SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES
(Autonomous)
DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING-DATA SCIENCE
I B.TECH II Sem

		L	T	P	C
23BSC121T	**DIFFERENTIAL EQUATIONS AND VECTOR (COMMON TO ALL BRANCHES OF ENGINEERING)**	**2**	**1**	**-**	**3**

COURSE EDUCATIONAL OBJECTIVES:

1. To enlighten the learners in the concept of differential equations and multivariable calculus.
2. To furnish the learners with basic concepts and techniques at plus two level to lead them into advanced level by handling various real-world applications

UNIT-I: DIFFERENTIAL EQUATIONS OF FIRST ORDER AND FIRST DEGREE (9)

Linear differential equations – Bernoulli's equations - Exact equations and equations reducible to exact form. Applications: Newton's Law of cooling – Law of natural growth and decay, Electrical circuits.

UNIT-II:LINEAR DIFFERENTIAL EQUATIONS OF HIGHER ORDER (CONSTANT COEFFICIENTS) (9)

Definitions,homogenous and non-homogenous, complimentary function, general solution, particular integral, Wronskean, Method of variation of parameters. Simultaneous linear equations, Applications to L-C-R Circuit problems and Simple Harmonic motion.

UNIT-III:PARTIAL DIFFERENTIAL EQUATIONS (9)

Introduction and formation of Partial Differential Equations by elimination of arbitrary constants and arbitrary functions, solutions of first order linear equations using Lagrange's method and Non-Linear (Standard forms) equations. Homogeneous Linear Partial differential equations with constant coefficients (Method of Separation of variables).

UNIT-IV: VECTOR DIFFERENTIATION (9)

Scalar and vector point functions, vector operator Del, Del applies to scalar point functions Gradient, Directional derivative, del applied to vector point functions-Divergence and Curl, vector identities.

UNIT-V:VECTOR INTEGRATION (9)

Line Integral-circulation-work done, surface integral-flux, Green's theorem in the plane (without proof), Stoke's theorem (without proof), volume integral, Divergence theorem (without proof) and related problems.

TOTAL HOURS: 45

COURSE OUTCOMES:

On successful completion of the course, students will be able to		Pos
CO1	Solve the first order differential equations related to various engineering fields	**PO1, PO2,PO3**
CO2	Solve the higher order differential equations related to various engineering fields.	**PO1, PO2,PO3**
CO3	Identify solution methods for partial differential equations that model physical processes.	**PO1, PO2,PO3**
CO4	Interpret the physical meaning of different operators such as gradient, curl and divergence.	**PO1, PO2,PO3**
CO5	Estimate the work done against a field, circulation and flux using vector calculus	**PO1, PO2,PO3**

TEXT BOOKS:

1. Higher Engineering Mathematics, B. S. Grewal, Khanna Publishers, 2017, 44th Edition
2. Advanced Engineering Mathematics, Erwin Kreyszig, John Wiley & Sons, 2018, 10th Edition.

REFERRENCE BOOKS:

1. Thomas Calculus, George B. Thomas, Maurice D. Weir and Joel Hass, Pearson Publishers, 2018, 14th Edition.
2. Advanced Engineering Mathematics, Dennis G. Zill and Warren S. Wright, Jones and Bartlett, 2018.
3. Advanced Modern Engineering Mathematics, Glyn James, Pearson publishers, 2018, 5th Edition.
4. Advanced Engineering Mathematics, R. K. Jain and S. R. K. Iyengar, Alpha Science International Ltd., 2021 5th Edition (9th reprint).
5. Higher Engineering Mathematics, B. V. Ramana, , McGraw Hill Education, 2017

REFERENCE WEBSITE:

1. https://nptel.ac.in/courses/111/106/111106100/
1. https://www.youtube.com/watch?v=OBhZvyhc8JQ&t=982s
2. https://nptel.ac.in/courses/111/106/111106100/
3. https://www.youtube.com/watch?v=3zCdNO2xp3s
4. https://www.youtube.com/watch?v=GFKggEkKtLM
5. https://www.youtube.com/watch?v=SZCsFS9izfQ
6. https://www.youtube.com/watch?v=ma1QmE1SH3I

CO-PO MAPPING:

CO\PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO.1	3	3	2	-	-	-	-	-	-	-	-	-
CO.2	3	3	2	-	-	-	-	-	-	-	-	-
CO.3	3	3	2	-	-	-	-	-	-	-	-	-
CO.4	3	3	2	-	-	-	-	-	-	-	-	-
CO.5	3	3	2	-	-	-	-	-	-	-	-	-
CO*	3	3	2	-	-	-	-	-	-	-	-	-

UNIT - I : Differential Equations of first order and first degree.

Differential Equation (DE):

An equation involving a dependent variable & its derivatives with respect to one (or) more independent variable is called DE.

Ordinary Differential Equation (ODE):

A DE is said to be ordinary if the derivatives in the equation have reference to only a single independent variable.

Ex: $\frac{dy}{dx}+1=0$, $\frac{d^2y}{dx^2}+\sin x=\log x$.

Partial Differential Equation (PDE):

A DE is said to be partial if the derivatives in the equation have reference to 2 or more independent variables.

Ex: $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}+\log x=1$, $\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial y}=\log x+\log y$.

Order of DE: The highest derivative present in the DE is called order of DE.

Ex: $\frac{dy}{dx}+\log x=1$, order $=1$

$\frac{d^2y}{dx^2}+\frac{dy}{dx}-\sin x=y$, order $=2$.

Degree of DE: The power of highest derivative is called degree of DE.

Ex: $\frac{d^2y}{dx^2}+\frac{dy}{dx}=x$, order $=2$, degree $=1$.

$\left(\frac{d^2y}{dx^2}\right)^2+\left(\frac{dy}{dx}\right)^3-y=\sin x$, order $=2$, degree $=2$.

Q: Find the order and degree of following DE's:

(a) $a\frac{d^2y}{dx^2}=\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

Given eqn is

$$a\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

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$$a^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

order = 2, degree = 2.

(b) $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$$\left(y - x\frac{dy}{dx}\right) = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$\left(y - x\frac{dy}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2$$

order = 1, degree = 2.

Formation of differential eqn:

In general an ODE is obtained by eliminating the arbitrary constant.

If the given eqns contain one constant we have to differentiate one time

If the given eqn contains two constants we have to differentiate two times.

Q: Form a differential eqn for

(1) $y = ax^2$, where 'a' is the arbitrary constant.

Sol: Given $y = ax^2$

$$\boxed{\frac{y}{x^2} = a} \quad -(1)$$

$$\frac{dy}{dx} = a(2x)$$

$$\frac{dy}{dx} = \frac{y}{x^2}(2x)$$

$$\Rightarrow \boxed{\frac{dy}{dx} - \frac{2y}{x} = 0}$$

(2) $y^2 = ax$ where a is the arbitrary constant.

$$\frac{y^2}{x} = a$$

$$\frac{dy}{dx}(2y) = a$$

$$2y\frac{dy}{dx} = \frac{y^2}{x}$$

$$\frac{dy}{dx} - \frac{y}{2x} = 0.$$

(3) $y = mx$ where 'm' is the arbitrary constant

$m = \frac{y}{x}$ $\quad \frac{dy}{dx} = m \quad \frac{dy}{dx} - \frac{y}{x} = 0$

(4) $y^2 = (c-x)^2$ where 'c' is the arbitrary constant.

$y = c - x$, $\quad c = y + x$

$$2y\frac{dy}{dx} = -2(c-x)$$

$$2y\frac{dy}{dx} = -(\cancel{x} + y - \cancel{x})$$

$$\frac{dy}{dx} + 1 = 0.$$

First order and first degree DE:

An eqn of the form $\frac{dy}{dx} = f(x,y)$ is called a DE of first order and of first degree.

In general first order DE can be classified as follows:

(1) Variable Seperable
(2) Homogeneous DE and Non-homogeneous DE
(3) Exact DE
(4) Linear DE
(5) Bernoulli's eqn.

Linear Differential eqn:

An eqn of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x)$ and $Q(x)$ are constants or functions of x only is called a linear DE of first order in y.

Working rule:

→ Compare the given eqn with $\frac{dy}{dx} + P(x)y = Q(x)$

→ Write the integrating factor, $IF = e^{\int P(x)dx}$.

→ General solution is

$$y(IF) = \int Q(x)\,(IF)\,dx + C$$

Note: Some times it may be convinient to put the differential equation in the form

$$\frac{dx}{dy} + P(y)x = Q(y)$$

$$IF = e^{\int P(y)dy}$$

$$x(IF) = \int Q(y)(IF)\,dy + c.$$

Q: Find the integrating factor of (1) $\frac{dy}{dx} + y = x$.

Sol: Given eqn is

$$\frac{dy}{dx} + y = x$$

The above eqn is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x) = 1$, $Q(x) = x$

$$IF = e^{\int P(x)dx}$$

$$= e^{\int 1\,dx} = e^{x}.$$

(2) $\frac{dy}{dx} - \frac{y}{x} = \cos x$.

$P(x) = -\frac{1}{x}$, $Q(x) = \cos x$.

$$IF = e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

(3) $\frac{dy}{dx} + \tan x\, y = \cos x$ $\qquad [e^{\log x} = x].$

$P(x) = \tan x$, $Q(x) = \cos x$

$$IF = e^{\int \tan x\,dx} = e^{\log|\sec x|} = \sec x.$$

(4) $\frac{dx}{dy} + yx = \sin y$

$\frac{dx}{dy} + P(y)x = Q(y)$, $P(y) = y$, $Q(y) = \sin y$

$$IF = e^{\int P(y)dy} = e^{\int y\,dy} = e^{\frac{y^2}{2}}.$$

Q: Solve the following:

(5) $$\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$$

Sol: Given eqn, $\frac{dy}{dx} + \frac{y}{x\log x} = \frac{\sin 2x}{\log x}$.

The above eqn is of the form

$$\frac{dy}{dx} + P(x)y = Q(x).$$

where $P(x) = \frac{1}{x\log x}$, $Q(x) = \frac{\sin 2x}{\log x}$. [Put $\log x = t$, $\frac{1}{x}dx = dt$]

$$IF = e^{\int P(x)dx} = e^{\int \frac{1}{x\log x}dx} = e^{\int \frac{1}{t}dt} = e^{\log t} = t = \log x.$$

General solution is

$$y(IF) = \int Q(x)(IF)dx + C.$$

$$y\log x = \int \frac{\sin 2x}{\log x}\cdot \log x \, dx + c$$

$$y\log x = \int \sin 2x \, dx + c$$

$$\boxed{y\log x = \frac{-\cos 2x}{2} + c}$$

(6) $(x+1)\frac{dy}{dx} - y = e^{3x}(x+1)^2$.

$$\frac{dy}{dx} - \frac{y}{x+1} = e^{3x}(x+1)$$

$$P(x) = -\frac{1}{x+1}, \quad Q(x) = e^{3x}(x+1)$$

$$IF = e^{\int P(x)dx} = e^{\int -\frac{1}{x+1}dx} = e^{-\log(x+1)} = (x+1)^{-1} = \frac{1}{x+1}$$

General solⁿ is

$$y(IF) = \int Q(x)(IF)dx + C$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x}\cancel{(x+1)}\cdot\frac{1}{\cancel{x+1}}dx + c$$

$$y\left(\frac{1}{x+1}\right) = \int e^{3x}dx + c$$

$$\boxed{\frac{y}{x+1} = \frac{e^{3x}}{3} + c}$$

(7) $(x+2y^3)\frac{dy}{dx} = y$

$$\frac{dy}{dx} = \frac{y}{x+2y^3} \Rightarrow \frac{dx}{dy} = \frac{x+2y^3}{y} \Rightarrow \frac{dx}{dy} = \frac{x}{y} + \frac{2y^3}{y}$$

$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

$$\frac{dx}{dy} + P(y)x = Q(y), \quad P(y) = -\frac{1}{y}, \quad Q(y) = 2y^2$$

$$IF = e^{\int P(y)dy} = e^{\int -\frac{1}{y}dy} = \frac{1}{y}.$$

$$x(IF) = \int Q(y)(IF)dy + c$$

$$x\left(\frac{1}{y}\right) = \int 2y^2\left(\frac{1}{y}\right)dy + c$$

$$\frac{x}{y} = \int 2y\,dy + c = \frac{2y^2}{2} + c$$

$$\boxed{\frac{x}{y} = y^2 + c}$$

⑧ $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy.$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$\frac{dx}{dy} + P(y)\,x = Q(y) \quad P(y) = \frac{1}{1+y^2},\; Q(y) = \frac{\tan^{-1}y}{1+y^2}.$$

$$IF = e^{\int P(y)dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$$x(IF) = \int Q(y)(IF)\,dy + c$$

$$x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy + c \quad \text{— ①}$$

consider

$$\int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy$$

Put $\tan^{-1}y = t$

$$\frac{1}{1+y^2}\,dy = dt$$

$$\int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy = \int t e^t\,dt = \cancel{te^t}\; e^t(t-1)+c$$

$$= e^{\tan^{-1}y}(\tan^{-1}y - 1)$$

① ⟹ $\boxed{x e^{\tan^{-1}y} = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c.}$

Note:

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int\left[f'(x)\int g(x)\,dx\right]dx.$$

$$\int te^t\,dt = te^t - \int 1.e^t\,dt = te^t - e^t = e^t(t-1)$$

$$\int t^2e^t\,dt = e^t(t^2 - 2t + 2)$$

$$\int t^3e^t\,dt = e^t(t^3 - 3t^2 + 6t - 6).$$

⑨ $\dfrac{dy}{dx} - \dfrac{2y}{x+1} = (x+1)^3$

$P(x) = -\dfrac{2}{x+1}$, $Q(x) = (x+1)^3$, $IF = e^{\int -\frac{2}{x+1}dx} = e^{-2\log(x+1)} = \dfrac{1}{(x+1)^2}$

$$y(IF) = \int Q(x)(IF)\,dx + c$$

$$y\left(\frac{1}{(x+1)^2}\right) = \int (x+1)^3 \cdot \frac{1}{(x+1)^2}\,dx + c = \int (x+1)\,dx + c$$

$$\boxed{\frac{y}{(x+1)^2} = \frac{x^2}{2} + x + c.}$$

(10) $x\frac{dy}{dx} + y = \log x$

$\frac{dy}{dx} + \frac{y}{x} = \frac{\log x}{x}$

$P(x) = \frac{1}{x}, \; Q(x) = \frac{\log x}{x}$.

$IF = e^{\int \frac{1}{x} dx} = e^{\log x} = x$

$y(IF) = \int Q(x)(IF)\,dx + c$

$y(x) = \int \frac{\log x}{\cancel{x}} \cdot \cancel{x}\, dx + C$

$xy = \int \log x \, dx + C$

$\boxed{xy = x\log x - x + C.}$

(11) $x\frac{dy}{dx} + y = 3x$

$\frac{dy}{dx} + \frac{y}{x} = 3$

$P(x) = \frac{1}{x}, \; Q(x) = 3$

$IF = e^{\int \frac{1}{x} dx} = x.$

$xy = \frac{3x^2}{2} + c.$

Bernoulli's equation:

An eqn of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's eqn if P and Q are constants or functions of x alone and n is a real constant.

Working rule:

→ Put the given eqn in the standard form

→ Divide throughout by y^n.

→ Put $y^{1-n} = u$ and get the linear eqn in u.

→ Solve the linear equation in u.

→ Put $u = y^{1-n}$ in the solution obtained to get the desired solution.

(1) Solve $x\frac{dy}{dx} + y = x^3y^6$.

Sol: Give DE is

$x\frac{dy}{dx} + y = x^3y^6 \Rightarrow \frac{dy}{dx} + \frac{y}{x} = x^2y^6$ — (1)

This is a Bernoulli's eqn, which is of the form,

$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$.

Here $P(x) = \frac{1}{x}, \; Q(x) = x^2, \; n = 6.$

Dividing by y^6, we get

$$\frac{1}{y^6}\frac{dy}{dx} + \frac{1}{y^5}\cdot\frac{1}{x} = x^2 \quad \text{—} \ (2)$$

let $\frac{1}{y^5} = u$ then $-\frac{5}{y^6}\frac{dy}{dx} = \frac{du}{dx}$.

$$\Rightarrow \frac{1}{y^6}\frac{dy}{dx} = -\frac{1}{5}\frac{du}{dx}$$

Substituting in eqn (2), we get

$$-\frac{1}{5}\frac{du}{dx} + \frac{u}{x} = x^2$$

$$\Rightarrow \frac{du}{dx} - \frac{5.u}{x} = -5x^2 \quad \text{—} \ (3)$$

This is a linear eqn in u, which is of the form

$$\frac{du}{dx} + P_1(x)\,u = Q_1(x)$$

where $P_1(x) = -\frac{5}{x}$, $Q_1(x) = -5x^2$.

$$IF = e^{\int -\frac{5}{x}dx} = e^{-5\log x} = x^{-5} = \frac{1}{x^5}.$$

General soln of (3) is

$$u\,(IF) = \int Q(x)\,(IF)\,dx + c$$

$$u\left(\frac{1}{x^5}\right) = \int -5x^2 \cdot \frac{1}{x^5}\,dx + c$$

$$\frac{u}{x^5} = -5\int \frac{1}{x^3}\,dx + c$$

$$u\,\frac{1}{x^5} = -5\left(-\frac{1}{2x^2}\right) + c \qquad \left[\because u = \frac{1}{y^5}\right]$$

$$\boxed{\frac{1}{y^5}\cdot\frac{1}{x^5} = \frac{5}{2x^2} + c}$$

which is the required soln.

(2) Solve $\frac{dy}{dx} + y\tan x = y^2\sec x$.

Bernoulli's eqn, $\frac{dy}{dx} + P(x)y = Q(x)y^n$.

$P(x) = \tan x$, $Q(x) = \sec x$, $n = 2$.

Dividing by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{\tan x}{y} = \sec x. \quad \text{—} \ (1)$$

$$\frac{1}{y} = u$$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{du}{dx}.$$

$$\frac{1}{y^2}\frac{dy}{dx} = \frac{-du}{dx}.$$

① $\Rightarrow$ $\frac{-du}{dx} + \tan x \, u = \sec x$

$\Rightarrow$ $\frac{du}{dx} - \tan x \, u = -\sec x$.

linear DE, $\frac{du}{dx} + P_1(x)u = Q_1(x)$

$$IF = e^{\int -\tan x \, dx} = e^{-\log(\sec x)} = (\sec x)^{-1} = \frac{1}{\sec x}.$$

- $u(IF) = \int Q(x)(IF)\,dx + c$

$u\left(\frac{1}{\sec x}\right) = \int \sec x \left(\frac{1}{\sec x}\right)dx + c$

$\Rightarrow$ $u \cdot \frac{1}{\sec x} = -\int dx + c$ $\quad\left(\because u = \frac{1}{y}\right.$

$\Rightarrow$ $\frac{1}{y\sec x} = -x + c$

$\Rightarrow$ $\frac{\cos x}{y} = -x + c$

③ Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$.

Sol: Given, $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$.

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{1}{\sec y}\frac{\tan y}{1+x} = (1+x)e^x$$

$$\frac{1}{\sec y}\frac{dy}{dx} - \cos y \cdot \frac{\sin y}{\cos y} \cdot \frac{1}{1+x} = (1+x)e^x.$$

$$\cos y\frac{dy}{dx} - \sin y \cdot \frac{1}{1+x} = (1+x)e^x.$$

Put $\sin y = z$.

$$\cos y\frac{dy}{dx} = \frac{dz}{dx}$$

$$\frac{dz}{dx} - \frac{z}{1+x} = (1+x)e^x.$$

$P(x) = -\frac{1}{1+x}$, $Q(x) = (1+x)e^x$

$$IF = e^{\int -\frac{1}{1+x}dx} = e^{-\log(1+x)} = \frac{1}{1+x}.$$

$$z\left(\frac{1}{1+x}\right) = \int (1+x)e^{x}\frac{1}{1+x}\,dx + c$$

$$z \cdot \frac{1}{1+x} = \int e^{x}\,dx + c \qquad [\because z = \sin y]$$

$$\frac{\sin y}{1+x} = e^{x} + c.$$

Exact equations and equations reducible to exact form:

Exact differential equations:

Let $M\,dx + N\,dy = 0$ be a first order and first degree DE, where M and N are real valued functions for some x, y. Then the equation $M\,dx + N\,dy = 0$ is said to be an exact DE if there exist a function f such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$.

The eqn $M\,dx + N\,dy = 0$ is said to be exact DE if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working rule:

→ Compare the given DE with $M\,dx + N\,dy = 0$

→ Verify $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (for exactness)

→ If it is exact, general solution is

$$\underset{(y\text{-constant})}{\int M\,dx} + \underset{(\text{terms not containing } x)}{\int N\,dy} = c.$$

where c is a constant.

① Discuss whether the DE is Exact or not
$(y^2+2xy)dx + (x^2+2xy)\,dy = 0$.

Sol: $M\,dx + N\,dy = 0$, $M = y^2+2xy$, $N = x^2+2xy$

$\frac{\partial M}{\partial y} = 2y+2x$, $\frac{\partial N}{\partial x} = 2x+2y \Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given eqn is exact.

(2) solve $e^y dx + (xe^y + 2y)dy = 0$

Sol: Given, $e^y dx + (xe^y + 2y)dy = 0$

which is of the form $Mdx + Ndy = 0$

where $M = e^y$, $N = xe^y + 2y$

$$\frac{\partial M}{\partial y} = e^y \quad , \quad \frac{\partial N}{\partial x} = e^y$$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. The given DE is exact.

The general solⁿ is

$$\underset{y\text{-const}}{\int M\,dx} + \underset{\text{terms independent of } x}{\int N\,dy} = c$$

$$\underset{y\text{-const}}{\int e^y\,dx} + \int 2y\,dy = c$$

$$\boxed{xe^y + y^2 = c}$$

(3) solve $\left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx + [x + \log x - x\sin y]dy = 0$.

Sol: $Mdx + Ndy = 0$,

$M = y\left(1+\frac{1}{x}\right) + \cos y$, $N = x + \log x - x\sin y$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y \quad , \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y.$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, exact DE

$$\underset{y\text{-const}}{\int M\,dx} + \underset{\text{terms not containing } x}{\int N\,dy} = c$$

$$\underset{y\text{-const}}{\int \left[y\left(1+\frac{1}{x}\right) + \cos y\right]dx} + \int 0\,dy = c$$

$$y(x + \log x) + \cancel{\sin y}\ x\cos y = c$$

$$xy + y\log x + x\cos y = c.$$

④ $(\sec x \tan x \tan y - e^x)dx + \sec x \sec^2 y\, dy = 0$

Sol: $M dx + N dy = 0$

$M = \sec x \tan x \tan y - e^x$, $N = \sec x \sec^2 y$

$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y$, $\frac{\partial N}{\partial x} = \sec x \tan x \sec^2 y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, exact DE

$$\underset{y-\text{const}}{\int M\,dx} + \underset{\substack{\text{terms not}\\ \text{containing } x}}{\int N\,dy} = c$$

$$\underset{y-\text{const}}{\int} (\sec x \tan x \tan y - e^x)dx + \int 0\,dy = c$$

$$\sec x \tan y - e^x = c.$$

⑤ Solve $(2x - y + 1)dx + (2y - x - 1)dy = 0$

Sol: $M dx + N dy = 0$

$M = 2x - y + 1$, $N = 2y - x - 1$

$\frac{\partial M}{\partial y} = -1$, $\frac{\partial N}{\partial x} = -1$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, exact DE

$$\underset{y-\text{const}}{\int} (2x - y + 1)dx + \int (2y - 1)dy = c$$

$$x^2 - xy + x + y^2 - y = c$$

$$x^2 + y^2 - xy + x - y = c.$$

⑥ Solve $(y^2 - 2xy)dx + (2xy - x^2)dy = 0$

Sol: $M dx + N dy = 0$

$M = y^2 - 2xy$, $N = 2xy - x^2$

$\frac{\partial M}{\partial y} = 2y - 2x$, $\frac{\partial N}{\partial x} = 2y - 2x$.

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, exact DE

$$\underset{y-\text{const}}{\int} (y^2 - 2xy)dx + \int 0\,dy = c$$

$$xy^2 - x^2y = c.$$

Integrating factor:

Let $Mdx + Ndy = 0$ be not an exact differential equation. If $Mdx+Ndy=0$ can be made exact by multiplying it with a suitable function $u(x,y) \neq 0$ then $u(x,y)$ is called an integrating factor of $Mdx+Ndy=0$

Eg: Let $ydx - xdy = 0$ —— ①

Here $M = y$, $N = -x$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. So given DE is not exact.

Multiply ① with $\frac{1}{x^2}$,

$$\frac{y}{x^2}dx - \frac{1}{x}dy = 0 \text{ —— ②}$$

In ②, $M_1 = \frac{y}{x^2}$, $N_1 = -\frac{1}{x}$

$$\frac{\partial M_1}{\partial y} = \frac{1}{x^2}, \quad \frac{\partial N_1}{\partial x} = \frac{1}{x^2}$$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$, ② is exact DE.

Hence $\frac{1}{x^2}$ is an integrating factor of ①.

Method 1: To find an integrating factor of $Mdx + Ndy = 0$

If $Mdx + Ndy = 0$ is a homogeneous differential eqn and $Mx + Ny \neq 0$ then $\frac{1}{Mx+Ny}$ is an integrating factor of $Mdx + Ndy = 0$.

Homogeneous
$y^4 + xy^3 - x^2y^2 + x^3y + x^4$
same degree

① Solve $x^2y\,dx - (x^3+y^3)dy = 0$.

Sol: Given DE is $x^2y\,dx - (x^3+y^3)dy = 0$ —— ①

which is of the form $Mdx + Ndy = 0$

where $M = x^2y$, $N = -x^3 - y^3$

$$\frac{\partial M}{\partial y} = x^2, \quad \frac{\partial N}{\partial x} = -3x^2$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ① is not exact DE

But ① is a homogeneous DE

$$Mx + Ny = (x^2y)x + (-x^3-y^3)y = x^3y - x^3y - y^4 = -y^4 \neq 0$$

$\therefore$ Integrating factor $= \dfrac{1}{Mx+Ny} = -\dfrac{1}{y^4}$.

Multiplying ① with IF $= -\dfrac{1}{y^4}$, we get.

$$-\frac{x^2}{y^3}dx + \left(\frac{x^3+y^3}{y^4}\right)dy = 0 \quad \text{— ②}$$

which is of the form

$$M_1 dx + N_1 dy = 0$$

where $M_1 = -x^2/y^3$, $N_1 = \dfrac{x^3+y^3}{y^4} = \dfrac{x^3}{y^4} + \dfrac{1}{y}$.

$$\frac{\partial M_1}{\partial y} = \frac{3x^2}{y^4}, \quad \frac{\partial N_1}{\partial x} = \frac{3x^2}{y^4}.$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow \text{② is exact DE.}$$

General soln is

$$\underset{(y\text{-const})}{\int M_1\, dx} + \underset{\substack{(\text{terms not} \\ \text{containing } x)}}{\int N_1\, dy} = c$$

$$\underset{(y\text{-const})}{\int -\frac{x^2}{y^3}dx} + \int \frac{1}{y}dy = c$$

$$\Rightarrow -\frac{x^3}{3y^3} + \log y = c.$$

② Solve $xy\,dx - (x^2+2y^2)dy = 0$

Sol: Given $xy\,dx - (x^2+2y^2)\,dy = 0$ — ①

which is of the form $M dx + N dy = 0$

where $M = xy$, $N = -x^2 - 2y^2$.

$$\frac{\partial M}{\partial y} = x, \quad \frac{\partial N}{\partial x} = -2x.$$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \Rightarrow$ ① is not exact. but ① is homogeneous,

$$Mx + Ny = x^2y - x^2y - 2y^3 = -2y^3 \neq 0.$$

$$\text{i.e. } IF = \frac{1}{Mx+Ny} = -\frac{1}{2y^3}.$$

$$① \times -\frac{1}{2y^3} \Rightarrow -\frac{x}{2y^2}dx + \left(\frac{x^2+2y^2}{2y^3}\right)dy = 0 \quad \text{— ②}$$

which is of the form $M_1 dx + N_1 dy = 0$

where $M_1 = -\dfrac{x}{2y^2}$, $N_1 = \dfrac{x^2}{2y^3} + \dfrac{1}{y}$.

$\frac{\partial M_1}{\partial y} = \frac{x}{y^3}$, $\frac{\partial N_1}{\partial x} = \frac{x}{y^3}$.

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ ② is exact

General soln is

$\int M_1 \, dx + \int N_1 \, dy = c$
(y-const) (terms not containing x)

$\int -\frac{x}{2y^2} dx + \int \frac{1}{y} dy = c$
(y-const)

$-\frac{x^2}{4y^2} + \log y = c$

Method 2: To find the integrating factor of $Mdx + Ndy = 0$

If the equation $Mdx + Ndy = 0$ is of the form $y f(x,y)dx + x g(xy)dy = 0$ and $Mx - Ny \neq 0$ then $\frac{1}{Mx-Ny}$ is an integrating factor of $Mdx + Ndy = 0$.

① Solve $y(x^2y^2+2)dx + x(2-x^2y^2)dy = 0$

Sol: Given eqn is $y(x^2y^2+2)dx + x(2-x^2y^2)dy = 0$ —①

which is of the form $Mdx + Ndy = 0$

where $M = y(x^2y^2+2)$, $N = x(2-x^2y^2)$
$= x^2y^3 + 2y$, $= 2x - x^3y^2$

$\frac{\partial M}{\partial y} = 3x^2y^2 + 2$, $\frac{\partial N}{\partial x} = 2 - 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ ① is not exact

But ① is of the form $y f(x,y)dx + x g(x,y)dy = 0$

$Mx - Ny = x^3y^3 + 2xy - 2xy + x^3y^3 = 2x^3y^3 \neq 0$

$\therefore$ I.F $= \frac{1}{Mx-Ny} = \frac{1}{2x^3y^3}$

① $\times \frac{1}{2x^3y^3} \Rightarrow \left(\frac{x^2y^2+2}{2x^3y^2}\right)dx + \left(\frac{2-x^2y^2}{2x^2y^3}\right)dy = 0$ —②

which is of the form $M_1dx + N_1dy = 0$.

where $M_1 = \frac{x^2y^2+2}{2x^3y^2} = \frac{1}{2x} + \frac{1}{x^3y^2}$, $N_1 = \frac{1}{x^2y^3} - \frac{1}{2y}$

$\frac{\partial M_1}{\partial y} = \frac{-2}{x^3y^3}$, $\frac{\partial N_1}{\partial x} = \frac{-2}{x^3y^3}$.

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ (2) is exact DE

General soln is

$$\int_{(y\text{-const})} M_1\, dx + \int_{(\text{terms not containing } x)} N_1\, dy = c$$

$$\int_{(y\text{-const})} \left(\frac{1}{2x} + \frac{1}{x^2y^2}\right) dx + \int \left(-\frac{1}{2y}\right) dy = c$$

$$\frac{1}{2}\log x - \frac{1}{2x^2y^2} - \frac{1}{2}\log y = c$$

$$\log(x/y) - \frac{1}{x^2y^2} = K.$$

(2) Solve $y(1+xy)dx + x(1-xy)dy = 0$

Sol: Given $y(1+xy)dx + x(1-xy)dy = 0$ — (1)

which is of the form $Mdx + Ndy = 0$

where $M = y(1+xy)$, $N = x(1-xy)$

$= y + xy^2$ $\qquad = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$, $\frac{\partial N}{\partial x} = 1 - 2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \Rightarrow$ (1) is not exact. But (1) is of the form

$y f(x,y)dx + x g(x,y)dy = 0$

$Mx - Ny = xy + x^2y^2 - xy + x^2y^2 = 2x^2y^2.$

$IF = \frac{1}{Mx - Ny} = \frac{1}{2x^2y^2}.$

(1) $\times \frac{1}{2x^2y^2} \Rightarrow \left(\frac{1+xy}{2x^2y}\right)dx + \left(\frac{1-xy}{2xy^2}\right)dy = 0$ — (2)

which is of the form, $M_1dx + N_1dy = 0$

where $M_1 = \frac{1}{2x^2y} + \frac{1}{2x}$, $N_1 = \frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M_1}{\partial y} = \frac{-1}{2x^2y^2}$, $\frac{\partial N_1}{\partial x} = -\frac{1}{2x^2y^2}.$

$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x} \Rightarrow$ (2) is exact

General soln is

$$\int_{(y\text{-const})} M_1\, dx + \int_{(\text{terms not containing } x)} N_1\, dy = c$$

$$\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=c.$$
(y-const)

$$-\frac{1}{2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=c$$

$$\log(x/y)-\frac{1}{xy}=k.$$

Method 3: To find the integrating factor of $Mdx+Ndy=0$

For a DE $Mdx+Ndy=0$, $\dfrac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=f(x)$ or constant,

then IF $=e^{\int f(x)dx}$.

① Solve $2xydy-(x^2+y^2+1)dx=0$

Sol: Given, $2xydy-(x^2+y^2+1)dx=0$ —— ①

which is of the form $Mdx+Ndy=0$

where $M=-x^2-y^2-1$, $N=2xy$.

$\frac{\partial M}{\partial y}=-2y$, $\frac{\partial N}{\partial x}=2y$, $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$ $\Rightarrow$ ① is not exact

$$\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N}=\frac{-2y-2y}{2xy}=\frac{-4y}{2xy}=-\frac{2}{x}.=f(x)\ (\text{say})$$

$$\therefore IF=e^{\int f(x)dx}=e^{\int-\frac{2}{x}dx}=e^{-2\log x}=x^{-2}=\frac{1}{x^2}.$$

① × IF $\Rightarrow$ $\dfrac{-(x^2+y^2+1)}{x^2}dx+\dfrac{2xy}{x^2}dy=0$ —— ②

which is of the form $M_1dx+N_1dy=0$

$M_1=\dfrac{-(x^2+y^2+1)}{x^2}=-1-\dfrac{y^2}{x^2}-\dfrac{1}{x^2}$, $N_1=\dfrac{2y}{x}$.

$\dfrac{\partial M_1}{\partial y}=-\dfrac{2y}{x^2}$, $\dfrac{\partial N_1}{\partial x}=\dfrac{-2y}{x^2}$, $\dfrac{\partial M_1}{\partial y}=\dfrac{\partial N_1}{\partial x}$ $\Rightarrow$ ② is exact

General soln is

$\int M_1dx$ (y-const) $+\int N_1dy$ (terms not containing x) $=c$

$$\int\left(-1-\frac{y^2}{x^2}-\frac{1}{x^2}\right)dx+\int 0\,dy=c$$
(y-const)

$$-x+\frac{y^2}{x}+\frac{1}{x}=c$$

② $(x^2+y^2+x)\,dx + xy\,dy = 0$

Sol: Given $(x^2+y^2+x)dx + xy\,dy = 0$ — ①

which is of the form $M\,dx + N\,dy = 0$

where $M = x^2+y^2+x$, $N = xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ ① is not exact

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x} = f(x) \text{ [say]}$$

$$\therefore IF = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$$

① × IF $\Rightarrow$ $(x^3+xy^2+x^2)dx + x^2y\,dy = 0$ — ②

which is of the form $M_1 dx + N_1 dy = 0$

where $M_1 = x^3+xy^2+x^2$, $N_1 = x^2y$

$\frac{\partial M_1}{\partial y} = 2xy$, $\frac{\partial N_1}{\partial x} = 2xy$, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\Rightarrow$ ② is exact

General soln is

$$\underset{(y\text{-const})}{\int M_1\,dx} + \underset{(\text{terms not containing } x)}{\int N_1\,dy} = c$$

$$\underset{(y\text{-const})}{\int (x^3+xy^2+x^2)\,dx} + \int 0\,dy = c$$

$$\frac{x^4}{4} + \frac{x^2y^2}{2} + \frac{x^3}{3} = c.$$

Method 4: To find the integrating factor of $M\,dx+N\,dy=0$

For a DE $M\,dx+N\,dy=0$, $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = f(y)$, then $IF = e^{\int f(y)dy}$.

① $y(2xy+e^x)\,dx - e^x\,dy = 0$

Sol: Given, $y(2xy+e^x)dx - e^x dy = 0$ — ①

which is of the form $M\,dx + N\,dy = 0$

where $M = 2xy^2 + ye^x$, $N = -e^x$.

$\frac{\partial M}{\partial y} = 4xy+e^x$, $\frac{\partial N}{\partial x} = -e^x$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ $\Rightarrow$ ① is not exact

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{4xy+e^x+e^x}{-(2xy^2+ye^x)} = \frac{4xy+2e^x}{-(2xy^2+ye^x)} = \frac{2(2xy+e^x)}{-y(2xy+e^x)} = -\frac{2}{y} = f(y)$$

$$\therefore IF = e^{\int f(y)dy} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}.$$

① × IF = $\frac{y(2xy+e^x)}{y^2}dx - \frac{e^x}{y^2}dy = 0$ — ②

which is of the form $M_1 dx + N_1 dy = 0$ where $M_1 = \frac{y(2xy+e^x)}{y^2} = 2x + \frac{e^x}{y}$

$N_1 = \frac{-e^x}{y^2}$, $\frac{\partial M_1}{\partial y} = \frac{-e^x}{y^2}$, $\frac{\partial N_1}{\partial x} = \frac{-e^x}{y^2}$, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ $\Rightarrow$ ② is exact.

General soln is $\underset{(y\text{-const})}{\int M_1\,dx} + \underset{(\text{terms not containing } x)}{\int N_1\,dy} = c \Rightarrow \underset{(y\text{-const})}{\int \left(2x + \frac{e^x}{y}\right)dx} + \int 0\,dy = c$

$$\Rightarrow x^2 + \frac{e^x}{y} = c.$$

② $(xy^3+y)dx + 2(x^2y^2+x+y^4)dy = 0$

Sol: Given $(xy^3+y)dx + 2(x^2y^2+x+y^4)\,dy = 0$ —— ①

which is of the form $Mdx + Ndy = 0$ where $M = xy^3+y$, $N = 2(x^2y^2+x+y^4)$

$\frac{\partial M}{\partial y} = 3xy^2+1$, $\frac{\partial N}{\partial x} = 2(2xy^2+1) = 4xy^2+2$, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ ⇒ ① is not exact.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{-M} = \frac{3xy^2+1-4xy^2-2}{-(xy^3+y)} = \frac{-xy^2-1}{-(xy^3+y)} = \frac{\cancel{+}(\cancel{xy^2+1})}{\cancel{+}y(\cancel{xy^2+1})} = \frac{1}{y} = f(y) \text{ [say]}$$

$$\therefore \text{IF} = e^{\int f(y)dy} = e^{\int \frac{1}{y}dy} = e^{\log y} = y.$$

① × IF ⇒ $(xy^4+y^2)dx + (2x^2y^3 + 2xy + 2y^5)\,dy = 0$ —— ②

which is of the form $M_1 dx + N_1 dy = 0$ where $M_1 = xy^4+y^2$, $N_1 = 2x^2y^3 + 2xy + 2y^5$

$\frac{\partial M_1}{\partial y} = 4xy^3+2y$, $\frac{\partial N_1}{\partial x} = 4xy^3+2y$, $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$ ⇒ ② is exact

General soln is $\underset{(y\text{-const})}{\int M_1\,dx} + \underset{(\text{terms not containing } x)}{\int N_1\,dy} = c$ ⇒ $\underset{(y\text{-const})}{\int (xy^4+y^2)dx} + \int 2y^5 dy = c$

$$\Rightarrow \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$$

Newton's law of cooling:

The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of its sorroundings medium.

$$\frac{d\theta}{dt} \propto (\theta-\theta_0) \Rightarrow \frac{d\theta}{dt} = -k(\theta-\theta_0)$$

where θ is the temperature at time t.

θ_0 is the temperature of sorroundings (usually air)

(1) Water temperature at $100°c$ cools ~~do~~ $80°c$ in 10 minutes in the room of temperature $25°c$. Find the Temperature of the water after 20 minutes.

Sol: Given,

$\theta_0 = 25°c$

$\theta_1 = 100°c$ $t=0$

$\theta_2 = 80°c$ $t=10$

$\theta = ?$ $t=20$

Temperature of sorroundings,

$\theta_0 = 25°c$

By Newton's law of cooling,

$$\frac{d\theta}{dt} \propto (\theta-\theta_0)$$

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta-\theta_0)$$

$$\Rightarrow \frac{d\theta}{dt} = -k(\theta-25)$$

Seperating the variables

$$\Rightarrow \frac{d\theta}{(\theta-25)} = -k\,dt$$

Integrating on both sides

$$\int\frac{d\theta}{(\theta-25)} = -k\int dt$$

$$\log(\theta-25) = -kt + \log c \quad \text{—(1)}$$

Given at $t=0$, $\theta = 100°c$

$$(1) \Rightarrow \log(100-25) = -k(0)+c$$

$$\Rightarrow c = \log 75, \quad \cancel{(1)' \Rightarrow \log\left[\frac{\theta-25}{75}\right] = -kt}$$

~~Given at $t=10$, $\theta=80°c$~~

$$\cancel{(1) \Rightarrow \log(80-25) = -k(10)+c}$$

$$\cancel{\Rightarrow \log 55 = -10k + \log 75}$$

$$(1) \Rightarrow \log(\theta-25) = -kt + \log 75$$

$$\Rightarrow \log\left(\frac{\theta-25}{75}\right) = -kt \quad \text{—(2)}$$

At $t = 10$, $\theta = 80°c$

$(2) \Rightarrow \log\left(\frac{80-25}{75}\right) = -k(10)$

$\Rightarrow \log\left(\frac{55}{75}\right) = -k(10)$

$\Rightarrow \log\left(\frac{11}{15}\right) = -10k$ —— (3)

$\frac{(2)}{(3)} \Rightarrow \frac{\log\left(\frac{\theta-25}{75}\right)}{\log(11/15)} = \frac{\cancel{+k}t}{+10\cancel{k}} = \frac{t}{10}$

$\Rightarrow \log\left(\frac{\theta-25}{75}\right) = \frac{t}{10}\log(11/15)$

So at $t = 20$,

$\log\left(\frac{\theta-25}{75}\right) = \frac{20}{10}\log(11/15) = \log(11/15)^2$

$\frac{\theta-25}{75} = \left(\frac{11}{15}\right)^2 = \frac{121}{225}$

$\theta = \frac{121}{3} + 25 = 65.33°C$.

∴ After 20 minutes, the temperature of the water is $65.33°c$.

(2) If the temperature of the air is 30°c and a substance cools from 100°c to 70°c in 15 minutes, find the time when the temperature is 40°c.

Sol: Given,

Temperature of air,

$\theta_0 = 30°c$.

$\theta_0 = 30°c$

$\theta_1 = 100°c$ $t = 0$

$\theta_2 = 70°c$ $t = 15$

$\theta = 40°c$ $t = ?$.

By Newton's of cooling,

$\frac{d\theta}{dt} = -k(\theta - \theta_0)$

$\Rightarrow \frac{d\theta}{dt} = -k(\theta - 30)$

Seperating the variables

$\Rightarrow \frac{d\theta}{\theta - 30} = -kdt$

Integrating on both sides

$\Rightarrow \int \frac{d\theta}{(\theta-30)} = -k\int dt$

$\Rightarrow \log(\theta - 30) = -kt + c$ —— (1)

At $t = 0$, $\theta = 100°c$

$(1) \Rightarrow \log(100-30) = -k(0) + c$

$\Rightarrow c = \log 70$

$① \Rightarrow \log(\theta-30) = -kt + \log 70$

$\Rightarrow \log\left[\frac{\theta-30}{70}\right] = -kt \quad — ②$

At $t = 15,\ \theta = 70°c$

$② \Rightarrow \log\left[\frac{70-30}{70}\right] = -k(15)$

$\Rightarrow \log\left[\frac{40}{70}\right] = -15k$

$\Rightarrow \log\left[\frac{4}{7}\right] = -15k \quad — ③$

$\frac{②}{③} \Rightarrow \frac{\log\left[\frac{\theta-30}{70}\right]}{\log\left[\frac{4}{7}\right]} = \frac{-kt}{-15k} = \frac{t}{15}$

$\Rightarrow \log\left[\frac{\theta-30}{70}\right] = \frac{t}{15}\log\left[\frac{4}{7}\right]$

At $\theta = 40°c$

$\log\left[\frac{40-30}{70}\right] = \frac{t}{15}\log\left[\frac{4}{7}\right]$

$\Rightarrow t = 15 \times \frac{\log\left(\frac{1}{7}\right)}{\log\left(\frac{4}{7}\right)}$

$\Rightarrow t = 52.15$ minutes.

∴ The temperature is $40°c$ when $t = 52.15$ minutes.

③ A body is originally at $80°c$ and cools down to $60°c$ in 20 min. If the temperature of the air is $40°c$, find the temperature of the body after 40 min?

Sol: Given,

Temperature of air is, $\theta_0 = 40°c$

$\theta_0 = 40°c$

$\theta_1 = 80°c \quad t=0$

$\theta_2 = 60°c \quad t=20$

$\theta = ? \quad t=40$.

By Newton's law of cooling,

$\frac{d\theta}{dt} = -k(\theta-\theta_0)$

$\Rightarrow \frac{d\theta}{dt} = -k(\theta-40)$

Seperating the variables

$\frac{d\theta}{\theta-40} = -k\,dt$

Integrating

$\log(\theta-40) = -kt + c \quad — ①$

At $t=0$, $\theta_0 = 80^\circ C$

$(1) \Rightarrow \log(80-40) = -K(0)+C$

$\Rightarrow C = \log 40$

$(1) \Rightarrow \log(\theta-40) = -Kt + \log 40$

$\Rightarrow \log\left[\frac{\theta-40}{40}\right] = -Kt \quad —(2)$

At $t = 20$, $\theta = 60^\circ C$

$(2) \Rightarrow \log\left[\frac{60-40}{40}\right] = -K(20)$

$\Rightarrow \log\left(\frac{20}{40}\right) = -20k$

$\Rightarrow \log\left(\frac{1}{2}\right) = -20k \quad —(3)$

$$\frac{(2)}{(3)} \Rightarrow \frac{\log\left(\frac{\theta-40}{40}\right)}{\log\left(\frac{1}{2}\right)} = \frac{+\cancel{K}t}{+20\cancel{K}} = \frac{t}{20}.$$

$$\Rightarrow \log\left(\frac{\theta-40}{40}\right) = \frac{t}{20}\log\left(\frac{1}{2}\right).$$

At $t = 40$

$$\Rightarrow \log\left(\frac{\theta-40}{40}\right) = \frac{\overset{2}{\cancel{40}}}{\underset{1}{\cancel{20}}}\log\left(\frac{1}{2}\right)$$

$$\Rightarrow \log\left(\frac{\theta-40}{40}\right) = \log\left(\frac{1}{2}\right)^2$$

$$\frac{\theta-40}{\underset{10}{\cancel{40}}} = \frac{1}{\cancel{4}}$$

$\theta - 40 = 10$

$\theta = 50^\circ C$

$\therefore$ The temperature of the body after 40 min is $50^\circ C$.

(4) If the temperature of a body drops from $100^\circ C$ to $60^\circ C$ in one minute when the surrounding temperature is $20^\circ C$, find the time when the temperature will come down to $30^\circ C$?

$\theta_0 = 20^\circ C$

$t=0 \quad \theta = 100^\circ C$

$t=1 \quad \theta = 60^\circ C$

$t=? \quad \theta = 30^\circ C$

Sol: Given,

Temperature of surroundings $\theta_0 = 20^\circ C$

By Newton's law of cooling

$$\frac{d\theta}{dt} = -K(\theta-\theta_0)$$

$$\Rightarrow \frac{d\theta}{dt} = -K(\theta-20)$$

Separating the variables

$$\Rightarrow \frac{d\theta}{\theta-20} = -K\,dt$$

Integrating

$$\int \frac{d\theta}{\theta - 20} = -k \int dt$$

$\Rightarrow \log(\theta - 20) = -kt + c$ — (1)

At $t = 0$, $\theta = 100°c$

(1) $\Rightarrow \log(100 - 20) = -k(0) + c$

$\Rightarrow c = \log 80$

(1) $\Rightarrow \log(\theta - 20) = -kt + \log 80$

$\Rightarrow \log\left[\frac{\theta - 20}{80}\right] = -kt$ — (2)

At $t = 1$, $\theta = 60°c$

(2) $\Rightarrow \log\left[\frac{60 - 20}{80}\right] = -k(1)$

$\log\left[\frac{40}{80}\right] = -k$

$\log\left[\frac{1}{2}\right] = -k.$ — (3)

$$\frac{(2)}{(3)} \Rightarrow \frac{\log\left[\frac{\theta - 20}{80}\right]}{\log\left[\frac{1}{2}\right]} = \frac{\cancel{k}t}{\cancel{k}} = t.$$

$\Rightarrow \log\left[\frac{\theta - 20}{30}\right] = t \log\left[\frac{1}{2}\right].$

At $\theta = 30°C$.

$\Rightarrow \log\left[\frac{30 - 20}{80}\right] = t \log\left(\frac{1}{2}\right]$

$\Rightarrow \log\left[\frac{10}{80}\right] = \log\left[\frac{1}{2}\right]^t$

$\Rightarrow \log\left[\frac{1}{8}\right] = \log\left[\frac{1}{2}\right]^t$

$\Rightarrow \frac{1}{8} = \frac{1}{2^t}.$

$\Rightarrow 8 = 2^t$

$\Rightarrow t = 3$ min

$\therefore$ At 3 min the temperature is $30°C$.

Law of Natural growth and decay:

The rate of change of a quantity of a substance is [illegible] proportional to the quantity of a substance at that time.

If $x(t)$ be the quantity of a substance at time t,

$$\frac{dx}{dt} \propto x$$

$$\frac{dx}{dt} = \begin{cases} Kx & \text{(growth)} \\ -kx & \text{(decay)} \end{cases}$$

Ex: For growth – (Bacteria when converting milk to curd)

For decay – (Bone, skeleton) (By the amount of carbon) (We can find the age of the skeleton).

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① The number N of the bacteria in a culture grow at a rate proportional to N. The value N was initially 100 and increased to 332 in 1 hr. What will be the value of N after $1\frac{1}{2}$ hour?

Sol: Given, the number N of the bacteria in a culture grow at a rate proportional to N. By law of natural growth,

$N = 100$, $t = 0$

$N = 332$, $t = 1\,hr$

$N = ?$ $t = 1\frac{1}{2}\,hr$

$$\frac{dN}{dt} \propto N \Rightarrow \frac{dN}{dt} = kN.$$

Seperating the variables

$$\frac{dN}{N} = k\,dt$$

Integrating

$$\log N = kt + c \quad \text{—— ①}$$

Given, $t = 0$, $N = 100$

$$① \Rightarrow \log 100 = k(0) + c$$

$\Rightarrow c = \log 100$

$(1) \Rightarrow \log N = kt + \log 100$

$\Rightarrow \log\left(\frac{N}{100}\right) = kt$ — (2)

At $t = 1$, $N = 332$

$(2) \Rightarrow \log\left(\frac{332}{100}\right) = k(1)$ — (3)

$\frac{(2)}{(3)} \Rightarrow \frac{\log\left(\frac{N}{100}\right)}{\log\left(\frac{332}{100}\right)} = \frac{kt}{k} = t$

$\Rightarrow \log\left(\frac{N}{100}\right) = t \log\left(\frac{332}{100}\right)$

At $t = 1\frac{1}{2}h = \frac{3}{2}h$

$\log\left(\frac{N}{100}\right) = \frac{3}{2}\log\left(\frac{332}{100}\right) = \log\left(\frac{332}{100}\right)^{3/2}$

$\Rightarrow \frac{N}{100} = \left(\frac{332}{100}\right)^{3/2}$

$\Rightarrow N = 100 \times \left(\frac{332}{100}\right)^{3/2}$

$\Rightarrow N = 604.932$

$\therefore$ After $1\frac{1}{2}$ hour, $N \simeq 605$.

Electrical circuits:

We shall consider circuits made up of
(i) three passive elements - resistance, inductance, capacitance and
(ii) an active element - voltage source which may be a battery or a generator

Basic relations

Let i be the current and q be the charge in the condenser plate at any time t. Then

(i) $i = \frac{dq}{dt}$ or $q = \int i\, dt$

(ii) Voltage drop across resistance $R = Ri = R\frac{dq}{dt}$ [ohm's law]

(iii) Voltage drop across inductance $L = L\frac{di}{dt} = L\frac{d^2q}{dt^2}$

(iv) Voltage drop across capacitance $C = \frac{q}{C}$.

The formulation of differential equations for an electrical circuit depends on the Kirchhoff's laws.

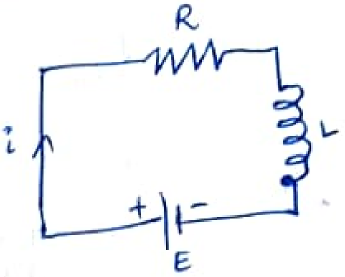
Kirchhoff's laws:

1) The algebraic sum of the voltage drops around any closed circuit is equal to the resultant electromotive force in the circuit ✓

2) The algebraic sum of the currents flowing into (or from) any node is zero.

Differential eqns:

1) R, L series circuit: Consider a circuit containing resistance R and inductance L in series with a voltage source (battery) V



Let i be the current flowing in the circuit at any time t,

Then by Kirchhoff's first law, we have sum of voltage drops across R and L = V

i.e., $Ri + L\frac{di}{dt} = V$

$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$

which is a linear eqn in i. ✓

2) R, L, C series circuit: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V$

① If the voltage of a battery in an LR circuit is 10 sint, Find the current I in the circuit under the initial condition $I(0) = 0$.

Sol: Given, $V = 10\sin t$, $I(0) = 0$.

WKT, For an LR circuit, by Kirchhoff's law

$Ri + L\frac{di}{dt} = V$

$\Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$

$\Rightarrow \frac{di}{dt} + \left(\frac{R}{L}\right)i = \frac{10}{L}\sin t$.

which is a linear eqn of the form,

$$\frac{di}{dt} + P(t)\, i = Q(t)$$

where $P(t) = \frac{R}{L}$, $Q(t) = \frac{10}{L}\sin t$

$$IF = e^{\int P(t)dt} = e^{\int \frac{R}{L}dt} = e^{\frac{R}{L}t}$$

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$$

General soln is

$$i(IF) = \int Q(IF)\,dt + c$$

$$i\left(e^{\frac{R}{L}t}\right) = \int \frac{10}{L}\sin t \cdot e^{\frac{R}{L}t}\,dt + c$$

$$= \frac{10}{L}\int e^{\frac{R}{L}t}\sin t\,dt + c$$

$$= \frac{10}{L}\left[\frac{e^{\frac{R}{L}t}}{\left(\frac{R}{L}\right)^2+1}\left\{\frac{R}{L}\sin t - \cos t\right\}\right] + c$$

$$i\,e^{\frac{R}{L}t} = \frac{10L}{R^2+L^2}\, e^{\frac{R}{L}t}\left[\frac{R}{L}\sin t - \cos t\right] + c \quad (1)$$

given $i = 0$ when $t = 0$

$$(1) \Rightarrow 0 = \frac{10L}{R^2+L^2}\, e^0 \cdot \left[\frac{R}{L}\sin 0 - \cos 0\right] + c$$

$$\Rightarrow 0 = \frac{10L}{R^2+L^2} \cdot 1\,[0-1] + c$$

$$\Rightarrow c = \frac{10L}{R^2+L^2}$$

$$(1) \Rightarrow i\,e^{\frac{R}{L}t} = \frac{10L}{R^2+L^2}\, e^{\frac{R}{L}t}\left[\frac{R}{L}\sin t - \cos t\right] + \frac{10L}{R^2+L^2}$$

$$\therefore i\,e^{\frac{R}{L}t} = \frac{10L}{R^2+L^2}\left[e^{\frac{R}{L}t}\left(\frac{R}{L}\sin t - \cos t\right) + 1\right].$$

current
i. ampere

Henrys ohm

(2) An inductance of 3 H and resistance of 12 Ω are connected in a series with an e.m.f of 90 V (Volt). If the current i is zero when $t = 0$, what is the current at the end of 1 sec.

Sol: Given, $L = 3H$, $R = 12\,\Omega$, $V = 90v$.

By Kirchhoff's law,

$$L\frac{di}{dt} + Ri = V$$

$$\Rightarrow \frac{di}{dt} + \frac{R}{L} i = \frac{V}{L}$$

$$\Rightarrow \frac{di}{dt} + \frac{12}{3} i = \frac{90}{3}$$

$$\Rightarrow \frac{di}{dt} + 4i = 30$$

which is a linear eqn of the form

$\frac{di}{dt} + P(t) i = Q(t)$, where $P(t) = 4$, $Q(t) = 30$

$$IF = e^{\int P(t) dt} = e^{\int 4 dt} = e^{4t}.$$

General soln is

$$i(IF) = \int Q(t)(IF) dt + c.$$

$$i(e^{4t}) = \int 30 e^{4t} dt + c$$

$$i e^{4t} = \frac{30}{4} e^{4t} + c \quad (1).$$

Given, at $t = 0$, $i = 0$

$$(1) \quad 0 = \frac{30}{4} e^0 + c$$

$$\Rightarrow c = \frac{-30}{4}$$

$$(2) \Rightarrow i e^{4t} = \frac{30}{4} e^{4t} - \frac{30}{4}.$$

At $t = 1$, $i e^4 = \frac{30}{4} e^4 - \frac{30}{4} = \frac{30}{4}[e^4 - 1].$

$$i = \frac{30}{4}[1 - e^{-4}] \text{ Amperes.}$$