# UNIT - II: Linear Differential Equations of Higher Order (Constant Coefficients)

**Linear DE of Higher Order:**

**Definition:** An eqn of the form,

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)y = Q(x),$$

where $P_1(x), P_2(x), P_3(x), \ldots, P_n(x)$ and $Q(x)$ are all continuous and real valued functions of $x$, is called a linear differential eqn of order $n$.

**Linear DE of higher order with constant coefficients:**

An eqn of the form

$$\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = Q(x) \quad \text{--- (1)}$$

where $P_1, P_2, \ldots, P_n$ are real constants and $Q(x)$ is a continuous function of $x$, is called an ordinary linear eqn of order $n$ with constant coefficients.

$$(1) \Rightarrow D^n y + P_1 D^{n-1} y + P_2 D^{n-2} y + \cdots + P_n y = Q(x)$$

$$(D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n)y = Q(x)$$

$$f(D)y = Q(x)$$

where $f(D) = D^n + P_1 D^{n-1} + P_2 D^{n-2} + \cdots + P_n$

and $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$, $D^3 = \frac{d^3}{dx^3}$, $\ldots$, $D^n = \frac{d^n}{dx^n}$.

**Homogeneous and Non-homogeneous:**

In $f(D)y = Q(x)$, if $Q(x) \neq 0$ it is called non-homogeneous eqn.

If $Q(x) = 0$, i.e., $f(D)y = 0$, it is called homogeneous eqn.

The complete ~~general~~ solution of (1) is $y = C.F + P.I$

where CF = complementary function

PI = Particular Integral.

For homogeneous, $y = CF$

For non-homogeneous, $y = PI$.

To find CF :

Let $f(D)y=0$ In eqn $f(D)y=Q(x)$

The algebraic eqn $f(m)=0$ is called auxiliary eqn (AE) of $f(D)y=Q(x)$. It is a polynomial eqn of degree $n$, it will have $n$ roots $m_1, m_2, \dots, m_n$.

| Roots | CF |
|---|---|
| ① $m_1, m_2, \dots, m_n$ All roots are real and distinct | $CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$ |
| ② $m_1, m_2 = m_1, m_3, \dots, m_n$ Two roots are real and equal and rest are real and distinct | $CF = (c_1+c_2x)e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$ |
| ③ $m_1, m_2 = m_1, m_3 = m_1, m_4, \dots, m_n$ Three roots are real and equal and rest are real and distinct | $CF = (c_1+c_2x+c_3x^2)e^{m_1 x} + c_4 e^{m_4 x} + \dots$ |
| ④ $m_1 = \alpha+i\beta,\ m_2 = \alpha-i\beta, m_3, \dots, m_n$ Two roots are complex and remaining roots are real and distinct | $CF = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$ |
| ⑤ $m_1 = \alpha+\sqrt{\beta},\ m_2 = \alpha-\sqrt{\beta}, m_3, \dots, m_n$ Two roots are irrational and remaining roots are real and distinct | $CF = e^{\alpha x}(c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$ (or) $e^{\alpha x}(c_1 e^{\sqrt{\beta}x} + c_2 e^{-\sqrt{\beta}x}) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$ |
| ⑤ $m_1 = \alpha+i\beta,\ m_2 = \alpha-i\beta,$ $m_3 = \alpha+i\beta,\ m_4 = \alpha-i\beta,$ $m_5, \dots, m_n$ ~~Four roots are~~ Four roots are complex and pairs are equal and rest are real and distinct | $CF = e^{\alpha x}\{(c_1+c_2x)\cos\beta x + (c_3+c_4x)\sin$ $+ c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$ |

* where $c_1, c_2, \dots, c_n$ are arbitrary constants.

Find the CF of

Ex: $(D^2+2D+1)y = \sin x$.

$f(D) = D^2+2D+1$, AE is $f(m) = m^2+2m+1 = 0$

$m = -1, -1$.

$CF = (c_1+c_2x)e^{-x}$.

Model 1: Homogeneous eqn

$f(D)y = 0$ (i.e., $Q(x) = 0$).

then general soln, $y = CF$

① Solve. $(D^2-4)y = 0$

Sol: Given, $f(D) = D^2 - 4$

AE is $f(m) = 0$

$\Rightarrow m^2 - 4 = 0$

$(m+2)(m-2) = 0$

$m = 2, -2.$

General soln is

$y = CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$

$y = c_1 e^{2x} + c_2 e^{-2x}$

② Solve $(4D^2+4D+1)y = 0$

AE is $f(m) = 0$

$\Rightarrow 4m^2+4m+1 = 0$

$m = -1/2, -1/2.$

General soln is

$y = CF = (c_1 + c_2 x)e^{-1/2 x}.$

③ Solve $(D-2)^2 y = 0$ (or) Solve $y'' - 4y' + 4y = 0.$

AE is $(m-2)^2 = 0$

$m = 2, 2$

$y = CF = (c_1 + c_2 x)e^{2x}.$

④ $\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0.$

Sol: Given eqn,

$\frac{d^3y}{dx^3} - 9\frac{d^2y}{dx^2} + 23\frac{dy}{dx} - 15y = 0$

$\Rightarrow D^3y - 9D^2y + 23Dy - 15y = 0$

$\Rightarrow (D^3 - 9D^2 + 23D - 15)y = 0$

$f(D)y = 0$

$f(D) = D^3 - 9D^2 + 23D - 15$

AE is $f(m) = 0 \Rightarrow m^3 - 9m^2 + 23m - 15 = 0$

$m = 1, 3, 5.$

$y = CF = c_1 e^{x} + c_2 e^{3x} + c_3 e^{5x}.$

⑤ Solve $\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 6x = 0.$

given $x(0) = 0$, $\frac{dx}{dt}(0) = 15$

i.e.,) 3s format

Solve $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = 0$

given $y(0) = 0$, $\frac{dy}{dx}(0) = 15$

$D^2y + 5Dy + 6y = 0$

$(D^2+5D+6)y = 0$

AE is $m^2 + 5m + 6 = 0$

$m = -2, -3.$

$y = CF = c_1 e^{-2x} + c_2 e^{-3x}.$

$y(0) = 0$

$\Rightarrow 0 = c_1 e^{-2(0)} + c_2 e^{-3(0)}$

$\Rightarrow c_1 + c_2 = 0. \Rightarrow c_1 = -c_2$

$\frac{dy}{dx}(0) = 15 \Rightarrow$ At $x=0$, $\frac{dy}{dx} = 15$.

$\frac{dy}{dx} = -2c_1 e^{-2x} - 3c_2 e^{-3x}.$

$\Rightarrow 15 = -2c_1 - 3c_2$

$15 = -2(-c_2) - 3c_2$

$15 = -c_2$

$c_2 = -15$

$c_1 = 15$

$\therefore y = 15e^{-2x} - 15e^{-3x}.$

⑥ Solve $(D^3 + D^2 + 4D + 4)y = 0$

AE is $m^3 + m^2 + 4m + 4 = 0$

$m = -1, +2i, -2i.$

$y = CF = c_1 e^{-x} + e^{0x}(c_2 \cos 2x + c_3 \sin 2x)$

$y = c_1 e^{-x} + c_2 \cos 2x + c_3 \sin 2x.$

## Model 2: PI when $Q(x) = e^{ax}$

Let $f(D)y = Q(x) = e^{ax}$

Then $PI = \frac{1}{f(D)} e^{ax}$

Put $D = a$.

$PI = \frac{1}{f(a)} e^{ax}$ when $f(a) \neq 0$.

If $f(a) = 0$, then,

$PI = \frac{x}{f'(D)} e^{ax}$.

Put $D = a$

$PI = \frac{x}{f'(a)} e^{ax}$ when $f'(a) \neq 0$.

If $f'(a) = 0$, then

$PI = \frac{x^2}{f''(D)} e^{ax}$

Put $D = a$

$PI = \frac{x^2}{f''(a)} e^{ax}$ when $f''(a) \neq 0$.

General soln is $y = CF + PI$ [CF from Mode 1].

Problems:

① Find the PI of $(D^2+7D+8)y = e^{4x}$.

Sol: Given eqn is

$(D^2+7D+8)y = e^{4x}$.

$f(D)\, y = Q(x)$

$f(D) = D^2+7D+8$.

$Q(x) = e^{4x}$.

$PI = \frac{1}{f(D)} e^{ax} = \frac{1}{D^2+7D+8} e^{4x}$.

Put $D = a = 4$.

$PI = \frac{1}{4^2+7(4)+8} e^{4x} = \boxed{\frac{1}{52} e^{4x}}$.

② Solve $(D^2-3D+2)y = e^{x}$.

Sol: Given eqn is $(D^2-3D+2)y = e^{x}$.

$f(D)\, y = Q(x)$

$f(D) = D^2-3D+2$

$Q(x) = e^{x}$.

To find CF:

AE is $f(m)=0$

$m^2-3m+2=0$

$m=1,2.$ (Distinct)

$CF = c_1e^{m_1x}+c_2e^{m_2x}$

$CF = c_1e^{x}+c_2e^{2x}$ —— ①

To find PI:

$PI = \frac{1}{f(D)}Q(x) = \frac{1}{D^2-3D+2}e^x$

Put $D=a=1$.

$PI = \frac{1}{1^2-3(1)+2}e^x = \frac{1}{0}\cdot e^x$.

Denominator is zero,

$PI = \frac{x}{f'(D)}e^x$.

$= \frac{x}{2D-3}e^x$

Put $D=a=1$.

$PI = \frac{x}{2(1)-3}e^x = \frac{x}{2-3}e^x$.

$PI = \frac{x}{-1}e^x = -xe^x$. —— ②

General soln is

$y = CF + PI$

$\therefore \boxed{y = c_1e^x+c_2e^{2x}-xe^x}$ [From ① & ②]

③ Solve $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y$

$= e^{-2x}+e^{-3x}$

Sol: Given, $\frac{d^3y}{dx^3}-6\frac{d^2y}{dx^2}+11\frac{dy}{dx}-6y$

$=e^{-2x}+e^{-3x}$

$\Rightarrow D^3y-6D^2y+11Dy-6y=e^{-2x}+e^{-3x}$

$\Rightarrow (D^3-6D^2+11D-6)y = e^{-2x}+e^{-3x}$

$f(D)\,y = Q(x)$

$f(D) = D^3-6D^2+11D-6$

$Q(x) = e^{-2x}+e^{-3x}$

To find CF:

AE is $m^3-6m^2+11m-6=0$

$m=1,2,3$ (Distinct)

$CF = c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}$

$CF = c_1e^{x}+c_2e^{2x}+c_3e^{3x}$ —— ①

To find PI:

$PI = \frac{1}{f(D)}Q(x)$

$= \frac{1}{D^3-6D^2+11D-6}(e^{-2x}+e^{-3x})$

$PI = \frac{e^{-2x}}{D^3-6D^2+11D-6} + \frac{e^{-3x}}{D^3-6D^2+11D-6}$

Let $PI = PI_1 + PI_2$ —— ②

$PI_1 = \frac{1}{D^3-6D^2+11D-6}\cdot e^{-2x}$.

Put $D=a=-2$.

$= \frac{1}{(-2)^3-6(-2)^2+11(-2)-6}e^{-2x}$

$PI_1 = \frac{1}{-60}e^{-2x}$.

$PI_2 = \frac{1}{(D^3-6D^2+11D-6)}\cdot e^{-3x}$.

Put $D=a=-3$.

$= \frac{1}{[(-3)^3-6(-3)^2+11(-3)-6]}e^{-3x}$

$PI_2 = -\frac{1}{120}e^{-3x}$.

② $\Rightarrow PI = -\frac{1}{60}e^{-2x}-\frac{1}{120}e^{-3x}$ —— ③

General soln is $y = CF + PI$.

From ① & ③,

$$\boxed{y = c_1e^x+c_2e^{2x}+c_3e^{3x}-\frac{1}{60}e^{-2x}-\frac{1}{120}e^{-3x}}$$

④ Solve $(D^2+4D+4)y = 18\cosh x$.

Sol: Given $(D^2+4D+4)y = 18\cosh x$

$f(D)y = Q(x)$

$f(D) = D^2+4D+4$

$Q(x) = 18\cosh x$.

$= 18\left(\frac{e^x+e^{-x}}{2}\right)$

$= 9(e^x+e^{-x})$

$Q(x) = 9e^x + 9e^{-x}$

To find CF:

AE is $f(m) = 0$

$m^2+4m+4 = 0$

$\Rightarrow m = -2, -2$. (Equal).

$CF = (c_1+c_2x)e^{m_1x}$

$\therefore CF = (c_1+c_2x)e^{-2x}$ —— ①

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2+4D+4} \cdot (9e^x + 9e^{-x})$

$PI = 9 \cdot \frac{e^x}{D^2+4D+4} + 9 \frac{e^{-x}}{D^2+4D+4}$

Let $PI = PI_1 + PI_2$ —— ②

$PI_1 = 9 \cdot \frac{e^x}{D^2+4D+4}$

Put $D = a = 1$

$= 9 \cdot \frac{e^x}{1^2+4(1)+4} = \frac{9}{9} e^x$.

$PI_1 = e^x$.

$PI_2 = 9 \cdot \frac{e^{-x}}{D^2+4D+4}$

Put $D = a = -1$.

$= 9 \cdot \frac{e^{-x}}{(-1)^2+4(-1)+4} = \frac{9 \cdot e^{-x}}{1-4+4}$

$PI_2 = 9e^{-x}$.

② ⇒ $PI = e^x + 9e^{-x}$ —— ③

General soln is

$y = CF + PI$.

From ① & ③

$\boxed{y = (c_1+c_2x)e^{-2x} + e^x + 9e^{-x}}$

⑤ Solve $(4D^2+4D+1)y = 100$

Sol: Given, $(4D^2+4D+1)y = 100$

$f(D)y = Q(x)$

$f(D) = 4D^2+4D+1$

$Q(x) = 100 \cdot e^{0x}$

To find CF:

AE is $f(m) = 0$

$4m^2+4m+1 = 0$

$\Rightarrow m = -1/2, -1/2$ (Equal)

$\therefore CF = (c_1+c_2x)e^{m_1x}$

$CF = (c_1+c_2x)e^{-1/2 x}$ —— ①

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{4D^2+4D+1} \cdot 100 \cdot e^{0x}$

Put $D = a = 0$

$PI = \frac{1}{4(0)+4(0)+1} \cdot 100 e^{0x}$

$PI = 100$ —— ②

General soln is

$y = CF + PI$

From ① & ②

$y = (c_1+c_2x)e^{-1/2 x} + 100$.

⑥ Solve $y'' - 4y' + 3y = 4e^{3x}$, Given that $y(0) = -1$, $y'(0) = 3$

Sol: Given $y'' - 4y' + 3y = 4e^{3x}$.

$(D^2 - 4D + 3)y = 4e^{3x}$

$f(D)y = Q(x)$

$f(D) = D^2 - 4D + 3$

$Q(x) = 4e^{3x}$.

To find CF:

AE is $f(m) = 0$

$\Rightarrow m^2 - 4m + 3 = 0$

$m = 1, 3$ (Distinct).

$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

$CF = c_1 e^{x} + c_2 e^{3x}$ — ①

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2 - 4D + 3} 4e^{3x}$.

Put $D = a = 3$.

$= \frac{1}{3^2 - 4(3) + 3} 4 \cdot e^{3x}$

$= \frac{1}{0} \cdot 4e^{3x}$.

Denominator is zero, then

$PI = \frac{x}{f'(D)} Q(x)$

$= \frac{x}{2D - 4} 4e^{3x}$.

Put $D = a = 3$.

$= \frac{x}{2(3) - 4} 4e^{3x}$.

$= \frac{x}{6 - 4} 4e^{3x} = \frac{x}{2} 4e^{3x}$

$PI = 2xe^{3x}$ — ②

General solⁿ is $y = CF + PI$

From ① & ②

$\boxed{y = c_1 e^{x} + c_2 e^{3x} + 2xe^{3x}}$ — ③

Given, $y(0) = -1$.

③ $\Rightarrow -1 = c_1 e^{0} + c_2 e^{3(0)} + 2(0)e^{3(0)}$

$\Rightarrow c_1 + c_2 = -1$ — ④

Also given $y'(0) = 3$

$y' = c_1 e^{x} + 3c_2 e^{3x} + 6xe^{3x} + 2e^{3x}$.

$\Rightarrow 3 = c_1 e^{0} + 3c_2 e^{0} + 6(0)e^{0} + 2e^{0}$

$\Rightarrow 3 = c_1 + 3c_2 + 2$

$\Rightarrow c_1 + 3c_2 = 1$ — ⑤

Solving ④ & ⑤ we get.

$c_1 = -2$, $c_2 = 1$

$\therefore$ From ③,

$y = -2e^{x} + e^{3x} + 2xe^{3x}$.

$\boxed{y = -2e^{x} + (1 + 2x)e^{3x}}$

⑦ Find the PI of $(D^2 - 1)y = e^{x}$

Sol: Given, $(D^2 - 1)y = e^{x}$

$f(D)y = Q(x)$

$Q(x) = e^{x}$

$f(D) = D^2 - 1$

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2 - 1} e^{x}$.

Put $D = a = 1$

$= \frac{1}{1 - 1} e^{x} = \frac{1}{0} e^{x}$

The denominator is zero, then

$PI = \frac{x}{f'(D)} Q(x) = \frac{x}{2D} e^{x}$

Put $D = a = 1$

$\boxed{PI = \frac{x}{2} e^{x}.}$

Model 3: PI when $Q(x) = \sin bx$ (or) $\cos bx$

Let $f(D)y = \sin bx$ (or) $\cos bx$.

$$PI = \frac{1}{f(D)} Q(x) = \frac{1}{f(D)} [\sin bx \text{ (or) } \cos bx]$$

Put $D^2 = -b^2$.

If the denominator is zero, then

$$PI = \frac{x}{f'(D)} [\sin bx \text{ (or) } \cos bx]$$

Put $D^2 = -b^2$

If the denominator is again zero then

$$PI = \frac{x^2}{f''(D)} [\sin bx \text{ (or) } \cos bx]$$

Put $D^2 = -b^2$.

General soln is $y = CF + PI$ [CF from Model 1].

① Find the PI of $(D^2+4)y = \sin 5x$.

Sol: Given $(D^2+4)y = \sin 5x$

$f(D)y = Q(x)$

$f(D) = D^2+4$

$Q(x) = \sin 5x$.

$$PI = \frac{1}{f(D)} Q(x) = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2+4} \sin 5x.$$

Put $D^2 = -b^2 = -5^2 = -25$.

$$PI = \frac{1}{-25+4} \sin 5x = \boxed{-\frac{1}{21}\sin 5x}$$

② Find the PI of $(D^2+9)y = \cos 3x$.

Sol: Given $f(D) = D^2+9$,

$Q(x) = \cos 3x$.

$$PI = \frac{1}{f(D)} \cos 3x.$$

$$= \frac{1}{D^2+9} \cos 3x$$

Put $D^2 = -b^2 = -9$.

$$PI = \frac{1}{-9+9} \cos 3x = \frac{1}{0} \cos 3x.$$

Denominator is zero, then

$$PI = \frac{x}{f'(D)} Q(x) = \frac{x}{2D} \cos 3x.$$

$$= \frac{x}{2D} \times \frac{D}{D} \cos 3x.$$

$$= \frac{xD}{2D^2} \cos 3x$$

Put $D^2 = -b^2 = -9$

$$= \frac{xD}{2(-9)} \cos 3x$$

$$= \frac{xD}{-18} \cos 3x$$

$$= \frac{x}{\cancel{-18}_{6}} (\cancel{-}\sin 3x \cdot (3))$$

$$\boxed{PI = \frac{x}{6}\sin 3x}$$

③ Solve $(D^2+3D+2)y = \sin 3x$.

Sol: Given, $(D^2+3D+2)y = \sin 3x$.

$f(D)y = Q(x)$

$f(D) = D^2+3D+2$,

$Q(x) = \sin 3x$.

To find CF:

AE is $f(m) = 0$

$m^2+3m+2 = 0$

$m = -1, -2$ (Distinct)

$CF = c_1e^{m_1x} + c_2e^{m_2x}$

$CF = c_1e^{-x} + c_2e^{-2x}$.

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2+3D+2} \sin 3x$.

Put $D^2 = -b^2 = -9$

$= \frac{1}{-9+3D+2} \sin 3x$

$= \frac{1}{3D-7} \sin 3x$

$= \frac{1}{3D-7} \times \frac{(3D+7)}{(3D+7)} \sin 3x$

$= \frac{3D+7}{(3D)^2-7^2} (\sin 3x)$

$= \frac{3D+7}{9D^2-49} (\sin 3x)$

$= \frac{3D+7}{9(-9)-49} (\sin 3x)$

$= \frac{3D+7}{-81-49} (\sin 3x)$

$= \frac{3D+7}{-130} (\sin 3x)$

$= \frac{3D \sin 3x + 7 \sin 3x}{-130}$

$= \frac{3\cos 3x (3) + 7 \sin 3x}{-130}$

$PI = \frac{9\cos 3x + 7\sin 3x}{-130}$

General soln is

$y = CF + PI$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-2x} + \frac{9\cos 3x + 7\sin 3x}{-130}}$$

4) Solve $(D^2+D+1)y = \sin 2x$.

Sol: Given, $(D^2+D+1)y = \sin 2x$.

$f(D)\, y = Q(x)$

$f(D) = D^2+D+1$

$Q(x) = \sin 2x$.

To find CF:

AE is $f(m) = 0$

$m^2+m+1 = 0$

$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)}$

$m = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3} i}{2}$

$m = -\frac{1}{2} \pm \frac{\sqrt{3} i}{2}$ (complex roots)

$CF = e^{\alpha x}[c_1 \cos \beta x + c_2 \sin \beta x]$

$CF = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right]$

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2+D+1} \sin 2x$.

Put $D^2 = -b^2 = -4$

$= \frac{1}{-4+D+1} \sin 2x$

$= \frac{1}{D-3} \sin 2x$

$= \frac{1}{D-3} \times \frac{D+3}{D+3} \cdot \sin 2x$.

$= \frac{(D+3)}{D^2-9} \sin 2x$

$= \frac{D+3}{-4-9} \sin 2x$.

$= \frac{D+3}{-13} \sin 2x$.

$= \frac{D \sin 2x + 3 \sin 2x}{-13}$

$PI = \frac{2\cos 2x + 3\sin 2x}{-13}$

General soln is

$y = CF + PI$

$$\boxed{y = e^{-x/2}\left[c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x\right] - \frac{1}{13}[2\cos 2x + 3\sin 2x]}$$

5) Solve $(D^2-1)y = e^x + \sin 3x + 2$.

Sol: Given, $(D^2-1)y = e^x + \sin 3x + 2$

$f(D)\, y = Q(x)$

$f(D) = D^2-1$

$Q(x) = e^x + \sin 3x + 2$.

To find CF:

AE is $f(m)=0$

$m^2-1=0$

$m=-1,1$ (Distinct)

$CF = c_1e^{m_1x}+c_2e^{m_2x}$

$$\boxed{CF = c_1e^{x}+c_2e^{-x}}$$

To find PI:

$PI = \frac{1}{f(D)}Q(x)$

$= \frac{1}{D^2-1}(e^x+\sin 3x+2)$

$PI = \frac{1}{D^2-1}e^x + \frac{1}{D^2-1}\sin 3x + \frac{2e^{0x}}{D^2-1}$

let $PI = PI_1 + PI_2 + PI_3$ — (1)

$PI_1 = \frac{1}{D^2-1}e^x$ Put $D=a=1$

$= \frac{1}{1-1}e^x = \frac{1}{0}e^x$

Denominator is zero, then

$PI_1 = \frac{x}{f'(D)}e^x = \frac{x}{2D}e^x$ Put $D=a=1$.

$PI_1 = \frac{x}{2(1)}e^x = \frac{x}{2}e^x$

$PI_2 = \frac{1}{D^2-1}\sin 3x$ Put $D^2=-b^2=-9$

$= \frac{1}{-9-1}\sin 3x$

$PI_2 = -\frac{1}{10}\sin 3x$.

$PI_3 = \frac{1}{D^2-1}\cdot 2e^{0x}$ Put $D=a=0$

$= \frac{1}{0-1}2e^{0x}$

$PI_3 = -2$.

From (1),

$PI = \frac{x}{2}e^x - \frac{1}{10}\sin 3x - 2$.

General soln is

$y = CF + PI$

$$\boxed{y = c_1e^{x}+c_2e^{-x}+\frac{x}{2}e^x-\frac{1}{10}\sin 3x-2.}$$

(6) Solve $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y = 4\cos x+3\sin x$

Sol: Given, $\frac{d^2y}{dx^2}+4\frac{dy}{dx}+3y = 4\cos x+3\sin x$

$(D^2+4D+3)y = 4\cos x+3\sin x$

$f(D)y = Q(x)$

$f(D) = D^2+4D+3$

$Q(x) = 4\cos x+3\sin x$.

To find CF:

AE is $f(m)=0$

$m^2+4m+3=0$

$m=-1,-3$ (Distinct)

$CF = c_1e^{m_1x}+c_2e^{m_2x}$

$CF = c_1e^{-x}+c_2e^{-3x}$

To find PI:

$PI = \frac{1}{f(D)}Q(x) = \frac{1}{D^2+4D+3}(4\cos x+3\sin x)$

$= \frac{4\cos x}{D^2+4D+3} + \frac{3\sin x}{D^2+4D+3}$.

$PI = PI_1 + PI_2$ — (1)

$PI_1 = \frac{4\cos x}{D^2+4D+3}$ Put $D^2=-b^2=-1$

$= \frac{1}{-1+4D+3}4\cos x$

$= \frac{1}{4D+2}4\cos x \times \frac{(4D-2)}{(4D-2)}$

$= \frac{(4D-2)}{16D^2-4}4\cos x$.

Put $D^2=-b^2=-1$

$= \frac{4(4D-2)\cos x}{16(-1)-4}$

$= \frac{4(4D-2)\cos x}{-16-4} = \frac{4(4D-2)\cos x}{-20}$

$= -\frac{4}{20}[4(-\sin x)-2\cos x]$

$PI_1 = \frac{(4\sin x+2\cos x)4}{20}$

$PI_2 = \frac{3\sin x}{D^2+4D+3}$ Put $D^2=-b^2=-1$

$= \frac{3\sin x}{-1+4D+3}$

$= \frac{1}{4D+2}\times\frac{4D-2}{4D-2}\cdot 3\sin x$.

$$= \frac{4D-2}{16D^2-4} \cdot 3\sin x$$

$$= \frac{3(4D-2)\sin x}{16(-1)-4}$$

$$PI_2 = \frac{3[4\cos x - 2\sin x]}{-20}$$

(1) $\Rightarrow$ $PI = PI_1 + PI_2$

$$= \frac{16\sin x + 8\cos x - 12\cos x + 6\sin x}{20}$$

$$= \frac{22\sin x - 4\cos x}{20}$$

$$= \frac{\cancel{2}(11\sin x - 2\cos x)}{\cancel{20}_{10}}$$

$$PI = \frac{11\sin x - 2\cos x}{10}.$$

General soln is

$y = CF + PI$

$$\boxed{y = c_1 e^{-x} + c_2 e^{-3x} + \frac{11\sin x - 2\cos x}{10}}$$

## Model - 4 : PI when $Q(x) = x^m$.

Let $f(D)y = x^m$, $m = 1, 2, 3, \ldots$

$$PI = \frac{1}{[1 \pm \phi(D)]} x^m$$

$$PI = [1 \pm \phi(D)]^{-1} x^m$$

General soln is $y = CF + PI$ [CF - from Model 1]

Note: $(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 - D^5 + \ldots$

$(1-D)^{-1} = 1 + D + D^2 + D^3 + D^4 + D^5 + \ldots$

① Find the PI of $(D^2 + 2D + 1)y = x$.

Sol: Given, $(D^2 + 2D + 1)y = x$

$f(D)\,y = Q(x)$

$f(D) = D^2 + 2D + 1$

$Q(x) = x$.

$$PI = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 + 2D + 1} x.$$

$$= \frac{1}{1 + (2D + D^2)} \cdot x$$

$$= [1 + (2D + D^2)]^{-1} x.$$

$$= [1 - (2D + D^2) + (2D + D^2)^2 - (2D + D^2)^3 + \cdots]x$$

$$= [1 - 2D - D^2 + (2D)^2 + 2 \cdot 2 \cdot D \cdot D^2 + (D^2)^2 + \cdots]x$$

$$= (1 - 2D)x \qquad \left[\because Dx = 1 = \frac{dx}{dx},\ D^2x = 0,\ D^3x = 0\right]$$

$$= x - 2Dx$$

$$= x - 2(1)$$

$\therefore \boxed{PI = x - 2}$

② Solve $(D^2 - 2D - 3)y = x^2$.

Sol: Given, $(D^2 - 2D - 3)y = x^2$.

$f(D)\,y = Q(x)$

$f(D) = D^2 - 2D - 3$

$Q(x) = x^2$.

To find CF:

AE is $f(m) = 0$

$m^2 - 2m - 3 = 0$

$m = -1, 3$ (Distinct)

$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$.

$CF = c_1 e^{3x} + c_2 e^{-x}$

To find PI:

$$PI = \frac{1}{f(D)} Q(x) = \frac{1}{D^2 - 2D - 3} x^2.$$

$$= \frac{1}{-3 - 2D + D^2} x^2$$

$$= \frac{1}{-3\left[1 + \left(\frac{2D - D^2}{3}\right)\right]} x^2$$

$$= \frac{1}{-3}\left[1 + \left(\frac{2D - D^2}{3}\right)\right]^{-1} x^2$$

$$= \frac{1}{-3}\left[1 - \left(\frac{2D - D^2}{3}\right) + \left(\frac{2D - D^2}{3}\right)^2 + \cdots\right] x^2$$

$$= \frac{1}{-3}\left[1 - \frac{2D}{3} + \frac{D^2}{3} + \frac{4D^2}{9}\right] x^2$$

$$= \frac{1}{-3}\left[x^2 - \frac{2}{3}(2x) + \frac{2}{3} + \frac{4}{9}(2)\right]$$

$$= \frac{-1}{3}\left[x^2 - \frac{4x}{3} + \frac{2}{3} + \frac{8}{9}\right] \qquad \left[\because D = \frac{d}{dx},\ Dx^2 = 2x,\ D^2x^2 = 2,\ D^3(x^2) = 0\right]$$

$$= -\frac{1}{27}[9x^2 - 12x + 6 + 8]$$

$$PI = -\frac{1}{27}[9x^2 - 12x + 14]$$

General soln is

$y = CF + PI$.

$$\boxed{y = c_1 e^{3x} + c_2 e^{-x} - \frac{1}{27}[9x^2 - 12x + 14]}$$

③ Solve $(D^2+3D+2)y = e^{-x}+x^2+\cos x$

Sol: Given, $(D^2+3D+2)y = e^{-x}+x^2+\cos x$

$f(D)y = Q(x)$

$f(D) = D^2+3D+2$

$Q(x) = e^{-x}+x^2+\cos x$

To find CF:

AE is $f(m)=0$

$m^2+3m+2=0$

$m=-1,-2$ (Distinct)

$CF = c_1e^{m_1x}+c_2e^{m_2x}$

$CF = c_1e^{-x}+c_2e^{-2x}$

To find PI:

$PI = \frac{1}{f(D)}Q(x)$

$= \frac{1}{D^2+3D+2}(e^{-x}+x^2+\cos x)$

$PI = \frac{e^{-x}}{D^2+3D+2} + \frac{x^2}{D^2+3D+2} + \frac{\cos x}{D^2+3D+2}$

let $PI = PI_1 + PI_2 + PI_3$ — ①

$PI_1 = \frac{1}{D^2+3D+2}e^{-x}$ Put $D=a=-1$

$= \frac{1}{(-1)^2+3(-1)+2}e^{-x} = \frac{1}{1-3+2}e^{-x}$

$= \frac{1}{0}e^{-x}$. Denominator is zero, then

$PI_1 = \frac{x}{f'(D)}e^{-x} = \frac{x}{2D+3}e^{-x}$ Put $D=a=-1$

$PI_1 = \frac{x}{2(-1)+3}e^{-x} = \frac{x}{-2+3}e^{-x} = xe^{-x}$

$PI_2 = \frac{1}{(D^2+3D+2)}x^2 = \frac{1}{1\pm\phi(D)}x^2$

$= \frac{1}{2+3D+D^2}x^2 = \frac{1}{2\left[1+\left(\frac{3D+D^2}{2}\right)\right]}x^2$

$= \frac{1}{2}\left[1+\left(\frac{3D+D^2}{2}\right)\right]^{-1}x^2$

$= \frac{1}{2}\left[1-\left(\frac{3D+D^2}{2}\right)+\left(\frac{3D+D^2}{2}\right)^2+\cdots\right]x^2$

$= \frac{1}{2}\left[1-\frac{3D}{2}-\frac{D^2}{2}+\frac{9D^2}{4}+\cdots\right]x^2$

$\left[\because D=\frac{d}{dx}\right.$

$Dx^2 = 2x$

$D^2(x^2) = 2$

$\left.D^3(x^2)=0\right]$

$= \frac{1}{2}\left[x^2-\frac{3}{2}(2x)-\frac{2}{2}+\frac{9}{4}(2)+0\right]$

$= \frac{1}{4}\left[2x^2-6x-2+9\right]$

$PI_2 = \frac{1}{4}\left[2x^2-6x+7\right]$

$PI_3 = \frac{1}{D^2+3D+2}\cos x$

Put $D^2=-b^2=-1$

$= \frac{1}{-1+3D+2}\cos x$

$= \frac{1}{3D+1}\cos x$

$= \frac{1}{3D+1}\times\frac{3D-1}{3D-1}\cos x$

$= \frac{(3D-1)}{9D^2-1}\cos x$ Put $D^2=-b^2=-1$

$= \frac{3D\cos x-\cos x}{9(-1)-1}$

$= \frac{-3\sin x-\cos x}{-10}$

$PI_3 = \frac{3\sin x+\cos x}{10}$

① ⇒ $PI = xe^{-x}+\frac{1}{4}\left[2x^2-6x+7\right] + \frac{3\sin x+\cos x}{10}$

General soln is $y = CF+PI$

$y = c_1e^{-x}+c_2e^{-2x}+xe^{-x} + \frac{1}{4}\left[2x^2-6x+7\right]+\frac{3\sin x+\cos x}{10}$

④ $(D^2-1)y = e^x+3x$

Sol: Given, $(D^2-1)y = e^x+3x$

$f(D)y = Q(x)$

$f(D) = D^2-1$

$Q(x) = e^x+3x$

To find CF:

AE is $f(m)=0$

$m^2-1=0$

$m=-1,+1$ (Distinct)

$CF = c_1e^{m_1x}+c_2e^{m_2x}$

$CF = c_1e^{-x}+c_2e^{x}$

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2-1}(e^x + 3x)$

$PI = \frac{e^x}{D^2-1} + \frac{3x}{D^2-1}$

let $PI = PI_1 + PI_2$ — (1)

$PI_1 = \frac{1}{D^2-1} e^x$.

Put $D = a = 1$

$= \frac{1}{1-1} e^x = \frac{1}{0} e^x$.

Denominator is zero, then

$PI_1 = \frac{x}{f'(D)} e^x = \frac{x}{2D} e^x$

Put $D = a = 1$

$PI_1 = \frac{x}{2} e^x$

$PI_2 = \frac{1}{D^2-1} 3x = \frac{1}{-1+D^2} \cdot 3x$

$= \frac{1}{-1(1-D^2)} 3x$

$= -(1-D^2)^{-1} 3x$

$= -[1+D^2+D^4+D^6+\dots]3x$

$= -3[1+D^2+\dots]x$  $\left[D = \frac{d}{dx},\ Dx = 1,\ D^2x = 0\right]$

$= -3[x+D^2x+\dots]$

$= -3[x+0+0]$

$PI_2 = -3x$

From (1),

$PI = \frac{x}{2}e^x - 3x$.

General solution is,

$y = CF + PI$

$\boxed{y = c_1e^{-x} + c_2e^x + \frac{x}{2}e^x - 3x}$

5) Solve $(D^2+2D+1)y = 2x + x^2$.

Sol: Given $(D^2+2D+1)y = 2x+x^2$

$f(D)\, y = Q(x)$

$f(D) = D^2+2D+1$

$Q(x) = 2x + x^2$

To find CF:

AE is $f(m) = 0$

$m^2+2m+1 = 0$

$m = -1, -1$. (Equal)

$CF = (c_1 + c_2x)e^{-x}$.

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2+2D+1}(2x+x^2)$

$PI = \frac{1}{D^2+2D+1} 2x + \frac{1}{D^2+2D+1} x^2$

let $PI = PI_1 + PI_2$ — (1)

$PI_1 = \frac{1}{D^2+2D+1} 2x$

$= \frac{1}{1+2D+D^2} 2x$

$= 2 \cdot \frac{1}{1+(2D+D^2)} x$

$= 2[1+(2D+D^2)]^{-1} x$

$= 2[1-(2D+D^2)+(2D+D^2)^2 - \dots]x$

$= 2[1-2D+D^2+\dots]x$  $\left[\because Dx = 1,\ D^2x = 0\right]$

$= 2[x-2Dx+D^2x+\dots]$

$= 2[x-2(1)+0 \dots]$

$PI_1 = 2x - 4$.

$PI_2 = \frac{1}{(D^2+2D+1)} x^2$

$= \frac{1}{1+(2D+D^2)} x^2$

$= [1+(2D+D^2)]^{-1} x^2$

$= [1-(2D+D^2)+(2D+D^2)^2 - \dots]x^2$

$= [1-2D-D^2+4D^2+D^4+4D^3+\dots]x^2$

$= [x^2-2Dx^2-D^2x^2+4D^2x^2+\dots]$  $\left[\because Dx^2 = 2x,\ D^2x^2 = 2,\ D^3x^2 = 0\right]$

$= [x^2-2(2x)-2+4(2)+0]$

$PI_2 = x^2-4x-2+8 = x^2-4x+6$

From (1) $PI = 2x-4+x^2-4x+6$

$PI = x^2-2x+2$.

General soln is $y = CF + PI$

$\boxed{y = (c_1+c_2x)e^{-x} + x^2 - 2x + 2}$

Model - 5 : PI when $Q(x) = e^{ax}[\sin bx \text{ (or) } \cos bx \text{ (or) } x^m]$

let $f(D)y = e^{ax}[\sin bx \text{ (or) } \cos bx \text{ (or) } x^m]$

$$PI = \frac{1}{f(D)} e^{ax}[\sin bx \text{ (or) } \cos bx \text{ (or) } x^m]$$

Put $D = D + a$.

$$PI = e^{ax}\left[\frac{1}{f(D+a)}[\sin bx \text{ (or) } \cos bx \text{ (or) } x^m]\right]$$

General soln is $y = CF + PI$

① Solve $(D^2-1)y = e^x \sin x$

Sol: Given $(D^2-1)y = e^x \sin x$

$f(D)y = Q(x)$

$f(D) = D^2 - 1$

$Q(x) = e^x \sin x$.

To find CF:

AE is $f(m) = 0$

$m^2 - 1 = 0$

$m = -1, 1$ (Distinct)

$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

$CF = c_1 e^{-x} + c_2 e^{x}$

To find PI:

$$PI = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2-1} e^x \sin x.$$

[Put $D = D + a = D + 1$]

$$= e^x \left[\frac{1}{(D+1)^2-1}\right] \sin x$$

$$= e^x \frac{1}{D^2+2D+1-1} \sin x.$$

$$= e^x \frac{1}{D^2+2D} \sin x$$

Put $D^2 = -b^2 = -1$

$$= e^x \frac{1}{-1+2D} \sin x$$

$$= e^x . \frac{1}{2D-1} \times \frac{2D+1}{2D+1} (\sin x)$$

$$= e^x . \frac{2D+1}{4D^2-1} \sin x \quad \text{Put } D^2 = -b^2 = -1$$

$$= e^x . \frac{(2D+1)\sin x}{4(-1)-1}$$

$$= e^x . \frac{(2D\sin x + \sin x)}{-4-1}$$

$$PI = e^x \frac{(2\cos x + \sin x)}{-5}.$$

General soln is $y = CF + PI$.

$$\boxed{y = c_1 e^{-x} + c_2 e^{x} - \frac{e^x}{5}(2\cos x + \sin x)}$$

② Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$

Sol: Given, $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x$

$\Rightarrow (D^2 - 6D + 13)y = 8e^{3x}\sin 2x$

$f(D)y = Q(x)$

$f(D) = (D^2 - 6D + 13)$

$Q(x) = 8e^{3x}\sin 2x$.

To find CF:

AE is $f(m) = 0$

$m^2 - 6m + 13 = 0$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{6 \pm \sqrt{36 - 4(1)(13)}}{2(1)}$$

$$= \frac{6 \pm \sqrt{36-52}}{2} = \frac{6 \pm \sqrt{-16}}{2}$$

$$m = \frac{6 \pm i4}{2} = 3 \pm 2i \text{ (complex)}$$

$CF = e^{\alpha x}(c_1\cos\beta x + c_2 \sin\beta x)$

$CF = e^{3x}(c_1\cos 2x + c_2 \sin 2x)$

To find PI:

$$PI = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 - 6D + 13} 8e^{3x} \sin 2x.$$

Put $D = D+a = D+3$

$= 8e^{3x}\left[\frac{1}{(D+3)^2-6(D+3)+13}\right]\sin 2x$

$= 8e^{3x}\frac{1}{D^2+\cancel{9}+6D-6D-18+13}\cdot\sin 2x$

$= 8e^{3x}\frac{1}{D^2+4}\cdot\sin 2x$

Put $D^2 = -b^2 = -4$

$= 8e^{3x}\frac{1}{-4+4}\sin 2x = 8.e^{3x}.\frac{1}{0}\sin 2x$

Denominator is zero, then

$PI = 8e^{3x}.\frac{x}{f'(D)}\sin 2x$

$= 8xe^{3x}.\frac{1}{2D}\sin 2x$

$= 8xe^{3x}.\frac{1}{2D}\times\frac{D}{D}\sin 2x.$

$= \frac{8xe^{3x}}{2}.\frac{D\sin 2x}{D^2}$ Put $D^2 = -b^2 = -4$

$= \frac{8xe^{3x}}{\cancel{2}}.\frac{2\cos 2x}{-\cancel{4}}$

$PI = -2xe^{3x}\cos 2x.$

General soln is $y = CF + PI$

$\boxed{y = e^{3x}(c_1\cos 2x + c_2\sin 2x) - 2xe^{3x}\cos 2x}$

3 Solve $(D^2-4D+3)y = 2xe^x + 3e^x\cos 2x.$

Sol: Given, $(D^2-4D+3)y = 2xe^x + 3e^x\cos 2x$

$f(D)y = Q(x)$

$f(D) = D^2-4D+3$

$Q(x) = 2xe^x + 3e^x\cos 2x.$

To find CF:

AE is $f(m) = 0$

$m^2-4m+3 = 0$

$m = 1, 3$ (Distinct)

$CF = c_1e^{m_1x} + c_2e^{m_2x}$

$CF = c_1e^x + c_2e^{3x}.$

To find PI:

$PI = \frac{1}{f(D)}Q(x)$

$= \frac{1}{D^2-4D+3}[2xe^x + 3e^x\cos 2x].$

$PI = \frac{2xe^x}{D^2-4D+3} + \frac{3e^x\cos 2x}{D^2-4D+3}$

Let $PI = PI_1 + PI_2$ — (1)

$PI_1 = \frac{1}{D^2-4D+3}\cdot 2xe^x.$

Put $D = D+a = D+1$

$= 2e^x.\frac{1}{(D+1)^2-4(D+1)+3}x.$

$= 2e^x\frac{1}{D^2+2D+1-4D-4+3}x$

$= 2e^x\frac{1}{D^2-2D}x.$

$= 2e^x\frac{1}{-2D+D^2}x$

$= 2e^x\frac{1}{-2D\left[1-\frac{D}{2}\right]}x$

$= \frac{2e^x}{-2}.\frac{1}{D}\left[1-\frac{D}{2}\right]^{-1}x$

$= \frac{2e^x}{-2}.\frac{1}{D}\left[1+\frac{D}{2}+\frac{D^2}{4}+\frac{D^3}{8}+\cdots\right]x$

$= -e^x.\frac{1}{D}\left[x+\frac{Dx}{2}+\frac{D^2x}{4}+\cdots\right]$

$= -e^x.\frac{1}{D}\left[x+\frac{1}{2}+0+\cdots\right]$

$= -e^x\frac{1}{D}\left[x+\frac{1}{2}\right]$

$= -e^x\int(x+\frac{1}{2})dx.$

$PI_1 = -e^x\left(\frac{x^2}{2}+\frac{x}{2}\right).$

$PI_2 = \frac{1}{D^2-4D+3}3e^x\cos 2x$

Put $D = D+a = D+1$

$= 3e^x\frac{1}{(D+1)^2-4(D+1)+3}.\cos 2x$

$= 3e^x.\frac{1}{D^2-2D}\cos 2x$

Put $D^2 = -b^2 = -4$

$= 3e^x.\frac{1}{-4-2D}\cos 2x$

$= 3e^x.\frac{1}{-2(2+D)}\cos 2x$

$= -\frac{3}{2}e^x.\frac{1}{2+D}\times\frac{2-D}{2-D}.\cos 2x$

$= -\frac{3}{2}e^x.\frac{(2-D)\cos 2x}{4-D^2}.$

Put $D^2 = -b^2 = -4$

$$= -\frac{3}{2}e^x \frac{(2\cos 2x - D\cos 2x)}{4-(-4)}$$

$$= -\frac{3}{2}e^x \left(\frac{2\cos 2x + 2\sin 2x}{8}\right)$$

$$= -\frac{3}{2}e^x \cdot \frac{2}{8}(\cos 2x + \sin 2x)$$

$$PI_2 = -\frac{3}{8}e^x(\cos 2x + \sin 2x)$$

From ①,

$$PI = PI_1 + PI_2$$

$$PI = -e^x\left(x^2+\frac{x}{2}\right) - \frac{3}{8}e^x(\cos 2x + \sin 2x)$$

General soln is $y = CF + PI$

$$y = c_1e^x + c_2e^{3x} - e^x\left(x^2+\frac{x}{2}\right) - \frac{3}{8}e^x(\cos 2x + \sin 2x)$$

④ Solve $(D^2+1)y = e^x \cdot x$

Sol: Given, $(D^2+1)y = xe^x$.

$$f(D)y = Q(x)$$

$$f(D) = D^2+1$$

$$Q(x) = xe^x.$$

To find CF:

AE is $f(m) = 0$

$$m^2+1 = 0$$

$$m = \pm i \quad \text{(complex)}$$

$$CF = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

$$= e^{0x}(c_1\cos x + c_2\sin x)$$

$$CF = c_1\cos x + c_2\sin x$$

To find PI:

$$PI = \frac{1}{f(D)}Q(x) = \frac{1}{D^2+1}xe^x.$$

Put $D = D+a = D+1$

$$PI = e^x \cdot \frac{1}{(D+1)^2+1}x.$$

$$= e^x \cdot \frac{1}{D^2+2D+1+1}x.$$

$$= e^x \cdot \frac{1}{1+(1+2D+D^2)}\cdot x.$$

$$= e^x \cdot \frac{1}{2+2D+D^2}\cdot x.$$

$$= e^x \cdot \frac{1}{2\left[1+\left(D+\frac{D^2}{2}\right)\right]}x.$$

$$= \frac{e^x}{2}\cdot\left[1+\left(D+\frac{D^2}{2}\right)\right]^{-1}x$$

$$= \frac{e^x}{2}\left[1-\left(D+\frac{D^2}{2}\right)+\left(D+\frac{D^2}{2}\right)^2-\cdots\right]x$$

$$= \frac{e^x}{2}\left[1-D-\frac{D^2}{2}+D^2+D^3+\frac{D^4}{4}+\cdots\right]x$$

$$= \frac{e^x}{2}\left[x-Dx-\frac{D^2}{2}x+D^2x+\cdots\right]$$

$$= \frac{e^x}{2}[x-1-0]$$

$$PI = \frac{e^x(x-1)}{2}.$$

General soln is $y = CF + PI$

$$\boxed{y = c_1\cos x + c_2\sin x + \frac{e^x(x-1)}{2}}$$

① Solve $(D^2+2D+1)y = x\cos x$

Sol: Given, $(D^2+2D+1)y = x\cos x$

$f(D)\, y = Q(x)$

$f(D) = D^2+2D+1$

$Q(x) = x\cos x$.

To find CF:

AE is $f(m)=0$

$m^2+2m+1=0$

$m=-1,-1$ (Equal)

$CF = (c_1+c_2x)e^{m_1 x}$

$CF = (c_1+c_2x)e^{-x}$.

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2+2D+1}(x\cos x)$

$= \frac{x}{f(D)}\cos x - \frac{f'(D)}{[f(D)]^2}\cos x$

$= \frac{x}{D^2+2D+1}\cos x - \frac{(2D+2)}{(D^2+2D+1)^2}\cos x$

Put $D^2 = -b^2 = -1$

$= \frac{x}{-1+2D+1}\cos x - \frac{(2D+2)}{(-1+2D+1)^2}\cos x$.

$= \frac{x}{2D}\cos x - \frac{(2D+2)}{4D^2}\cos x$.

$= \frac{x}{2}\left[\frac{1}{D}\cos x\right] - \frac{(2D+2)\cos x}{4(-1)}$

$= \frac{x}{2}\left[\int \cos x\, dx\right] - \frac{(2D\cos x+2\cos x)}{-4}$

$= \frac{x}{2}\sin x + \frac{\cancel{2}(-\sin x+\cos x)}{\cancel{4}2}$

$PI = \frac{x}{2}\sin x - \frac{\sin x}{2} + \frac{\cos x}{2}$.

General soln is $CF+PI$

$$\boxed{y = (c_1+c_2x)e^{-x} + \frac{x}{2}\sin x - \frac{\sin x}{2} + \frac{\cos x}{2}}$$

② Solve $(D^2-2D+1)y = xe^{x}\cos x$

Sol: Given, $(D^2-2D+1)y = xe^{x}\cos x$

$f(D)y = Q(x)$

$f(D) = D^2-2D+1$

$Q(x) = xe^{x}\cos x$

To find CF:

AE is $f(m)=0$

$m^2-2m+1=0$

$m=1,1$ (Equal)

$CF = (c_1+c_2x)e^{m_1 x}$

$CF = (c_1+c_2x)e^{x}$

To find PI:

$PI = \frac{1}{f(D)} Q(x)$

$= \frac{1}{D^2-2D+1} xe^{x}\cos x$

Put $D = D+a = D+1$

$= e^{x}\cdot \frac{1}{(D+1)^2-2(D+1)+1}\cdot x\cos x$.

$= e^{x}\cdot \frac{1}{D^2+2D+1-2D-2+1}\, x\cos x$.

$= e^{x}\cdot \frac{1}{D^2} x\cos x$.

$= e^{x}\left[\frac{x}{f(D)}\cos x - \frac{f'(D)}{[f(D)]^2}\cos x\right]$

$= e^{x}\left[\frac{x}{D^2}\cos x - \frac{2D}{D^4}\cos x\right]$

Put $D^2 = -b^2 = -1$

$= e^{x}\left[\frac{x}{-1}\cos x - \frac{2D}{(-1)^2}\cos x\right]$

$= e^{x}\left[-x\cos x - 2D\cos x\right]$

$PI = e^{x}\left[-x\cos x + 2\sin x\right]$

General soln is $y = CF+PI$

$$\boxed{y = (c_1+c_2x)e^{x} + e^{x}[-x\cos x + 2\sin x]}$$

③ Solve $(D^2+9)y = x\sin(2x)$.

Sol: Given, $(D^2+9)y = x\sin 2x$.

$f(D)y = Q(x)$

$f(D) = D^2+9$

$Q(x) = x\sin 2x$.

To find CF:

AE is $f(m)=0$

$m^2+9=0$

$m = \pm 3i$ (Complex)

$CF = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$

$CF = e^{0x}(c_1\cos 3x + c_2\sin 3x)$

$CF = c_1\cos 3x + c_2\sin 3x$.

To find PI:

$PI = \frac{1}{f(D)}Q(x)$

$= \frac{1}{D^2+9}x\sin 2x$.

$= \frac{x}{f(D)}\sin 2x - \frac{f'(D)}{[f(D)]^2}\sin 2x$

$= \frac{x}{D^2+9}\sin 2x - \frac{2D}{(D^2+9)^2}\sin 2x$.

Put $D^2 = -b^2 = -4$.

$= \frac{x}{-4+9}\sin 2x - \frac{2D}{(-4+9)^2}\sin 2x$

$= \frac{x}{5}\sin 2x - \frac{2D}{25}\sin 2x$

$= \frac{x}{5}\sin 2x - \frac{2}{25}[2\cos 2x]$

$PI = \frac{x}{5}\sin 2x - \frac{4}{25}\cos 2x$

General soln is $y = CF + PI$

$$\boxed{y = c_1\cos 3x + c_2\sin 3x + \frac{x}{5}\sin 2x - \frac{4}{25}\cos 2x}$$

④ Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x$

Sol: Given, $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x$.

$\Rightarrow (D^2-2D+1)y = xe^x\sin x$.

$f(D)y = Q(x)$

$f(D) = D^2-2D+1$

$Q(x) = xe^x\sin x$

To find CF:

AE is $f(m)=0$

$m^2-2m+1=0$

$m = 1, 1$ (Equal)

$CF = (c_1+c_2x)e^{m_1x}$

$CF = (c_1+c_2x)e^x$.

To find PI:

$PI = \frac{1}{f(D)}Q(x)$

$= \frac{1}{D^2-2D+1}x\cdot e^x\cdot\sin x$. Put $D = D+a = D+1$

$= e^x\cdot\frac{1}{(D+1)^2-2(D+1)+1}x\sin x$

$= e^x\cdot\frac{1}{D^2+2D+1-2D-2+1}x\sin x$

$= e^x\cdot\frac{1}{D^2}x\sin x$.

$= e^x\cdot\left[\frac{x}{f(D)}\sin x - \frac{f'(D)}{[f(D)]^2}\sin x\right]$

$= e^x\left[\frac{x}{D^2}\sin x - \frac{2D}{D^4}\sin x\right]$

Put $D^2 = -b^2 = -1$

$= e^x\left[\frac{x}{-1}\sin x - \frac{2D}{(-1)^2}\sin x\right]$

$PI = e^x[-x\sin x - 2\cos x]$

General soln is $y = CF + PI$

$$\boxed{y = (c_1+c_2x)e^x + e^x[-x\sin x - 2\cos x]}$$

## Method of variation of Parameter:

Standard form is $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$ where $P(x)$, $Q(x)$ and $R(x)$ are functions of $x$ (or) constants.

General soln is $y = CF + PI$.

We find CF using model-1.

$PI = AU + BV$

Here $A = -\int \frac{VR}{W} dx$ & $B = \int \frac{UR}{W} dx$

$W = UV' - VU' = \begin{vmatrix} U & V \\ U' & V' \end{vmatrix}$

Suppose $CF = c_1 U(x) + c_2 V(x)$

$U(x)$ = coefficient of $c_1$.

$V(x)$ = coefficient of $c_2$.

$W = UV' - VU'$ is called as Wronskean.

~~Wronskean~~.

① Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$. by method of variation of parameter

Sol: Given, $\frac{d^2y}{dx^2} + 4y = \tan 2x$

~~$(D^2+4)y = \tan 2x$~~.

Comparing with standard form

$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R(x)$

$R(x) = \tan 2x$.

$\frac{d^2y}{dx^2} + 4y = \tan 2x \Rightarrow (D^2+4)y = \tan 2x$

$f(D) = D^2 + 4$.

To find CF:

AE is $f(m) = 0$

$m^2 + 4 = 0$

$m = \pm 2i$ (complex)

$CF = e^{\alpha x}[c_1 \cos\beta x + c_2 \sin\beta x]$

$= e^{0x}[c_1 \cos 2x + c_2 \sin 2x]$

$CF = c_1 \cos 2x + c_2 \sin 2x$.

let $CF = c_1 u(x) + c_2 v(x)$

$\Rightarrow u = \cos 2x \qquad v = \sin 2x$

$u' = -2\sin 2x \qquad v' = 2\cos 2x$.

Wronskean:

$W = UV' - VU'$

$= \cos 2x(2\cos 2x) - \sin 2x(-2\sin 2x)$

$= 2\cos^2 2x + 2\sin^2 2x$

$= 2(\cos^2 2x + \sin^2 2x)$

$= 2(1)$

$W = 2$

To find PI:

$PI = AU + BV$.

where $A = -\int \frac{VR}{W} dx$

$= -\int \frac{\sin 2x \cdot \tan 2x}{2} dx$

$= -\frac{1}{2}\int \sin 2x \cdot \frac{\sin 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x} dx$

$= -\frac{1}{2}\int \left(\frac{1}{\cos 2x} - \frac{\cos^2 2x}{\cos 2x}\right) dx$

$= -\frac{1}{2}\int (\sec 2x - \cos 2x) dx.$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x+\tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$A = -\frac{1}{4}\left[\log(\sec 2x+\tan 2x) - \sin 2x\right].$$

$$B = \int \frac{UR}{W}\,dx.$$

$$= \int \frac{\cos 2x \cdot \tan 2x}{2}\,dx$$

$$= \frac{1}{2}\int \cos 2x \cdot \frac{\sin 2x}{\cos 2x}\,dx$$

$$= \frac{1}{2}\int \sin 2x\,dx = \frac{-1}{2}\left[\frac{\cos 2x}{2}\right]$$

$$B = -\frac{1}{4}\cos 2x.$$

$$PI = AU + BV$$

$$PI = -\frac{1}{4}\cos 2x\left[\log(\sec 2x+\tan 2x) - \sin 2x\right]$$
$$-\frac{1}{4}\cos 2x\ \sin 2x.$$

General soln is $y = CF + PI$

$$\boxed{\begin{aligned} y &= c_1\cos 2x + c_2\sin 2x \\ &-\frac{1}{4}\cos 2x\left[\log(\sec 2x+\tan 2x) - \sin 2x\right] \\ &-\frac{1}{4}\cos 2x\sin 2x \end{aligned}}$$

② Solve $\frac{d^2y}{dx^2} + a^2y = \tan ax$ by method of variation of parameter

Soln: Given, $\frac{d^2y}{dx^2} + a^2y = \tan ax.$

$$(D^2 + a^2)y = \tan ax$$

Here $f(D) = D^2 + a^2$

$R = \tan ax.$

To find CF:

AE is $f(m) = 0$

$m^2 + a^2 = 0$

$m = \pm ai$ (Complex)

$$CF = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x)$$

$$= e^{0x}(c_1\cos ax + c_2\sin ax)$$

$$CF = c_1\cos ax + c_2\sin ax.$$

Let $CF = c_1 U + c_2 V$

$U = \cos ax$ $\quad V = \sin ax$

$U' = -a\cos ax$ $\quad V' = a\cos ax.$

Wronskian:

$$W = UV' - VU'$$

$$= \cos ax(a\cos ax) - (\sin ax)(-a\sin ax)$$

$$= a\cos^2 ax + a\sin^2 ax$$

$$= a(\cos^2 ax + \sin^2 ax) = a(1)$$

$$W = a$$

To find PI:

$$PI = AU + BV$$

$$A = -\int \frac{VR}{W}\,dx.$$

$$= -\int \frac{\sin ax \cdot \tan ax}{a}\,dx$$

$$= -\frac{1}{a}\int \sin ax \cdot \frac{\sin ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}\,dx$$

$$= -\frac{1}{a}\int(\sec ax - \cos ax)\,dx$$

$$= -\frac{1}{a}\left[\frac{\log(\sec ax+\tan ax)}{a} - \frac{\sin ax}{a}\right]$$

$$A = -\frac{1}{a^2}\left[\log(\sec ax+\tan ax) - \sin ax\right]$$

$$B = \int \frac{UR}{W}\,dx$$

$$= \int \frac{\cos ax \cdot \tan ax}{a}\,dx$$

$$= \frac{1}{a}\int \cos ax \cdot \frac{\sin ax}{\cos ax}\,dx$$

$$= \frac{1}{a}\int \sin ax\,dx$$

$$= \frac{1}{a}\left[-\frac{\cos ax}{a}\right]$$

$$B = -\frac{1}{a^2}\cos ax.$$

$$PI = AU + BV$$

$$PI = -\frac{1}{a^2}\cos ax\left[\log(\sec ax+\tan ax) - \sin ax\right]$$
$$-\frac{1}{a^2}\cos ax\sin ax$$

General soln is $y = CF + PI$

$$\boxed{\begin{aligned} y &= c_1\cos ax + c_2\sin ax \\ &-\frac{1}{a^2}\cos ax\left[\log(\sec ax+\tan ax) - \sin ax\right] \\ &-\frac{1}{a^2}\cos ax\sin ax \end{aligned}}$$

(3) Solve $\frac{d^2y}{dx^2}+9y=\sec 3x$ by method of variation of parameter

Sol: Given, $\frac{d^2y}{dx^2}+9y=\sec 3x$

$(D^2+9)y=\sec 3x$

Here $f(D)=D^2+9$

$R=\sec 3x$

To find CF:

AE is $f(m)=0$

$m^2+9=0$

$m=\pm 3i$ (complex)

$CF=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$

$CF=e^{0x}(c_1\cos 3x+c_2\sin 3x)$

$CF=c_1\cos 3x+c_2\sin 3x$.

Let $CF=c_1U+c_2V$

$U=\cos 3x$ $\quad V=\sin 3x$

$U'=-3\sin 3x$ $\quad V'=3\cos 3x$.

Wronskian:

$W=UV'-VU'$

$=(\cos 3x)(3\cos 3x)-(\sin 3x)(-3\sin 3x)$

$=3\cos^2 3x+3\sin^2 3x$

$W=3$

To find PI:

$PI=AU+BV$

$A=-\int\frac{VR}{W}dx$

$=-\int\frac{\sin 3x.\sec 3x}{3}dx$

$=-\frac{1}{3}\int\frac{\sin 3x}{\cos 3x}dx$

$=\frac{-1}{3}\int\tan 3x\,dx$

$=-\frac{1}{3}\frac{\log|\sec 3x|}{3}$

$=-\frac{1}{9}\log|\sec 3x|$

$B=\int\frac{UR}{W}dx=\int\frac{\cos 3x.\sec 3x}{3}dx$

$=\frac{1}{3}\int\cos 3x.\frac{1}{\cos 3x}dx=\frac{1}{3}\int dx$

$B=\frac{x}{3}$

$PI=-\frac{1}{9}\cos 3x\log|\sec 3x|+\frac{x}{3}\sin 3x$

General soln is $y=CF+PI$

$$\boxed{y=c_1\cos 3x+c_2\sin 3x-\frac{1}{9}\cos 3x\log|\sec 3x|+\frac{x}{3}\sin 3x}$$

(4) Solve $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$ by the method of variation of parameter

Sol: Given, $(D^2-6D+9)y=\frac{e^{3x}}{x^2}$.

Here $f(D)=D^2-6D+9$

$R(x)=\frac{e^{3x}}{x^2}$.

To find CF: AE is $f(m)=0$

$m^2-6m+9=0$

$m=3,3$ (equal)

$CF=(c_1+c_2x)e^{m_1x}$

$CF=(c_1+c_2x)e^{3x}=c_1e^{3x}+c_2xe^{3x}$

Let $CF=c_1U+c_2V$

$U=e^{3x}$ $\quad V=xe^{3x}$

$U'=3e^{3x}$ $\quad V'=e^{3x}+3xe^{3x}$.

Wronskian

$W=UV'-VU'$

$=e^{3x}.(e^{3x}+3xe^{3x})-xe^{3x}(3e^{3x})$

$=e^{6x}+3xe^{6x}-3xe^{6x}$

$W=e^{6x}$

To find PI:

$PI=AU+BV$

$A=-\int\frac{VR}{W}dx=-\int\frac{xe^{3x}}{e^{6x}}\cdot\frac{e^{3x}}{x^2}dx$

$A=-\int\frac{1}{x}dx=-\log x$.

$B=\int\frac{UR}{W}dx=\int\frac{e^{3x}}{e^{6x}}\cdot\frac{e^{3x}}{x^2}dx$

$B=\int\frac{1}{x^2}dx=-\frac{1}{x}$

$PI=(-\log x)(e^{3x})+(-\frac{1}{x})(xe^{3x})$

$PI=-e^{3x}(\log x+1)$

General soln is $y=CF+PI$

$$\boxed{y=(c_1+c_2x)e^{3x}-e^{3x}(\log x+1)}$$

(5) Solve $(D^2+1)y = \cos x$ by the method of variation of parameter.

Sol: Given, $(D^2+1)y = \cos x$

here $f(D) = D^2+1$

$R = \cos x$.

To find CF:

AE is $f(m)=0$

$m^2+1=0$

$m = \pm i$ (complex)

$CF = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$

$= e^{0x}(c_1\cos x + c_2 \sin x)$

$CF = c_1\cos x + c_2 \sin x$.

~~let CF = Au + Bv~~

let $CF = c_1 u + c_2 v$

$u = \cos x$ $\quad v = \sin x$

$u' = -\sin x$ $\quad v' = \cos x$

Wronskean:

$w = uv' - vu' = (\cos x)(\cos x) - (\sin x)(-\sin x)$

$w = \cos^2 x + \sin^2 x = 1$

To find PI:

$PI = Au + Bv$

$A = -\int \frac{vR}{w} = -\int \frac{\sin x . \cos x}{1} dx$

$= -\int \sin x \cos x \, dx$. Put $\sin x = t$, $\cos x \, dx = dt$

$= -\int t \, dt = -\frac{t^2}{2}$

$A = -\frac{\sin^2 x}{2}$.

$B = \int \frac{uR}{w} = \int \frac{\cos x . \cos x}{1} dx$

$= \int \cos^2 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right) dx$

$= \frac{1}{2}\int (1+\cos 2x) dx$

$B = \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]$

$PI = -\frac{\sin^2 x . \cos x}{2} + \frac{1}{2}\sin x\left[x + \frac{\sin 2x}{2}\right]$

General soln is $y = CF + PI$.

$$\boxed{y = c_1\cos x + c_2 \sin x - \frac{\sin^2 x \cos x}{2} + \frac{1}{2}\sin x\left[x + \frac{\sin 2x}{2}\right]}$$

# Simultaneous Linear Differential Equations:

A set of two or more differential equations having one same independent variable and two or more dependent variables is called Simultaneous Linear DEs.

$$\left.\begin{aligned} \frac{dx}{dt} + ax + by &= f_1(t) \\ \frac{dy}{dt} + cx + dy &= f_2(t) \end{aligned}\right\} - (1)$$

In eqn (1), if $f_1(t) = f_2(t) = 0$, then (1) is called homogeneous simultaneous linear DE. otherwise it is called non-homogeneous simultaneous linear DE.

These eqns are solved by eliminating all but one of the dependent variables and then solving the resulting equations as before.

(1) Solve $\frac{dx}{dt} - 7x + y = 0$,

$\frac{dy}{dt} - 2x - 5y = 0$

Sol: Given simultaneous eqns are,

$\frac{dx}{dt} - 7x + y = 0$, — (1) $\quad \frac{dy}{dt} - 2x - 5y = 0$ — (2)

let $\frac{d}{dt} = D$

(1) $\Rightarrow Dx - 7x + y = 0$

$(D-7)x + y = 0$ — (3)

(2) $\Rightarrow Dy - 2x - 5y = 0$

$-2x + (D-5)y = 0$ — (4)

Solving (3) & (4),

(3) $\times (D-5) \Rightarrow (D-5)(D-7)x + (D-5)y = 0$

(4) $\Rightarrow \quad -2x + (D-5)y = 0$

$\quad (+) \quad (-)$

$(D-5)(D-7)x + 2x = 0.$

$\Rightarrow (D-5)(D-7)x + 2x = 0$

$\Rightarrow [(D-5)(D-7) + 2]x = 0$

$\Rightarrow (D^2 - 7D - 5D + 35 + 2)x = 0$

$\Rightarrow (D^2 - 12D + 37)x = 0$

To find CF:

AE is $f(m) = 0$

$m^2 - 12m + 37 = 0$

$m = 6 \pm i$ (complex)

$CF = e^{\alpha t}[c_1 \cos\beta t + c_2 \sin\beta t]$

$CF = e^{6t}[c_1 \cos t + c_2 \sin t]$

General soln is $x = CF$

$x = e^{6t}[c_1 \cos t + c_2 \sin t]$

$\frac{dx}{dt} = 6e^{6t}[c_1 \cos t + c_2 \sin t] + e^{6t}[-c_1 \sin t + c_2 \cos t]$

(1) $\Rightarrow \frac{dx}{dt} - 7x + y = 0$

$\Rightarrow y = 7x - \frac{dx}{dt}$

$\Rightarrow y = 7e^{6t}[c_1 \cos t + c_2 \sin t] - 6e^{6t}[c_1 \cos t + c_2 \sin t] - e^{6t}[-c_1 \sin t + c_2 \cos t]$

$\Rightarrow y = e^{6t}[7c_1 \cos t + 7c_2 \sin t - 6c_1 \cos t - 6c_2 \sin t + c_1 \sin t - c_2 \cos t]$

$\Rightarrow y = e^{6t}[c_1(\cos t + \sin t) + c_2(\sin t - \cos t)]$

$$\therefore \boxed{\begin{aligned} x &= e^{6t}[c_1 \cos t + c_2 \sin t] \\ y &= e^{6t}[c_1(\cos t + \sin t) + c_2(\sin t - \cos t)] \end{aligned}}$$

(2) Solve $\frac{dx}{dt} + 2x + 3y = 0$,

$\frac{dy}{dt} + 2y = 2e^{2t}$.

Sol: Given eqns are

$\frac{dx}{dt} + 2x + 3y = 0$ —(1)

$\frac{dy}{dt} + 2y = 2e^{2t}$ —(2)

let $D = \frac{d}{dt}$

(1) $\Rightarrow Dx + 2x + 3y = 0$

$\Rightarrow (D+2)x + 3y = 0$ —(3)

(2) $\Rightarrow Dy + 2y = 2e^{2t}$.

$\Rightarrow (D+2)y = 2e^{2t}$ —(4)

Solving (3) & (4)

(3)$\times(D+2) \Rightarrow (D+2)(D+2)x + \cancel{3(D+2)y} = 0$

(4)$\times 3 \Rightarrow \cancel{3(D+2)y} = 6e^{2t}$

(−) (−)

$(D+2)^2 x = -6e^{2t}$

To find CF:

AE is $f(m) = 0$

$(m+2)^2 = 0$

$m = -2, -2$ (Equal)

$CF = (c_1 + c_2 t)e^{-2t}$

To find PI:

$PI = \frac{1}{f(D)}(-6e^{2t})$

$= \frac{1}{(D+2)^2}(-6e^{2t})$.

Put $D = a = 2$.

$= \frac{1}{(2+2)^2}(-6e^{2t})$

$PI = \frac{1}{16}(-6e^{2t}) = -\frac{3}{8}e^{2t}$.

General soln is $x = CF + PI$

$\boxed{x = (c_1 + c_2 t)e^{-2t} - \frac{3}{8}e^{2t}}$

$\frac{dx}{dt} = -2(c_1 + c_2 t)e^{-2t} + c_2 e^{-2t} - \frac{3}{4}e^{2t}$.

(1) $\Rightarrow \frac{dx}{dt} + 2x + 3y = 0$

$\Rightarrow -2(c_1+c_2t)e^{-2t} + c_2e^{-2t} - \frac{3}{4}e^{2t} + 2(c_1+c_2t)e^{-2t} - \frac{3}{4}e^{2t} + 3y = 0$

$\Rightarrow c_2e^{-2t} - \frac{3}{2}e^{2t} + 3y = 0$

$\Rightarrow \boxed{y = -\frac{c_2}{3}e^{-2t} + \frac{1}{2}e^{2t}}$

(3) Solve $\frac{dx}{dt} + 2y + \sin t = 0$

$\frac{dy}{dt} - 2x - \cos t = 0$

Sol: Given eqns are

$\frac{dx}{dt} + 2y + \sin t = 0$ —(1)

$\cancel{\frac{dy}{dt} + 2y = 2e^{2t}},$

$\frac{dy}{dt} - 2x - \cos t = 0$ —(2)

let $D = \frac{d}{dt}$

(1) $\Rightarrow Dx + 2y + \sin t = 0$

$\Rightarrow Dx + 2y = -\sin t$ —(3)

(2) $\Rightarrow Dy - 2x - \cos t = 0$

$\Rightarrow Dy - 2x = \cos t$ —(4)

(3)$\times 2 \Rightarrow \cancel{2Dx} + 4y = -2\sin t$

(4)$\times D \Rightarrow \cancel{-2Dx} + D^2y = D\cos t = -\sin t$.

$(D^2+4)y = -3\sin t$.

To find CF:

AE is $f(m) = 0$

$m^2 + 4 = 0$

$m = \pm 2i$

$CF = c_1\cos 2t + c_2 \sin 2t$

To find PI:

$PI = \frac{1}{f(D)}(-3\sin t)$

$= \frac{1}{D^2+4}(-3\sin t)$

Put $D^2 = -b^2 = -1$

$= \frac{1}{-1+4}(-3\sin t) = \frac{1}{3}(-3\sin t)$

$PI = -\sin t$

General soln is $x = CF + PI$

$\boxed{y = c_1\cos 2t + c_2 \sin 2t - \sin t}$

$\frac{dy}{dt} = -2c_1\sin 2t + 2c_2\cos 2t - \cos t$

(2) $\Rightarrow \frac{dy}{dt} - 2x - \cos t = 0$

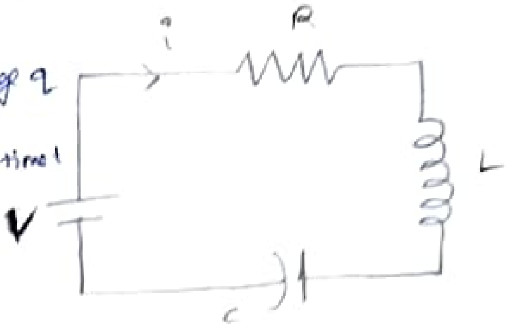
$\Rightarrow -2c_1\sin 2t + 2c_2\cos 2t - \cos t - 2x - \cos t = 0$

$\Rightarrow 2x = -2c_1\sin 2t + 2c_2\cos 2t - 2\cos t$

$\Rightarrow \boxed{x = -c_1\sin 2t + c_2\cos 2t - \cos t}$

## Applications to L-C-R circuits:

Consider a circuit containing Resistance R, inductance L, charge q and capacitance C all in series with constant emf V at time t




then by Kirchhoff's law,

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=V(t)$$

① A series RLC circuit has R = 80 ohms, L = 240 mH, C = 5 mF. If the input voltage is $v(t)=10\cos(2t)$, find the current flowing through the circuit.

Sol: Given, R = 80 ohms

$L = 240\text{ mH} = 240\times10^{-3}\text{H}$

$C = 5\text{ mF} = 5\times10^{-3}\text{F}$.

The differential eqn is given by

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=v(t)$$

$$\Rightarrow \frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=\frac{1}{L}v(t)$$

$$\Rightarrow \frac{d^2q}{dt^2}+\frac{80}{240\times10^{-3}}\frac{dq}{dt}+\frac{q}{5\times10^{-3}\times240\times10^{-3}}=\frac{10\cos2t}{240\times10^{-3}}$$

$$\Rightarrow \frac{d^2q}{dt^2}+\frac{1000}{3}\frac{dq}{dt}+\frac{2500}{3}q=\frac{125}{3}\cos2t$$

Let $D=\frac{d}{dt}$

$$\Rightarrow D^2q+\frac{1000}{3}Dq+\frac{2500}{3}q=\frac{125}{3}\cos2t$$

$$\Rightarrow (3D^2+1000D+2500)q=125\cos2t$$

$f(D)=3D^2+1000D+2500$

$Q(t)=125\cos2t$

To find CF:

DE is $f(m)=0$

$3m^2+1000m+2500=0$

$m=-2.519,\ -330.81$

$CF=C_1e^{-2.519x}+C_2e^{-330.81x}$

To find PI:

$$PI=\frac{1}{f(D)}125\cos2t$$

$$=\frac{1}{3D^2+1000D+2500}\cdot125\cos2t$$

Put $D^2=-b^2=-4$

$$=\frac{1}{3(-4)+1000D+2500}125\cdot\cos2t$$

$$=\frac{1}{1000D+2488}\cdot125\cos2t$$

$$=125\times\frac{1}{1000D+2488}\times\frac{1000D-2488}{1000D-2488}\cos2t$$

$$=(125)\frac{(1000D-2488)}{1000000D^2-6190144}\cos2t$$

Put $D^2=-b^2=-4$

$$=(125)\frac{(1000D\cos2t-2488\cos2t)}{-4000000-6190144}$$

$$=(125)\frac{(1000(-2\sin2t)-2488\cos2t]}{-10190144}$$

$$=(125)\frac{[2000\sin2t+2488\cos2t]}{10190144}$$

$PI = 16304\,2304\sin2t + 40564\,9252400$

$q = CF + PI$

$q=C_1e^{-2.519x}+C_2e^{-330.81x}+163042304\sin2t+405649252400$

Current, $i=\frac{dq}{dt}$

$$\boxed{i=-2.519C_1e^{-2.519x}-330.81C_2e^{-330.81}+326084608\cos2t}$$