UNIT-①

ENERGY METHODS

Introduction:-

When an external load act on a structure, the structure undergoes deformation & hence, the work is done. To resist these external forces, the internal forces develop gradually from zero to their final value & work is done This internal workdone is stored as energy in the structure & it helps the structure to spring back to the original shape & size, whenever the external loads are removed, provided the material of the structure is still within elastic limit.

When equilibrium is reached, as per the well known law of Conservation of energy, the work done by the external forces must equal the strain energy stored. This Concept of energy balance is utilized in structural analysis to develop a number of methods to find deflection of structures. The following methods are finding the deflection of beams & frames.

① Strain Energy / Real work method

② Virtual work / Unit load Method

③ Castigliano's Method.

Strain Energy:-

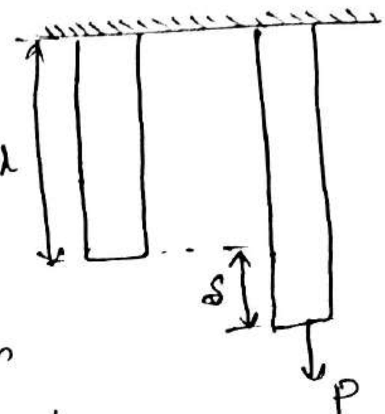
When an elastic body is subjected to external forces it will deform, if the elastic limit is not ~~reachable~~ exceeded, the work done in straining the material is stored in the form of resilience of internal energy. This is known as "strain Energy".

(or)

Internal energy stored in a body within elastic limit of elastic body is known as strain Energy.

Strain Energy due to axial loading:-

Let l = length of member,
A = Area of c/s of member
δ = Extension of member
P = External load



Since the load is applied gradually from zero to 'P'. the member is also gradually extended.

$\therefore$ External workdone by force (W_e) = Average force $\times$ distance

$$= \left(\frac{0+P}{2}\right) \times \delta$$

$$W_e = \frac{1}{2} \times P \times \delta \longrightarrow ①$$

Let Internal workdone (or) Strain Energy = $W_i = U \longrightarrow ②$

From law of Conservation of Energy

Internal workdone = External work done

i.e. eq① = eq②

$$U = \frac{1}{2} \times P \times \delta \longrightarrow ③$$

But we know $\delta = \frac{Pl}{AE} \longrightarrow ④$ $\left(\because \sigma \propto e, \quad \frac{P}{A} = E \cdot \frac{\delta}{l}, \quad \delta = \frac{Pl}{AE} \right)$

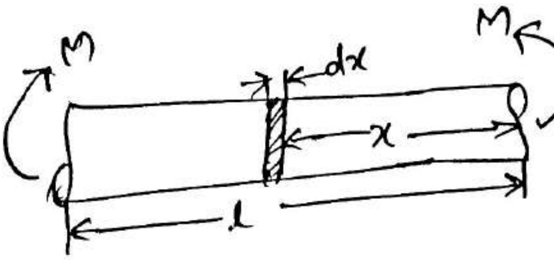
From eq③ & eq④ $U = \frac{1}{2} \times P \times \frac{Pl}{AE}$

$$\boxed{U = \frac{P^2 l}{2AE}}$$

If, however, the bar has variable area of c/s, Consider a small section of length dx & area of c/s A. The strain Energy in small element of length dx, is $dU = \frac{P^2 \cdot dx}{2AE}$

Total strain Energy $\boxed{U = \int_0^l \frac{P^2 \cdot dx}{2AE}}$

Strain Energy due to Bending Moment:-



Consider a member of length(l) Subjected to uniform bending moment(M). In that Consider a small element of length "dx". Let '$d\theta$' is the change in slope.

So Strain Energy stored in the element

$$du = \frac{1}{2} \times M \times d\theta \longrightarrow (1)$$

But we know $\frac{M}{EI} = \frac{d^2y}{dx^2}$

$$\frac{M}{EI} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\frac{M}{EI} = \frac{d}{dx}(\theta) = \frac{d\theta}{dx} \qquad \left(\because \theta = \frac{dy}{dx}\right)$$

$$\Rightarrow d\theta = \frac{M}{EI} \times dx \longrightarrow (2)$$

From eq (1) & eq (2)

$$du = \frac{1}{2} \times M \times \frac{M}{EI} dx$$

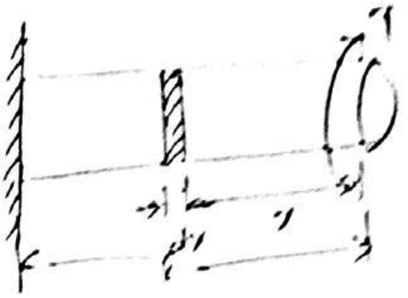
$$\boxed{du = \frac{M^2 dx}{2EI}}$$

The total Strain Energy

$$\boxed{\begin{gathered} U = \int_0^l \frac{M^2 dx}{2EI} \\ \text{(or)} \\ U = \frac{1}{2EI} \cdot \int_0^l M^2 dx \end{gathered}}$$

Strain Energy due to Torsion:-

Consider a shaft of length (l) subjected to twisting moment (T). When torsion is subjected to shaft it will produce twist. Let 'θ' be the angle of twist.



$\therefore$ Work done by External force (W_e) $= \frac{1}{2} \times T \times \theta$ ⟶ ①

Internal work done (or) Strain Energy (W_i) $= U$ ⟶ ②

From law of Conservation of energy

eq ① = eq ②

$$U = \frac{1}{2} \times T \times \theta \longrightarrow ③$$

But we know from torsion Equation

$$\frac{T}{J} = \frac{G\theta}{l} = \frac{\tau}{r}$$

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$\theta = \frac{Tl}{GJ} \longrightarrow ④$$

From eq ③ & eq ④

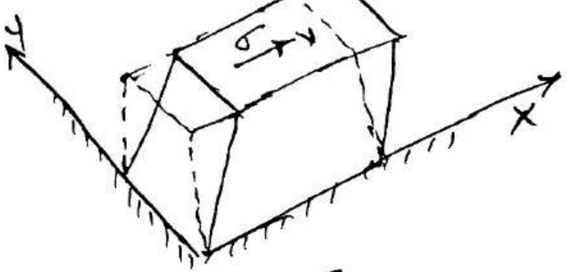
$$U = \frac{1}{2} \times T \times \frac{Tl}{GJ}$$

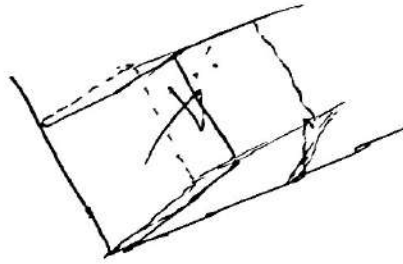
$$\boxed{U = \frac{1}{2} \times \frac{T^2 l}{GJ}}$$

(or)

$$\boxed{U = \int_0^l \frac{T^2\,dx}{2GJ}}$$

Strain Energy due to transverse Shear:-





The shear stress on a c/s of beam of rectangular c/s may be found out by the relation

$$\tau = \frac{VQ}{b I_{zz}}$$

where Q = first moment of portion of c/s above the point where shear stress is reqd about NA

V = Transverse shear force

b = width of section.

I_{zz} = MOI of the section about NA.

due to shear stress, the angle b/w the lines of right angle will change. The shear stress varies across the height in a parabolic manner in the case of rectangular c/s. Also, the shear stress distribution is different for different shape of c/s. However, to simplify the computation of shear stress is assumed to be uniform across the c/s. Consider segment of length "dx" subjected to shear stress τ. The shear stress across the c/s may be taken as

$$\tau = k \times \frac{V}{A} \rightarrow (1)$$ where k = factor, depend on shape of c/s

deformation $d\delta = \Delta\gamma \cdot dx \rightarrow (2)$

But we know $G = \frac{\tau}{\Delta\gamma} = \frac{\text{shear stress}}{\text{shear strain}}$

$$\Rightarrow \Delta\gamma = \frac{\tau}{G} \rightarrow (3)$$

from eq ① & ⑧

$$\Delta\gamma = K \times \frac{V}{AG}$$

From eq ②, deformation $d\delta = K \cdot \frac{V}{AG} \times dx$

Total deformation $\delta = \int_0^l K \cdot \frac{V}{AG} \cdot dx$.

Strain Energy $U = \frac{1}{2} \times V \times \delta$

$$U = \frac{1}{2} \times V \times \int_0^l K \cdot \frac{V^2}{AG} \cdot dx$$

$$\boxed{U = \int_0^l \frac{KV^2}{2AG} \cdot dx}$$

Strain Energy :-

① Due to axial loading $= \int_0^l \frac{P^2}{2AE} \cdot dx$

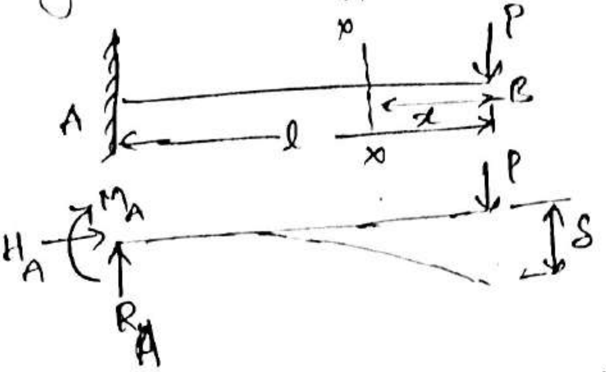
② Due to Bending $= \int_0^l \frac{M^2}{2EI} \cdot dx$

③ Due to twisting $= \int_0^l \frac{T^2}{2GJ} \cdot dx$

④ Due to shear $= \int_0^l \frac{V^2}{2AG} \cdot dx$

Prob① find the deflection at free end of Cantilever carrying a point load at free end using Strain Energy principle.

Sol:-



Now BM at section x-x from free end $M = Px$

$$\text{strain Energy } U = \int_0^l \frac{M^2 dx}{2EI}$$

$$= \int_0^l \frac{(Px)^2 dx}{2EI} = \int_0^l \frac{P^2 x^2 dx}{2EI}$$

$$= \frac{P^2}{2EI} \cdot \int_0^l x^2 \cdot dx$$

$$= \frac{P^2}{2EI}\left(\frac{x^3}{3}\right)_0^l = \frac{P^2}{2EI}\left(\frac{l^3}{3} - 0\right)$$

$$U = \frac{P^2 l^3}{6EI} \longrightarrow ①$$

$$\text{workdone by External load} = \frac{1}{2} \times P \times \delta \longrightarrow ②$$

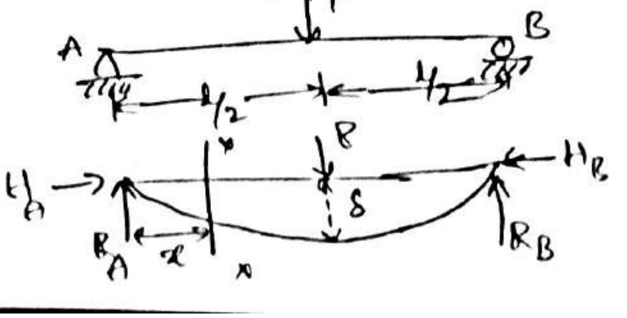
eq① = eq②

$$\frac{P^{\cancel{2}} l^3}{\cancel{6}EI} = \frac{1}{\cancel{2}} \times P \times \delta$$

$$\boxed{\delta = \frac{Pl^3}{3EI}}$$

Prob②:- A Beam of span 'l' carries a Concentrated load 'P' at midspan find Central deflection using strain energy principle.

Sol:-



Reaction at each end $R_A = R_B = \frac{P}{2}$

Bending Moment at section X–X, $M = R_A \times x = \frac{P}{2} \times x$

Strain Energy stored by half of the beam $= \int_0^{l/2} \frac{M^2}{2EI} dx$

Total Strain Energy $U = 2\int_0^{l/2} \frac{M^2}{2EI} . dx$

$$= 2 \int_0^{l/2} \left(\frac{P}{2}x\right)^2 \times \frac{1}{2EI} dx$$

$$= \frac{\cancel{2}}{\cancel{2}EI} \int_0^{l/2} \frac{P^2}{4} \times x^2 . dx$$

$$= \frac{P^2}{4EI} \int_0^{l/2} x^2 . dx = \frac{P^2}{4EI} \left(\frac{x^3}{3}\right)_0^{l/2}$$

$$= \frac{P^2}{4EI} \left[\frac{(l/2)^3}{3} - 0\right]$$

$$= \frac{P^2}{4EI} \left[\frac{l^3}{8 \times 3}\right]$$

$$U = \frac{P^2 l^3}{96EI} \longrightarrow (1)$$

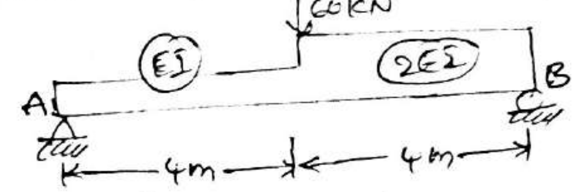
work done by External force $= \frac{1}{2} \times P \times \delta \longrightarrow (2)$

eq (1) = eq (2)

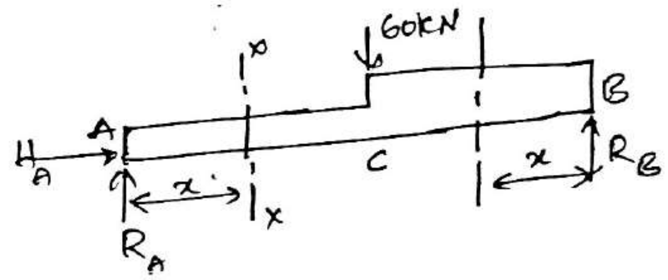
$$\frac{P^{\cancel{2}} l^3}{\cancel{96}_{48} EI} = \frac{1}{\cancel{2}} \times \cancel{P} \times \delta$$

$$\boxed{\delta = \frac{Pl^3}{48EI}}$$

Problem③ :- Determine deflection under 60KN load by using Strain Energy Method.



Sol:-



Reactions at each support $R_A = R_B = \frac{60}{2} = 30KN$

BM at Section x–x from Support A = $R_A x = 30x$

BM at Section x–x from Support B = $R_B x = 30x$

Strain Energy Stored in portion AC = $\int_0^4 \frac{M^2}{2EI} dx = \int_0^4 \frac{(30x)^2}{2EI} dx$

Strain Energy Stored in portion BC = $\int_0^4 \frac{M^2}{2(2EI)} dx = \int_0^4 \frac{(30x)^2}{4EI} dx$

Total Strain Energy stored in member

$$U = \int_0^4 \frac{(30x)^2}{2EI} dx + \int_0^4 \frac{(30x)^2}{4EI} dx$$

$$= \frac{\overset{450}{\cancel{900}}}{\cancel{2}EI} \int_0^4 x^2 dx + \frac{900}{4EI} \int_0^4 x^2 dx$$

$$= \frac{450}{EI} \left(\frac{x^3}{3}\right)_0^4 + \frac{225}{EI} \left(\frac{x^3}{3}\right)_0^4$$

$$= \frac{450}{EI}\left(\frac{64}{3}\right) + \frac{225}{EI}\left(\frac{64}{3}\right)$$

$$U = \frac{14400}{EI} \longrightarrow ①$$

But External work done $W_e = \frac{1}{2} \times P \times \delta = \frac{1}{2} \times \overset{30}{\cancel{60}} \times \delta = 30\delta \longrightarrow ②$

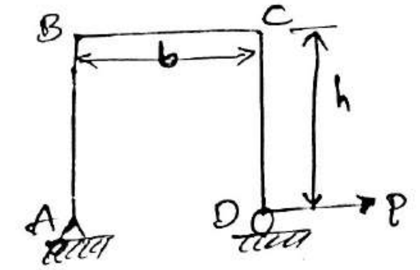
eq① = eq③

$$\frac{14400}{EI} = 30\delta$$

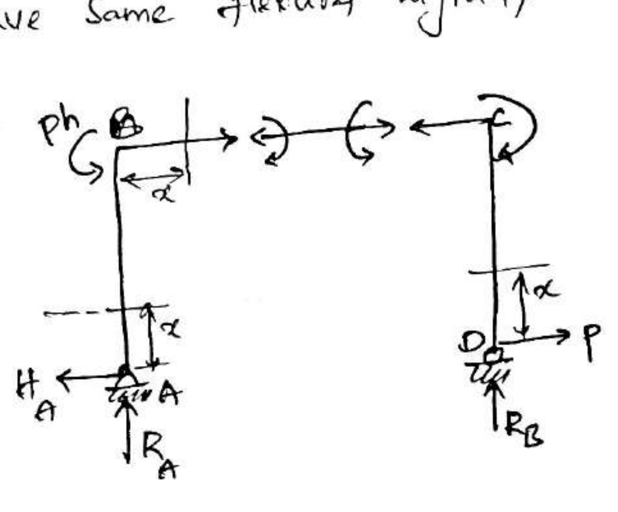
$$\delta = \frac{14400}{30EI}$$

$$\boxed{\delta = \frac{480}{EI}}$$

Prob(4) :- A portal frame ABCD has it's end 'A' is hinged while end 'D' is ~~pinned~~ placed on roller a horizontal force 'P' is applied on the end 'D' as shown in fig. Determine horizontal movement of 'D'. Assume the members have same flexural rigidity.



Sol:-



$R_A = R_B = 0,$

$H_A = P$

From above fig. the BM expression for various portion are

portion	AB	BC	CD
origin	A	B	D
Limit	0–h	0–b	0–h
M_x	$H_A x = Px$	Ph	Px

Total strain Energy stored in frame $U = \int_0^h \frac{M^2}{2EI}dx + \int_0^b \frac{M^2}{2EI}dx + \int_0^h \frac{M^2}{2EI}dx$

$$U = \int_0^h \frac{(Px)^2}{2EI}dx + \int_0^b \frac{(Ph)^2}{2EI}dx + \int_0^h \frac{(Px)^2}{2EI}dx.$$

$$= 2\int_0^h \frac{P^2x^2}{2EI}.dx + \int_0^b \frac{P^2h^2}{2EI}dx$$

$$= \frac{\cancel{2}P^2}{\cancel{2}EI}\int_0^h x^2 dx + \frac{P^2h^2}{2EI}\int_0^b 1 dx.$$

$$= \frac{P^2}{EI}\left(\frac{x^3}{3}\right)_0^h + \frac{P^2h^2}{2EI}(x)_0^b$$

$$= \frac{P^2}{3EI}(h^3) + \frac{P^2h^2}{2EI}(b)$$

$$= \frac{P^2h^2}{EI}\left(\frac{h}{3} + \frac{b}{2}\right) = \frac{P^2h^2}{EI}\left(\frac{2h+3b}{6}\right)$$

$$U = \frac{P^2h^2}{6EI}(2h+3b) \longrightarrow (1)$$

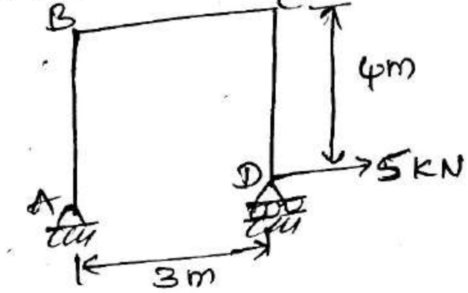
External workdone $(W_e) = \frac{1}{2} \times P \times \delta \longrightarrow ②$

$eq ① = eq ②$

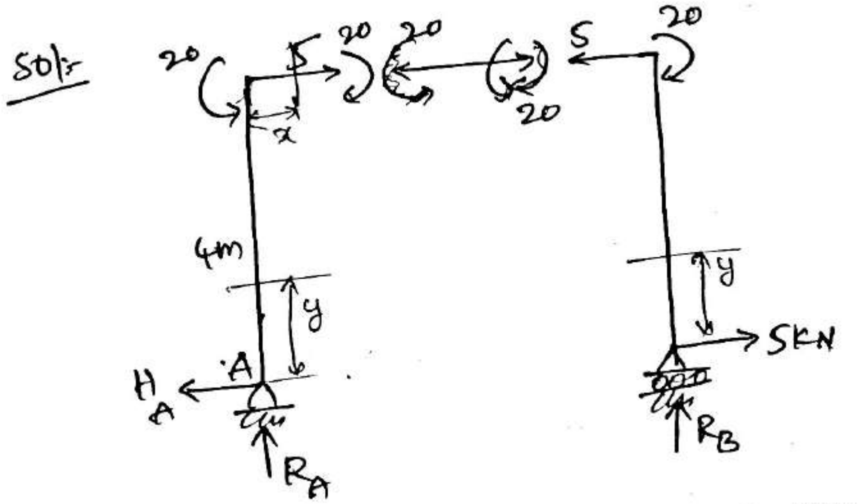
$$\frac{1}{\cancel{2}} \times \cancel{P} \times \delta = \frac{\cancel{P}h^2}{\cancel{6}EI}(2h+3b)$$

$$\boxed{\delta = \frac{Ph^2}{3EI}(2h+3b)}$$

Prob: Determine the horizontal displacement of the roller end 'D' of the portal frame shown in fig. $EI = 8000\ kN\text{-}m^2$ throughout.



Sol:



$R_A = R_B = 0.$

$H_A = 5KN.$

From the above the BM expression for various portions are

portion	AB	BC	CD
origin	A	B	D
Limit	0–4	0–3	0–4
M_x	$5x$	20	$5x$

Total strain Energy stored in frame $U = \int_0^4 \frac{M^2}{2EI}dx + \int_0^3 \frac{M^2}{2EI}dx + \int_0^4 \frac{M^2}{2EI}dx$

$$U = \int_0^4 \frac{(5x)^2}{2EI}dx + \int_0^3 \frac{(20)^2}{2EI}dx + \int_0^4 \frac{(5x)^2}{2EI}dx$$

$$= 2\int_0^4 \frac{(5x)^2 dx}{2EI} + \int_0^3 \frac{400}{2EI}dx$$

$$= \cancel{2} \times \frac{25}{\cancel{2}EI}\int_0^4 x^2 dx + \frac{\overset{200}{\cancel{400}}}{\cancel{2}EI}\int_0^3 1\,dx$$

$$= \frac{25}{EI}\left(\frac{x^3}{3}\right)_0^4 + \frac{200}{EI}(x)_0^3$$

$$U = \frac{25}{EI}\left(\frac{64}{3}\right) + \frac{200}{EI}(3)$$

$$= \frac{25}{EI}\left(\frac{64}{3} + 24\right) = \frac{25}{EI}\left(\frac{64+72}{3}\right)$$

$$= \frac{25}{EI}\left(\frac{136}{3}\right)$$

$$U = \frac{1133.33}{EI} \rightarrow ①$$

External workdone $W_e = \frac{1}{2} \times P \times \delta \longrightarrow ②$

eq① = eq②

$$\frac{1}{2} \times P \times \delta = \frac{1133.33}{EI}$$

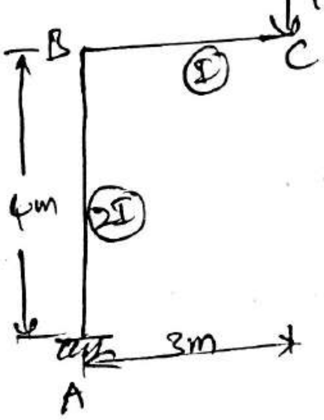
$$\frac{1}{\cancel{2}} \times \overset{2.5}{\cancel{5}} \times \delta = \frac{1133.33}{EI}$$

$$\delta = \frac{1133.33}{2.5EI} = \frac{1133.33}{2.5 \times 8000}$$

$$\delta = 0.0567 m$$

$$\boxed{\delta = 56.67 mm}$$

prob 6 Determine the vertical deflection at point 'C' in the frame shown in fig. Take ~~E = 200 KN/KN-mm²~~ $E = 200\ KN/mm^2$, $I = 30 \times 10^6\ mm^4$.



Strain Energy method can be conveniently used for finding deflection in structures only under the following conditions.

(1) The structure is subjected to single concentrated load.

(2) Deflection reqd is at the ~~end~~ loaded point end only & is in direction of the load.

Castigliano's first theorem:-

Statement:- In a linear elastic structure, partial derivative of the strain energy with respect to a load is equal to the deflection of the point where the load is acting, the deflection being measured in the direction of load.

The load may be force (or) moment. Mathematically this theorem may be represented by

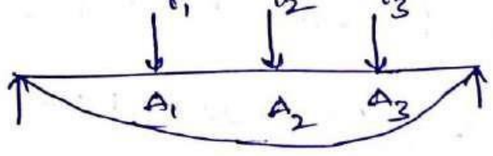
$$\boxed{\frac{\partial U}{\partial P_i} = \Delta_i \; ; \; \frac{\partial U}{\partial M_j} = \theta_j}$$

where U = Total Strain Energy

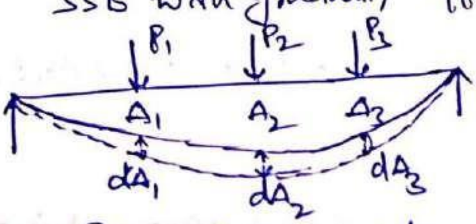
P_i, M_j = load.

Δ_i, θ_j = deflections.

Proof:- Consider a SSB shown in fig on which loads P_1, P_2 & P_3 are applied gradually. Let deflections under the loads P_1, P_2, P_3 be Δ_1, Δ_2 & Δ_3 respectively.



SSB with gradually applied loads.



Beam subjected to additional load.

Total Strain Energy $U = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 \longrightarrow (1)$

Let, the additional load dP_1 be added after the loads P_1, P_2 & P_3

Let the additional deflection be $d\Delta_1, d\Delta_2$ & $d\Delta_3$

Additional Strain Energy $dU = \frac{1}{2}dP_1 d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \longrightarrow (2)$

$$dU = P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \longrightarrow (2)$$

Total Strain Energy

$$U + dU = \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}dP_1 d\Delta_1 + P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 \longrightarrow (3)$$

If $(P_1 + dP_1)$, P_2 & P_3 were applied simulatiniously strain Energy stored.

$$= \frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1) + \frac{1}{2}P_2(\Delta_2 + d\Delta_2) + \frac{1}{2}P_3(\Delta_3 + d\Delta_3) \longrightarrow (4)$$

eq (3) = eq (4)

$$U + dU = \left[\frac{1}{2}(P_1 + dP_1)(\Delta_1 + d\Delta_1)\right] + \frac{1}{2}\left[P_2\Delta_2 + d\Delta_2 P_2\right] + \frac{1}{2}\left[P_3\Delta_3 + P_3 d\Delta_3\right]$$

$$= \frac{1}{2}\left[P_1\Delta_1 + P_1 d\Delta_1 + dP_1\Delta_1 + dP_1 d\Delta_1\right] + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}P_3 d\Delta_3$$

$$= \frac{1}{2}P_1\Delta_1 + \frac{1}{2}P_1 d\Delta_1 + \frac{1}{2}dP_1\Delta_1 + \frac{1}{2}dP_1 d\Delta_1 + \frac{1}{2}P_2\Delta_2 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3\Delta_3 + \frac{1}{2}P_3 d\Delta_3$$

$$\cancel{U} + dU = \cancel{U} + \frac{1}{2}P_1 d\Delta_1 + \frac{1}{2}dP_1\Delta_1 + \frac{1}{2}P_2 d\Delta_2 + \frac{1}{2}P_3 d\Delta_3$$

$$dU = \frac{1}{2}\left[P_1 d\Delta_1 + P_2 d\Delta_2 + P_3 d\Delta_3 + dP_1\Delta_1\right]$$

$$2dU = \left[dU + dP_1\Delta_1\right]$$

$$dU = dP_1\Delta_1$$

$$\boxed{\frac{dU}{dP_1} = \Delta_1}$$ simillarly

$$\frac{dU}{dP_2} = \Delta_2 \; ----$$

Example 3.11 A simply supported beam of span L, carries a concentrated load P at a distance a from left hand side support as shown in Figure 3.22. Using castigliano's theorem determine the deflection under the load. Assume uniform flexural rigidity.

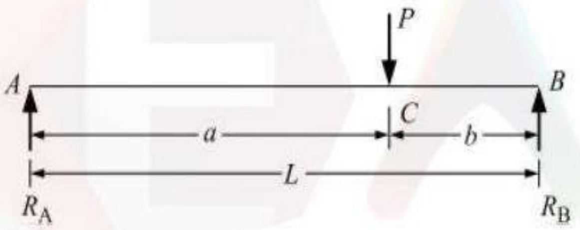


Figure 3.22 Example 3.11

Solution Reaction at A,

$$R_A = \frac{Pb}{L}$$

and Reaction at B,

$$R_B = \frac{Pa}{L}$$

Table 3.9 Calculation table for Example 3.11

Portion	*AC*	*CB*
Origin	A	B
Limit	$0–a$	$0–b$
M	$\frac{Pb}{L}x$	$\frac{Pa}{L}x$
Flexural Rigidity	EI	EI

Therefore, Shear Energy of the beam

$$U = \int_0^a \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx + \int_0^b \left(\frac{Pb}{L}x\right)^2 \times \frac{1}{2EI}dx$$

$$= \left[\frac{P^2b^2}{L^2} \times \frac{1}{6EI}x^3\right]_0^a + \left[\frac{P^2a^2}{L^2} \times \frac{1}{6EI}x^3\right]_0^b$$

$$= \frac{P^2b^2a^3}{6EIL^2} + \frac{P^2a^2b^3}{6EIL^2}$$

$$= \frac{P^2a^2b^2}{6EIL^2}(a+b)$$

$$= \frac{P^2a^2b^2}{6EIL}, \text{ Since, } a + b = L$$

$$\Delta_C = \frac{\delta U}{\delta P} = \frac{Pa^2b^2}{3EIL}$$

Example 3.12 Determine the vertical deflection at the free end and rotation at A in the overhanging beam shown in Figure 3.23(a). Assume constant EI. Use Castigliano's method.

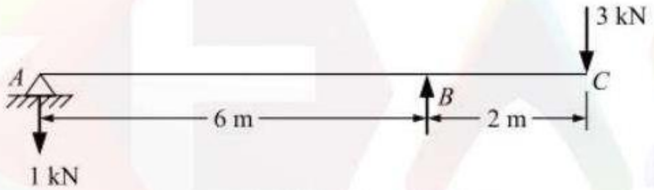


Figure 3.23(a) Example 3.12

Solution (1) Deflection at C: Taking 3 kN force as p,

$$R_B \times 6 = P \times 8$$

$$R_B = \frac{4}{3}P\uparrow$$

$$\therefore \quad R_A = \frac{P}{3}\downarrow$$

Figure 3.23(b) Reaction if 3 kN load is taken as ?

Bending moment expressions are noted, in the tabular form.

Table 3.10 Calculation table for Example 3.12

Portion	*AB*	*BC*
Origin	A	C
Limit	0–6	0–2
M	$\frac{-P}{3}x$	$-Px$
Flexural Rigidity	EI	EI

$$U = \int \frac{M^2}{2EI}dx$$

$$= \int_0^6 \frac{P^2x^2}{9} \times \frac{1}{2EI}dx + \int_0^2 \frac{P^2x^2}{2EI}dx$$

$$= \frac{P^2}{18EI}\left[\frac{x^3}{3}\right]_0^6 + \left[\frac{P^2x^3}{6EI}\right]_0^2$$

$$= \frac{4P^2}{EI} + \frac{4}{3} \times \frac{P^2}{EI}$$

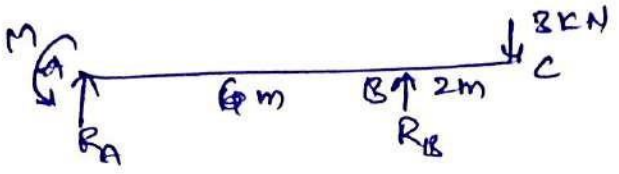
$$= \frac{5.333P^2}{EI}$$

$$\Delta_C = \frac{dU}{dP} = \frac{10.667P}{EI}$$

Substituting $P = 3$ kN, we get

$$\Delta_C = \frac{32}{EI}$$

For rotation at 'A', apply a dummy moment at 'A'.



$$\Sigma M_B = 0 \Rightarrow R_A \times 6 - M + 6 = 0$$

$$R_A = \frac{M-6}{6}$$

Portion	AB	BC
origin	A	C
limit	0-6	0-2
M	$\left(\frac{M-6}{6}\right)x - M$	$3x$

Total Strain Energy $U = \int_0^6 \left[\left(\frac{M-6}{6}\right)x - M\right]^2 \frac{1}{2EI}\,dx + \int_0^{6} \frac{(3x)^2}{2EI}\,dx$

$$\theta_A = \frac{\partial U}{\partial M} = \int_0^6 2\left[\left(\frac{M-6}{6}\right)x - M\right]\left(\frac{x}{6}-1\right)\frac{1}{2EI}\,dx + 0$$

Put $M=0$ ($\because$ dummy)

$$\cancel{\theta = \int_0^6 2(-x)\left(\frac{x}{6}-1\right)\frac{1}{EI}\,dx + 0}$$

$$\theta = \int_0^6 (-x)\left(\frac{x}{6}-1\right)\frac{1}{EI}\,dx$$

$$= \frac{1}{EI}\int_0^6 \left(\frac{-x^2}{6}+x\right)dx$$

$$= \frac{1}{EI}\left[\frac{1}{6}\left(\frac{-x^3}{3}\right)_0^6 + \left(\frac{x^2}{2}\right)_0^6\right]$$

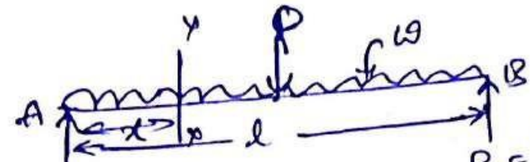
$$= \frac{1}{EI}\left[\frac{1}{6\times 3}(-6)^3 + \frac{6^2}{2}\right]$$

$$= \frac{1}{EI}\left[\frac{-36}{3} + \frac{36}{2}\right]$$

$$\boxed{\theta_A = \frac{6}{EI}}$$

Calculate the central deflection & slope at ends of SSB carrying UDL w per unit length over the whole span. (4)

Sol:- Central deflection



$R_A = \frac{wl+P}{2}$ $\quad R_B = \frac{wl+P}{2}$

due to symmetry $R_A = R_B = \frac{wl+P}{2}$

~~Bending moment at section x-x~~ $\cancel{= \int \frac{M^2}{2EI} dx}$

$\cancel{U = 2\int_0^{l/2} \left(\frac{wx+P}{2}\right)^2}$

Bending moment at section x-x $= \left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}$

Total strain energy stored $= \int \frac{M^2}{2EI} dx$

$$U = \cancel{2}\int_0^{l/2} \left[\left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}\right]^2 \times \frac{1}{\cancel{2}EI} dx$$

$$U = \frac{1}{EI} \int_0^{l/2} \left[\left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}\right]^2 dx$$

central deflection

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^{l/2} \cancel{2}\left[\left(\frac{wl+P}{2}\right)x - \frac{wx^2}{2}\right]\cdot\frac{x}{\cancel{2}} dx$$

$$\Delta_C = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_0^{l/2} \left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\cdot x \cdot dx \quad (\because P=0)$$

$$\Delta_C = \frac{1}{EI}\int_0^{l/2} \left(\frac{wlx^2}{2} - \frac{wx^3}{2}\right) dx$$

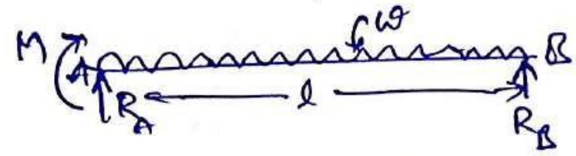
$$= \frac{1}{EI}\left[\frac{wl}{2}\left(\frac{x^3}{3}\right)_0^{l/2} - \frac{w}{2}\left(\frac{x^4}{4}\right)_0^{l/2}\right]$$

$$= \frac{1}{EI}\left[\frac{wl}{6}\left(\frac{l^3}{8}\right) - \frac{w}{8}\left(\frac{l^4}{16}\right)\right]$$

$$= \frac{1}{EI}\left[\frac{wl^4}{48} - \frac{wl^4}{128}\right]$$

$$\boxed{\Delta_C = \frac{5wl^4}{384EI}}$$

To find rotation at 'A', apply dummy moment at 'A'.



$\Sigma V = 0 \Rightarrow R_A + R_B = wl \rightarrow (1)$

$\Sigma M_A = 0 \Rightarrow -R_B l + wl \cdot \frac{l}{2} + M = 0$

$R_B l = \frac{wl^2}{2} + M$

$R_B = \frac{wl^2}{2l} + \frac{M}{l}$

$R_B = \frac{wl}{2} + \frac{M}{l}$

From eq (1)

$R_A = wl - R_B$

$= wl - \frac{wl}{2} - \frac{M}{l}$

$R_A = \frac{wl}{2} - \frac{M}{l}$

~~Total Strain Energy stored~~ $\cancel{U = \int_0^l \left(\frac{wl}{2} + \frac{M}{l}\right)^2 \frac{1}{2EI} dx}$

$\cancel{\theta_A = \frac{\partial U}{\partial M} = \int_0^l 2\left(\frac{wl}{2} + \frac{M}{l}\right)x}$

Bending moment section x-x $= +R_B x = -\left(\frac{wl}{2} + \frac{M}{l}\right)x + \frac{wx^2}{2}$
from support 'B'.

Total Strain Energy stored $\cancel{U = \int_0^l \left[\left(-\frac{wl}{2} + \frac{M}{l}\right)x\right]^2 \frac{1}{2EI} dx}$.

$\theta_A = \frac{\partial U}{\partial M} = \frac{1}{EI}\int_0^l M_x \frac{\partial M_x}{\partial M} dx \rightarrow (2)$

where $M_x = -\left(\frac{wl}{2} + \frac{M}{l}\right)x + \frac{wx^2}{2}$

$\frac{\partial U}{\partial M} = \frac{-x}{l}$

from eq (2) $\Rightarrow \theta_A = \frac{1}{EI}\int_0^l \left[-\left(\frac{wl}{2} + \frac{M}{l}\right)x + \frac{wx^2}{2}\right]\left(\frac{-x}{l}\right) dx$

$\theta_A = \frac{1}{EI}\int_0^l \left[\left(\frac{wl}{2} + \frac{M}{l}\right)x - \frac{wx^2}{2}\right]\left(\frac{x}{l}\right) dx$

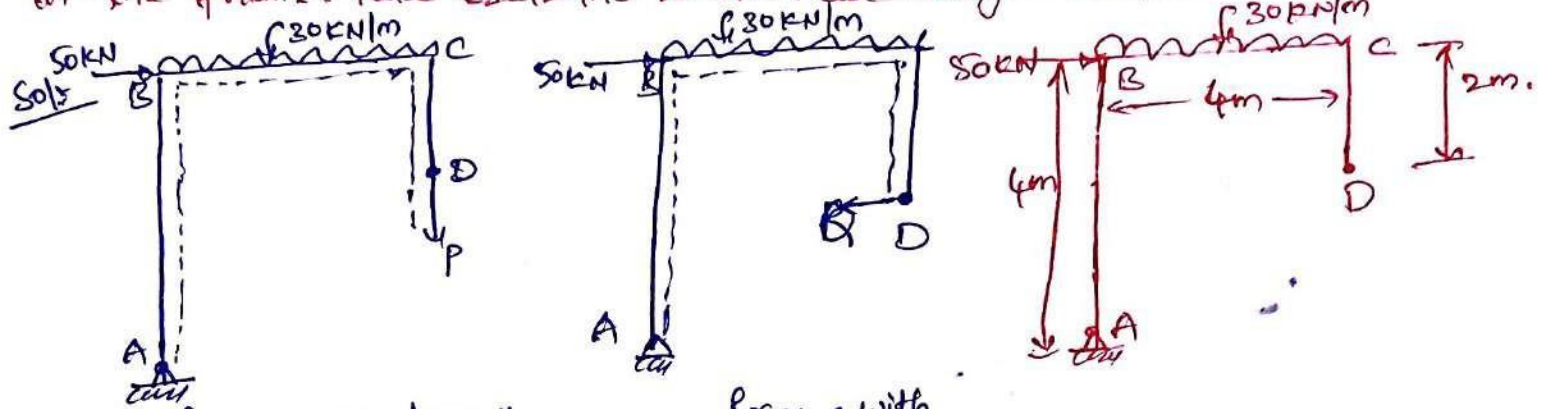
But $M = 0 \Rightarrow \theta = \frac{1}{EI}\int_0^l \left(\frac{wlx}{2} - \frac{wx^2}{2}\right)\left(\frac{x}{l}\right) dx$

$\theta_A = \frac{1}{EI}\int_0^l \left(\frac{w\cancel{l}x^2}{2\cancel{l}} - \frac{wx^3}{2l}\right) dx = \frac{1}{EI}\left(\frac{w x^3}{2 \cdot 3} - \frac{wx^4}{2l(4)}\right)_0^l$

$\theta_A = \frac{1}{EI}\left(\frac{wl^3}{6} - \frac{wl^4}{8l}\right)_0^l = \frac{wl^3}{24EI}$ $\boxed{\therefore \theta_A = \frac{wl^3}{24EI}}$

Prob: Determine the vertical & horizontal displacement of the free end 'D' in the frame. Take $EI = 12\times10^{13}$ N-mm². Use Castigliano's first theorem. (10)

Sol:



Vertical Deflection:-

Portion	Origin	Limit	Moment
AB	B	0-4	$-(4P+30\times4\times2+50x)$
BC	C	0-4	$-(Px+30x\cdot\frac{x}{2}) = -(Px+15x^2)$
CD	D	0-2	0

∵ 'P' is dummy load (P=0)

$$\text{Strain Energy } U = \int_0^4 \frac{(4P+240+50x)^2}{2EI}dx + \int_0^4 \frac{(Px+15x^2)^2}{2EI}dx + 0$$

$$\Delta = \frac{\partial U}{\partial P} = \int_0^4 \frac{\cancel{2}(4P+240+50x)4}{\cancel{2}EI}dx + \int_0^4 \frac{\cancel{2}(Px+15x^2)x}{\cancel{2}EI}dx$$

'P' is dummy so, P=0.

$$\Delta = \int_0^4 \frac{4(240+50x)}{EI}dx + \int_0^4 \frac{15x^3}{EI}dx$$

$$= \frac{4}{EI}\left(240x + \frac{50x^2}{2}\right)_0^4 + \frac{15}{EI}\left(\frac{x^4}{4}\right)_0^4$$

$$= \frac{4}{EI}\left[240\times4 + \frac{50\times4^2}{2}\right] + \frac{15}{EI}\left(\frac{4^4}{4}\right)$$

$$= \frac{5440}{EI} + \frac{960}{EI} = \frac{6400}{EI} = \frac{6400}{12\times10^4}$$

$EI = 12\times10^{13}$ N-mm² $= \frac{12\times10^{13}}{10^3\times10^6} = 12\times10^4$ KN-m² $\Delta = 0.053$m

$\boxed{\Delta = 53.33 \text{ mm}}$

Horizontal deflection

Portion	origin	limit	Moment
AB	B	0-4	$-[Q(2-x)+240+50x]$
BC	C	0-4	$-[2Q+15x^2]$
CD	D	0-2	Qx

$$\text{Strain Energy } U = \int_0^4 \frac{[Q(2-x)+240+50x]^2}{2EI}\,dx + \int_0^4 \frac{(2Q+15x^2)^2}{2EI}\,dx + \int_0^2 \frac{(Qx)^2}{2EI}\,dx$$

$$\Delta_{DH} = \frac{\partial U}{\partial Q} = \int_0^4 \frac{\cancel{2}[Q(2-x)+240+50x]}{\cancel{2}EI}(2-x)\,dx + \int_0^4 \frac{\cancel{2}(2Q+15x^2)}{\cancel{2}EI}\,2dx + \int_0^2 \frac{\cancel{2}Qx}{\cancel{2}EI}\,x\,dx$$

substitute $Q = 0$.

$$\Delta_{DH} = \int_0^4 \frac{(240+50x)(2-x)}{EI} + \int_0^4 \frac{30x^2}{EI}\,dx + 0$$

$$= \int_0^4 \frac{(480-240x+100x-50x^2)}{EI} + \int_0^4 \frac{30x^2}{EI}\,dx$$

$$= \int_0^4 \frac{(480-140x-50x^2)}{EI} + \int_0^4 \frac{30x^2}{EI}\,dx$$

$$= \frac{1}{EI}\left[480x - 140\frac{x^2}{2} - 50\frac{x^3}{3}\right]_0^4 + \frac{1}{EI}\left[\frac{\overset{10}{\cancel{30}}\,x^3}{\cancel{3}}\right]_0^4$$

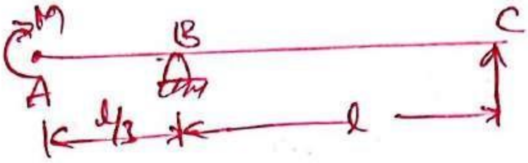
$$= \frac{1}{EI}\left[480\times4 - 140\times\frac{4^2}{2} - 50\times\frac{4^3}{3}\right] + \frac{10}{EI}\left[4^3\right]$$

$$= \frac{1}{EI}\left[1920 - 1120 - \frac{3200}{3}\right] + \frac{640}{EI}$$

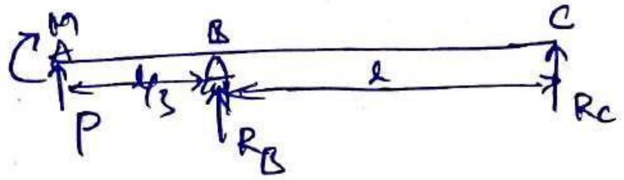
$$= \frac{-800}{3EI} + \frac{640}{EI} = \frac{373.33}{EI} = \frac{373.33}{12\times10^4}$$

$$\boxed{\begin{aligned}\Delta_{DH} &= 0.00311\ m\\ \Delta_{DH} &= 3.11\ mm\end{aligned}}$$

Using Castigliano's first theorem, determine the deflection & rotation of the overhanging end 'A' of the beam loaded as shown in fig. (13)



Sol:- For vertical deflection:-



$$\Sigma V = 0 \Rightarrow R_B + R_C + P = 0 \longrightarrow (1)$$

$$\Sigma M_C = 0 \Rightarrow P\left(l + \frac{l}{3}\right) + R_B l + M = 0.$$

$$R_B l = -M - P\left(\frac{3l+l}{3}\right)$$

$$R_B = \frac{-M}{l} - \frac{4Pl}{3l} = \frac{-M}{l} - \frac{4P}{3}$$

$$R_B = -\left(\frac{M}{l} + \frac{4P}{3}\right) = \left(\frac{M}{l} + \frac{4P}{3}\right)\downarrow$$

From eq(1) $\Rightarrow$ $R_C = -P - R_B$

$$R_C = -P + \left(\frac{M}{l} + \frac{4P}{3}\right) = \left(\frac{M}{l} + \frac{P}{3}\right)\uparrow$$

Deflection of 'A'.

Then $$\delta_A = \frac{\partial U}{\partial P} = \frac{1}{EI}\int_A^B M_x \cdot \frac{\partial M_x}{\partial M} \cdot dx + \frac{1}{EI}\int_B^C M_x \frac{\partial M_x}{\partial M} dx.$$

For portion AB, $x = 0$ at 'A' & $x = \frac{l}{3}$ at B.

$$M_x = (-M - Px),$$

$$\frac{\partial M_x}{\partial P} = -x$$

For portion CB, $x = 0$ at 'C' & $x = l$ at B.

$$M_x = -R_C \cdot x = -\left(\frac{M}{l} + \frac{P}{3}\right)x$$

$$\frac{\partial M_x}{\partial P} = \frac{-x}{3}$$

Subtituting above values

$$\delta_A = \frac{1}{EI}\int_0^{l/3} (M+Px).x.dx + \frac{1}{EI}\int_0^{l} \left(\frac{M}{l}+\frac{Pl}{3}\right)x\left(\frac{x}{3}\right).dx$$

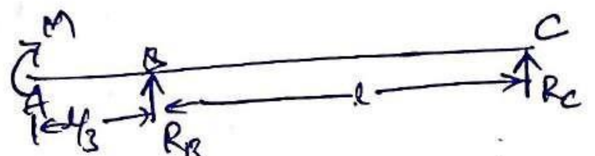
$$\delta_A = \frac{1}{EI}\int_0^{l/3} (Mx)dx + \frac{1}{EI}\int_0^{l} \frac{Mx^2}{3l}.dx \quad (\because P=0)$$

$$\delta_A = \frac{1}{EI}\left(\frac{Mx^2}{2}\right)_0^{l/3} + \frac{1}{EI}\left(\frac{M}{3l}\left(\frac{x^3}{3}\right)\right)_0^{l}$$

$$\delta_A = \frac{1}{EI}\left[\frac{M}{2}\left(\frac{l^2}{9}\right)\right] + \frac{1}{EI}\left[\frac{Ml^{3}}{9l}\right) = \frac{Ml^2}{18EI} + \frac{Ml^2}{9EI}$$

$$\boxed{\delta_A = \frac{Ml^2}{6EI}}$$

Rotation at 'A':-



$$\Sigma V=0 \Rightarrow R_B + R_C = 0. \rightarrow (1)$$

$$\Sigma M_B = 0 \Rightarrow M - R_C \times l = 0. \Rightarrow R_C l = M \Rightarrow R_C = \frac{M}{l} (\uparrow)$$

$$\text{from eq (1)} \Rightarrow R_B = -R_C = -\frac{M}{l} = \frac{M}{l} (\downarrow)$$

Now Rotation at 'A', $\theta_A = \frac{\partial U}{\partial M} = \frac{1}{EI}\int_A^B M_x . \frac{\partial M_x}{\partial M} dx + \frac{1}{EI}\int_C^B M_x . \frac{\partial M_x}{\partial M} dx$

for portion AB, at A, x=0, at B x=l/3.

$$M_x = -M, \quad \frac{\partial M_x}{\partial M} = -1,$$

for portion CB, at C, x=0, at B, x=l.

$$M_x = -R_C x = -\frac{M}{l}x, \quad \frac{\partial M_x}{\partial M} = -\frac{x}{l}.$$

Substituting above values in the eq

$$\theta_A = \frac{1}{EI}\int_0^{l/3} (-M)(-1)dx + \frac{1}{EI}\int_0^{l}\left(\frac{-M}{l}x\right)\left(\frac{-x}{l}\right)dx$$

$$= \frac{M}{EI}(x)_0^{l/3} + \frac{M}{EI.l^2}\left(\frac{x^3}{3}\right)_0^{l} = \frac{Ml}{3EI} + \frac{Ml}{3EI}$$

$$\boxed{\theta_A = \frac{2Ml}{3EI}} \text{ (Clockwise)}$$

Castigliano's Theorems:
The theorem of least work derives from what is known as Castigliano's second theorem. So, let's first state the two theorems of Carlo Alberto Castigliano (1847-1884) who was an Italian railroad engineer. In 1879, Castigliano published two theorems.

Castigliano's first theorem
The first partial derivative of the total internal energy (strain energy) in a structure with respect to any particular deflection component at a point is equal to the force applied at that point and in the direction corresponding to that deflection component.

This first theorem is applicable to linearly or nonlinearly elastic structures in which the temperature is constant and the supports are unyielding.

Castigliano's second theorem
The first partial derivative of the total internal energy in a structure with respect to the force applied at any point is equal to the deflection at the point of application of that force in the direction of its line of action.

The second theorem of Castigliano is applicable to linearly elastic (Hookean material) structures with constant temperature and unyielding supports.

Note that in the above statements, *force* may mean point force or couple (moment) and *displacement* may mean translation or angular rotation. Proofs of Castigliano's theorems are given at the end of this document.

Without further due, here is the theorem of least work, a.k.a. **Castigliano's theorem of least work:**

The redundant reaction components of a statically indeterminate structure are such that they make the internal work (strain energy) a minimum.

$$\therefore \text{ Total strain energy stored by the frame} = U = \sum \frac{S_1^2 l_1}{2A_1E}$$

$$= \sum (P_1 + XK_1)^2 \frac{l_1}{2A_1E}$$

According to least work principle $\dfrac{\partial U}{\partial X} = 0$

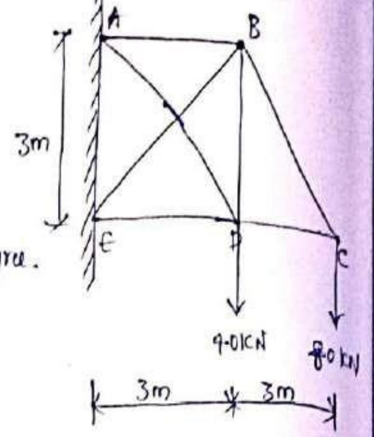
$$\Rightarrow \quad \sum 2\,(P_1 + XK_1)\frac{K_1 l_1}{2A_1E} = 0$$

$$\text{or,} \quad \sum \frac{P_1K_1l_1}{A_1E} + X\sum\frac{K_1^2 l_1}{A_1E} = 0$$

$$\text{or,} \quad X = -\frac{\sum \dfrac{P_1K_1l_1}{A_1E}}{\sum \dfrac{K_1^2 l_1}{A_1E}}$$

* Find the forces in the members of truss shown in figure. The cross area and young's modulus of all the members are the same.

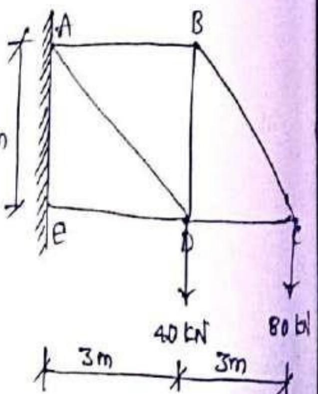
-A) The verti



In this truss there is internal indeterminacy of 1 degree.

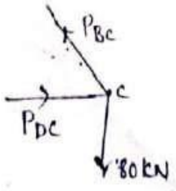
Force in the member BC is taken as the redundant force 'R'. The basic determinate structure with the given loading is shown in figure 1(A) and with the unit load in the direction of the redundant force is shown in figure 1(B)

* P-forces :-



1 (A)

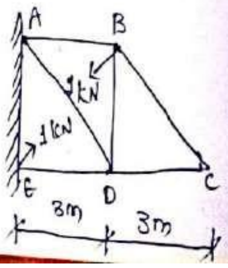
At joint C :-



$\Sigma V = 0$

$P_{BC} \sin 45' = 80$

$P_{BC} = 80 \times \frac{1}{\sin 45'}$



$= 80 \times \sqrt{2}$

$= 113.13$ kN [Tension]

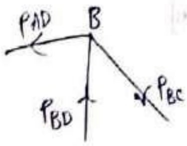
$\Sigma H = 0$

$P_{DC} = P_{BC} \cos 45'$

$= 113.13 \times \frac{1}{\sqrt{2}}$

$= 80$ KN [Compression]

* At joint B :-



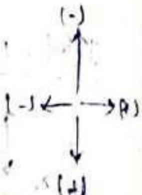
$\Sigma V = 0$

$P_{BD} = P_{BC} \sin 45'$

$= 113.13 \times \frac{1}{\sqrt{2}}$

$\boxed{P_{BD} = 80\text{KN}}$ [Compression]

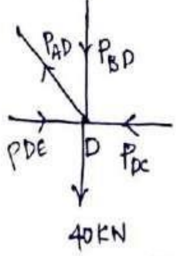
$\Sigma H = 0$



$P_{AB} = P_{BC} \cos 45' = 113.13 \times \frac{1}{\sqrt{2}}$

$\boxed{P_{AB} = 80\text{ kN}}$ [Tension]

- At joint D :-



$\Sigma V = 0$

$40 + P_{BD} - P_{AD} \sin 45' = 0$

$40 + 80 - P_{AD} \sin 45' = 0$

$\boxed{P_{AD} = 169.71\text{ kN}}$ (Tension)

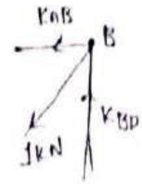
$\Sigma H = 0$

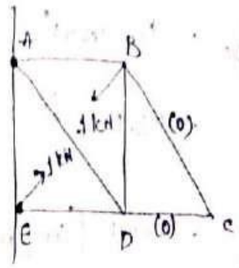
$- P_{DC} - P_{AD} \cos\theta + P_{DE} = 0$

$\boxed{P_{DE} = 200\text{ kN}}$ [Compression]

* K-forces :- [Removing external loads and applying unit load at which the deflection is to be determined.]

Joint B :-





$\sum V = 0$

$k_{BD} = 1 \sin 45^\circ$

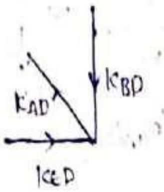
$= \frac{1}{\sqrt{2}} = 0.707$ [Compression]

$\sum H = 0$

$K_{AB} = 1 \cos 45^\circ$

$= 0.707$ [Compression]

Joint - D:-



$\sum V = 0$

$K_{BD} = K_{AD} \sin 45^\circ$

$\boxed{K_{AD} = 1 \text{ KN}}$ [Tension]

$\sum H = 0$

$K_{ED} = K_{AD} \cos 45^\circ$

$K_{ED} = \frac{1}{\sqrt{2}} =$

$\boxed{K_{ED} = 0.707}$ [Compression]

Member	P	K	l	PKl	$K^2 l$	S = P+RK
AB	-80	0.707	3	-169.68	1.49	-192.92
BC	-113.13	0	4.2	0	0	-113.13
CD	80	0	3	0	0	80
DE	200	0.707	3	424.8	1.49	137.67
BD	80	0.707	3 ~~4.2~~	169.68	1.49	17.67
AD	-169.71	-1	4.2	718.78	4.2	-81.55
~~BC~~ BE	0	1	4.2	0	4.2	88.14

$\sum PKl = 1136.94$ $\quad \sum K^2 l = \cancel{88.69415}$ 12.897.

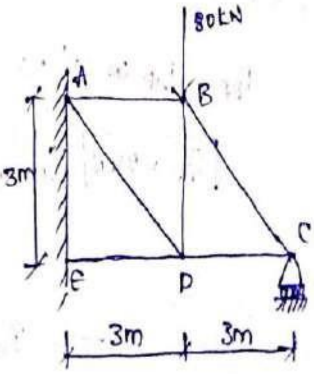
$$R = \frac{\sum \frac{PKl}{AE}}{\sum \frac{K^2 l}{AE}} = -\frac{1136.94}{\cancel{88.69415}\ 12.897}$$

88.15

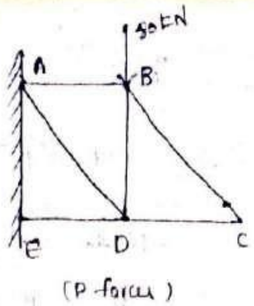
$= -$ ~~130.12~~ kN.

* Find the forces in the members of the truss shown in figure. The cross sectional area and young's modulus of all the members are same.

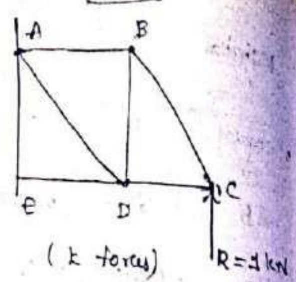
Ans)



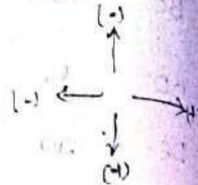
$\boxed{R = 40}$



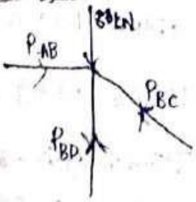
(P forces)



(k forces)



At point B :-



$\sum V = 0$

~~8 - $P_{BC} \sin\theta$ = 0~~

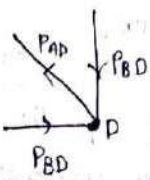
~~P_{BC} =~~

$\boxed{80 = P_{BD}}$ [Compression]

$\sum H = 0$

$\boxed{P_{AB} = 0}$

At Joint D :-



$\sum V = 0$

$P_{BD} - P_{AD} \sin 45^\circ = 0$

$\boxed{P_{AD} = 113.137 \text{ kN}}$ [Tension]

$\sum H = 0$

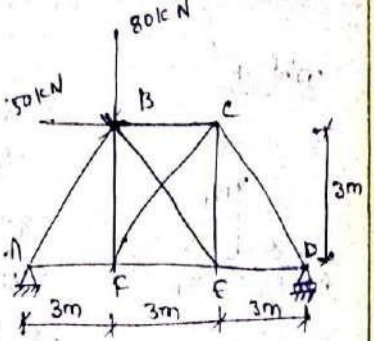
$P_{BD} - P_{AD} \cos 45^\circ = 0$

$\boxed{P_{BD} = 80 \text{kN}}$ [Compression]

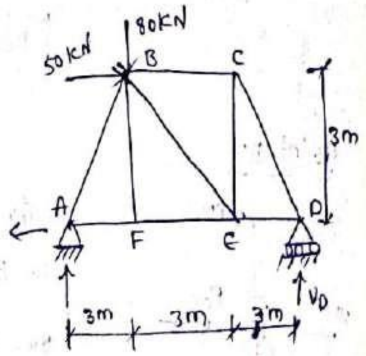
— Incomplete

* Find the forces in the members BE and CF as shown in figure. Assume same c/s area and young's modulus ~~of~~ for all members.

1) The structure is having 1 degree of internal indeterminacy. Let us consider, CF is redundant member.



P forces :-



Reactions :-

$\sum V = 0$:

$V_A + V_D = 80 \text{ kN}$

$\sum H = 0$:

$H_A = 50 \text{ kN}$

$\sum M_A = 0$:

$-V_D \times 9 + (8 \times 3) + (50 \times 3) = 0$

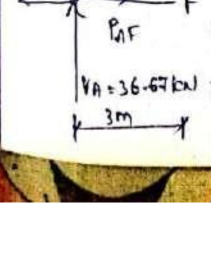
$V_D \times 9 = +390$

$\boxed{V_D = 43.33 \text{ kN}}$

$V_A + 43.33 = 80 \Rightarrow V_A = 80 - 43.33$

$\boxed{V_A = 36.67 \text{kN}}$

Joint A :-



$\sum V = 0$

$-V_A + P_{AB} \sin 45^\circ = 0$

$-36.67 = -P_{AB} \sin 45^\circ$

$P_{AB} = 36.67 \times \frac{1}{\sin 45}$

$\therefore \boxed{P_{AB} = 51.86 \text{ kN}}$ (Compression)

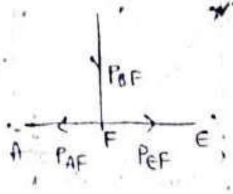
$\Sigma H = 0$

$-H_A + P_{AF} - P_{AB}\cos 45^\circ = 0$

$-50 + P_{AF} - 51.86\cos 45^\circ = 0$

$\boxed{P_{AF} = 86.67\ kN}$ [Tension]

Joint F:-



$\Sigma V = 0$

$P_{BF} = 0$

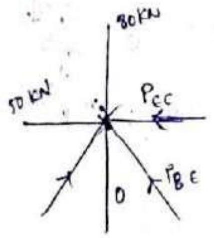
$\Sigma H = 0$

$-P_{AF} + P_{EF} = 0$

$-86.67 + P_{EF} = 0$

$\boxed{P_{EF} = 86.67\ kN}$ [Tension]

Joint B:-



$\Sigma V = 0$

$80 - P_{AB}\sin 45^\circ + P_{Be}\sin 45^\circ = 0$

$80 - 51.86\sin 45^\circ = P_{Be}\sin 45^\circ$

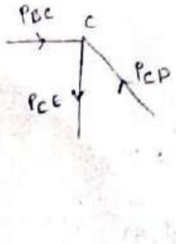
$\boxed{P_{Be} = 61.28\ kN}$ Compression

$\Sigma H = 0$

$50 - P_{BC} + P_{AB}\cos 45^\circ - P_{Be}\cos 45^\circ = 0$

$\boxed{P_{BC} = 43.34\ kN}$ [Compression]

* Joint C:-



$\Sigma V = 0$

$+P_{CE} - P_{CD}\sin 45^\circ = 0$

$P_{CE} = 61.28\sin 45^\circ = 43.34\ kN$ [Tension]

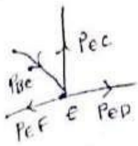
$\Sigma H = 0$

$P_{BC} - P_{CD}\cos 45^\circ = 0$

$43.34 - P_{CD}\cos 45^\circ$

$\boxed{P_{CD} = 61.29\ kN}$

Joint E:-



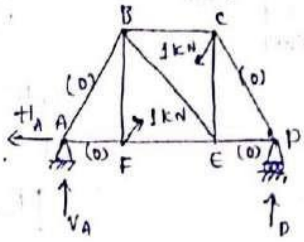
$\Sigma H = 0$

$-P_{EF} + P_{ED} + P_{BE}\cos 45^\circ = 0$

$-86.67 + P_{ED} + 61.28\cos 45^\circ = 0$

$\boxed{P_{ED} = 43.34\ kN}$ (Tension)

* K-forces:-



$\Sigma V = 0$

$V_A + V_D + 1\sin 45^\circ - 1\sin 45^\circ = 0$

$\boxed{V_A + V_D = 0}$

$\Sigma H = 0$

$-H_A + 1\cos 45^\circ - 1\cos 45^\circ = 0$

$\boxed{H_A = 0}$

$\Sigma M_A = 0$

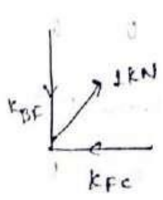
$-V_D \times 9 + [1\sin 45^\circ \times 6] - (1\sin 45^\circ \times 3) - (1\cos 45^\circ \times 3) = 0$

$\boxed{V_D = 0}$

$\boxed{V_A = 0}$

$\boxed{H_A = 0}$

* At joint F:-



$\Sigma V = 0$

$K_{BF} - 1\sin 45^\circ = 0$

$\boxed{K_{BF} = 0.707\ kN}$ [Compression]

$\Sigma H = 0$

$-K_{FE} + 1\cos 45^\circ = 0$

$\boxed{K_{FE} = 0.707\ kN}$ [Compression]