20ECE361 - Digital Signal Processing

Unit-I - Discrete Fourier Transform

Syllabus: Discrete Signals and Systems - A Review. Introduction to DFT - Properties of DFT - Circular Convolution - FFT Algorithms - Decimation in time Algorithms, Decimation in Frequency Algorithms - use of FFT in Linear Filtering.

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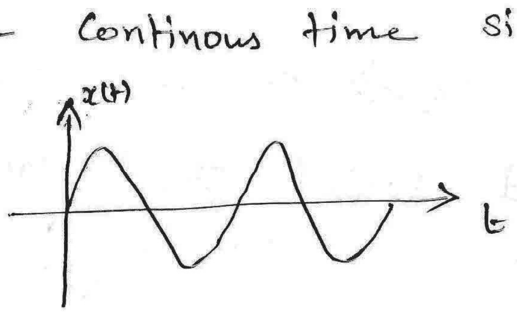
* Signal : A signal is a physical quantity that carries some information which varies with time, space or any other independent variable.

Ex:- Current, volt. etc

* Types:-

① Continous Time Signal: The signals that are defined for every value of time is called continous time signal (Analog Signal)

Ex:-

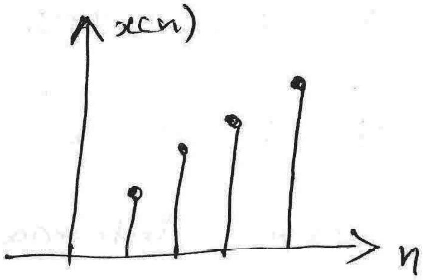


$x(t)$ = signal / amplitude.

t = time / continous time Interval

② Discrete Time Signal:- The signals that are defined only at specific value of time. These time instants need not be equal ditent.

The signal is said to be a discrete time signal if the amplitude is continous and the time is discrete (certain specific ~~#~~ Interval of time).



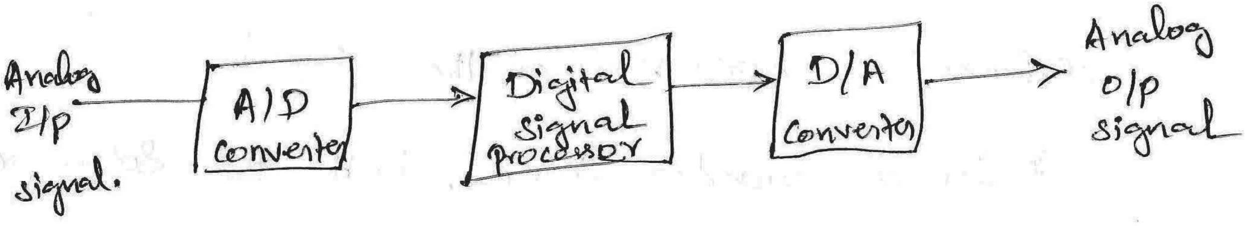
Ex:- Speed of vehicle at certain Interval.

* Signal processing:-

* signal processing is a method of extracting information from the signal which in turn depends on the type of signal and the information it carries.

* Digital signals can be processed by a processor called Digital signal Processors (DSP).

Basic Elements of DSP:-



* Most of signal is analog in nature, such may not be processed directly.

* Hence we need to convert the analog signal to digital by a converter called A/D converter.

* DSP processor may be a hardware processor which is controlled by software. It is to perform operation on the given I/p signal.

* D/A converter, ~~Con~~ which is used to convert the digital signal to analog signal. ~~Co~~

* Difference b/w Analog & Digital Signal Processing.

S.No.	Analog Signal	Digital Signal.
1.	It ~~ha~~ has less flexibility	It has more flexibility.
2.	Accuracy is not good	Accuracy is high
3.	It has high cost for Processing	Lower cost for Processing
4.	ADC and DAC converters are not required	ADC & DAC converters are required

System:-

* A system is a physical device that performs an operation on the signal.

* In a broader sense, both the software and hardware which processes the signal can be called as a system.

* Filter, that is used to reduce the noise in the informations bearing signal.

* The operations performed by the system on the signal is called processing.

* The system operation is Linear, the system is called Linear system.

* If the operation on the signal is non-linear, the system is said to be

Adv

① flexibility

② Accuracy

③ Easy storage.

④ processing

⑤ Cost effective

Application

① Speech processing

② Communication (spectrum Analysis, Echo, noise cancellati

③ Biomedical (ECG, EEG)

④ Consumer electronics

⑤ Seismology (eachquake, Volcanic, etc)

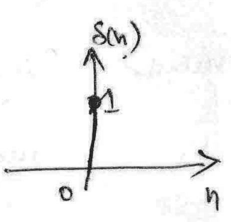
⑥ Image Processing (compress, Resize the Image)

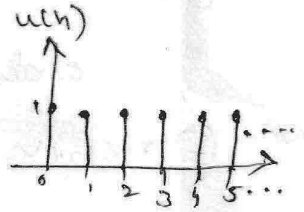
Standard Discrete Time Signals:-

① Digital impulse signal (or) unit sample sequence.

$\delta(n) = 1 \; ; \; n = 0$

$\quad\; = 0 \; ; \; n \neq 0$





② unit step signal:

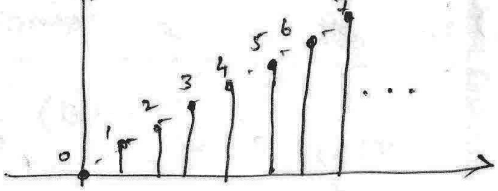
$u(n) = 1 \; ; \; n \geq 0$

$\quad\; = 0 \; ; \; n < 0$

③ Ramp signal:

$u_r(n) = n \; ; \; n \geq 0$

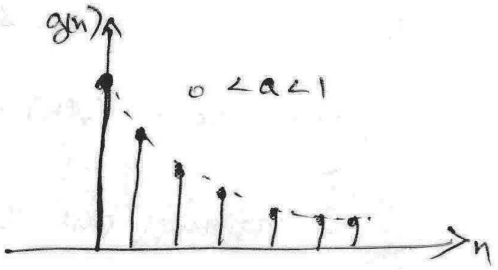
$\quad\; = 0 \; ; \; n < 0$

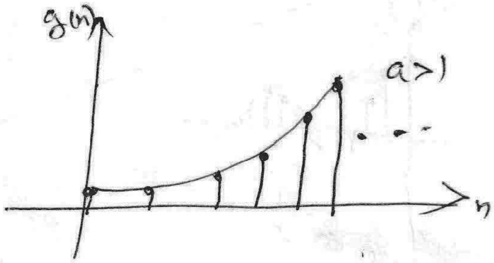


④ Exponential Signal:-

$g(n) = a^n \; ; \; n \geq 0$

$\quad\; = 0 \; ; \; n < 0$





⑤ Discrete time Sinusoidal Signal:-

$x(n) = A \cos(\omega_0 n + \theta) \; ; \; -\infty < n < +\infty$

$x(n) = A \sin(\omega_0 n + \theta) \; ; \; -\infty < n < +\infty$

where, ω_0 = freq. in radiance/sample ; θ - phase in radians ; $f_0 = \frac{\omega_0}{2\pi}$ = freq in cycle/sample

$x(n) = A \cos\left(\frac{\pi}{6} n + \frac{\pi}{3}\right) \; ; \; \omega_0 = \frac{\pi}{6} \; ; \; \theta = \frac{\pi}{3}$

Properties of Discrete Time Sinusoid:-

① A discrete time sinusoid is periodic only if its freq. f_0 is a radianal number. (ie; ratio of two integers).

② Discrete time sinusoids whose freq are separed by integer multiples of 2π are identical.

$$\therefore x(n) = A\cos[(\omega_0 + 2\pi k)n + \theta], \quad \text{for } k = 0, 1, 2 \dots\dots$$

⑥ Discrete time Complex exponential signal.

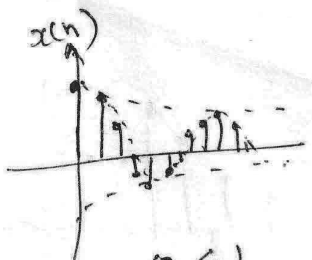
$$x(n) = a^n e^{j(\omega_0 n + \theta)} = a^n\left[\cos(\omega_0 n + \theta) + j\sin(\omega_0 n + \theta)\right]$$

$$= a^n \cos(\omega_0 n + \theta) + j\, a^n \sin(\omega_0 n + \theta)$$

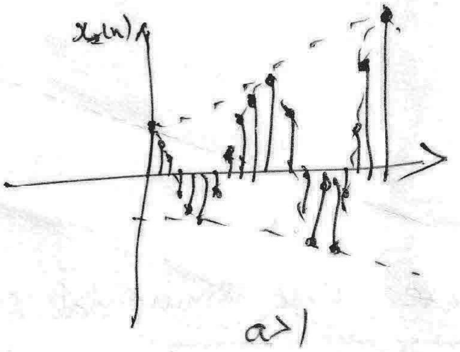
$$= x_r(n) + j\, x_i(n)$$

real part $x_r(n) = a^n \cos(\omega_0 n + \theta)$

Imaginary part $x_i(n) = a^n \sin(\omega_0 n + \theta)$



$0 < a < 1$



$a > 1$

Discrete Time Signals :-

* A discrete time signals $x(n)$, is a fun. of an independent Variable where the independent Variable is an integer.

* A discrete time signals is defined for every integer Value of 'n' in the range $-\infty < n < \infty$

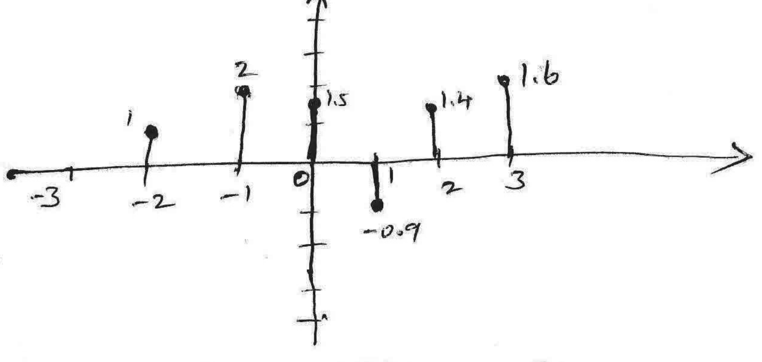
* discrete time signal is represented by a set of numbers it is also called a sequence.

Method of Representing Discrete-Time Signals :-

① Functional representation :-

$$x(n) = \begin{cases} +1 & ; n = -2 \\ 2 & ; n = -1 \\ 1.5 & ; n = 0 \\ -0.9 & ; n = 1 \\ 1.4 & ; n = 2 \\ 1.6 & ; n = 3 \\ 0 & ; \text{other } n \end{cases}$$

② Graphical Representation :-



③ Tabular representation:-

n	-2	-1	0	1	2	3
x(n)	·1	2	1.5	-0.9	1.4	1.6

④ Sequence Representation:-

An infinite duration signal or sequence with the time origin (n=0) indicated by the symbol ↑ is represented as,

$$x(n) = \{\ldots\ldots -1, 2, \underset{\uparrow}{1.5}, -0.9, 1.4, 1.6, \ldots\}$$

* An infinite sequence x(n), which is zero for n<0, may be represented as,

$$x(n) = \{\underset{\uparrow}{1.5}, -0.9, 1.4, 1.6 \ldots\}$$

(or)

$$x(n) = \{1.5, -0.9, 1.4, 1.6, \ldots\}$$

Mathematical operations on Discrete time Signals:-

① Shifting in time :-

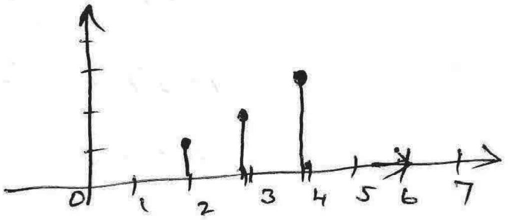
* A Signal $x(n)$ may be Shifted in time by replacing the independent variable 'n' by $n-k$,

⇒ k - is an integer.

⇒ If k - is +ve, the time Shift results in a delay by k-units of time. (ie; $x(n)$ Shifts to right)

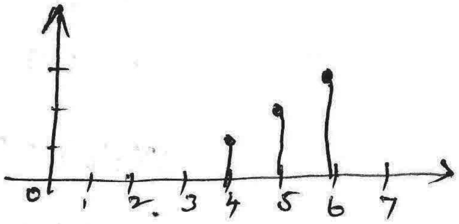
⇒ If k is -ve, the time Shift results is an advance of signal by $|k|$ units in time. (ie; $x(n)$ Shifts to left)

Ex:- Let $x(n) = 1 \; ; n=2$
$2 \; ; n=3$
$3 \; ; n=4$



* Let $x_1(n) = x(n-k) = x(n-\overset{k}{\overset{\downarrow}{2}})$, ∴ $x_1(n)$ is the delayed signal of $x(n)$.

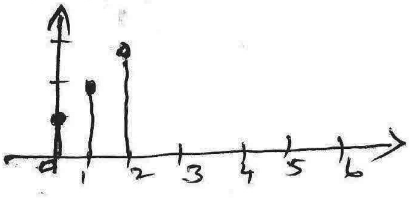
when $n=4 \; ; x_1(4) = x(4-2) = x(2) = 1$
$n=5 \; ; x_1(5) = x(5-2) = x(3) = 2$
$n=6 \; ; x_1(6) = x(6-2) = x(4) = 3$



* Let $x_2(n) = \cancel{x(n+2)}\; x(n-(-k)) = x(n+k) = x(n+\overset{k}{2})$

∴ $x_2(n)$ is the advanced of signal $x(n)$.

when $n=0 \; ; x_2(0) = x(0+2) = x(2) = 1$
$n=1 \; ; x_2(0) = x(1+2) = x(3) = 2$
$n=2 \; ; x_2(0) = x(2+2) = x(4) = 3$



② Folding (or) Reflection (or) Transpose:-

* The folding of a signal $x(n)$ is performed by changing the sign of the time base 'n' in the signal $x(n)$.

* folding operation produces a signal $x(-n)$ which is a mirror image of $x(n)$ with respect to time origin $n=0$.

Ex:- Let $x(n) = n$; $-3 \leq n \leq 3$.

$\therefore x(-3) = -3$

$x(-2) = -2$

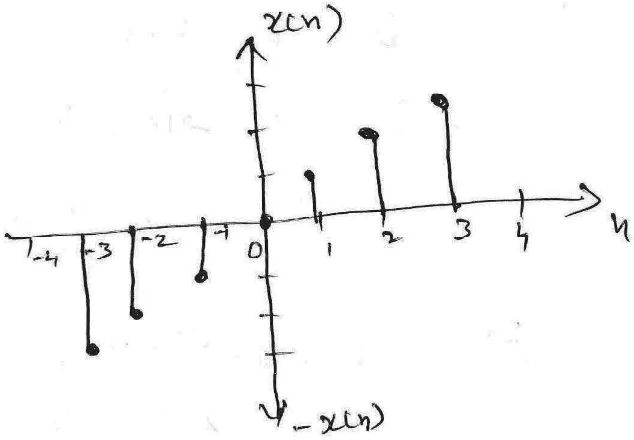
$x(-1) = -1$

$x(0) = 0$

$x(1) = 1$

$x(2) = 2$

$x(3) = 3$



Now the folding the signal $x_1(n) = x(-n) = -n$; $-3 \leq n \leq 3$

$x_1(-3) = x(-(\overset{n}{-3})) = x(3) = 3$

$x_1(-2) = x(-(-2)) = x(2) = 2$

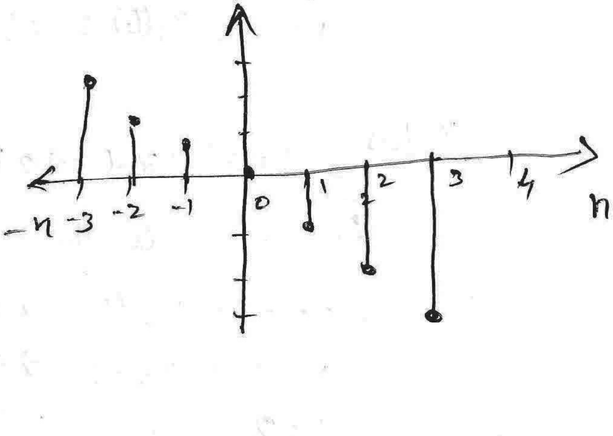
$x_1(-1) = x(-(-1)) = x(1) = 1$

$x_1(0) = x(-(0)) = x(0) = 0$

$x_1(1) = x(-1) = -1$

$x_1(2) = x(-2) = -2$

$x_1(3) = x(-3) = 3$



③ Amplitude Scaling (or) Scalar Multiplication:-

Amplitude scaling of a signal by a constant 'A' is accomplished by multiplying the value of every signal $x(n)$.

Let; $y(n) = A\,x(n)$ [where A - is constant].

Ex:- ~~$y(n)$~~ $x(n) = 20$; $n=0$ and $A = 0.1$

$= 36$; $n=1$

$= 40$; $n=2$

$= -15$; $n=3$

$y(n) = A * x(n)$

$y(0) = 0.1 * 20 = 2.0$

$y(1) = 0.1 \times 36 = 3.6$

$y(2) = 0.1 \times 40 = 4.0$

$y(3) = 0.1 \times (-15) = -1.5$.

④ Time Scaling (or) down Sampling:-

In a signal, $x(n)$, if n is replaced by μn, where μ - is an integer, then it is called time scaling or down sampling.

Ex:- If $x(n) = b^n$; $n \geq 0$

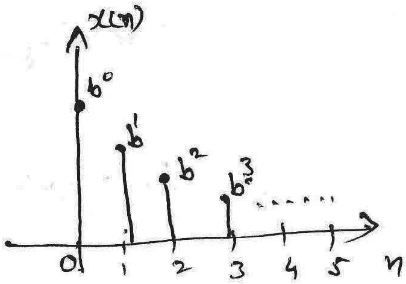
∴ when $x(0) = b^0$

$x(1) = b^1$

$x(2) = b^2$

$x(3) = b^3$

⋮



~~Ex:- If $x(n) = a^n$; $n \geq 0$~~

~~∴ $x(0) = a^0$~~

~~Let, $x(n) = 3$; $n=2$~~

~~2 ; $n=3$~~

~~1 ; $n=4$~~

~~0 ; for other n~~

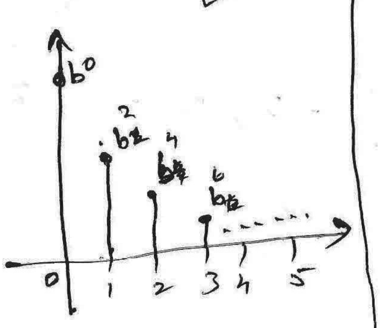
If $x_1(n) = x(2n) = b^n$

$x_1(0) = x(2\times0) = x(0) = b^0$

$x_1(1) = x(2\times1) = x(2) = b^2$

$x_1(2) = x(2\times2) = x(4) = b^4$

$x_1(3) = x(3\times2) = x(6) = b^6$



5. Signal (vector) Addition

The sum of two signal $x_1(n)$ and $x_2(n)$ is a signal $y(n)$, whose value of any instant is equal to the sum of the samples of these two signals at that instant.

i.e; $y(n) = x_1(n) + x_2(n)$; $-\infty \leq n \leq \infty$

Ex $x_1(n) = \{1, 2, -1, 2\}$; $x_2(n) = \{-2, 1, 3, 1\}$

$y(n) = ?$

$y(0) = x_1(0) + x_2(0) = 1 + (-2) = -1$

$y(1) = x_1(1) + x_2(1) = 2 + 1 = 3$

Similarly, $y(2) = 2$; $y(3) = 3$.

$\therefore$ $y(n) = \{-1, 3, 2, 3\}$

⑥ Product or vector Multiplication:-

$$y(n) = x_1(n) * x_2(n)$$

Ex:- $x_1(n) = \{1, 2, -1, 2\}$; $x_2(n) = \{-2, 1, 3, 1\}$.

$y(n) = ?$

Sol $y(0) = x_1(0) * x_2(0) = 1 * -2 = -2$

$y(1) = x_1(1) * x_2(1) = 2 * 1 = 2$

Similarly, $y(2) = -3$; $y(3) = 2$

$\therefore$ $y(n) = \{-2, 2, -3, 2\}$ //

Clarification of Discrete Time Signals:-

(1) Deterministic and non-deterministic signal

(2) periodic and aperiodic signal

(3) Symmetric and Antisymmetric signal.

(4) Energy & power signals.

(5) Causal and non-causal Signals. → Causal : signals that are zero for all negative time

non-causal signal : signals that have non-zeros values in both positive and negative time.

(1)(a) Deterministic signal : Signal that can be completely specified by mathematical equ. are called deterministic Signal

Ex:- Step, ramp, exponential.

(b) non-Deterministic : signal where characteristic are random in nature are called non-deterministic signals.

Ex:- noise signal from various Sources.

(2) periodic & Aperiodic Signals:-

The signal $x(n)$ is repeats after some time is called periodic signal, Otherwise, it is known as aperiodic signal.

⇒ A signal $x(n)$ is periodic with period N (where $N>0$)

If and only if, $x(n+N) = x(n)$ for all n

Ex:

~~① $x(n) = e^{j7\pi n/3}$~~

① $\cos(0.01\pi n)$

sol

$\cos(\cancel{2\pi}\,\omega n) = \cos(0.01\pi n)$

$\because \omega n = 0.01\pi n$

$2\cancel{\pi} f \cancel{n} = 0.01 \cancel{\pi} \cancel{n}$

$f = \frac{0.01}{2} = \frac{1}{200} = \frac{K}{N}$ - is a rational no.

where $K = 1$ & $N = 200$, both are Integer so given value is periodic

② ~~Cos~~ $\sin(3n)$

sol

$\sin(\omega n) = \sin(3n)$

$\therefore \omega n = 3n$

$2\pi f \cancel{n} = 3 \cancel{n}$

$f = \frac{3}{2\pi} = \frac{K \leftarrow \text{Integer}}{N \leftarrow \text{not Integer}} \Big\} \because$ not a rational no.

So given value is non-periodic.

③ $\cos 3\pi n$

sol

$\because \omega n = 3\pi n$

$2\cancel{\pi} f \cancel{n} = 3 \cancel{\pi} \cancel{n}$

$f = \frac{3}{2}$

$\therefore$ given value is periodic

④ $x(n) = \cos\left(\frac{2\pi n}{5}\right) + \cos\left(\frac{2\pi n}{7}\right)$

(a) $\cancel{2\pi} f_1 \cancel{n} = \frac{\cancel{2\pi}\cancel{n}}{5}$

$\therefore f = \frac{\cancel{2}}{5\times\cancel{2}} = \cancel{\frac{1}{10}} = \frac{1}{5} = \frac{K}{N_1} =$ It is a rational no.

∴ periodic.

(b) $\cancel{2\pi} f_2 n = \frac{2\cancel{\pi} n}{7}$

$f_2 = \frac{1}{7} = \frac{K}{N_2} =$ It is a rational no.

∴ f_1 & f_2 = are Integer, so periodic.

⑤ $x(n) = e^{\left(j\frac{\pi}{4}\right)n}$ $\quad [e^{j\theta} = \cos\theta + j\sin\theta$

Sol) $x(n) = \cos\frac{\pi}{4}n + j\sin\frac{\pi}{4}n$

where $2\cancel{\pi} f_1 \cancel{n} = \frac{\cancel{\pi}}{4} n$ | $2\pi f_2 n = \frac{\pi}{4} n$

$f_1 = \frac{1}{8} = \frac{K}{N}$ | $f_2 = \frac{1}{8} = \frac{K}{N}$

K & N are Integer.

∴ periodic

⑥ $x(n) = \cos\left(\frac{n}{8}\right)\cos\left(\frac{n\pi}{8}\right)$

Sol) ~~x(n)~~ $2\pi f_1 n = \frac{n}{8}$ | $2\cancel{\pi} f_2 \cancel{n} = \frac{\cancel{n}\cancel{\pi}}{8}$

$f_1 = \frac{1}{16\pi} = \frac{K}{N}$ | $f_2 = \frac{1}{16} = \frac{K}{N}$

N - is not Integer. | K & N = Integer.

f_1 - non-periodic

f_2 - periodic

∴ - Result is non-periodic.

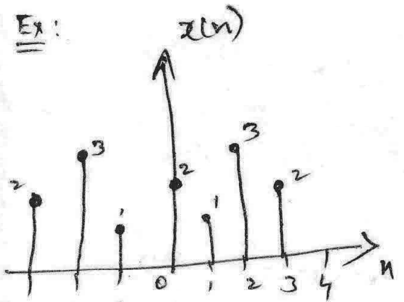
(3) Symmetric (Even) and Antisymmetric (odd) Signals:-

A real-valued signal $x(n)$ is,

Symmetric if, $x(n) = x(-n)$ (with respect to $n=0$)

Anti-Symmetric if, $x(-n) = -x(n)$ (with respect to $n=0$)

Ex:



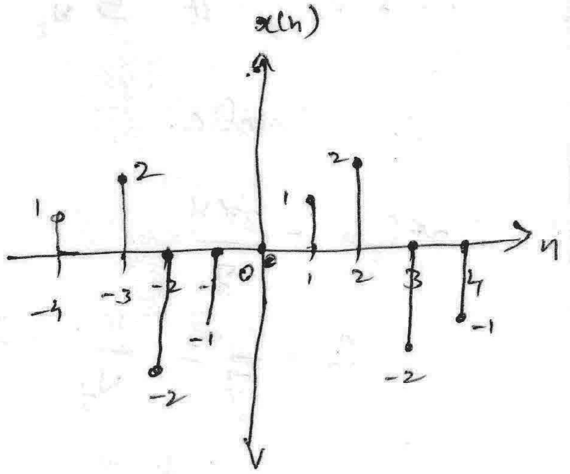
$x(1) = x(-1)$

$x(2) = x(-2)$

$x(3) = x(-3)$

Symmetric Signal

$x(n) = \{2, 3, 1, \underset{\uparrow}{2}, 1, 3, 2\}$



$x(-1) = -x(1)$

$x(n) = \{1, 2, -2, -1, \underset{\uparrow}{0}, 1, 2, -2, -1\}$

Let,

* $x(n) = x_e(n) + x_o(n)$

$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$x_o(n) = \frac{1}{2}[x(n) + x(-n)]$

[where, $x_e(n)$ - even part of $x(n)$

$x_o(n)$ - odd part of $x(n)$

→ If $x(n)$ is even then its odd part will be zero.

→ If $x(n)$ is odd then its even part will be zero.

#9

* Determine the even & odd part of the signal

① $x(n) = \{\underset{\uparrow}{2}, -2, 6, -2\}$

sol

$x(n) = \{\underset{\uparrow}{2}, -2, 6, -2\}$

$\therefore\ x(0) = 2\ ;\ x(1) = -2\ ;\ x(2) = 6;\ x(3) = -2$

$x(-n) = \{-2, 6, -2, \underset{\uparrow}{2}\}$

$\therefore\ x(-3) = -2\ ;\ x(-2) = 6;\ x(-1) = -2\ ;\ x(0) = 2$

even part

$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

At $n=-3$; ~~$x(n$~~ $x(n) + x(-n) = 0 + (-2) = -2$

At $n=-2$; $x(n) + x(-n) = 0 + 6 = 6$

At $n=-1$; $x(n) + x(-n) = 0 + -2 = -2$

At $n=0$; $x(n) + x(-n) = 2 + 2 = 4$

At $n=1$; $x(n) + x(-n) = -2 + 0 = -2$

At $n=2$; $x(n) + x(-n) = 6 + 0 = 6$

At $n=3$; $x(n) + x(-n) = -2 + 0 = -2$

$\therefore\ x(n) + x(-n) = \{-2, 6, -2, \underset{\uparrow}{4}, 2, 6, 2\}$

$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$

$= \frac{1}{2}\{-2, 6, -2, \underset{\uparrow}{4}, 2, 6, 2\}$

$= \{-1, 3, -1, \underset{\uparrow}{2}, 1, 3, 1\}$

odd part

$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

At $n=-3$; $x(n) - x(-n) = 0 - (-2) = 2$

At $n=-2$; $x(n) - x(-n) = 0 - 6 = -6$

At $n=-1$; $= 0-(-2) = 2$

At $n=0$; $= 2-2 = 0$

At $n=1$; $= -2-0 = -2$

At $n=2$; $= 6-0 = 6$

At $n=3$; $= -2-0 = -2$

~~$x(n) =$~~ $x(n) - x(-n) = \{2, -6, 2, \underset{\uparrow}{0}, -2, 6, -2\}$

$\therefore x_o(n) = \frac{1}{2}[x(n) - x(-n)]$

$= \{1, -3, 1, \underset{\uparrow}{0}, -1, 3, -1\}$

② $x(n) = 3^n$

sol

$x(n) = 3^n$

$x(-n) = 3^{-n}$

Even part, $x_e(n) = \frac{1}{2}[x(n) + x(-n)] = \frac{1}{2}[3^n + 3^{-n}]$

odd part, $x_o(n) = \frac{1}{2}[x(n) - x(-n)] = \frac{1}{2}[3^n - 3^{-n}]$

<u>System</u>:- is a device (or) Algorithm that operates on a ~~discrete time~~ signal $x(n)$ according to some well-defined rule to produce another ~~discr~~ signal $y(n)$ is called o/p signal

Discrete time ~~Signal~~ System : I/p $x(n)$ and o/p $y(n)$ are discrete time signal

$$y(n) = T[x(n)] \qquad [\because \text{T- Transformation (or) operation.}]$$

Classification:-

(1) Static § dynamic system.

(2) Causal and non-causal system.

(3) Linear and non-Linear system.

(4) Time - Variant § Time - Invariant Systems

(5) Stable and unstable system.

(6) FIR and IIR system.

① Static § Dynamic systems:

A discrete time system is called Static (or) memory less if its o/p at any instant 'n' depends on the input samples at the same time, but does not depend on past and future samples of the I/p.

Otherwise the system is known as dynamic (or) to have memory.

<u>Ex</u>:-

Static : $y(n) = ax(n)$; $y(n) = ax^2(n)$

dynamic : $y(n) = x(n-3) + x(n-2)$; $y(n) = x(n+2) + x(n)$.

② Causal and Non-causal systems:-

A system is said to be causal if the o/p of the system at any time 'n' depends only at present and past I/p's, but does not depend on future I/p's.

In the system, if the o/p depends upon future I/ps, then that system is known as non-causal system.

Ex:- $y(n) = x(n) + x(n-3)$ - Causal

$y(n) = x(3n)$ - non causal.

③ Linear and non-Linear Systems:-

The system is linear when it satisfies the superposition principles. It states that the response of the system to a weighted sum of signals should be equal to the corresponding weighted sum of the o/p of the system to each of the individual I/p signals.

A system is linear if and only if,

$$T\left[a_1 x_1(n) + a_2 x_2(n)\right] = a_1 T\left[x_1(n)\right] + a_2 T\left[x_2(n)\right]$$

where a_1, a_2 - Arbitrary constants.

A system does not satisfies the superposition principle is called non-linear.

④ Time Variant and Time Invariant Systems:

A System is said to be time invariant if the Char. of the System do not Change with time.

Ex:- $y(n,k) = T[x(n-k)]$ - Time invariant.

⑤ Stable and unstable Systems:-

If an LTI (Linear time Invariant) System with impulse reshone $h(n)$ is bounded I/p sequence to bounded O/p sequence, then it is called Stable System.

If the O/p is unbounded, then the system is Classified as unstable.

⑥ FIR & IIR system:-

If the impulse response($h(n)$) of the system is of finite duration, then the System is called a Finite Impulse Response (FIR System)

Ex: $$h(n) = \begin{cases} 1 & \text{for } n = -1, 3 \\ 3 & \text{for } n = 2 \\ 2 & \text{for } n = 1, 0 \\ 0 & \text{otherwise.} \end{cases}$$

An Infinite Impulse response (IIR) System has an impulse response of infinite duration.

Ex:- $h(n) = a^n u(n)$

→ Determine the following Systems are time-Invariant or not,

① $y(n) \neq x(n) + x(n-1)$

sol

I/p delayed by k units, we have

$$y(n,k) = T[x(n-k)]$$
$$= x(n-k) + x(n-k-1)$$

Delay the o/p by k units,

$$y(n-k) = x(n-k) + x(n-k-1)$$

∴ $y(n,k) = y(n-k)$, then the system is time-Invariant.

② $y(n) = x(-n+4)$

sol I/p is delayed by k samples,

$$y(n,k) = x(-n+4-k)$$

O/p is delayed by k samples;

$$y(n-k) = x[-(n-k)+4]$$
$$= x[-n+k+4]$$

∴ $y(n,k) = y(n-k)$, then this system is time ~~Invariant~~ Variant

③ $y(n) = x(4n)$.

$$y(n,k) = x(4n-k)$$
$$y(n-k) = x(4(n-k))$$

∴ $y(n,k) \neq y(n-k)$, then the system is time Variant

→ Determine if the following systems are stable (or) unstable system.

(i) $y(n) = n x(n)$

Sol)

If $n \to \infty$ then $y(n) \to \infty$, Therefore the O/p is not bounded for all the I/p. Hence the system is unstable.

(2) $y(n) = x(n)$

$x(n)$ I/p is bounded and $y(n)$ also bounded. That is

$y(0) = x(0)$ for $n=0$

$y(1) = x(1)$ for $n=1$

So, it is stable.

(3) $y(n) = x(n^2)$

If $n=1$ $y(1) = x(1)$

$n=2$ $y(2) = x(4)$

$n=3$ $y(3) = x(9)$

$x(n)$ is bounded and $y(n)$ also bounded. So, it is stable.

(4) $y(n) = x(n) + u(n)$

$\left[u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{otherwise.} \end{cases} \right.$

$n=1$; $y(1) = x(1) + 1$

$n=2$; $y(2) = x(2) + 1$

$x(n)$ bounded and $y(n)$ also bounded. So it is stable.

→ Determine the following Systems are linear (or) non-Linear.

① $y(n) = x(n) + x(n+1)$

Sol

Two I/p Sequ. is are $x_1(n)$ and $x_2(n)$.

$$y_1(n) = x_1(n) + x_1(n+1)$$

$$y_2(n) = x_2(n) + x_2(n+1)$$

The o/p due to weighted Sum of I/p is

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$$

$$= a_1 x_1(n) + a_1 x_1(n+1) + a_2 x_2(n) + a_2 x_2(n+1) \quad \text{——} ①$$

The Linear Combination of the two o/p,

$$a_1 y_1(n) + a_2 y_2(n) = a_1 x_1(n) + a_1 x_1(n+1) + a_2 x_2(n) + a_2 x_2(n) \quad \text{——} ②$$

∴ equ ① & ② are equal, so the System is linear System.

② $y(n) = n\, x(n)$

Sol

$$y_1(n) = n\, x_1(n)$$

$$y_2(n) = n\, x_2(n)$$

The o/p due to weighted Sum of I/p is.

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)] = a_1 n\, x_1(n) + a_2 n\, x_2(n) \quad \text{——} ①$$

The Linear Combination of the o/p.

$$a_1 y_1(n) + a_2 y_2(n) = a_1 n\, x_1(n) + a_2 n\, x_2(n) \quad \text{——} ②$$

∴ equ ① = ②, so the System is linear System.

③ $y(n) = x^2(n)$.

sol

$$y_1(n) = x_1^2(n)$$

$$y_2(n) = x_2^2(n)$$

weighted sum of I/p is,

$$y_3(n) = T[a_1x_1(n) + a_2x_2(n)] = [a_1x_1^2(n) + a_2x_2(n)]^2$$

$$= a_1^2x_1^2(n) + a_2^2x_2^2(n) + 2a_1a_2x_1(n)x_2(n). \longrightarrow ①$$

weighted sum of o/p is,

$$a_1y_1(n) + a_2y_2(n) = a_1x_1^2(n) + a_2x_2^2(n) \longrightarrow ②$$

∴ equ ① ≠ ②, then the System is non-Linear.

④ $y(n) = x(n) + \frac{1}{x(n+1)}$

sol

$$y_1(n) = x_1(n) + \frac{1}{x_1(n+1)} \quad ; \quad y_2(n) = x_2(n) + \frac{1}{x_2(n+1)}$$

* $y_3(n) = T[a_1x_1(n) + a_2x_2(n)]$

$$= a_1x_1(n) + \frac{1}{a_1x_1(n+1)} + a_2x_2(n) + \frac{1}{a_2x_2(n-1)} \longrightarrow ①$$

* Linear combination of o/p is,

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n) + \frac{a_1}{x_1(n+1)} + a_2x_2(n) + \frac{a_2}{x_2(n-1)} \longrightarrow ②$$

∴ equ ① ≠ ②, so System is non-Linear.

~~So~~

→ Find whether the following systems are static (or) dynamic.

① $y(n) = x(n)\,x(n-4)$

y(n) depends on present and past I/p. So the system is dynamic

② $y(n) = x(n) + x^2(n)$.

y(n) depends on present I/p only, so system is static

③ $y(n) = x(2n) + x(n-1)$

y(n) depends on ~~present~~ Past and future I/p. So, the system is dynamic.

→ Check the following systems are causal and non-causal.

① $y(n) = x(n) + x(n-3)$

Sol y(n) depends on past and present I/p, then the system is causal.

② $y(n) = x(n^2)$

y(n) depends on future I/p signal, then system is non-causal.

③ $y(n) = a\,x(n) + b\,x(n-2)$

y(n) depends on present I/p & past I/p Then the system is Causal.

④ $y(n) = x(2n)$

Sol
$n=0$; $y(0) = x(2\times0) = x(0)$
$n=1$; $y(1) = x(2\times1) = x(2)$
$n=2$; $y(2) = x(2\times2) = x(4)$

∴ y(n) depends on present and future I/p. ~~then~~ the system is non-causal.

Energy & Power Signals:-

① Energy 'E' of Discrete time signal $x(n)$ is defined as,

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

* Energy signal may be finite (or) Infinite and can be applied to complex valued and real valued signal.

If energy signal(E) is finite and non-zero, then the discrete time signal is called an energy signal.

Ex:- Exponential signal.

② Power signal:-

The Average power Signal ($x(n)$) is defined as

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

If power P of a signal is finite and ~~(or)~~ ~~Infinite~~ non-zero, then the signal is called power signal.

Ex:- periodic signal.

Note:-

For energy signal, $0 < E < \infty$ and $P = 0$

For power signal $0 < P < \infty$ and $E = \infty$

→ Determine whether the following signals are Energy or power Signals.

① $x(n) = \left(\frac{1}{4}\right)^n u(n)$

Sol

$$x(n) = \left(\frac{1}{4}\right)^n u(n) \quad \text{for all } n$$

$$x(n) = \left(\frac{1}{4}\right)^n = (0.25)^n \quad ; n \geq 0$$

$$\because \; E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} |((0.25)^n)|^2 = \sum_{n=0}^{\infty} (0.25^2)^n = \sum_{n=0}^{\infty} (0.0625)^n$$

$$= \frac{1}{1-0.625} = 1.067 \text{ joules} \quad \left[\because \sum_{n=0}^{\infty} c^n = \frac{1}{1-c}\right.$$

Infinite geometric series sum formula.

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} |(0.25)^n|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.25^2)^n = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} (0.0625)^n$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \frac{(0.0625)^{N+1} - 1}{0.0625 - 1} \quad \left[\because \sum_{n=0}^{\infty} c^n = \frac{c^{N+1} - 1}{c - 1}\right.$$

$$= \frac{1}{\infty} \times \frac{0.0625^{\infty} - 1}{0.0625 - 1}$$

using finite geometric series sum formula.

$$= 0$$

② $x(n) = \sin\left(\frac{\pi}{3}n\right)$

Sol

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=-\infty}^{\infty} \sin^2\left(\frac{\pi}{3}n\right) = \sum_{n=-\infty}^{\infty} \frac{1-\cos\frac{2\pi}{3}n}{2} \quad \left[\because \sin^2\theta = \frac{1-\cos\theta}{2}\right.$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(1-\cos\frac{2\pi}{3}n\right) = \frac{1}{2}\left(\sum_{n=\infty}^{\infty} 1^n - \sum_{n=-\infty}^{\infty}\cos\frac{2\pi n}{3}\right) = \frac{1}{2}(\infty - 0)$$

$$= \infty$$

Notes:- Sum of infinite 1's is infinity.

sum of one period of cosinusoidal signal is zero.

$$P = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}|x(n)|^2 = \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}\sin^2\frac{\pi n}{3}$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\sum_{n=-N}^{N}\frac{\left(1-\cos\frac{2\pi}{3}n\right)}{2}$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\times\frac{1}{2}\sum_{n=-N}^{N}\left(1-\cos\frac{2\pi}{3}n\right)$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\times\frac{1}{2}\left[\sum_{n=-\infty}^{N}1^n - \sum_{n=-N}^{N}\cos\frac{2\pi}{3}n\right]$$

where,

$\cos\frac{2\pi}{3}n$ is,

1st period

$n = 0;\ \cos\frac{3\pi}{3}n = 1$

$n = 1;\ \cos\frac{2\pi}{3}n = -0.5$

$n = 2;\ \cos\frac{2\pi}{3}n = -0.5$

2nd period

$n = 3;\ \cos\frac{2\pi}{3}n = 1$

$n = 4;\ \cos\frac{2\pi}{3}n = -0.5$

$n = 5;\ \cos\frac{2\pi}{3}n = -0.5$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\times\frac{1}{2}\left[\underbrace{1+1+\cdots+1}_{N\ \text{terms}} + 1 + \underbrace{1+\cdots+1+1}_{N\ \text{terms.}} - 0\right]$$

$$= \underset{N\to\infty}{Lt}\ \frac{1}{2N+1}\times\frac{1}{2}\left[2N+1\right] = \underset{N\to\infty}{lt}\ \frac{1}{2} = \frac{1}{2}\ \text{watts.}$$

sine P is ~~infite~~ finite and E is Infinite.

∴ $x(n)$ is a power signal.

③ $x(n) = u(n)$

sol

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} (u(n))^2$$

$$= \sum_{n=0}^{+\infty} u(n) = 1+1+1+1+1+ \dots \infty = \infty$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{\infty} |u(n)|^2 = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{\infty} u(n)$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \left[\underbrace{1+1+1+\dots+1}_{N+1 \text{ terms}} \right]$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} (N+1)$$

$$= \underset{N\to\infty}{Lt} \frac{N\left(1+\frac{1}{N}\right)}{N\left(2+\frac{1}{N}\right)}$$

$$= \frac{1+0}{2+0}$$

$$= \frac{1}{2} \text{ watts}$$

∴ E - Infine , P = finite.

x(n) is a power signal.

Sampling :-

It is the process of conversion of a Continous time signal into a discrete time signal.

- Time Interval b/w two successive samples will be same and such type of sampling is called periodic (or) uniform sampling.

- The time interval b/w successive samples is called sampling time' (sampling period (or) sampling Interal). ('T').

- Inverse of sampling period is called 'Sampling freq.'

i.e; denotes $(F_s) = \frac{1}{T}$

Let, $x_a(t)$ - Continuous time / Analog Signal.

$x(n)$ - Discrete time signal obtained by sampling $x_a(t)$.

→ Relation b/w $x(n)$ & $x_a(t)$ can be expressed as.

$$x(n) = x_a(t)\Big|_{t=nT} = x_a(nT) = x_a\left(\frac{n}{F_s}\right) \;;\; -\infty < n < \infty$$

where, T - sampling period (or) Interval.

$F_s = \frac{1}{T}$ = Sampling rate (or) Sampling freq. in (Hz)

Sampling Theorem:-

A band limited continous time signal with highest freq (bandwidth) F_m hertz. Can be uniquely recovered from its samples provided that the Sampling rate F_s is greater than (or) equal to $2F_m$ samples per second.

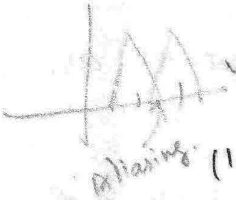
~~for~~)

ie, $F_s \geq 2F_m$ [where, F_s - Sampling freq.
F_m - maxi. freq of analog signal.

Aliasing:-

when the sampling freq. is less than twice of the highest freq content of the signal, then Aliasing in ~~freq~~ freq. domain takes place.

✓ In Aliasing, the high freq. of the signal mix with lower freq. and create distortion in freq. spectrum.



✓ Aliasing can be avoided by two ways,

(1) Sampling freq. must be higher than twice of highest freq present in the signal. (ie, $F_s \geq 2F_m$)

(2) A low pass filter must be used before Sampling to band limit the Signal to some specific freq.

Nyquist rate:- when Sampling freq. F_s is equal to $2F_M$, the sampling rate is called Nyquist rate ($F_s = 2F_m$)

Recussive System:-

A System whose o/p $y(n)$ at time 'n' dependes on any no. of past o/p value as well as present and past I/p is called Recussive system.

Ex: $y(n) = 2y(n-2) + x(n-1) + 5x(n)$

non - Recursive System:-

A System whose o/p does not depend on past o/p but depends only on the present and past I/p is called a non Recursive System.

Ex: $y(n) = 2x(n) + 4x(n-1)$

(may / June 2012)

→ Check the following systems are Linear, Causal, time-Invariant, stable, static.

(1) $y(n) = x\left(\frac{1}{2n}\right)$ - Linear, Causal, time-variant, Stable, dynamic.

(2) $y(n) = \sin x(n)$ - non-Linear, Causal, time-Invariant, Stable, Static

(3) $y(n) = x(n) \cos x(n)$ - non-Linear, ~~non~~-causal, time invariant, stable, ~~dynamic~~ Static

(4) $y(n) = x(-n+5)$ - Linear, non-causal, time Variant, stable, dynamic

(5) $y(n) = x(n) + n.x(n+2)$ - Linear, non-causal, time variant, unstable, dynamic.

(6) $y(n) = x(n^2)$ - Linear, time Variant, dynamic, non-Causal.

(7) $y(n) = (n-1)x^2(n) + c$ → non-Linear, time Variance.

⑧ $T[x(n)] = e^{x(n)}$ → Stable, Causal, non-linear, Shift Invariant

⑨ $T[x(n)] = a x(n) + b$ → Stable, causal, non-Linear, Shift Invariant.

⑩ $h(n) = 2^n u(-n)$ → stable, non-causal.

⑪ $h(n) = \sin \frac{n\pi}{2}$ → non-causal, Stable

⑫ $h(n) = \delta(n) + \sin n\pi$ → Causal and stable

⑬ $h(n) = p^{2n} u(n-1)$ → stable,

sol (i) $u(n-1)$ for $n \geq 1$, $h(n) = 0$ for $n < 1$ → causal systems

(ii) $\sum_{k=-\infty} |h(k)| = \sum_{k=0}^{\infty} p^{2n} u(k-1) = \sum_{k=1}^{\infty} p^{2n} = \sum_{n=0}^{\infty} p^{2n} - 1 = \frac{1}{1-p^2} - 1 = \frac{p^2}{1-p^2}$ if $p^2 < 1$

system stable
otherwise unstable.

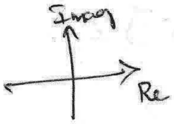
UNIT - II

Discrete Time System Analysis

Z-transform and its properties, Inverse - Z transforms difference equation, Solution by Z-transform, Application of discrete System, - Stability analysis, freq. response - Convolution - Discrete time Fourier transform, magnitude and phase representation

Introduction:-

- Z - transform is an best tools for analysing systems
- Z transform is a complex variable (real & imag. value)



plot the graf versus real & imag. - is called "Z - plane"

∴ poles and zeros are plotted in the Complex plane.

- It is one of the transformation tech., It is used to analyse discrete time Signal and LTI - System.

- Definition:

$$X(Z) \overset{Z}{\longleftrightarrow} x(n)$$
(or)
$$X(Z) = Z\{x(n)\}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

Properties:-

(1) Linearity (2) Time Shifting (3) Time ~~Seve~~ reversal

(4) Scaling (5) Differentiation (6) Convolution.

(7) parseval's theorem (8) Initial value theorem (9) Final value theorem.

(1) Linearity:-

If $X_1(z) = Z\{x_1(n)\}$

$X_2(z) = Z\{x_2(n)\}$

they

$$Z\{a x_1(n) + b x_2(n)\} = a X_1(z) + b X_2(z)$$

proof

$$Z\{a x_1(n) + b x_2(n)\} = \sum_{n=-\infty}^{\infty} [a x_1(n) + b x_2(n)] z^{-n}$$

$$= a \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + b \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}$$

$$= a X_1(z) + b X_2(z) \,//$$

(2) Time Shifting:-

If $X(z) = Z\{x(n)\}$ and the initial conditions for $x(n)$ are zeros, then

$$Z\{x(n-m)\} = Z^{-m} X(z) \quad [\, m - \text{+ve (or) -ve Integer.}$$

proof:-

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m) Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-m) Z^{-n}$$

$$= \sum_{l=-\infty}^{\infty} x(l) Z^{-(l+m)}$$

$$= Z^{-m} \sum_{l=-\infty}^{\infty} x(l) Z^{-l} = Z^{-m} X(Z).$$

$$\cancel{= Z^{-m} \sum_{n=-\infty}^{\infty} x(l) Z^{-l}}$$

Let, $n - m = l$

$\therefore \; n = l + m$

when $n \to -\infty$; $l \to -\infty$

$n \to +\infty$; $l \to +\infty$

③ Time Reversal:-

If $x(n) \overset{Z}{\longleftrightarrow} X(Z)$

then $x(-n) \overset{Z}{\longleftrightarrow} X(Z^{-1})$

proof

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$$

$$\cancel{= \sum_{m=-\infty}^{\infty} x(m) Z^{-(l+m)}}$$

$$= \sum_{m=-\infty}^{\infty} x(m) (Z^{-1})^{-m} = X(Z^{-1})$$

~~$n - m = l$~~

~~$n = l + m$~~

$m = -n \Rightarrow m = \infty$ to $-\infty$

$n = -m \Rightarrow m = -\infty$ to ∞

④ Scaling :-

Let $x(n) \xleftrightarrow{z} X(z)$

then $a^n x(n) \longleftrightarrow X(z/a)$

proof

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} x(n) (a^{-1}z)^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{a}\right)^{-n} = X(z/a)$$

⑤ Differentiation property :-

If $x(n) \xleftrightarrow{z} X(z)$ then, $n x(n) \xleftrightarrow{z} -z \frac{d}{dz}\{X(z)\}$

proof

$$Z(x(n)) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = X(z)$$

$$\frac{d}{dz}(X(z)) = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right] = \sum_{n=-\infty}^{\infty} \frac{d}{dz}(x(n) z^{-n})$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{d}{dz}(z^{-n}) = \sum_{n=-\infty}^{\infty} x(n)(-n) z^{-n-1}$$

$$= \sum_{n=-\infty}^{\infty} x(n)(-n) z^{-n} z^{-1} = -\frac{1}{z} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$\therefore \frac{d}{dz}(X(z)) = -\frac{1}{z} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$ ~~$\frac{d}{dz}X(z) = -z\frac{d}{dz}(X(z))$~~

$$-z\frac{d}{dz}(X(z)) = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$-z\frac{d}{dz}(X(z)) = Z\{n x(n)\}$$

⑥ Convolution Property:-

$$Z[x_1(n) * x_2(n)] = X_1(z)\,X_2(z).$$

where, $x_1(n) \overset{Z}{\longleftrightarrow} X_1(z)$

$x_2(n) \overset{Z}{\longleftrightarrow} X_2(z)$

Proof

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k) . x_2(n-k)$$

$$\therefore\ Z[x_1(n) * x_2(n)] = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k) \cdot x_2(n-k) \right] z^{-n}$$

change the order of summation

$$= \sum_{k=-\infty}^{k=\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\ z^{-n} . z^{-k} . z^{+k}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\, z^{-k} . \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-(n-k)}$$

$$= X_1(z) . \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-(n-k)}$$

where, $n-k = m$; for $-\infty < m < \infty$

$$= X_1(z) \sum_{n=-\infty}^{\infty} x_2(m)\ z^{-m}$$

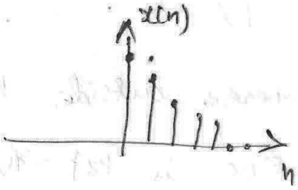
$$= X_1(z) . X_2(z) . //$$

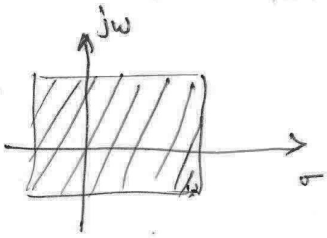
Region of Convergence (ROC):-

ROC of $X(z)$ is the set of all value of z for which. $X(z)$ attains finite value.

① Finite-Duration Signals:-

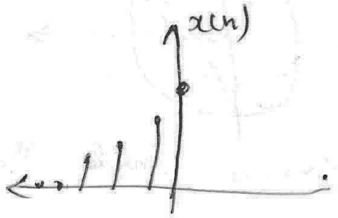
(a) causal (or) right sided sequence:

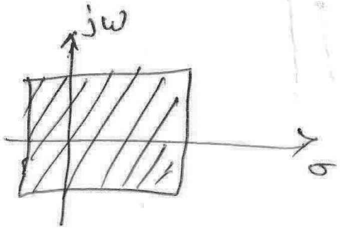




It converge entire Z-plane except $Z=0$

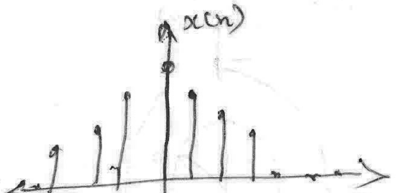
(b) Anticausal (or) left side sequence:

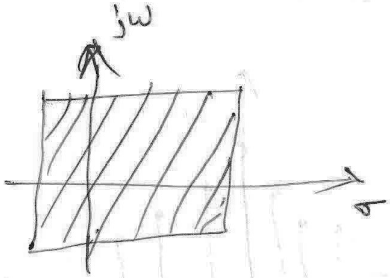




It converge entire Z-plane except $Z=\infty$

(c) Two sided sequence:-



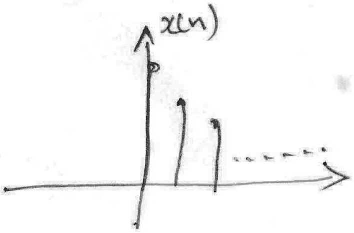


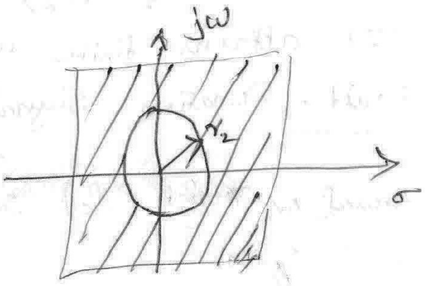
It converge entire Z-plane except at $Z=0$ and $Z=\infty$

Notes: In the Z-transform, the value of Z can be evaluated that is called the Region of Convergence (ROC). The ROC must always be mentioned along ~~the~~ with Z-transform.

For Infinite duration :-

④ Causal (or) right sided sequence :-

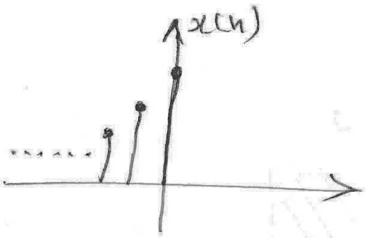


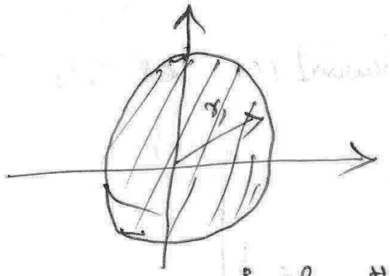


It converge outside the r_2 circle

i.e; ROC is $|z| > r_2$

⑤ ~~So~~ Anticausal (or) left sided Seq;

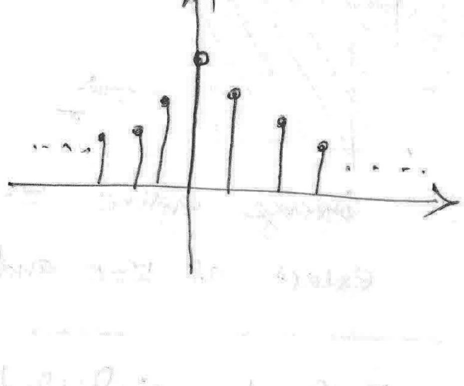


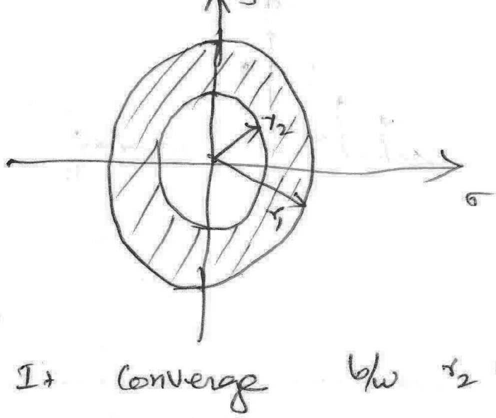


It Converge inside the r_1-circle.

i.e; ROC is $|z| < r_1$

⑥ Two-sided Sequences :-





It Converge b/w r_2 and r_1

i.e; ROC is ~~$r_2 > |z|$~~

$r_2 < |z| < r_1$

① $x(n) = \{\underset{\uparrow}{1}, 2, 5, 6, 1\}$

Sol

In this seq, all the values converge in the right side.

Hence ROC: Is Convergen entire z-plance except $z=0$.

$$x(n) = \{\underset{x(0)}{1}, \underset{x(1)}{2}, \underset{x(2)}{5}, \underset{x(3)}{6}, \underset{x(4)}{1}\}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=0}^{4} x(n)\, z^{-n} = x(0)z^{-0} + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3} + x(4)z^{-4}$$

$$= 1\cdot(1) + 2z^{-1} + 5z^{-2} + 6z^{-3} + 1\cdot z^{-4}$$

② $x(n) = \{1, 3, 5, 2, \underset{\uparrow}{1}\}$

Sol

$$X(z) = \sum_{n=-4}^{n=0} x(n)\, z^{-n} = x(-4)z^{+4} + x(-3)z^{+3} + x(-2)z^{+2} + x(-1)z^{+1} + x(0)z^{0}$$

~~$+x(0)$~~

$$= z^{4} + 3z^{3} + 5z^{2} + 2z^{1} + 1$$

∴ ROC is Converge entire z-plance except $z=\infty$

③ $x(n) = \{1, 3, \underset{\uparrow}{5}, 4, 1\}$

Result: $X(z) = z^{2} + 3z^{1} + 5 + 4z^{-1} + 1z^{-2}$

ROC: Converge entire z-plance except $z=0$ & $z=\infty$

Determine the z-transforms of the following seq.

④ $x(n) = u(n)$

$u(n) = 1$ for $n \geq 0$

$= 0$ for $n < 0$

so)

given signal have Infinite duration, $\therefore |z| > r_2$

ROC: ~~Converge entire z-plane except z=0~~ (or) $|z| > 1$

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n} \quad \text{~~}\ \cancel{x(0) z^{-0} + x(1)}$$

$x(n) = u(n)$] $= 1$

$$= \sum_{n=0}^{\infty} z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

[using finite geometric Series Sum formula.

$$\sum_{n=0}^{\infty} a^{n} = \frac{1}{1-a}$$

⑤ $x(n) = \left(\frac{1}{2}\right)^{n} u(n)$

so)

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

[$u(n) = 1$ for $n \geq 0$

$$= \sum_{n=0}^{\infty} (1/2)^{n} u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} \overset{(1/2\, z^{-1})^{n}}{\cancel{(1/2)^{n} z^{-n}}} = \cancel{\left(\frac{1}{2}\right)^{0} z^{-0} + (1/2)^{1} z^{-1} + (1/2) z^{-2} + \dots \infty}$$

[$\because 1 + r + r^{2} + r^{3} + \dots \infty = \frac{1}{1-r}$ if $|r| < 1$

$$= \cancel{1 + (1/2)^{1} z^{-1} + (1/2)^{2} z^{-2} + \dots + \infty}$$

$$= \frac{1}{1 + (1/2) z^{-1}} = \frac{1}{1 + \frac{1}{2} \cdot \frac{1}{z}} = \frac{1}{1 + \frac{1}{2z}} = \frac{2z}{2z + 1}$$

If ROC $|z| > 1$

Properties of Region of Convergence:- (ROC)

① The ROC is a ring or disk in the z-plane centered at the origin

② The ROC cannot contain any poles.

③ If $x(n)$ is a causal seq then the ROC is the entire Z-plane except at $Z=0$

④ If $x(n)$ is a non-causal seq. then the ROC is the entire Z-plane except at $Z = \infty$

⑤ If $x(n)$ is a finite duration, two-sided seq. the ROC is entire Z-plane except at $Z=0$ and $Z=\infty$

⑥ If $x(n)$ is an infinite duration, two-sided sequ the ROC will consist of a ring in the Z-plane, bound on the interior and exterior by a pole, not containg any poles.

⑦ The ROC of a LTI stable system contains the unit circle

⑧ The ROC must be a connected region.

~~Ramesh Babu~~
~~Page (2.18)~~

Page 3.26 (Nagoorkani)

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Some Common Z-transform pairs

Sequence	Transform	ROC
① $\delta(n)$	1	All z
② $u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
③ $-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
④ $\delta(n-m)$	z^{-m}	All z except 0 (if $m>0$) All z except ∞ (if $m<0$)
⑤ $a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
⑥ $-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
⑦ $n a^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
⑧ $-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
⑨ $[\cos\omega_0 n]u(n)$	$\frac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}}$	$\|z\|>1$
⑩ $[\sin\omega_0 n]u(n)$		
⑪ $r^n \cos\omega_0 n\, u(n)$		
⑫ $r^n \sin\omega_0 n\, u(n)$		

⑥ $x(n) = a^n u(n)$

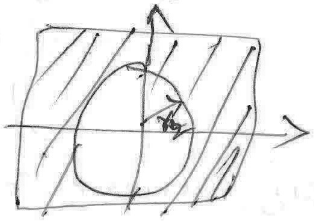
Sol $$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \cancel{a^0 z^{-0} + a^1 z^{-1} + a^2 z^{-2} + + + + \infty}$$

$$= 1 + (a^1 z^{-1})^1 + (a^2 z^{-1})^2 + \cdots \infty$$

$\because 1 + r + r^2 + \cdots = \frac{1}{1-r}$ If $|r| < 1$]

$$= \frac{1}{1 - az^{-1}} = \frac{1}{1 - \frac{a}{z}} = \frac{z}{z-a}$$



ROC ; $|z| > \cancel{a}_z$

⑦ $x(t) = t^2$

[T- sampling time period

Sol $x(\cancel{t}) = x(nT) = t^2 = n^2 T^2 = n^2 g(n)$

$$\therefore G(z) = Z\{g(n)\} = Z\{T^2\} = \sum_{n=0}^{\infty} T^2 z^{-n} = T^2 \sum_{n=0}^{\infty} (\cancel{z} z^{-1})^n$$

$\because 1 + r + r^2 + \cdots = \frac{1}{1-r}$]

$$= T^2 \cdot \frac{1}{1 - 1/z} = \frac{T^2 z}{z-1}$$

by the property of z transform (diff. property)

$$z\{x(n)\} = X(z) = \left(-z \frac{d}{dz}\right)^2 G(z) = -z \frac{d}{dz}\left(-z \frac{d}{dz} G(z)\right)$$

$$\frac{u\,dv - v\,du}{v^2}$$

$\cancel{z}T^2 - T^2 - T^2 z =$

$$= -z \frac{d}{dz}\left(-z \frac{d}{dz} \cdot \frac{T^2 z}{z-1}\right) = -z \frac{d}{dz}\left(-z \times \left(\frac{(z-1)T^2 - T^2 z}{(z-1)^2}\right)\right)$$

$$= -z \frac{d}{dz}\left(\frac{zT^2}{(z-1)^2}\right) = -z \times \frac{(z-1)^2 T^2 - zT^2 \times 2(z-1)}{(z-1)^4} = -z \times \frac{\cancel{(z-1)}(zT^2 - T^2 - 2zT^2)}{(z-1)^{\cancel{4}3}}$$

$$= -z \times \frac{-zT^2 - T^2}{(z-1)^3} = \frac{zT^2(z+1)}{(z-1)^3}$$

⑧ $x(n) = -a^n u(-n-1)$

$u(n) = 1$ for $n \geq 0$

$u(-n) = 1$ for $n \leq 0$

$u(-n-1) = 1$ for $n \leq -1$

$$\therefore X(z) = \sum_{n=-\infty}^{-1} -a^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -a^n z^{-n} = + \cancel{\sum_{n=+\infty}^{+1} (a z^{-1})^n} = -\sum_{n=1}^{\infty} a^{-n} z^n = -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

$$= -\sum_{n=1}^{\infty} (a^{-1} z)^n$$

By geometric series,

$$\sum_{n=1}^{\infty} a^n = (a + a^2 + a^3 + \cdots + a^{\infty})$$

$$= a(1 + a + a^2 + a^3 \cdots + a^{\infty})$$

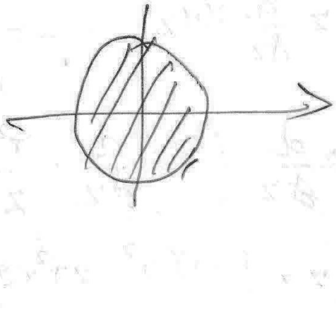
$$= a \cdot \frac{1}{1-a}$$

$\therefore \Sigma$

$$\therefore X(z) = -\frac{a^{-1} z}{1 - a^{-1} z}$$

$$= -\frac{1}{1 - a z^{-1}}$$

ROC; $|z| < r$



⑨ $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{3}\right)^n u(n)$

Sol

$$X(z) = \sum_{n=0}^{\infty} \left(\tfrac{1}{2}\right)^n z^{-n} + \sum_{n=0}^{\infty} \left(\tfrac{1}{3}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\tfrac{1}{2} z^{-1}\right)^n + \sum_{n=0}^{\infty} \left(\tfrac{1}{3} z^{-1}\right)^n$$

$$\left[\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right.$$

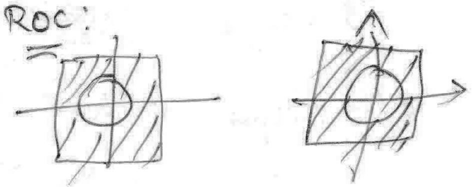
$$= \frac{1}{1-\frac{1}{2z}} + \frac{1}{1-\frac{1}{3z}} = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-\frac{1}{3}}$$

(or)

$$= \frac{2z}{2z-1} + \frac{3z}{3z-1}$$

$|z| > \frac{1}{2}$; $|z| > \frac{1}{3}$

ROC:



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⑩ $x(n) = -na^n u(-n-1)$

Sol

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \cancel{= \sum_{n=-\infty}^{-1} -n a^n z^{-n}} \cancel{= \sum_{n=1}^{\infty} n a^{-n} z^{n}}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2}$$

$$\left[-na^n u(-n-1) = \frac{az^{-1}}{(1-az^{-1})^2} \right.$$

$$= \frac{a}{z\left(1-\frac{a}{z}\right)^2} = \frac{a}{z\left(\frac{z-a}{z}\right)^2}$$

$$= \frac{az^2}{z(z-a)^2} = \frac{az}{(z-a)^2}$$

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Class Test - 2

① Define System.

② Calculate the minimum sampling freq. required for $x(t) = 0.5 \sin 50\pi t + 0.25 \sin 25\pi t$, so as to avoid aliasing.

Nyq. rate $\geq 2 f_{max}$

$\omega_1 = 50\pi$

Here $f_1 =$ ~~50~~ 25 Hz. $f_2 =$ ~~25~~ 12.5 Hz.

$f_1 = \frac{50\pi}{2\pi} = 25$

maxi. freq = 25 Hz.

$f_2 = \frac{25\pi}{2\pi} = 12.5$

Nyq. rate $\geq 2 \times 25 = 50$.

maxi. freq. required to avoid aliasing is ~~100~~ 50 Hz.

③ Define Symmetric and antisymmetric signal

✗ ④ Define system transfer fun.
- is the ratio of Z-transform of the o/p to Z-transform of the I/p.

⑤ Define Z-transform ; ⑤ Define time variant & Invariant system.

Part-B

⑥ proof : Linearity, Multiplication by n (or Differentiation in Z-domain)

⑦ $x(n) = u(n)$, $x(n) = 0.8^n u(-n-1)$, $x(n) = 0.3^n u(n) + 0.8^n u(-n-1)$

⑧ ~~$y(n) = x(n^2)$~~

$$h(n) = 2^n u(-n)$$
$$h(n) = \delta(n) + \sin n\pi$$
$$h(n) = \sum_{k=0}^{\infty} \left(\tfrac{3}{4}\right)^k x(n-k)$$

→ causal & stable. ~~Linear~~

→ $x(t) = \sin \Omega_0 t$

sol

$x(n) = \sin(\Omega_0 n T) = \sin \omega n$ $[\because \Omega_0 T = \omega$

using one sided z-transform, T- sampling time period

$$= \sum_{n=0}^{\infty} \frac{e^{j\omega n} - e^{-j\omega n}}{2j} z^{-n} \qquad \left[\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \right.$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j\omega n} - e^{-j\omega n}) z^{-n}$$

$$= \frac{1}{2j} \sum_{n=0}^{\infty} e^{j\omega n} z^{-n} - \frac{1}{2j} \sum_{n=0}^{\infty} e^{-j\omega n} z^{-n}$$

$$\left. \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \right] \qquad = \frac{1}{2j} \sum_{n=0}^{\infty} (e^{j\omega} z^{-1})^n - \frac{1}{2j} \sum_{n=0}^{\infty} (e^{-j\omega} z^{-1})^n \qquad \frac{3+2+1}{6}$$

$$= \frac{1}{2j} \cdot \frac{1}{1 - e^{j\omega} z^{-1}} - \frac{1}{2j} \frac{1}{1 - e^{-j\omega} z^{-1}}$$

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

2 | 2, 3, 6
3 | 1, 3, 3
  | 1, 1, 1

$$= \frac{1}{2j} \cdot \frac{z}{(z - e^{j\omega})} - \frac{1}{2j} \cdot \frac{z}{(z - e^{j\omega})}$$

$$\left. e^{j\theta} e^{-j\theta} = e^0 = 1 \right] \qquad = \frac{\cancel{2j}\, z(z - e^{-j\omega}) - \cancel{2j}\, z(z - e^{j\omega})}{\cancel{2j}\; 2j[(z - e^{j\omega})(z - e^{-j\omega})]} = \frac{\cancel{z^2} - ze^{-j\omega} - \cancel{z^2} + ze^{+j\omega}}{2j\,[z^2 - ze^{-j\omega} - ze^{j\omega} + e^{j\omega} e^{-j\omega}]}$$

$$\because \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j} \qquad = \frac{z(e^{j\omega} - e^{-j\omega})/2j}{z^2 - z(e^{j\omega} + e^{-j\omega}) + 1} = \frac{z \sin j\omega}{z^2 - z2\cos j\omega + 1}$$

$$\because \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

//

→ $\cancel{x(t}$ $x(n) = e^{-at} \cos \Omega_0 t$

sol

$$x(n) = e^{-anT} \cos \Omega_0 nT = e^{-anT} \cos wn$$

$$X(z) = z\{x(n)\} = \sum_{n=0}^{\infty} e^{-anT} \cos wn \cdot z^{-n} = \sum_{n=0}^{\infty} e^{-anT} \left(\frac{e^{jwn} + e^{-jwn}}{2} \right) \cancel{z}^{-n}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-a\cancel{n}T} e^{jw\cancel{n}} z^{-1} \right)^n + \frac{1}{2} \sum_{n=0}^{\infty} \left(e^{-aT} e^{-jw} z^{-1} \right)^n$$

$$= \frac{1}{2} \cdot \frac{1}{1 - e^{-aT} e^{jw} z^{-1}} + \frac{1}{2} \frac{1}{1 - e^{-aT} e^{-jw} z^{-1}}$$

$$= \frac{1}{2} \cdot \left[\frac{1}{1 - \frac{e^{jw}}{z \cdot e^{aT}}} + \frac{1}{1 - \frac{e^{-jw}}{e^{aT} z}} \right]$$

$$= \frac{1}{2} \left[\frac{z e^{aT}}{z e^{aT} - e^{jw}} + \frac{z e^{aT}}{z e^{aT} - e^{-jw}} \right]$$

$$= \frac{1}{2} \left[\frac{z e^{aT} (z e^{aT} - e^{-jw}) + z e^{aT} (z e^{aT} - e^{jw})}{(z e^{aT} - e^{jw})(z e^{aT} - e^{-jw})} \right]$$

$$= \frac{z e^{aT}}{2} \left[\frac{z e^{aT} - e^{-jw} + z e^{aT} - e^{jw}}{(z e^{aT})^2 - z e^{aT} e^{-jw} - z e^{aT} e^{jw} + e^{jw} e^{-jw}} \right]$$

$$= \frac{z e^{aT}}{2} \left[\frac{2 z e^{aT} - (e^{jw} + e^{-jw})}{z^2 e^{2aT} - z e^{aT} (e^{jw} + e^{jw}) + 1} \right] = \frac{z e^{aT} (z e^{aT} - \cos w)}{\cancel{2} z^2 e^{2aT} - 2 z e^{aT} \cos w + 1}$$

//

Inverse Z- Transform:-

It is converted to Z-domain into time domain.

ie; $X(z) \longrightarrow x(n)$ is called Inverse Z- transform

(or)

* 3- different ~~X(z)~~ Methods are using,

(1) power Series expansion ← (Long Division Methods) using in large class function

(2) partial fraction expansion - X(z) is the form of a partial function

(3) Contour Integration. ← - using in large class function.

(4) Residue Method

(1) Long Division Method :-

steps to be followed to find Inverse Z-transform.

(a) Always express the numerator and denominator (or) X(z) as a polynomial in z (or) z^{-1}

(b) for a left-sided signal, the numerator & demoninator are to be arranged according to increasing power of z (or) decreasing power of z^{-1} ie; in the order $z, z^2, z^3, \ldots$

(c) For a right-sided signal, the numerator & demoniator polynomials are to be arranged according to decreasing Power of z (or) Increasing power of z^{-1} ie; $z^{-1}, z^{-2}, z^{-3}, \ldots$

→ By using long division method, find the Inverse Z-transform.

$$X(z) = \frac{z+0.2}{(z+0.5)(z-1)}$$

Sol

$$X(z) = \frac{z+0.2}{(z+0.5)(z-1)} = \frac{z+0.2}{z^2 - z + 0.5z - 0.5} = \frac{z+0.2}{z^2 - 0.5z - 0.5}$$

$$\begin{array}{r|l}
 & z^{-1} + 0.7z^{-2} \\
\hline
z^2 - 0.5z - 0.5 & z + 0.2 \\
 & z + 0.5 - 0.5z^{-1} \\
 & (-)\quad (+)\quad (+) \\
\cline{2-2}
 & 0.7 + 0.5z^{-1} \\
 & 0.7 - 0.35z^{-1} - 0.35z^{-2} \\
 & (-)\quad (+)\quad (+) \\
\cline{2-2}
 & 0.85z^{-1} + 0.35z^{-2} \\
 & 0.85z^{-1} - 0.425z^{-2} - 0.425z^{-3} \\
 & (-)\quad (+)\quad (+) \\
\cline{2-2}
 & 0.775z^{-2} + 0.425z^{-3}
\end{array}$$

$$X(z) = z^{-1} + 0.7z^{-2} + 0.85z^{-3} + \cdots\cdots = \sum_{n=0}^{\infty} x(n) z^{-n}$$

$x(0) = 0$, $x(1) = 1$, $x(2) = 0.7)$, $x(3) = 0.85$ so on.

→ Given $X(z) = \frac{1}{1-az^{-1}}$, $|z| > |a|$, Find $x(n)$ by long division method.

Sol

$$X(z) = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$$

$$z-a \;\big)\; z$$

$$\overset{(-)}{z} \overset{(+)}{-a}$$

$$a$$

$$\overset{(-)}{a} \overset{(+)}{-a^2z^{-1}}$$

$$a^2z^{-1}$$

$$\overset{(-)}{a^2z^{-1}} \overset{(+)}{-a^3z^{-2}}$$

$$a^3z^{-2}$$

$$a^3z^{-2} - a^4z^{-3}$$

$$\cdots$$

$$\cdots$$

It is right side seq, ~~is $x(0) = 1$~~

$$X(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots = \sum_{n=0}^{\infty} x(n)z^{-n}$$

$$x(0) = 1, \quad x(1) = a, \quad x(2) = a^2, \quad x(3) = a^3 = \cdots$$

→ $X(z) = \dfrac{z}{(z-3)(z-4)}$

sol) $X(z) = \dfrac{z}{(z-3)(z-4)} = \dfrac{z}{z^2-7z+12}$

$$\frac{z}{12} + \frac{7z^2}{144} + \frac{37}{1728}z^3$$

$12-7z+z^2$ ~~$z^2-7z+12$~~

$$z$$

$$\overset{(-)}{z} \overset{(+)}{-\frac{7z^2}{12}} \overset{(-)}{+\frac{z^3}{12}}$$

$$\frac{7z^2}{12} - \frac{z^3}{12}$$

$$\overset{(-)}{\frac{7z^2}{12}} \overset{(+)}{-\frac{49z^3}{144}} \overset{(-)}{+\frac{7z^4}{144}}$$

$$\frac{37z^3}{144} - \frac{7z^4}{144}$$

$$\frac{37z^3}{144} - \frac{259z^4}{1729} + \frac{37}{1728}z^5$$

$$X(z) = \frac{1}{12}z + \frac{7}{144}z^2 + \frac{37}{1728}z^3 + \cdots$$

$x(0) = 0$

$x(-1) = \dfrac{1}{12}$

$x(-2) = 7/144$

$x(-3) = \dfrac{37}{1728}$ so on.

(2) Partial fraction Method:-

steps to be followed,

(1) Multiply the numerater and denominater of X(z) by the same power of z to elimate the negative power of Z.

(2) Dividad each co-efficient of the numerater and denominater of X(z) by the co-effi. of the highest power of z in the denominater.

(3) Multiply X(z) by $\frac{1}{z}$

(4) Factor the denominater.

(5) Find the partial fraction expansion of X(z)/z.

(6) Multiply X(z)/z by z to get back X(z). then find Inverse z- transform.

Note:- Partial function
- (1) Function with seperate pole.
- (2) Function with multiple pole.
- (3) Function with complex conjugate pole.

(1) Fun. with seperate poles:

$$X(z) = \frac{K}{z(z+P_1)(z+P_2)} = \frac{A}{z} + \frac{B}{z+P_1} + \frac{C}{z+P_2}$$

$$\therefore A = X(z)z\Big|_{z=0} \;;\; B = X(z)(z+P_1)\Big|_{z=-P_1} \;;\; B = X(z)(z+P_2)\Big|_{z=-P_2}$$

(2) Function with Multiple poles.

$$X(z) = \frac{K}{z(z+P_1)(z+P_2)^2} = \frac{A}{z} + \frac{B}{(z+P_1)} + \frac{C}{(z+P_2)^2} + \frac{D}{(z+P_2)}$$

$$A = X(z)z\Big|_{z=0} \;;\; B = X(z)(z+P_1)\Big|_{z=-P_1} \;;\; C = X(z)(z+P_2)^2\Big|_{z=-P_2} \;;\; D = X(z)(z+P_2)\Big|_{z=-P_2}$$

(3) Fun. with Complex Conjugate poles:

$$X(z) = \frac{K}{(z-P_1)(z^2+bz+c)} = \frac{A}{(z-P_1)} + \frac{Bs+C}{z^2+bz+C}$$

$$\rightarrow \quad X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{N(z)}{D(z)} = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} \quad \begin{matrix} \text{ie; } M=1 \\ \text{ie; } N=2 \end{matrix} \Bigg\} \text{Polynomial form.}$$

sol)

To Elimate negative power of z, we multiply both numerator & denominator by z^2, thus,

$$X(z) = \frac{z^2+3z}{z^2+3z+2} = \frac{z(z+3)}{(z+1)\ (z+2)}$$

2
1 2
$(z-1)$ $(z \cdot 2)$
denominator,
ie; $z^2 + z + 2z + 2$
z^2+3z+2

denominator in factor form,

2
$2 \times 1 = 2$
$2 + 1 = 3$

÷ z (both side)

$$\frac{X(z)}{z} = \frac{(z+3)}{(z+1)(z+2)}$$

$$\frac{X(z)}{z} = \frac{A}{(z+1)} + \frac{B}{(z+2)}$$

$$A = (z+1)\left.\frac{X(z)}{z}\right|_{z=-1} = (z+1) \cdot \frac{(z+3)}{(z+1)(z+2)}$$

Put $z=-1$

$$A = \frac{2}{1} = 2$$

$$B = (z+2)\frac{X(z)}{z}\Bigg|_{z=-2}$$

$$= (z+2)\,\frac{(z+3)}{(z+1)(z+2)}$$

Put, $z=-2$

$$B = \frac{1}{-1} = -1$$

$$\frac{X(z)}{z} = \frac{2}{(z+1)} + \frac{1}{(z+2)}$$

Multiply $X(z)/z$ by z,

$$X(z) = \frac{2z}{(z+1)} - \frac{z}{z+2}$$

$$a^n u(n) \longrightarrow \frac{z}{z-a}$$

$$u(n) \longrightarrow \frac{z}{z-1}$$

$$= 2.\frac{z}{(z+1)} - \frac{z}{z+2}$$

$$x(n) = 2.(-1)^n u(n) - (-2)^n u(n)$$

$$\boxed{x(n) = 2(-1)^n u(n) - (-2)^n u(n)}$$

→ Determine the inverse Z-transform of the following seq.

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

For following ROC : (1) ROC : $|z|>1$

(2) ROC : $|z|<0.5$

(3) ROC : $0.5<|z|<1$

Note:-

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0+b_1z^{-1}+\cdots\cdots+b_Mz^{-M}}{a_0+a_1z^{-1}+\cdots\cdots+a_Nz^{-N}}$$

Here, $a_0=1$ ⇒ (1) If $a_0 \neq 1$ - Polynomials are adjusted Properly.

(2) If $a_N \neq 0$ & $M<N$ - It is called proper form

Otherwise $X(z)$ is adjusted proper.

Sol

$N(z)=1$, $M=0$

$D(z) = 1-1.5z^{-1}+0.5z^{-2}$ ⇒ $N=2$, $a_0=1$ & $a_N=0.5$

Hence, $M<N$, $a_N \neq 0$, $a_0=1$, then it is proper form.

* Multiply Numerator & denominator by z^2, thus (ie; $N=2 \Rightarrow z^2$)

$$X(z) = \frac{z^2}{z^2} \times \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

$$= \frac{z^2}{z^2-1.5z+0.5}$$

* Hence the denominator can be factored as

$$z^2-1.5z+0.5 = (z-1)(z-0.5)$$

write in equ, $X(z) = \frac{z^2}{(z-1)(z-0.5)}$

0.5

1 0.5

(z-1) (z-0.5)

partial fraction expansion,

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)} = \frac{A_1}{z-1} + \frac{A_2}{z-0.5}$$

$$A_1 = (z-1) \times \frac{X(z)}{z}\Bigg|_{z=1} = (z-1) \cdot \frac{z}{(z-1)(z-0.5)} = \frac{1}{0.5} = 2_{//}$$

$$A_2 = (z-0.5) \times \frac{X(z)}{z}\Bigg|_{z=0.5} = (z-0.5) \cdot \frac{z}{(z-1)(z-0.5)} = \frac{0.5}{-0.5} = -1_{//}$$

$$\therefore \frac{X(z)}{z} = \frac{2}{z-1} - \frac{1}{z-0.5}$$

$$X(z) = \frac{2z}{z-1} - \frac{z}{z-0.5} = \frac{2}{1-z^{-1}} - \frac{1}{1-0.5z^{-1}}$$

Now, calculate x(n) obtained on the basic of ROC seq.

(i) $x(n)$ for ROC : $|z| > 1$

i.e, it is causal seq $\Rightarrow a^n u(n) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$

$$\therefore x(n) = IZT[X(z)] = 2\, IZT\left[\frac{1}{1-z^{-1}}\right] - IZT\left[\frac{1}{1-0.5z^{-1}}\right]$$

$$= 2(1)^n u(n) - (0.5)^n u(n)_{//}$$

= ~~E2~~

(ii) $x(n)$ for ROC : $|Z| < 0.5$, anti causal sequence, i.e; $|Z| < 1$

Formula: $-a^n u(-n-1) \xleftrightarrow{z} \frac{1}{1-az^{-1}}$; ROC : $|Z| < |a|$

$$x(n) = IZT[X(Z)] = 2\, IZT\left[\frac{1}{1-z^{-1}}\right] - IZT\left[\frac{1}{1-0.5Z^{-1}}\right]$$

$$= 2\left[-(+1)^n u(-n-1)\right] - \left[-(0.5)^n u(-n-1)\right] //$$

(iii) $x(n)$ for ROC : $0.5 < |Z| < 1$

$$x(n) = IZT\{X(Z)\} = 2\, IZT\left\{\frac{1}{1-(1.)Z^{-1}}\right\} - IZT\left\{\frac{1}{1-0.5Z^{-1}}\right\}$$

i.e; $|Z|$ is now annual causal i.e; having causal & non causal,

It is overlapping of ROC in $0.5 < |Z|$ and $|Z| < 1$,

So you can write,

$$2\, IZT\left\{\frac{1}{1-(1)Z^{-1}}\right\} = 2(-1)^n u(-n-1) \quad \text{for ROC } |Z| < 1$$

$$IZT\left\{\frac{1}{1-0.5Z^{-1}}\right\} = (0.5)^n u(n)$$

$$\therefore x(n) = 2(-1)^n u(-n-1) - (0.5)^n u(n) //$$

→ $X(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}}$

sol

$$X(z) = \frac{3+2z^{-1}+z^{-2}}{1-3z^{-1}+2z^{-2}} = \frac{z^{-2}(3z^2+2z+1)}{z^{-2}(z^2-3z+2)} = \frac{3z^2+2z+1}{(z-1)\ (z-2)}$$

$$\frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z-1)\ (z-2)}$$

partial Fraction Expansion Method:-

$$\frac{X(z)}{z} = \frac{3z^2+2z+1}{z(z-1)(z-2)} = \frac{A_1}{z} + \frac{A_2}{z-1} + \frac{A_3}{z-2}$$

Now,

$$A_1 = z \times \frac{X(z)}{z} = \cancel{z} \cdot \left.\frac{3z^2+2z+1}{\cancel{z}(z-1)(z-2)}\right|_{z=0} = \frac{0+0+1}{(0-1)\ (0-2)} = 0.5$$

$$A_2 = (z-1)\left.\frac{X(z)}{z}\right|_{z=1} = (z-1)\cdot\frac{3z^2+2z+1}{z(z-1)(z-2)} = \frac{3+2+1}{1\times(-1)} = -6$$

$$A_3 = (z-2)\left.\frac{X(z)}{z}\right|_{z=2} = 8.5$$

$$\therefore \frac{X(z)}{z} = \frac{0.5}{z} - \frac{6}{z-1} + \frac{8.5}{z-2}$$

$$X(z) = 0.5 - 6\frac{z}{z-1} + 8.5\frac{z}{z-2}$$

$$x(n) = 0.5\ \delta(n) - 6\ u(n) + 8.5\ \frac{z}{z-2}$$

$\because$ $\delta(n) = 1$

$u(n) \xrightarrow{z} \frac{z}{z-1}$

$a^n u(n) \xrightarrow{z} \frac{z}{z-a}$

P. N - 4.7

Fourier Transform of Discrete-Time Signal (DTFT)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}.$$

$e^{j\omega n}$

$F\{x(n)\}$

where,

$x(n)$ - Discrete signal is time domain

$X(\omega)$ - Discrete signal is freq. domain.

Where, freq. ω is continuous freq. range from $-\pi$ to π (or) 0 to 2π

* The Inverse Fourier Transform is given as

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)\, e^{j\omega n}\, d\omega.$$

Discrete Fourier series :-

The Discrete time "periodic" signals are represented by discrete Fourier Series. The discrete time periodic signal with period 'N' ~~have~~ is defined as,

$$x(n+N) = x(n)$$

The discrete Fourier series of $x(n)$ is given as

$$x(n) = \sum_{k=0}^{N-1} C(k)\, e^{\frac{j2\pi kn}{N}}$$

where, $C(k)$ - are the Co-effi. of Series (called as Fourier Co-effi)

$$C(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

→ Determine Fourier transform of following signal,

(i) $x(n) = a^n u(n)$

Sol)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= \frac{1}{1-ae^{-j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-a}$$

(ii) $x(n) = u(n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=0}^{\infty} e^{-j\omega n} = \frac{1}{1-e^{-j\omega n}}$$

$x(n) = \delta(n)$

(iii) $$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n}$$

$$= \delta(0) e^{0} = 1,$$

(iv) $x(n) = a^n u(-n)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = \sum_{n=-\infty}^{\infty} a^n u(-n) e^{-j\omega n} = \sum_{n=-\infty}^{0} a^n e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{-n} e^{j\omega n} = \sum_{n=0}^{\infty} (a^{-1} e^{-j\omega})^n = \frac{1}{1-a^{-1}e^{j\omega}} = \frac{a}{a-e^{j\omega}}$$

Inverse DFFT

* The IDFFT can be defined as $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\cancel{e}^{j\omega}) e^{j\omega n} d\omega$

→ Determine the discrete time seq. where Fourier transform is given as,

$$X(\omega) = \frac{1}{1-a\cos\omega + ja\sin\omega}$$

Sol

$$X(\omega) = \frac{1}{1-a\cos\omega + ja\sin\omega} = \frac{1}{1-a(\cos\omega + j\sin\omega)} = \frac{1}{1-ae^{j\omega}}$$

$$X(\omega) = \frac{1}{1-ae^{j\omega}} = a^n u(n)$$

Hence $x(n) = a^n u(n)$,

→ ~~The Inver~~ Determine the discrete time seq. where Fourier transform is given as. $X(\omega) = \begin{cases} 1 & \text{for} \quad -\omega_c \leq \omega \leq \omega_c \\ 0 & \text{for} \quad \omega_c < \omega < \pi \end{cases}$

Sol

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} (1) e^{j\omega n} d\omega = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn}\right] = \frac{1}{n\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{j2}\right]$$

by Euler identify we can write the above eqn. as

$$x(n) = \frac{1}{n\pi} \sin(\omega_c n) \quad \text{for } n \neq 0$$

for $n=0$, $\cancel{x(0) = \frac{1}{(0)\pi} \sin}$

$$x(0) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} (1) e^{0} \, d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} d\omega = \frac{1}{2\pi} [\omega]_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi} [\omega_c - (-\omega_c)]$$

$$= \frac{1}{\cancel{2}\pi} [\cancel{2}\omega_c]$$

$$= \frac{\omega_c}{\pi}$$

$$\boxed{x(0) = \frac{\omega_c}{\pi}}$$

Hence,

$$\boxed{\begin{aligned} x(n) &= \frac{\omega_c}{\pi} \quad \text{for } n=0 \\ x(n) &= \frac{\sin \omega_c n}{n\pi} \quad \text{for } n \neq 0 \end{aligned}}$$

$\frac{e^{+j2\omega} + e^{-j2\omega}}{2}$ ↗

→ Determine the Inverse DTFT of $X(\omega) = 2\cos 2\omega$ for $-\pi < \omega < \pi$

→ Find the IDTFT of $X(\omega) = \frac{5}{1+6e^{-j\omega}+8e^{-2\omega}}$

sol:

$$X(\omega) = \frac{5}{1+6e^{-j\omega}+8e^{-2\omega}} = \frac{5}{(1+2e^{-j\omega})(1+4e^{-j\omega})}$$

$$= \frac{A}{1+2e^{-j\omega}} + \frac{B}{1+4e^{-j\omega}}$$

$\longleftarrow$ $A = -5$, $B = 10$

$x(n) = [-5(-2)^n + 10(-4)^n] u(n) \longleftarrow$