UNIT-V: MULTIPLE INTEGRALS
(MULTIPLE VARIABLE CALCULUS)

Syllabus: Double integrals, Triple integrals, change of order of integration, change of variables to polar, cylindrical and spherical co-ordinates. Finding areas (by double integrals) and volumes (by double integrals and triple integrals).

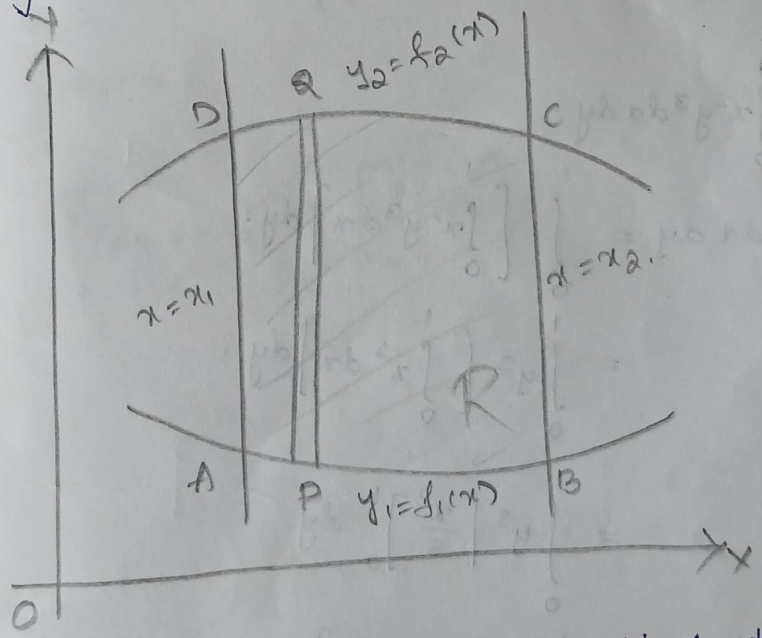
Application: We use double integrals to find the area and triple integrals to find the volume of the region. Tell about definite integrals how to apply limits

Double integrals: $\int$ integration $\int_0 dx \cdot \int_0 dy$

The double integral of $f(x,y)$ over the region R of the xy-plane is written as $\iint_R f(x,y)\, dR$ and expressed as

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\, dx\, dy.$$

(1) When y_1, y_2 are functions of x, and x_1, x_2 are constants then $f(x,y)$ is first integrated with respect to y keeping x fixed between the limits y_1, y_2 and the resulting expression is integrated w.r.t. x within the limits x_1 and x_2.

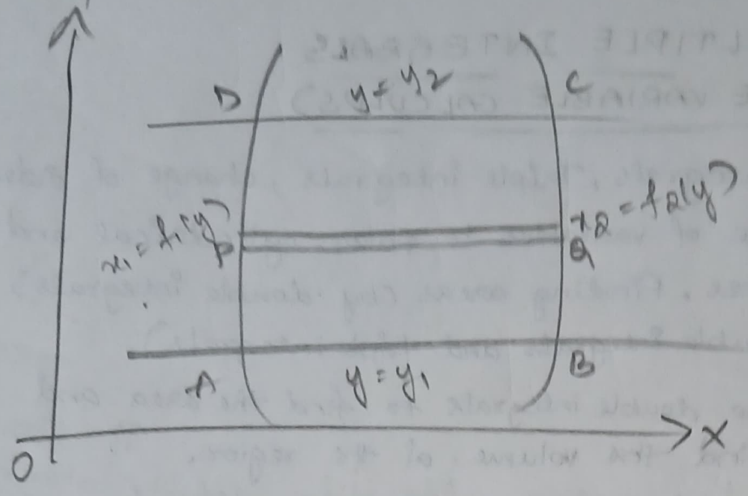


[First from P to Q along the strip. then moving the strip from AD to BC i.e., x_1 to x_2]

The above figure gives the geometrical illustration of the process.

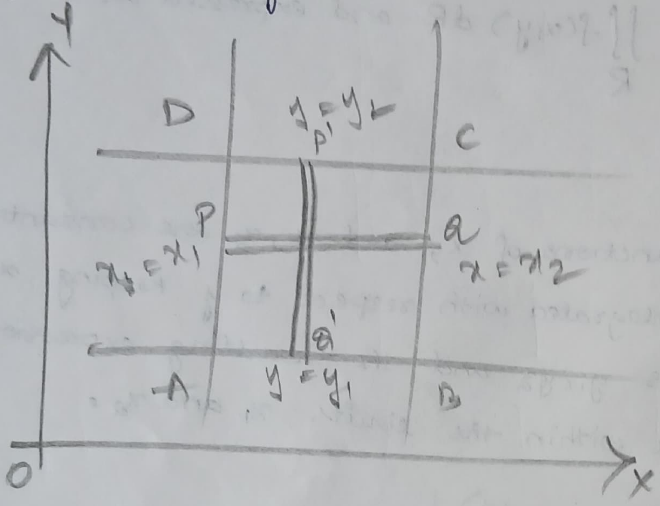
(2) When x_1, x_2 are functions of y and y_1, y_2 are constants, $f(x,y)$ is first integrated w.r.t x keeping y fixed between the limits x_1, x_2 and the resulting expression is integrated w.r.t to y within the limits y_1 and y_2.

This process is geometrically illustrated in the following figure.



First from P to Q by fixing the strip then move the strip from y_1 to y_2.

(3) when both pairs of limits are constants, then we integrate according to the given order.



$\iint dx\,dy$ first dx.

$\iint dy\,dx$ first dy.

① Evaluate $\int_0^1\int_0^1 x^2y^3\,dx\,dy$.

Sol: $\int_0^1\int_0^1 x^2y^3\,dx\,dy = \int_0^1\left[\int_0^1 x^2y^3\,dx\right]dy$.

$$= \int_0^1 y^3\left[\int_0^1 x^2\,dx\right]dy.$$

$$= \int_0^1 y^3\left[\frac{x^3}{3}\right]_0^1 dy$$

$$= \int_0^1 y^3\left[\frac{1}{3} - \frac{0}{3}\right]dy$$

$$= \int_0^1 \frac{y^3}{3}\,dy$$

$$= \frac{1}{3}\left[\frac{y^4}{4}\right]_0^1$$

$$\therefore \int_0^1\int_0^1 x^2y^3\,dx\,dy = \frac{1}{3}\left[\frac{1}{4} - \frac{0}{4}\right] = \frac{1}{12}.$$

(2) $\int_0^3 \int_0^{-2} xy\,dx\,dy$

Sol: $\int_0^3 \int_0^{-2} xy\,dx\,dy = \int_0^3 \left[\int_0^{-2} xy\,dx\right] dy$

$= \int_0^3 y\left[\int_0^{-2} x\,dx\right] dy$

$= \int_0^3 y\left[\frac{x^2}{2}\right]_0^{-2} dy$

$= \int_0^2 y\left[\frac{(-2)^2}{2} - \frac{(0)^2}{2}\right] dy$

$= \int_0^3 y\left[2-0\right] dy$

$= 2\left[\frac{y^2}{2}\right]_0^3 = 2\left[\frac{3^2}{2} - \frac{0^2}{2}\right]$

$\therefore \int_0^3 \int_0^{-2} xy\,dx\,dy = 2\left[\frac{9}{2}\right] = 9.$

(3) $\int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx.$

Sol: $\int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx = \int_{-1}^{2} \left[\int_{x^2}^{x+2} dy\right] dx$

$= \int_{-1}^{2} [y]_{x^2}^{x+2}\,dx$

$= \int_{-1}^{2} [x+2-x^2]\,dx$

$= \left[\frac{x^2}{2}\right]_{-1}^{2} + 2[x]_{-1}^{2} - \left[\frac{x^3}{3}\right]_{-1}^{2}$

$= \frac{2^2}{2} - \frac{(-1)^2}{2} + 2[2-(-1)] - \left[\frac{2^3}{3} - \frac{(-1)^3}{3}\right]$

$= 2 - \frac{1}{2} + 6 - \frac{8}{3} - \frac{1}{3} = \frac{3}{2} + 6 - 3 = \frac{3}{2} + 3.$

$\int_{-1}^{2} \int_{x^2}^{x+2} dy\,dx = \frac{9}{2}.$

4. Evaluate $\int_0^1 \int_y^{y^2} x \, dy \, dx$

Sol: $\int_0^1 \int_y^{y^2} x \, dy \, dx = \int_0^1 \left[\int_y^{y^2} x \, dx \right] dy.$

$$= \int_0^1 \left[\frac{x^2}{2} \right]_y^{y^2} dy$$

$$= \int_0^1 \left[\frac{y^4}{2} - \frac{y^2}{2} \right] dy.$$

$$= \frac{1}{2} \int_0^1 \left[y^4 - y^2 \right] dy$$

$$= \frac{1}{2} \left[\left[\frac{y^5}{5} \right]_0^1 - \left[\frac{y^3}{3} \right]_0^1 \right]$$

$$= \frac{1}{2} \left[\frac{1}{5} - \frac{1}{3} \right]$$

$$\therefore \int_0^1 \int_y^{y^2} x \, dy \, dx. = \frac{1}{2} \left[\frac{3-5}{15} \right] = \frac{1}{2} \left[\frac{-2}{15} \right] = -\frac{1}{15}$$

(5) $\int_0^{\infty} \int_0^{\pi/2} e^{-r} \, d\theta \, dr.$

Sol: $\int_0^{\infty} \int_0^{\pi/2} e^{-r} \, d\theta \, dr = \int_0^{\infty} e^{-r} \left[\int_0^{\pi/2} d\theta \right] dr.$

$$= \int_0^{\infty} e^{-r} \left[\theta \right]_0^{\pi/2} dr$$

$$= \frac{\pi}{2} \int_0^{\infty} e^{-r} \, dr = \frac{\pi}{2} \left[\frac{e^{-r}}{-1} \right]_0^{\infty}$$

$$= -\pi/2 \left[e^{-r} \right]_0^{\infty}$$

$$= -\pi/2 \left[e^{-\infty} - e^{-0} \right]$$

$$= -\pi/2 \left[\frac{1}{e^{\infty}} - \frac{1}{e^0} \right]$$

$$= -\pi/2 \left[\frac{1}{\infty} - \frac{1}{1} \right]$$

$$= -\pi/2 \left[0 - 1 \right]$$

$$\therefore \int_0^{\infty} \int_0^{\pi/2} e^{-r} \, d\theta \, dr = \pi/2.$$

(6) $\int_0^3\int_0^5 (x+y)\,dy\,dx = \int_0^3 \left[\int_0^5 (x+y)\,dy\right]dx = \int_0^3\left[x[y]_0^5 + \left[\frac{y^2}{2}\right]_0^5\right]dx$

$= \int_0^3\left[x(5-0)+\frac{25}{2}\right]dx = \int_0^3\left[5x+\frac{25}{2}\right]dx = 5\left[\frac{x^2}{2}\right]_0^3 + \frac{25}{2}[x]_0^3$

$= 5\left[\frac{9}{2}-\frac{0}{2}\right]+\frac{25}{2}[3-0] = \frac{45}{2}+\frac{75}{2} = \frac{120}{2} = 60$

(7) $\int_0^2\int_0^3 dy\,dx = \int_0^2\left[\int_0^3 dy\right]dx = \int_0^2 [y]_0^3\,dx = \int_0^2 [3-0]\,dx$

$= 3[x]_0^2 = 3[2-0] = 6.$

(8) $\int_0^{\pi/2}\int_0^1 x^2y^2\,dx\,dy = \int_0^{\pi/2}\left[\int_0^1 x^2y^2\,dx\right]dy = \int_0^{\pi/2} y^2\left[\frac{x^3}{3}\right]_0^1 dy$

$= \int_0^{\pi/2} y^2\left[\frac{1}{3}-\frac{0}{3}\right]dy = \frac{1}{3}\left[\frac{y^3}{3}\right]_0^{\pi/2} = \frac{1}{3}\left[\frac{\pi^3}{2^3\times 3}-\frac{0}{3}\right]$

$= \frac{1}{3}\cdot\frac{\pi^3}{8\times 3} = \frac{\pi^3}{72}$

(9) $\int_0^{\pi/2}\int_{-1}^1 x^2y^2\,dx\,dy = \int_0^{\pi/2}\left[\int_{-1}^1 x^2y^2\,dx\right]dy = \int_0^{\pi/2} y^2\left[\frac{x^3}{3}\right]_{-1}^1 dy.$

$= \int_0^{\pi/2} y^2\left[\frac{1^3}{3}-\frac{(-1)^3}{3}\right]dy = \frac{2}{3}\int_0^{\pi/2} y^2\,dy = \frac{2}{3}\left[\frac{y^3}{3}\right]_0^{\pi/2}$

$= \frac{2}{3}\left[\frac{\pi^3}{2^3\times 3}-0\right] = \frac{\cancel{2}}{3}\cdot\frac{\pi^3}{\cancel{8}_4\times 3} = \frac{\pi^3}{36}.$

(10) Evaluate $\int_0^5\int_0^{x^2}(x^2+y^2)\,dy\,dx$

$\int_0^5\int_0^{x^2}(x^2+y^2)\,dy\,dx = \int_0^5\left[\int_0^{x^2}(x^2+y^2)\,dy\right]dx = \int_0^5\left[\int_0^{x^2}x^2\,dy+\int_0^{x^2}y^2\,dy\right]dx$

$= \int_0^5\left[x^2[y]_0^{x^2}+\left[\frac{y^3}{3}\right]_0^{x^2}\right]dx.$

$= \int_0^5\left[x^2[x^2-0]+\left[\frac{x^6}{3}-\frac{0}{3}\right]\right]dx = \int_0^5\left(x^4+\frac{x^6}{3}\right)dx = \left[\frac{x^5}{5}\right]_0^5+\frac{1}{3}\left[\frac{x^7}{7}\right]_0^5$

$= \left[\frac{5^5}{5}-\frac{0}{5}\right]+\frac{1}{3}\left[\frac{5^7}{7}-\frac{0}{7}\right] = \frac{5^5}{5}+\frac{5^7}{21} = 5^4+\frac{5^7}{21}$

$= 5^4\left[1+\frac{5^3}{21}\right] = 5^4\left(1+\frac{125}{21}\right)$

(11) $\int_0^{\pi}\int_0^{1} r\,dr\,d\theta = \int_0^{\pi}\left[\int_0^{1} r\,dr\right]d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{1} d\theta = \frac{1}{2}\left[\theta\right]_0^{\pi} = \pi/2.$

(12) $\int_0^{3}\int_0^{2}(x+y)^2\,dx\,dy = \int_0^{3}\int_0^{2}(x^2+y^2+2xy)\,dx\,dy.$

$= \int_0^{3}\left[\frac{x^3}{3} + y^2[x] + 2y\left[\frac{x^2}{2}\right]\right]_0^{2} dy = \int_0^{3}\left[\frac{8}{3} + 2y^2 + 4y\right]dy$

$= \frac{8}{3}\int_0^{3} dy + 2\int_0^{3} y^2\,dy + 4\int_0^{3} y\,dy = \frac{8}{3}[y]_0^{3} + 2\left[\frac{y^3}{3}\right]_0^{3} + 4\left[\frac{y^2}{2}\right]_0^{3}$

$= \frac{8}{3}(3) + 2\left(\frac{27}{3}\right) + 4\left(\frac{9}{2}\right)\Big] = 8+18+18 = 44.$

(13) $\int_0^{5}\int_0^{x^2} x(x^2+y^2)\,dx\,dy = \int_0^{5}\left[\int_0^{x^2}(x^3+xy^2)\,dy\right]dx.$

$= \int_0^{5}\left[x^3[y]_0^{x^2} + x\left[\frac{y^3}{3}\right]_0^{x^2}\right]dx$

$= \int_0^{5}\left\{x^3[x^2-0] + x\left[\frac{x^6}{3} - \frac{0}{3}\right]\right\}dx$

$= \int_0^{5}\left[x^5 + \frac{x^7}{3}\right]dx = \left(\frac{x^6}{6}\right]_0^{5} + \frac{1}{3}\left[\frac{x^8}{8}\right]_0^{5}$

$= \frac{5^6}{6} + \frac{1}{3}\left(\frac{5^8}{8}\right)$

$= 5^6\left[\frac{1}{6} + \frac{25}{24}\right]$

$= 5^6\left[\frac{4}{24} + \frac{25}{24}\right] = 5^6\left[\frac{29}{24}\right].$

(14) $\int_0^{a}\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy$

$\int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

$\int_0^{a}\int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,dx\,dy = \int_0^{a}\int_0^{\sqrt{a^2-y^2}} \sqrt{(\sqrt{a^2-y^2})^2-x^2}\,dx\,dy.$

$= \int_0^{a}\left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{a^2-y^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{a^2-y^2}}\right)\right]_0^{\sqrt{a^2-y^2}} dy.$

$= \int_0^{a}\left[\frac{\sqrt{a^2-y^2}}{2}\sqrt{a^2-y^2-a^2+y^2} + \frac{a^2-y^2}{2}\sin^{-1}\left(\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}}\right)\right]dy$

$\int_0^{a}\left[-\left[0 + \frac{a^2-y^2}{2}\sin^{-1}(0)\right]\right]dy$

$$= \int_0^a \frac{a^2-y^2}{2}\left[\sin^{-1}(1) - \sin^{-1}(0)\right] dy$$

$$= \frac{\pi}{4}\int_0^a \left[\frac{a^2}{\cancel{2}} - \frac{y^2}{\cancel{2}}\right] dy = \frac{\pi}{4}\left[a^2 y - \frac{y^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^2 \cdot a - \frac{a^3}{3} - 0\right]$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right] = \frac{\pi}{4}\cdot\frac{2a^3}{3}$$

$$= \frac{\pi a^3}{6}$$

(15) $$\int_a^{2a}\int_0^{\sqrt{2ax-x^2}} xy \, dy\, dx = \int_a^{2a} x\left[\int_0^{\sqrt{2ax-x^2}} y \, dy\right] dx.$$

$$= \int_a^{2a} x\left[\frac{y^2}{2}\right]_0^{\sqrt{2ax-x^2}} dx = \int_a^{2a}\left(\frac{2ax-x^2}{2} - 0\right] x \, dx.$$

$$= \int_a^{2a}\left[ax^2 - \frac{x^3}{2}\right] dx = a\left[\frac{x^3}{3}\right]_a^{2a} - \frac{1}{2}\left[\frac{x^4}{4}\right]_a^{2a}.$$

$$= a\left[\frac{8a^3}{3} - \frac{a^3}{3}\right] - \frac{1}{2}\left[\frac{16a^4}{4} - \frac{a^4}{4}\right]$$

$$= a\left[\frac{7a^3}{3}\right] - \frac{1}{2}\left[\frac{15a^4}{4}\right] = \frac{7}{3}a^4 - \frac{15}{8}a^4 = a^4\left[\frac{56-45}{24}\right]$$

$$= \frac{11}{24}a^4.$$

(16) $$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2} = \int_0^1\left[\int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2} dx\right] dy$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \int_0^1\left[\frac{1}{\sqrt{1+x^2}} \tan^{-1}\left(\frac{y}{\sqrt{1+x^2}}\right)\right]_0^{\sqrt{1+x^2}} dy$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left[\tan^{-1}(1) - \tan^{-1}(0)\right] = \frac{\pi}{4}\left[\log\left(x+\sqrt{x^2+1}\right)\right]_{x=0}^1$$

$$= \frac{\pi}{4}\left[\log(1+\sqrt{2}) - \log 1\right] = \frac{\pi}{4}\log(1+\sqrt{2})$$

$$(8) = \frac{\pi}{4}\left(\sinh^{-1}x\right)_0^1 = \pi/4 \sinh^{-1}(1)$$

(17) $\int_0^{\pi}\int_0^{a(1+\cos\theta)} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{r^2}{2}\right]_0^{a(1+\cos\theta)} d\theta$

$\cos 2\theta = 2\cos^2\theta - 1$

$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$= \int_0^{\pi} \frac{a^2(1+\cos\theta)^2}{2} d\theta = \frac{a^2}{2}\int_0^{\pi}\left[1+\cos^2\theta + 2\cos\theta\right] d\theta$$

$$= \frac{a^2}{2}\left[\int_0^{\pi} d\theta + \int_0^{\pi}\frac{1+\cos 2\theta}{2} d\theta + \int_0^{\pi} 2\cos\theta\, d\theta\right]$$

$$= \frac{a^2}{2}\left[[\theta]_0^{\pi} + \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} + 2(\sin\theta)_0^{\pi}\right]$$

$$= \frac{a^2}{2}\left[(\pi - 0) + \frac{1}{2}\left[\pi + \frac{\sin 2\pi}{2} - 0 - \frac{\sin 2(0)}{2}\right] + 2[\sin\pi - \sin 0]\right]$$

$$= \frac{a^2}{2}\left[\pi + \frac{1}{2}[\pi + 0] + 2[0]\right]$$

$$= \frac{a^2}{2}\left[\pi + \pi/2\right] = \frac{a^2}{2}\left(\frac{3\pi}{2}\right) = \frac{3\pi a^2}{4}.$$

(18) $\int_0^{\pi/2}\int_0^{a\cos\theta} r\sin\theta\, dr\, d\theta = \int_0^{\pi/2} \sin\theta \left[\frac{r^2}{2}\right]_0^{a\cos\theta} d\theta$

$$= \int_0^{\pi/2} \sin\theta\left[\frac{a^2\cos^2\theta}{2} - \frac{0}{2}\right] d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \sin\theta\cos^2\theta\, d\theta$$

let $\cos\theta = t$

$-\sin\theta\, d\theta = dt$

$\sin\theta\, d\theta = -dt$

$\cos 0 = 1$

$\cos\pi/2 = 0$

$$= \frac{a^2}{2}\int_1^0 -t^2\, dt$$

$$= -\frac{a^2}{2}\left[\frac{t^3}{3}\right]_1^0 = -\frac{a^2}{2}\left[\frac{0}{3} - \frac{1}{3}\right] = -\frac{a^2}{2}\left(-\frac{1}{3}\right) = \frac{a^2}{6}$$

(19) Evaluate $\iint_R xy\,dx\,dy$ where R is the region bounded by x-axis, $x = 2a$ and the curve $x^2 = 4ay$.

Sol: Given, $\iint_R xy\,dx\,dy$ — (1) where R is the region bounded by x-axis, $x = 2a$ — (2) and the curve $x^2 = 4ay$ — (3)

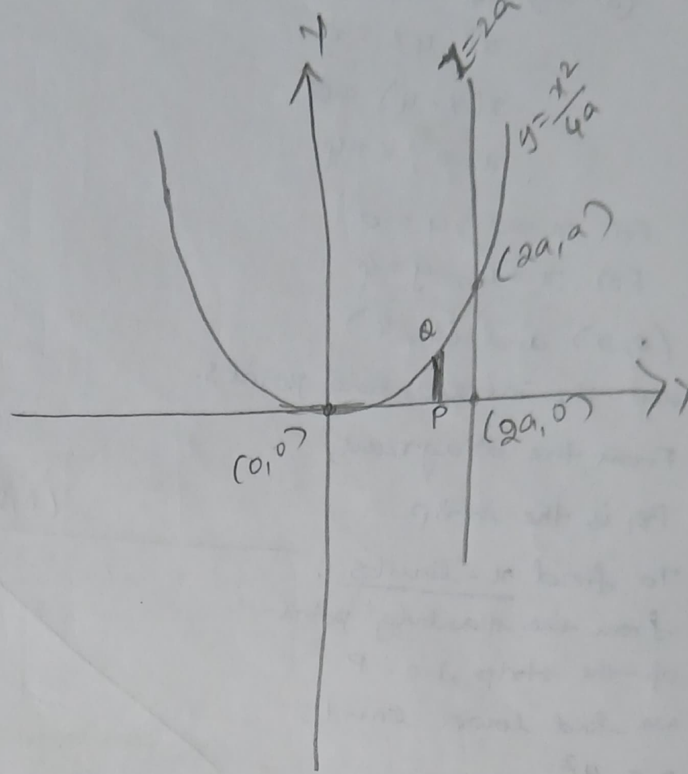
Substituting ② in ③

$(2a)^2 = 4ay$

$4a^2 = 4ay$

$y = a$

$(x,y) = (2a, a)$



From the diagram PQ is the strip.

y- limits:

To find y-limits,

from the starting point of the strip, i.e., P we find the lower limit,

$y=0$.

From the ending point of the strip, i.e., Q we find the upper limit

$y=\frac{x^2}{4a}$.

To find x-limits,

we move the strip in the region bounded parallel to y-axis,

so lower limit of $x=0$, upper limit of $x=2a$.

Now,

① ⇒ $\int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\,dx\,dy = \int_0^{2a} x\left[\int_0^{\frac{x^2}{4a}} y\,dy\right]dx$

$= \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{\frac{x^2}{4a}} dx = \int_0^{2a} \frac{x}{2}\left[\frac{x^4}{16a^2} - 0\right]dx$

$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \left[\frac{2^6a^6}{6} - 0\right]\frac{1}{32a^2}$

$= \frac{1}{32a^2} \times \frac{16\times4\times a^6}{6} = \frac{a^4}{3}$.

(20) Evaluate $\iint_R (x^2+y^2)\,dx\,dy$ where R is the region bounded by $y=x$ and $y^2=4x$.

Sol: Given, $\iint_R (x^2+y^2)\,dx\,dy$ —① where R is the region bounded

by $y=x$ —② and $y^2=4x$ —③

Solving ② & ③

③ $\Rightarrow x^2 = 4x$

$x^2 - 4x = 0$

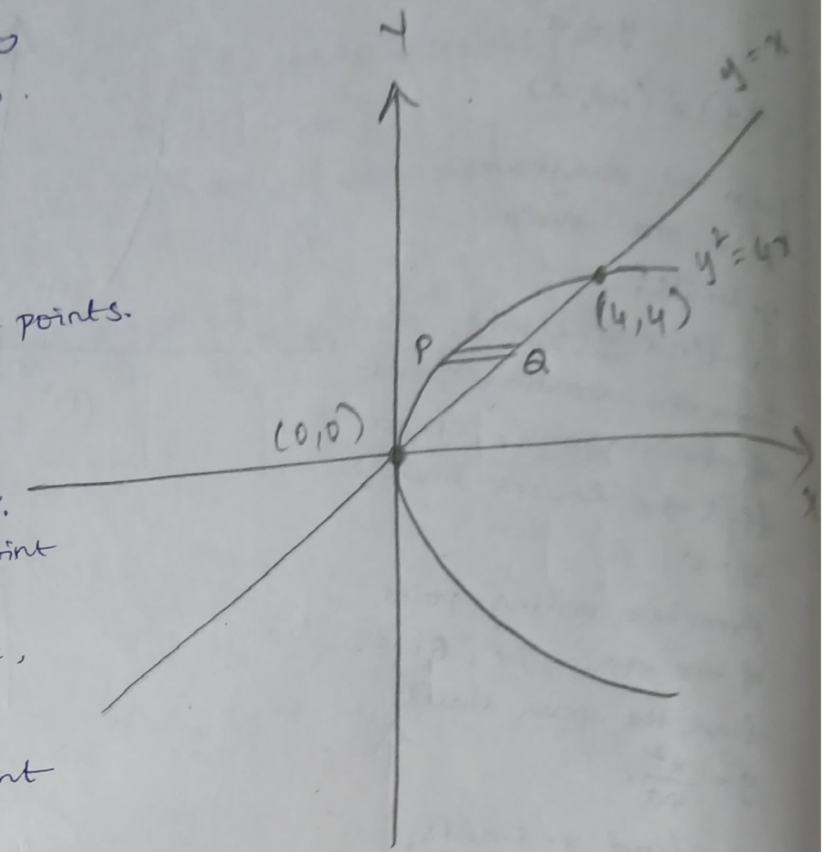
$x(x-4) = 0$

$x = 0,\ x = 4.$

For $x = 0,\ y = 0$

For $x = 4,\ y = 4.$

$(0,0)$ and $(4,4)$ are the intersecting points.



From the diagram, PQ is the strip.

To find x-limits:

from the starting point of the strip, i.e., P we find lower limit,

$x = \frac{y^2}{4}$

From the ending point of the strip, i.e., Q we find the upper limit

$x = y.$

To find y-limits

we move the strip in the region bounded, parallel to x-axis.

So lower limit $y = 0$, upper limit, $y = 4$.

Now, ① $\Rightarrow \int_0^4 \int_{y^2/4}^{y} (x^2+y^2)\,dx\,dy$

$$= \int_0^4 \left[\int_{y^2/4}^{y} (x^2+y^2)\,dx \right] dy \cdot = \cancel{\int_0^4 \left[x^2 \left[y\right]_{y^2/4}^{y} + \left[\frac{y^3}{3}\right]_{y^2/4}^{y} \right] dy}$$

$$\cancel{\int_0^4 x^2 \left[y - y^2\right]} = \int_0^4 \left[\left[\frac{x^3}{3}\right]_{y^2/4}^{y} + y^2 \left[x\right]_{y^2/4}^{y} \right] dy.$$

$$= \int_0^4 \left[\frac{y^3}{3} - \frac{y^6}{4^3 \times 3} + y^2 \left[y - \frac{y^2}{4} \right] \right] dy$$

$$= \int_0^4 \left[\frac{y^3}{3} - \frac{y^6}{4^3 \times 3} + y^3 - \frac{y^4}{4} \right] dy$$

$$= \int_0^4 \left[\frac{4y^3}{3} - \frac{y^6}{4^3 \times 3} - \frac{y^4}{4}\right] dy$$

$$= \frac{4}{3}\left[\frac{y^4}{4}\right]_0^4 - \left[\frac{y^7}{7}\right]_0^4 \frac{1}{4^3 \times 3} - \frac{1}{4}\left[\frac{y^5}{5}\right]_0^4$$

$$= \frac{\cancel{4} \times 4^4}{3 \times \cancel{4}} - \frac{4^{\cancel{7}4}}{7 \times \cancel{4^3} \times 3} - \frac{4^{\cancel{5}4}}{\cancel{4} \times 5} - 0$$

$$= 4^4\left[\frac{1}{3} - \frac{1}{21} - \frac{1}{5}\right]$$

$$= 4^4\left[\frac{7}{21} - \frac{1}{21} - \frac{1}{5}\right] = 4^4\left[\frac{\cancel{6}^{2}}{\cancel{21}_{7}} - \frac{1}{5}\right]$$

$$= 4^4\left[\frac{10-7}{35}\right] = \frac{256 \times 3}{35} = \frac{768}{35}$$

Change of order of integration:

1. If the given integral is $\int_{x=a}^{b} \int_{y=f_1(x)}^{y=f_2(x)} f(x,y)\, dy\, dx$, then changing the strip to horizontal, we evaluate $\int_{y=a}^{b} \int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\, dx\, dy$.

2. If the given integral is $\int_{y=a}^{b} \int_{x_1=f_1(y)}^{x_2=f_2(y)} f(x,y)\, dx\, dy$, then changing the strip to vertical, we evaluate $\int_{x=a}^{b} \int_{y_1=f_1(x)}^{y_2=f_2(x)} f(x,y)\, dy\, dx$.

① Change the order of integration and evaluate $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$.

Sol: Given, $\int_0^1 \int_x^{\sqrt{x}} xy\, dy\, dx$ — ①

Here y limits : $y = x$, $y = \sqrt{x}$.

$\Rightarrow y = x$ ②, $y^2 = x$ ③

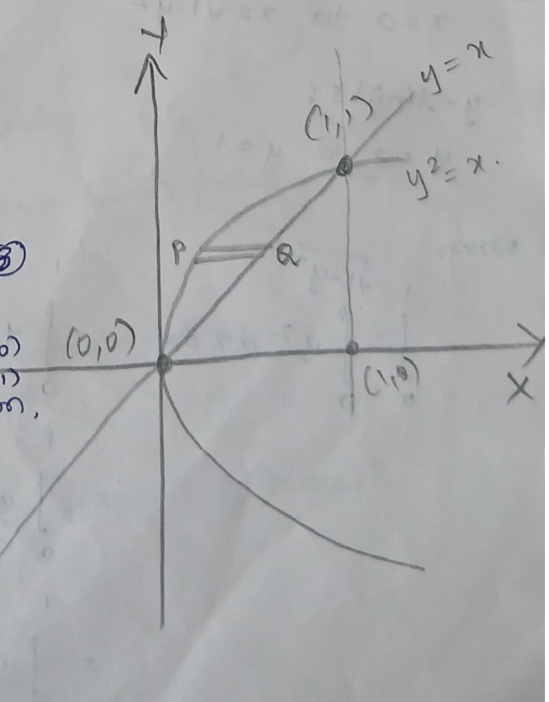
x limits : $x = 0$, $x = 1$.

[Solve for point of intersection ② & ③] (0,0) (1,1)

By changing the order of integration,

x limits : $x = y^2$ to $x = y$

y limits : $y = 0$ to $y = 1$



Now, $\int_0^1 \int_{y^2}^{y} xy\,dy\,dx = \int_0^1 y\left(\int_{y^2}^{y} x\,dx\right)dy$

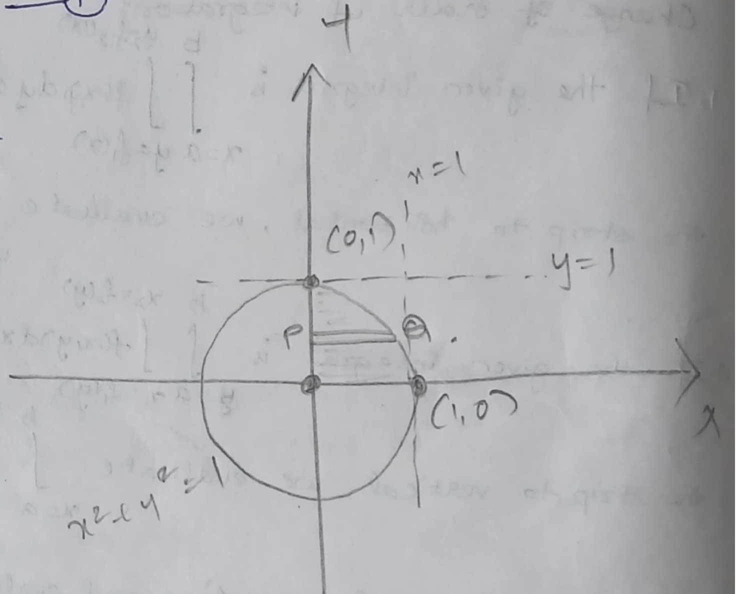
$= \int_0^1 y\left(\frac{x^2}{2}\right]_{y^2}^{y} dy = \int_0^1 y\left[\frac{y^4}{2} - \frac{y^2}{2}\right]dy$

$= \frac{1}{2}\int_0^1 (y^5 - y^3)\,dy = \frac{1}{2}\left\{\left[\frac{y^6}{6}\right]_0^1 - \left[\frac{y^4}{4}\right]_0^1\right\}$

$= \frac{1}{2}\left[\frac{1}{6} - \frac{1}{4}\right] = \frac{1}{2}\left[\frac{4-6}{24}\right] = \frac{1}{\cancel{2}}\left[\frac{-\cancel{2}}{24}\right] = -\frac{1}{24}$

(2) Change the order of integration and evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$.

Sol: Given, $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$ —(1)



Here y-limits:
$y = 0$ to $y = \sqrt{1-x^2}$
$x^2 + y^2 = 1$

x-limits:
$x = 0$ to $x = 1$.

By changing the order of integration,

x-limits:
$x = 0$ to $x = \sqrt{1-y^2}$

y-limits:
$y = 0$ to $y = 1$.

Now,

$\int_0^1 \int_0^{\sqrt{1-y^2}} y^2\,dx\,dy = \int_0^1 y^2\left(\int_0^{\sqrt{1-y^2}} dx\right)dy$

$= \int_0^1 y^2\,[x]_0^{\sqrt{1-y^2}}\,dy.$

$= \int_0^1 y^2\sqrt{1-y^2}\,dy.$

$$= \int_0^{\pi/2} \sin^2\theta \sqrt{1-\sin^2\theta}\, \cos\theta \, d\theta$$

let $y = \sin\theta$

$dy = \cos\theta \, d\theta$

$y = 0 \Rightarrow \theta = 0$

$y = 1 \Rightarrow \theta = \pi/2$.

$$= \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta$$

$$= \int_0^{\pi/2} \left[\frac{\sin 2\theta}{2}\right]^2 d\theta$$

$\cos 2A = 1 - 2\sin^2 A$

$\sin^2 A = \frac{1-\cos 2A}{2}$

$\sin^2 2\theta = \frac{1-\cos 4\theta}{2}$

$$= \frac{1}{4}\int_0^{\pi/2} \sin^2 2\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} \frac{1-\cos 4\theta}{2} \, d\theta$$

$$= \frac{1}{8}\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2}$$

$$= \frac{1}{8}\left[\frac{\pi}{2} - 0\right]$$

$$= \frac{\pi}{16}.$$

Triple Integrals:

It is evaluated as the repeated integral.

$$\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\, dx\, dy\, dz.$$

If x_1, x_2 are constants

y_1, y_2 are functions of x.

z_1, z_2 are functions of x and y.

Then we first integrate z, then y and finally x.

$$\int_{x_1}^{x_2}\int_{y_1=f_1(x)}^{y_2=f_2(x)}\int_{z_1=\phi_1(x,y)}^{z_2=\phi_2(x,y)} f(x,y,z)\, dz\, dy\, dx.$$

If all the limits are constants, then it is done in the order given.

① Evaluate $\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\,dz$.

Sol: Given $\int_{-1}^{1}\int_{-2}^{2}\int_{-3}^{3} dx\,dy\,dz = \int_{-1}^{1}\int_{-2}^{2}[x]_{-3}^{3}\,dy\,dz$

$$= \int_{-1}^{1}\int_{-2}^{2}[3+3]\,dy\,dz$$

$$= \int_{-1}^{1} 6[y]_{-2}^{2}\,dz$$

$$= 24\,[z]_{-1}^{1}$$

$$= 48.$$

② Evaluate $\int_0^1\int_0^2\int_0^3 y\,dx\,dy\,dz$.

Sol: Given, $\int_0^1\int_0^2\int_0^3 y\,dx\,dy\,dz = \int_0^1\int_0^2 y\,[x]_0^3\,dy\,dz$

$$= \int_0^1\int_0^2 3y\,dy\,dz$$

$$= \int_0^1 3\left[\frac{y^2}{2}\right]_0^2 dz$$

$$= \frac{3}{2}\int_0^1 [4-0]\,dz$$

$$= 3(2)\,[z]_0^1$$

$$= 6.$$

③ $\int_0^1\int_0^1\int_0^1 x\,dx\,dy\,dz = \int_0^1\int_0^1\left[\frac{x^2}{2}\right]_0^1 dy\,dz = \frac{1}{2}\int_0^1 [y]_0^1\,dz = \frac{1}{2}[z]_0^1 = \frac{1}{2}.$

④ $\int_0^1\int_1^2\int_1^3 xyz\,dx\,dy\,dz = \int_0^1\int_1^2 yz\left[\frac{x^2}{2}\right]_1^3 dy\,dz$

$$= \int_0^1\int_1^2 yz\left[\frac{9}{2}-\frac{1}{2}\right] dy\,dz.$$

$$= \int_0^1 4z\left[\frac{y^2}{2}\right]_1^2 dz = \int_0^1 4z\left[\frac{4}{2}-\frac{1}{2}\right] dz$$

$$= 4 \times \frac{3}{2} \int_0^1 z\,dz = 6\left[\frac{z^2}{2}\right]_0^1 = 6\left[\frac{1}{2} - \frac{0}{2}\right] = 3.$$

5) Evaluate $\int_1^2 \int_1^z \int_0^{4z} (xyz)\,dx\,dy\,dz.$

Sol: Given, $\int_1^2 \int_1^z \int_0^{4z} (xyz)\,dx\,dy\,dz.$

$$= \int_1^2 \int_1^z yz\left[\frac{x^2}{2}\right]_0^{yz} dy\,dz.$$

$$= \int_1^2 \int_1^z yz\left[\frac{y^2z^2}{2} - 0\right] dy\,dz$$

$$= \frac{1}{2} \int_1^2 \int_1^z y^3z^3\,dy\,dz.$$

$$= \frac{1}{2} \int_1^2 z^3\left[\frac{y^4}{4}\right]_1^z dz.$$

$$= \frac{1}{2} \int_1^2 z^3\left[\frac{z^4}{4} - \frac{1}{4}\right] dz$$

$$= \frac{1}{8} \int_1^2 (z^7 - z^3)\,dz$$

$$= \frac{1}{8}\left[\frac{z^8}{8} - \frac{z^4}{4}\right]_1^2$$

$$= \frac{1}{8}\left[\frac{2^8}{8} - \frac{2^4}{4} - \frac{1}{8} + \frac{1}{4}\right]$$

$$= \frac{1}{8}\left[\frac{\cancel{2^3} \times 2^5}{\cancel{8}} - \frac{\cancel{2^2} \times 2^2}{\cancel{4}} - \frac{1}{8} + \frac{2}{8}\right]$$

$$= \frac{1}{8}\left[32 - 4 + \frac{1}{8}\right] = \frac{1}{8}\left[28 + \frac{1}{8}\right] = \cancel{\frac{56+1}{8 \times 8}} = \cancel{\frac{57}{64}}$$

$$= \frac{1}{8}\left[\frac{224+1}{8}\right] = \frac{225}{64}$$

⑥ Evaluate $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$.

Sol: Given $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z}(x+y+z)\,dy\,dx\,dz$

$$= \int_{-1}^{1}\int_{0}^{z}\left[x[y] + \frac{y^2}{2} + z[y]\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[x[x+z-x+z] + \frac{(x+z)^2}{2} - \frac{(x-z)^2}{2} + z[x+z-x+z]\right] dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}\left[x[2z] + \frac{x^2+z^2+2zx-x^2-z^2+2zx}{2} + z[2z]\right] dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}[2zx + 2zx + 2z^2]\,dx\,dz$$

$$= \int_{-1}^{1}\int_{0}^{z}(4xz + 2z^2)\,dx\,dz$$

$$= \int_{-1}^{1}\left[4z\left[\frac{x^2}{2}\right]_0^z + 2z^2[x]_0^z\right] dz$$

$$= \int_{-1}^{1}\left\{4z\left[\frac{z^2}{2}\right] + 2z^2[z-0]\right\} dz$$

$$= \int_{-1}^{1}(2z^3+2z^3)\,dz = \int_{-1}^{1}4z^3\,dz = 4\left[\frac{z^4}{4}\right]_{-1}^{1} = 0$$

⑦ $\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx = \int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} xy\left[\frac{z^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$.

$$= \frac{1}{2}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} xy(1-x^2-y^2)\,dy\,dx = \frac{1}{2}\int_{0}^{1}\int_{0}^{\sqrt{1-x^2}}(xy - x^3y - xy^3)\,dy\,dx$$

$$= \frac{1}{2}\int_{0}^{1}\left\{x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} - x^3\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} - x\left[\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}}\right\} dx.$$

$$= \frac{1}{2}\int_0^1 \left\{ x\left(\frac{1-x^2}{2}\right) - x^3\left(\frac{1-x^2}{2}\right) - x\left(\frac{(1-x^2)^2}{4}\right) \right\} dx.$$

$$= \frac{1}{2}\int_0^1 \left\{ \frac{x - x^3 - x^3 + x^5}{2} - x\left[\frac{1+x^4-2x^2}{4}\right] \right\} dx.$$

$$= \frac{1}{2}\times\frac{1}{4}\int_0^1 \left[2x - 2x^3 - \cancel{2x^3} + 2x^5 - x - x^5 + \cancel{2x^3} \right] dx.$$

$$= \frac{1}{8}\int_0^1 (x - 2x^3 + x^5)\, dx.$$

$$= \frac{1}{8}\left\{ \left[\frac{x^2}{2}\right]_0^1 - 2\left[\frac{x^4}{4}\right]_0^1 + \left[\frac{x^6}{6}\right]_0^1 \right\}$$

$$= \frac{1}{8}\left[\frac{1}{2} - \frac{2}{4} + \frac{1}{6}\right]$$

$$= \frac{1}{8}\left[\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right] = \frac{1}{8}\times\frac{1}{6} = \frac{1}{48}.$$

(8) Evaluate $\iiint_V (x^2+y^2+z^2)\, dz\, dy\, dx$ where V is the volume of the cube bounded by $x=0, y=0, z=0, x=a, y=b, z=c$.

Sol: Given, $\iiint_V (x^2+y^2+z^2)\, dz\, dy\, dx \rightarrow (1)$ where V is the volume of the cube bounded by $x=0, y=0, z=0, x=a, y=b, z=c$

$$(1) \Rightarrow \int_0^a\int_0^b\int_0^c (x^2+y^2+z^2)\, dz\, dy\, dx.$$

$$= \int_0^a\int_0^b \left\{ x^2[z]_0^c + y^2[z]_0^c + \left[\frac{z^3}{3}\right]_0^c \right\} dy\, dx$$

$$= \int_0^a\int_0^b \left\{ x^2(c-0) + y^2(c-0) + \frac{c^3}{3} \right\} dy\, dz$$

$$= \int_0^a\int_0^b \left\{ cx^2 + cy^2 + \frac{c^3}{3} \right\} dy\, dz.$$

$$= \int_0^a \left\{ c x^2 [y]_0^b + c\left[\frac{y^3}{3}\right]_0^b + \frac{c^3}{3}[y]_0^b \right\} dx$$

$$= \int_0^a \left\{ c x^2 (b-0) + c\left(\frac{b^3}{3}-0\right) + \frac{c^3}{3}(b-0) \right\} dx$$

$$= \int_0^a \left\{ bcx^2 + \frac{cb^3}{3} + \frac{c^3 b}{3} \right\} dx$$

$$= bc\left[\frac{x^3}{3}\right]_0^a + \frac{cb^3}{3}[x]_0^a + \frac{c^3 b}{3}[x]_0^a$$

$$= \frac{a^3bc}{3} + \frac{ab^3c}{3} + \frac{abc^3}{3}$$

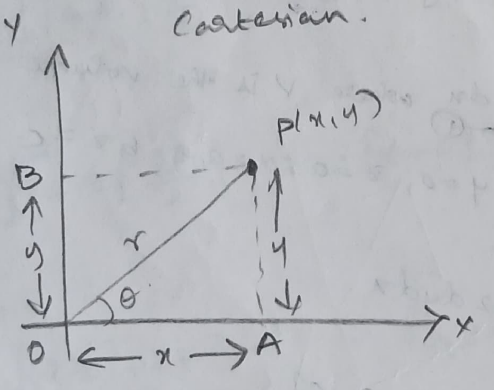
$$= \frac{abc(a^2+b^2+c^2)}{3}$$

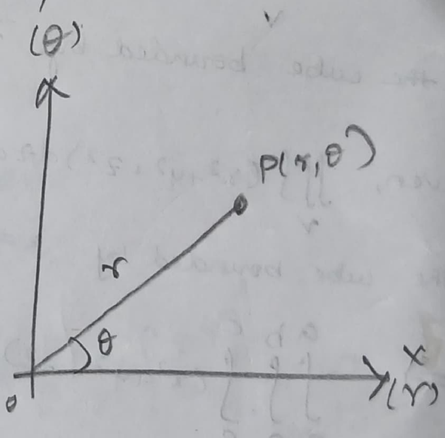
Change of variable of double integral from Cartesian co-ordinates to polar co-ordinates

Cartesian co-ordinates $= (x,y)$

Polar co-ordinates $= (r,\theta)$

Cartesian.





From the above figure.

$\cos\theta = \frac{x}{r} \Rightarrow x = r\cos\theta$

$\sin\theta = \frac{y}{r} \Rightarrow y = r\sin\theta$.

$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2$

$x^2+y^2 = r^2$.

Now, $\frac{\partial x}{\partial r} = \cos\theta$, $\frac{\partial x}{\partial \theta} = -r\sin\theta$

$\frac{\partial y}{\partial r} = \sin\theta$, $\frac{\partial y}{\partial \theta} = r\cos\theta$

Jacobian, $J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$

$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$

$= r.$

We have,

$dx\,dy = |J|\,dr\,d\theta$

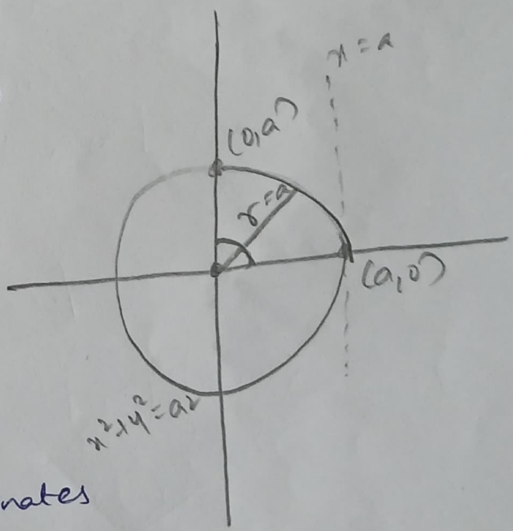
$dx\,dy = r\,dr\,d\theta.$

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,[r\,dr\,d\theta]$$

$$= \iint_R F(r,\theta)\,r\,dr\,d\theta.$$

① Evaluate the following integral by transforming into polar-co-ordinates $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy.$

Sol: Given, $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\,dx\,dy.$ —①



y-limits:

$y = 0$ to $y = \sqrt{a^2-x^2}$

$y^2 = a^2 - x^2$

$x^2 + y^2 = a^2.$

x-limits:

$x = 0$ to $x = a.$

Transforming to polar co-ordinates

r-limits : $r = 0$ to $r = a.$

θ-limits : $\theta = 0$ to $\theta = \pi/2.$

$$① \Rightarrow \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} (r\sin\theta)\sqrt{r^2}\,r\,dr\,d\theta = \int_0^{\pi/2}\int_0^a r^3\sin\theta\,dr\,d\theta$$

$$= \int_0^{\pi/2} \sin\theta \left[\frac{r^4}{4}\right]_0^a d\theta = \frac{a^4}{4} \int_0^{\pi/2} \sin\theta \, d\theta$$

$$= \frac{a^4}{4} \left[-\cos\theta\right]_0^{\pi/2}$$

$$= \frac{a^4}{4} \left[-\cos\pi/2 + \cos 0\right]$$

$$= \frac{a^4}{4} \left[-0+1\right]$$

$$= \frac{a^4}{4}$$