UNIT - III - CALCULUS

Syllabus: Mean Value Theorems: Rolle's Theorem, Lagrange's mean value theorem with their geometrical interpretation, Cauchy's mean value theorem, Taylor's and Maclaurin's theorems with remainders (without proof), Problem and applications on the above theorems.

Introduction:

Limit:

At $x = a$

$$LHL = \lim_{x \to a^-} f(x) = \lim_{h \to 0} f(a-h)$$

$$RHL = \lim_{x \to a^+} f(x) = \lim_{h \to 0} f(a+h)$$

If $LHL = RHL = l$ then $\lim_{x \to a} f(x) = l$.

Continuity: $LHL = RHL = f(a) = l$, then $f(x)$ is continuous.

Differentiable (or) Derivable:

$$LHD = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{+h} = \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h}$$

$$RHD = \lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Eg: $f(x) = \begin{cases} x-2 \text{ at } x \leq 2 \\ x+2 \text{ at } x > 2 \end{cases}$ at $x = 2$

Limit: $LHL = \lim_{x \to a^-} f(x) = \lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x-2) = 2-2 = 0.$

$$RHL = \lim_{x \to 2^+} (x+2) = 2+2 = 4$$

$LHL \neq RHL.$

Eg: $f(x) = |x|$ at $x = 0$

$$LHL = \lim_{x \to a^-} f(x) = \lim_{x \to 0^-} (-x) = 0.$$

$$RHL = \lim_{x \to a^+} f(x) = \lim_{x \to 0^+} (x) = 0.$$

$LHL = RHL = 0$

Limit exists.

$f(0) = |0| = 0 = LHL = RHL$

$f(x)$ is continuous.

$$LHD = \lim_{h \to 0^-} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0^-} \frac{f(h) - 0}{h} = \lim_{h \to 0^-} \frac{|h|}{h}$$

$$= \lim_{h \to 0^-} \frac{(-h)}{h} = \lim_{h \to 0^-} -(1) = -1.$$

$$RHD = \lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0^+} \frac{f(h) - 0}{h} = \lim_{h\to 0^+} \frac{|h|}{h}$$

$$= \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} (1) = 1.$$

$LHD \neq RHD$

$f(x)$ is not differentiable at $x = 0$.

ROLLE'S THEOREM:

Let $f(x)$ be a function. If the function

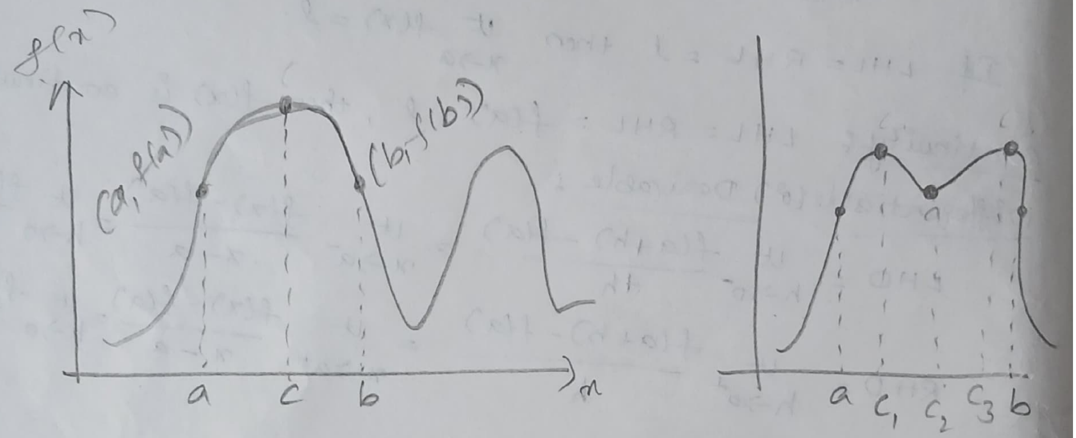
(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b)

(iii) $f(a) = f(b)$.

then $\exists\, c \in (a, b)$ such that $f'(c) = 0$.

Note:



1. nth degree polynomial where $n \in Z^+$ is continuous at every point.
2. Exponential function is continuous at every point.
3. sin, cos, sinh, cosh are continuous at every point.
4. Logarithmic functions are continuous over positive values.
5. If f, g are continuous functions over an interval then
   a) $f+g$ b) $f-g$ c) fg d) f/g $(g \neq 0)$
   are continuous over the given interval.
6. nth degree polynomial, exponential, sin, cos, sinh, cosh are differentiable at every point.
7. If f, g are differentiable over an interval then
   a) $f+g$ b) $f-g$ c) fg d) f/g $(g \neq 0)$
   are differentiable over the given interval.

① Discuss the applicability of Rolle's theorem for $f(x) = x^2$ on $[-3, 3]$.

Sol: Given, $f(x) = x^2$

(i) WKT, polynomial function is continuous at every point.

$\therefore f(x) = x^2$ is continuous on $[-3, 3]$.

(ii) WKT, polynomial function is differentiable at every point.

$\therefore f(x) = x^2$ is differentiable on $(-3, 3)$.

(iii) $f(a) = f(-3) = (-3)^2 = 9$.

$f(b) = f(3) = (3)^2 = 9$.

$\therefore f(a) = f(b)$.

Since $f(x)$ satisfies all three conditions of Rolle's theorem, Rolle's theorem is applicable for $f(x) = x^2$ on $[-3, 3]$.

Then $\exists\, c \in (-3, 3)$ such that $f'(c) = 0$.

$f(x) = x^2$

$f'(x) = 2x$

$f'(c) = 2c = 0$

$\therefore c = 0 \in (-3, 3)$.

Hence rolle's theorem is verified.

② verify Rolle's theorem for $f(x) = x^3 + x^2 - x + 1$ over $[-1, 1]$.

Sol: Given, $f(x) = x^3 + x^2 - x + 1$.

(i) WKT, Polynomial function is continuous at every point.

$\therefore f(x) = x^3 + x^2 - x + 1$ is continuous over $[-1, 1]$.

(ii) WKT, polynomial function is differentiable at every point.

$\therefore f(x) = x^3 + x^2 - x + 1$ is ~~cont~~ differentiable over $(-1, 1)$.

(iii) $f(a) = f(-1) = (-1)^3 + (-1)^2 - (-1) + 1 = \cancel{-1} + \cancel{1} + 1 + 1 = 2$.

$f(b) = f(1) = (1)^3 + (1)^2 - (1) + 1 = 1 + 1 \cancel{-1} + \cancel{1} = 2$.

$\therefore f(a) = f(b)$

Since $f(x)$ satisfies all three conditions of Rolle's theorem, Rolle's theorem is applicable for $f(x) = x^3 + x^2 - x + 1$ over $[-3, 3]$.

Then $\exists\, c \in (-3, 3)$ such that $f'(c) = 0$.

$f(x) = x^3 + x^2 - x + 1$

$f'(x) = 3x^2 + 2x - 1$

$f'(c) = 3c^2 + 2c - 1 = 0$.

$3c^2 + 3c - c - 1 = 0$

$3c(c + 1) - 1(c + 1) = 0$

$(3c - 1)(c + 1) = 0$

$\therefore c = 1/3, -1 \in (-3, 3)$.

Hence rolle's theorem is verified.

③ Verify Rolle's theorem for $f(x) = \frac{\sin x}{e^x}$ over $[0, \pi]$.

Sol: Given, $f(x) = \frac{\sin x}{e^x}$.

(i) WKT, $\sin x$ and e^x are continuous functions.

$\therefore f(x) = \frac{\sin x}{e^x}$ is continuous over $[0, \pi]$.

(ii) WKT, $\sin x$ and e^x are differentiable at every point.

$\therefore f(x) = \frac{\sin x}{e^x}$ is differentiable over $(0, \pi)$.

(iii) $f(a) = f(0) = \frac{\sin 0}{e^0} = \frac{0}{1} = 0$.

$f(b) = f(\pi) = \frac{\sin \pi}{e^\pi} = \frac{0}{e^\pi} = 0$.

$\therefore f(a) = f(b)$.

Since, $f(x)$ satisfies all three ~~contions~~ conditions of Rolle's theorem, Rolle's theorem is applicable for $f(x) = \frac{\sin x}{e^x}$ over $[0, \pi]$.

Then $\exists\, c \in (0, \pi)$ such that $f'(c) = 0$.

$f(x) = \frac{\sin x}{e^x}$

$f'(x) = \frac{e^x \cos x - \sin x\, e^x}{(e^x)^2}$ $\qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$.

$f'(c) = \frac{e^c \cos c - e^c \sin c}{(e^c)^2} = 0$

$\frac{\cancel{e^c}(\cos c - \sin c)}{(e^c)^{\cancel{2}}} = 0$.

$\cos c - \sin c = 0$

$\cos c = \sin c$

$1 = \tan c$

$\tan \pi/4 = \tan c$

$\therefore c = \pi/4 \in (0, \pi)$.

Hence Rolle's theorem is verified.

④ What is the 'c' value in Rolle's theorem for $f(x) = x^2 - x + 1$ on $[-1, 1]$.

Sol: Given $f(x) = x^2 - x + 1$.

(i) WKT, polynomial functions are continuous at every point.

$\therefore f(x) = x^2 - x + 1$ is continuous on $[-1,1]$.

(ii) WKT, polynomial functions are differentiable at every point

$\therefore f(x) = x^2 - x + 1$ is differentiable on $(-1,1)$.

(iii) $f(a) = f(-1) = (-1)^2 - (-1) + 1 = 1+1+1 = 3.$

$f(b) = f(1) = (1)^2 - (1) + 1 = \cancel{1} \cancel{-1} + 1 = 1.$

$\therefore f(a) \neq f(b).$

Hence Rolle's theorem is not applicable.

(5) Verify Rolle's theorem for $f(x) = e^x(\sin x - \cos x)$ on $[\pi/4, 5\pi/4]$

Sol: Given, $f(x) = e^x(\sin x - \cos x)$

(i) WKT, e^x, $\sin x$ and $\cos x$ functions are continuous at every point.

$\therefore f(x) = e^x(\sin x - \cos x)$ is continuous on $[\pi/4, 5\pi/4]$.

(ii) WKT, e^x, $\sin x$ and $\cos x$ functions are differentiable at every point.

$\therefore f(x) = e^x(\sin x - \cos x)$ is differentiable on $(\pi/4, 5\pi/4)$.

(iii) $f(a) = f(\pi/4) = e^{\pi/4}[\sin \pi/4 - \cos \pi/4]$

$= e^{\pi/4}[1/\sqrt{2} - 1/\sqrt{2}]$

$= 0.$

$f(b) = f(5\pi/4) = e^{5\pi/4}[\sin 5\pi/4 - \cos 5\pi/4]$

$= e^{5\pi/4}[-1/\sqrt{2} - (-1/\sqrt{2})]$

$= 0.$

$\therefore f(a) = f(b).$

Since, $f(x)$ satisfies all three conditions of Rolle's theorem, Rolle's theorem is applicable for $f(x) = e^x(\sin x - \cos x)$ on $[\pi/4, 5\pi/4]$.

Then $\exists\, c \in (\pi/4, 5\pi/4)$ such that $f'(c) = 0$.

$f(x) = e^x(\sin x - \cos x)$ $\qquad \frac{d}{dx}(uv) = uv' + vu'$.

$f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x)$

$= e^x \cancel{\cos x} + e^x \sin x + e^x \sin x - \cancel{e^x \cos x}.$

$f'(x) = 2e^x \sin x.$

$f'(c) = 2e^c \sin c = 0$

$\sin c = 0 = \sin \pi$

$\therefore c = \pi \in (\pi/4, 5\pi/4)$.

Hence Rolle's theorem is verified.

(6) Verify Rolle's theorem for $f(x) = x^2$ on $[0, 2]$.

Sol: Given $f(x) = x^2$.

(i) WKT, polynomial functions are continuous at every point.

$\therefore f(x) = x^2$ is continuous on $[0, 2]$.

(ii) WKT, polynomial functions are differentiable at every point.

$\therefore f(x) = x^2$ is differentiable on $(0, 2)$.

(iii) $f(a) = f(0) = 0^2 = 0$.

$f(b) = f(2) = 2^2 = 4$

$\therefore f(a) \neq f(b)$.

Hence, Rolle's theorem is not applicable.

(7) Verify Rolle's theorem for $f(x) = x^2$ on $[-1, 1]$. Ans. $c = 0$

(8) Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$.

Sol: Given, $f(x) = x(x+3)e^{-x/2}$.

(i) WKT, polynomial and exponential functions are continuous at every point.

$\therefore f(x) = x(x+3)e^{-x/2}$ is continuous on $[-3, 0]$.

(ii) WKT, polynomial and exponential functions are differentiable at every point.

$\therefore f(x) = x(x+3)e^{-x/2}$ is differentiable on $(-3, 0)$.

(iii) $f(a) = f(-3) = (-3)(-3+3)e^{-(-3)/2}$

$= -3(0)e^{3/2} = 0$.

$f(b) = f(0) = 0(0+3)e^{-0/2} = 0$.

$\therefore f(a) = f(b)$.

Since, $f(x)$ satisfies all three conditions, Rolle's theorem is applicable for $f(x) = x(x+3)e^{-x/2}$.

Then $\exists\, c \in (-3, 0)$ such that $f'(c) = 0$.

$f(x) = x(x+3)e^{-x/2} = (x^2+3x)e^{-x/2}$.

$f'(x) = (x^2+3x)e^{-x/2}(-1/2) + e^{-x/2}(2x+3)$.

$= e^{-x/2}\left[-x^2/2 - 3x/2 + 2x + 3\right] = e^{-x/2}\left[-\frac{x^2}{2} - \frac{3x}{2} + \frac{4x}{2} +\right.$

$= e^{-x/2}\left[-\frac{x^2}{2} + \frac{x}{2} + \frac{6}{2}\right] = e^{-x/2}\left[\frac{-x^2+x+6}{2}\right]$

$$f'(c) = e^{-c/2}\left[\frac{-c^2+c+6}{2}\right] = 0$$

$\Rightarrow -c^2+c+6 = 0$

$\Rightarrow c^2-c-6=0$

$\Rightarrow c^2-3c+2c-6=0$

$\Rightarrow c(c-3)+2(c-3)=0$

$\Rightarrow (c-3)(c+2)=0$

$c = 3, -2$

Here $c=-2 \in (-3,0)$.

Hence Rolle's theorem is verified.

(9) Verify Rolle's theorem for $f(x) = (x-a)^m(x-b)^n$ on $[a,b]$, $b>a$; $m \& n \in Z^+$. Ans. $c = \frac{mb+na}{m+n}$.

(10) verify Rolle's theorem for the function $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right]$ in $[a,b]$, $(x \neq 0)$, $a>0, b>0$. Ans. $c=\sqrt{ab}$

(11) Discuss the applicability of Rolle's theorem for $f(x) = \frac{1}{x^2}$ in $[-1,1]$.

Sol: Given, $f(x) = \frac{1}{x^2}$.

We find that $x=0 \in [-1,1]$, $f(0) = \frac{1}{0^2} = \infty$.

$f(x) = \frac{1}{x^2}$ is not continuous on $[-1,1]$.

$\therefore$ Rolle's theorem is not applicable.

LAGRANGES THEOREM:

Let $f(x)$ be a function such that

(1) $f(x)$ is continuous in $[a,b]$

(2) $f(x)$ is differentiable in (a,b).

Then $\exists$ at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

GEOMETRICAL INTERPRETATION OF LAGRANGE'S MEAN VALUE THEOREM:

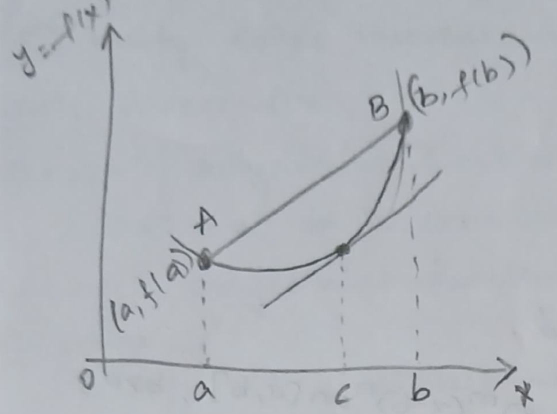
Geometrically Lagrange's mean value theorem interprets that

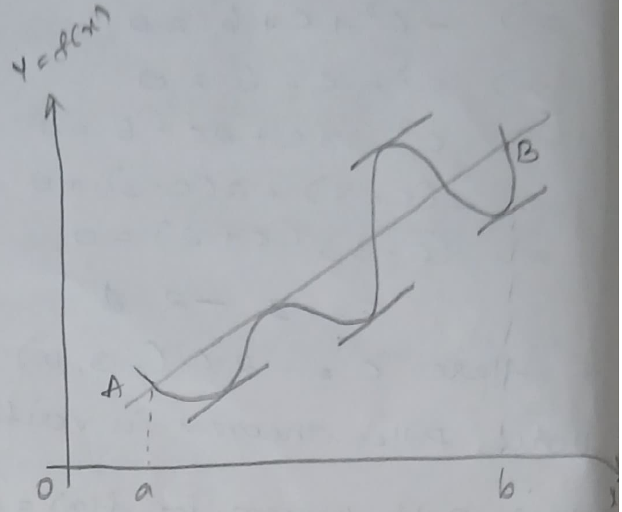
(1) The curve $y=f(x)$ is continuous in $[a,b]$.

(2) At every point $x=c$, where $a<c<b$, at the point $(c, f(c))$ on the curve $y=f(x)$ there is a unique tangent to the curve.

Then Lagrange's mean value theorem says that there is atleast one point on the curve where the tangent to the curve is parallel to the chord joining the end points.

$A(a, f(a))$ and $B(b, f(b))$ on the curve. Since the slope at c, $f'(c)$ is equal to the slope of the chord $AB = \frac{f(b) - f(a)}{b-a}$.





① Discuss the applicability of Lagrange's theorem for $f(x) = x^4$ on $[0, 2]$.

Sol: Given, $f(x) = x^4$ on $[0, 2]$

(i) $f(x) = x^4$ is continuous on $[0, 2]$.

(ii) $f(x) = x^4$ is differentiable on $(0, 2)$.

It satisfies two conditions of Lagrange's mean value theorem.

Then $\exists\ c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$. — ①

$f(x) = x^4$

$f'(x) = 4x^3$

$f'(c) = 4c^3$.

$f(a) = f(0) = 0^4 = 0$

$f(b) = f(2) = 2^4 = 16$.

Now ① $\Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$

$\Rightarrow 4c^3 = \frac{16 - 0}{2 - 0} = \frac{16}{2} = 8$.

$\Rightarrow c^3 = 2$.

$\Rightarrow c = \sqrt[3]{2} \in (0, 2)$.

Hence Lagrange's mean value theorem is verified.

② Verify Lagrange's theorem for $f(x) = (x-1)(x-2)(x-3)$ on $[0, 4]$.

Sol: Given, $f(x) = (x-1)(x-2)(x-3)$ on $[0, 4]$.

$\Rightarrow f(x) = [x^2 - 2x - x + 2](x-3)$

$= (x^2 - 3x + 2)(x-3)$

$= x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$

$f(x) = x^3 - 6x^2 + 11x - 6$

(i) WKT, Polynomial function are continuous at every point

$\therefore f(x) = x^3 - 6x^2 + 11x - 6$ is continuous in $[0,4]$.

(ii) WKT, polynomial function is differentiable at every point

$\therefore f(x) = x^3 - 6x^2 + 11x - 6$ is differentiable in $(0,4)$.

$f(x)$ satisfies two conditions of Lagrange's mean value theorem. Then $\exists\ c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ —(1)

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11.$

$f(a) = f(0) = (0)^3 - 6(0)^2 + 11(0) - 6 = -6.$

$f(b) = f(4) = 4^3 - 6(4)^2 + 11(4) - 6 = 64 - 96 + 44 - 6 = 6$

(1) $\Rightarrow 3c^2 - 12c + 11 = \dfrac{6-(-6)}{4-0} = \dfrac{\cancel{12}^{3}}{\cancel{4}} = 3.$

$3c^2 - 12c + 11 - 3 = 0$

$3c^2 - 12c + 8 = 0$

$c = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$c = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4(3)(8)}}{2(3)}$

$= \dfrac{12 \pm \sqrt{144 - 96}}{6}$

$= \dfrac{12 \pm \sqrt{48}}{6} = \dfrac{12 \pm 2\sqrt{12}}{6}$

$= \dfrac{2(6 \pm \sqrt{12})}{6} = \dfrac{6 \pm \sqrt{12}}{3}.$

$c = \dfrac{6 + \sqrt{12}}{3}$ (or) $\dfrac{6 - \sqrt{12}}{3}.$

$c = 3.154$ (or) $0.845 \in (0,4).$

Hence Lagrange's theorem is verified.

(3) verify Lagrange's theorem for $f(x) = e^{-x}$ on $[-1,1]$.

Sol: Given, $f(x) = e^{-x}$.

(i) $f(x)$ is continuous on $[-1,1]$

(ii) $f(x) = e^{-x}$ is differentiable on $(-1,1)$.

Hence, Lagrange's mean value theorem is applicable.

Then, $\exists\, c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ — ①

$f(x) = e^{-x}$

$f'(x) = -e^{-x}$

$f'(c) = -e^{-c}$

$f(a) = f(-1) = e^{-(-1)} = e^{1}$

$f(b) = f(1) = e^{-(1)} = e^{-1}$

① $\Rightarrow$ $-e^{-c} = \dfrac{e^{-1} - e^{1}}{1-(-1)}$

$\Rightarrow e^{-c} = \dfrac{e^{1} - e^{-1}}{2} = \sinh(1)$

$\log e^{-c} = \log[\sinh 1]$

$-c = \log[\sinh 1]$

$c = -\log[\sinh 1] = -0.07011 \in (-1, 1)$.

Hence Lagrange's theorem is verified.

④ Verify Lagrange's theorem for $f(x) = \log x$ in $[1, e]$.

Sol: Given, $f(x) = \log x$.

(i) WKT, logarithmic function is continuous over positive values.

$\therefore f(x) = \log x$ is continuous on $[1, e]$.

(ii) We observe that $f(x)$ is differentiable in $(1, e)$.

Hence Lagrange's theorem is applicable.

Then $\exists\, c \in (1, e)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ — ①

$f(x) = \log x$

$f'(x) = \dfrac{1}{x}$

$f'(c) = \dfrac{1}{c}$

$f(a) = f(1) = \log 1 = 0$

$f(b) = f(e) = \log e = 1$.

① $\Rightarrow \dfrac{1}{c} = \dfrac{1-0}{e-1} = \dfrac{1}{e-1}$

$c = e - 1 = 2.718 - 1 = 1.718 \in (1, e)$.

Hence Lagrange's mean value theorem is verified.

⑤ Show that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+b^2}$, $a<b$

and hence deduce that

(i) $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\left(\frac{4}{3}\right) < \pi/4 + \frac{1}{6}$.

(ii) $\frac{5\pi+4}{20} < \tan^{-1}(2) < \frac{\pi+2}{4}$.

Sol: Let $f(x) = \tan^{-1}(x)$ and

(i) $f(x)$ is continuous in $[a,b]$

(ii) $f(x)$ is differentiable in (a,b).

Hence by Lagrange's theorem, $\exists\, c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} \quad \text{—①}$$

$$f(x) = \tan^{-1}x$$

$$f'(x) = \frac{1}{1+x^2}$$

$$f'(c) = \frac{1}{1+c^2}$$

$$f(a) = \tan^{-1}a$$

$$f(b) = \tan^{-1}b.$$

$$① \Rightarrow \frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}. \quad \text{—②}$$

$\because c \in (a,b) \Rightarrow a < c < b$

$\Rightarrow a^2 < c^2 < b$

$\Rightarrow 1+a^2 < 1+c^2 < 1+b^2$

$\Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$.

$\Rightarrow \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$

$\Rightarrow \frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$ ($\because$ From ②).

$\Rightarrow \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ —③

Hence shown.

(i) Put $a=1$, $b=4/3$ in ③.

$$\frac{\frac{4}{3}-1}{1+(4/3)^2} < \tan^{-1}(4/3) - \tan^{-1}(1) < \frac{4/3-1}{1+(1)^2}.$$

$$\frac{\frac{4-3}{3}}{\frac{9+16}{9}} < \tan^{-1}(4/3) - \tan^{-1}(\tan \pi/4) < \frac{\frac{4-3}{3}}{2}.$$

$$\frac{1}{3} \times \frac{9}{25} < \tan^{-1}(4/3) - \pi/4 < \frac{1}{3\times 2}.$$

$$\therefore \pi/4 + \frac{3}{25} < \tan^{-1}(4/3) < \frac{\pi}{4} + \frac{1}{6}.$$

(ii) Put $a=1$, $b=2$ in (3), we get

$$\frac{2-1}{1+2^2} < \tan^{-1}2 - \tan^{-1}1 < \frac{2-1}{1+1^2}.$$

$$\frac{1}{5} < \tan^{-1}2 - \pi/4 < \frac{1}{2}$$

$$\frac{\pi}{4}+\frac{1}{5} < \tan^{-1}2 < \pi/4+\frac{1}{2}$$

$$\frac{5\pi+4}{20} < \tan^{-1}2 < \frac{\pi+2}{4}.$$

(6) Show that $1+x<e^x<1+xe^x$ for $x>0$ by Lagrange's theorem.

Sol: Let $f(x)=e^x$ defined on $[0,x]$.

(i) $f(x)$ is continuous on $[0,x]$

(ii) $f(x)$ is differentiable on $(0,x)$.

Then by Lagrange's mean value theorem,

$\exists\, c\in(a,b)$ i.e., $c\in(0,x)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

$$e^c = \frac{e^b-e^a}{b-a} = \frac{e^x-e^0}{x} \quad (\because f'(x)=e^x)$$

$$e^c = \frac{e^x-1}{x} \quad \text{—(1)}$$

We have $c\in(0,x) \Rightarrow 0<c<x$.

$$e^0<e^c<e^x \quad [\because e^x \text{ is an increasing function}]$$

$$1<\frac{e^x-1}{x}<e^x \quad [\because \text{From (1)}]$$

$$x<e^x-1<xe^x$$

$$1+x<e^x<1+xe^x.$$

(7) Show that $\frac{x}{1+x}<\log(1+x)<x$.

Sol: Let $f(x)=\log(1+x)$ defined on $[0,x]$

(i) $f(x)$ is continuous on $[0,x]$

(ii) $f(x)$ is differentiable on $(0,x)$.

Then by Lagrange's mean value theorem,

$\exists\, c\in(a,b)$ i.e., $c\in(0,x)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a} \Rightarrow \frac{1}{1+c}=\frac{\log(1+x)-\log(1+0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c}=\frac{\log(1+x)}{x} \quad \text{—(1)}$$

We have $c\in(0,x) \Rightarrow 0<c<x$

$\Rightarrow 1<1+c<1+x$

$\Rightarrow 1>\frac{1}{1+c}>\frac{1}{1+x}$

$\Rightarrow \frac{1}{1+x}<\frac{1}{1+c}<1 \Rightarrow \frac{1}{1+x}<\frac{\log(1+x)}{x}<1 \Rightarrow \frac{x}{1+x}<\log(1+x)<x$

$[\because \text{From (1)}]$

CAUCHY'S MEAN VALUE THEOREM:

If two functions $f(x)$, $g(x)$ are

(i) continuous in the closed interval $[a,b]$ i.e., $a \le x \le b$

(ii) Differentiable in the open interval (a,b) i.e., $a < x < b$.

and (iii) $g'(x) \neq 0$ for any $x \in (a,b)$.

Then there exists atleast one $c \in (a,b)$ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}.$$

① verify cauchy's mean value theorem for $f(x) = e^x$ and $g(x) = e^{-x}$ in $[a,b]$.

Sol: Given, $f(x) = e^x$, $g(x) = e^{-x}$.

(i) WKT, exponential function is continuous at every point.

$\therefore f(x) = e^x$, $g(x) = e^{-x}$ are continuous in $[a,b]$.

(ii) WKT, exponential function are differentiable ~~at~~ at every point.

$\therefore f(x) = e^x$, $g(x) = -e^{-x}$ are differentiable in (a,b).

(iii) We observe that $g'(x) \neq 0$ for $x \in (a,b)$.

Hence cauchy's mean value theorem is applicable.

Then $\exists\, c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$ — ①

$f(x) = e^x$	$g(x) = e^{-x}$
$f'(x) = e^x$	$g'(x) = -e^{-x}$
$f'(c) = e^c$	$g'(c) = -e^{-c}$.

Now, $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{e^b - e^a}{e^{-b} - e^{-a}} = \frac{-(e^a - e^b)}{\frac{1}{e^b} - \frac{1}{e^a}} = \frac{-\cancel{(e^a - e^b)}}{\frac{\cancel{(e^a - e^b)}}{e^a e^b}}$.

$= -e^a e^b$.

① $\Rightarrow$ $-e^a e^b = \frac{e^c}{-e^{-c}}$

$\Rightarrow e^{a+b} = e^{2c}$

$\Rightarrow 2c = a+b$

$\Rightarrow c = \frac{a+b}{2} \in (a,b)$.

Hence cauchy's mean value theorem is verified.

(2) Verify Cauchy's mean value theorem for

$f(x) = 3x^2+4x+5$, $g(x) = x^2 - x + 25$ over $[1, 2]$.

Sol: (i) WKT, polynomial function are continuous everywhere

$\therefore f(x), g(x)$ are continuous over $[1, 2]$.

(ii) WKT, polynomial function are differentiable at every point.

$\therefore f(x), g(x)$ are differentiable over $(1, 2)$.

(iii) We observe that $g(x) \neq 0$ for $x \in (1, 2)$

Hence, Cauchy's mean value theorem is applicable.

Then $\exists\, c \in (a, b)$ such that $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$ — (1)

$f(x) = 3x^2+4x+5$

$f'(x) = 6x+4$

$f'(c) = 6c+4$

$f(a) = f(1) = 3(1)^2+4(1)+5$

$= 3+4+5$

$= 12$

$f(b) = f(2) = 3(2)^2+4(2)+5$

$= 3(4)+8+5$

$= 12+13$

$= 25$.

$g(x) = x^2-x+25$

$g'(x) = 2x-1$

$g'(c) = 2c-1$

$g(a) = g(1) = (1)^2-1+25$

$= 25$

$g(b) = g(2) = (2)^2-2+25$

$= 4-2+25$

$= 27$.

(1) $\Rightarrow \dfrac{25-12}{27-25} = \dfrac{6c+4}{2c-1}$.

$\Rightarrow \dfrac{13}{2} = \dfrac{6c+4}{2c-1} \Rightarrow 13(2c-1) = 2(6c+4)$.

$\Rightarrow 26c - 13 = 12c+8$. $\Rightarrow 26c - 12c = 8+13$

$\Rightarrow 14c = 21$

$\Rightarrow c = \dfrac{21}{14} = \dfrac{3}{2} = 1.5$

$\therefore c = 1.5 \in (1, 2)$.

Hence Cauchy's mean value theorem is verified.

(3) Verify Cauchy's mean value theorem for $f(x) = \sin x$, $g(x) = \cos x$ in $[0, \pi/2]$.

Given, $f(x) = \sin x$, $g(x) = \cos x$

Sol: (i) WKT, $\sin x$, $\cos x$ are continuous at every point.

$\therefore f(x), g(x)$ are continuous on $[0, \pi/2]$.

(ii) WKT, $\sin x, \cos x$ are differentiable at every point.

$\therefore f(x), g(x)$ are differentiable on $(0, \pi/2)$.

(iii) We observe that $g(x) = \cos x \neq 0$ for $x \in (0, \pi/2)$.

Hence Cauchy's mean value theorem is applicable.

Then $\exists\, c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ —— (1).

$f(x) = \sin x$ $\qquad$ $g(x) = \cos x$

$f'(x) = \cos x$ $\qquad$ $g'(x) = -\sin x$.

$f'(c) = \cos c$ $\qquad$ $g'(c) = -\sin c$

$f(a) = f(0) = \sin 0 = 0$ $\qquad$ $g(a) = g(0) = \cos 0 = 1$

$f(b) = f(\pi/2) = \sin \pi/2 = 1$ $\qquad$ $g(b) = g(\pi/2) = \cos \pi/2 = 0$.

(1) $\Rightarrow \dfrac{1-0}{0-1} = \dfrac{\cos c}{-\sin c}$.

$\Rightarrow -1 = -\cot c$

$\Rightarrow \cot c = 1 = \cot \pi/4$

$\Rightarrow c = \pi/4 \in (0, \pi/2)$

Hence Cauchy's mean value theorem is verified.

(4) Verify Cauchy's mean value theorem for $f(x) = x^3$ and $g(x) = x^2$ in $[1, 2]$.

Sol: Given, $f(x) = x^3$, $g(x) = x^2$, we have,

(i) WKT, $f(x)$ and $g(x)$ are continuous in $[1, 2]$.

(ii) $f(x)$ and $g(x)$ are differentiable in $(1, 2)$

(iii) $g(x) \neq 0$ for $x \in (1, 2)$.

Hence Cauchy's Mean Value Theorem is applicable.

Then $\exists\, c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{g(b) - g(a)} = \dfrac{f'(c)}{g'(c)}$ —— (1)

$f(x) = x^3$ $\qquad$ $g(x) = x^2$

$f'(x) = 3x^2$ $\qquad$ $g'(x) = 2x$

$f'(c) = 3c^2$ $\qquad$ $g'(c) = 2c$

$f(a) = f(1) = 1^3 = 1$ $\qquad$ $g(a) = g(1) = 1^2 = 1$

$f(b) = f(2) = 2^3 = 8$ $\qquad$ $g(b) = g(2) = 2^2 = 4$.

(1) $\Rightarrow \dfrac{8-1}{4-1} = \dfrac{3c^{\cancel{2}}}{2\cancel{c}} \Rightarrow \dfrac{7}{3} = \dfrac{3c}{2} \Rightarrow \dfrac{2 \times 7}{3 \times 3} = c$

$\Rightarrow c = \dfrac{14}{9} \simeq 1.55 \in (1, 2)$.

Hence Cauchy's Mean Value Theorem is verified.

TAYLOR'S AND MACLAURIN...

Taylor's theorem:

If $f:[a,b] \to R$ such that

① $f^{(n-1)}$ is continuous on $[a,b]$

② $f^{(n-1)}$ is derivable on (a,b) (or) $f^{(n)}$ exist on (a,b).

and $p \in Z^+$ then there exist a point $c \in (a,b)$ such that

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$$
$$+ \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n.$$

where $R_n = \dfrac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}.$

① Schlomilch-Roche's form of remainder:

$$R_n = \frac{(b-a)^p (b-c)^{n-p} f^{(n)}(c)}{(n-1)!\, p}$$

② Lagrange's form of remainder:

Put $p=n$ we get

$$R_n = \frac{(b-a)^n f^{(n)}(c)}{n!}$$

③ Cauchy's form of remainder:

Put $p=1$ we get

$$R_n = \frac{(b-a)(b-c)^{n-1} f^{(n)}(c)}{(n-1)!}$$

Taylor's series of the function $f(x)$ in powers of $(x-a)$:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^{(n)}(a) \cdots$$

is called Taylor's series expansion of $f(x)$ about $x=a$.

Note: $f(a+h) = f(a) + \dfrac{h}{1!} f'(a) + \dfrac{h^2}{2!} f''(a) + \dfrac{h^3}{3!} f'''(a) + \cdots$

Maclaurin's theorem:

If $f:[0,x]\to R$ is such that

① $f^{(n-1)}$ is continuous on $[0,x]$

② $f^{(n-1)}$ is derivable on $(0,x)$

and $p\in Z^+$ then there exist a real number $\theta\in(0,1)$ such that

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0)+R_n$$

① Schlomilch form of remainder:

$$R_n=\frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f^n(\theta x).$$

② Lagrange's form of remainder:

Put $p=n$, we get

$$R_n=\frac{x^n(1-\theta)^{n-n}}{n(n-1)!}f^n(\theta x)=\frac{x^n}{n!}f^{(n)}(\theta x).$$

② Cauchy's form of remainder:

Put $p=1$, we get

$$R_n=\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^n(\theta x).$$

① Obtain the Taylor's Series Expansion of e^x about $x=-1$.

Sol: Let $f(x)=e^x$, $a=-1$.

The taylor's series expansion of $f(x)$ in powers of $(x-a)$ is

$$f(x)=f(a)+\frac{(x-a)}{1!}f'(a)+\frac{(x-a)^2}{2!}f''(a)+\frac{(x-a)^3}{3!}f'''(a)+\cdots \quad —①$$

$f(x)=e^x$ $\qquad f(a)=e^{-1}$

$f'(x)=e^x$ $\qquad f'(a)=e^{-1}$

$f''(x)=e^x$ $\qquad f''(a)=e^{-1}$

$f'''(x)=e^x$ $\qquad f'''(a)=e^{-1}$

$f^{IV}(x)=e^x$ $\qquad f^{IV}(a)=e^{-1}$.

$$①\Rightarrow f(x)=f(-1)+\frac{(x+1)}{1!}f'(-1)+\frac{(x+1)^2}{2!}f''(-1)+\cdots$$

$$\boxed{e^x=f(x)=e^{-1}+\frac{(x+1)}{1!}(e^{-1})+\frac{(x+1)^2}{2!}(e^{-1})+\cdots}$$

(2) Obtain the Taylor's series expansion of $\cos x$ about $x = \pi/2$.

Sol: Let $f(x) = \cos x$, $a = \pi/2$.

The taylor's series expansion for ~~per~~ $f(x)$ in powers of $(x-a)$ is

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots \quad \text{--- (1)}$$

$f(x) = \cos x$ $\quad f(a) = f(\pi/2) = \cos \pi/2 = 0$

$f'(x) = -\sin x$ $\quad f'(\pi/2) = -\sin \pi/2 = -1.$

$f''(x) = -\cos x$ $\quad f''(\pi/2) = -\cos \pi/2 = 0$

$f'''(x) = \sin x.$ $\quad f'''(\pi/2) = \sin \pi/2 = 1$

$f^{iv}(x) = \cos x.$ $\quad f^{iv}(\pi/2) = \cos \pi/2 = 0.$

$$(1) \Rightarrow f(x) = f(\pi/2) + \frac{(x-\pi/2)^1}{1!} f'(\pi/2) + \frac{(x-\pi/2)^2}{2!} f''(\pi/2) + \cdots$$

$$f(x) = 0 + \frac{(x-\pi/2)}{1!}(-1) + \frac{(x-\pi/2)^2}{2!}(0) + \frac{(x-\pi/2)^3}{3!}(1) + \frac{(x-\pi/2)^4}{4!}(0) + \cdots$$

$$\boxed{\cos x = -(x-\pi/2) + \frac{(x-\pi/2)^3}{3!} - \frac{(x-\pi/2)^5}{5!} + \cdots}$$

(3) Express $\sin 2x$ as power series in $(x-\pi/4)$.

Sol: Let $f(x) = \sin 2x$, and $a = \pi/4$. We have $f(a) = \sin \pi/2 = 1$.

$f'(x) = 2\cos 2x$ $\quad f'(\pi/4) = 2\cos 2(\pi/4) = 2\cos \pi/2 = 0$

$f''(x) = -4\sin 2x.$ $\quad f''(\pi/4) = -4\sin \pi/2 = -4.$

$f'''(x) = -8\cos 2x$ $\quad f'''(\pi/4) = -8\cos \pi/2 = 0$

$f^{iv}(x) = 16\sin 2x.$ $\quad f^{iv}(\pi/4) = 16\sin \pi/2 = 16.$

By Taylor's series expansion we have.

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \frac{(x-a)^4}{4!} f^{iv}(a) + \cdots$$

$$\sin 2x = f(\pi/4) + \frac{x-\pi/4}{1!} f'(\pi/4) + \frac{(x-a)^2}{2!} f''(\pi/4) + \cdots$$

$$= 1 + (x-\pi/4)(0) + \frac{(x-a)^2}{2!}(-4) + \frac{(x-a)^3}{3!}(0) + \frac{(x-a)^4}{4!}(16) + \cdots$$

$$\therefore \boxed{\sin 2x = 1 - 4\frac{(x-a)^2}{2!} + 16\frac{(x-a)^4}{4!} + \cdots}$$

(4) Express the polynomial $2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$ using Taylor's series.

Sol: Let $f(x) = 2x^3 + 7x^2 + x - 6$, $a = 2$, $f(a) = f(2) =$

$2(2)^3 + 7(2)^2 + 2 - 6 = 16 + 28 + 2 - 6 = 40.$

$f'(x) = 6x^2+14x+1$ $\qquad$ $f'(2) = 24+28+1 = 53.$

$f''(x) = 12x+14$ $\qquad$ $f''(2) = 24+14 = 38$

$f'''(x) = 12$ $\qquad$ $f'''(2) = 12.$

$f^{IV}(x) = 0$ $\qquad$ $f^{IV}(2) = 0.$

By Taylor series expansion, at $a=2$.

$$f(x) = f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \frac{(x-2)^4}{4!}f^{IV}(2) + \cdots$$

$$f(x) = 40 + (x-2)(53) + \frac{(x-2)^2}{2!}(38) + \frac{(x-3)^3}{3!}(12) + \frac{(x-4)^4}{4!}(0) + \cdots$$

$$\boxed{2x^3+7x^2+x-6 = 40+53(x-2) + 38\frac{(x-2)^2}{2!} + 12\frac{(x-2)^3}{3!} + \cdots}$$

⑤ Expand $\log x$ in powers of $(x-1)$ and hence evaluate $\log 1.1$ correct to 4 decimal places using Taylor's theorem.

Sol: The taylor's series expansion is

$$f(x) = f(a) + \frac{(x-a)}{1!}f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \quad \text{—①}$$

Here let $f(x) = \log x$, $a=1$, then $f(a) = \log 1 = 0$.

$f'(x) = \frac{1}{x}$, $\qquad$ $f'(1) = \frac{1}{1} = 1.$

$f''(x) = -\frac{1}{x^2}$ $\qquad$ $f''(1) = -1.$

$f'''(x) = \frac{2}{x^3}$ $\qquad$ $f'''(1) = 2.$

$f^{IV}(x) = -\frac{6}{x^4}$ $\qquad$ $f^{IV}(1) = -6.$

$$① \Rightarrow f(x) = f(1) + \frac{x-1}{1!}f'(1) + \frac{(x-1)^2}{2!}f''(1) + \cdots$$

$$\Rightarrow f(x) = 0 + (x-1)(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \cdots$$

$$\therefore \boxed{\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots \quad \text{—②}}$$

To find $\log 1.1$: Put $x = 1.1$ in ②.

$$\log 1.1 = (1.1-1) - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4} + \cdots$$

$$= (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \cdots$$

$$\boxed{\log 1.1 = 0.0953}$$

(6) Calculate the approximate value of $\sqrt{10}$ correct to 4 decimal places using Taylor's theorem.

Sol: ~~let $f(x+h) = \sqrt{x+h}$, where $x=9, h=1$~~

let $f(x) = \sqrt{x}$ ~~in $[a, a+h]$~~.

$f(x+h) = \sqrt{x+h}$ i.e., $\sqrt{10} = \sqrt{9+1}$.

let $x = 9$, $h = 1$.

Taylor series expansion is

$$f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots \quad -(1)$$

$f(x) = \sqrt{x}$, $f(a) = f(9) = \sqrt{9} = 3$.

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2x^{1/2}} = \frac{1}{2} x^{-1/2}$, $f'(9) = \frac{1}{2} 9^{-1/2} = \frac{1}{2\times 3} = \frac{1}{6}$.

$f''(x) = -\frac{1}{4} x^{-3/2}$, $f''(9) = \frac{-1}{108}$.

$f'''(x) = \frac{3}{8} x^{-5/2}$, $f'''(9) = \frac{1}{648}$.

$$(1) \Rightarrow f(x+h) = f(9) + \frac{1}{1!} f'(9) + \frac{1^2}{2!} f''(9) + \frac{1^3}{3!} f'''(9) + \cdots$$

$$\Rightarrow \sqrt{10} = 3 + 1\left(\frac{1}{6}\right) + \frac{1}{2}\left(-\frac{1}{108}\right) + \frac{1}{3}\left(\frac{1}{648}\right) + \cdots$$

$\sqrt{10} \simeq 3.1625514403$

$$\boxed{\sqrt{10} \simeq 3.1625}$$

(7) Find Maclaurin's series for

a) e^x

Sol: Maclaurin's series expansion is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad -(1)$$

let $f(x) = e^x$, $f(0) = e^0 = 1$

$f'(x) = e^x$, $f'(0) = e^0 = 1$

$f''(x) = e^x$, $f''(0) = e^0 = 1$

$f'''(x) = e^x$, $f'''(0) = e^0 = 1$.

$$(1) \Rightarrow f(x) = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(1) + \cdots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$$

b) $\sin x$.

let $f(x) = \sin x$, $f(0) = \sin 0 = 0$.

$f'(x) = \cos x$, $f'(0) = \cos 0 = 1$.

$f''(x) = -\sin x$, $f''(0) = -\sin 0 = 0$.

$f'''(x) = -\cos x$, $f'''(0) = -\cos 0 = -1$.

$f^{iv}(x) = \sin x$, $f^{iv}(0) = \sin 0 = 0$

$f^{v}(x) = \cos x$, $f^{v}(0) = \cos 0 = 1$

Meclaurin's series expansion is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \text{-----}$$

$$\sin x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \text{----}$$

$$\therefore \boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \text{----}}$$

c) $\cos x$.

let $f(x) = \cos x$, $f(0) = \cos 0 = 1$

$f'(x) = -\sin x$, $f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$, $f''(0) = -\cos 0 = -1$

$f'''(x) = \sin x$, $f'''(0) = \sin 0 = 0$

$f^{iv}(x) = \cos x$, $f^{iv}(0) = \cos 0 = 1$

$f^{v}(x) = -\sin x$, $f^{v}(0) = -\sin 0 = 0$.

Meclaurin's series expansion is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0) + \text{----}$$

$$\cos x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \frac{x^5}{5!}(0) + \text{----}$$

$$\therefore \boxed{\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{----}}$$

d) $\sinh x$.

Given $f(x) = \sinh x$, $f(0) = \sinh(0) = 0$

$f'(x) = \cosh x$, $f'(0) = \cosh(0) = 1$.

$f''(x) = \sinh x$, $f''(0) = \sinh(0) = 0$

$f'''(x) = \cosh x$, $f'''(0) = \cosh(0) = 1$.

Meclaurin's series expansion is

$$\cancel{\sinh x = 0 +} \; f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \text{----}$$

$$\sinh x = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(1) + \text{----}$$

$$\therefore \boxed{\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \text{----}}$$

e) $\cosh x$.

Given, $f(x) = \cosh x$, $f(0) = \cosh 0 = 1$

$f'(x) = \sinh x$, $f'(0) = \sinh 0 = 0$

$f''(x) = \cosh x$, $f''(0) = \cosh 0 = 1$

$f'''(x) = \sinh x$, $f'''(0) = \sinh 0 = 0$.

Meclaurin's series expansion is,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\cosh x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \cdots$$

$$\boxed{\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}$$

(8) Expand $\log(1+e^x)$ in ascending powers of x.

Sol: Let $f(x) = \log(1+e^x)$, $f(0) = \log(1+e^0) = \log(1+1) = \log 2$
$= 0.301$

$f'(x) = \frac{1}{1+e^x} \cdot e^x$, $f'(0) = \frac{1}{1+e^0} \cdot e^0 = \frac{1}{1+1} = \frac{1}{2}$.

$f''(x) = \frac{(1+e^x)e^x - e^x e^x}{(1+e^x)^2}$, $f''(0) = \frac{(1+e^0)e^0 - e^0 \cdot e^0}{(1+e^0)^2}$

$= \frac{e^x + e^{2x} - e^{2x}}{(1+e^x)^2}$ $\quad = \frac{(1+1) \cdot 1 - 1 \cdot 1}{(1+1)^2}$

$= \frac{e^x}{(1+e^x)^2}$. $\quad = \frac{2-1}{2^2} = \frac{1}{4}$.

Meclaurin series expansion is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$\log(1+e^x) = 0.301 + \frac{x}{2} + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \cdots$$

$$\boxed{\log(1+e^x) = 0.301 + \frac{x}{2} + \frac{x^2}{8} + \cdots}$$

(9) Expand $e^{\sin^{-1}(x)}$ as a power series of x.

Sol: Let $f(x) = e^{\sin^{-1}x}$, $f(0) = e^{\sin^{-1}(0)} = e^{\sin^{-1}(\sin 0)} = e^0 = 1$.

$f'(x) = \frac{1}{\sqrt{1-x^2}} e^{\sin^{-1}x}$, $f'(0) = \frac{1}{\sqrt{1-0^2}} e^{\sin^{-1}0} = 1$.

Meclaurin's series expansion is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\boxed{e^{\sin^{-1}x} = 1 + x + \frac{x^2}{2!} + \cdots}$$

(10) Expand $\tan x$ upto fifth power of x using Maclaurin's series expansion and hence evaluate the expansion of $\log(\sec x)$.

Sol: Maclaurin's series expansion is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots \quad (1)$$

Let $f(x) = \tan x$, $\quad f(0) = \tan(0) = 0$.

$f'(x) = \sec^2 x$, $\quad f'(0) = 1+0 = 1$.

$f'(x) = 1+\tan^2 x$

$f''(x) = 2\tan x(\sec^2 x)$, $\quad f''(0) = 2\tan 0 + 2\tan^3 0$

$= 2\tan x(1+\tan^2 x)$ $\quad = 0$.

$= 2\tan x + 2\tan^3 x$.

$f'''(x) = 2\sec^2 x + 6\tan^2 x \sec^2 x$.

$= 2(1+\tan^2 x) + 6\tan^2 x(1+\tan^2 x)$

$= 2 + 2\tan^2 x + 6\tan^2 x + 6\tan^4 x$.

$= 2 + 8\tan^2 x + 6\tan^4 x$, $\quad f'''(0) = 2+0+0 = 2$.

$f^{iv}(x) = 16\tan x(\sec^2 x) + 24\tan^3 x(\sec^2 x)$

$= 16\tan x(1+\tan^2 x) + 24\tan^3 x(1+\tan^2 x)$

$= 16\tan x(1+\tan^2 x) + 24\tan^3 x + 24\tan^5 x$.

$= 16\tan x + 16\tan^3 x + 24\tan^3 x + 24\tan^5 x$, $\quad f^{iv}(0) = 0+0+0 = 0$.

$= 16\tan x + 40\tan^3 x + 24\tan^5 x$

$f^{v}(x) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x$.

$= 16(1+\tan^2 x) + 120\tan^2 x(1+\tan^2 x) + 120\tan^4 x(1+\tan^2 x)$

$= 16 + 16\tan^2 x + 120\tan^2 x + 120\tan^4 x + 120\tan^4 x + 120\tan^6 x$

$f^{v}(0) = 16 + 0+0+0+0+0 = 16$.

$$(1) \Rightarrow f(x) = 0 + \frac{x}{1!}(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(16) + \cdots$$

$$\therefore \boxed{\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots} \quad (2)$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}.$$

Deduction of $\log(\sec x)$:

Integrating (2) on both sides,

$$\int \tan x\,dx = \int x\,dx + \frac{1}{3}\int x^3 dx + \frac{2}{15}\int x^5 dx + \cdots$$

$$\log(\sec x) = \frac{x^2}{2} + \frac{1}{3}\left(\frac{x^4}{4}\right) + \frac{2}{15}\left(\frac{x^6}{6}\right) + \cdots$$

$$\therefore \boxed{\log(\sec x) = \frac{x^2}{2} + \frac{x^4}{12} + \frac{x^6}{45} + \cdots}$$

(11) Verify Taylor's theorem for the function $f(x) = (1-x)^{5/2}$ by Lagrange's form of remainder upto the second term in the interval $[0, 1]$.

Sol: Given, $f(x) = (1-x)^{5/2}$.

Taylor's series expansion in $[a, b]$ is

$$f(x) = f(a) + \frac{(b-a)}{1!} f'(a) + \cdots + R_n$$

In $[0, x]$ is

$$f(x) = f(0) + \frac{x}{1!} f'(0) + R_n.$$

With Lagrange's remainder it becomes

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(c) \quad \text{—(1)}$$

~~At $x = x_1$.~~

$$f(x) = (1-x)^{5/2}, \quad f(0) = (1-0)^{5/2} = 1$$

$$f'(x) = 5/2\,(1-x)^{5/2-1}(-1), \quad f'(0) = -5/2\,(1-0)^{3/2} = -5/2.$$

$$= \frac{5}{2}(1-x)^{3/2}(-1)$$

$$f''(x) = -5/2 \cdot 3/2\,(1-x)^{3/2-1}(-1).$$

$$= \frac{15}{4}(1-x)^{1/2}, \quad f''(c) = \frac{15}{4}(1-c)^{1/2}.$$

$$(1) \Rightarrow (1-x)^{5/2} = 1 + \frac{x}{1!}(-5/2) + \frac{x^2}{2!}(1-c)^{1/2} \cdot \frac{15}{4}$$

Put $x = 1$.

$$\Rightarrow 0 = 1 - 5/2 + \frac{(1-c)^{1/2}}{2} \cdot \frac{15}{4}$$

$$\Rightarrow 0 = -3/2 + \frac{15}{8}(1-c)^{1/2}$$

$$\Rightarrow \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{8}^{4}}{\cancel{15}_{5}} = (1-c)^{1/2}$$

$$\Rightarrow \frac{4}{5} = (1-c)^{1/2}$$

$$\Rightarrow \frac{16}{25} = 1 - c$$

$$\Rightarrow c = 1 - \frac{16}{25} = \frac{25-16}{25} = \frac{9}{25}$$

$$\Rightarrow c = 0.36 \in (0, 1).$$

Hence Taylor's theorem is verified.