# UNIT 16 PLASTIC ANALYSIS

## Structure



## 16.1 INTRODUCTION

The basis for computing the ultimate load (or maximum plastic strength) is the strength of steel in the plastic range. As shown previously, structural steel has the ability to deform plastically after the yield point is reached. The resulting flat stress strain characteristic assures dependable plastic strength, on the one hand, and provides an effective 'limit' to the strength of a given cross section, theoretically making it dependent of further deformation. Thus, when certain parts of a structure reach the yield stress, they maintain that same stress under increasing deformation while other less highly stressed parts deform elastically until they too reach the yield condition, the analysis is considerably simplified because only this fact need be considered. The *continuity* condition is no longer applicable.

The ductility, elastic and plastic behaviour is discussed in Section 16.2. The analysis of axial load is presented in Section 16.3. The plastic analysis under flexural loading is described in Section 16.4. The statical method and mechanism method applied to beams and frames are dicussed in Section 16.5.

### Objectives

After studying this unit, you should be able to

- decide the reserve strength of structure after attaining first yield point,
- determine the shape factor of various cross section,
- decide the collapse load of the structure by statical method and mechanism method, and
- design the beam/frame using plastic method of analysis.

## 16.2 IDEALISED STRESS STRAIN CURVE

### 16.2.1 The Ductility of Steel

Steel possesses *ductility* a unique property that no other structural material exhibits in quite the same way. Through ductility, structural steel is able to absorb large deformations beyond the elastic limit without danger of fracture. It is this characteristic feature of structural steel that makes possible the application of plastic analysis of structural design (Figure 16.1).

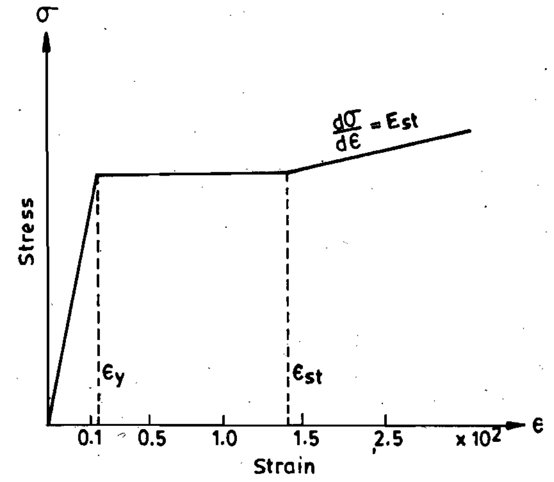



**Figure 16.1 : Idealised Stress-Strain Curve**

Average value,

$$E = 2.0 \times 10^5 \text{ MPa}$$

$$\sigma_y = 250 \text{ MPa}$$

$$\varepsilon_{st} = 0.015$$

This ductility is evident from Figure 16.1, which shows in somewhat idealized form the stress-strain properties of steel in the initial portion of the curve. This idealization is a very close approximation to the actual behavior of structural steel. The compressive and tensile stress-strain relationships are found to be practically identical, and are assumed so in the plastic theory.

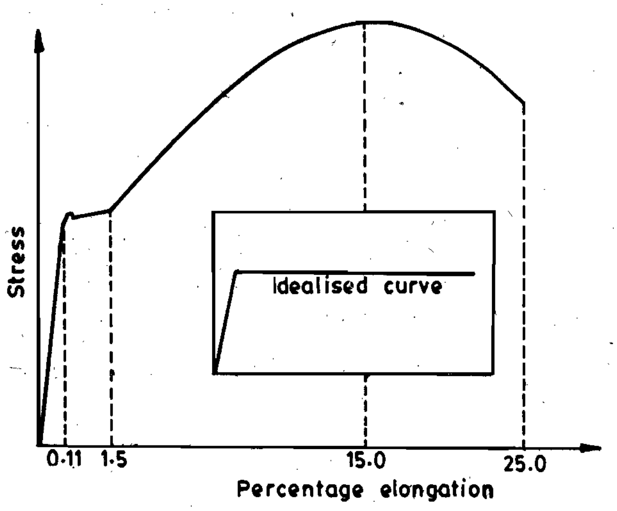



**Figure 16.2 : Stress-Strain Curve for Structural Steel**

As seen from Figure 16.2, for ordinary structural steel, final failure by rupture occurs only after a specimen has stretched some 15 to 25 times the maximum strain that is encountered in plastic design. Even in plastic design, at ultimate load the critical strains will not have exceeded about 1.5% elongation. Thus the use of ultimate load as the design criterion still leaves available major portion of the reserve ductility of steel which can be used as an added margin of safety. This maximum strain of 1.5% is a strain at ultimate load in the structure and not at working load. Further, this strain does not exist through out the structure but only at a few critical sections and for a limited length in each section.

### 16.2.2 Elastic and Plastic Behaviour of a Material

The material is said to behave elastically when the strains caused in a material by application of load disappear with removal of load. The largest value of the stress for which the material behaves elastically is called the elastic limit of the material. In other words, the material behaves elastically and linearly as long as the stress is kept below the yield point. If the stress is reached beyond yield point and, when the load is removed, the stress and strain decrease linearly alongside *CD* parallel to the straight line portion *AB* of the loading curve. The fact that strain ($\varepsilon$) does not return to zero after the load has been removed indicates that a plastic deformation of the material has taken place.

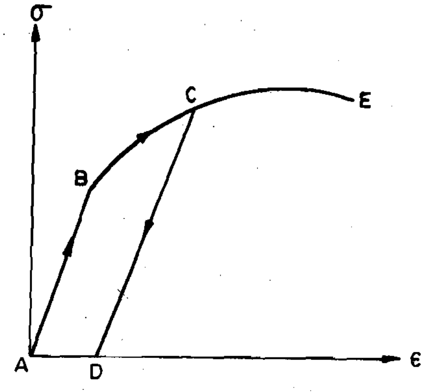



**Figure 16.3 : Plastic Deformation**

Plastic behavior can be studied by considering an idealized elasto-plastic material for which the stress-strain diagram consists of the two straight-line segments shown in Figure 16.4.

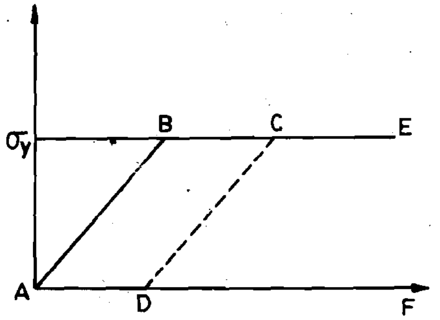



**Figure 16.4 : Idealised Stress-Strain Diagram for an Elasto-Plastic Material**

It can be noted that the stress-strain diagram for mild steel in the elastic and plastic ranges is similar to this idealisation as long as the stress $\sigma$ is less than the yield strength $\sigma_y$ , and the material behaves elastically and obeys Hooke's law, $\sigma = E\,\varepsilon$. When $\sigma$ reaches the value $\sigma_y$, the material starts yielding and keeps deforming plastically under the constant load. However, no actual material behaves exactly as shown in Figure 16.4.

## 16.3 PLASTIC ANALYSIS FOR AXIAL LOAD

Plastic design is an advantageous replacement for conventional elastic design as applied to loaded structural steel frame of certain types. These are rigid jointed frames, continuous or restrained beams and girders and statically indeterminate structures in general which are stressed primarily in bending.

It is not suggested that plastic design be applied to steel frames with statically determinate beams and girders, nor to simple structures with effectively pin-connected members.

Statically determinate structure has one yield while statically indeterminate structure have more than one yield points. Load deflection curve of statically determinate structure has one point of maximum load, and thus, its true or ultimate load carrying capacity is little above the yield load $P_y$.

On other hand, load-deflection curve of statically indeterminate structure has two point of maximum load ($P_{y_1}$, $P_{y_2}$). Quite the contrary, there is a considerable reserve of load carrying capacity beyond the first yield load ($P_{y_1}$).

Thus, in the case of statically indeterminate structure, the failure load is not reached until yield developed at central rod and at end rods. Through plastic design, this reserve strength beyond the elastic limit may be utilised.

### Example 16.1

For the statically indeterminate structure shown in Figure 16.5 (a), following data is available :

$\sigma_y = 250$ MPa $\qquad E = 2 \times 10^5$ MPa $\qquad A = 400$ mm$^2$

Plastically analyse the structure.

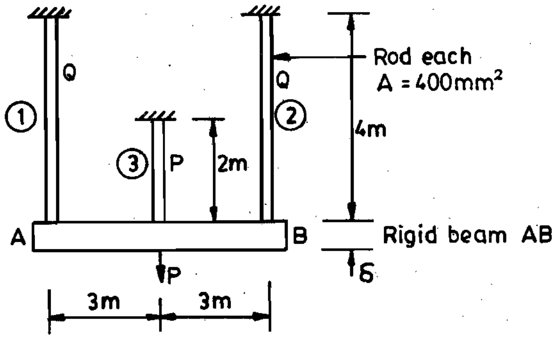



Figure 16.5 (a)

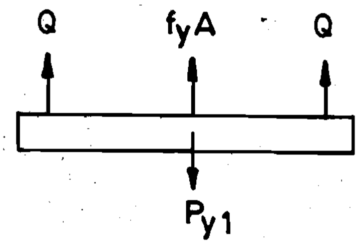



Figure 16.5 (b)

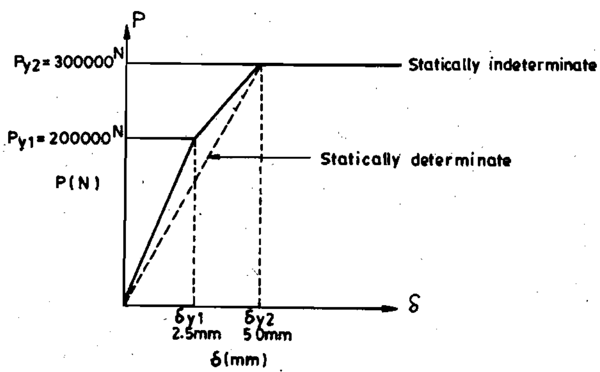



Figure 16.5 (d)

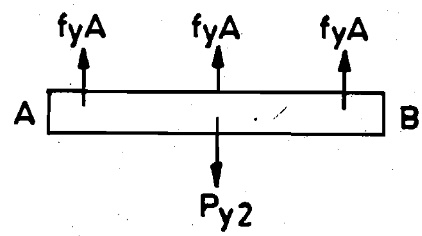



Figure 16.5 (c)

### Solution

*First yield*

Comparing the deflection of rods 1 and 3, the load $P$ carried by short rod is more. Therefore, yielding of rod 3 occurs first. Rods 1 and 2 have same internal force as they have same length and area.

*Equilibrium Condition*

$$P_{y_1} = 2Q + P$$

$$P_{y_1} = 2Q + 250 \times 400$$

$$= 2Q + 100000 \qquad (16.1)$$

*Compatibility Condition*

$$\delta_1 = \delta_2 = \delta_3 = \delta_{y_1}$$

$$\frac{Q \times 4000}{400 \times 2 \times 10^5} = \frac{250 \times 400 \times 2000}{400 \times 2 \times 10^5}$$

$$Q = 50000 \text{ N}$$

Substituting value of $Q$ in Eq. (16.1)

$$P_{y_1} = 2 \times 50000 + 250 \times 400$$

$$= 200000 \text{ N}$$

$$\delta_{y_1} = \frac{50000 \times 4000}{400 \times 2 \times 10^5}$$

$$= 2.5 \text{ mm}$$

*Plastic Stage*

Due to increase in load $P$ there will be plastic deformation of rod (3) and forces in rod (1) and (2) will increase to yield load.

*Equilibrium Condition*

$$P_{y_2} = 3 \times \sigma_y \times A$$

$$= 3 \times 250 \times 400 = 300000 \text{ N}$$

Deflection of rigid beam $AB$ is equal to elongation of rod (1) or (2).

$$\delta_{y_2} = \frac{(\sigma_y \times A) \times l_1}{AE} = \frac{250 \times 400 \times 4000}{400 \times 2 \times 10^5}$$

$$= 5 \text{ mm}$$

After yield in all three rods, there is increase in deflection of rigid beam $AB$ at constant yield load $P_{y_2}$ till system collapses.

## SAQ 1

If rod 3 was a bent, what will be load at yield and corresponding deflection ? A vertical force $P$ is supported by three rods arranged in a vertical plance as shown in Figure 16.6, calculate the collapse load.

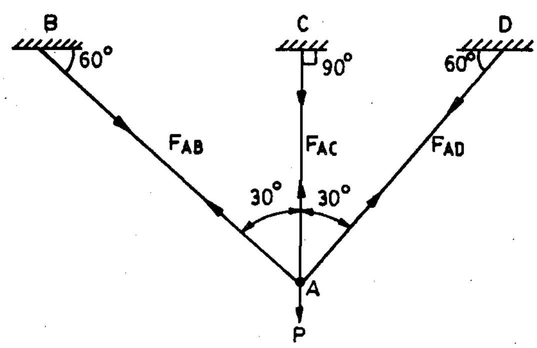



Figure 16.6

# 16.4 PLASTIC ANALYSIS OF A SECTION UNDER FLEXURE

## 16.4.1 Plastic Moment of Resistance

**Assumptions**

The simple plastic theory of beams takes advantage of the ductility of steel. The principal assumptions and conditions are used in the development of the moment-curvature ($M$–$\phi$) relationship as follows :

(a) Strains are proportional to the distance from the neutral axis (plane sections under bending remain plane after deformation).

(b) The stress strain relationship is idealized to consist of two straight lines

$$\sigma = E\varepsilon \, (0 < \varepsilon < \varepsilon_y)$$

$$\sigma = \sigma_y \, (\varepsilon_y < \varepsilon < \infty)$$

The properties in compression are assumed to be the same as those in tension. Also the behavior of fibers in bending is assumed to be the same as in tension or compression.

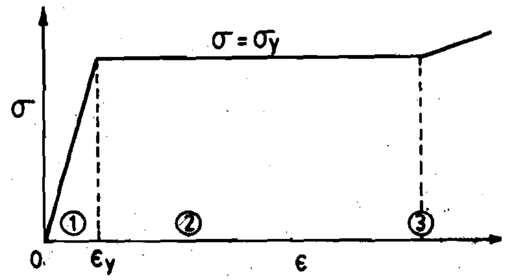



**Figure 16.7 (a) : Idealised Stress-Strain Diagram**

(c) Deformation are sufficiently small so that

$$\theta = \tan\theta \quad (\theta = \text{curvature}).$$

(d) The equilibrium condition are given by

Normal force $P = \int_{\text{area}} \sigma \, dA$

Moment $M = \int_{\text{area}} (\sigma . \, dA) \, y$

where $\sigma$ = stress at distance $y$ from the neutral axis.

**Moment Curvature Relationship**

As a background and for later comparison with inelastic case, the equations for elastic bending are :

$$\phi = \frac{1}{R} = \frac{\varepsilon}{y} = \frac{\sigma}{Ey} = \frac{M}{EI}$$

$$M = EI\,\phi$$

$$My = \sigma_y \, S$$

where,

$\theta$ = Curvature,

$R$ = Radius of curvature,

$\varepsilon$ = Strain,

$y$ = Distance from neutral axis of fiber,

$\sigma$ = Stress at distance $y$ from neutral axis,

$\sigma_y$ = Yield stress level,

$E$ = Modulus of elasticity,

$I$ = Moment of inertia, and

$S$ = Section modulus $\left(\frac{I}{C}\right)$.

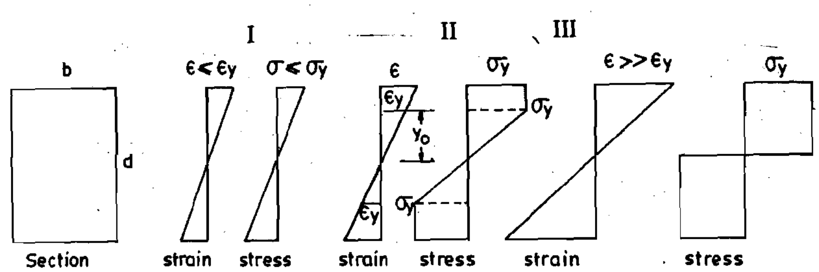



**Figure 16.7 (b) : Theory of Plastic Bending**

(a) *Elastic Stage*

Strain $\varepsilon$ at extreme fibre is less than yield strain $\varepsilon_y$ and stress $\sigma$ is less than yield stress, i.e. $\sigma < \sigma_y$. Refer Figure 16.7 (b) I.

According to elastic theory

$$M = \sigma Z \quad \text{where } \sigma = \text{extreme fibre stress}$$
$$Z = \text{section modulus}$$

When $\varepsilon$ reaches a value of $\varepsilon_y$, $\sigma = \sigma_y$ and section develops yield moment $M_y$

$$M_y = \sigma_y \,.\, Z$$

(b) *Elasto-Plastic Stage*

Strain $\varepsilon > \varepsilon_y$, let yield strain $\varepsilon_y$ occur at $y_0$ from neutral axis then. Refer Figure 16.7 (b) II.

$$\sigma = \sigma_y \text{ at } y_0 \text{ and above.}$$

Taking moment of forces about neutral axis,

$$M = 2\left[\sigma_y\, b\left(\frac{d}{2}\right)\left(\frac{d}{4}\right) - \sigma_y\left(\frac{b\,y_0}{2}\right)\left(\frac{y_0}{3}\right)\right]$$
$$= \sigma_y\left(\frac{bd^2}{4}\right)\left[1 - \left(\frac{4}{3}\right)\left(\frac{y_0^2}{d^2}\right)\right]$$

Substituting $M_y = \sigma_y \,.\, Z = \sigma_y \dfrac{bd^2}{6}$ in the above expression,

$$M = M_y\left[\left(\frac{3}{2}\right) - \frac{2\,y_0^2}{d^2}\right]$$

The value of $y_0$ decreases as the extreme fibre strain increases. The value of $M$ is greater than $M_y$ and it increases as $y_0$ decreases.

(c) *Plastic Stage*

The fully plastic stage in the section occurs when stresses from top and bottom reaches a value of $\sigma_y$ and moment corresponding to this is called fully plastic moment. At this stage, section is devided into two zones of equal area. Refer Figure 16.7 (b) III.

$$M_p = \text{force} \times \text{lever arm}$$
$$= \left(\sigma_y\, b\left(\frac{d}{2}\right) \times \left(\frac{d}{2}\right)\right) = \sigma_y\left(\frac{bd^2}{4}\right)$$

which can be written as

$$M_p = \left(\frac{3}{2}\right)\left(\sigma_y \times \frac{bd^2}{6}\right), \text{ i.e. } 1.5\, M_y$$

and the ratio of $\dfrac{M_p}{M_y} = \dfrac{(\sigma_y Z_p)}{(\sigma_y Z)} = \dfrac{Z_p}{Z} = K$ is called shape factor of the section.

In elasto-plastic stage, we have established a relation

$$\left(\frac{M}{M_y}\right) = \left[\left(\frac{3}{2}\right) - \left(\frac{y_0^2}{d^2}\right)\right]$$

at any point $\dfrac{1}{R} = \dfrac{\varepsilon}{y} = \phi$ as shown in Figure 16.8 (a).

Taking first yield curvature as $\phi_y = \dfrac{\varepsilon_y}{\left(\dfrac{d}{2}\right)}$

$$\frac{y_0}{d} = \frac{y_0}{\varepsilon_y}\frac{\varepsilon_y}{d} = \frac{1}{\phi}\frac{\phi_y}{2} = \frac{\phi_y}{2\phi}$$

Above equation can be reduced to

$$\left(\frac{M}{M_y}\right) = \left[\left(\frac{3}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\phi_y}{\phi}\right)^2\right]$$

$$\left(\frac{M}{M_y}\right) = \left(\frac{3}{2}\right)\left[1 - \left(\frac{1}{3}\right)\left(\frac{\phi_y}{\phi}\right)^2\right]$$

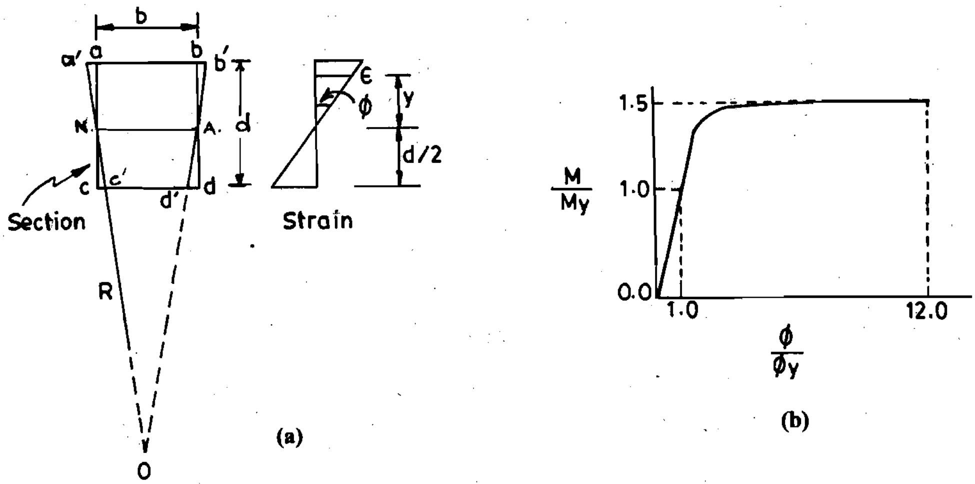



Figure 16.8 : Moment-Curvature Relationship

The relation between $\frac{M}{M_y}$ and $\frac{\phi}{\phi_y}$ can be established and can be shown as in Figure 16.8 (b).

## SAQ 2

A member of uniform rectangular cross section 50 mm by 100 mm (Figure 16.9) is subjected to a bending moment of $M = 24$ kN m. Assuming that the member is made of an elasto-plastic material with a yield stress of 240 MPa and a modulus of elasticity of 200 GPa, determine

(a) depth of elastic core (Figure 16.9),

(b) plastic moment $M_p$, and

(c) the radius of curvature of the neutral surface.

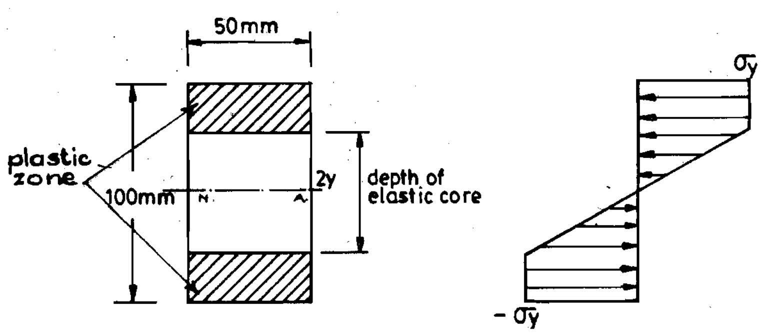



Figure 16.9

### 16.4.2 Plastic Modulus and Shape Factor

For sections which only have symmetry about an axis in the plane of bending, the position of the neutral axis at the plastic moment condition must first be computed before $Z_p$ can be determined. Since $\sum p = 0$ and $\sigma = \sigma_y$, then the area above the neutral axis must equal that below, if equilibrium of horizontal forces is to be maintained. In other words the neutral axis (equal area axis) divides the section into two equal areas. Therefore, $Z_p$ may be defined more generally as the combined statical moment of cross sectional area above and below the equal area axis, and the equation may be written

$$Z_p = \left(\frac{A}{2}\right) y_c + \left(\frac{A}{2}\right) y_t$$

$$Z_p = \left(\frac{A}{2}\right)\left(y_c + y_t\right)$$

where $y_c$ = distance of centroid of upper half area from equal area axis,

$y_t$ = distance of centroid of lower half area from equal area axis.

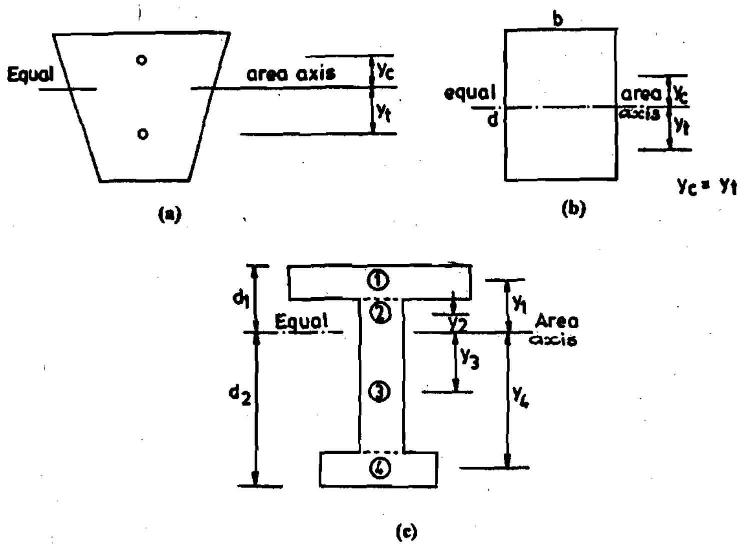



**Figure 16.10 : Plastic Modulus : Different Sections**

In general,

$$Z_p = \sum_{i=1}^{NP} A_i\, y_i$$

where $A_i$ is part area of a section and $y_i$ is corresponding distance of centroid measured from equal area axis. $NP$ denotes number of parts made.

For symmetrical sections equal area axis and neutral axis coincide and hence plastic modulus is twice the statical moment taken about the neutral axis of the half sectional area.

For non-rectangular cross-section $K$ is generally not equal to 3/2. For structural shapes such as wide flanges beams (I-section), this ratio varies approximately from 1.08 to 1.14.

Because it depends only upon the shape of the cross-section, the ratio $K = \dfrac{M_p}{M_y}$ is referred to as the shape factor of the cross section. We note that, if the shape factor $K$ and the maximum elastic moment $M_y$ of a beam are known, the plastic moment $M_p$ of the beam may be obtained by

$$M_p = K\, M_y$$

The ratio $\dfrac{M_p}{\sigma_y}$ obtained by dividing the plastic moment $M_p$ of a member by the yield strength $\sigma_y$ of its material is called the plastic section modulus of the member and is denoted by $Z_p$. When the plastic section modulus $Z_p$ and the yield strength $\sigma_y$ of a beam are known, the plastic moment $M_p$ of the beam may be computed as $M_p = Z_p\, \sigma_y$.

Similarly,

$$M_y = Z\sigma_y$$

where $Z$ is elastic section modulus, $\left(\frac{M_y}{\sigma_y}\right)$. We note that the shape factor $K = \left(\frac{M_p}{M_y}\right)$ of a given cross-section may be expressed as

$$K = \frac{M_p}{M_y} = \frac{Z_p\,\sigma_y}{Z\,\sigma_y} = \frac{Z_p}{Z}$$

Here,
$$Z_p = \frac{M_p}{\sigma_y} = \frac{\left(\frac{bd^2}{4\sigma_y}\right)}{\sigma_y} = \frac{bd^2}{4} \text{ and}$$

$$Z = \frac{M_y}{y} = \frac{\left(\frac{bd^2}{6\sigma_y}\right)}{\sigma_y} = \frac{bd^2}{6}$$

Hence,
$$K = \frac{Z_p}{Z} = \frac{\frac{1}{4}bd^2}{\frac{1}{6}bd^2} = \frac{3}{2}$$

**Example 16.2**

The section shown in Figure 16.11 is ISMB 400. Calculate plastic moment $M_p$ and shape factor. Take $\sigma_y = 250$ MPa.

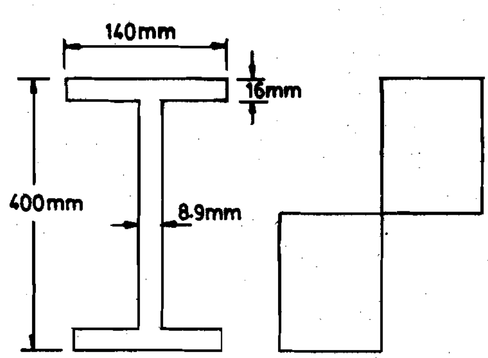



**Figure 16.11**

**Solution**

The moment of inertia about centroidal axis,

$$I_x = \frac{1}{12}\left(140 \times 400^3 - 131.1 \times 368^3\right) = 2.02 \times 10^8 \text{ mm}^4$$

Section modulus, $Z = \frac{I_x}{y} = \frac{2.02 \times 10^8}{200} \text{ mm}^3$

The section is symmetrical, therefore, centroidal axis and equal area axis concides.

$$Z_p = 2\left[a_1 y_1 + a_2 y_2\right]$$

$$= 2\left[140 \times 16 \times 192 + 184 \times 8.9 \times \frac{184}{2}\right] = 1.16 \times 10^6 \text{ mm}^3$$

Shape factor,

$$K = \frac{Z_p}{Z} = \frac{1.16 \times 10^6}{1.01 \times 10^6} = 1.148$$

Thus, plastic moment,

$$M_p = \sigma_y Z_p = 250 \times 1.16 \times 10^6 \text{ N mm} = 2900 \text{ kN m}$$

SAQ 3

Obtain plastic modulus and shape factor for section shown in Figure 16.12 to 16.15.

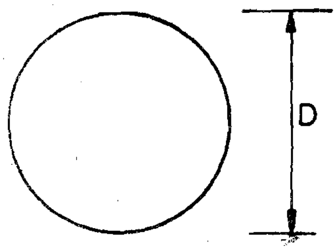



Figure 16.12

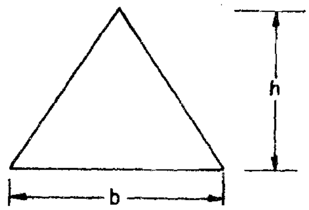



Figure 16.13

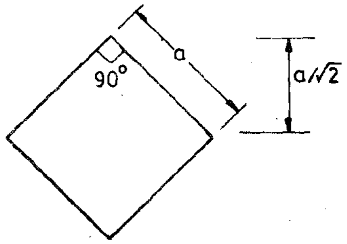



Figure 16.14

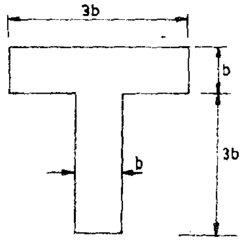



Figure 16.15

## 16.4.3 Concept of Plastic Hinge

From the above $M$-$\phi$ characteristic of the plastic hinge, the following two features are particularly important :

(a) After the elastic limit is reached, the curve approaches very rapidly to the horizontal line corresponding to the plastic moment value.

(b) There is an infinite increase in curvature at constant moment.

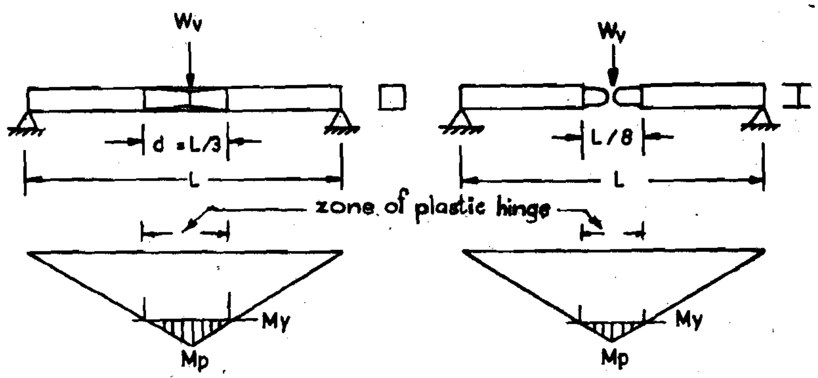
These two features are expressed in the following idealization of Figure 16.2 through the use of two straight lines,

$$M = EI\phi \ (0 < \phi < \phi_p)$$

$$M = M_p \ (\phi > \phi_p)$$

where $\phi_p = \dfrac{M_p}{EI}$.

According to above, the member remains elastic until the moment reaches $M_p$. Thereafter rotation occurs at constant moment, i.e. the member acts as if it were hinged except with a constant restraining moment $M_p$.




(a) Rectangular Beam ($K = 1.50$)

(b) Wide-flange Beam ($K = 1.14$)

Figure 16.16 : Length of Plastic Hinge

### Distribution of the Plastic Hinge

For the idealised $M$-$\phi$ curve, the plastic hinges form at discrete points at which all plastic rotation occurs. Thus, the length of the hinge approaches zero. Actually, extent of the hinge over a length of member is dependent on the loading and the geometry. For example, in the rectangular beam for $M_y = 0.67\ M_p$ the hinge length is equal to one-third of the span. For wide flange beam with a shape factor of 1.14 and loaded as shown in Figure 16.16 the hinge length is $L/8$. In other words, the hinge length $L$ is the length of the beam over which the moment is greater than the yield moment $M_y$.

### Example 16.3

Evaluate the length of a plastic hinge for the beam loaded as shown in Figure 16.17.

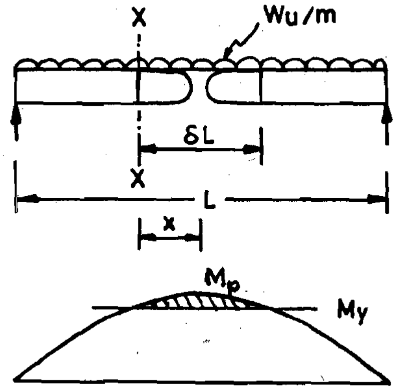



**Figure 16.17**

### Solution

Here,
$$M_y = \frac{W_u L}{2}\left(\frac{L}{2} - x\right) - \frac{W_u}{2}\left(\frac{L}{2} - x\right)^2$$

$$M_y = \frac{W_u L^2}{8} - \frac{4x^2}{L^2} \times \frac{W_u L^2}{8}$$

We know that
$$\frac{W_u L^2}{8} = M_p$$

Thus,
$$M_y = M_p - 4\,M_p \frac{x^2}{L^2}$$

$$\therefore\ 4\,M_p \frac{x^2}{L^2} = M_p - M_y$$

Dividing both side by $M_p$

$$4\frac{x^2}{L^2} = 1 - \frac{M_y}{M_p} = 1 - \frac{1}{K}$$

where $K$ = shape factor.

$$x^2 = \frac{L^2}{4}\left(1 - \frac{1}{K}\right)$$

$$x = \frac{L}{2}\sqrt{\left(1 - \frac{1}{K}\right)}$$

Hinge length
$$\delta L = 2x = L\sqrt{\left(1 - \frac{1}{K}\right)}$$

For rectangular section, $K = \dfrac{3}{2}$

$$\delta L = L\sqrt{\left(1 - \frac{2}{3}\right)} = 0.577\,L$$

**Characteristics of Plastic Hinge**

- A plastic hinge is a zone of yielding due to flexure in a structural member. Even though its length depends on the geometry and loading, it is assumed that all plastic rotation occurs at a point. At those sections where plastic hinge are formed, the member acts as if it were hinged except with a constant restraining moment $M_p$. In other words, capacity of the plastic hinge is equal to $M_p$.
- In a framed structure with prismatic members plastic hinges may occur at points of concentrated loads, at the ends of each member meeting at a connection involving change in geometry, and at the point of zero shear in a span under distributed load.
- Where two members meet, a hinge will form in the member whose moment of resistance is lesser of the two.
- Plastic hinges are reached first at sections subjected to greatest curvature. Formation of plastic hinges allows a subsequent redistribution of moment until $M_p$ is reached at each critical section. Redistribution will be effective if the plastic hinges form with a phase difference between each other.

### 16.4.4 Redistribution of Moments

Besides the modest increase of load that results from the formation of a plastic hinge, a second factor contributing to the reserve strength of statically indeterminate structure is called "*Redistribution of Moment*". It is a consequence of the action of plastic hinges.

Consider the fixed ended beam under the action of a uniformly distribution load.

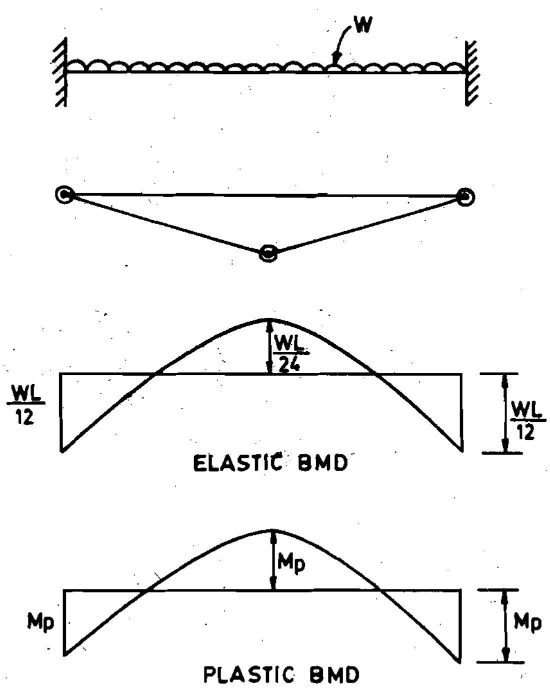



**Figure 16.18 : Redistribution of Moments**

From the elastic bending moment diagram it is seen that the elastic moments are maximum at the two ends. As the load is gradually increased on the structure plastic moment capacity $M_p$ is reached at these two critical sections that are most highly stressed in the elastic range. Because of symmetry, plastic hinges will form simultaneously at the two ends.

As load further increases, this plastic moment $M_p$ is maintained at these two sections. These sections continue to rotate offering at the same time a resistance equal to $M_p$. Now the other less highly stressed section is the middle section. With the increase in load, a plastic hinge is developed at this section. With the development of three hinges a mechanism is formed. In this particular case, with the formation of third hinge in the middle, the ultimate load is reached converting the "*structure*" into a "*Mechanism*". Thus, mechanism is a situation in which sufficient number of *plastic hinges* alongwith the *real or structural hinges* are developed so as to cause the collapse of the structure without out any further increase in load. The load corresponding to the mechanism formation is known as ultimate load.

It will be of interest to compute reserve strength which is defined as the ratio of the the ultimate load $W_u$ to the load at first yield $W_y$ of the structure

$$\text{Reserve strength} = \frac{W_u}{W_y}$$

The yield load and the ultimate load may be computed from the elastic and plastic bending moment diagrams respectively.

First yield will occur in the structure at the section where the elastic moment is maximum. Then from the elastic bending moment diagram

$$\frac{W_y L}{12} = M_y \quad \longrightarrow \quad W_y = \frac{12 M_y}{L}$$

and from the plastic bending moment diagram

$$\frac{W_u L}{8} = M_p + M_p$$

Giving

$$W_u = \frac{16 M_p}{L}$$

Now the reserve strength

$$= \frac{W_u}{W_y} = \left(\frac{16 M_p}{L}\right) \times \left(\frac{L}{12 M_y}\right)$$

$$= \frac{4 M_p}{3 M_y} = \frac{4}{3} K$$

The quantity 4/3 is called "Redistribution factor" is designated as $K_r$.

Hence Reserve Strength $= K_r \times K$

For rectangular section $K = \frac{3}{2}$,

$$\text{Reserve Strength } K_r.K = \left(\frac{4}{3}\right) \times \left(\frac{3}{2}\right) = 2$$

This means, the structure is able to carry 100% more load beyond first yield.

## 16.5 PLASTIC ANALYSIS OF STRUCTURE

### 16.5.1 Fundamental Principles

**Virtual Displacement**

The **principle of virtual displacement** is useful in expressing the equilibrium condition. It may be stated as follows.

*If a system of forces in equilibrium is subjected to a virtual displacement, the work done by the external forces equal the work done by the internal forces.*

$$W_e = W_i$$

where $W_e$ = external work done, and $W_i$ = Internal work done.

**Upper and Lower Bound Theorems**

These important upper and lower bound theorems or principles were proved by Greenberg and Prager. When both theorems are satisfied in any given problem, then the solution is in fact the '*unique*' one.

*Upper Bound Theorem*

The upper bound theorem may be stated as :

*"A load computed on the basis of an assumed mechanism will always be greater than or at best equal to the true ultimate load."*

Although equilibrium condition will always be satisfied by a solution arrived on the basis of an assumed mechanism which will give a loading that is either correct or too high. This will be explained as follows.

Consider the fixed beam shown in Figure 16.19 (a).

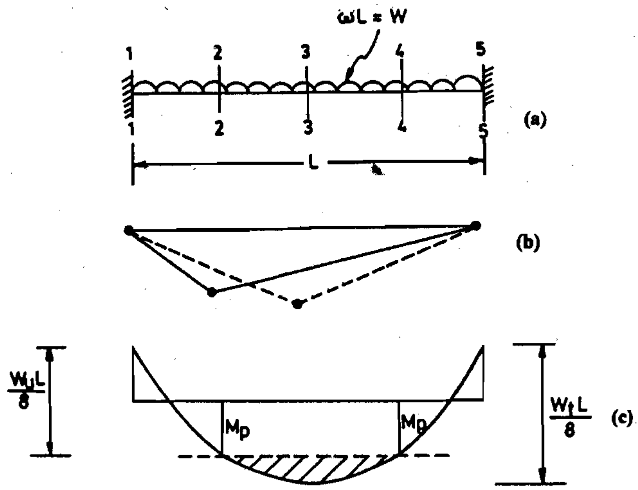



**Figure 16.19 : Upper Bound Theorem**

If a mechanism is assumed as shown by solid lines in Figure 16.19 (b) on the basis of guess that the plastic hinge forms at the point 2, then equilibrium moment diagram would be shown by the solid line in Figure 16.19 (c). Since $M_p$ is exceeded from the point 2 to the point 4, the beam would have to be reinforced over this length in order to carry trial load $W_t$; the load is too great. Only when the mechanism is selected in such a way that the plastic moment value is nowhere exceeded, is the correct load obtained.

*Lower Bound Theorem*

The Lower Bound Theorem may be stated as :

*"A load computed on the basis of an assumed equilibrium moment diagram in which the moments are not greater than $M_p$ is less than or best equal to the true ultimate load."*

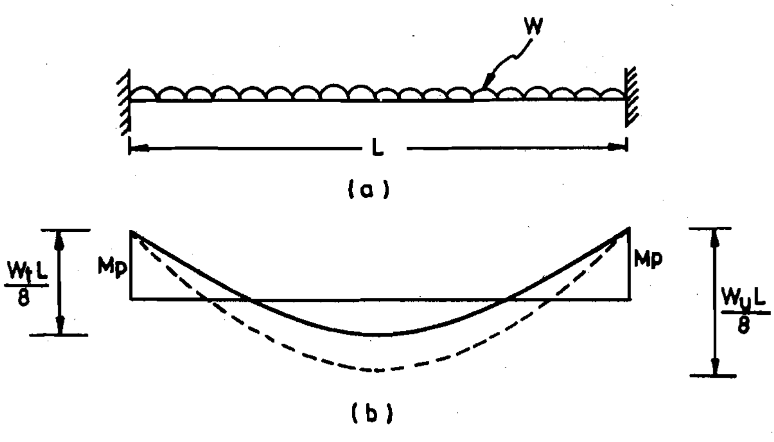



**Figure 16.20 : Lower Bound Theorem**

The load obtained using the assumed moment diagram that does not violate the plastic moment condition will be either correct or too low. This will be explained as follows :

The statical moment diagram (solid curves) is drawn such that the moment is not greater than $M_p$ anywhere. Hinges are formed only at ends. A mechanism is not developed, then the corresponding trial load, $W_t$ may be less than $W_u$.

Because the full load capacity has not been used and the center line moment is less than $M_p$. Only when the load is increased to the point that a mechanism is formed (dotted curves) will the correct load be obtained.

Thus, depending on how the solution to the problem is started, one will obtain upper bound *below* which correct answer must lie or one will determine a lower bound *above* which true value must lie.

**Further Assumptions**

(a) The deformation are assumed to be sufficiently small so that equilibrium conditions can be formulated for the undeformed structure (as in case of elastic analysis).

(b) Instability of the structure will not occur prior to the attainment of the ultimate load.

(c) The connections provide full continuity so that the plastic moment $M_p$ can be transmitted.

(d) The influence of normal and shearing forces on the plastic moment $M_p$ are neglected.

(e) The loading is proportional, that is all loads are such that they increase in fixed proportions to another. However independent increase can be allowed, provided no load failure occurs.

These assumptions, coupled with a moment-curvature relationship that asymptotically approaches a limiting or plastic moment, are the essence of what has been termed as the "simple plastic theory".

### 16.5.2 Comparison of Plastic and Elastic Methods

An analysis according the plastic method must satisfy three conditions stated below.

(a) *Mechanism condition* : the ultimate load is reached when a mechanism forms.

(b) *Equilibrium condition* : summation of forces and moments is equal to zero.

(c) *Plastic moment condition* : the moment may nowhere be greater than $M_p$.

Actually, these conditions are similar to those in elastic analysis which requires a condition of continuity, equilibrium and the limiting stress conditions. This similarity is demonstrated in Figure 16.21.

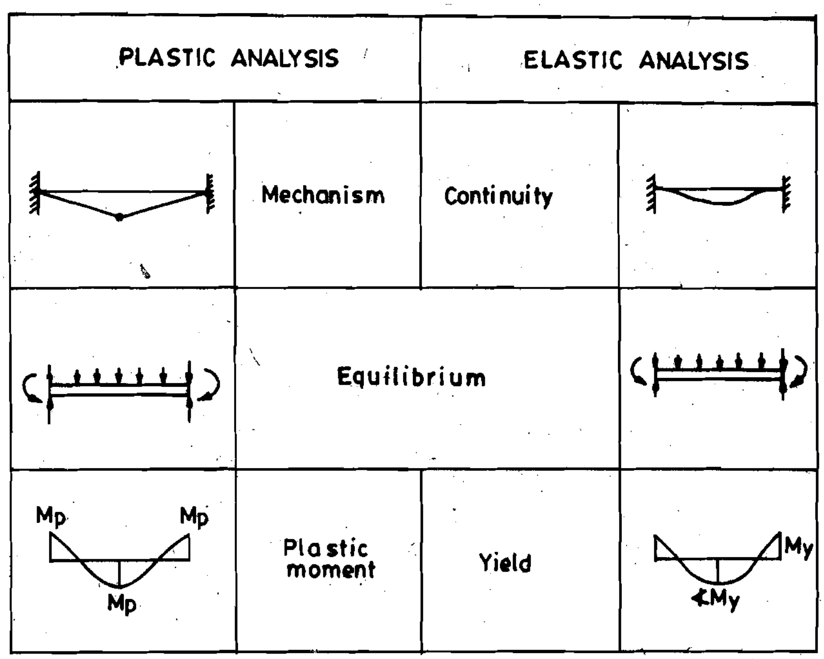



**Figure 16.21 : Necessary Conditions for Plastic Analysis as Compared with Elastic Analysis**

With regard to continuity, the situation in plastic analysis is just the reverse of that exists in elastic analysis. Theoretically plastic hinges interrupt continuity. So the requirement is that sufficient plastic hinges be formed to allow the structure (or part of it) to deform as a mechanism. This is termed a mechanism condition. The equilibrium condition is the same in both the case, namely, the load must be supported. Instead of initial yield, the limit of usefulness is the attainment of plastic hinge moment, not only at one cross section but at each of the critical sections; this is termed as plastic moment condition.

There are two alternative methods for plastic analysis designated from the particular conditions being satisfied.

(a) Mechanism Method : satisfies — Mechanism condition; Equilibrium condition

(b) Statical method : satisfies — Equilibrium condition; Plastic moment condition

In the first method, a mechanism is assumed and the resulting equilibrium equations are solved for the ultimate load. This value is only correct if the plastic moment condition is also satisfied. On the other hand, in statical or "equilibrium" method, equilibrium moment diagram is drawn in such a manner that $M \leq M_P$. The resulting ultimate load is the only correct value if sufficient plastic hinges were assumed to create mechanism.

### 16.5.3 Statical Method

The objective is to find an equilibrium moment diagram in which $M \leq M_P$ such that a mechanism is formed. The procedure is as follows :

(a) Select redundant(s).

(b) Draw bending moment diagram for determinate structure.

(c) Draw bending moment diagram for structure due to unknown redundants.

(d) Sketch the composite moment diagram in such a way that a mechanism is formed (sketch mechanism).

(e) Compute value of ultimate load by solving equilibrium equation.

(f) Check to see that $M \leq M_P$.

**Example 16.4**

The beam shown in Figure 16.22 (a) is statically indeterminate to the second degree. End moments may be selected as redundants. The resulting determinate structure and the beam subjected to the redundant actions are shown in Figure 16.22 (b) and (c). The moment diagram for the determinate structures is shown in Figure 16.22 (d), the moment at the center being given by

$$M_s = \frac{WL}{8}$$

Further, the moment diagram for the structure due to the redundants is shown in Figure 16.22 (e), the moment $M_1$ being an unknown.

The next step is to combine the two moment diagrams [Figure 16.22 (d) and (e)] in such a way that a mechanism is formed. This will be accomplished if the 'fixing line' (designated as A) is drawn in such a way that the moment at sections (1) and (3) is equal to that at section (2). The resulting composite moment diagram is drawn in Figure 16.22 (f).

The equilibrium equation from Figure 16.22 (f) is

$$\frac{W_u L}{8} = M_p + M_p$$

and the ultimate load is given by

$$w_u = \frac{16 M_p}{L^2}; \quad (W_u = w_u L)$$

### 16.5.4 Kinematic Method

**General Procedure**

As the number of redundants increases, the number of possible failure mechanisms also increases. Thus it may become more difficult to construct the correct equilibrium moment diagram. For such cases, the mechanism method of plastic analysis may be used, and various "upper bounds" to the correct load will be obtained for the different possible mechanisms. The correct mechanism will be the one which results in the lowest possible load (upper bound theorem) and for which the moment does not exceed the plastic moment at any section of the structure (lower bound theorem). Thus, the objective is to find the mechanism such that the plastic moment condition is not violated.

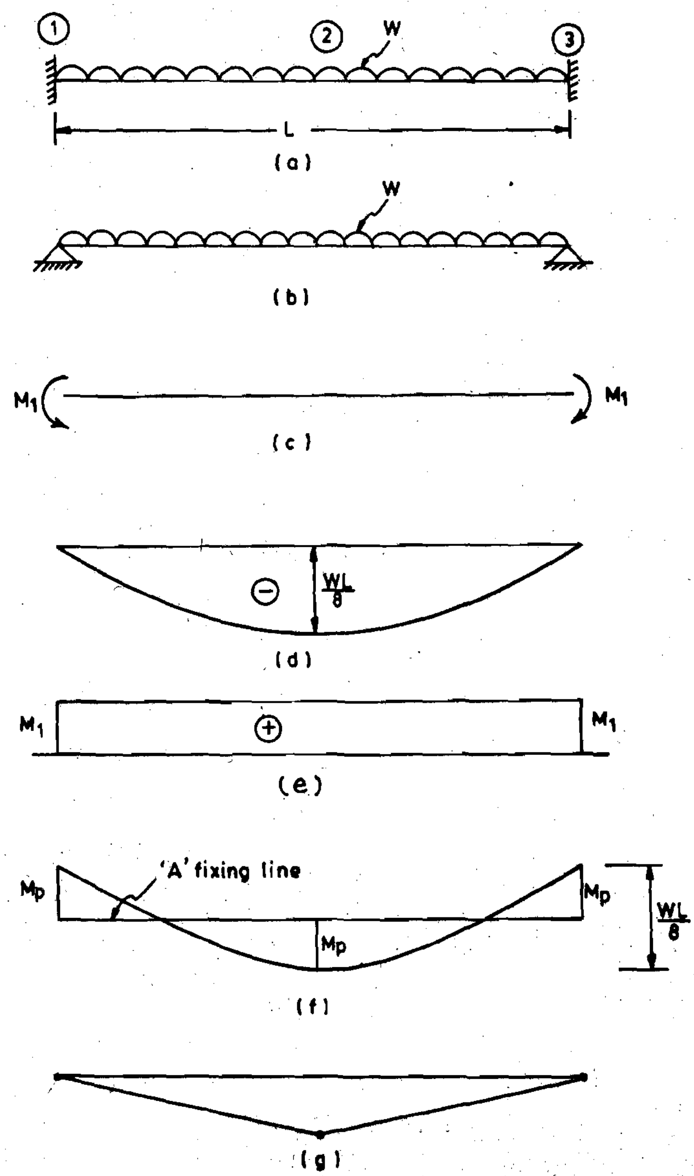



**Figure 16.22 : Plastic Analysis of Fixed-ended Uniformly-Loaded Beam**

### Steps in Mechanism Method (Kinematic Method)

Find a mechanism (independent or composite) such that $M \leq M_p$. Further,

(a) Determine location of possible plastic hinges (load points, connections, point of zero shear in a beam span under distributed load).

(b) Select possible "independent" and "composite" mechanisms.

(c) Solve equilibrium equation (virtual displacement method) for the lowest load.

(d) Check to see that $M \leq M_p$ at all sections.

### Example 16.5

Let the external work done be $W_e$ for the fixed beam shown in Figure 16.23.

Work done by the small load $w_u\, dx$ is as follows :

$$w_u\, dx \times \left( \frac{x\, \delta_{cr}}{\frac{L}{2}} \right)$$

Total work done will be obtained by integration

$$W_e = 2\int_0^{L/2} (w_u\, dx) \left[ \delta_{cr} \cdot \left( \frac{x}{(L/2)} \right) \right]$$

$$W_e = \frac{w_u L\, \delta_{cr}}{2} \qquad \text{but } \delta_{cr} = \frac{L\theta}{2}$$

$$W_e = \frac{w_u L^2 \theta}{4}$$

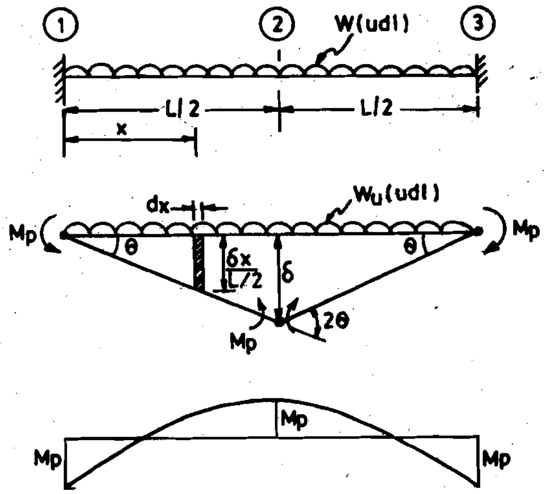



**Figure 16.23**

Internal work done; $W_i$

| **Mechanism Section** | **Plastic Moment at Section (a)** | **Mechanism Angle at Section (b)** | **Internal Work Done = (a) × (b)** |
|---|---|---|---|
| 1 | $M_p$ | $\theta$ | $M_p\theta$ |
| 2 | $M_p$ | $2\theta$ | $2M_p\theta$ |
| 3 | $M_p$ | $\theta$ | $M_p\theta$ |
| | | | $W_i = \Sigma = 4M_p\theta$ |

Now, we know $\qquad W_e = W_i$

On putting the values, $\qquad \dfrac{w_u L^2 \theta}{4} = 4M_p\theta$

Thus, we get $\qquad w_u = \dfrac{16\, M_p}{L^2}$

**Example 16.6**

Find the collapse load using (a) mechanism method, and (b) statical method for the propped cantilever shown in Figure 16.24 (a).

**Solution**

*Mechanism Method*

Strictly speaking, there are infinite number of mechanisms in this problem. However, we can locate the right one in the very beginning.

As the loading on the beam is increased gradually, first plastic hinge will be formed at the fixed end whose capacity is $M_p$. With the formation of this hinge, the structure becomes statically determinate. The second hinge will be formed at the place where the shear is zero (and not at the place where the elastic BM is maximum). To locate the position of zero shear, the following procedure is carried out.

With the known plastic hinge at the fixed end [Figure 16.24 (c)], take moments of loads about A;

$$R_B \times L = \frac{w_u L^2}{2} - M_p$$

$$R_B = \left( \frac{w_u L}{2} - \frac{M_p}{L} \right)$$

The BM at section $X$-$X$ from the end A,

$$M_x = \left( \frac{w_u L}{2} - \frac{M_p}{L} \right) x - \frac{w_u x^2}{2}$$

Thus, shear force,

$$\frac{dMx}{dx} = \frac{w_u L}{2} - \frac{M_p}{L} - \frac{2 w_u x}{2} = 0$$

Hence, at $x = \left( \frac{L}{2} - \frac{M_p}{w_u L} \right)$, the shear is zero. This is the location of second plastic hinge. Thus, with the formation of two plastic hinges, mechanism is formed.

Now $$W_e = W_i$$

$$W_e = \text{Intensity of loading} \times \text{Area swept under the mechanism diagram}$$

$$= \frac{1}{2} w_u \times L \times \delta$$

$$W_i = M_p \, \theta_A + M_p \left( \theta_A + \theta_B \right)$$

$$= M_p \left( 2\,\theta_A + \theta_B \right)$$

$$= M_p \left[ \frac{2\delta}{\left( L - \frac{L}{2} + \frac{M_p}{w_u L} \right)} + \frac{\delta}{\left( \frac{L}{2} - \frac{M_p}{w_u L} \right)} \right]$$

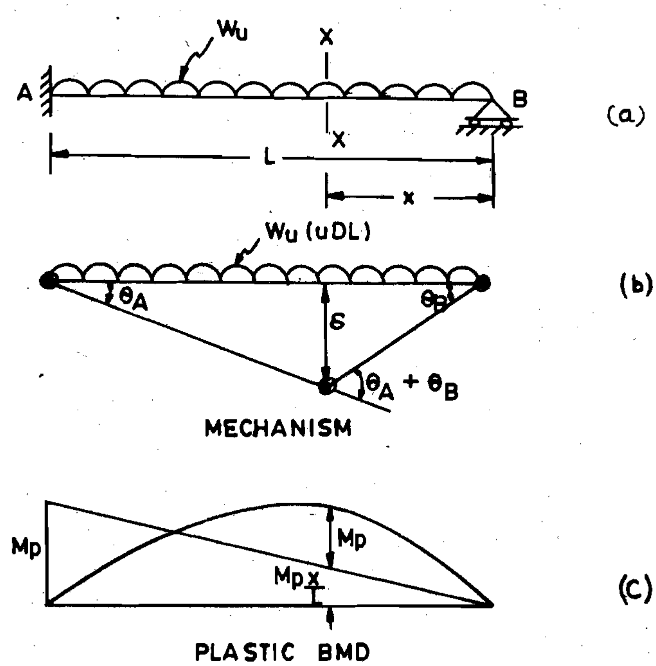



Figure 16.24

Equating external work to internal work, i.e. $W_e = W_i$, we get

$$w_u^2 L^4 - 12 M_p w_u L^2 + 4 M_p^2 = 0$$

On solving the quadratic equation, we get

$$w_u = \frac{11.656\, M_p}{L^2}$$

Position of the second hinge (from the B end) is as follows,

$$x = \frac{L}{2} - \frac{M_p L^2}{11.656\, M_p\, L} = 0.5\, L - 0.0856\, L$$

$$x = 0.4140\, L \quad \text{from end B}$$

*Statical Method*

Moment at $C$

$$M_C = \left( \frac{W_u x}{2} - \frac{W_u x^2}{2L} \right)$$

$$= M_p + \frac{M_p x}{L}$$

$$M_p \left( 1 + \frac{x}{L} \right) = \frac{W_u x}{L} \left( \frac{L}{2} - \frac{x}{2} \right)$$

$$\therefore \quad M_p = \frac{W_u}{2} \times \frac{(Lx - x^2)}{(L + x)}$$

Now for maxima,

$$\frac{dM_p}{dx} = \frac{\left[ (L + x)(L - 2x) - (Lx - x^2) \right]}{(L + x)^2} = 0$$

$$x^2 + 2Lx - L^2 = 0$$

On solving the equation, we get

$$x = L(\sqrt{2} - 1) = 0.414\, L$$

Putting the value of $x$

$$M_p = \frac{W_u}{2} \left[ \frac{[x(L - x)]}{(L + x)} \right]$$

$$= \frac{W_u}{2} \left[ \frac{L(2 - 1)\; L(2 - \sqrt{2})}{\sqrt{2}\, L} \right]$$

$$= \frac{W_u\, L(3 - 2\sqrt{2})}{2}$$

Thus, we get $\quad W_u = \dfrac{2\, M_p}{L(3 - 2\sqrt{2})} = \dfrac{11.656\, M_p}{L}$

It gives, $\quad w_u = \dfrac{W_u}{L} = \dfrac{11.656\, M_p}{L^2}$

which is same as per mechanism method.

**Example 16.7**

Find the collapse load using (a) statical method, and (b) mechanism method for the fixed beam shown in Figure 16.25 (a).

**Solution**

*Statical Method*

Referring Figure 16.25 (b), maximum free bending moment occurs at point D, and which is

$$M_D = \frac{5PL}{9}.$$

As per static method, the sum of plastic moments at A and B must be equal to maximum free bending moment at D,

$$M_p + M_p = \frac{5PL}{9}$$

$$M_p = \frac{5PL}{18}, \quad \text{or } P = \frac{18\,M_p}{5L}$$

Therefore, the collapse load,

$$\boldsymbol{P = \frac{3.6\,M_P}{L}}$$

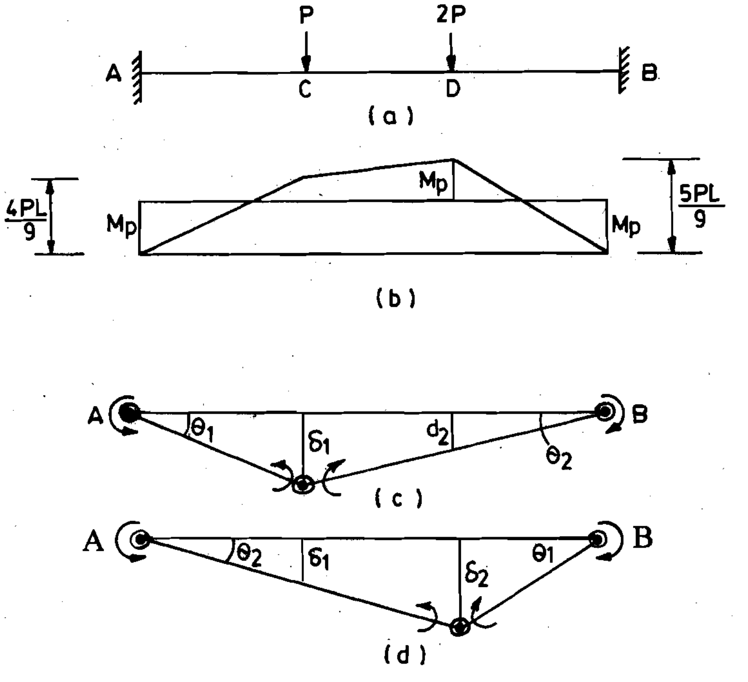



**Figure 16.25**

*Mechanism Method*

Trying first the mechanism of Figure 16.25 (c).

For mechanism I, $\theta_2 = \frac{\theta_1}{2}$

External work done $W_e = \frac{PL\theta_1}{3} + \frac{2PL\theta_2}{3} = \frac{2PL\theta_1}{3}$

Internal work done $W_i = M_p\,\theta_1 + M_p\,\theta_2 + M_p\,(\theta_1 + \theta_2) = 3\,M_P\,\theta_1$

Equating $W_e = W_i$

$$\frac{2PL\theta_1}{3} = 3M_p\,\theta_1$$

Thus,

$$\boldsymbol{P = \frac{4.5\,M_p}{L}}$$

Trying next the mechanism of Figure 16.25 (d).

In mechanism II, $\theta_2 = \frac{\theta_1}{2}$

External work done $W_e = \frac{2PL\theta_1}{3} + \frac{PL\theta_2}{3} = \frac{5PL\theta_1}{6}$

Internal work done $W_i = M_p\,\theta_2 + M_p\,(\theta_1 + \theta_2) + M_p\,\theta_1 = 3\,M_P\,\theta_1$

Equating $W_e = W_i$

$$\frac{5PL\theta_1}{6} = 3\,M_p\,\theta_1$$

$$P = \frac{18\,M_p}{5L} = \frac{3.6\,M_p}{L} \qquad \text{(b)}$$

Therefore, the true collapse load is $\frac{3.6\,M_p}{L}$ which is smaller of the two values obtained, i.e. (a) and (b).

**Example 16.8**

Find the collapse load for the three-span continuous beam shown in Figure 16.26 (a). The plastic moment capacity of the end-span sections is two-third of the plastic moment capacity ($M_p$) of the mid-span section.

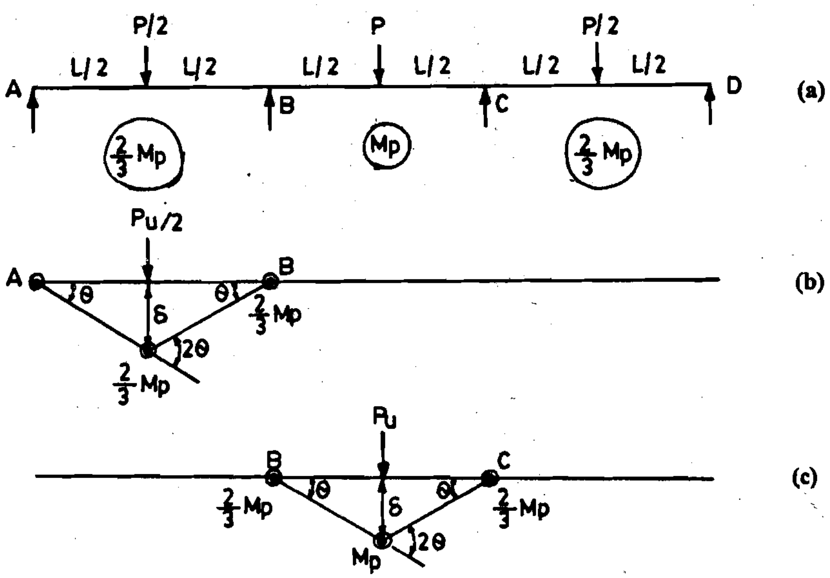



**Figure 16.26**

**Solution**

Here,

$n$ = number of possible location of hinges = 5

$r$ = degree of indeterminacy = 2

$i$ = number of possible mechanism = $(n - r) = (5 - 2) = 3$

Consider only the mechanism-formation in first and second span only because of symmetry. At junction B and C, plastic hinge will occur on the side in which the plastic moment is smaller.

*Hinge Formation in First Span*

External work done, $W_e = \frac{P_u}{2}\delta = \frac{P_u}{2} \times \frac{L}{2}\theta = \frac{P_u L\theta}{4}$

Internal work done, $W_i = \frac{2}{3} M_p (2\theta) + \frac{2}{3} M_p (\theta) = 2\,M_p\,\theta$

Equating $W_e = W_i$,

$$\frac{P_u\,L\,\theta}{4} = 2\,M_p\,\theta$$

It gives

$$P_u = \frac{8\,M_p}{L}$$

*Hinge Formation in Second Span*

Here, the external work done, $W_e = P_u \, \delta = P_u \times \frac{L}{2} \times \theta$

Internal woed done, $$W_i = \frac{2}{3} M_p (\theta) + M_p (2\theta) + \frac{2}{3} M_p (\theta) = \frac{10 \, M_p \, \theta}{3}$$

Equating $W_e = W_i$,

$$\frac{P_u \, L \, \theta}{2} = \frac{10 \, M_p \, \theta}{3}$$

It gives, $$P_u = \frac{20 \, M_p}{3L}$$

$$\boldsymbol{P_u = \frac{6.67 \, M_p}{L}}$$

Mechanism formation in the third (RHS) span will give the same value of $P_u$ as in the first span because of summetry.

Therefore, the collapse load is smallar of the two values, i.e., $P_u = \frac{6.67 \, M_p}{L}$

## SAQ 4

Using statical method and or mechanism method find the collapse load for the beams shown in Figure 16.27 to Figure 16.33.

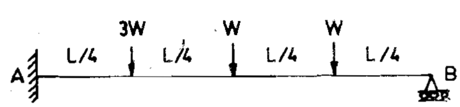



**Figure 16.27**

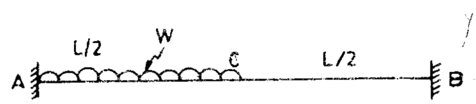



**Figure 16.28**

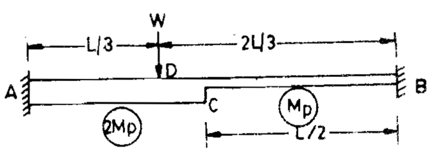



**Figure 16.29**

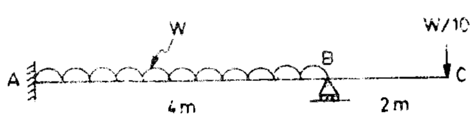



**Figure 16.30**

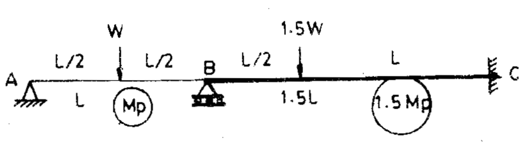



**Figure 16.31**

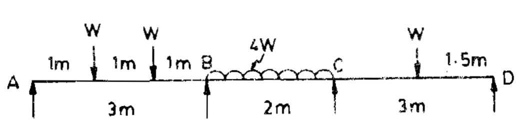



**Figure 16.32**

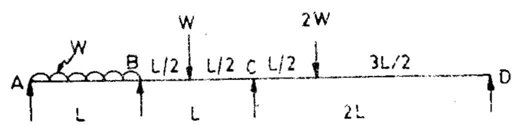



**Figure 16.33**

### 16.5.5 Application to Portal Frames

In the case of portal frames, the number of collapse mechanisms to be investigated can be higher. This is because the number of composite mechanisms can be obtained by systematically combining a number of elementary mechanisms. If one recognizes this fact, then the work involved in finding the true collapse mechanism and corresponding collapse load, is very much reduced. After finding the lowest critical load, the bending moment distribution is drawn to check that the plastic moment condition is not violated anywhere in the structure.

**Sign Convention**

In portal frame analysis, dotted lines will be drawn to indicate the side on which tension occurs. In the case where two members meet, a hinge is assumed theoretically to develop exactly at the meeting point. In practice, it will form a little away from the meeting point on the member for which the moment capacity is lesser. For combining mechanism purpose, it is assumed to develop a little away from the meeting point, so that identification of the tension side will be easier. A rotation θ is positive, if it produces tension on the side on which the dotted line is shown, otherwise it will be negative.

**Example 16.9**

Find the collapse load and draw BM diagram at collapse load for the portal frame shown in Figure 16.34 (a). Assume constant plastic moment capacity $M_p$ throughout.

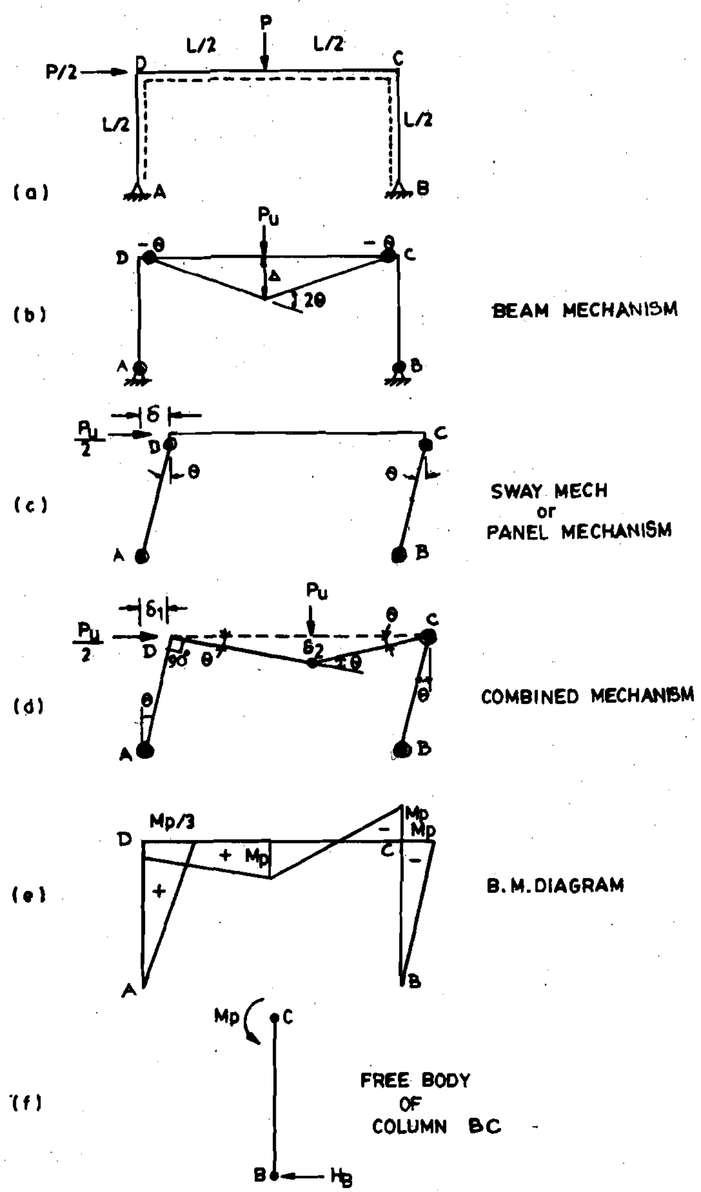



**Figure 16.34**

## Solution

There are three possible locations at which the plastic hinges can form. The frame is statically indeterminate to first degree.

Therefore, (number of independent mechanisms) $i = n - r = 3 - 1 = 2$

There are two independent mechanisms. One is the *beam mechanism* and the other is *sway or panel mechanism*. These two are shown in Figure 16.34 (b) and (c) respectively.

The two mechanisms should possess as common coordinate a rotation $\theta$ so that combination could be made by adding them together.

**Collapse Load Computation**

*Mechanism* (1) (*Beam Mechanism*)

$$W_e = W_i$$

$$P_u \Delta = P_u \times \frac{L\theta}{2} = M_p [\theta + 2\theta + \theta] = 4M_p \theta$$

$$P_u = \frac{8 M_p}{L}$$

*Mechanism* (2) (*Sway Mechanism*) :

$$W_e = W_i$$

$$\frac{P_u \delta}{2} = M_p (\theta + \theta)$$

$$\frac{P_u}{2} \times \frac{L\theta}{2} = 2M_p \theta$$

$$P_u = \frac{8 M_p}{L}$$

*Mechanism* (3)

Mechanism (1) and Mechanism (2) will be combined now. Add these two mechanisms. The hinge at left top point in panel mechanism and the hinge at left side in the beam mechanism will get cancelled because they are of opposite signs where as on the right side they remain as such because they are of same sign. Because of cancellation of hinge on the left side, the original angle between the two members remains the same, i.e. the angle 90° is maintained.

$$W_e = W_i$$

$$\frac{P_u \delta_1}{2} + P_u \delta_2 = M_p (2\theta) + M_p \theta + M_p \theta$$

Here, $\delta_1 = \frac{L\theta}{2}$ and $\delta_2 = \frac{L\theta}{2}$

$$\frac{P_u}{2} \times \frac{L\theta}{2} + P_u \left(\frac{L\theta}{2}\right) = 4 M_p \theta$$

Giving, $P_u = \frac{16 M_p}{3L}$

Obviously, the Mechanism (3) gives the lowest load. This is because of the elimination of hinges and thus, reducing the internal work.

**Moment Diagram**

In the composite mechanism there are two plastic hinges at failure. Therefore remaining redundancy at failure is zero.

The structure becomes statically determinate at collapse such a collapse is known as "complete mechanism". All that we need to determine the reactions using statical conditions. As a first step, isolate the right column BC as shown in Figure 16.34 (f) along with the moments.

Then taking moment about $C$,

$$H_B \times \frac{L}{2} = M_p$$

$$H_B = \frac{2\,M_p}{L}$$

Then, considering horizontal equilibrium,

$$\sum H = 0$$

we get

$$\frac{P_u}{2} = H_A + H_B$$

$$H_A = \frac{P_u}{2} - H_B$$

$$H_A = \frac{16\,M_p}{3L} \times \left(\frac{1}{2}\right) - \frac{2\,M_p}{L} = \frac{2\,M_p}{3L}$$

Moment at D

$$M_D = H_A \times \frac{L}{2} = \frac{2}{3} \times \frac{M_p}{L} \times \frac{L}{2} = \frac{M_p}{3}$$

### Example 16.10

Find the collapse load and draw BM diagram at collapse for the portal frame shown in Figure 16.35 (a).

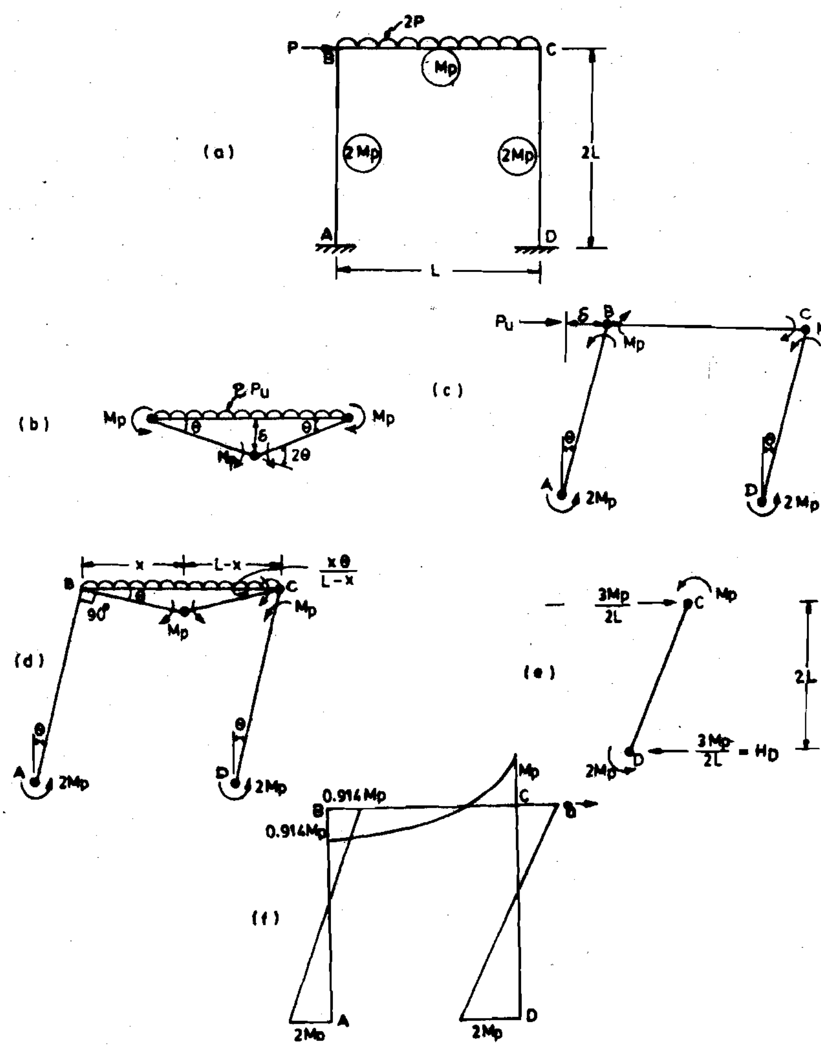



**Figure 16.35**

**Solution**

Here, we have,

$n$ = number of possible location of hinges = 5

$r$ = degree of indeterminacy = 3

$i$ = number of possible mechanism = $(n - r) = (5 - 2) = 2$

There are following two independent mechanisms :

(a) Beam mechanism, and

(b) Panel mechanism.

**Beam Mechanism** [Refer Figure 16.35 (b)]

The load $2P_u$ on the beam deflects from zero at ends to maximum $\delta$ at midspan.

$$\text{External work done, } W_e = 2\,P_u \times \delta_{av} = 2\,P_u \times \frac{\delta_{cr}}{2} = 2\,P_u \times \frac{\left(\frac{L}{2}\right)\theta}{2} = \frac{P_u L\theta}{2}$$

$$\text{Internal work done, } W_i = M_p\,\theta + M_p\,(2\theta) + M_p\,\theta = 4\,M_p\,\theta$$

$$\text{Now because, } \quad W_e = W_i \longrightarrow \frac{P_u L\,\theta}{2} = 4\,M_p\,\theta \text{ or } \boldsymbol{P_u = \frac{8\,M_p}{L}}$$

**Panel (Sway) Mechanism** [Refer Figure 16.35 (c)]

$$\text{External work done, } W_e = P_u \times \delta = P_u\,(2L\,\delta)$$

Internal work done,

$$W_i = \underset{\text{at B}}{M_p\,\theta} + \underset{\text{at C}}{M_p\,\theta} + \underset{\text{at A}}{(2\,M_p)\,\theta} + \underset{\text{at D}}{(2\,M_p)\,\theta} = 6\,M_p\,\theta$$

$$\text{Now because, } \quad W_e = W_i \longrightarrow 2\,P_u\,L\,\theta = 6\,M_p\,\theta$$

$$\text{It gives, } \quad \boldsymbol{P_u = \frac{3\,M_p}{L}}$$

**Composite Mechanism** [Refer Figure 16.35 (d)]

On equating the external work done to internal work done, we have

$$2P_u\,L\,\theta + \left(\frac{2P_u}{L} \times \frac{x\theta\,L}{2}\right) = \underset{\text{at A}}{2\,M_p\,\theta} + \underset{\text{at D}}{2\,M_p\,\theta} + \underset{\text{at C}}{M_p\left[\theta + \frac{x\,\theta}{L - x}\right]} + \underset{\text{at beam hinge}}{M_p\left[\theta + \frac{x\,\theta}{L - x}\right]}$$

On simplifying, we get

$$P_u = \frac{2\,M_p\,(3L - 2x)}{(2L + x)\,(L - x)} = \frac{2\,M_p\,(3L - 2x)}{2L^2 - xL - x^2}$$

Now,

$$\frac{dP_u}{dx} = 2x^2 - 6xL + L^2 = 0 \quad \text{(For maximum value of } P_u\text{)}$$

Solving the quadratic equation, we get

$$x = 0.1771\,L$$

On substituting the value of $x$ in equation of $P_u$

$$P_u = \frac{2\,M_p\,[3L - 2 \times 0.1771\,L]}{[2\,L^2 - 0.1771\,L^2 - 0.03136\,L^2]}$$

$$\boldsymbol{P_u = \frac{2.957\,M_p}{L}}$$

Note that this lowest load is almost equal to the ultimate load obtained in panel mechanism. This is an example for an over-complete mechanism.

## Moment Diagram

Refer Figure 16.35 (e). Taking moment about $C$

$$H_D \times 2L = M_p + 2M_p = 3M_p$$

$$\therefore \quad H_D = \frac{3M_p}{2L} = \frac{1.5M_p}{L}$$

Now

$$(\sum Fx = 0) \quad P_u = H_A + H_D = \frac{2.957 M_p}{L}$$

It gives,
$$H_A = \frac{1.457 M_p}{L}$$

Bending moment at B,

$$M_B = -2M_p + \frac{1.457 M_p}{L} \times 2L = 0.914 M_p$$

## Example 16.11

Find the collapse load and draw B.M diagram at collapse for the portal shown in Figure 16.36 (a).

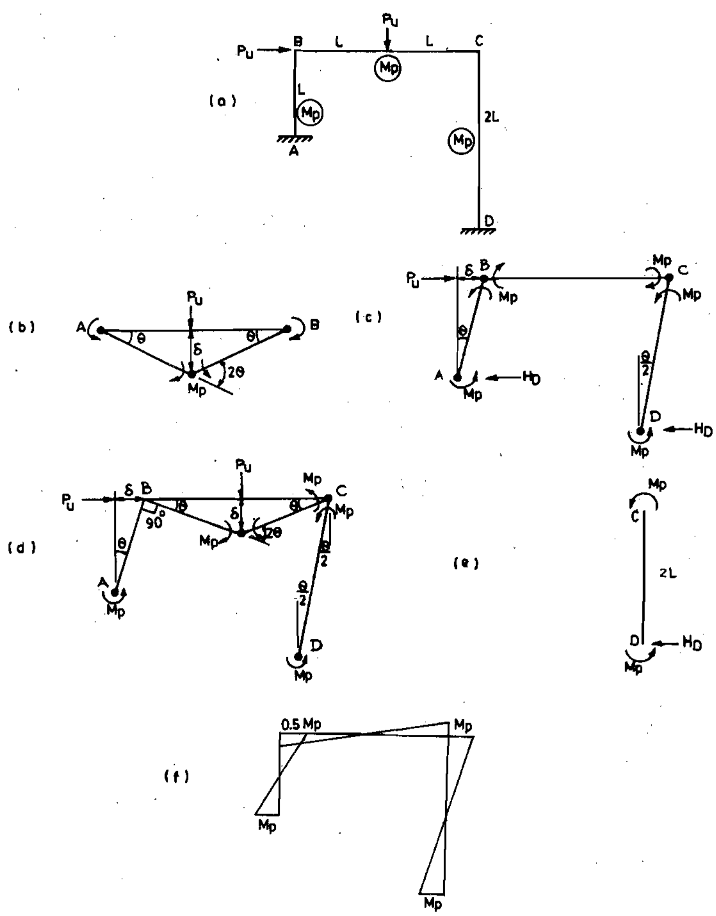



Figure 16.36

## Solution

**Beam Mechanism** [Refer Figure 16.36 (b)]

$$W_e = W_i \Rightarrow P_u \times L\theta = M_p\theta + 2M_p\theta + M_p\theta$$

$$P_u \times L\theta = 4M_p\theta$$

$$P_u = \frac{4M_p}{L}$$

**Panel Mechanism** [Refer Figure 16.36 (c)]

$$W_e = W_i \;\Rightarrow\; P_u\,\delta = \underset{\text{at A}}{M_p\,\theta} + \underset{\text{at B}}{M_p\,\theta} + \underset{\text{at C}}{\frac{M_p\,\theta}{2}} + \underset{\text{at D}}{\frac{M_p\,\theta}{2}}$$

$$P_u\,L\,\theta = 3\,M_p\,\theta$$

$$P_u = \frac{\mathbf{3\,M_p}}{\mathbf{L}}$$

**Composite Mechanism** [Refer Figure 16.36 (d)]

$$W_e = W_i \;\Rightarrow\; P_u\,\delta + P_u\,\delta = M_p\,\theta + M_p\,2\theta + M_p\,\theta + \frac{M_p\,\theta}{2} + \frac{M_p\,\theta}{2}$$

$$2\,P_u\,L\,\theta = 5\,M_p$$

$$P_u = \frac{\mathbf{2.5\,M_p}}{\mathbf{L}}$$

Lowest load in mechanisms is true collapse load, i.e $\boldsymbol{P_u} = \dfrac{\mathbf{2.5\,M_p}}{\mathbf{L}}$

**Moment Diagram** [Refer Figure 16.36 (f)]

Taking moment about $C$ in Figure 16.36 (e), we get

$$2\,M_p = 2\,L\,H_D$$

Therefore, $$H_D = \frac{M_p}{L}$$

and $$H_A = P_u - \frac{M_p}{L} = \frac{2.5\,M_p}{L} - \frac{M_p}{L} = \frac{1.5\,M_p}{L}$$

Bending moment at B,

$$M_B = -\,M_p + \frac{1.5\,M_p\,L}{L} = 0.5\,M_p$$

## SAQ 5

Calculate the collapse load for the frame shown in Figure 16.37.

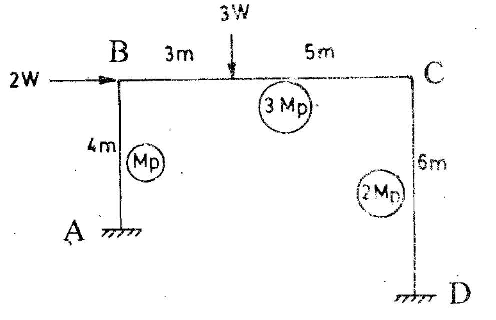



**Figure 16.37**

### 16.5.6 Design of Continuous Beam

The following steps are followed in the design of continuous beams :

(i) Determine the possible loading conditions.

(ii) Compute the ultimate load by multiplying working loads with appropriate load factor.

(iii) Determine the plastic moment for each span by beam mechanism equation also find span moments.

(iv) Find plastic modulus or each span and provide separate section, or alternately find the plastic modulus and hence sectionfor east plastic moment and provide this section throughout, cover plate may be provided in outer spans for additional moments.

(v) As a secondary consideration, following checks may be carried out :

(a) Shear stress,
(b) Deflection,
(c) Buckling of flanges.
(d) Web buckling, and
(e) Design of connection.

**Example 16.12**

A continuous beam has three spans and working loads as shown in Figure 16.38 (a). Design the beam with cover plates where necessary load factor = 1.70

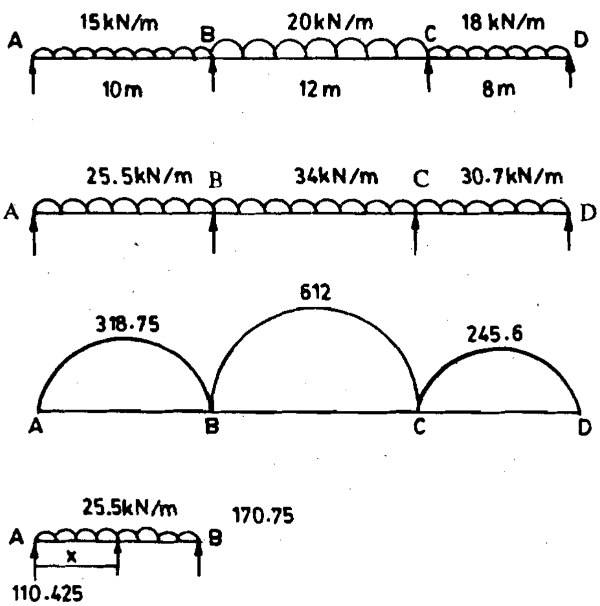



**Figure 16.38**

**Solution**

Ultimate load for span AB, $w_u$ = load factor × working load

$= 1.7 \times 15 = 25.5$ kN/m

Ultimate load for span BC, $w_u = 1.7 \times 20 = 34$ kN/m

Ultimate load for span CD, $w_u = 1.7 \times 18 = 30.7$ kN/m

$$\text{Span moments for AB} = \frac{(25.5 \times 10^2)}{8} = 318.75 \text{ kN m}$$

$$\text{Span moments for BC} = \frac{(34 \times 12^2)}{8} = 612.00 \text{ kN m}$$

$$\text{Span moments for CD} = \frac{(35.7 \times 8^2)}{8} = 245.6 \text{ kN m}$$

$$\text{Plastic moment for AB} = \frac{(25.5 \times 10^2)}{11.656} = 218.8 \text{ kN m} \qquad \text{(Ex. 16.6)}$$

$$\text{Plastic moment for BC} = \frac{(34 \times 12^2)}{16} = 306 \text{ kN m} \qquad \text{(Exs. 16.4 and 16.5)}$$

$$\text{Plastic moment for CD} = \frac{(30.7 \times 8^2)}{11.656} = 168.6 \text{ kN m} \qquad \text{(Ex. 16.6)}$$

Minimum $M_p$ occuring in span CD is equal to 168.6 kN m. Therefore, provide a single rolled section in span CD and additional plates in span AB and BC.

**Span CD**

Plastic Section modulus required, $Z_p = \frac{M_p}{\sigma_y} = \frac{168.6 \times 10^6}{250} = 0.674 \times 10^6 \text{ mm}^3$

Use ISMB 300 @ 46.1 kg/m for which $Z_p$ provided is $0.683 \times 10^6 \text{ mm}^3$

$M_p$ of this section = $Z_p \times \sigma_y = 0.683 \times 10^6 \times 250 = 170.75$ kN m

*Cover Plate for Span BC,*

Extra $M_p$ required $= (612 - 168.60 - 170.75) = 272.65$ kN m

$$Z_p \text{ for the cover plate } = \frac{M_p}{\sigma_y} = \frac{(272.65 \times 10^6)}{250}$$

$$= 1.09 \times 10^6 \text{ mm}^3$$

Area of cover plate, $A = \frac{Z_p}{d} = \frac{(1.09 \times 10^6)}{300} = 3633.33 \text{ mm}^2$

Provide plate of $180 \times 20$ on each flange over ISMB 300.

$Z_p$ provided by these plates = $180 \times 20 \times (300 + 20) = 1.152 \times 10^6 \text{ mm}^3$ (O.K.)

*Cover Plate for Span AB*

Let shear force be zero at $x$ from A.

The reaction at A, $V_A = \frac{(25.5 \times 10)}{2} - \frac{170.75}{10} = 110.425$ kN

SF at $x$ is zero (where moment is maximum), $V_A - w_u x = 0$

$$x = \frac{110.425}{25.5} = 4.33 \text{ m from A}$$

Plastic moment at this section,

$$M_p = 110.425 \times 4.33 - \frac{(25.5 \times 4.33^2)}{2} = 239.09 \text{ kN m}$$

Excess $M_p = 239.09 - 170.75 = 68.34$ kN m

$Z_p$ for the cover plate = $\frac{Z_p}{\sigma_y} = \frac{(68.34 \times 10^6)}{250} = 0.273 \times 10^6 \text{ mm}^3$

Area of cover plate, $A = \frac{Z_p}{d} = \frac{(0.273 \times 10^6)}{300} = 910 \text{ mm}^2$

Use cover plate 100 mm × 9 mm on each flange.

$Z_p$ provided by this plate = $100 \times 9\ (300 + 9) = 0.278 \times 10^6 \text{ mm}^3$ (O.K.)

## SAQ 6

Design a continuous beam as shown in Figure 16.39. Take load factor = 1.70.

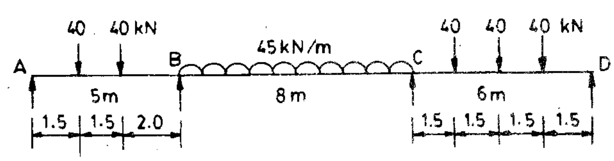



Figure 16.39

## 16.6 SUMMARY

- Steel is a structural material having ductile property. Load analysis is carried out considerring the fully plastic section which shows that there is considerable reserve strength beyond first yield point.
- The stress strain relationship in flexure is idealised to consists of two parts only

  $$\sigma = E\ (0 < \varepsilon < \varepsilon_y) \text{ and } \sigma = \sigma_y\ (\varepsilon_y < \varepsilon < \infty)$$

- The equilibrium conditions are

  $$\text{Normal force } P = \int \sigma\, dA$$

  and $$\text{Moment } M = \int \sigma\, dAy$$

- At first yield, $\varepsilon$ reaches a value $\varepsilon_y$, $\sigma = \sigma_y$ and moment $M_y = \sigma_y Z$.
- At fully plastic stage. $M_y = \sigma_y Z_p$.
- The ratio of $\frac{M_p}{M_y} = \frac{Z_p}{Z} =$ shape factor $= K$.
- Plastic modulus, $Z_p = \sum_{i=1}^{N_p} A_i\ Y_i$
- Shape factor of rectangular section = 1.5; for I-section, it is around 1.12.
- Plastic hinge is a zone of yielding due to flexure in a structural member and carries constant moment equal to $M_p$.
- Plastic hinges are reached first at sections of greatest moment (curvature) and this then allows subsequent redistribution of moments till the structure is converted into a mechanism.
- *Upper Bound Theorem* : A load computed on the basis of assumed mechanism will always be greater than or at best equal to the ultimate load.
- *Lower Bound Theoerm* : A load computed on the basis of an assumed equilibrium moment diagram in which moment is not greater than *Mp* and is less than or best equal to the true ultimate load.
- Mechanism method satisfies mechanism condition and equilibrium condition. Statical method satisfied equilibrium condition and plastic moment condition. In both the methods, it is to be checked that $M \le M_p$ at all sections.
- In portal frames, there are beam mechanism, sway or portal mechanism and combined mechanisms.
- The number of independent mechanisms ($i$),

  $$i = n - r$$

  where, $n$ = number of possible location of hinges, and

  $r$ = number of redundancy.

## 16.7 KEY WORDS

**Elastic Design** : A design method which defines the limit of structural usefulness as the load at which a stress equal to the yield point of the material is first attained at any point.

**Factor of Safety** : As used in elastic design, it is a factor by which the yield point is divided to determine a working or allowable stress for the most highly stressed fibre.

**Load Factor** : As used in plastic design, it is a factor by which the working load is multiplied to determine the ultimate load.

**Mechanism** : A system of members containing a sufficient number of plastic (or real) hinges so that it is able to deform without a finite increase in load is called mechanism.

**Mechanism Angle** : It is the virtual angle of rotation at a plastic hinge corresponding to the virtual displacement of the mechanism.

**Plastic Design** : This is a design method which defines the limit of structural usefulness as the "ultimate load". The term "plastic" comes from the fact that the ultimate load is computed from a knowledge of the strength of steel in the plastic range.

**Plastic Hinge** : It is a yielded zone found in a beam when the plastic moment is applied. The beam rotates as if hinged, except for the constant restraining moment $M_p$.

**Plastic Modulus** : Plastic modulus is the resisting modulus of a completely yielded cross section. It is the combined statical moment about the neutral axis of the cross-section areas above and below that axis.

**Plastic Moment** : Maximum moment of resistance of a fully yielded cross section is called plastic moment.

**Plastification** : Gradual penetration of yield stress from the outer fiber towards the centroid of a section under increase of moment. Plastification process is completed when the plastic moment $M_p$ is reached.

**Proportional Loading** : A system of loading in which all loads increase in a constant ratio, one to the other.

**Redistribution of Moment** : It is a process which results in the successive formation of plastic hinges until the ultimate load is reached. Through transfer of moment which results from formation of "hinges", the less highly stressed portions of a structure also may reach the $M_p$ value.

**Shape Factor** : It is the ratio $\frac{M_p}{M_y}$, or $\frac{Z_p}{Z}$, for a cross section.

**Yield Moment** : In a member subjected to bending, it is the moment at which an outer fiber attains yield point stress.

## 16.8 ANSWERS TO SAQs

**SAQ 1**

$$P_u = (1 + \sqrt{3})\ \sigma_y A$$

**SAQ 2**

Depth of elastic core = 77.4 mm.

Plastic moment $M_p = 30 \times 10^6$ N mm, and $R = 32.25$ m.

**SAQ 3**

(a) 1.70 (b) 2.34 (c) 2.00 (d) 1.76

**SAQ 4**

(a) $W = \frac{2M_p}{L}$ (b) $W = \frac{128M_p}{9L}$

(c) $W = \frac{15M_p}{L}$ (d) $W = 3.74\,M_p$

(e) $W = \frac{16M_p}{3L}$ (f) $W = \frac{4M_p}{3}$

(g) $W = \frac{7M_p}{3L}$

**SAQ 5**

$$W = 0.5686\,M_p$$