IIR Filter:-

It is used to reducing the Error (ripple) in the requirements O/P signals. Then given the good result.

* Typical requirements which are considered in the design process are,
  (1) The filter should have a specific freq. fun.
  (2) The filter should have a specific impulse response
  (3) The filter should be causal.
  (4) The filter should be stable.
  (5) The computational complexity of the filter should be low.
  (6) The filter should be implemented in a particular hardware (or) software.

Design of IIR filter:-

* given system response is converted to analog filter transfer function of $H_a(s)$. with approximation tech. of Chebyshev & Butterworth filter.

* Next, $H_a(s)$ is transfer to digital filter transfer funct. for any one of tech. (Bilinear (or) Impulse Invariant transformations)

i.e, $H_a(s) \longrightarrow H(z)$

* Requirement for an analog filter to be stable and Causal:-

(1) The Analog filter transfer fun. $H_a(s)$ should be a rational fun. of s and the Co-effi. of s should be real.

(2) The poles should lie on the left half of s-plane.

(3) The no. of zeros should be less than or equal to no. of poles.

* Requirement for a digital filter to be Stable and Causal:-

(1) The digital filter transfer fun. H(z) should be a rational fun. of Z and the Co-effi. of z should be real.

(2) The poles should be inside the unit circle in Z-plane.

(3) The no. of zeros should be less than (or) equal to no. of poles.

Types of Transformation in IIR Filter:-

(1) Bilinear Transformation.

(2) Impulse Invariant Transformation.

Types of Approximation Tech.

(1) Butterworth Filter

(2) Chebyshev filter

IIR Filter Design by Impulse Invariant Method:-

The Transformation of analog filter to digital filter without modifying the Impulse response of the filter is called impulse Invariant Method.

In this Tech, the desired impulse response of the digital filter is obtained by uniformly sample the impulse response of the equivalent analog filter,

i.e; $h(n) = h_a(nT)$ —— ① [∵ T - Sampling Interval

Analog filter System functions,

$$H_a(s) = \sum_{i=1}^{M} \frac{A_i}{s-P_i} \quad \text{—— ②}$$

Take Inverse Laplace Transform,

$$h_a(t) = \sum_{i=1}^{M} A_i e^{P_i t} u_a(t) \quad \text{— ③}$$

[$u_a(t)$ - unit step fun. in continuous time.

Sub in equ ①,

$$h(n) = \sum_{i=1}^{M} A_i e^{P_i nT} u_a(nT)$$

taking Z- transform,

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{M} A_i e^{P_i nT} u_a(nT) \right] z^{-n}$$

Interchanging the order of summation,

$$H(z) = \sum_{i=1}^{M} \left[\sum_{n=0}^{\infty} A_i e^{P_i nT} u_a(nT) \right] z^{-n}$$

$$\left[Z\{a^n u(n)\} = \frac{z}{z-a} \right]$$

$$H(z) = \sum_{i=1}^{M} \frac{A_i}{1 - e^{P_i \cancel{n} T} z^{-1}} \quad \text{———} \quad (4)$$

Compare equ (2) & (4),

$$\boxed{\frac{1}{s - P_i} \longrightarrow \frac{1}{1 - e^{P_i T} z^{-1}}} \quad \text{———} \quad (5)$$

analog pole at $s = P_i$

digital pole $z = e^{P_i T}$

$\therefore \quad z = e^{sT}$

Similiarly,

some of the properties of the Impulse Invariant transformation are given below,

$$\frac{1}{(s+s_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} \left[\frac{1}{1 - e^{-sT} z^{-1}} \right] ; \quad s \rightarrow s_i$$

$$\frac{s+a}{(s+a)^2 + b^2} \longrightarrow \frac{1 - e^{-aT} (\cos bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

$$\frac{b}{(s+a)^2 + b^2} \longrightarrow \frac{e^{-aT} (\sin bT) z^{-1}}{1 - 2e^{-aT} (\cos bT) z^{-1} + e^{-2aT} z^{-2}}$$

//

→ For the analog transfer function,

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Sol: using partial fractions,

$$H(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Cross Multiplying of left & Right side

$$1 = A(s+2) + B(s+1)$$

$s=-2$, we get

$$1 = A(-2+2) + B(-2+1)$$

$$B = -1$$

$s=-1$, we get

$$A = 1$$

$$\therefore H(s) = \frac{1}{s+1} - \frac{B}{s+2}$$

$$\left[\because \frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T} z^{-1}}\right.$$

$$= \frac{1}{1-e^{-T}z^{-1}} - \frac{\cancel{B}}{1-e^{-2T}z^{-1}}$$

$$= \frac{1}{(1-1.718z^{-1})} - \frac{1}{(1-0.718z^{-1})} = \frac{(1-0.718z^{-1})-(1-1.718z^{-1})}{(1-1.718z^{-1})(1-0.718z^{-1})} = \frac{0.718z^{-1}}{1-0.718z^{-1}-1.718z^{-1}+1.233z^{-2}}$$

$$= \frac{(1-e^{-2T}z^{-1})-(1-e^{-T}z^{-1})}{(1-e^{-T}z^{-1})(1-e^{-2T}z^{-1})} = \frac{(1-e^{-2T}z^{-1})-(1-e^{-T}z^{-1})}{1-e^{-2T}z^{-1}-e^{-T}z^{-1}+e^{-T}z^{-1}e^{-2T}z^{-1}}$$

$$= \frac{(1-e^{-2T}z^{-1})-(1-e^{-T}z^{-1})}{1-(e^{-T}+e^{-2T})z^{-1}+e^{-3T}z^{-2}}$$

since, $T = 1s$

$$H(z) = \frac{0.236z^{-1}}{1-0.5032z^{-1}+0.0498z^{-2}}$$ //

→ Convert the analog filter into a digital filter whose System function is,

$$H(s) = \frac{s+0.2}{(s+0.2)^2+9}$$

use the Impulse Invariant tech. Assume T=1 s

Sol:

$$H(s) = \frac{s+a}{(s+a)^2+b^2} \Rightarrow a = 0.2 \ \& \ b=3$$

$$H(z) = \frac{1-e^{-0.2T}(\cos 3T)z^{-1}}{1-2e^{-0.2T}(\cos 3T)z^{-1} + e^{-0.4T}z^{-2}}$$

Sub. T=1 s

$$H(z) = \frac{1-0.8105z^{-1}}{1+1.6210z^{-1} + 0.625z^{-2}}$$

→ $$H_a(s) = \frac{(s+0.1)}{(s+0.1)^2+9}$$

Sol

$$H(z) = \frac{1-e^{-0.1T}(\cos 3T)z^{-1}}{1-2e^{-0.1T}(\cos 3T)z^{-1} + e^{-2(0.1)T}z^{-2}}$$

$$\left[\because \frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}(\cos bT)z^{-1}}{1-2e^{-aT}(\cos bT)z^{-1}+e^{-2aT}z^{-2}} \right.$$

$$= \frac{1+0.8959z^{-1}}{1+1.7915z^{-1}+0.8187z^{-2}}$$

IIR Filter Design by Bilinear Transformation:-

* Impulse Invariant Method is used to design only low pass & bandpass filter where resonant freq. are low.
* And not possible to design highpass (or) band-reject filters. This problem overcome in bilinear transformation.
* bilinear Transformation is obtained by using the trapezoidal formula for numerical Integration.

Let, the System fun. of the analog filter,

$$H(s) = \frac{b}{s+a} \quad \text{———} \quad (1)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

$$Y(s)\,s + a\,Y(s) = b\,X(s)$$

taking Inverse Laplace Transform,

$$\frac{dy(t)}{dt} + a y(t) = b\,x(t)$$

Integrated b/w the limits $(nT-T)$ and nT.

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{nT} x(t)\,dt \quad \text{———} \quad (2)$$

$$\int_{nT-T}^{nt} f(v)\,dv = F(v) = F(nT) - F(nT-T)$$

The Trapezoidal rule for numeric Integration is given by

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}\left[a(nT) + a(nT-T)\right]$$

Applying equ ② we get,

$$y(nT) - y(nT-T) + \frac{aT}{2}y(nT) + \frac{aT}{2}y(nT-T) = \frac{bT}{2}x(nT) + \frac{bT}{2}x(nT-T)$$

Taking Z- transform,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a} \quad \text{——— ③}$$

Comparing equ ① & ③ we get,

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right) \quad \text{——— ④}$$

where, $z = e^{sT}$

$s = \sigma + j\Omega$ [Ω - Analog freq in rad/sec.

* complex variable z in the polar forms is $z = re^{j\omega}$

so we can write equ ④.

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right) = \frac{2}{T}$$

Relation b/w Analog & Digital Freq:

Put $s=j\Omega$ and $z=e^{j\omega}$

$$\therefore \quad s = \frac{2}{T}\,\frac{1-z^{-1}}{1+z^{-1}} = \frac{2}{T}\,\frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{2}{T}\,\frac{\left(e^{j\omega/2}e^{-j\omega/2}-e^{-j\omega}\right)}{\left(e^{j\omega/2}e^{-j\omega/2}+e^{-j\omega}\right)}$$

$[\because 1 = e^{j\theta}e^{-j\theta}$

$$= \frac{2}{T}\,\frac{e^{-j\omega/2}\left(e^{j\omega/2}-e^{-j\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2}+e^{-j\omega/2}\right)}$$

$s=j\Omega]$

$$j\Omega = \frac{2}{T}\,\frac{\left(e^{j\omega/2}-e^{-j\omega/2}\right)/2}{\left(e^{j\omega/2}+e^{-j\omega/2}\right)/2}$$

$$\Omega = \frac{2}{T}\,\frac{\left(e^{j\omega/2}-e^{-j\omega/2}\right)/2j}{\left(e^{j\omega/2}+e^{-j\omega/2}\right)/2}$$

$$= \frac{2}{T}\cdot\frac{\sin\omega/2}{\cos\omega/2} = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\therefore \text{Analog freq, } \Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$\therefore \text{digital freq, } \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

//

$s=\sigma+j\Omega$

$z=$

→ Apply the bilinear transformation to $H_a(s) = \frac{2}{(s+1)(s+3)}$ with $T = 0.1$ sec and find $H(z)$.

Sol

$$H(z) = H(s)\Big|_{s = \frac{2}{T}\frac{(z-1)}{(z+1)}} = \frac{2}{\left(\frac{2}{T}\frac{z-1}{z+1} + 1\right)\left(\frac{2}{T}\frac{(z-1)}{(z+1)} + 3\right)}$$

$T = 0.1$ sec.

$$H(z) = \frac{2}{\left(\frac{20(z-1)}{(z+1)} + 1\right)\left(\frac{20(z-1)}{(z+1)} + 3\right)} = \frac{2}{\left(\frac{20z-20+z+1}{(z+1)}\right) \times \left(\frac{20z-20+3z+3}{z+1}\right)}$$

$$= \frac{2(z+1)^2}{(21z-19)(23z-17)} = \frac{2(z+1)^2}{483z^2 - 357z - 399z + 323}$$

$$= \frac{2(z+1)^2}{483z^2 - 756z + 323} \neq \cancel{\frac{2(z+1)^2}{483z^2}}$$

$\div z^2$

$$= \frac{2(1+z^{-1})^2}{483 - 756z^{-1} + 323} \neq$$

$(z+1)^2 = z^2 + 1 + 2z$

$= z^2 + 2z + 1$

$\div z^2$

$= 1 + 2z^{-1} + z^{-2}$

In form of $(a+b)^2$

$= (1+z^{-1})^2$

$\div 483$

$$= \frac{0.0041(1+z^{-1})^2}{1 - 1.644z^{-1} + 0.668z^{-2}}$$

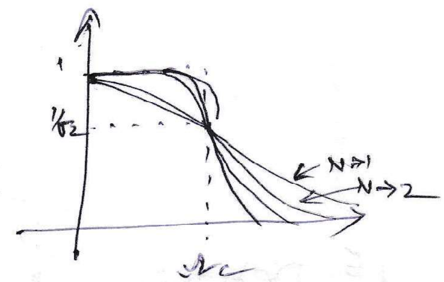
→ Design of Lowpass Digital Butterworth filter :-

Let, A_1 - gain at a passband freq. ω_1,

A_2 - gain at a stopband freq. ω_2,

Ω_1 - Analog freq. Corresponding to ω_1,

Ω_2 - Analog freq. Corresponding to ω_2,



(1) Choose either bilinear (or) Impulse Invariant transformation.

(2) Calculate the ratio of Ω_2/Ω_1

For bilinear transformation, $\dfrac{\Omega_2}{\Omega_1} = \dfrac{\tan \omega_2/2}{\tan \omega_1/2}$

For Impulse Invariant transformation, $\dfrac{\Omega_2}{\Omega_1} = \dfrac{\omega_2}{\omega_1}$

(3) Decide the order N of the filter, The order N should be greater than (or) equal to N_1, where N_1 is given by

$$N_1 = \frac{1}{2} \frac{\log\left\{\left[\frac{1}{A_2^2}-1\right] \Big/ \left[\frac{1}{A_1^2}-1\right]\right\}}{\log \frac{\Omega_2}{\Omega_1}}$$

Note:- Choose N such that, $N \geq N_1$

(4) Calculate the analog cut-off freq, Ω_c.

For bilinear transformation, $\Omega_c = \frac{2}{T} \frac{\tan \omega_1/2}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$

For Impulse Invariant transformation,

$$\Omega_c = \frac{\omega_1/T}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$$

(5) Determine the Analog transfer fun. of the filter.

Let, $H_a(s)$ = Analog filter transfer fun.

when, N is even, $H_a(s) = \prod_{K=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_K \Omega_c s + \Omega_c^2}$

N is odd, $H_a(s) = \frac{\Omega_c}{s+\Omega_c} \prod_{K=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_K \Omega_c s + \Omega_c^2}$

- The co-effi. $b_K = 2 \sin\left(\frac{(2K-1)\pi}{2N}\right)$

- For normalized case take $\Omega_c = 1$ rad/sec.

(6) using the chosen transformation, transform $H_a(s)$ to $H(Z)$, where $H(Z)$ is the transfer fun. of the digital filter.

(7) Realize the digital filter transfer fun. $H(z)$ by a suitable structure.

→ The Specification of the desired lowpass filter is,

$$\frac{1}{\sqrt{2}} \le |H(\omega)| \le 1.0 \; ; \quad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.08 \; ; \quad 0.4\pi \le \omega \le \pi$$

Design a Butterworth digital filter using Bilinear transformation

Sol :

$$A_1 = \frac{1}{\sqrt{2}} = 0.707 \; ; \quad \omega_1 = 0.2\pi$$

$$A_2 = 0.08 \; ; \quad \omega_2 = 0.4\pi$$

1) Billnear transformation,

$$\frac{\Omega_2}{\Omega_1} = \frac{\tan(\omega_2/2)}{\tan(\omega_1/2)} = \frac{\tan(0.4\pi/2)}{\tan(0.2\pi/2)} = \frac{0.7265}{0.3249} = 2.236$$

2) $$N_1 = \frac{1}{2} \frac{\log\left\{\left[\frac{1}{A_2^2}-1\right] \Big/ \left[\frac{1}{A_1^2}-1\right]\right\}}{\log \Omega_2/\Omega_1} = \frac{1}{2} \frac{\log\left\{\left[\frac{1}{0.08^2}-1\right] \Big/ \left[\frac{1}{0.707^2}-1\right]\right\}}{\log 2.236}$$

$$= \frac{1}{2} \frac{\log\{155.25/1.0006\}}{\log 2.236} = \frac{1}{2} \cdot \frac{2.1908}{0.3495} = 3.1342$$

Choose N such that $N \ge N_1$.

Let the order of the filter, $N = 4$. Also let, $T = 1$ sec.

$$\Omega_c = \frac{2}{T} \times \frac{\tan(\omega_1/2)}{\left(\frac{1}{A_1^2}-1\right)^{1/2N}} = \frac{2 \times \tan(0.2\pi/2)}{\left(\frac{1}{0.707^2}-1\right)^{1/2\times 4}} = \frac{0.6498}{(1.0006)^{1/8}} = \cancel{0.65} \; 0.65$$

Where, $N=4$, $\therefore K=1,2$.

(3) $$H_a(s) = \prod_{K=1}^{N/2} \frac{\Omega_c^2}{s^2+b_K\Omega_c s+\Omega_c^2} = \frac{\Omega_c^2}{s^2+b_1\Omega_c s+\Omega_c^2} \times \frac{\Omega_c^2}{s^2+b_2\Omega_c s+\Omega_c^2}$$

When $K=1$, $b_K = b_1 = 2\sin\left(\frac{(2-1)\pi}{2\times 4}\right) = 2\sin\left(\frac{\pi}{8}\right) = 0.765$

$K=2$, $b_K = b_2 = 2\sin\left(\frac{(4-1)\pi}{2\times 4}\right) = 2\sin\left(\frac{3\pi}{8}\right) = 1.848$.

$$\therefore H_a(s) = \frac{0.65^2}{s^2+0.765\times 0.65s+0.65^2} \times \frac{0.65^2}{s^2+1.848\times 0.65s+0.65^2}$$

$$= \frac{0.179}{(s^2+\cancel{0.765}\,0.497s+0.423)(s^2+1.\cancel{201}s+0.423)}$$

In Bilinear transformation we have to put, $S = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ in $H_a(s)$ to get the digital filter transfer fn. $H(z)$. Here $T=1$ sec.

$$\therefore H(z) = \frac{0.179}{\left[\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right)^2+0.497\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right)+0.423\right]\left[\left(\frac{2(1-z^{-1})}{(1+z^{-1})}\right)^2+1.201\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right)+0.423\right]}$$

$$= \frac{0.179}{\left[\frac{4(1-z^{-1})^2+0.994(1-z^{-1})(1+z^{-1})+0.423(1+z^{-1})^2}{(1+z^{-1})^2}\right]}$$

$1+z^{-1}-z^{-1}-z^{-2}$

$(1-z^{-2})$

$$\left[\frac{4(1-z^{-1})^2+2.402(1-z^{-1})(1+z^{-1})+0.423(1+z^{-1})^2}{(1+z^{-1})^2}\right]$$

$$= \frac{0.179(1+z^{-1})^4}{\left[4(1-2z^{-1}+z^{-2})+0.994(1-z^{-2})+0.423(1+2z^{-1}+z^{-2})\right]\left[4(1-2z^{-1}+z^{-2})+2.402(1-z^{-2})+0.423(1+2z^{-1}+z^{-2})\right]}$$

$$= \frac{0.179(1+z^{-1})^4}{\left[4-8z^{-1}+4z^{-2}+0.994-0.994z^{-2}+0.423+0.846z^{-1}+0.423z^{-2}\right]\left[4-8z^{-1}+4z^{-2}+2.402\cancel{z}-2.402z^{-2}+0.423+0.86z^{-1}+0.423z^{-2}\right]}$$

$$= \frac{0.179(1+z^{-1})^4}{\left[5.417-7.154z^{-1}+3.429z^{-2}\right]\left[6.825-7.154z^{-1}+2.021z^{-2}\right]}$$

$$= \frac{0.179(1+z^{-1})^4}{5.417\left[1-\frac{7.154}{5.417}z^{-1}+\frac{3.429}{5.417}z^{-2}\right]6.825\left[1-\frac{7.154}{6.825}z^{-1}+\frac{2.021}{6.825}z^{-2}\right]}$$

$$= \frac{0.005(1+z^{-1})^4}{\left[1-1.321z^{-1}+0.633z^{-2}\right]\left[1-1.048z^{-1}+0.296z^{-2}\right]}$$

Design Procedure for lowpass digital Chebyshev IIR Filter:-

A_1 - gain at a passband freq. ω_1,

A_2 - " " " " " ω_2

Ω_1 - Analog freq corresponding to ω_1

Ω_2 - " " " " ω_2.

① Choose either bilinear or impulse invariant transformation

② Calculate the attenuation constant ϵ

$$\epsilon = \left[\frac{1}{A_1^2} - 1\right]^{1/2}$$

③ Calculate the ratio Ω_2/Ω_1

For bilinear, $\dfrac{\Omega_2}{\Omega_1} = \dfrac{\tan \omega_2/2}{\tan \omega_1/2}$

For Impulse, $\dfrac{\Omega_2}{\Omega_1} = \dfrac{\omega_2}{\omega_1}$

④ Decide the order N of the filter. Choose N such that $N \geq N_1$,

where N_1 is

$$N_1 = \frac{\cosh^{-1}\left\{\frac{1}{\epsilon}\left[\frac{1}{A_2^2} - 1\right]^{1/2}\right\}}{\cosh^{-1}\left\{\frac{\Omega_2}{\Omega_1}\right\}}$$

⑤ calculate the analog cutoff freq. Ω_c.

bilinear, $$\Omega_c = \frac{2}{T}\frac{\tan\frac{\omega_1}{2}}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}}$$

Impulse, $$\Omega_c = \frac{\omega_1/T}{\left[\frac{1}{A_1^2}-1\right]^{1/2N}}$$

⑥ Determine the analog transfer fun. $H_a(s)$ of the filter,

N is even, $$H_a(s) = \prod_{k=1}^{N/2}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+C_k\Omega_c^2}$$

N is odd, $$H_a(s) = \frac{B_0\Omega_c}{s+C_0\Omega_c}\prod_{k=1}^{\frac{N-1}{2}}\frac{B_k\Omega_c^2}{s^2+b_k\Omega_c s+C_k\Omega_c^2}$$

where, $$b_k = 2y_N\sin\left(\frac{(2k-1)\pi}{2N}\right)\ ;\ C_k = y_N^2+\cos^2\frac{(2k-1)\pi}{2N}$$

$$C_0 = y_N\ ;\ y_N = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2}+\frac{1}{\epsilon}\right]^{1/N} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2}+\frac{1}{\epsilon}\right]^{-1/N}\right\}$$

For even value of N and unity dc gain filter find B_k such that $$H_a(0) = \frac{1}{(1+\epsilon^2)^{1/2}}$$

For odd value of N and unity. $H_a(0)=1$

⑦ $H_a(s) \rightarrow H(z)$.

① ~~Chebyshev~~ The specification of the desired lowpass filter is,

$$0.8 \le |H(\omega)| \le 1.0 \quad ; \quad 0 \le \omega \le 0.2\pi$$

$$|H(\omega)| \le 0.2 \quad ; \quad 0.32\pi \le \omega \le \pi$$

Design chebyshev digital filter using bilinear transformation

Sol

$A_1 = 0.8$; $\omega_1 = 0.2\pi$ rad/sec

$A_2 = 0.2$; $\omega_2 = 0.32\pi$ rad/sec.

① For bilinear transformation

$$\frac{\Omega_2}{\Omega_1} = \frac{\tan \omega_2/2}{\tan \omega_1/2} = \frac{\tan 0.32\pi/2}{\tan 0.2\pi/2} = 1.692$$

② The Atten- Constant

$$\epsilon = \left(\frac{1}{A_1^2} - 1\right)^{1/2} = \left(\frac{1}{0.8^2} - 1\right)^{1/2} = 0.75$$

③

$$N_1 = \frac{\cosh^{-1}\left\{\frac{1}{\epsilon}\left[\frac{1}{A_2^2} - 1\right]^{1/2}\right\}}{\cosh^{-1}\frac{\Omega_2}{\Omega_1}} = \frac{\cosh^{-1}\left\{\frac{1}{0.75}\left[\frac{1}{0.2^2} - 1\right]^{1/2}\right\}}{\cosh^{-1} 1.692}$$

$$= 2.295$$

$$\therefore N = 3$$

④ If $T = 1$ sec.

$$\Omega_c = \frac{2}{T} \frac{\tan\frac{\omega_1}{2}}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}} = 0.715 \text{ rad/sec}$$

use radians mode in cal.

$$H_a(s) = \frac{B_0\Omega_c}{s+c_0\Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{B_k \Omega_c^2}{s^2+b_k\Omega_c s+\Omega_c^2}$$

$\underline{\underline{N=3 \therefore k=1}}$

$$H_a(s) = \frac{B_0\Omega_c}{s+c_0\Omega_c} \times \frac{B_1\Omega_c^2}{s^2+b_1\Omega_c s+c_1\Omega_c^2}$$

$$Y_n = \frac{1}{2}\left\{\left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2}+\frac{1}{\epsilon}\right]^{1/N} - \left[\left(\frac{1}{\epsilon^2}+1\right)^{1/2}+\frac{1}{\epsilon}\right]^{-1/N}\right\}$$

$$= \frac{1}{2}\left[\left(\frac{1}{0.75^2}+1\right)^{1/2}+\frac{1}{0.75}\right]^{1/3} - \frac{1}{2}\left[\left(\frac{1}{0.75^2}+1\right)^{1/2}+\frac{1}{0.75}\right]^{-1/3}$$

$$= 0.7211 - 0.3467 = 0.3744$$

$$c_0 = Y_N = 0.3743$$

$$c_k = Y_N^2 + \cos^2\frac{(2k-1)\pi}{2N}$$

When, $k=1$, $c_1 = Y_N^2 + \cos^2\left(\frac{\pi}{6}\right) = 0.3744^2 + \cos^2\left(\frac{\pi}{6}\right) = 0.8902$

$$b_k = 2Y_n \sin\frac{(2k-1)\pi}{2N} \Rightarrow b_1 = 2 \times 0.3744 \sin\frac{\pi}{6} = 0.377$$

$$H_a(s) = \frac{0.715 B_0}{s+0.3744\times0.715} \cdot \frac{(0.715)^2 B_1}{s^2+0.3744\times0.715 s+0.8902\times0.715^2}$$

$$= \frac{0.3655 B_0 B_1}{(s+0.2677)(s^2+0.2677s+0.4551)}$$

when, $S=0$, $H_a(0) = \dfrac{0.3655\, B_0 B_1}{0.2677 \times 0.4557} = 3 B_0 B_1$

Let $H_a(0) = 1$, $\therefore 3 B_0 B_1 = 1$

Let, $B_0 = B_1$, $\therefore B_0^2 = \frac{1}{3}$

$B_0 = \frac{1}{\sqrt{3}} = 0.577$; $\therefore B_0 = B_1 = 0.577$

$$H_a(s) = \frac{0.3655 \times 0.577^2}{(S+0.2677)(s^2+0.2677s+0.4557)} = \frac{0.1217}{(s+0.2677)(s^2+0.2677s+0.4557)}$$

for bilinear transformation, put $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$

Here, $T=1$ $\therefore s = \dfrac{2(1-z^{-1})}{(1+z^{-1})}$

$$H(z) = \frac{0.1217}{\left(\frac{2(1-z^{-1})}{1+z^{-1}} + 0.2677\right)\left(\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right)^2 + 0.2677\left(\frac{2(1-z^{-1})}{1+z^{-1}}\right) + 0.4557\right)}$$

$$= \frac{0.1217}{\left(\frac{2(1-z^{-1}) + 0.2677(1+z^{-1})}{(1+z^{-1})}\right)\left(\frac{4(1-z^{-1})^2 + \cancel{0.2677}\,0.5354(1-z^{-1})(1+z^{-1}) + 0.4557(1+z^{-1})^2}{(1+z^{-1})^2}\right)}$$

$$= \frac{0.1217(1+z^{-1})^3}{[2-2z^{-1}+0.2677+0.2677z^{-1}][4(1+z^{-2}-2z^{-1})+0.5354(1-z^{-2})+0.4557(1+z^{-2}+2z^{-1})]}$$

$$= \frac{0.1217(1+z^{-1})^3}{(2.2677-1.7323z^{-1})(4.4557-7.0898z^{-1}+3.9197z^{-2})}$$

$$= \frac{0.1217(1+z^{-1})^3}{2.2677\left(1-\frac{1.7323}{2.2677}z^{-1}\right)\times 4.4557\left(1-\frac{7.0898}{4.4557}z^{-1}+\frac{3.9197}{4.4557}z^{-2}\right)}$$

$$= \frac{\cancel{0.1217}\ 0.012(1+z^{-1})^3}{(1-0.7639z^{-1})(1-1.5913z^{-1}+0.8798z^{-2})}$$

→ **Frequency Warping:-**

In bilinear transformation the relation b/w analog and digital freq is non-linear. When the s-plane is mapped into z-plane using bilinear transformation, this nonlinear relationship Introduces distortion in freq. axis, which is called freq. warping.

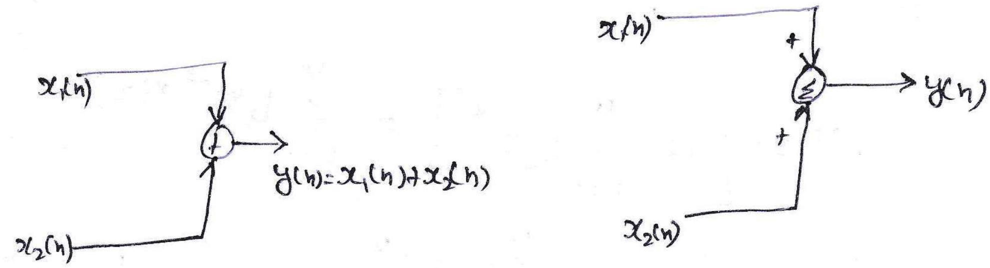
→ **Prewarping:-** In IIR filter design using bilinear transformation, the conversion of the specified digital freq to ~~al~~ analog freq is called prewarping.

The prewarping is necessary to eliminate the effect of warping on amplitude respon impulse invariant and bilinear

Introduction:-

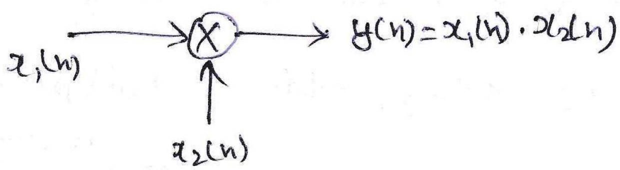
In Discrete time System have following basic building block. They are.

(1) Adder:-

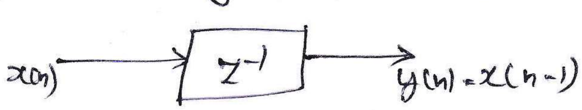


(2) Constant Multiplier:-

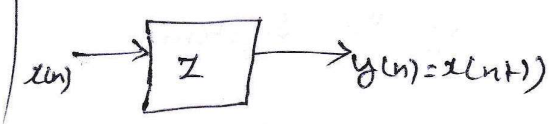
(3) Signal Multiplier:-



(4) unit delay block: (delay by one same)



(5) unit Advace block:-



Structures For Realization of IIR Systems:-

In general, the O/P of a finite order linear time Invariant System at time 'n' can be expressed as a linear Combination of the,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

where, a_k & b_k - Constants with $a_0 \neq 0$ and $M \leq N$

taking z. transforms of equ,

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z)\left[1 + \sum_{k=1}^{N} a_k z^{-k}\right] = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

* This system function, consisting by different equation in Numerater and denominater.

* From this set of each equ, can be constructed by block diagram consisting of delays, adder & Multiplier.

This block diagram arrangement is called realization structure of the System.

In IIR system, different type of Structure realizations are,

(1) Direct form - I Structure

(2) Direct form - II Structure

(3) ~~Di~~ cascade form structure

(4) parallel form Structure.

(1) Direct form - I Structure:-

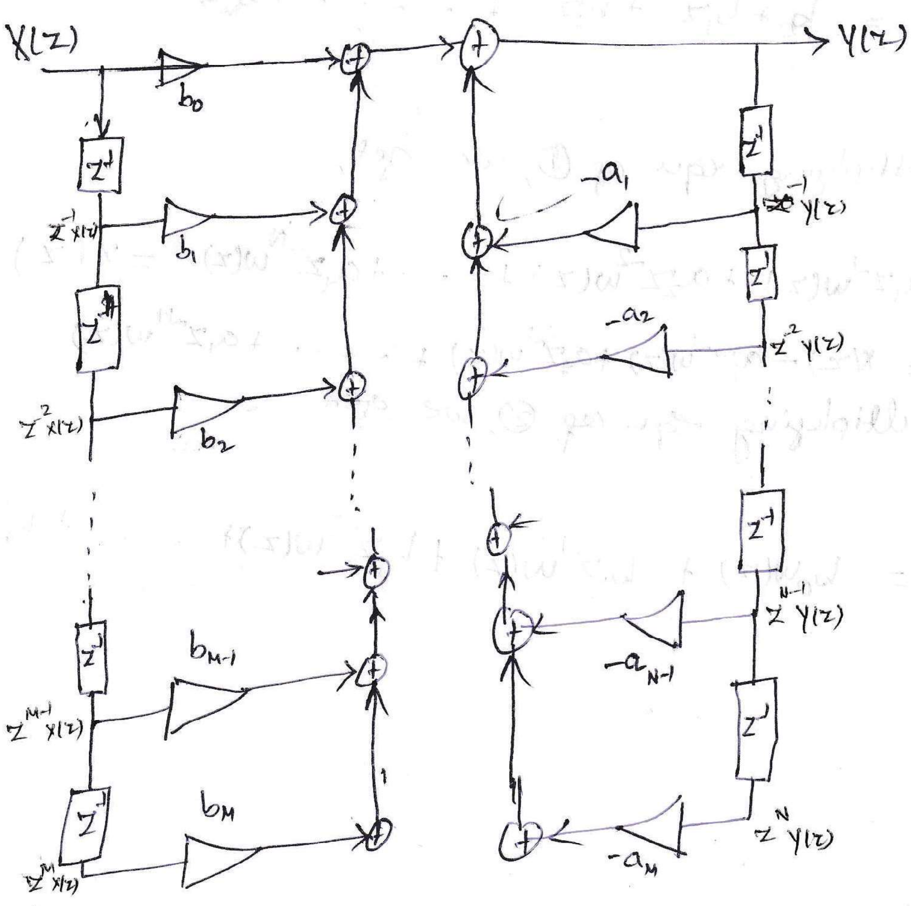
Let,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) - \dots - a_N y(n-N) + b_0 x(n) + b_1 x(n-1) + \dots + b_M x(n-M)$$

on taking Z- transform,

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) - \dots - a_N z^{-N} Y(z) + b_0 X(z) + b_1 z^{-1} X(z) + \dots + b_M z^{-M} X(z).$$



* In this structure, using seperate delay(z^{-1}) for I/p and o/p samples.

* So, they required more memory.

This is provide direct relation b/w time domain and Z-domain equ.

(2) Direct Form - II

It is reduced No. of delay elements,

* generation difference equ, is,

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k).$$

After simplification, the result of H(z) is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

Let,

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \cdot \frac{Y(z)}{W(z)}$$

where,

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}} \quad \text{—— (1)}$$

$$\frac{Y(z)}{W(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M} \quad \text{—— (1)}$$

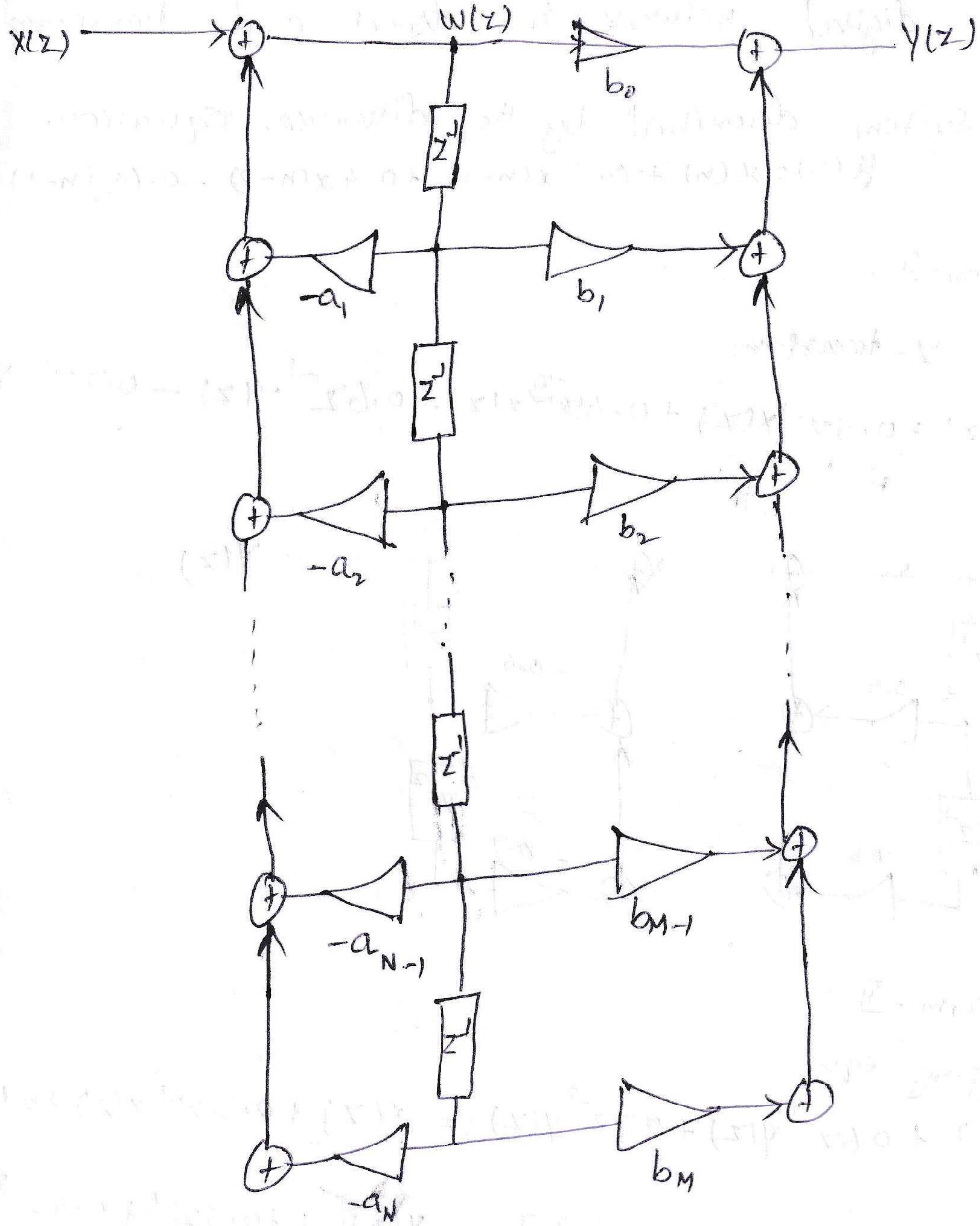
On cross multiplying equ of (1), we get,

$$W(z) + a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + \cdots + a_1 z^{-N} W(z) = X(z)$$

$$\therefore W(z) = X(z) - a_1 z^{-1} W(z) + a_2 z^{-2} W(z) + \cdots + a_1 z^{-N} W(z)$$

On cross multiplying equ of (2), we get

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) + \cdots + b_M z^{-M} W(z).$$



X(Z)
W(Z)
Y(Z)
b_0
z^{-1}
$-a_1$
b_1
$-a_2$
b_2
$-a_{N-1}$
b_{M-1}
$-a_N$
b_M

→ Find the digital network in direct and transposed form of the system described by the difference equation.

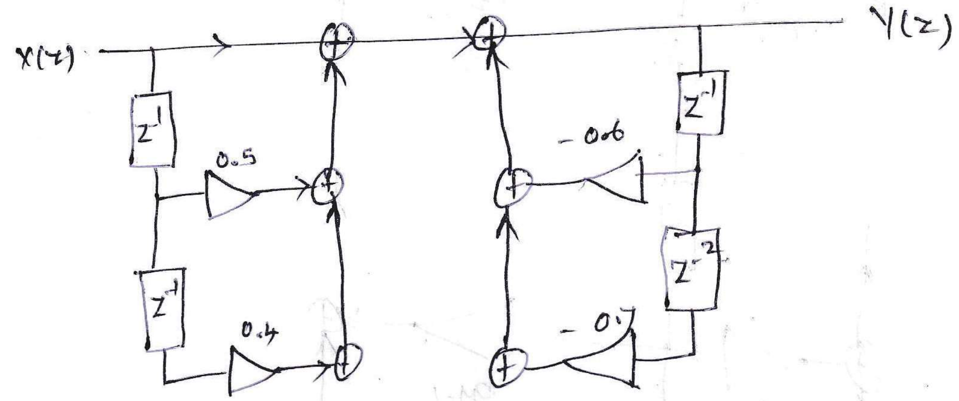
$$y(n) = x(n) + 0.5x(n-1) + 0.4x(n-2) - 0.6y(n-1) - 0.7y(n-2)$$

sol

Direct form - I :

taking z-transform:

$$Y(z) = X(z) + 0.5z^{-1}X(z) + 0.4z^{-2}X(z) - 0.6z^{-1}Y(z) - 0.7z^{-2}Y(z)$$



Direct Form - II

Rearranging equ,

$$Y(z) + 0.6z^{-1}Y(z) + 0.7z^{-2}Y(z) = X(z) + 0.5z^{-1}X(z) + 0.4z^{-2}X(z)$$

$$Y(z)\left[1 + 0.6z^{-1} + 0.7z^{-2}\right] = X(z)\left[1 + 0.5z^{-1} + 0.4z^{-2}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 0.5z^{-1} + 0.4z^{-2}}{1 + 0.6z^{-1} + 0.7z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{W(z)}{X(z)} \times \frac{Y(z)}{W(z)}$$

$$\therefore \frac{W(z)}{X(z)} = \frac{1}{1 + 0.6z^{-1} + 0.7z^{-2}} \quad \text{—— (1)}$$

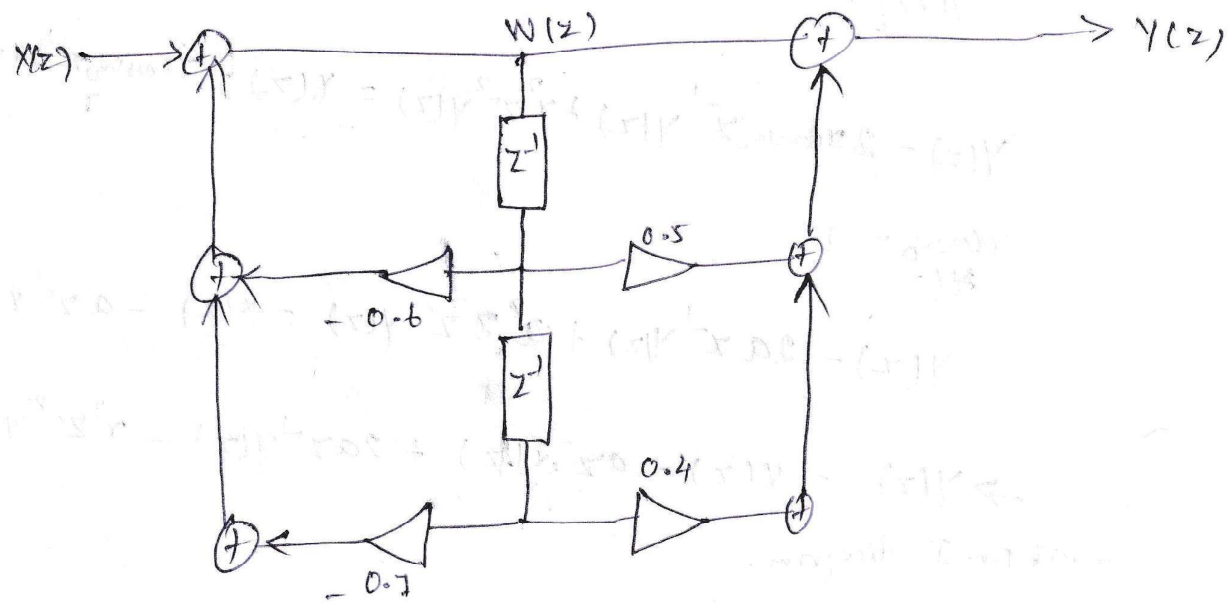
$$\frac{Y(z)}{X(z)} = 1 + 0.5z^{-1} + 0.4z^{-2} \qquad \text{—— (2)}$$

On cross multiplying equ. (1) & (2) we get.

$$W(z) + 0.6z^{-1}W(z) + 0.7z^{-2}W(z) = X(z)$$

$$W(z) = X(z) - 0.6z^{-1}W(z) - 0.7z^{-2}W(z)$$

$$Y(z) = X(z) + 0.5z^{-1}X(z) + 0.4z^{-2}X(z).$$



→ Realize the digital N/w described by H(z) in two ways,

$$H(z) = \frac{1 - r\cos w_0 z^{-1}}{1 - 2r\cos w_0 z^{-1} + r^2 z^{-2}}$$

sol

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - r\cos w_0 z^{-1}}{1 - 2r\cos w_0 z^{-1} + r^2 z^{-2}}$$

on the Cross Multipying we get.

$$Y(z)\left[1 - 2r\cos w_0 z^{-1} + r^2 z^{-2}\right] = X(z)\left[1 - r\cos w_0 z^{-1}\right]$$

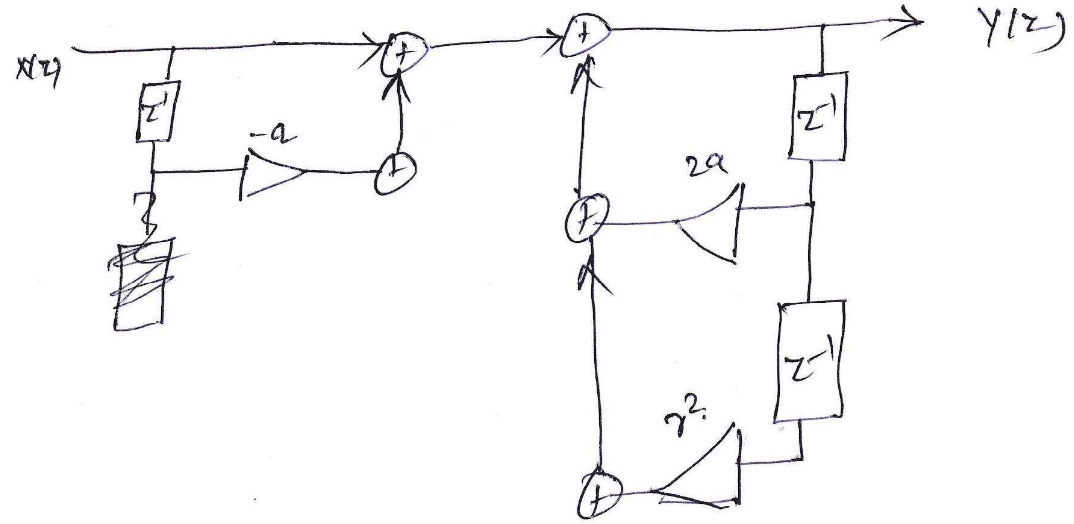
$$Y(z) - 2r\cos w_0 z^{-1} Y(z) + r^2 z^{-2} Y(z) = X(z) - r\cos w_0 z^{-1} X(z)$$

$r\cos w_0 = a$

$$Y(z) - 2a z^{-1} Y(z) + r^2 z^{-2} Y(z) = X(z) - a z^{-1} X(z)$$

$$\rightarrow Y(z) = X(z) - a z^{-1} X(z) + 2a z^{-1} Y(z) - r^2 z^{-2} Y(z).$$

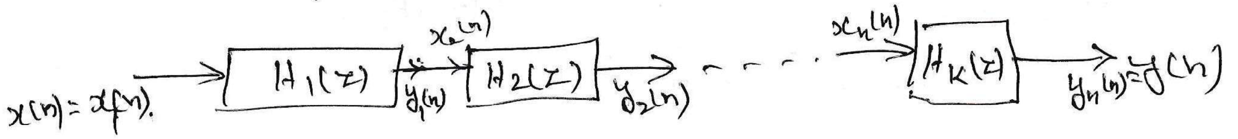
direct For. I diagram.



Cascade form

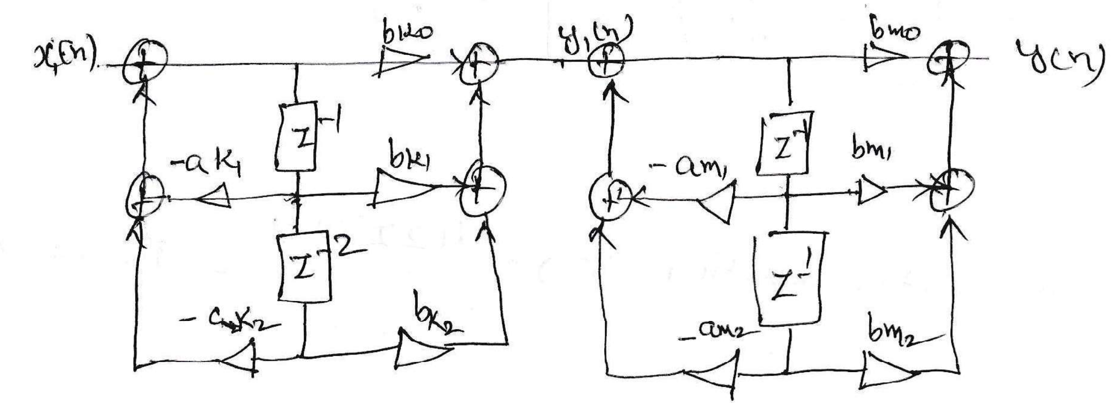
Let us consider an IIR system with fun.

$H(z) = H_1(z) \cdot H_2(z) \cdots H_K(z)$. This can be represented using block diagram as,



* Now realize each $H_K(z)$ in direct form-II and cascade all structure.

$$H(z) = \frac{(b_{K0} + b_{K1} z^{-1} + b_{K2} z^{-2})\,(b_{m0} + b_{m1} z^{-1} + b_{m2} z^{-2})}{(1 + a_{K1} z^{-1} + a_{K2} z^{-2})\,(1 + a_{m1} z^{-1} + a_{m2} z^{-2})} = H_1(z) \cdot H_2(z)$$



→ Realize the system with different equation.

$$y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{3} x(n-1) \text{ in cascade form.}$$

Sol)

take Z-transform,

$$Y(z) = \frac{3}{4} z^{-1} Y(z) - \frac{1}{8} z^{-2} Y(z) + X(z) + \frac{1}{3} z^{-1} X(z)$$

$$Y(z) - \frac{3}{4} z^{-1} Y(z) + \frac{1}{8} z^{-2} Y(z) = X(z) + \frac{1}{3} z^{-1} X(z)$$

$$Y(z)\left[1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}\right] = X(z)\left[1 + \frac{1}{3} z^{-1}\right]$$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}}$$

~~take~~ apply polynomial equ. both numerator & denominator

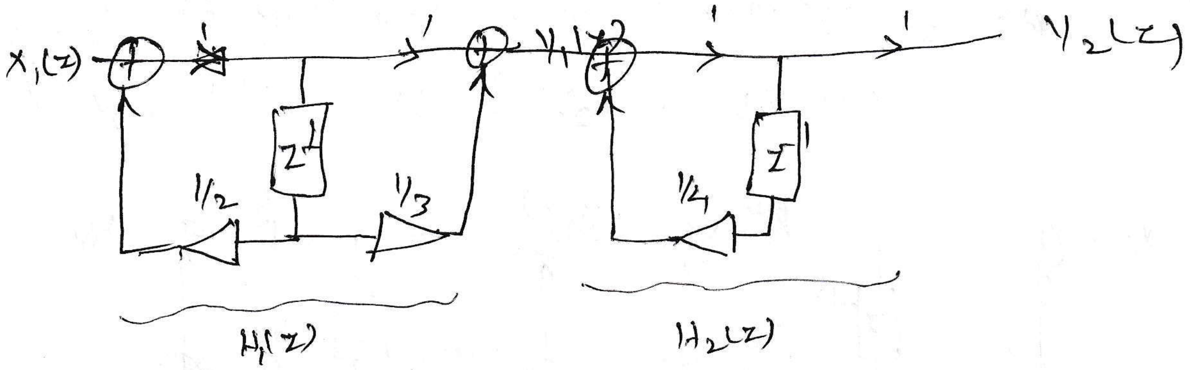
$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

$\frac{1}{8}$

$-\frac{1}{4}$ $-\frac{1}{2}$

$$H(z) = \frac{1+\frac{1}{3}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = H_1(z)\cdot H_2(z)$$

$$H_1(z) = \frac{1+\frac{1}{3}z^{-1}}{1-\frac{1}{2}z^{-1}} \quad ; \quad H_2(z) = \frac{1}{\left(1-\frac{1}{4}z^{-1}\right)}$$



→ for the system function $H(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$ in cascade

$$H(z) = \frac{1+2z^{-1}+z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)} = H_1(z) * H_2(z)$$

numerator polynomial simplification,

$= 1+2z^{-1}+z^{-2}$

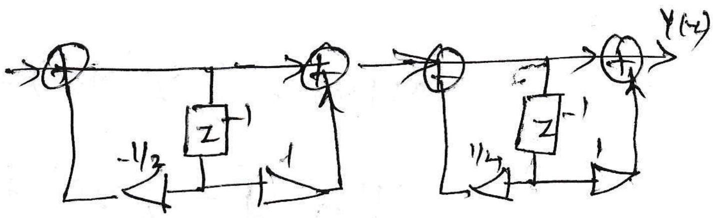
$= z^{-2}\left(\frac{1}{z^{-2}}+\frac{2}{z^{-1}}+1\right)$

$= \frac{1}{z^2}\left(z^2+2z+1\right)$

$= \frac{1}{z^2}(z+1)(z+1)$

$= \frac{z+1}{z} \times \frac{z+1}{z} = (1+z^{-1})(1+z^{-1})$

$\therefore H_1(z) = \dfrac{1+z^{-1}}{1+\frac{1}{2}z^{-1}}$ & $H_2(z) = \dfrac{1+z^{-1}}{1-\frac{1}{4}z^{-1}}$



Parallel Form Realization :-

$$H(z) = \frac{Y(z)}{X(z)} = C + \sum_{i=1}^{K} H_i(z)$$

where, $$H_i(z) = \frac{c_{0i} + c_{1i} z^{-1}}{d_{0i} + d_{1i} z^{-1} + d_{2i} z^{-2}}$$ (second order section)

(or) $$H_i(z) = \frac{c_{0i}}{d_{0i} + d_{1i} z^{-1}}$$ (1^{st} order section)

Parallel form

→ Realize the system given by different equation,

$$y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$$

in parallel form.

Sol take Z-transform,

$$Y(z) = -0.1z^{-1}Y(z) + 0.72z^{-2}Y(z) + 0.7X(z) - 0.252z^{-2}X(z)$$

$$Y(z) + 0.1z^{-1}Y(z) - 0.72z^{-2}Y(z) = 0.7X(z) - 0.252z^{-2}X(z)$$

$$Y(z)[1 + 0.1z^{-1} - 0.72z^{-2}] = X(z)[0.7 - 0.252z^{-2}]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.7 - 0.252z^{-2}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$\begin{array}{r|l} & \quad 0.35 \\ \hline -0.72z^{-2} + 0.1z^{-1} + 1 & \cancel{0.7 - 0.252z^{-2}} \\ & -0.252z^{-2} + 0.7 \\ & -0.252z^{-2} + 0.35 + 0.035z^{-1} \\ & (+) \qquad (-) \qquad (-) \\ \hline & 0.35 - 0.035z^{-1} \end{array}$$

$$H(z) = 0.35 + \frac{0.35 - 0.35z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}$$

$$= 0.35 + \frac{0.35 - 0.35z^{-1}}{(1 - 0.8z^{-1})(1 + 0.9z^{-1})}$$

$$\begin{array}{c} -0.72 \\ / \quad \backslash \\ -0.8 \quad +0.9 \end{array}$$

take partial fraction for above polynomial step,

$$\frac{0.35 - 0.35z^{-1}}{(1 - 0.8z^{-1})(1 + 0.9z^{-1})} = \frac{A}{1 - 0.8z^{-1}} + \frac{B}{1 + 0.9z^{-1}}$$

$$0.35 - 0.35z^{-1} = A(1 + 0.9z^{-1}) + B(1 - 0.8z^{-1})$$

$$0.35 - 0.35z^{-1} = A + 0.9z^{-1}A + B - 0.8z^{-1}B$$

take to arrange above equ,

$$0.35 = A + B \quad \text{--- (1)}$$

$$-0.035z^{-1} = 0.9z^{-1}A - 0.8z^{-1}B$$

(or)

$$-0.035 = 0.9A - 0.8B \quad \text{--- (2)}$$

equ (1) Multiply by 0.8

$$0.28 = 0.8A + 0.8B$$

$$-0.035 = 0.9A - 0.8B$$

$$0.245 = 1.7A$$

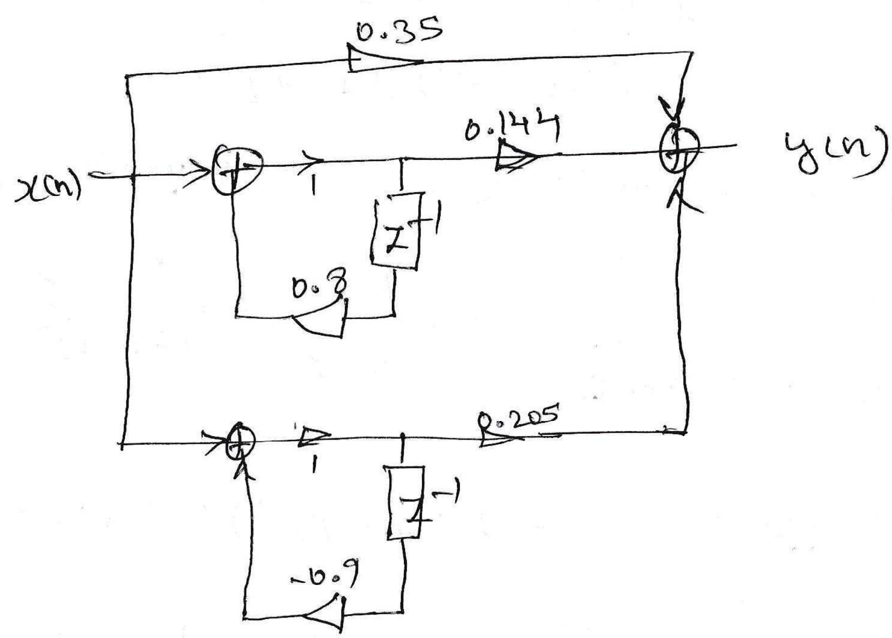
$$\therefore A = 0.144$$

Sub A value to equ (1),

$$0.35 = 0.144 + B$$

$$\therefore B = 0.205$$

$$\therefore H(z) = 0.35 + \frac{0.144}{1-0.8z^{-1}} + \frac{0.205}{1+0.9z^{-1}}$$



① Obtain the direct form I & II realization for the System described by

(a) $y(n) = 0.5y(n-1) - 0.25y(n-2) + x(n) + 0.4x(n-1)$

(b) $y(n) = 2y(n-1) + 3y(n-2) + x(n) + 2x(n-1) + 3x(n-2)$

(c) $y(n) = -0.1y(n-1) + 0.72y(n-2) + 0.7x(n) - 0.252x(n-2)$

(d) $y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{2}x(n-1)$

② (a) for the sys. fun. $H(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$ in Cascade

(b) $$H(z) = \frac{\left(1+\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}\right)\left(1-\frac{3}{2}z^{-1}+z^{-2}\right)}{\left(1+z^{-1}+\frac{1}{4}z^{-2}\right)\left(1+\frac{1}{4}z^{-1}+\frac{1}{2}z^{-2}\right)}$$

③ for the System fun. $H(z) = \dfrac{1+z^{-1}+z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{6}z^{-1}\right)}$.

obtain parallel structure.