is the required eqn.

(7) What are the eigen values of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 9 \end{bmatrix}$

WKT, Eigen values of a triangular matrix is its diagonal elements.

$\therefore$ Eigen values of $A = 1, 4, 9$.

(8) What are the eigen values of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 7 \end{bmatrix}$

WKT, Eigen values of a diagonal matrix are its diagonal elements.

$\therefore$ Eigen value of $A = 1, -2, 7$.

(9) If the eigen values of A are $1, -2, 3$ then eigen values of A^{-1} are?

Sol: WKT, If λ is the eigen value of A, then $\lambda^{-1} = \frac{1}{\lambda}$ is the eigen value of A^{-1}.

If eigen values of A are $1, -2, 3$, then eigen values of A^{-1} are $1, -\frac{1}{2}, \frac{1}{3}$.

(10) If the eigen values of A are 3, 4 then eigen values of A^3 are?

Sol: WKT, If λ is the eigen value of A, then λ^n is the eigen value of A^n.

If eigen values of A are 3, 4 then eigen values of A^3 are $3^3 = 27$, $4^3 = 64$.

(11) What is the eigen value of $3A^3 + 5A^2 - 6A + 2I$ if 2 is the eigen value of A?

Sol: WKT, If $\lambda = 2$ is the eigen value of A, then is the eigen value of $3A^3 + 5A^2 - 6A + 2I$ is

$$3\lambda^3 + 5\lambda^2 - 6\lambda + 2 = 3(2)^3 + 5(2)^2 - 6(2) + 2$$
$$= 24 + 20 - 12 + 2$$
$$= 34.$$

DIAGONALISATION OF A MATRIX:

A matrix A is said to be diagonalizable if there exists an invertible matrix P such that $P^{-1}AP = D$ where D is a diagonal matrix. Here P is called the modal matrix and the resulting diagonalize matrix D is known as spectral matrix.

Procedure to diagonalize the matrix:

Step 1: To a given matrix A, find eigen values $\lambda_1, \lambda_2, \lambda_3$.

Step 2: Find eigen vectors X_1, X_2, X_3.

Step 3: Consider modal matrix $P = [X_1 \; X_2 \; X_3]$

Step 4: Find P^{-1}.

Step 5: Compute $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$.

Procedure to find powers of a matrix:

Let A be a square matrix. Then

$$D = P^{-1}AP$$

$$D^2 = D \cdot D$$

$$= (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A(PP^{-1})AP$$

$$= P^{-1}A(I)AP$$

$$D^2 = P^{-1}A^2P$$

Similarly $D^3 = P^{-1}A^3P$

$\vdots \qquad \vdots$

$$D^n = P^{-1}A^nP.$$

$$PD^nP^{-1} = P(P^{-1}A^nP)P^{-1}$$

$$= (PP^{-1})A^n(PP^{-1})$$

$$PD^nP^{-1} = IA^nI = A^n$$

$$\therefore \boxed{A^n = PD^nP^{-1}}$$

(1) Diagonalize the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

Sol: Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$

To find eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where S_1 = Sum of diagonal elements of A.

$$= 1 + 2 + 3 = 6,$$

S_2 = Sum of the minors of diagonal elements of A.

$$= \begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$$

$$= (6-4) + (3+4) + (2-0)$$

$$= 2+7+2 = 11.$$

$$S_3 = |A| = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{vmatrix} = 1(6-4) - 1(0+4) + 1(0+8)$$

$$= 2 - 4 + 8 = 6.$$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$

$\lambda=1$	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0.$

$\lambda = 1, \quad \lambda^2 - 2\lambda - 3\lambda + 6 = 0$

$\lambda(\lambda-2) - 3(\lambda-2) = 0$

$(\lambda-2)(\lambda-3) = 0.$

$\therefore \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3.$

To find eigen vectors:

$(A - \lambda I)X = 0.$

$$\begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ -4 & 4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{--- (1)}$$

For $\lambda_1 = 1$:

$$\begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 2-1 & 1 \\ -4 & 4 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -4 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_3.$

$$\begin{bmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} -4 & 4 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-4x_1 + 4x_2 + 2x_3 = 0$ --- (2)

$x_2 + x_3 = 0$ --- (3)

let $x_3 = K_1$

(3) $\Rightarrow x_2 + K_1 = 0$

$\Rightarrow x_2 = -K_1$

(2) $\Rightarrow -4x_1 + 4(-K_1) + 2(K_1) = 0$

$-4x_1 - 4K_1 + 2K_1 = 0$

$-4x_1 - 2K_1 = 0$

$-4x_1 = 2K_1$

$x_1 = -K_1/2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K_1/2 \\ -K_1 \\ K_1 \end{bmatrix} = K_1 \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} = K_1 \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

$\therefore X_1 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$ is the eigen ~~value~~ vector corresponding to the eigen value $\lambda_1 = 1$

For $\lambda_2 = 2$:

$$① \Rightarrow \begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 2-2 & 1 \\ -4 & 4 & 3-2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ -4 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 3R_1$

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_1 + x_2 + x_3 = 0$ — ④

$x_3 = 0$ — ⑤

Let $x_2 = K_2$

④ $\Rightarrow -x_1 + K_2 + 0 = 0$

$\therefore x_1 = K_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} K_2 \\ K_2 \\ 0 \end{bmatrix} = K_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector corresponding to the eigen value $\lambda_2 = 2$.

For $\lambda_3 = 3$:

$$① \Rightarrow \begin{bmatrix} 1-3 & 1 & 1 \\ 0 & 2-3 & 1 \\ -4 & 4 & 3-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ -4 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} -2 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 + x_3 = 0$ — (6)

$-x_2 + x_3 = 0$ — (7)

let $x_3 = k_3$

(7) $\Rightarrow -x_2 + k_3 = 0$

$x_2 = k_3$

(6) $\Rightarrow -2x_1 + k_3 + k_3 = 0$

$\Rightarrow x_1 = k_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_3 \\ k_3 \\ k_3 \end{bmatrix} = k_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is the eigen value of A corresponding to the eigen vector $\lambda_3 = 3$.

Modal matrix $P = [X_1 \; X_2 \; X_3]$

$$\therefore P = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}.$$

To find P^{-1}:

$$|P| = \begin{vmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -1(1-0) - 1(-2-2) + 1(0-2)$$

$= -1 + 4 - 2$

$= 1$

$$\text{adj } P = \begin{bmatrix} +(1) & -(-4) & +(-2) \\ -(1) & +(-3) & -(-2) \\ +(0) & -(1) & +(1) \end{bmatrix}^T$$

$$\text{adj } P = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{|P|} \text{adj } P = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix}.$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -1 \\ -2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 8 & -6 & -2 \\ -6 & 6 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

Hence $P^{-1}AP$ is a diagonal matrix.

(2) Diagonalize the matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

To find eigen values:

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

where $S_1 = 1+2+3 = 6$.

$$S_2 = \begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = (6-2)+(3+2)+(2-0)$$

$$= 4+5+2 = 11.$$

$$S_3 = |A| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = 1(6-2) - 0 - 1(2-4)$$

$$= 1(4) - 1(-2)$$

$$= 4+2$$

$$= 6.$$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\begin{array}{r|rrrr} \lambda=1 & 1 & -6 & 11 & -6 \\ & 0 & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$$(\lambda-1)(\lambda^2 - 5\lambda + 6) = 0$$

$$\therefore \lambda = 1, 2, 3.$$

i.e., $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$.

To find eigen vectors:

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

For $\lambda_1 = 1$:

$$(1) \Rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 2R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 + x_2 + x_3 = 0$ —— (2)

$-x_3 = 0$

$\Rightarrow x_3 = 0$

Let $x_2 = k_1$

$x_1 + k_1 + 0 = 0$

$x_1 = -k_1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_1 \\ k_1 \\ 0 \end{bmatrix} = k_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ ~~the~~ $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ is the eigen vector of $\lambda_1 = 1$.

For $\lambda_2 = 2$:

$$(1) \Rightarrow \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_1$

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \rightarrow 2R_2 - R_1$

$$\begin{bmatrix} 2 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + 2x_2 + x_3 = 0$ —— (3)

$-2x_2 + x_3 = 0$ —— (4)

Let $x_3 = k_2$

(4) $\Rightarrow$ $-2x_2 + k_2 = 0$

$x_2 = \frac{k_2}{2}$

(3) $\Rightarrow$ $2x_1 + k_2 + k_2 = 0$

$x_1 = -k_2$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K_2 \\ K_2/2 \\ K_2 \end{bmatrix} = K_2 \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \end{bmatrix} = K_2 \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector of $\lambda_2 = 2$.

For $\lambda_3 = 3$:

$$① \Rightarrow \begin{bmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$
$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 + R_2$

$$\begin{bmatrix} -2 & 0 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 - x_3 = 0$ —— ⑤
$-2x_2 + x_3 = 0$ —— ⑥

let $x_3 = k_3$.

$⑥ \Rightarrow -2x_2 + k_3 = 0$
$x_2 = k_3/2$.

$⑤ \Rightarrow -2x_1 - k_3 = 0$
$x_1 = -k_3/2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -k_3/2 \\ k_3/2 \\ k_3 \end{bmatrix} = k_3 \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} = k_3 \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore X_3 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$ is the eigen vector of $\lambda_3 = 3$.

Modal matrix $P = [X_1 \ X_2 \ X_3]$

$$P = \begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

To find P^{-1}: $|P| = -1(2-2) + 2(2-0) - 1(2-0)$
$= -1(0) + 2(2) - 1(2)$
$= 0 + 4 - 2$
$= 2$

$$\text{adj } A = \begin{bmatrix} +(-0) & -(2) & +(2) \\ +(-2) & +(-2) & -(-2) \\ +(-1) & -(0) & +(1) \end{bmatrix}^T$$

$$\text{adj } A = \begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\therefore P^{-1} = \frac{1}{|P|} \text{adj } A = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Now,

$$P^{-1}AP = \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -2 & -2 & 0 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2 & -1 \\ -4 & -4 & 0 \\ 6 & 6 & 3 \end{bmatrix}\begin{bmatrix} -1 & -2 & -1 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

$\therefore$ Hence $P^{-1}AP$ is the diagonal matrix.

(3) Diagonalize the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

Sol: [Note: Follow upto finding eigen vectors from last topic 5th problem].

$$\therefore X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \lambda_1 = 5, \quad X_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, X_3 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \lambda_2 = \lambda_3 = -3.$$

Modal matrix $P = \begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

To find P^{-1}:

$$|P| = -1(0-1) - 3(0-1) - 2(-2-0)$$
$$= -1(-1) - 3(-1) - 2(-2)$$
$$= 1 + 3 + 4 = 8.$$

$$\operatorname{adj} P = \begin{bmatrix} -1 & +1 & -2 \\ -2 & 2 & +4 \\ 3 & +5 & 6 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix}$$

$$P^{-1} = \frac{1}{|P|} \operatorname{adj} P = \frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix}.$$

$$\therefore P^{-1}AP = \frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ 1 & 2 & 5 \\ -2 & 4 & 6 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} -5 & -10 & 15 \\ -3 & -6 & -15 \\ 6 & -12 & -18 \end{bmatrix}\begin{bmatrix} -1 & 3 & -2 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 40 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & -24 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

Hence $P^{-1}AP$ is a diagonal matrix.

CAYLEY-HAMILTON THEOREM:

Statement: Every square matrix satisfies its own characteristic equation.

Application: 1. To find the inverse of a matrix
2. To find higher powers of a matrix.

① Verify Cayley-Hamilton theorem and hence find A^{-1} and A^4 where $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$.

To find the characteristic equation $|A - \lambda I| = 0$

we write $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where S_1 = Sum of the diagonal elements of A.

$= 1 + 1 + 1 = 3$.

S_2 = Sum of minors of the diagonal elements of A:

$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$= (1-4)+(1+2)+(1-4)$

$= -3 + 3 - 3.$

$= -3$

$$S_3 = |A| = \begin{vmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{vmatrix} = 1(1-4)-2(2+4)-1(-4-2)$$

$= 1(-3)-2(6)-1(-6)$

$= -3-12+6 = -9.$

Now $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\lambda^3 - 3\lambda^2 - 3\lambda + 9 = 0$

is the required characteristic eqn.

To verify Cayley-Hamilton theorem, we have to show that.

$A^3 - 3A^2 - 3A + 9I = 0$ — (1)

Now, $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix}$$

$\therefore A^3 - 3A^2 - 3A + 9I =$

$$= \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} - 3\begin{bmatrix} 3 & 6 & -6 \\ 0 & 9 & -6 \\ 0 & 0 & 3 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 24 & -21 \\ 6 & 21 & -24 \\ 6 & -6 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 18 & -18 \\ 0 & 27 & -18 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -3 \\ 6 & 3 & -6 \\ 6 & -6 & 3 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A satisfies its own characteristic equation. Hence cayley - Hamilton theorem is verified.

To find A^{-1}:

$$A^{-1} \times (1) \Rightarrow A^{-1}A^3 - 3A^{-1}A^2 - 3A^{-1}A + 9A^{-1}I = 0$$

$$\Rightarrow A^2 - 3A - 3I + 9A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{9}[3A + 3I - A^2]$$

$$\Rightarrow A^{-1} = \frac{1}{9}\left\{3\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix} + 3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix}\right\}$$

$$= \frac{1}{9}\left\{\begin{bmatrix}3 & 6 & -3\\ 6 & 3 & -6\\ 6 & -6 & 3\end{bmatrix} + \begin{bmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{bmatrix} - \begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix}\right\}$$

$$\therefore A^{-1} = \frac{1}{9}\begin{bmatrix}3 & 0 & 3\\ 6 & -3 & 0\\ 6 & -6 & 3\end{bmatrix} = \begin{bmatrix}\frac{1}{3} & 0 & \frac{1}{3}\\ \frac{2}{3} & -\frac{1}{3} & 0\\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3}\end{bmatrix}$$

To find A^4:

$$A \times (1) \Rightarrow A\cdot A^3 - 3A\cdot A^2 - 3A\cdot A + 9 A\cdot I = 0$$

$$\Rightarrow A^4 - 3A^3 - 3A^2 + 9A = 0$$

$$\Rightarrow A^4 = 3A^3 + 3A^2 - 9A.$$

$$= 3\begin{bmatrix}3 & 24 & -21\\ 6 & 21 & -24\\ 6 & -6 & 3\end{bmatrix} + 3\begin{bmatrix}3 & 6 & -6\\ 0 & 9 & -6\\ 0 & 0 & 3\end{bmatrix} - 9\begin{bmatrix}1 & 2 & -1\\ 2 & 1 & -2\\ 2 & -2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}9 & 72 & -63\\ 18 & 63 & -72\\ 18 & -18 & 9\end{bmatrix} + \begin{bmatrix}9 & 18 & -18\\ 0 & 27 & -18\\ 0 & 0 & 9\end{bmatrix} - \begin{bmatrix}9 & 18 & -9\\ +18 & 9 & -18\\ 18 & -18 & 9\end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix}9 & 72 & -72\\ 0 & 81 & -72\\ 0 & 0 & 9\end{bmatrix}$$

(2) Show that the matrix $A = \begin{bmatrix}1 & -2 & 2\\ 1 & +2 & 3\\ 0 & -1 & 2\end{bmatrix}$ satisfies its characteristic equation. Hence find A^{-1}.

Sol: Given, $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

To find characteristic eqn $|A - \lambda I| = 0$

we write, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1 = 1+2+2 = 5$.

$S_2 = \begin{vmatrix} 2 & 3 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & 2 \end{vmatrix}$

$= (4+3) + (2-0) + (2+2)$

$= 7+2+4 = 13$.

$S_3 = |A| = 1(4+3) + 2(2-0) + 2(-1-0)$

$= 1(7) + 2(2) + 2(-1)$

$= 7+4-2 = 9$.

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 13\lambda - 9 = 0$

is the required characteristic equation.

To verify Cayley-Hamilton theorem, we have to show that

$A^3 - 5A^2 + 13A - 9I = 0$ — (1)

Now, $A^2 = A \cdot A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

$A^2 = \begin{bmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

$= \begin{bmatrix} -9 & -14 & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{bmatrix}$

$\therefore A^3 - 5A^2 + 13A - 9I$

$= \begin{bmatrix} -9 & -14 & -26 \\ 2 & -22 & 31 \\ -5 & -7 & -12 \end{bmatrix} - 5\begin{bmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{bmatrix} + 13\begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

∴ A satisfies its own chara...
Hence Cayley-Hamilton theorem is verified.

To find A^{-1}:

$A^{-1} \times ① \Rightarrow A^{-1}A^3 - 5A^{-1}A^2 + 13A^{-1}A - 9A^{-1}I = 0$

$\Rightarrow A^2 - 5A + 13I - 9A^{-1} = 0$

$\Rightarrow 9A^{-1} = A^2 - 5A + 13I$

$A^{-1} = \frac{1}{9}[A^2 - 5A + 13I]$

$$A^{-1} = \frac{1}{9}\left\{\begin{bmatrix} -1 & -8 & 0 \\ 3 & -1 & 14 \\ -1 & -4 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & -2 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + 13\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\}$$

$$\therefore A^{-1} = \frac{1}{9}\begin{bmatrix} 7 & 2 & -10 \\ -2 & 2 & -1 \\ -1 & 1 & 4 \end{bmatrix}.$$

③ Verify Cayley-Hamilton theorem and hence find A^{-1} where $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

To find characteristic eqn $|A - \lambda I| = 0$.
we write
~~where~~ $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$,

where $S_1 = 1 - 2 + 2 = 1$

$S_2 = \begin{vmatrix} -2 & 3 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 0 & 2 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 1 & -2 \end{vmatrix}$

$= (-4+3) + (2-0) + (-2+2)$

$= -1 + 2 + 0 = 1$

$S_3 = |A| = 1(-4+3) + 2(2-0) + 2(-1-0)$

$= 1(-1) + 2(2) + 2(-1)$

$= -1 + 4 - 2 = 1.$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - \lambda^2 + \lambda - 1 = 0$

is the required characteristic equation.

To verify Cayley-Hamilton theorem, we have to show that

$A^3 - A^2 + A - I = 0$ — ①.

Now, $A^2 = A \cdot A = \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

Now, $A^3 - A^2 + A - I$

$$\Rightarrow \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A satisfies its own characteristic equation.

Hence Cayley-Hamilton theorem is verified.

To find A^{-1}:

$A^{-1} \times (1) \Rightarrow A^{-1}A^3 - A^{-1}A^2 + A^{-1}A - A^{-1}I = 0$

$\Rightarrow A^2 - A + I - A^{-1} = 0$

$\Rightarrow A^{-1} = A^2 - A + I$

$$= \begin{bmatrix} -1 & 0 & 0 \\ -1 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} +1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -1 & 2 & -2 \\ -2 & 2 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

(4) Verify Cayley-Hamilton theorem and hence find A^4

where $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

To find characteristic eqn $|A - \lambda I| = 0$

we write $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where S_1 = Sum of the diagonal elements of A.

$= 1 - 1 - 1 = -1.$

S_2 = Sum of minors of the diagonal elements of A.

$= \begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$

$= (1-0) + (-1-0) + (-1-4)$

$= 1 - 1 - 5 = -5.$

$S_3 = |A| = \begin{vmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1(1-0) - 2(-2-0) + 0$

$= 1(1) - 2(-2)$

$= 1 + 4 = 5.$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 + \lambda^2 - 5\lambda - 5 = 0$

which is the required characteristic eqn.

To verify Cayley Hamilton theorem we have to prove that

$A^3 + A^2 - 5A - 5I = 0$ — (1)

Now, $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$\therefore A^3 + A^2 - 5A - 5I = \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A satisfies its own characteristic equation.
Hence Cayley-Hamilton theorem is verified.

To find A^4:

$$A \times (1) \Rightarrow A\cdot A^3 + A\cdot A^2 - 5A\cdot A - 5\cdot A\cdot I = 0$$

$$\Rightarrow A^4 + A^3 - 5A^2 - 5A = 0$$

$$\Rightarrow A^4 = 5A + 5A^2 - A^3$$

$$= 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} + 5\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 25 & 0 & 0 \\ 0 & 25 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(5) Verify Cayley-Hamilton theorem and hence find A^{-1} and A^4

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

Sol: Given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

To find characteristic eqn $|A - \lambda I| = 0$

we write $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where S_1 = Sum of the diagonal elements of A

$= 1 - 1 + 2 = 2.$

S_2 = Sum of minors of diagonal elements of A.

$$= \begin{vmatrix} -1 & 3 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix}$$

$$= (-2-3) + (2-3) + (-1-4)$$

$$= -5 - 1 - 5 = -11.$$

$$S_3 = |A| = 1(-2-3) - 2(4-3) + 3(2+1)$$

$$= 1(-5) - 2(1) + 3(3)$$

$$= -5 - 2 + 9 = 2.$$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$

$\Rightarrow \lambda^3 - 2\lambda^2 - 11\lambda - 2 = 0$

which is the required characteristic eqn.

To verify Cayley-Hamilton theorem we have to prove that

$$A^3 - 2A^2 - 11A - 2I = 0 \quad (1)$$

Now $A^2 = A\cdot A = \begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 3\\ 1 & 1 & 2\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 3\\ 1 & 1 & 2\end{bmatrix}$

$$A^2 = \begin{bmatrix}8 & 3 & 15\\ 3 & 8 & 9\\ 5 & 3 & 10\end{bmatrix}$$

$$A^3 = A^2\cdot A = \begin{bmatrix}8 & 3 & 15\\ 3 & 8 & 9\\ 5 & 3 & 10\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\ 2 & -1 & 3\\ 1 & 1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}29 & 28 & 63\\ 28 & 7 & 51\\ 21 & 17 & 44\end{bmatrix}$$

$$A^3 - 2A^2 - 11A - 2I = \begin{bmatrix}29 & 28 & 63\\ 28 & 7 & 51\\ 21 & 17 & 64\end{bmatrix} - \begin{bmatrix}16 & 6 & 30\\ 6 & 16 & 18\\ 10 & 6 & 20\end{bmatrix} - \begin{bmatrix}11 & 22 & 33\\ 22 & -11 & 33\\ 11 & 11 & 22\end{bmatrix} - \begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore$ A satisfies its own characteristic eqn.
Hence Cayley-Hamilton theorem is verified.

To find A^{-1}:

$A^{-1} \times ① \Rightarrow A^{-1}\cdot A^3 - 2A^{-1}A^2 - 11A^{-1}A - 2A^{-1}I = 0.$

$\Rightarrow A^2 - 2A - 11I - 2A^{-1} = 0$

$\Rightarrow 2A^{-1} = A^2 - 2A - 11I$

$\Rightarrow A^{-1} = \frac{1}{2}\left[A^2 - 2A - 11I\right]$

$$= \frac{1}{2}\left\{\begin{bmatrix}8 & 3 & 15\\ 3 & 8 & 9\\ 5 & 3 & 10\end{bmatrix} - \begin{bmatrix}2 & 4 & 6\\ 4 & -2 & 6\\ 2 & 2 & 4\end{bmatrix} - \begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}\right\}$$

$$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix}-5 & -1 & 9\\ -1 & -1 & 3\\ 3 & 1 & -5\end{bmatrix}$$

To find A^4:

$A \times ① \Rightarrow A\cdot A^3 - 2A\cdot A^2 - 11A\cdot A - 2A\cdot I = 0$

$\Rightarrow A^4 = 2A^3 + 11A^2 + 2A.$

$$= \begin{bmatrix}58 & 56 & 126\\ 56 & 14 & 102\\ 42 & 34 & 88\end{bmatrix} + \begin{bmatrix}88 & 33 & 165\\ 33 & 88 & 99\\ 55 & 33 & 110\end{bmatrix} + \begin{bmatrix}2 & 4 & 6\\ 4 & -2 & 6\\ 2 & 2 & 4\end{bmatrix}$$

$$\therefore A^4 = \begin{bmatrix}148 & 93 & 297\\ 93 & 100 & 207\\ 99 & 69 & 202\end{bmatrix}$$

⑥ Find the characteristic eqn of the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and verify the cayley hamilton theorem and hence find A^{-1}.

Sol: Given $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

To find characteristic eqn $|A - \lambda I| = 0$

we write $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where S_1 = sum of the diagonal elements of A.

$= 2 + 2 + 2 = 6$.

S_2 = sum of the minors of diagonal elements of A.

$= \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$

$= (4-1) + (4-1) + (4-1)$

$= 3 + 3 + 3 = 9$.

$S_3 = |A| = 2(4-1) + 1(-2+1) + 1(1-2)$

$= 2(3) + 1(-1) + 1(-1)$

$= 6 - 1 - 1 = 6 - 2 = 4$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4 = 0$

which is the required characteristic eqn.

To verify Cayley-Hamilton theorem, we have to prove that

$A^3 - 6A^2 + 9A - 4I = 0$ — ①

Now, $A^2 = A \cdot A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$A^3 = A^2 \cdot A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

$A^3 - 6A^2 + 9A - 4I = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 22 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\therefore$ A satisfies its own characteristic eqn.

Hence Cayley-Hamilton theorem is verified.

To find A^{-1}: $A^{-1} \times$ ① $\Rightarrow A^2 - 6A + 9I - 4A^{-1} = 0$

$\Rightarrow 4A^{-1} = A^2 - 6A + 9I$

$\Rightarrow A^{-1} = \frac{1}{4}\{A^2 - 6A + 9I\} = \frac{1}{4}\left\{\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right\}$

$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

QUADRATIC FORM:

An expression of the form $Q = X^T A X = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$

where a_{ij}'s are constants is called a quadratic form in n variables $x_1, x_2, \ldots, x_n$.

If the constants a_{ij}'s are real numbers, it is called a real quadratic form.

We have $Q = X^T A X = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j$

which can be written as

$$Q = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij} x_i x_j = [x_1 \; x_2 \; \cdots \; x_n]\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix}$$

$$Q = X^T A X$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n\end{bmatrix}$ and A is known as the matrix of quadratic form.

Ex: If $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c\end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z\end{bmatrix}$,

then $X^T = [x \; y \; z]$

and $Q = X^T A X = [x\; y\; z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c\end{bmatrix}\begin{bmatrix} x \\ y \\ z\end{bmatrix}$

A	x	y	z
x	a	h	g
y	h	b	f
z	g	f	c

$= ax^2 + hxy + gxz + hxy + by^2 + fyz + gxz + fyz + cz^2$

$\therefore Q = ax^2 + by^2 + cz^2 + 2hxy + 2gxz + 2fyz$.

① Write the matrix of the quadratic form $2x^2 + 4xy + 2y^2$.

Sol: Given, $Q = 2x^2 + 4xy + 2y^2$

$$\therefore A = \begin{bmatrix} 2 & 2 \\ 2 & 2\end{bmatrix}$$

	x	y
x	2	4/2
y	4/2	2

② Write the matrix of the quadratic form $x^2 + y^2 + z^2 + 2xy + 4yz + 2zx$

Sol: Given, $Q = x^2 + y^2 + z^2 + 2xy + 4yz + 2zx$.

The matrix of the quadratic form

$$\therefore A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1\end{bmatrix}$$

	x	y	z
x	1	2/2	2/2
y	2/2	1	4/2
z	2/2	4/2	1

(3) Convert the symmetric matrix into quadratic form

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$$

Sol: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix}$

$$Q = x^T A x = [x\ y\ z] \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

	x	y	z
x	1	2	3
y	2	1	3
z	3	3	1

$$= x^2 + 2xy + 3xz + 2xy + y^2 + 3yz + 3zx + 3yz + z^2$$

$$= x^2 + y^2 + z^2 + 4xy + 6xz + 6yz.$$

(4) Find the symmetric matrix corresponding to QF $ax^2 + 2hxy + by^2$

Sol: Given $Q = ax^2 + 2hxy + by^2$

$$A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$$

	x	y
x	a	2h/2
y	2h/2	b

Linear transformation of Quadratic form:

Let $P(x, y)$ and $P(x', y')$ are any two points related by $x' = a_1 x + a_2 y$

$y' = b_1 x + b_2 y.$

i.e., $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\Rightarrow X = AY$

Such transformation is called linear transformation in two dimension. Similarly the linear transformation in n-dimension can be defined.

Canonical form (or) Normal form of a quadratic form:

Let X^TAX be a quadratic form in n variables. Then, there exists a real non-singular linear transformation $X = PY$ which transforms X^TAX to another quadratic form of type $Y^TDY = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$, then Y^TDY is called the canonical form of X^TAX. Here $D = \text{diag}[\lambda_1, \lambda_2, \cdots, \lambda_n]$.

Rank of a Quadratic form:

Let X^TAX be a quadratic form over $\mathbb{R}$. The rank 'r' of A is called rank of the QF, which is equal to the no. of terms in the canonical form.

If $r < n$, where n = no. of variables (or) $|A| = 0$ (or)

A is singular, then Q.F is singular otherwise non-singular.

Index of a Quadratic form:

The no. of positive terms in a normal form (or) canonical form of a quadratic form is known as the index of a quadratic form. It is denoted by 's'.

Signature of a Quadratic form:

Signature of a quadratic form is defined as the difference between the number of positive terms and number of negative terms in its canonical form.

Nature of a Quadratic form:

The quadratic form X^TAX in n variables is said to be

1. Positive definite, if $r = n$ and $s = n$ (or) if all the eigen values of A are positive.
2. Negative definite, if $r = n$ and $s = 0$ (or) if all the eigen values of A are negative.
3. Positive semi definite, If $r < n$ and $s = r$ (or) if all the eigen values of $A \geq 0$ and at least one eigen value is zero.
4. Negative semi definite, if $r < n$ and $s = 0$ (or) if all the eigen values of $A \leq 0$ and at least one eigen value is zero.
5. Indefinite, in all other cases (or) if A has positive as well as negative eigen values.

Q.
① Identify the nature of the quadratic form
$-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$.

Sol: The given quadratic form is

$$-3x_1^2 - 3x_2^2 - 3x_3^2 - 2x_1x_2 - 2x_1x_3 + 2x_2x_3$$

	x_1	x_2	x_3
x_1	-3	$-2/2$	$-2/2$
x_2	$-2/2$	-3	$2/2$
x_3	$-2/2$	$2/2$	-3

The corresponding matrix of QF is

$$A = \begin{bmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{bmatrix}$$

The characteristic eqn of A is

$$|A - \lambda I| = 0.$$

$$\begin{vmatrix} -3-\lambda & -1 & -1 \\ -1 & -3-\lambda & 1 \\ -1 & 1 & -3-\lambda \end{vmatrix} = 0.$$

To find eigen values,

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where,

S_1 = Sum of diagonal elements of $A = -3-3-3 = -9$.

S_2 = Sum of minor of diagonal elements of A.

$= \begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ -1 & -3 \end{vmatrix}$

$= (9-1)+(9-1)+(9-1) = 8+8+8 = 24.$

$S_3 = |A| = \begin{vmatrix} -3 & -1 & -1 \\ -1 & -3 & 1 \\ -1 & 1 & -3 \end{vmatrix} = -3(9-1)+1(3+1)-1(-1-3)$

$= -3(8)+1(4)-1(-4)$

$= -24+4+4 = -16.$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 + 9\lambda^2 + 24\lambda + 16 = 0$

$\lambda = -1, -4, -4 < 0.$

-1	1	9	24	16
	0	-1	-8	-16
-4	1	8	16	$\boxed{0}$
	0	-4	-16	
	1	4	$\boxed{0}$	

All the eigen values are negative.

$\therefore$ The Quadratic form is negative definite.

(2) State the nature of QF $2x_1x_2 + 2x_1x_3 + 2x_2x_3$.

Sol: Given, $Q = 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

The matrix is, $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

	x_1	x_2	x_3
x_1	0	$2/2$	$2/2$
x_2	$2/2$	0	$2/2$
x_3	$2/2$	$2/2$	0

The characteristic eqn is

$|A - \lambda I| = 0$

$\begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$

To find eigen values,

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$S_1 = 0+0+0 = 0$

$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = (0-1)+(0-1)+(0-1)$

$= -3.$

$S_3 = 0(0-1) - 1(0-1) + 1(1-0) = 0+1+1 = 2.$

Now $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 3\lambda - 2 = 0$

$\therefore \lambda = -1, -1, 2.$

-1	1	0	-3	-2
	0	-1	$+1$	2
-1	$+1$	-1	-2	$\boxed{0}$
	0	-1	2	
	1	-2	$\boxed{0}$	

We see that some eigen values are positive and some are negative.

$\therefore$ Given QF is indefinite.

Orthogonal matrix: A square matrix A is orthogonal if
$A^T A = A A^T = I$.

If A is a orthogonal matrix, then

- $A^{-1} = A^T$.
- $|A| = +1$ or -1
- Rows and columns of an orthogonal matrix are orthonormal vectors (mutually perpendicular unit vectors)

Orthogonal transformation: If A is an orthogonal matrix and X, Y are two column vectors, then the transformation $Y = AX$ is called an orthogonal transformation.

REDUCTION OF QUADRATIC FORM TO CANONICAL FORM BY ORTHOGONAL TRANSFORMATION

Step 1: write the corresponding matrix of the given QF.

Step 2: Find eigen values $(\lambda_1, \lambda_2, \lambda_3)$, eigen vectors (X_1, X_2, X_3) and hence find normalised eigen vectors

$$\left(e_1 = \frac{X_1}{|X_1|}, e_2 = \frac{X_2}{|X_2|}, e_3 = \frac{X_3}{|X_3|}\right)$$

Step 3: Form the orthogonal modal matrix $P = [e_1\ e_2\ e_3]$

Step 4: Compute Diagonal matrix, $D = P^{-1}AP$, where $P^{-1} = P^T$.

Step 5: Compute the required canonical form given by

$$Y^T D Y = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2.$$

Here $X = PY$ is the orthogonal transformation, which reduces quadratic form to canonical form.

Q

① Reduce the quadratic form $3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ into canonical form by orthogonal transformation. And find Nature, Rank, Index and Signature of the canonical form.

Sol: Let given quadratic form is

$$Q = 3x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3.$$

The corresponding matrix is

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

	x_1	x_2	x_3
x_1	3	$2/2$	$2/2$
x_2	$2/2$	3	$-2/2$
x_3	$2/2$	$-2/2$	3

The characteristic eqn is

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

To find eigen values.

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where, S_1 = Sum of diagonal elements of $A = 3+3+3 = 9$.

S_2 = Sum of minors of diagonal elements of A

$$= \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix}$$

$= (9-1) + (9-1) + (9-1) = 8+8+8 = 24.$

$$S_3 = |A| = \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3(9-1) - 1(3+1) + 1.(-1-3)$$

$= 3(8) - 1(4) + 1.(-4)$

$= 24 - 4 - 4 = 16.$

$$\begin{array}{c|cccc} 1 & 1 & -9 & 24 & -16 \\ & 0 & 1 & -8 & 16 \\ \hline +4 & 1 & -8 & 16 & \boxed{0} \\ & 0 & +4 & -16 & \\ \hline & 1 & -4 & \boxed{0} & \end{array}$$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$\Rightarrow \lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0.$

$\therefore \lambda_1 = 1, \lambda_2 = 4, \lambda_3 = 4.$

To find eigen vectors,

$(A - \lambda I)X = 0$

$$\begin{pmatrix} 3-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda = 1$:

$$\begin{bmatrix} 3-1 & 1 & 1 \\ 1 & 3-1 & -1 \\ 1 & -1 & 3-1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$2x_1 + x_2 + x_3 = 0$

$x_1 + 2x_2 - x_3 = 0$

$x_1 - x_2 + 2x_3 = 0$

$$\begin{array}{cccccc} & \underline{x_1} & & \underline{x_2} & & \underline{x_3} \\ 1 & & 1 & & 2 & & 1 \\ 2 & & -1 & & 1 & & 2 \end{array}$$

$$\frac{x_1}{-1-2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{-3} = \frac{x_2}{3} = \frac{x_3}{3}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix} = 3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 4$:

$$\begin{bmatrix} 3-4 & 1 & 1 \\ 1 & 3-4 & -1 \\ 1 & -1 & 3-4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x_1 + x_2 + x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$
$$x_1 - x_2 - x_3 = 0$$

let $x_3 = 0$, then
$x_1 = x_2$.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

let $x_2 = 0$ then
$x_1 = x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

we observe that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \end{bmatrix} = 1+0+0 = 1$.

are not orthogonal to each other.

let us choose α and β such that

$$\alpha\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ is orthogonal to } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha+\beta \\ \alpha \\ \beta \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = 0$$

$$\alpha + \beta + \alpha = 0$$
$$2\alpha + \beta = 0$$
$$\beta = -2\alpha$$

Now, $$\begin{bmatrix} \alpha+\beta \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -\alpha \\ \alpha \\ -2\alpha \end{bmatrix} = \alpha\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}.$$

Now ~~let~~ $x_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $x_3 = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$ are eigen vectors.

To check orthogonality:

$$x_1 x_2^T = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = -1+1+0 = 0$$

$$x_2 x_3^T = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cancel{\begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}} \begin{bmatrix} -1 & 1 & -2 \end{bmatrix} = -1+1+0 = 0$$

$$x_3 x_1^T = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} = 1+1-2 = 0.$$

$\therefore$ Modal matrix $P = [e_1 \ e_2 \ e_3] = \left[\frac{x_1}{|x_1|} \ \frac{x_2}{|x_2|} \ \frac{x_3}{|x_3|} \right]$

$$P = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$$

$$\therefore D = P^{-1}AP = P^{T}AP = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

The required canonical form $= Y^T D Y = \cancel{(y_1 \ y_2 \ y_3)}$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= y_1^2 + 4y_2^2 + 4y_3^2.$$

Rank of QF (r) = No. of terms in canonical form = 3.

Singnature of QF ~~(s)~~ = 3 − 0 = 3.

Index of QF (s) = 3.

Since $r = 3 = n = s$, the given quadratic form is positive definite.

② Reduce the Quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ into canonical form by orthogonal transformation.

Sol: Given QF $= 3x^2+5y^2+3z^2-2yz+2zx-2xy$.

The corresponding matrix is.

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

	x	y	z
x	3	$-2/2$	$2/2$
y	$-2/2$	5	$-2/2$
z	$2/2$	$-2/2$	3

The characteristic eqn is

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix} = 0$$

To find eigen values,

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0.$$

where $S_1 =$ Sum of diagonal elements of $A = 3+5+3 = 11$.

$S_2 =$ Sum of minors of diagonal elements of A.

$$= \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 3 & -1 \\ -1 & 5 \end{vmatrix}$$

$$= (15-1)+(9-1)+(15-1) = 14+8+14$$

$$S_2 = 36$$

$$S_3 = |A| = \begin{vmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{vmatrix} = 3(15-1)+1(-3+1)+1(1-5)$$

$$= 3(14)+1(-2)+1(-4)$$

$$= 42-2-4$$

$$S_3 = 36.$$

Now, $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$\Rightarrow \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0.$$

For $\lambda = 2$.

$$(2)^3 - 11(2)^2 + 36(2) - 36 = 0$$

$$8 - 44 + 36 = 0$$

$$0 = 0$$

$\lambda = 2$	1	-11	36	-36
	0	2	-18	36
	1	-9	18	$\boxed{0}$

$$(\lambda-2)(\lambda^2 - 9\lambda + 18) = 0$$

$$(\lambda-2)(\lambda^2 - 3\lambda - 6\lambda + 18) = 0$$

$(\lambda-2)(\lambda(\lambda-3)-6(\lambda-3))=0$

$(\lambda-2)(\lambda-3)(\lambda-6)=0$.

$\therefore \lambda = 2, 3, 6$.

$\therefore$ The eigen values are $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$.

To find eigen vectors:

$(A-\lambda I)X = 0$.

$$\begin{bmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 2$:

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x - y + z = 0$

$-x + 3y - z = 0$

$x - y + z = 0$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & -1 \\ -1 & 3 \end{vmatrix}}$$

$$\frac{x}{1-3} = \frac{y}{-1+1} = \frac{z}{3-1}$$

$$\frac{x}{-2} = \frac{y}{0} = \frac{z}{2}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore X_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 3$:

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-y + z = 0$

$-x + 2y - z = 0$

$x - y = 0$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & 0 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 0 & -1 \\ -1 & 2 \end{vmatrix}}$$

$$\frac{x}{1-2} = \frac{y}{-1-0} = \frac{z}{0-1}$$

$$\frac{x}{-1} = \frac{y}{-1} = \frac{z}{-1}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} = -\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda_3 = 6$:

$$\begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3x - y + z = 0$$
$$-x - y - z = 0$$
$$x - y - 3z = 0.$$

$$\frac{x}{\begin{vmatrix} -1 & 1 \\ -1 & -1 \end{vmatrix}} = \frac{y}{\begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix}} = \frac{z}{\begin{vmatrix} -3 & -1 \\ -1 & -1 \end{vmatrix}}$$

$$\frac{x}{1+1} = \frac{y}{-1-3} = \frac{z}{3-1}$$

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{2}.$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\therefore x_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

To check orthogonality :

$$x_1 x_2^T = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} [1 \quad 1 \quad 1] = -1 + 0 + 1 = 0.$$

$$x_2 x_3^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} [1 \quad -2 \quad 1] = 1 - 2 + 1 = 0.$$

$$x_3 x_1^T = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} [-1 \quad 0 \quad 1] = -1 + 0 + 1 = 0.$$

$\therefore$ Modal matrix $P = [e_1 \quad e_2 \quad e_3] = \left[\frac{x_1}{|x_1|} \quad \frac{x_2}{|x_2|} \quad \frac{x_3}{|x_3|} \right]$

$$= \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$\therefore D = P^{-1}AP = P^{T}AP = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

The required canonical form $= Y^T D Y$

$$= \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$= 2y_1^2 + 3y_2^2 + 6y_3^2.$$

(3) Reduce the Quadratic form $2x^2+2y^2+2z^2-2xy-2yz-2zx$ to canonical form and state its nature.

Sol: Given quadratic form is

$2x^2+2y^2+2z^2-2xy-2yz-2zx$

The corresponding matrix is

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

$$\begin{array}{c|ccc} & x & y & z \\ \hline x & 2 & -2/2 & -2/2 \\ y & -2/2 & 2 & -2/2 \\ z & -2/2 & -2/2 & 2 \end{array}$$

The characteristic eqn is

$|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = 0$$

To find eigen values,

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where $S_1 = 2+2+2 = 6$

$S_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = (4-1)+(4-1)+(4-1)$
$= 3+3+3 = 9.$

$S_3 = |A| = 2(4-1)+1(-2-1)-1(1+2)$
$= 2(3)+1(-3)-1(3)$
$= 6-3-3 = 0.$

Now, $\lambda^3 - 6\lambda^2 + 9\lambda = 0$

$\lambda(\lambda^2 - 6\lambda + 9) = 0$

$\lambda(\lambda^2 - 3\lambda - 3\lambda + 9) = 0$

$\lambda(\lambda-3)(\lambda-3) = 0$

$\lambda = 0, 3, 3$.

$\therefore$ The eigen values are $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 3$.

To find eigen vectors:

$$(A - \lambda I)X = 0$$

$$\begin{bmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_1 = 0$:

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x - y - z = 0$$
$$x + 2y - z = 0$$
$$-x - y + 2z = 0$$

$$\begin{array}{cccccc} & x & & y & & z \\ -1 & & -1 & & 2 & & -1 \\ 2 & & -1 & & -1 & & 2 \end{array}$$

$$\frac{x}{1+2} = \frac{y}{1+2} = \frac{z}{4-1}$$

$$\frac{x}{3} = \frac{y}{3} = \frac{z}{3}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$$\therefore X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = 3$:

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-x - y - z = 0$$
$$\Rightarrow x + y + z = 0$$

let $z = 0$, then $\frac{x}{1} = \frac{y}{-1} = \frac{z}{0}$.

$$X_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

let $y = 0$, then $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$.

Here $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ are not orthogonal to each other.

Let $\alpha \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ be orthogonal to X_2.

$$\Rightarrow \begin{pmatrix} \alpha+\beta \\ -\alpha \\ -\beta \end{pmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = 0$$

$$\alpha + \beta + \alpha = 0.$$
$$2\alpha + \beta = 0$$
$$\beta = -2\alpha.$$

$$\begin{bmatrix} \alpha+\beta \\ -\alpha \\ -\beta \end{bmatrix} = \begin{bmatrix} \alpha-2\alpha \\ -\alpha \\ -(-2\alpha) \end{bmatrix} = \begin{bmatrix} -\alpha \\ -\alpha \\ +2\alpha \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

$$\therefore X_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

To check orthogonality:

$$X_1 X_2^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} = 1-1+0=0$$

$$X_2 X_3^T = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \end{bmatrix} = -1+1+0=0$$

$$X_3 X_1^T = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = -1-1+2=0.$$

Now, modal matrix $P = \begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} \frac{X_1}{|X_1|} & \frac{X_2}{|X_2|} & \frac{X_3}{|X_3|} \end{bmatrix}$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$\therefore D = P^{-1}AP = P^TAP = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The required canonical form $= Y^T D Y = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$= 3y_2^2 + 3y_3^2.$$

Rank $(r) = 2$, Index $(s) = 2$, Signature $= 2$.

Nature is positive semi definite $(r < n$ and $s = r)$.