UNIT-IV: PARTIAL DIFFERENTIATION AND APPLICATIONS (MULTI VARIABLE CALCULUS)

Syllabus: Functions of several variables: Continuity and Differentiability, Partial derivatives, total derivatives, chain rule, Taylor's and Maclaurin's series expansion of functions of two variables. Jacobians, Functional dependence, maxima and minima of functions of two variables, method of Lagrange multipliers.

Introduction:

Function of several variables: $y = f(x_1, x_2, x_3, ----)$.

Limit:

$\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = l$ (or) $\lim\limits_{(x,y) \to (a,b)} f(x,y) = l$.

$\lim\limits_{(x,y) \to (a,b)} f(x,y) = l \Rightarrow \lim\limits_{x \to a} f(x,b) = l = \lim\limits_{y \to b} f(a,y)$.

Ex: $\lim\limits_{\substack{x \to 1 \\ y \to 2}} \left[\frac{2x^2 y}{x^2+y^2+1}\right] = \lim\limits_{x \to 1} \left[\lim\limits_{y \to 2} \frac{2x^2 y}{x^2+y^2+1}\right]$

$= \lim\limits_{x \to 1} \left[\frac{4x^2}{x^2+5}\right]$

$= \frac{4}{6} = \frac{2}{3}$.

Continuity:

$\lim\limits_{\substack{x \to a \\ y \to b}} f(x,y) = f(a,b)$.

Differentiability: $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^ny}{dx^n}, ----$

Partial derivative: The partial derivative of a function with respect to a variable is the derivative with other variables treated as constant.

Let $K = f(x,y,z)$, then

- partial derivative of K with respect to x is denoted by $\frac{\partial f}{\partial x}$ or $\frac{\partial K}{\partial x}$ or f_x which is obtained by differentiating f treating y and z as constant.

- partial derivative of k with respect to y is denoted by $\frac{\partial f}{\partial y}$ or $\frac{\partial k}{\partial y}$ or f_y which is obtained by differentiating f treating x and z as constant.
- partial derivative of k with respect to z is denoted by $\frac{\partial f}{\partial z}$ or $\frac{\partial k}{\partial z}$ or f_z which is obtained by differentiating f treating x and y as constant.

Ex:

① let $z = x + y + xy = f(x, y)$.

Partially differentiate with respect to x

$$\frac{\partial z}{\partial x} = 1 + 0 + y(1) = 1 + y.$$

Partially differentiate w.r.to y

$$\frac{\partial z}{\partial y} = 0 + 1 + x(1) = 1 + x.$$

② let $k = x + y + z + xyz$

$$\frac{\partial k}{\partial x} = 1 + 0 + 0 + (1)yz = 1 + yz.$$

$$\frac{\partial k}{\partial y} = 1 + xz$$

$$\frac{\partial k}{\partial z} = 1 + xy.$$

③ let $z = \sin(xy)$

$$\frac{\partial z}{\partial x} = \cos(xy) \frac{\partial (xy)}{\partial x}$$

$$= \cos(xy) \cdot y(1)$$

$$= y\cos(xy)$$

$$\frac{\partial z}{\partial y} = \cos(xy) \frac{\partial (xy)}{\partial y}$$

$$= \cos(xy) \cdot x(1)$$

$$= x\cos(xy).$$

Note: let $z = f(x, y)$

→ Partial derivative w.r.t x is $\frac{\partial z}{\partial x}$

→ Partial derivative w.r.t y is $\frac{\partial z}{\partial y}$.

Here $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are called first order derivatives.

$\rightarrow \dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right)$

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right)$

$\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right)$.

are the second ~~~~ order derivatives.

Notations: The standard notations are

$p = \dfrac{\partial z}{\partial x}$

$q = \dfrac{\partial z}{\partial y}$

$r = \dfrac{\partial^2 z}{\partial x^2}$

$s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$

$t = \dfrac{\partial^2 z}{\partial y^2}$.

① Find the first and second order derivatives if $z = ax^2 + 2hxy + by^2$.

Sol: Given, $z = ax^2 + 2hxy + by^2$.

$\dfrac{\partial z}{\partial x} = 2ax + 2hy$

$\dfrac{\partial z}{\partial y} = 2hx + 2by$

$\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}(2ax + 2hy) = 2a.$

$\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}(2hx + 2by) = 2b.$

$\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}(2hx + 2by) = 2h.$

② Let $f(x,y) = x^3 + y^3 - 3axy$, verify $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

Sol: Given, $f = x^3 + y^3 - 3axy$.

$\dfrac{\partial f}{\partial x} = 3x^2 - 3ay$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}(3y^2 - 3ax)$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3a$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}(3x^2 - 3ay)$$

$$\frac{\partial^2 f}{\partial y \partial x} = -3a.$$

$$\therefore \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

(3) If $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$ then ~~the~~ prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Sol: Given, $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$.

$$\frac{\partial u}{\partial x} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\frac{\partial}{\partial x}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2} \cdot \frac{(x^2-y^2)(2y) - 2xy(2x)}{\cancel{(x^2-y^2)^2}}.$$

$$= \frac{2y[x^2-y^2-2x^2]}{x^4+y^4-2x^2y^2+4x^2y^2}$$

$$= \frac{-2y\cancel{[x^2+y^2]}}{(x^2+y^2)^{\cancel{2}}}$$

$$\frac{\partial u}{\partial x} = \frac{-2y}{x^2+y^2}.$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-2y}{x^2+y^2}\right)$$

$$= -2y \frac{\partial}{\partial x}(x^2+y^2)^{-1}$$

$$= -2y(-1)(x^2+y^2)^{-1-1}(2x).$$

$$= (2y)(2x)(x^2+y^2)^{-2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{4xy}{x^2+y^2}.$$

Similarly $\frac{\partial^2 u}{\partial y^2} = \frac{-4xy}{x^2+y^2}.$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4xy}{x^2+y^2} - \frac{4xy}{x^2+y^2} = 0.$$

(4) If $u = \log(x^3+y^3+z^3-3xyz)$, prove that

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}.$$

Sol: Given, $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}.$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3[x^2+y^2+z^2-xy-yz-zx]}{(x+y+z)[x^2+y^2+z^2-xy-yz-zx]}$$

$$= \frac{3}{x+y+z}.$$

Now, $\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)u$

$$= \left(\frac{\partial}{\partial x}+\frac{\partial}{\partial y}+\frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) + \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right)$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$\therefore \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

Composite function: A composite function is formed when we substitute one function into another.

Let $z = f(x, y)$ be a function in two variables.

and $x = f_1(t)$, $y = f_2(t)$.

Then we can form the composite function,

$$z = f(f_1(t), f_2(t)).$$

Total derivative:

Let $z = f(x, y)$, $x = f_1(t)$ and $y = f_2(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

1) If $z = x+y$, $x = t^2$, $y = t^3$, then find $\frac{dz}{dt}$.

Sol: Given, $z = x+y$, $x = t^2$, $y = t^3$.

By total derivative

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x+y) = 1.$$

$$\frac{\partial z}{\partial y} = 1.$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^2) = 2t$$

$$\frac{dy}{dt} = \frac{d}{dt}(t^3) = 3t^2.$$

$$(1) \Rightarrow \frac{dz}{dt} = (1)(2t) + (1)(3t^2)$$

$$\frac{dz}{dt} = 2t + 3t^2.$$

(2) If $u = x^2+y^2$; $x = at^2$, $y = 2at$ then find $\frac{du}{dt}$.

Sol: Given, $u = x^2+y^2$, $x = at^2$, $y = 2at$.

By total derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial x} = \frac{\partial (x^2+y^2)}{\partial x} = 2x.$$

$$\frac{dx}{dt} = \frac{d}{dt}(at^2) = 2at$$

$$\frac{\partial u}{\partial y} = \frac{d}{\partial y}(x^2+y^2) = 2y.$$

$$\frac{dy}{dt} = \frac{d}{dt}(2at) = 2a.$$

$$(1) \Rightarrow \frac{du}{dt} = (2x)(2at) + (2y)(2a)$$

$$\frac{du}{dt} = 4axt + 4ay.$$

(3) If $u = \sin\left(\frac{x}{y}\right)$; $x = e^t$, $y = t^2$ then find $\frac{du}{dt}$.

Sol: Given, $u = \sin\left(\frac{x}{y}\right)$, $x = e^t$, $y = t^2$.

By total derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial x} = \cos\left(\frac{x}{y}\right) \frac{\partial}{\partial x}\left(\frac{x}{y}\right) = \frac{1}{y} \cos \frac{x}{y}.$$

$$\frac{\partial u}{\partial y} = \cos\left(\frac{x}{y}\right) \frac{\partial}{\partial y}\left(\frac{x}{y}\right) = -\frac{x}{y^2} \cos \frac{x}{y}.$$

$$\frac{dx}{dt} = e^t, \quad \frac{dy}{dt} = 2t.$$

$$(1) \Rightarrow \frac{du}{dt} = \frac{1}{y}\cos\left(\frac{x}{y}\right) \cdot e^t + \left(-\frac{x}{y^2}\right) \cos\left(\frac{x}{y}\right) \cdot 2t$$

$$\frac{du}{dt} = \frac{e^t}{y} \cos\left(\frac{x}{y}\right) - 2t \frac{x}{y^2} \cos\left(\frac{x}{y}\right).$$

(4) If $u = x^2+y^2+z^2$; $x = e^{2t}$, $y = e^{2t}\cos(3t)$, $z = e^{2t}\sin(3t)$ then find $\frac{du}{dt}$.

Sol: Given, $u = x^2+y^2+z^2$; $x = e^{2t}$, $y = e^{2t}\cos(3t)$, $z = e^{2t}\sin(3t)$

By total derivative,

$$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt} \quad \text{—— (1)}$$

$\frac{\partial u}{\partial x} = 2x$, $\frac{\partial u}{\partial y} = 2y$, $\frac{\partial u}{\partial z} = 2z$.

$\frac{dx}{dt} = 2e^{2t}$, $\frac{dy}{dt} = 2e^{2t}\cos(3t) + 3(-\sin 3t)e^{2t}$,

$\frac{dz}{dt} = 2e^{2t}\sin(3t) + 3e^{2t}\cos(3t)$

① $\Rightarrow$ $\frac{du}{dt} = (2x)(2e^{2t}) + 2y\left[2e^{2t}\cos 3t - 3e^{2t}\sin 3t\right]$
$+ (2z)\left[2e^{2t}\sin(3t) + 3e^{2t}\cos(3t)\right]$

$\therefore \frac{du}{dt} = 4xe^{2t} + 4ye^{2t}\cos 3t - 6ye^{2t}\sin 3t + 4ze^{2t}\sin 3t$
$+ 6ze^{2t}\cos 3t$.

Chain rule : let $z = f(x, y)$; $x = f_1(u, v)$, $y = f_2(u, v)$,

then the chain rule in two variables is

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u}.$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v}.$$

Ex : If $z = x^2 + y^2$; $x = uv$, $y = u + v$, then find $\frac{\partial z}{\partial u}$, $\frac{\partial z}{\partial v}$.

Sol: Given, $z = x^2 + y^2$; $x = uv$, $y = u + v$.

By chain rule,

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} \quad \text{—①}.$$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$, $\frac{\partial z}{\partial y} = 2y$.

$\frac{\partial x}{\partial u} = v$, $\frac{\partial y}{\partial u} = 1$

① $\Rightarrow$ $\frac{\partial z}{\partial u} = (2x)(v) + (2y)(1)$

$\frac{\partial z}{\partial u} = 2xv + 2y$.

By chain rule.

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial v} \quad \text{—②}$$

$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2) = 2x$, $\frac{\partial z}{\partial y} = 2y$

$\frac{\partial x}{\partial v} = u$, $\frac{\partial y}{\partial v} = 1$

② $\Rightarrow$ $\frac{\partial z}{\partial v} = 2xu + 2y$.

Jacobian: Let $u = u(x, y)$ & $v = v(x, y)$, then the transformation (x, y) to (u, v) is denoted by

$\frac{\partial(u, v)}{\partial(x, y)}$ (or) $J\left(\frac{u, v}{x, y}\right)$ and is defined as

$$\frac{\partial(u, v)}{\partial(x, y)} = J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$$

Jacobian for 3 variables:

Let $u = u(x, y, z)$, $v = v(x, y, z)$ and $w = w(x, y, z)$, then

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Note: 1. If the Jacobian value is equal to zero then the given functions are functionally dependent, i.e., there is a relation between them.

2. If the Jacobian value is not equal to zero then the given functions are functionally independent, i.e., there is no relation between them.

(1) If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$ then find the value of $\frac{\partial(u, v, w)}{\partial(x, y, z)}$.

Sol: Given, $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$.

$\frac{\partial u}{\partial x} = yz$, $\frac{\partial u}{\partial y} = xz$, $\frac{\partial u}{\partial z} = xy$

$\frac{\partial v}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = 2y$, $\frac{\partial v}{\partial z} = 2z$

$\frac{\partial w}{\partial x} = 1$, $\frac{\partial w}{\partial y} = 1$, $\frac{\partial w}{\partial z} = 1$

Now Jacobian is

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = J\left(\frac{u, v, w}{x, y, z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} yz & zx & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= yz(2y-2z) - zx(2x-2z) + xy(2x-2y)$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 2y^2z - 2yz^2 - 2x^2z + 2xz^2 + 2x^2y - 2xy^2.$$

(2) If $x = r\cos\theta$, $y = r\sin\theta$, find $\frac{\partial(x,y)}{\partial(r,\theta)}$

Sol: Given, $x = r\cos\theta$, $y = r\sin\theta$.

The Jacobian is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = J\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}.$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = r$$

(3) If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$ find $\frac{\partial(u,v)}{\partial(x,y)}$. Verify whether u, v are functionally dependent, if yes find the relation.

Sol: Given, $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$.

Now Jacobian is

$$\frac{\partial(u,v)}{\partial(x,y)} = J\left(\frac{u,v}{x,y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \quad -(1)$$

$$d\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}.$$

$$\frac{\partial u}{\partial x} = \frac{(1-xy)(1) - (x+y)(-y)}{(1-xy)^2}$$

$$= \frac{1 - xy + xy + y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial x} = \frac{1+y^2}{(1-xy)^2}.$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy)(1) - (x+y)(-x)}{(1-xy)^2} = \frac{1 - xy + x^2 + xy}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{1+x^2}{(1-xy)^2}.$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+x^2}, \quad \frac{\partial v}{\partial y} = \frac{1}{1+y^2}$$

(1) $\Rightarrow$ $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \left[\frac{\cancel{1+y^2}}{(1-xy)^2} \cdot \frac{1}{\cancel{1+y^2}} - \frac{\cancel{1+x^2}}{(1-xy)^2} \cdot \frac{1}{\cancel{1+x^2}} \right]$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$\frac{\partial(u,v)}{\partial(x,y)} = 0$$

i-e-) Jacobian value = 0.

∴ The given functions are dependent.

Relation between u & v:

$$v = \tan^{-1}x + \tan^{-1}y$$

$$v = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$$

$$\boxed{v = \tan^{-1}(u)}$$

(4) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

Sol: Given, $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$.

$$\frac{\partial u}{\partial x} = -\frac{yz}{x^2}, \quad \frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial u}{\partial z} = \frac{y}{x}.$$

$$\frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial v}{\partial y} = -\frac{zx}{y^2}, \quad \frac{\partial v}{\partial z} = \frac{x}{y}.$$

$$\frac{\partial w}{\partial x} = \frac{y}{z}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}, \quad \frac{\partial w}{\partial z} = \frac{-xy}{z^2}.$$

Now Jacobian is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & -\frac{zx}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= -\frac{yz}{x^2}\left(\frac{zx}{y^2} \cdot \frac{xy}{z^2} - \frac{x}{y} \cdot \frac{x}{z}\right) - \frac{z}{x}\left(\frac{z}{y} \cdot \frac{(-xy)}{z^2} - \frac{x}{y} \cdot \frac{y}{z}\right)$$

$$+ \frac{y}{x}\left(\frac{z}{y} \cdot \frac{x}{z} + \frac{zx}{y^2} \cdot \frac{y}{z}\right)$$

$$= \frac{-yz}{x^2}\cdot\frac{zx}{y^2}\cdot\frac{xy}{z^2} + \frac{yz}{x^2}\cdot\frac{x}{y}\cdot\frac{x}{z} + \frac{z}{x}\cdot\frac{z}{y}\cdot\frac{xy}{z^2} + \frac{z}{x}\cdot\frac{x}{y}\cdot\frac{y}{z}$$

$$+ \frac{y}{x}\cdot\frac{z}{y}\cdot\frac{x}{z} + \frac{zx}{y^2}\cdot\frac{y}{z}\cdot\frac{y}{x}$$

$$= -1+1+1+1+1+1 = 4.$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 4.$$

(5) Prove that $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$ are functionally ~~[illegible]~~ dependent and find the relation between them.

Sol: Given, $u = x+y+z$, $v = xy+yz+zx$, $w = x^2+y^2+z^2$.

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = 1, \quad \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = y+z, \quad \frac{\partial v}{\partial y} = x+z, \quad \frac{\partial v}{\partial z} = x+y$$

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 2y, \quad \frac{\partial w}{\partial z} = 2z.$$

The Jacobian is

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = J\left(\frac{u,v,w}{x,y,z}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ y+z & x+z & x+y \\ 2x & 2y & 2z \end{vmatrix}$$

$$= 1(2xz+2z^2-2xy-2y^2)$$
$$-1(2yz+2z^2-2x^2-2xy)$$
$$+1(2y^2+2yz-2x^2-2xz)$$

$$= 2xz+2z^2-2xy-2y^2-2yz-2z^2$$
$$+2x^2+2xy+2y^2+2yz-2x^2-2xz$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$$

i.e.) Jacobian value $=0$

$\therefore$ The given u, v, w are functionally dependent.

Relation b/w u, v, w:

$$u^2 = (x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$$

$$\therefore u^2 = 2v+w.$$

⑥ If $x = u - uv$, $y = uv$ then find $\frac{\partial(x,y)}{\partial(u,v)}$.

Sol: Given, $x = u - uv$, $y = uv$.

$\frac{\partial x}{\partial u} = 1 - v$, $\frac{\partial x}{\partial v} = -u$

$\frac{\partial y}{\partial u} = v$, $\frac{\partial y}{\partial v} = u$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = (1-v)u + uv$$

$$= 1 - vu + uv$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = 1.$$

⑦ If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$.

Sol: $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

$$= \begin{vmatrix} 2x & -2 & 0 \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix}$$

$$= 2x(3+2) + 2(3-1) + 0(-2-1)$$

$$= 6x + 4x + 2(2) + 0$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = 10x + 4$$

⑧ Verify if $u = 2x - y + 3z$, $v = 2x - y - z$, $w = 2x - y + z$ are functionally dependent, if so find the relation between them.

Sol: $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 2 & -1 & 3 \\ 2 & -1 & -1 \\ 2 & -1 & 1 \end{vmatrix} = 2(-1-1) + 1(2+2) + 3(-2+2)$$

$$= 2(-2) + 1(4) + 3(0)$$

$$= -4 + 4 + 0 = 0.$$

$\therefore$ Given functions are dependent.

Relation between u, v and w:

$$u + v = 2x - y + 3z + 2x - y - z$$

$$= 4x - 2y + 2z$$

$$= 2(2x - y + z)$$

$$\boxed{u + v = 2w} \Rightarrow \boxed{u + v - 2w = 0}$$

Maxima and minima of functions of two variables:

Working rule:

Let the given function be $f(x,y)$.

1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.

2. Take $\frac{\partial f}{\partial x} = 0$ —— ① and $\frac{\partial f}{\partial y} = 0$ —— ②

3. Solve ② and ③, we get

$(x,y) = (a_1,b_1), (a_2,b_2), (a_3,b_3), \text{------}$

These points are stationary points.

4. Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$ at stationary points.

5. Find $rt - s^2$:

a) If $rt - s^2 > 0$ and $r < 0$ at (a_1,b_1), then f has maximum value at (a_1,b_1).
i.e., $f(a_1,b_1)$ is a maximum value.

b) If $rt - s^2 > 0$ and $r > 0$ at (a_1,b_1), then f has minimum value at (a_1,b_1).
i.e., $f(a_1,b_1)$ is a minimum value.

c) If $rt - s^2 < 0$ at (a_1,b_1), then $f(a_1,b_1)$ is neither maximum nor minimum,
i.e., (a_1,b_1) is a saddle point).

d) If $rt - s^2 = 0$ at (a_1,b_1), then (the case) we can't say (is doubtful) and needs further investigation.

Similarly we ~~fi~~ analyse maxima and minima at other stationary points $(a_2,b_2), (a_3,b_3), \text{---}$

① Find the stationary points of $x^2+y^2+6x+12$. And discuss the maxima and minima.

Sol: Let $f(x,y) = x^2+y^2+6x+12$.

$\frac{\partial f}{\partial x} = 2x+6$, $\frac{\partial f}{\partial y} = 2y$

Taking $\frac{\partial f}{\partial x} = 0 \Rightarrow 2x+6 = 0 \Rightarrow x = -3$.

Taking $\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$.

$(x,y) = (-3,0)$. is the stationary point.

Now, $r = \frac{\partial^2 f}{\partial x^2} = 2$, $s = \frac{\partial^2 f}{\partial x \partial y} = 0$, $t = \frac{\partial^2 f}{\partial y^2} = 2$.

At $(-3,0)$, $r = 2 > 0$, $s = 0$, $t = 2$.

$rt - s^2 = 2(2) - 0^2 = 4 > 0.$

Here $rt - s^2 > 0$ and $r > 0$.

$\therefore f$ has minimum value at $(-3, 0)$.

$\therefore$ Minimum value $= f(-3, 0)$

$= (-3)^2 + (0)^2 + 6(-3) + 12.$

$= 9 + 0 - 18 + 12$

$= 21 - 18$

Minimum value $= 3$.

(2) Find the maximum and minimum values of $x^3 + y^3 - 3axy$, $a > 0$.

Sol: Let $f(x, y) = x^3 + y^3 - 3axy$.

$\frac{\partial f}{\partial x} = 3x^2 - 3ay$, $\frac{\partial f}{\partial y} = 3y^2 - 3ax$.

Taking $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 3ay = 0$

$\Rightarrow x^2 = ay$ — (1)

Taking $\frac{\partial f}{\partial y} = 0 \Rightarrow 3y^2 - 3ax = 0$

$\Rightarrow y^2 = ax$ — (2)

$\Rightarrow y^4 = a^2(ay) \Rightarrow y^4 - a^3y = 0 \Rightarrow y(y^3 - a^3) = 0$

$\Rightarrow y^3 = a^3$ or $y = 0$

$\Rightarrow y = a$. or $y = 0$

(1) $\Rightarrow x^2 = a(a) \Rightarrow x = a$. or $x = 0$.

$(x, y) = (0, 0)$

$(x, y) = (a, a)$ are the stationary point.

Now, $r = \frac{\partial^2 f}{\partial x^2} = 6x$

$s = \frac{\partial^2 f}{\partial x \partial y} = -3a$

$t = \frac{\partial^2 f}{\partial y^2} = 6y$.

At (a, a), $r = 6a > 0$, $s = -3a$, $t = 6a$.

$rt - s^2 = (6a)(6a) - (-3a)^2 = 36a^2 - 9a^2 = 27a^2 > 0$.

Here $rt - s^2 > 0$ and $r > 0$.

$\therefore f$ has minimum value at (a, a).

$\therefore$ Minimum value $= f(a, a)$

$= a^3 + a^3 - 3a(a)(a)$

$= a^3 + a^3 - 3a^3$

Minimum value $= -a^3$.

At $(0, 0)$, $r = 0$, $s = -3a$, $t = 0$; $rt - s^2 = -9a^2 < 0$.

$\therefore f(0, 0)$ is neither maximum nor minimum.

③ Examine the following function for extreme values:

$f(x,y) = x^4+y^4-2x^2+4xy-2y^2$.

Sol: Given, $f(x,y) = x^4+y^4-2x^2+4xy-2y^2$

$\frac{\partial f}{\partial x} = 4x^3-4x+4y$; $\frac{\partial f}{\partial y} = 4y^3-4y+4x$.

$\frac{\partial f}{\partial x} = 0 \Rightarrow x^3-x+y=0$ — ①

$\frac{\partial f}{\partial y} = 0 \Rightarrow y^3+x-y=0$ — ②

① + ② $\Rightarrow x^3+y^3=0$

$(x+y)(x^2-xy+y^2)=0$

$x+y=0$ or $x^2-xy+y^2=0$

$\therefore x=-y$. $\quad b^2-4ac = 1-4=-3$

(No real values)

② $\Rightarrow y^3-y-y=0 \Rightarrow y^3-2y=0$

$y(y^2-2)=0 \Rightarrow y=0$ or $y=\pm\sqrt{2}$.

$y=0 \Rightarrow x=0$

$y=\sqrt{2} \Rightarrow x=-\sqrt{2}$

$y=-\sqrt{2} \Rightarrow y=\sqrt{2}$.

$(x,y) = (0,0), (\sqrt{2},-\sqrt{2}), (-\sqrt{2},\sqrt{2})$

are the stationary points.

$r = \frac{\partial^2 f}{\partial x^2} = 12x^2-4$

$s = \frac{\partial^2 f}{\partial x \partial y} = 4$

$t = \frac{\partial^2 f}{\partial y^2} = 12y^2-4$.

At $(0,0)$: $r=-4$, $s=4$, $t=-4$

$rt-s^2 = (-4)(-4)-4^2 = 16-16=0$

We cant say about minima & maxima. Further investigation is needed.

At $(\sqrt{2},-\sqrt{2})$:

$r=20$, $s=4$, $20=t$

$rt-s^2 = 384>0$, $r>0$.

$\therefore f$ has minimum value at $(\sqrt{2},-\sqrt{2})$.

Minimum value $= f(\sqrt{2},-\sqrt{2})$

$= 4+4-4-8-4$

$= -8$.

At $(-\sqrt{2},\sqrt{2})$:

$r=20$, $S=4$, $t=20$

$rt-s^2 = 384>0$, $r>0$.

$\therefore f$ has minimum value at $(-\sqrt{2},\sqrt{2})$

Minimum value $= f(-\sqrt{2},\sqrt{2})$

$= 4+4-4-8-4$

$= -8$.

Method of Lagrange multipliers (or) Lagrange's method of undetermined multipliers:

This method is also called as Lagrange's method of undetermined multipliers or Lagrange's method of undetermined coefficients.

Working rule (or) Process:

① Let the given function be $f(x,y,z)$ — ①

and constraint function be $\phi(x,y,z)=0$ — ②

② Write the Lagrangian function as

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$ — ③

Here, λ is called Lagrangian multiplier.

③ Partially differentiating F w.r.t x, y, z and equating to zero. i.e.,

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \quad — ④$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \quad — ⑤$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \quad — ⑥$$

④ Find the values of x, y, z using equations ②, ④, ⑤, ⑥.

④ And hence get stationary points.

$(x,y,z) = (a_1, b_1, c_1), (a_2, b_2, c_2)$ ----

⑤ Substitute the stationary points in ① to get maximum & minimum values.

① Find the minimum value of $x^2+y^2+z^2$ given that $xyz=a^3$.

Sol): Let $f(x,y,z) = x^2+y^2+z^2$ — ①

and $\phi(x,y,z) = xyz - a^3 = 0$ — ②

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z$$

$$\frac{\partial \phi}{\partial x} = yz, \quad \frac{\partial \phi}{\partial y} = xz, \quad \frac{\partial \phi}{\partial z} = xy$$

Consider the Lagrangian function,

$F = f + \lambda\phi$

$F(x,y,z) = x^2+y^2+z^2+\lambda(xyz-a^3)$ — ③

Partially differentiating above eqn and equating to zero, we get

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 2x + \lambda yz = 0 \quad — ④ \Rightarrow \lambda = \frac{-2x}{yz}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 2y + \lambda xz = 0 \quad — ⑤ \Rightarrow \lambda = \frac{-2y}{xz}$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 2z + \lambda xy = 0 \quad — ⑥ \Rightarrow \lambda = \frac{-2z}{xy}.$$

From ④, ⑤, ⑥,

$$\lambda = \frac{-2x^2}{xyz} = \frac{-2y^2}{xyz} = \frac{-2z^2}{xyz}.$$

$$\lambda = \frac{-2x^2}{a^3} = \frac{-2y^2}{a^3} = \frac{-2z^2}{a^3} \quad [\because \text{From } ②]$$

$$-\frac{a^3\lambda}{2} = x^2 = y^2 = z^2 \text{ ——— } ⑦$$

$② \Rightarrow \quad xyz = a^3$

$$x^2y^2z^2 = a^6$$

$$-\frac{a^3\lambda}{2} \cdot -\frac{a^3\lambda}{2} \cdot -\frac{a^3\lambda}{2} = a^6$$

$$-\frac{\lambda^3 a^3}{8} = 1$$

$$\lambda^3 = -\frac{8}{a^3}$$

$$\lambda = -\frac{2}{a}.$$

From ⑦,

$$x^2 = -\frac{a^3}{2}\left(-\frac{2}{a}\right) = a^2$$

$$x = \pm a$$

Similarly $y = \pm a$, $z = \pm a$.

The points satisfying ② are

(a, a, a), $(-a, -a, a)$, $(-a, a, -a)$, $(a, -a, -a)$.

which are the stationary points.

Minimum value of $f = x^2 + y^2 + z^2$ at the stationary points.

$\therefore$ Minimum value of $f = 3a^2$ for all the points.

③ Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Sol: ~~let A(x,y,z)~~

Let $2x, 2y, 2z$ be the length, breadth and height of the rectangular parallelopiped.

Then volume $= lbh = (2x)(2y)(2z) = 8xyz$.

Let $f(x, y, z) = 8xyz$ ——— ①

and $\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$ ②

$$\frac{\partial f}{\partial x} = 8yz, \quad \frac{\partial f}{\partial y} = 8xz, \quad \frac{\partial f}{\partial z} = 8xy$$

$$\frac{\partial \phi}{\partial x} = \frac{2x}{a^2}, \quad \frac{\partial \phi}{\partial y} = \frac{2y}{b^2}, \quad \frac{\partial \phi}{\partial z} = \frac{2z}{c^2}.$$

~~Consider the lagrangian function,~~

Consider,

$$\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 8yz + \lambda\frac{2x}{a^2} = 0 \Rightarrow -\frac{8yzx}{2\lambda} = \frac{x^2}{a^2} \text{ ——— } ③$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 8xz + \lambda\frac{2y}{b^2} = 0 \Rightarrow \frac{-8xzy}{2\lambda} = \frac{y^2}{b^2} \text{ ——— } ④$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 8xy + \lambda\frac{2z}{c^2} = 0 \Rightarrow \frac{-8xyz}{2\lambda} = \frac{z^2}{c^2}. \text{ ——— } ⑤$$

From ③, ④ and ⑤

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}.$$

② $\Rightarrow$ $3\frac{x^2}{a^2} = 1$

$\Rightarrow x = \frac{a}{\sqrt{3}}$. Similarly $y = \frac{b}{\sqrt{3}}$, $z = \frac{c}{\sqrt{3}}$.

$(x, y, z) = \left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is the stationary point.

Maximum value of $f = f\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$

$$= 8\left(\frac{a}{\sqrt{3}} \cdot \frac{b}{\sqrt{3}} \cdot \frac{c}{\sqrt{3}}\right)$$

$\therefore$ Required volume $= \frac{8abc}{3\sqrt{3}}$.

③ Find the minima of x^2yz^3 under the condition $2x+y+3z=a$.

Sol: Let $f(x,y,z) = x^2yz^3$ —— ①

and $\phi(x,y,z) = 2x+y+3z-a = 0$ —— ②

$\frac{\partial f}{\partial x} = 2xyz^3$, $\frac{\partial f}{\partial y} = x^2z^3$, $\frac{\partial f}{\partial z} = 3x^2yz^2$.

$\frac{\partial \phi}{\partial x} = 2$, $\frac{\partial \phi}{\partial y} = 1$, $\frac{\partial \phi}{\partial z} = 3$.

Take $\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2xyz^3 + 2\lambda = 0 \Rightarrow \lambda = -xyz^3$ —— ③

$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow x^2z^3 + \lambda = 0 \Rightarrow \lambda = -x^2z^3$ —— ④

$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 3x^2yz^2 + 3\lambda = 0 \Rightarrow \lambda = -x^2yz^2$ —— ⑤

~~From ③, ④ and ⑤~~

$\cancel{\lambda = -xyz^3 = -x^2z^3 = -x^2yz^2}$.

$\frac{③}{④} \Rightarrow \frac{\cancel{\lambda}}{\cancel{\lambda}} = \frac{-xyz^3}{-x^2z^3} \Rightarrow x = y$.

$\frac{④}{⑤} \Rightarrow \frac{\cancel{\lambda}}{\cancel{\lambda}} = \frac{-x^2z^3}{-x^2yz^2} \Rightarrow y = z$.

$\therefore x = y = z$.

② $\Rightarrow 2x + x + 3x = a$

$\Rightarrow 6x = a$

$\Rightarrow x = \frac{a}{6} = y = z$.

$(x,y,z) = \left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right)$ is the stationary point.

Minimum value of $f = f\left(\frac{a}{6}, \frac{a}{6}, \frac{a}{6}\right)$

$$= \left(\frac{a}{6}\right)^2\left(\frac{a}{6}\right)\left(\frac{a}{6}\right)^3$$

Minimum value $= \frac{a^6}{6^6} = \frac{a^6}{46,656}$.

④ Find the minimum value of $x^2+y^2+z^2$ given $x+y+z=3a$.

Sol: Let $f(x,y,z) = x^2+y^2+z^2$ — ①

and $\phi(x,y,z) = x+y+z-3a=0$ — ②

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 2z$

$\frac{\partial \phi}{\partial x} = 1$, $\frac{\partial \phi}{\partial y} = 1$, $\frac{\partial \phi}{\partial z} = 1$

Take, $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$\Rightarrow 2x + \lambda = 0$

$x = -\frac{\lambda}{2}$.

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0$

$\Rightarrow y = -\frac{\lambda}{2}$

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0$

$\Rightarrow z = -\frac{\lambda}{2}$

② $\Rightarrow -\frac{\lambda}{2} - \frac{\lambda}{2} - \frac{\lambda}{2} - 3a = 0$

$-3\lambda - 6a = 0$

$\lambda = -2a$

$x = -\frac{(-2a)}{2} = a$

$y = a$

$z = a$

$(x,y,z) = (a,a,a)$

Minimum value

$= f(a,a,a)$

$= a^2+a^2+a^2 = 3a^2$.

⑤ Find the point on the plane $3x+2y+z-12=0$ which is nearest to the origin.

Sol: Let $P(x,y,z)$ be any point on the plane.

Distance of P from origin, $d = \sqrt{x^2+y^2+z^2}$.

We have to minimize d.

Let $f(x,y,z) = x^2+y^2+z^2$ — ①

and $\phi(x,y,z) = 3x+2y+z-12=0$ — ②.

$\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = 2y$, $\frac{\partial f}{\partial z} = 2z$.

$\frac{\partial \phi}{\partial x} = 3$, $\frac{\partial \phi}{\partial y} = 2$, $\frac{\partial \phi}{\partial z} = 1$.

Take $\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0$

$\Rightarrow 2x + 3\lambda = 0 \Rightarrow x = -\frac{3\lambda}{2}$

$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow y = -\lambda$

$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow z = -\frac{\lambda}{2}$

② $\Rightarrow 3\left(\frac{-3\lambda}{2}\right) + 2(-\lambda) + \left(-\frac{\lambda}{2}\right) - 12 = 0$

$\Rightarrow -\frac{9\lambda}{2} - \frac{4\lambda}{2} - \frac{\lambda}{2} = 12$.

$\Rightarrow -14\lambda = 24$

$\Rightarrow \lambda = \frac{-12}{7}$

$x = -\frac{3}{2}\left(\frac{-12}{7}\right) = \frac{18}{7}$.

$y = -\left(-\frac{12}{7}\right) = \frac{12}{7}$

$z = -\frac{1}{2}\left(-\frac{12}{7}\right) = \frac{6}{7}$.

$P(x,y,z) = \left(\frac{18}{7}, \frac{12}{7}, \frac{6}{7}\right)$ is the point nearest to the origin

⑥ A rectangular box open at the top has a capacity of 32 cubic feet. Find the dimensions of the box requiring least material for its construction.

Sol: Let x, y, z ft. be the edges (length, breadth and height) of the box.

Given, volume of the box = 32 cubic feet

$$xyz = 32$$

$$\text{let } \phi = xyz - 32 = 0 \quad \text{—— (1)}$$

Now, Surface area of the box which is open at the top,

$$S = 2xy + 2yz + zx$$

$$\text{let } f(x,y,z) = 2xy + 2yz + zx \quad \text{—— (2)}$$

$$\frac{\partial f}{\partial x} = 2y + z \qquad \frac{\partial \phi}{\partial x} = yz$$

$$\frac{\partial f}{\partial y} = 2x + 2z \qquad \frac{\partial \phi}{\partial y} = xz$$

$$\frac{\partial f}{\partial z} = 2y + x \qquad \frac{\partial \phi}{\partial z} = xy$$

Consider the lagrangian function,

$$F = f + \lambda\phi \Rightarrow F = 2xy + 2yz + zx + \lambda(xyz - 32)$$

Partially differentiating above eqn we consider,

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow (2y + z) + \lambda(yz) = 0 \Rightarrow \lambda = \frac{-(2y+z)}{yz} \quad \text{—— (3)}$$

$$\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow (2x + 2z) + \lambda(xz) = 0 \Rightarrow \lambda = \frac{-(2x+2z)}{xz} \quad \text{—— (4)}$$

$$\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow (2y + x) + \lambda(xy) = 0 \Rightarrow \lambda = \frac{-(2y+x)}{xy} \quad \text{—— (5)}$$

From (3), (4) & (5), $\lambda = \frac{-(2y+z)}{yz} = \frac{-(2x+2z)}{xz} = \frac{-(2y+x)}{xy}$

$$\Rightarrow \frac{2}{z} + \frac{1}{y} = \frac{2}{z} + \frac{2}{x} = \frac{2}{x} + \frac{1}{y}.$$

$$\frac{2}{z} + \frac{1}{y} = \frac{2}{z} + \frac{2}{x} \Rightarrow \frac{1}{y} = \frac{2}{x}. \Rightarrow x = 2y$$

$$\frac{2}{z} + \frac{2}{x} = \frac{2}{x} + \frac{1}{y} \Rightarrow \frac{2}{z} = \frac{1}{y} \Rightarrow 2y = z.$$

$$\therefore x = 2y = z.$$

(1) $\Rightarrow xyz - 32 = 0 \Rightarrow (2y)y(2y) - 32 = 0 \Rightarrow y^3 = 8 \Rightarrow y = 2$ ft.

$x = 2y = 2(2) = 4$ ft.

$z = 2y = 4$ ft.

$\therefore$ The required dimensions of the box are 4 ft, 2 ft, 4 ft.

Taylor's and Maclaurin's series expansion of functions of two variables:

Taylor's series expansion of $f(x,y)$:

The Taylor series expansion of $f(x,y)$ at (a,b) is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right]$$
$$+ \frac{1}{3!}\left[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b) + (y-b)^3 f_{yyy}\right]$$
$$+ \cdots$$

Maclaurin's series expansion of $f(x,y)$:

The Maclaurin's series expansion of $f(x,y)$ at $(0,0)$ is

$$f(x,y) = f(0,0) + \frac{1}{1!}\left[x f_x(0,0) + y f_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + \frac{1}{3!}\left[x^3 f_{xxx}(0,0) + 3x^2 y f_{xxy}(0,0) + 3xy^2 f_{xyy}(0,0) + y^3 f_{yyy}(0,0)\right]$$
$$+ \cdots$$

1) Expand $f(x,y) = e^x \cdot \cos y$ in powers of $(x-1)$ and $(y-\pi/4)$ by Taylors series.

Sol: Given $f(x,y) = e^x \cos y$

Take $(a,b) = (1, \pi/4)$

Taylor's series expansion of $f(x,y)$ at (a,b) is

$$f(x,y) = f(a,b) + \frac{1}{1!}\left[(x-a) f_x(a,b) + (y-b) f_y(a,b)\right]$$
$$+ \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + \cdots$$

At $(a,b) = (1, \pi/4)$

$$f(x,y) = f(1,\pi/4) + 1\left[(x-1) f_x(1,\pi/4) + (y-\pi/4) f_y(1,\pi/4)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 f_{xx}(1,\pi/4) + 2(x-1)(y-\pi/4) f_{xy}(1,\pi/4) + (y-\pi/4)^2 f_{yy}(1,\pi/4)\right] + \cdots \quad \text{--- (1)}$$

$f(x,y) = e^x \cos y$, $f(1,\pi/4) = e^1 \cos \pi/4 = \frac{e}{\sqrt{2}}$

$f_x = e^x \cos y$, $f_x(1, \pi/4) = \frac{e}{\sqrt{2}}$

$f_y = -e^x \sin y$, $f_y(1, \pi/4) = -\frac{e}{\sqrt{2}}$.

$f_{xx} = e^x \cos y$, $f_{xx}(1, \pi/4) = \frac{e}{\sqrt{2}}$.

$f_{xy} = -e^x \sin y$, $f_{xy}(1, \pi/4) = -\frac{e}{\sqrt{2}}$.

$f_{yy} = -e^x \cos y$, $f_{yy}(1, \pi/4) = -\frac{e}{\sqrt{2}}$.

$$① \Rightarrow f(x,y) = \frac{e}{\sqrt{2}} + \frac{1}{1!}\left[(x-1)\frac{e}{\sqrt{2}} + (y-\pi/4)\left(-\frac{e}{\sqrt{2}}\right)\right]$$
$$+ \frac{1}{2!}\left[(x-1)^2 \frac{e}{\sqrt{2}} + 2(x-1)(y-\pi/4)\left(-\frac{e}{\sqrt{2}}\right) + (y-\pi/4)^2\left(-\frac{e}{\sqrt{2}}\right)\right] + \cdots$$

② Expand $f(x,y) = e^x \cdot \sin y$ in powers of x and y, by Maclaurin's theorem.

Sol: Given, $f(x,y) = e^x \sin y$

Maclaurin's series expansion of $f(x,y)$ is

$$f(x,y) = f(0,0) + \frac{1}{1!}[x f_x(0,0) + y f_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0,0) + 2xy f_{xy}(0,0) + y^2 f_{yy}(0,0)] + \cdots \quad ①$$

$f(x,y) = e^x \sin y$, $f(0,0) = e^0 \sin 0 = 0$.

$f_x = e^x \sin y$, $f_x(0,0) = 0$

$f_y = e^x \cos y$, $f_y(0,0) = 1$

$f_{xx} = e^x \sin y$, $f_{xx}(0,0) = 0$

$f_{xy} = e^x \cos y$, $f_{xy}(0,0) = 1$.

$f_{yy} = -e^x \sin y$, $f_{yy}(0,0) = 0$.

$$① \Rightarrow f(x,y) = 0 + \frac{1}{1!}[x(0) + y(1)] + \frac{1}{2!}[x^2(0) + 2xy(1) + y^2(0)] + \cdots$$
$$= y + xy + \cdots$$

$f_{xxy} = f_{yxx} = f_{xyx}$

$f_{yyx} = f_{xyy} = f_{yxy}$

③ Expand e^{xy} in the neighbourhood of $(1,1)$

$$f(x,y) = e + \frac{1}{1!}[(x-1)e + (y-1)e] + \frac{1}{2!}[(x-1)^2 e + 2(x-1)(y-1)2e + (y-1)^2 e] + \cdots$$

④ Expand $x^2y + 3y - 2$ in the powers of $(x-1)$ & $(y+2)$ upto the terms of third degree.

~~$f(x,y)$~~ z

⑤ Expand $x \sin y$ in powers of x and y.