UNIT- 3

Discrete Fourier Transform & Computation

Discrete Time Fourier Transform:- (DTFT (or) DFT)

It is used to convert the N-point time domain sequences [$x(n)$] to an N-point freq. domain sequences [$X(k)$].

The N-point DFT of a finite duration sequence $x(n)$ of length L, where $L \leq N$ is defined as,

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \quad ; \text{ for } k = 0, 1, 2 \ldots (N-1)$$

IDFT (Inverse DFT)

IDFT is used to convert N-point frequency domain sequence $X(k)$ to an N-point time domain sequences. The IDFT of the sequence $X(k)$ of length N is defined as.

$$x(n) = \frac{1}{N} \sum_{n=0}^{N-1} X(k) e^{j2\pi kn/N} \quad ; \text{ for } k = 0, 1, 2, \ldots (N-1).$$

→ Calculate the DFT of the Sequence $x(n) = \{1, 1, -2, -2\}$

Sol

The N - point DFT of $x(n)$ is given by,

$$\text{DFT}\{x(n)\} = X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi nk/N} \quad ; \text{ for } k = 0, 1, 2, \ldots (N-1)$$

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{-j2\pi nk/4} = \sum_{n=0}^{3} x(n)\, e^{-j2\pi nk/4}$$

$e^{-j\theta} = \cos\theta - j\sin\theta$

$e^{j\theta} = \cos\theta + j\sin\theta$

$$= x(0)\, e^{0} + x(1)\, e^{-j\frac{\pi}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{-j\frac{3\pi k}{2}}$$

; for $k = 0, 1, 2, 3.$

→ Compute the DFT of the Sequence $x(n) = \{0, 1, 2, 3\}$, sketch the magnitude and phase Spectrums.

Sol

$N = 4 \quad ; \quad 0 \le k \le N-1 \quad$ ie; $k = 0$ to 3

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi nk/N} = x(0)\, e^{0} + x(1)\, e^{-j\frac{\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{-j\frac{3\pi k}{2}}$$

$$= 0 + e^{-j\frac{\pi k}{2}} + 2e^{-j\pi k} + 3e^{-j\frac{3\pi}{2}k}$$

$$= \left(\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right) + 2\left(\cos\pi k - j\sin\pi k\right) + 3\left(\cos\frac{3\pi}{2}k - j\sin\frac{3\pi}{2}\right.$$

when,

real → Polar

$k=0$, $X(0) = (\cos 0 - j\sin 0) + 2(\cos 0 - j\sin 0) + 3(\cos 0 - j\sin 0) = 1 + 2(1) + 3(1) = 6\angle 0$

$k=1$, $X(1) = \left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right) + 2\left(\cos\pi - j\sin\pi\right) + 3\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$

$= (0 - j(1)) + 2(-1 - j(0)) + 3(0 - (-j(1))) = -j - 2 + 3j = -2 + 2j$

$= -2 + 2j = 2.8\angle 135^\circ \times \frac{\pi}{180^\circ} = 2.8\angle 0.75\pi$

real → Polar

Similiarly. $= (\cos\pi - j\sin\pi) + 2(\cos 2\pi - j\sin 2\pi) + 3(\cos 3\pi - j\sin 3\pi) = -1+2-3 = -2$

$= 2\angle 180^\circ$

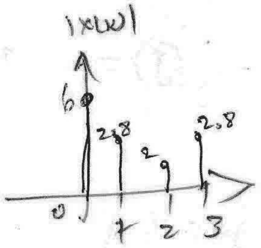
$k=2, \quad X(2) = 2\angle 180^\circ \times \frac{\pi}{180^\circ} = 2\angle\pi$

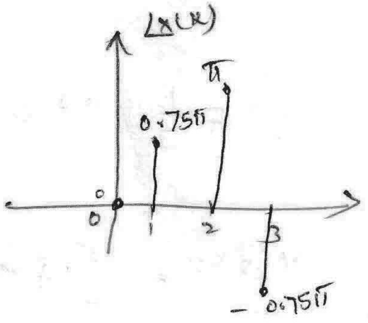
$k=3, \quad X(3) = 2.8\angle -135 \times \frac{\pi}{180^\circ} = 2.8\angle -0.75\pi \quad (-j-2-j3 = -2-j2$

$= 2.8\angle -135$

magn. fun, $|X(k)| = \{6, 2.8, 2, 2.8\}$

phase fun, $\angle X(k) = \{0, 0.75\pi, \pi, -0.75\pi\}$





→ Compute 4-point DFT ~~and 8-point DFT of causal~~ three sample sequences given by,

$x(n) = \frac{1}{3}; \quad 0 \le n \le 2$

$= 0; \quad$ else

$\therefore x(n) = \frac{1}{3}$; for $n = 0, 1, 2$

Show that DFT co-efficients are samples of Fourier transform of $x(n)$,

Sol → by the definition of N-point DFT, the k^{th} complex co-effi: of $X(k)$, for $0 \le k \le N-1$, is

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi k n}{N}} \quad ; \text{ for } k = 0, 1, 2 \ldots N-1$$

where, $N=4$, therefore the 4-point DFT is,

$$= \sum_{n=0}^{4-1} \left(\frac{1}{3}\right) e^{-j\frac{2\pi k n}{4}} = \frac{1}{3}\sum_{n=0}^{2} e^{-j\frac{2\pi k n}{4}} + (0)\, e^{-j\frac{2\pi k n}{4}}$$

$$= \frac{1}{3}\left[e^{0} + e^{-j\frac{\pi}{2}k} + e^{-j\pi k}\right] + (0)\, e^{-j\,3\pi k/2}$$

$$= \frac{1}{3}\left[1 + \cos\frac{\pi}{2}k - j\sin\frac{\pi}{2}k + \cos\pi k - j\sin\pi k\right] + 0$$

for $k = 0, 1, 2, 3$

~~Circular Convolution:-~~

when $K=0$, $X(0) = \frac{1}{3}\left[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0\right]$

$$= \frac{1}{3}(1 + 1 - j0 + 1 - j0) = \frac{3}{3} = 1 = 1\angle 0$$

when $K=1$, $X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi\right]$

$$= \frac{1}{3}\left[1 + 0 - j(1) + 1 - j(0)\right] = \frac{1}{3}(-j) = \frac{1}{3}\angle -\pi/2$$

when $K=2$, $X(2) = \frac{1}{3}\left[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi\right]$

$$= \frac{1}{3}\left[1 - 1 - j(0) + 1 - j(0)\right] = \frac{1}{3}\cancel{\angle 0} = \frac{1}{3}\angle 0$$

when $K=3$, $X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi\right]$

$$= \frac{1}{3}\left[1 + 0 + j - 1 - j0\right] = j\frac{1}{3} = \frac{1}{3}\angle\pi/2$$

∴ The 4-point DFT seq. of $x(n)$ is,

$$X(K) = \left\{1\angle 0, \frac{1}{3}\angle -\pi/2, \frac{1}{3}\angle 0, \frac{1}{3}\angle \pi/2\right\}$$

mag. $\Rightarrow |X(K)| = \left\{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$

phase $\Rightarrow \angle X(K) = \left\{0, -\pi/2, 0, \pi/2\right\}$

→ Find out the IDFT of the sequ. X(k).

$$X(K) = \{2, 1-j, 0, 1+j\}$$

Sol) $N = 4$.

$$x(n) = \frac{1}{N}\sum_{K=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad n = 0, 1, 2 \cdots N-1$$

$$x(n) = \frac{1}{4}\sum_{K=0}^{3} X(k)\, e^{\frac{j2\pi kn}{4}}$$

Put $k = 0, 1, 2, 3$, we get.

$$x(n) = \frac{1}{4}\left[x(0)e^{0} + x(1)\, e^{\frac{j\pi n}{2}} + x(2)\, e^{j\pi n} + x(3)\, e^{\frac{j3\pi n}{2}}\right]$$

~~we ha~~ $= \frac{1}{4}\left[2e^{0} + (1-j)e^{\frac{j\pi n}{2}} + 0\, e^{j\pi n} + (1+j)e^{\frac{j3\pi n}{2}}\right]$

$$= \frac{1}{4}\left[2 + (1-j)e^{\frac{j\pi n}{2}} + (1+j)e^{\frac{j3\pi n}{2}}\right]$$

Now put, $n = 0, 1, 2, 3$ we get,

Put $n = 0 \Rightarrow x(0) = \frac{1}{4}\left[2 + (1-j)\, e^{0} + (1+j)e^{0}\right]$

$$= \frac{1}{4}\left[2 + 1 - \cancel{j} + 1 + \cancel{j}\right]$$

$$= \frac{4}{4} = 1$$

Put $n=1$,

$$x(1) = \frac{1}{4}\left[2 + (1-j)e^{j\frac{\pi}{2}} + (1+j)e^{j\frac{3\pi}{2}}\right]$$

$$= \frac{1}{4}\left[2 + (1-j)\left[\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right] + (1+j)\left(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}\right)\right]$$

$$= \frac{1}{4}\left[2 + (1-j)(0+j) + (1+j)(0-j)\right]$$

$$= \frac{1}{4}\left[2 + j - j^2 - j - j^2\right] \qquad \left[\because j^2 = -1\right.$$

$$= \frac{1}{4}\left[2 \cancel{+j} + 1 \cancel{-j} + 1\right] = \frac{4}{4} = 1$$

$$n=2,\ x(2) = \frac{1}{4}\left[2 + (1-j)e^{j\pi} + (1+j)e^{j3\pi}\right]$$

$$= \frac{1}{4}\left[2 + (1-j)(\cos\pi + j\sin\pi) + (1+j)(\cos 3\pi + j\sin 3\pi)\right]$$

$$= \frac{1}{4}\left[2 + (1-j)(-1 + j(0)) + (1+j)(-1 + j0)\right]$$

$$= \frac{1}{4}\left[2 - 1 \cancel{+j} - 1 \cancel{-j}\right] = 0$$

$$n=3,\ x(3) = \frac{1}{4}\left[2 + (1-j)e^{j\frac{3\pi}{2}} + (1+j)e^{j\frac{9\pi}{2}}\right]$$

$$= \frac{1}{4}\left[2 + (1-j)\left(\cos\frac{3\pi}{2} + j\sin\frac{3\pi}{2}\right) + (1+j)\left(\cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2}\right)\right]$$

$$= \frac{1}{4}\left[2 + (1-j)(0-j) + (1+j)(0+j)\right]$$

$$= \frac{1}{4}\left[2 \cancel{-j} + j^2 \cancel{+j} + j^2\right] = \frac{1}{4}\left[2 - 1 - 1\right] = \frac{1}{4}\left[2-2\right] = 0$$

$$\therefore x(n) = \{1, 1, 0, 0\}$$

*. Properties of DFT:-

① periodicity:-

If a seq. $x(n)$ is periodic with periodicity of N samples then N-point DFT, of the Sequ. is also periodic with periodicity of N-samples.

Hence, if $x(n)$ and $X(k)$ are an N-point DFT pair, then,

$x(n+N) = x(n)$; for all n

and $X(k+N) = X(k)$; for all k.

Proof: by definition of DFT, the $(k+N)^{th}$ co-effi. of $X(k)$ is,

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi n(k+N)/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} e^{-j2\pi nN/N}$$

$$= \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N}$$

$$= X(k)_{\parallel}$$

$\left[e^{-j2\pi n} = \cos 2\pi n - j\sin 2\pi n \right.$

when, n - Integer

$\cos 2\pi n = 1$

$\sin 2\pi n = 0$

Note:- $X(k) = X_r(k) + jX_i(k)$ $[X_r$ - real part of $X(k)$, $X_i(k)$ - Imag. Part of $X(k)$.

mag. fun:

$|X(k)|^2 = X(k)X^*(k)$

(or)

$= \sqrt{X(k)X^*(k)}$

Alternative.

$|X(k)|^2 = X(k)X^*(k) = [X_r(k) + jX_i(k)][X_r(k) - jX_i(k)]$

$= X_r^2(k) + X_i^2(k) = \sqrt{X_r^2(k) + X_i^2(k)}$

phase fun:

$\angle X(k) = \text{Arg}[X(k)]$

$= \tan^{-1}\left[\frac{X_i(k)}{X_r(k)}\right]$

$$DFT\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=0}^{N-1} \{a_1 x_1(n) + a_2 x_2(n)\} e^{-j2\pi kn/N}$$

$$= a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j2\pi kn/N} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{-j2\pi kn/N} = a_1 X_1(k) + a_2 X_2(k)$$

(2) Linearity:-

If $DFT\{x_1(n)\} = X_1(K)$ and $DFT\{x_2(n)\} = X_2(K)$

Then for any real-valued or complex valued constant a_1 & a_2.

$$DFT\{a_1 x_1(n) + a_2 x_2(n)\} = a_1 X_1(K) + a_2 X_2(K)$$

(3) DFF of even and odd Sequence:-

The DFT of an even sequence is purely real. The DFT of an odd sequence is purely Imaginary. Therefore DFT can be evaluated using cosine and sine transforms for even and odd seq respectively.

For even sequ, $X(K) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi nK}{N}\right)$

For odd sequ, $X(K) = \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi nK}{N}\right)$

(4) DFT of Complex Conjugate Sequence:

Let $x(n)$ be a complex N-point discrete sequence and $x^*(n)$ be its conjugate sequence.

Let, $DFT\{x(n)\} = X(K)$

Now $DFT\{x^*(n)\} = X^*((-K))_N$ (or) $X^*(N-k)$

(5) Magnitude and phase function:-

The DFT of a Seq. is a complex quantity and so it can be ~~Expr~~ represented as

$$X(K) = X_r(K) + j X_i(K)$$

where, $X_r(K)$ = Real part of $X(K)$

$X_i(K)$ = Imaginary part of $X(K)$

Now the magnitude function $|X(k)|$ is defined as,

$$|X(k)|^2 = X(k)\, X^*(k) \quad (\text{or}) \quad |X(k)|^2 = X_r^2(k) + X_i^2(k)$$

The Phase function $\angle X(k)$ is defined as,

$$\angle X(k) = \text{Arg}[X(k)] = \tan^{-1}\left[\frac{X_i(k)}{X_r(k)}\right]$$

(6) DFT of delayed Sequence (or Shifted Sequence)

Let $x(n)$ be a discrete sequ. and $x'(n)$ be a Shifted sequ. of $x(n)$ by n_0 units of time.

Let $\text{DFT}\{x(n)\} = X(k)$

Now! $\text{DFT}\{x'(n)\} = \text{DFT}\{x(n-n_0)\} = X(k)\, e^{-j\frac{2\pi k n_0}{N}}$

Proof

by the definition of IDFT,

$$\text{IDFT}\{X(k)\} = x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi k n}{N}}$$

replace n by $n-n_0$,

$$\therefore x(n-n_0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi k(n-n_0)}{N}} = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{-j\frac{2\pi k n_0}{N}}\, e^{j\frac{2\pi k n}{N}}$$

$$\therefore x(n-n_0) = \text{IDFT}\left\{X(k)\, e^{-j\frac{2\pi k n_0}{N}}\right\}$$

on taking DFT of equ, we get,

$$\text{DFT}\{x(n-n_0)\} = X(k)\, e^{-j\frac{2\pi k n_0}{N}}$$

7. DFT of real valued Sequence:-

Let $x(n)$ be a real sequence, by the definition of DFT,

$$DFT\{x(n)\} = X(K) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{N-1} x(n)\left(\cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}\right)$$

$$= \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi kn}{N} - j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi kn}{N}$$

Also $X(K) = X_r(K) + jX_i(K)$

Real part, $X_r(K) = \sum_{n=0}^{N-1} x(n)\cos\left(\frac{2\pi kn}{N}\right)$; for $0 \le K \le N-1$

Imag. part, $X_i(K) = \sum_{n=0}^{N-1} x(n)\sin\left(\frac{2\pi kn}{N}\right)$; for $0 \le K \le N-1$

when $x(n)$ is real then $X(K)$ will have the following features,

(a) $X(K)$ has complex-conjugate symmetry ie; $X(K) = X^*(N-K)$

(b) Real component is even function. ie; $X_r(K) = X_r(N-K)$

(c) Imaginary component is odd function; ie; $X_i(K) = -X_i(N-K)$ ~~$|X(K)| = |X(N-K)|$~~

(d) Magnitude function is even function; ie; $|X(K)| = |X(N-K)|$

(e) Phase function is odd function; ie; $\angle X(K) = -\angle X(N-K)$

(f) If $x(n) = x(-n)$ (even Seq.), then $X(K)$ is purely real.

(g) If $x(n) = -x(-n)$ odd sequence, then $X(K)$ is purely Imaginary.

(8) DFT of Time Reversed Sequence :-

If DFT$\{x(n)\} = X(k)$ then DFT$\{x(N-n)\} = X(N-K)$

Hence reversing the N-point seq. in time is equivalent to reversing the DFT values.

Proof

$$DFT\{x(N-n)\} = \sum_{n=0}^{N-1} x(N-n)\, e^{-j2\pi kn/N}$$

Let us change the index from n to m, where $m=N-n$,

$\therefore n = N-m$

now, $$DFT\{x(N-n)\} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi k(N-m)}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi kN}{N}}\, e^{\frac{j2\pi km}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} \cdot e^{-j2\pi k} \cancel{= \sum_{m=0}^{N-1} x(m)\, e^{-j2\pi m}}$$

If k is Integer.
$e^{-j2\pi k} = 1$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}} = \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{-j2\pi m}$$

Since m is an Integer
$e^{-j2\pi m} = 1$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{j2\pi km}{N}}\, e^{\frac{-j2\pi mN}{N}}$$

$$= \sum_{m=0}^{N-1} x(m)\, e^{\frac{-j2\pi m(N-k)}{N}}$$

$$= X(N-k) \;//$$

(9) Circular time shift of a sequence.

If $DFT\{x(n)\} = X(k)$

then $DFT\{x(n-l, (\text{mod } N))\} = X(k)\, e^{-j\frac{2\pi k l}{N}}$

$DFT\{x((n-m))_N\} =$

(10) Circular freq. shift:-

If $DFT\{x(n)\} = X(k)$

then $DFT\left\{x(n)\, e^{j\frac{2\pi \overset{m}{\cancel{l}} n}{N}}\right\} = X((k-m))_N \leftarrow X(k-l, (\text{mod } N))$

Hence the multiplication of the seq. $x(n)$ with the complex exponential sequ. $e^{j\frac{2\pi k n}{N}}$ is equ. to the circular shift of the ~~(10)~~ DFT by l units in freq. This is the dual to the circular time shifting property.

(11) Circular Convolution:-

If $DFT\{x_1(n)\} = X_1(k)$ and $DFT\{x_2(n)\} = X_2(k)$

then $DFT\{x_1(n) \circledast x_2(n)\} = X_1(k) X_2(k)$.

(12) Multiplication of Two Sequence:-

$$DFT\{x_1(n) * x_2(n)\} = \frac{1}{N} X_1(k) \circledast X_2(k).$$

(13) Parseval's Relation:-

Let $\text{DFT}\{x_1(n)\} = X_1(k)$ and $\text{DFT}\{x_2(n)\} = X_2(k)$

Then by parseval's relation,

$$\sum_{n=0}^{N-1} x_1(n) x_2^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2^*(k)$$

Proof

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi nk}{N}} \; ; \; \cancel{X_2(k)} =$$

$$\text{IDFT}\{x_2(n)\} = \frac{1}{N} \sum_{k=0}^{N-1} X_2(k) e^{j\frac{2\pi nk}{N}}$$

Consider the right-hand side term of parsevel's relation,

$$\frac{1}{N} \sum_{k=0}^{N-1} X_1(k) X_2^*(k) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi nk}{N}} \right] X_2^*(k)$$

$$= \sum_{n=0}^{N-1} x_1(n) \left[\frac{1}{N} \sum_{k=0}^{N-1} X_2^*(k) e^{-j\frac{2\pi nk}{N}} \right]$$

$$= \sum_{n=0}^{N-1} x_1(n) \left[\frac{1}{N} \sum_{k=0}^{N-1} X_2(k) e^{j\frac{2\pi nk}{N}} \right]^*$$

$$= \sum_{n=0}^{N-1} x_1(n) x_2^*(n)$$

→ Circular Convolution:-

$$x_1(n) \circledast x_2(n) = \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N$$ ← Circular Convolution of two N-point seq. $x_1(n)$ & $x_2(n)$ is defined as

Let, $\mathrm{DFT}\{x_1(n)\} = X_1(K)$ and $\mathrm{DFT}\{x_2(n)\} = X_2(K)$, then

$$\mathrm{DFT}\{x_1(n) \circledast X_2(n)\} = X_1(K) X_2(K).$$

Proof:

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi nK}{N}} = \sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi mK}{N}} \quad ; K = 0,1,2,\ldots N-1 \quad [\because n=m$$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi nK}{N}} = \sum_{p=0}^{N-1} x_2(p) e^{-j\frac{2\pi pK}{N}} \quad ; K = 0,1,2,\ldots N-1 \quad [n=p$$

Consider the product $X_1(K)X_2(K)$. The IDFT of the product is,

$$\mathrm{DFT}^{-1}\{X_1(K)X_2(K)\} = \frac{1}{N}\sum_{K=0}^{N-1} X_1(K)X_2(K) e^{j\frac{2\pi nK}{N}}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1}\left[\sum_{m=0}^{N-1} x_1(m) e^{-j\frac{2\pi mK}{N}}\right]\left[\sum_{p=0}^{N-1} x_2(p) e^{-j\frac{2\pi pK}{N}}\right] e^{j\frac{2\pi nK}{N}}$$

$$= \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{K=0}^{N-1} e^{+j\frac{2\pi K(n-m-p)}{N}}$$

Consider the Summation $\sum_{K=0}^{N-1} e^{j\frac{2\pi K(n-m-p)}{N}}$

Let, $n-m-p = qN$, where, q- Integer]

$$\therefore \sum_{K=0}^{N-1} e^{j\frac{2\pi K(n-m-p)}{N}} = \sum_{K=0}^{N-1} e^{j\frac{2\pi KqN}{N}} = \sum_{K=0}^{N-1} \left(e^{j2\pi q}\right)^K = \sum_{K=0}^{N-1} 1^K = N$$

[since q is an Integer $e^{j2\pi q} = 1$.

Consider the Summation. $\sum_{p=0}^{N-1} x_2(p)$,

Since, $n-m-p = qN$, $p = n-m-qN$

$$\therefore \sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN) = \sum_{m=0}^{N-1} x_2(n-m, \text{mode } N) = \sum_{m=0}^{N-1} x_2((n-m))_N$$

Result: $$\mathrm{DFT}^{-1}\{X_1(K)X_2(K)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m)\sum_{m=0}^{N-1} x_2((n-m))_N N = \sum_{m=0}^{N-1} x_1(m) x_2((n-m))_N = x_1(n) \circledast x_2(n)$$

↖ $\therefore X_1(K) X_2(K) = \mathrm{DFT}\{x_1(n) \circledast x_2(n)\}$

Proof:

→ Conjugation:-

If $DFT\{x(n)\} = X(K)$

then $DFT\{x^*(n)\} = X^*(N-K)$

proof:

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\, e^{-j\frac{2\pi kn}{N}} = \left[\sum_{n=0}^{N-1} x(n)\, e^{+j\frac{2\pi kn}{N}}\right]^*$$

$$\left[\because e^{j2\pi n} = 1\right] \qquad = \left[\sum_{n=0}^{N-1} x(n)\, e^{+j\frac{2\pi kn}{N}}\, e^{-j2\pi n}\right]^* = \left[\sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi kn}{N}}\, e^{-j\frac{2\pi nN}{N}}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi n(N-K)}{N}}\right]^* = \left[X(n-k)\right]^* = X^*(N-K)$$

→ Circular freq. Shift:-

If $DFT\{x(n)\} = X(K)$ then $DFT\left\{x(n)\, e^{j\frac{2\pi mn}{N}}\right\} = X((k-m))_N$

Proof:

$$DFT\left\{x(n)\, e^{j\frac{2\pi mn}{N}}\right\} = \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi mn}{N}}\, e^{-j\frac{2\pi kn}{N}}$$

$$= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi(k-m)n}{N}}$$

$$= X((k-m))_N$$

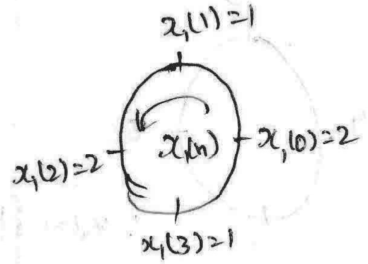
→ perform Circular Convolution of the two Sequence:-

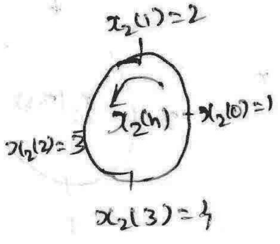
$x_1(n) = \{2, 1, 2, 1\}$, $x_2(n) = \{1, 2, 3, 4\}$

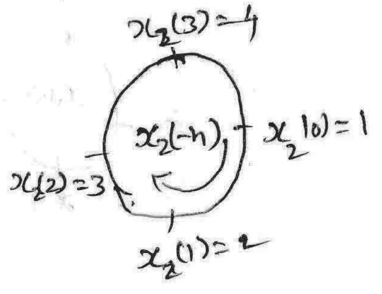
sol

(1) Graphical Method:-

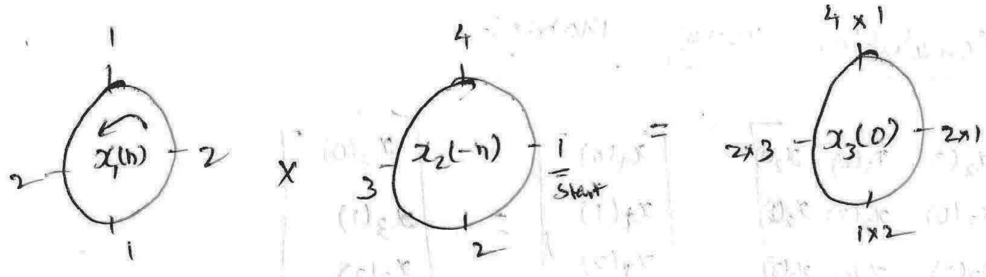
Formula : $x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n) = \sum_{n=0}^{N-1} x_1(n)\, x_{2,m}(n)$ $\left[x_{2,m}(n) = x_2(m-n) \right.$





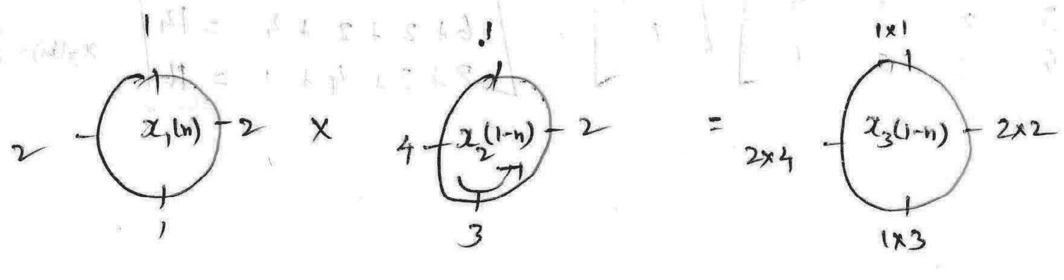


when $m=0$, $x_3(0) = \sum_{n=0}^{N-1} x_1(n)\, x_2(-n) = \sum_{n=0}^{3} x_1(n)\, x_{2,0}(n)$



$\therefore x_3(n) = 2+4+6+2 = 14,$

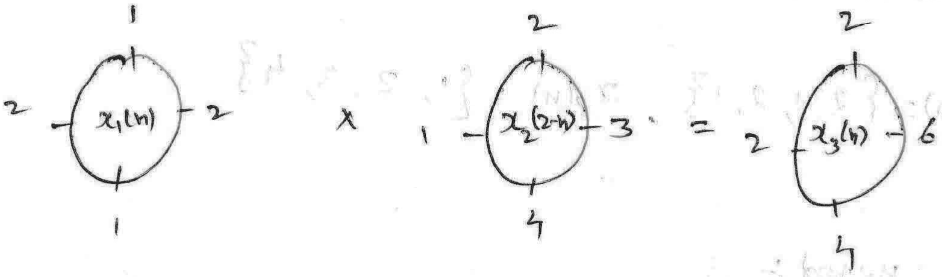
When $m=1$, $x_3(1) = \sum_{n=0}^{N-1} x_1(n)\, x_2(1-n) = \sum_{n=0}^{3} x_1(n)\, x_{2,1}(n)$



$\therefore x_3(1) = 4 + 1, 8 + 3$

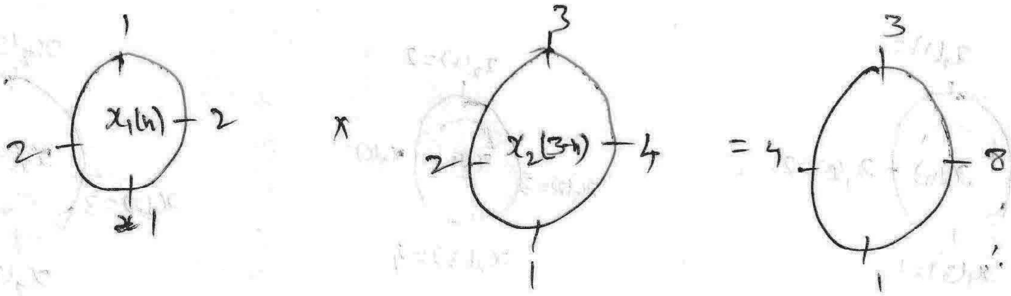
$= 16,$

when, $m=2$, $x_3(2) = \sum_{n=0}^{N-1} x_1(n)\, x_2(2-n) = \sum_{n=0}^{N-1} x_1(n)\, x_{2,2}(n)$



$\therefore x_3(2) = 14$,

when, $m=3$



$\therefore x_3(3) = 16$

$\therefore x_3(m) = \{\underset{\uparrow}{14}, 16, 14, 16\}$

Circular Convolution using Matrix:-

$$\begin{bmatrix} x_2(0) & x_2(3) & x_2(2) & x_2(1) \\ x_2(1) & x_2(0) & x_2(3) & x_2(2) \\ x_2(2) & x_2(1) & x_2(0) & x_2(3) \\ x_2(3) & x_2(2) & x_2(1) & x_2(0) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_1(1) \\ x_1(2) \\ x_1(3) \end{bmatrix} = \begin{bmatrix} x_3(0) \\ x_3(1) \\ x_3(2) \\ x_3(4) \end{bmatrix}$$

Result

$$\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+4+6+2 = 14 \\ 4+1+8+3 = 16 \\ 6+2+2+4 = 14 \\ 8+3+4+1 = 16 \end{bmatrix},$$

$x_3(n) = \{14, 16, 14, 16\}$

Circular convolution using DFT

→ $x_1(n) = \{2, 1, 2, 1\}$; $x_2(n) = \{1, 2, 3, 4\}$

<u>sol)</u>

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{3} x_1(n)\, e^{-j\frac{2\pi nk}{4}}$$

$$= x_1(0) e^{0} + x_1(1) e^{-j\frac{\pi k}{2}} + x_1(2) e^{-j\pi k} + x_1(3) e^{-j\frac{3\pi k}{2}}$$

$$X_1(k) = 2 + \left[\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right] + 2\left[\cos\pi k - j\sin\pi k\right] + \left[\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right]$$

$$k=0 \Rightarrow X_1(0) = 2 + [\cos 0 - j\sin 0] + 2[\cos 0 - j\sin 0] + [\cos 0 - j\sin 0]$$

$$= 2 + [1 - j0] + 2[1 - j0] + [1 - j0] = 6$$

$$k=1 \quad X_1(k) = 2 + [0 - j1] + 2[-1 - j(0)] + [0 + j(1)]$$

$$= 2 + j1 - 2 + j1 = 0$$

$$k=2, \quad X_1(2) = 2 + [\cos\pi - j\sin\pi] + 2[\cos 2\pi - j\sin 2\pi] + [\cos 3\pi - j\sin 3\pi]$$

$$= 2 + [-1 - j0] + 2[1 - j(0)] + [-1 - j0]$$

$$= 4 - 2 = 2$$

$$k=3, \quad X_1(3) = 2 + \left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right] + 2[\cos 3\pi - j\sin 3\pi] + \left[\cos\frac{9}{2}\pi - j\sin\frac{9\pi}{2}\right]$$

$$= 2 + [0 - j(-1)] + 2[-1 - j0] + [0 - j1]$$

$$= 2 + j - 2 - j = 0$$

$\therefore X_1(k) = \{6, 0, 2, 0\}$

$$X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j\frac{2\pi nk}{N}}$$

$$X_2(K) = x_1(0)e^{0} + x_1(1)\, e^{-j\frac{\pi k}{2}} + x_1(2)\, e^{-j\pi k} + x_1(3)\, e^{-j\frac{3\pi k}{2}}$$

$$= 1e^{0} + 2e^{-j\frac{\pi k}{2}} + 3e^{-j\pi k} + 4e^{-j\frac{3\pi k}{2}}$$

$$= 1 + 2\left[\cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right] + 3\left[\cos\pi k - j\sin\pi k\right] + 4\left[\cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}\right]$$

sub $k = 0, 1, 2, 3$

$k=0 \Rightarrow X(0) = 1 + 2[\cos 0 - j\sin 0] + 3[\cos 0 - j\sin 0] + 4[\cos 0 - j\sin 0]$

$= 1 + 2(1-0) + 3(1-0) + 4(1-0) = 10$

$k=1 \Rightarrow X(1) = 1 + 2\left[\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right] + 3[\cos\pi - j\sin\pi] + 4\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$= 1 + 2(0 - j(1)) + 3(-1 - j0) + 4(0 - j(-1))$

$= 1 - j2 - 3 + j4 =$ ~~$-2+j6$~~ $-2 + j2.$

$k=2 \Rightarrow X(2) = 1 + 2[\cos\pi - j\sin\pi] + 3[\cos 2\pi - j\sin 2\pi] + 4[\cos 3\pi - j\sin 3\pi]$

$= 1 + 2(-1 - j0) + 3(1 - j0) + 4(-1 - j0)$

$= 1 - 2 + 3 - 4 = -2$

$k=3 \Rightarrow X(3) = 1 + 2\left[\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right] + 3[\cos 3\pi - j\sin 3\pi] + 4\left[\cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}\right]$

$= 1 + 2[0 - j(-1)] + 3[-1 - j0] + 4[0 - j(1)]$

$= 1 + j2 - 3 - j4 = -2 - j2$

$$Y(k) = \sum_{k=0}^{N-1} X_1(k) \cdot X_2(k)$$

$$= \{X_1(0) \cdot X_2(0) , X_1(1) X_2(1) , X_1(2) X_2(2) , X_1(3) X_2(3)\}$$

$$Y(k) = \{6 \times 10 , 0 \times (-2+j2) , 2 \times (-2) , 0 \times (-2-j2)\}$$

$$Y(k) = \{60 , 0 , -4 , 0\}$$

IDFT$[Y(k)] = y(n)$

$$\therefore\ y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{\frac{j2\pi nk}{N}} = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{\frac{j2\pi nk}{4}}$$

$$= \frac{1}{4}\left[60 \cdot e^{0} + 0 + (-4) e^{j\pi n} + 0\right]$$

$$y(n) = \frac{1}{4}\left[60 e^{0} \mp 4 e^{j\pi n}\right] = \frac{1}{4}\left[60 - 4 e^{j\pi n}\right]$$

$$n = 0 \Rightarrow y(0) = \frac{1}{4}\left[60 - 4(\cos\pi 0 + j\sin\pi 0)\right] = \frac{1}{4}[60 - 4] = \frac{56}{4} = 14.$$

$$n = 1 \Rightarrow y(1) = \frac{1}{4}\left[60 - 4(\cos\pi + j\sin\pi)\right] = \frac{1}{4}\left[60 - 4(-1 + j0)\right]$$

$$= \frac{1}{4}[60 + 4] = \cancel{\frac{56}{4}}\ \cancel{= 14}\ \frac{64}{4} = 16.$$

$$n = 2 \Rightarrow y(2) = \frac{1}{4}\left[60 - 4(\cos 2\pi + j\sin 2\pi)\right]$$

$$= \frac{1}{4}\left[60 - 4(1 + j0)\right] = \frac{56}{4} = 14$$

$$n=3 \Rightarrow y(3) = \frac{1}{4}\left[60 - 4\left(\cos 3\pi + j\sin 3\pi\right)\right]$$

$$= \frac{1}{4}\left[60 - 4(-1 + j0)\right]$$

$$= \frac{1}{4}\left[60 + 4\right] = \frac{64}{4} = 16$$

$$\therefore\ y(n) = \{14, 16, 14, 16\}$$

→ Find the Linear and Circular convolution of the sequences x(n) and h(n) using DFT method.

$$x(n) = \{1,\ 0.5\} \quad \text{and} \quad h(n) = \{\underset{\uparrow}{0.5},\ 1\}$$

Sol

Linear Convolution :-

~~N~~= To avoid time aliasing, convert 2 sample I/p to 3 sample sequences.

$$1.\quad x(n) = \{\underset{\uparrow}{1},\ 0.5,\ 0\} \ ; \ h(n) = \{0.5,\ 1,\ 0\}$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}} = \sum_{n=0}^{3-1} x(n)\, e^{-j\frac{2\pi kn}{3}} = \sum_{n=0}^{2} x(n)\, e^{-j\frac{2\pi kn}{3}}$$

$$= x(0)\, e^{0} + x(1)\, e^{-j\frac{2\pi k}{3}} + x(2)\, e^{j\frac{4\pi k}{3}}$$

$$= 1 + 0.5\, e^{-j\frac{2\pi k}{3}} + 0 \quad = \quad 1 + 0.5\, e^{j\frac{2\pi k}{3}}$$

$e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$$= 1 + 0.5\,(\cos\overset{2\pi k/3}{\theta} + j\sin\overset{2\pi k/3}{\theta})$$

When,

$K=0,\ X(0) = 1 + 0.5(1) = 1.5$

$k=1,\ X(1) = 1 + 0.5(-0.5 - j0.866) = 0.75 - j0.433$

$K=2,\ X(2) = 1 + 0.5\,(-0.5 + j0.866) = 0.75 + j0.433$

Similiarly,

$$H(K) = \sum_{n=0}^{2} h(n)\, e^{-j\frac{2\pi k n}{3}} = h(0)e^{0} + h(1)e^{-j\frac{2\pi k}{3}} + h(2)\,e^{-j\frac{4\pi k}{3}}$$

$$= 0.5 + e^{-j\frac{2\pi k}{3}} + 0$$

When, $K=0$, $H(0) = 0.5 + (\cos\theta - j\sin\theta) = 0.5 + 1 = 1.5$

$K=1$, $H(1) = 0.5 + (-0.5 - j0.866) = -j0.866$

$K=2$, $H(2) = 0.5 + (-0.5 + j0.866) = j0.866.$

Let, $Y(k) = X(K)\, H(K)$ ~~*~~ ; for $K = 0, 1, 2$

when $k=0$, $Y(0) = X(0)\, H(0) = 1.5 \times 1.5 = 2.25$

$K=1$, $Y(1) = X(1)\, H(1) = (0.75 - j0.433) \times (-j0.866) = -0.375 - j0.6495$

$K=2$, $Y(2) = X(2)\, H(2) = (0.75 + j0.433) \times (j0.866) = -0.375 + j0.6495$

$$Y(K) = \{2.25, -0.375 - j0.6495, -0.375 + j0.6495\}$$

The sequence $y(n)$ is obtained from IDFT of $Y(K)$.

$$Y(K) = \frac{1}{N} \sum_{K=0}^{N-1} Y(K)\, e^{j\frac{2\pi k n}{N}} \; ; \text{ for } n = 0, 1, 2, \ldots, (N-1)$$

$$y(n) = \frac{1}{3} \sum_{K=0}^{2} Y(K)\, e^{j\frac{2\pi k n}{3}} = \frac{1}{3}\left[Y(0)\, e^{0} + Y(1)\, e^{j\frac{2\pi n}{3}} + Y(2)\, e^{j\frac{4\pi n}{3}}\right]$$

~~for n=0,1,2~~

when, $n=0$,

$$y(0) = \frac{1}{3}\left[Y(0) + Y(1) + Y(2)\right]$$

$$= \frac{1}{3}\left[2.25 - 0.375 - j0.6495 - 0.375 + j0.6495\right]$$

$$= \frac{1}{3}\left[1.5\right] = 0.5$$

$n=1, \; y(1) = \frac{1}{3}\left[Y(0) + Y(1)e^{j\frac{2\pi}{3}} + Y(2)\, e^{j\frac{4\pi}{3}}\right]$

$= \frac{1}{3}\left[2.25 + (-0.375 - j0.6495)(0.5 + j0.866) + (-0.375 + j0.6495)(-0.5 - j0.866)\right]$

$= \frac{1}{3}\left[2.25 + 0.75 + j0 + 0.75 + j0\right] = 1.25$

when $n=2$,

$y(2) = \frac{1}{3}\left[Y(0) + Y(1)e^{j\frac{4\pi}{3}} + Y(2)\, e^{j\frac{8\pi}{3}}\right]$

$= \frac{1}{3}\left[2.25 + (-0.375 - j0.6495)(-0.5 - j0.866) + (-0.375 + j0.6495)(-0.5 + j0.866)\right]$

$= \frac{1}{3}\left[2.25 - 0.375 + j6495 - 0.375 - j0.6495\right]$

$= \frac{1}{3}[1.5) = 0.5$

$\therefore y(n) = \{\underset{\uparrow}{0.5}, 1.25, 0.5\}$

Circular Convolution by DFT:-

given seq. is two point seq. Hence 2-point DFT of the seq are obtained as follows.

$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi nk}{N}} = \sum_{n=0}^{2-1} x(n)\, e^{-j\frac{2\pi nk}{2}} = \sum_{n=0}^{1} x(n)\, e^{-j\pi Kn}$

$= x(0)\, e^{0} + x(1)\, e^{-j\pi K} = 1 + 0.5\, e^{-j\pi K}$; for $K = 0, 1$

when, $K=0$; $X(0) = 1 + 0.5(\cos\pi K - j\sin\pi K) = 1 + 0.5(1) = 1.5$

$K=1$; $X(1) = 1 + 0.5(-1) = 1 - 0.5 = 0.5$

The 2-point DFT of h(n) is given by

$$H(k) = \sum_{n=0}^{1} h(n) e^{-j\frac{2\pi k n}{2}} = \sum_{n=0}^{1} h(n) e^{-j\pi k n} = h(0) e^{0} + h(1) e^{-j\pi k}$$

for $k = 0, 1$

when, $k=0$, $H(0) = 0.5 + \overset{j}{\cancel{e}} (\cos \pi k - j \sin \pi k) = 0.5 + 1 = 1.5$

$k=1$, $H(1) = 0.5 + (\cos \pi k - j \sin \pi k) = 0.5 - 1 = -0.5$

$\therefore H(k) = \{1.5, -0.5\}$

$Y(k) = X(k)\, H(k) = \{1.5 \times 1.5 , 0.5 \times (-0.5)\} = \{2.25, -0.25\}$

taking IDFT of $Y(k)$,

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j\frac{2\pi k m}{N}} \quad ; \text{ for } m = 0, 1, 2, \ldots (N-1)$$

Here, $N = 2$

$$y(m) = \frac{1}{2} \sum_{k=0}^{1} y(k) e^{j\pi k m} = \frac{1}{2} \left[y(0) e^{0} + y(1) e^{j\pi m} \right)$$

$$= \frac{1}{2} \left[2.25 - 0.25 e^{j\pi m} \right]$$

when, $m=0$; $y(0) = \frac{1}{2} \left[2.25 - 0.25 (\cos [illegible] + j \sin [illegible] \right] = \frac{1}{2} [2.25 - 0.25] = 1$

$m=1$; $y(1) = \frac{1}{2} \left(2.25 - 0.25 (\cos \pi + j \sin \pi) \right] = \frac{1}{2} [2.25 + 0.25)$

$= 1.25$.

$\therefore y(m) = \{\underset{\uparrow}{1}, 1.25\}$

Fast Fourier Transform:-

* This method of computing used to Deduce no. of calculation in DFT.

* In this Method, decomposed a no. of DFT in successfuly smaller DFT.

* In an N-point sequence, expressed N as $N = r^m$

where, r - no. of sequences (called radix of the FFT Algori)

m - no. of stages of computation.

* Radix -2 FFT : N-point seq. is decimated into 2-point sequences and calculated 2-point DFT values.

where, From the result of 2-point DFTs, the 4-point DFT is computed

From the result of 4-point DFTs, the 8-Point DFT is computed

From the result of 8-point DFTs, the 16-point DFT is compute

* In Radix-2, value of N should be such that $N = 2^m$

where, m- times decimated ($m = \log_2 N$)

* In direct computation of N-point DFT,

The total no. of complex Additions are $N(N-1)$

The total no. of complex Multiplication are N^2.

* In Radix-2, total no. of complex Addition are $= N \log_2 N$

total no. of complex Multiplication are $= \frac{N}{2} \log_2 N$

The 8-point DFT using Radix-2 DIF FFT (DIT- Decimation In Time)

The 8-point DFT using radix-2 FFT,

\# where, $N = 8 = 2^3$, therefore $r=2$ & $m=3$

* 8-point seq. is decimated to 2-point sequence.

Then, 2-point sequence computed to 2-point DFT

Next, 2-point DFT is computed to 4-point DFT

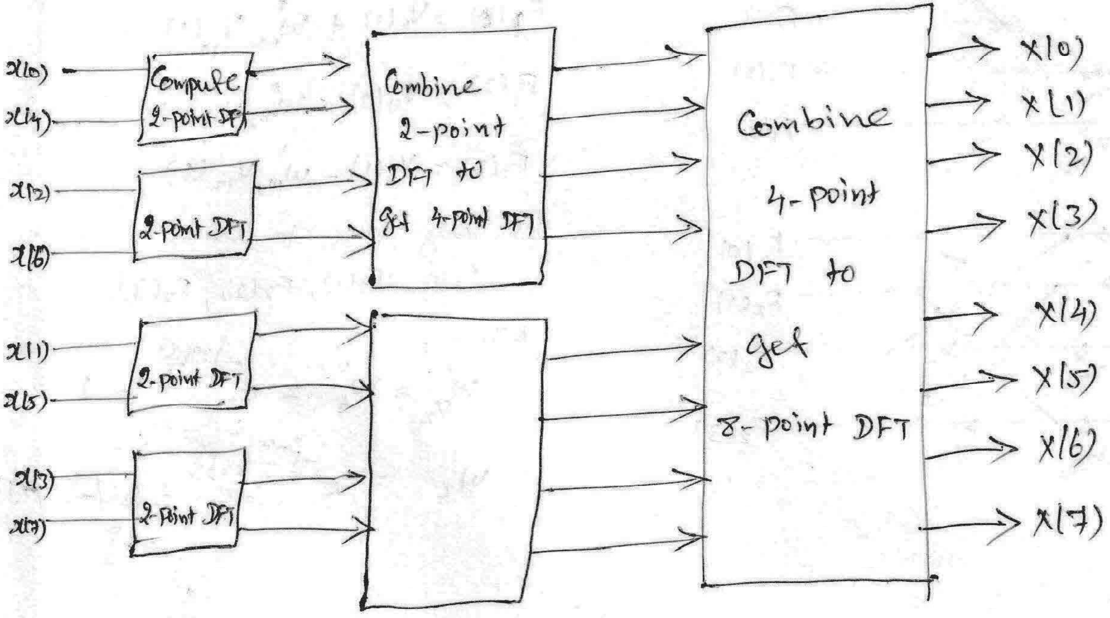
and, 4-point DFT is computed to 8-point DFT.

* Let, the seq, $x(0), x(1), x(2), x(3), x(4), x(5), x(6)$ and $x(7)$. which consist of 8 samples. This 8-samples is decimated to 2 samples. before they want to arrange in bit reversal order.

Normal order		Bit reversal order	
$x(n)$	bits	bits	$x(n)$
$x(0)$	000	000	$x(0)$
$x(1)$	001	100	$x(4)$
$x(2)$	010	010	$x(2)$
$x(3)$	011	110	$x(6)$
$x(4)$	100	001	$x(1)$
$x(5)$	101	101	$x(5)$
$x(6)$	110	011	$x(3)$
$x(7)$	111	111	$x(7)$

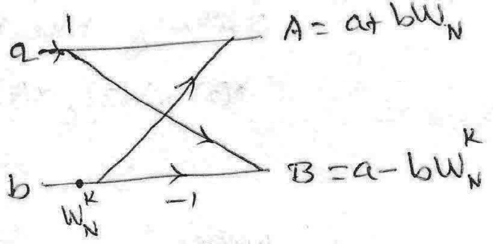
Next, decimated to 2 samples

they are,
(1) $x(0)$ & $x(4)$
(2) $x(2)$ & $x(6)$
(3) $x(1)$ & $x(5)$
(4) $x(3)$ & $x(7)$.



Graph for 8-point DIT Radix-2 FFT:-

Basic Graph :-



Note:-

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

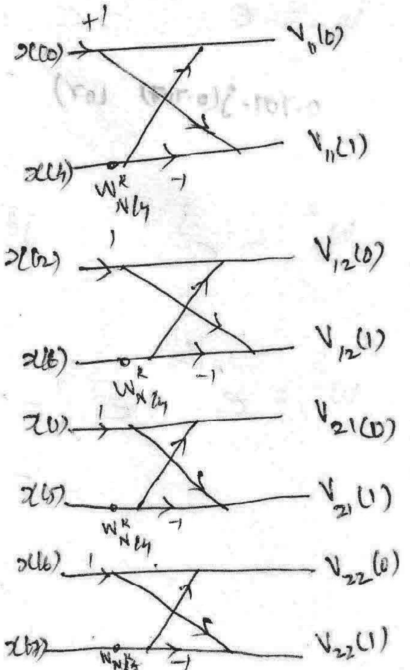
where, $e^{-j2\pi/N} = W_N$

$$e^{-j\frac{2\pi nk}{N}} = W_N^{nk}$$

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk}$$

W_N - complex value phase factor

First stage of Computation:-



$$V_{11}(k) = \sum_{n=0,1} x(n) e^{-j\frac{2\pi nk}{N/4}} = \sum_{n=0,1} x(n) W_{N/4}^{nk} = x(0)W_2^{k(0)} + x(1)W_2^{k(1)}$$

$$V_{11}^{k=0}(0) = x(0) + W_{N/4}^{0} x(4)$$

$$V_{11}^{k=1}(1) = x(0) - W_{N/4}^{0} x(4)$$

similarly,

$$V_{12}(0) = x(2) + W_{N/4}^{0} x(6)$$

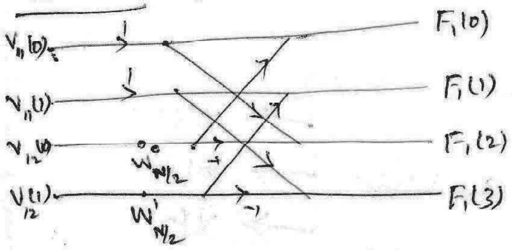
$$V_{12}(1) = x(3) - W_{N/4}^{0} x(6)$$

where, $W_{N/4} = \cancel{e}^{j} W_{8/4} = W_2 = e^{-j\frac{2\pi}{2}k}$

$$W_2^{nk} = e^{-j\frac{2\pi nk}{2}} = e^{-j\pi nk}$$

If $W_2^{0} = e^{-j\pi(0)} = 1$

Second stage:-



$F_1(0) = V_{11}(0) + W^0_{N/2} V_{12}(0)$

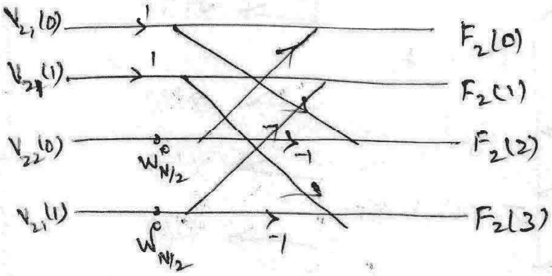
$F_2(0) = V_{11}(1) + W^0_{N/2} V_{12}(1)$

$F_1(2) = V_{11}(0) - W^0_{N/2} V_{12}(0)$

$F_1(3) = V_{11}(1) - W^1_{N/2} V_{12}(1)$

Similiarly take to write,

$F_2(0)$, $F_2(1)$, $F_2(2)$, $F_2(3)$

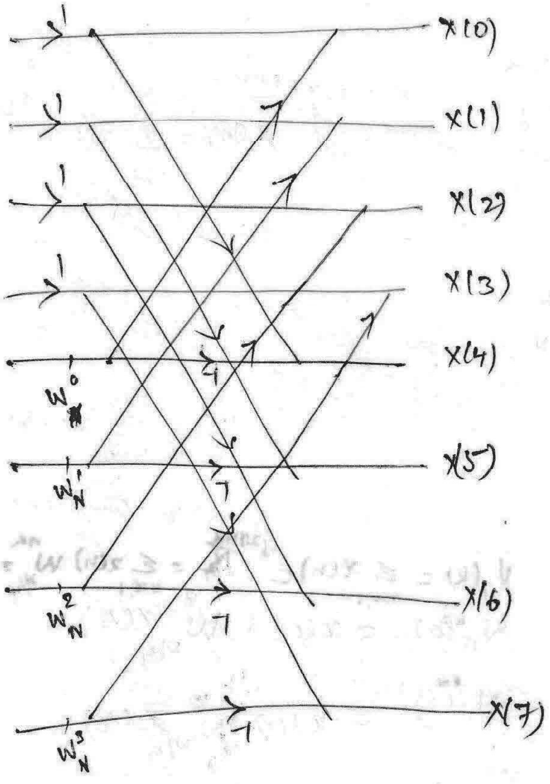


where,

$$W^0_{8/2} = W^0_4 = e^{-j2\pi\frac{0}{4}} = 1$$

$$W^1_4 = e^{-j\frac{2\pi 1}{4}} = e^{-j\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) \mp j\sin\left(\frac{+\pi}{2}\right)$$
$$= -j$$

3rd Stage



$X(0) = F_1(0) + W^0_N F_2(0)$

$X(1) = F_1(0) + W^1_N F_2(1)$

Similiarly take to write,

X(2), X(3), X(4), X(5), X(6) & X(7)

Where,

$$W^0_N = W^0_8 = e^{-j\frac{2\pi 0}{8}} = 1$$

$$W^1_8 = e^{-j\frac{2\pi(1)}{8}} = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4}$$

$0.707 - j(0.707)$ (or) $= \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$

$$= \frac{1}{\sqrt{2}}(1-j)$$

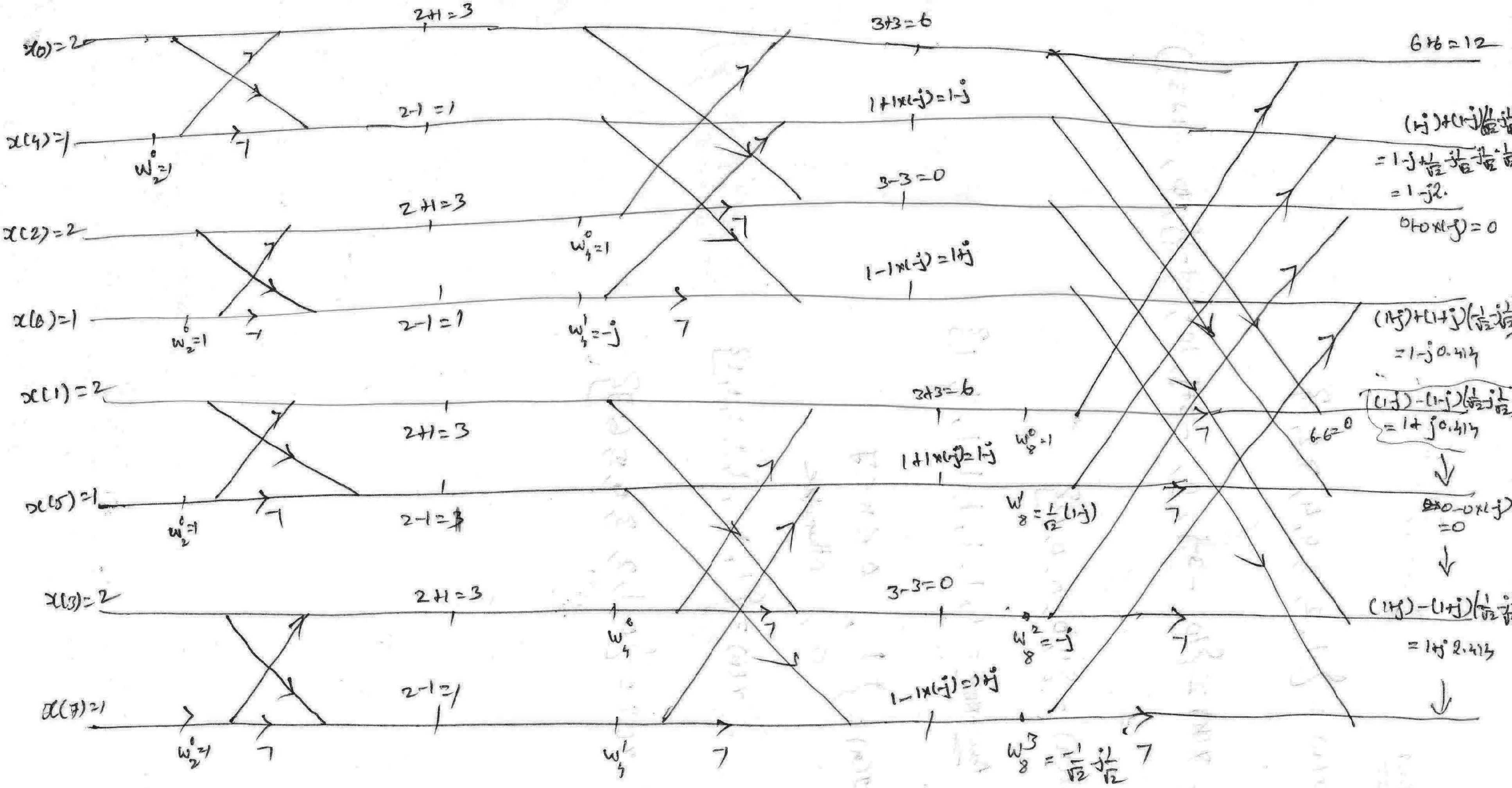
$$W^2_8 = e^{-j\frac{4\pi}{8}} = e^{-j\frac{\pi}{2}} = -j$$

$$W^3_8 = e^{-j\frac{6\pi}{8}} = e^{-j\frac{3\pi}{4}} = \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4}$$

$$= -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}(1+j)$$

① An 8-point seq. is given by $x(n) = \{2, 2, 2, 2, 1, 1, 1, 1\}$. Compute 8-point DFT of $x(n)$ by radix-2 DIT FFT.



H.W

① $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$

Ans:- $X(k) = \{10, -3j, 0, -3+j, 10, -1-3j, 0, -1+3j)$

② $x(n) = \{1, 0, 0, 0, 0\}$

Ans:- $X(k) = \{ = 1, 1, 1, 1, 1, 1, 1, 1\}$

③ $x(n) = \begin{cases} 1, & 0 \leq n \leq 7 \\ 0, & \text{otherwise} \end{cases}$

$\therefore x(n) = \{1, 1, 1, 1, 1, 1, 1, 1\}$

④ $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$H(w = x(n) = \{-1, 1, 2, 1, -1\}$

$h(n) = \{-1, 1, -1, 1\}$

→ In an LTI System the I/p $x(n) = \{1, 1, 1\}$ and the impulse response $h(n) = \{-1, -1\}$, Determine the response of LTI system by radix-2 DIT FFT.

Sol

$x(n) = \{1, 1, 1, 0\}$ and $h(n) = \{-1, -1, 0, 0\}$

$y(n) = x(n) \circledast h(n).$

Let, $DFT\{x(n)\} = X(K)$, $DFT\{h(n)\} = H(K)$, $DFT\{y(n)\} = Y(K).$

$DFT\{x(n) \circledast h(n)\} = X(K)H(K)$

$\therefore y(n) = IDFT\{X(K)\,H(K)\} = IDFT\{Y(K)\}$

The Various steps in Computing $y(n)$ are

(1) Determine $X(K)$ using radix-2 DIT algorithm.

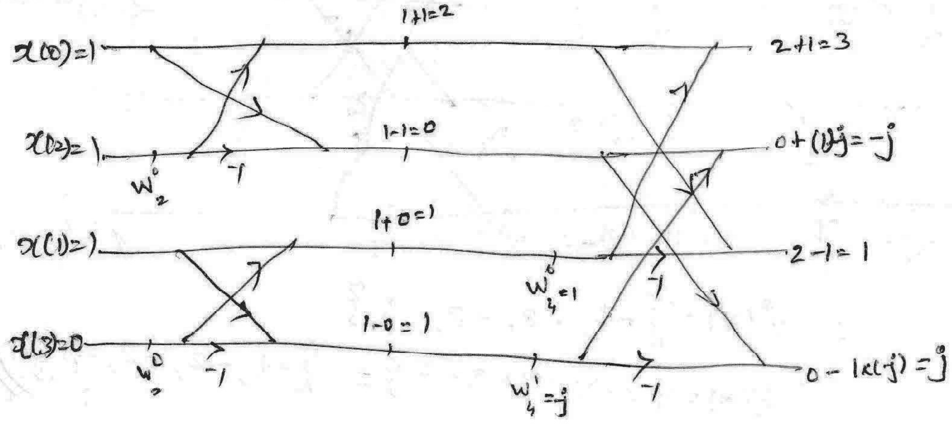
(2) Determine $H(K)$ using radix-2 DIT algorithm

(3) Determine the product $X(K)\ H(K)$.

(4) Take IDFT of the product $X(K)\ H(K)$ using radix-2 DIT Algorithm.

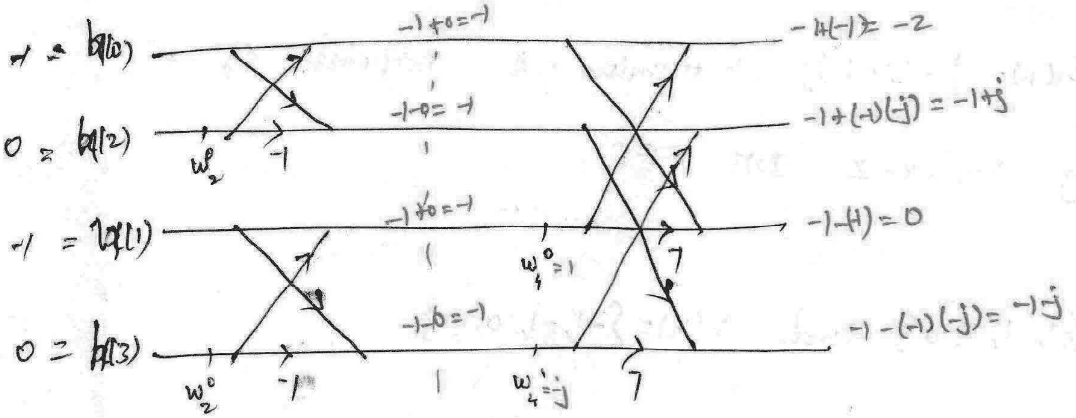
1st Step ~~Stage~~

Normal order	Reverse bit
00	00
01	10
10	01
11	11



$\therefore X(K) = \{3, -j, 1, +j\}$

H(k)



Step-3

$X(k)\,H(k) = Y(k)$ where $k = 0, 1, 2, 3$

$X(0)\,H(0) = Y(0) = 3\times(-2) = -6$

$X(1)\,H(1) = Y(1) = -j\times(-1+j) = j + j + 1 = 1+j$

$X(2)\,H(2) = Y(2) = 1\times 0 = 0$

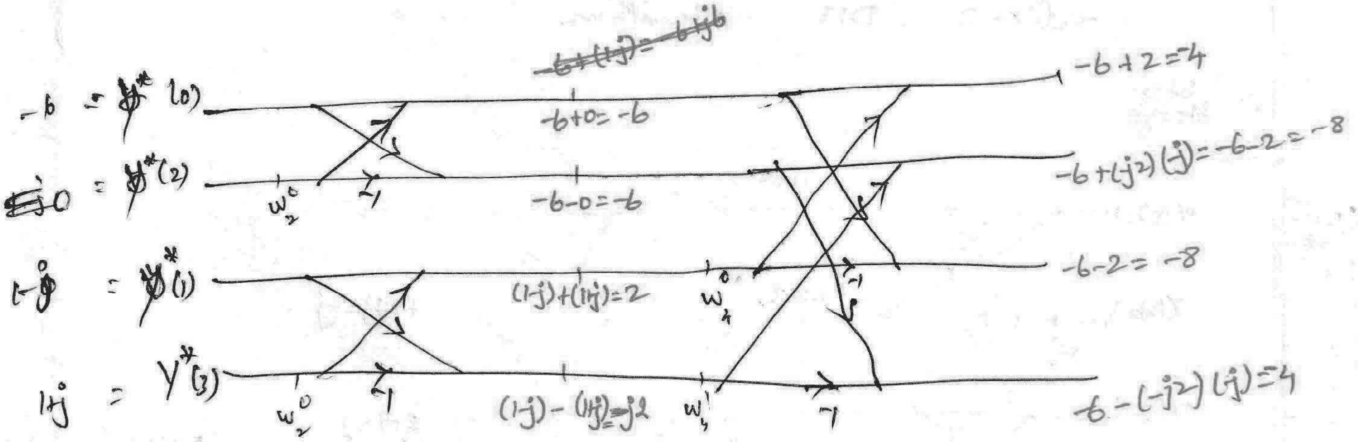
$X(3)\,H(3) = Y(3) = j\times(-1-j) = 1-j$

$\therefore Y(k) = \{-6, 1+j, 0, 1-j)$

Step-4

$$y(n) = IDFT\{Y(k) = \frac{1}{N}\left[\sum_{k=0}^{N-1} Y^*(k)\,W_N^{nk}\right]^*$$

$Y(k) = \{-6, 1+j, 0, 1-j\}$ but $Y^*(k) = \{-6, 1-j, 0, 1+j\}$



$\therefore Y^*(k) = \{-4, -8, -8, -4\}$

$\therefore y(n) = \frac{1}{4}\{-4, -8, -8, -4\} = \{-1, -2, -2, -1\}$ //

8-point DFT using Radix-2 DIF FFT:-

~~B~~

$$N = 8 = 2^3 = r^m$$

Here, $r = 2$ and $m = 3$.

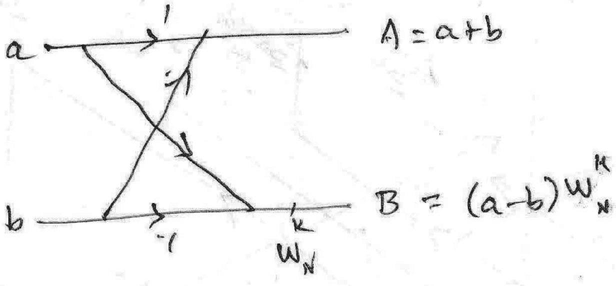
ie; 8-point DFT using radix-2 FFT involves 3-stages of computation.

* At the computation, (following points are considered.)

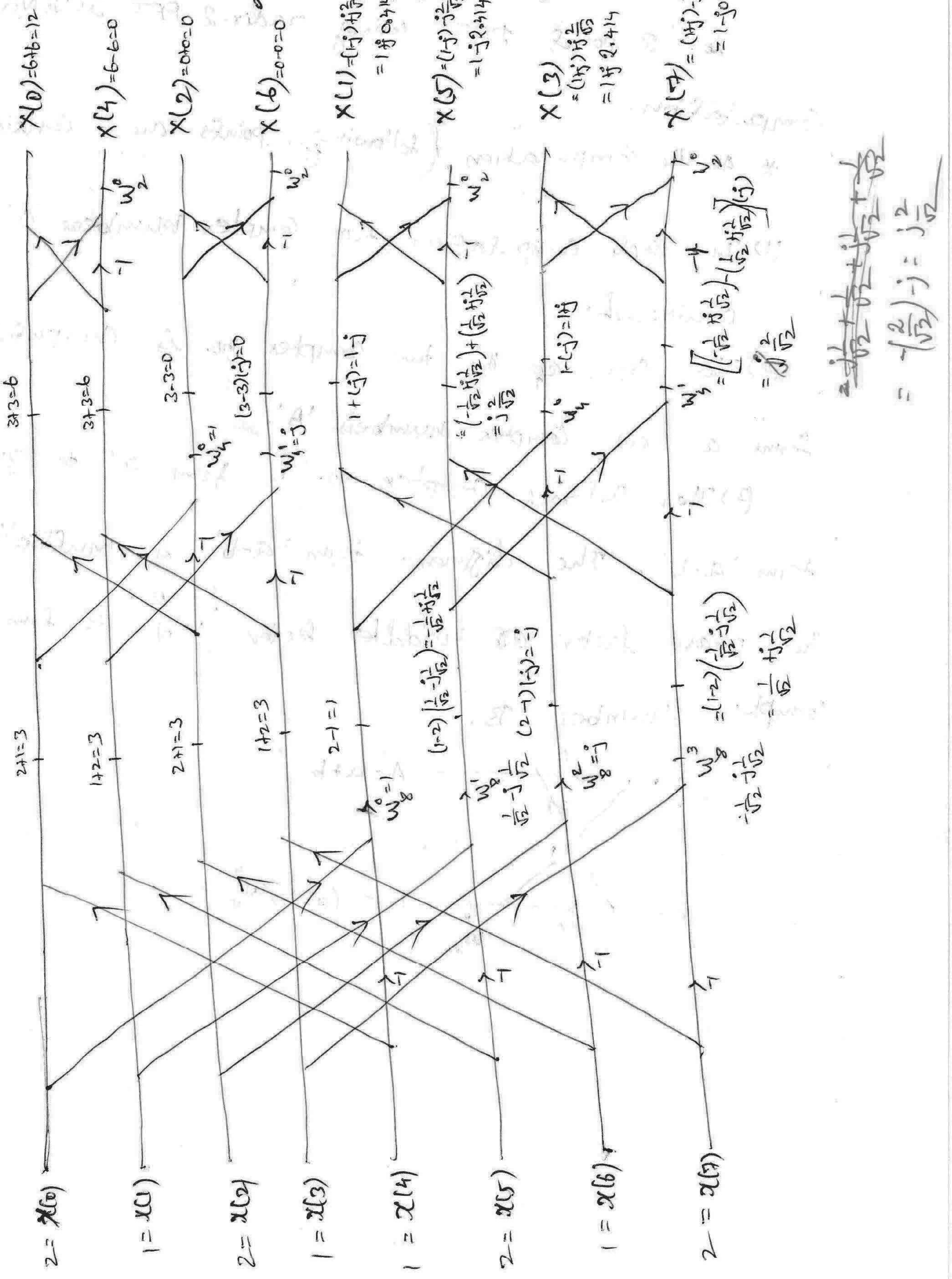
(1) In each computation two complex numbers 'a' & 'b' are considered.

(2) The sum of the two complex no. is computed which forms a new complex numbers 'A'.

(3) Then subtract complex no. 'b' from 'a' to get the term 'a-b'. The difference term 'a-b' is multiplied with the phase factor ~~or~~ twiddle factor 'W_N^k' to form a new complex number 'B'.



→ An - 8 point sequence is given by $x(n) = \{2, 1, 2, 1, 1, 2, 1, 2\}$. Compute 8 - point DFT of $x(n)$ by radix-2 DIF-FFT. Also Sketch the mag. and phase function.



$$W_8^0 = e^{j2\pi \times \frac{0}{8}} = 1$$

$$W_8^1 = e^{-j2\pi \times \frac{1}{8}} = e^{j\frac{\pi}{4}} = \cos\left(-\frac{\pi}{4}\right) + j\sin\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_8^2 = e^{j2\pi \times \frac{2}{8}} = e^{j\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right) = j$$

$$W_8^3 = e^{-j2\pi \times \frac{3}{8}}$$

$$= e^{j\frac{3\pi}{4}} = \cos\left(-\frac{3\pi}{4}\right) + j\sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$

$$W_4^0 = e^{j2\pi \times \frac{0}{4}} = 1$$

$$W_4^1 = e^{j2\pi \times \frac{1}{4}} = e^{j\frac{\pi}{2}} = \cos\left(-\frac{\pi}{2}\right) + j\sin\left(-\frac{\pi}{2}\right)$$

$$= j$$

$$W_2^0 = e^{-j2\pi \times \frac{0}{2}} = 1$$

$$x(k) = \{12, 0, 1+j2.414, 0, 1-j2.414, 0, 1-j0.414\}$$

Difference in DIT and DIF

* In DIT the time domain seq. is decimated, whereas in DIF the freq domain seq. is decimated.

*In DIT the I/p should be in bit-reversed order and the O/p will be in normal order. For DIF the reverse in true, ie; I/p is normal order, while O/p is bit reversed.

* Considering the butterfly diagram in DIT the complex multiplication takes place before the add-subtract operation, whereas in DIF the complex multiplication takes places after the add-subtract operation.

Similiarities in DIT and DIF:-

* $N = 2^m$

* Total no. of complex additions = $N\log_2 N$

Total no. of complex Multi = $(N/2)\log_2 N$

* Both algorithms require bit reversal at some place during computation.

Assignment-I

Assignment Question

① DFT

① $x(n) = \{1, 1, 0, 0\}$

② $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

③ $x(n) = \{1, 1, 1, 1, 1, 1, 0, 0\}$

IDFT

④ $X(K) = \{1, 0, 1, 0\}$

⑤ $X(K) = \{-2, 3j3, 0, 3+j3\}$

⑥ $X(K) = \{3, 1.707-j1.707, -j, 0.293+j0.2931, 0.293-j0.293,$
$j, 1.707+j1.707\}$ and $N=8$.

⑦ Circular Convolution for the following sequence.

$x_1(n) = \{\underset{\uparrow}{1}, 1, 2, 2\}$; $x_2(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$

DIT - FFT

① ~~find~~ tt $x(n) = \{1, 0, 0, 0, 0\}$

② $x(n) = \{1, 2, 3, 4, 4, 3, 2, 1\}$.

DIF - FFT

① $x(n) = \{1, 2, 0, 0, 0, 2, 1, 1\}$

② $x(n) = \{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0, 0, 0\}$

③ Compute the FFT of the sequ. $x(n) = n+1$ where $N = 8$, using the inplace radix-2 decimation in freq. algorithm,

Note ⇒ $x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$

④ Compute the 8-point DFT of the seq.

$$x(n) = \begin{cases} 1 \; ; & 0 \leq n \leq 7 \\ 0 \; ; & \text{Otherwise.} \end{cases}$$