SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES

(AUTONOMOUS)

DEPARTMENT OF MECHANICAL ENGINEERING

COURSE MATERIAL

Subject Name	Fluid Mechanics and Hydraulic Machines
Subject Code	23MEC241T
Semester	IV Semester
Academic Year	2025-26
Regulation	R23

Unit-III

Boundary Layer Theory and Dimensional Analysis

INTRODUCTION

When a real fluid flows past a solid body or a solid wall, the fluid particles adhere to the boundary and condition of no slip occurs. This means that the velocity of fluid close to the boundary will be same as that of the boundary. If the boundary is stationary, the velocity of fluid at the boundary will be zero. Farther away from the boundary, the velocity will be higher and as a result of this variation of velocity, the velocity gradient $\frac{du}{dy}$ will exist. The velocity of fluid increases from zero velocity on the stationary boundary to free-stream velocity (U) of the fluid in the direction normal to the boundary. This variation of velocity from zero to free-stream velocity in the direction normal to the boundary takes place in a narrow region in the vicinity of solid boundary. This narrow region of the fluid is called boundary layer. The theory dealing with boundary layer flows is called boundary layer theory.

According to boundary layer theory, the flow of fluid in the neighbourhood of the solid boundary may be divided into two regions as shown in Fig. 13.1.

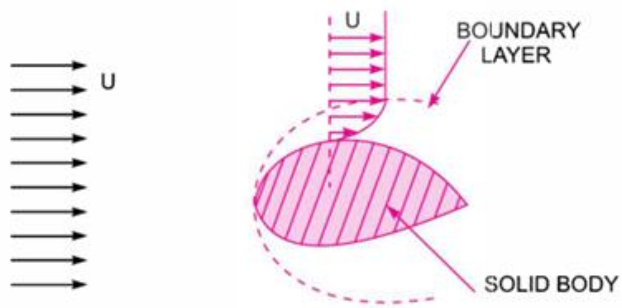


Fig. 13.1 *Flow over solid body.*

1. A very thin layer of the fluid, called the boundary layer, in the immediate neighbourhood of the solid boundary, where the variation of velocity from zero at the solid boundary to free-stream velocity in the direction normal to the boundary takes place. In this region, the velocity gradient $\frac{du}{dy}$ exists and hence the fluid exerts a shear stress on the wall in the direction of motion. The value of shear stress is given by

$$\tau = \mu \frac{du}{dy}.$$

2. The remaining fluid, which is outside the boundary layer. The velocity outside the boundary layer is constant and equal to free-stream velocity. As there is no variation of velocity in this region, the velocity gradient $\frac{du}{dy}$ becomes zero. As a result of this the shear stress is zero.

DEFINITIONS

Laminar Boundary Layer. For defining the boundary layer (*i.e.*, laminar boundary layer or turbulent boundary layer) consider the flow of a fluid, having free-stream velocity (U), over a smooth thin plate which is flat and placed parallel to the direction for free stream of fluid as shown in Fig. 13.2. Let us consider the flow with zero pressure gradient on one side of the plate, which is stationary.

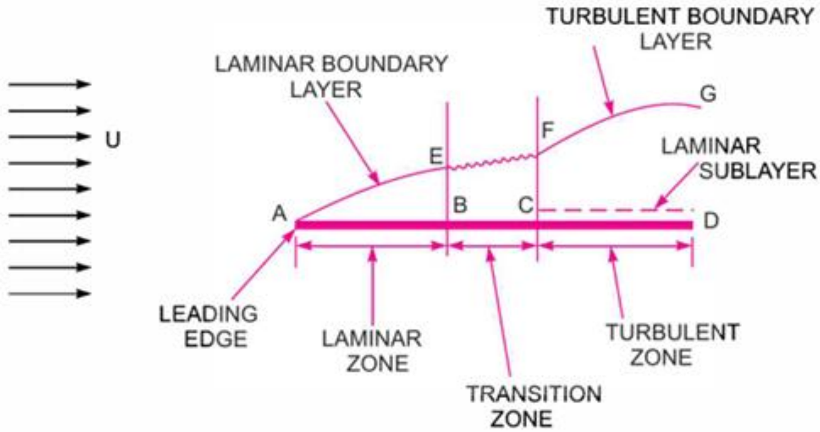


Fig. 13.2 *Flow over a plate.*

The velocity of fluid on the surface of the plate should be equal to the velocity of the plate. But plate is stationary and hence velocity of fluid on the surface of the plate is zero. But at a distance away from the plate, the fluid is having certain velocity. Thus a velocity gradient is set up in the fluid near the surface of the plate. This velocity gradient develops shear resistance, which retards the fluid. Thus the fluid with a uniform free stream velocity (U) is retarded in the vicinity of the solid surface of the plate and the boundary layer region begins at the sharp leading edge. At subsequent points downstream the leading edge, the boundary layer region increases because the retarded fluid is further retarded. This is also referred as the growth of boundary layer. Near the leading edge of the surface of the plate, where the thickness is small, the flow in the boundary layer is laminar though the main flow is turbulent. This layer of the fluid is said to be laminar boundary layer. This is shown by AE in Fig. 13.2. The length of the plate from the leading edge, upto which laminar boundary layer exists, is called laminar zone. This is shown by distance AB. The distance of B from leading edge is obtained from Reynold number equal to 5×10^5 for a plate. Because upto this Reynold number the boundary layer is laminar. The Reynold number is given by $(R_e)_x = \dfrac{U \times x}{\nu}$

where x = Distance from leading edge,
U = Free-stream velocity of fluid,
ν = Kinematic viscosity of fluid,

Hence for laminar boundary layer, we have $5 \times 10^5 = \dfrac{U \times x}{\nu}$ ...(13.1)

If the values of U and ν are known, x or the distance from the leading edge upto which laminar boundary layer exists can be calculated.

Turbulent Boundary Layer. If the length of the plate is more than the distance x, calculated from equation (13.1), the thickness of boundary layer will go on increasing in the down-stream direction. Then the laminar boundary layer becomes unstable and motion of fluid within it, is disturbed and irregular which leads to a transition from laminar to turbulent boundary layer. This short length over which the boundary layer flow changes from laminar to turbulent is called transition zone. This is shown by distance BC in Fig. 13.2. Further downstream the transition zone, the boundary layer is turbulent and continues to grow in thickness. This layer of boundary is called turbulent boundary layer, which is shown by the portion FG in Fig. 13.2.

Laminar Sub-layer. This is the region in the turbulent boundary layer zone, adjacent to the solid surface of the plate as shown in Fig. 13.2. In this zone, the velocity variation is influenced only by viscous effects. Though the velocity distribution would be a parabolic curve in the laminar sub-layer zone, but in view of the very small thickness we can reasonably assume that velocity variation is linear and so the velocity gradient can be considered constant. Therefore, the shear stress in the laminar sub-layer would be constant and equal to the boundary shear stress τ_0. Thus the shear stress in the sub-layer is

$$\tau_0 = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = \mu \frac{u}{y} \qquad \left\{\because \text{ For linear variation, } \frac{\partial u}{\partial y} = \frac{u}{y}\right\}$$

Boundary Layer Thickness (δ). It is defined as the distance from the boundary of the solid body measured in the y-direction to the point, where the velocity of the fluid is approximately equal to 0.99 times the free stream velocity (U) of the fluid. It is denoted by the symbol δ. For laminar and turbulent zone it is denoted as :

1. δ_{lam} = Thickness of laminar boundary layer,
2. δ_{tur} = Thickness of turbulent boundary layer, and
3. δ' = Thickness of laminar sub-layer.

Displacement Thickness (δ^*). It is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in flow rate on account of boundary layer formation. It is denoted by δ^*. It is also defined as :

"The distance perpendicular to the boundary, by which the free-stream is displaced due to the formation of boundary layer".

Expression for δ^*.

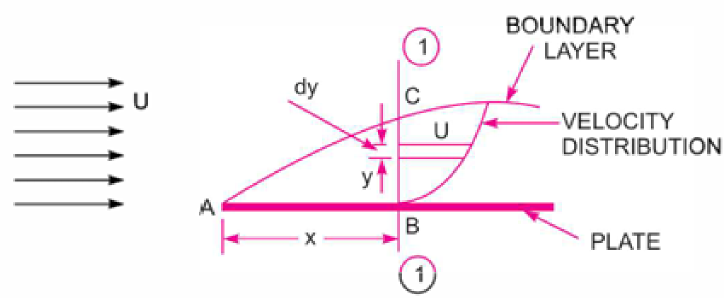


Fig. 13.3 *Displacement thickness.*

Consider the flow of a fluid having free-stream velocity equal to U over a thin smooth plate as shown in Fig. 13.3. At a distance x from the leading edge consider a section 1-1. The velocity of fluid at B is zero and at C, which lies on the boundary layer, is U. Thus velocity varies from zero at B to U at C, where BC is equal to the thickness of boundary layer *i.e.*,

Distance $BC = \delta$

At the section 1-1, consider an elemental strip.

Let y = distance of elemental strip from the plate,

dy = thickness of the elemental strip,

u = velocity of fluid at the elemental strip,

b = width of plate.

Then area of elemental strip, $dA = b \times dy$

Mass of fluid per second flowing through elemental strip

$$= \rho \times \text{Velocity} \times \text{Area of elemental strip}$$

$$= \rho u \times dA = \rho u \times b \times dy \qquad ...(i)$$

If there had been no plate, then the fluid would have been flowing with a constant velocity equal to free-stream velocity (U) at the section 1-1. Then mass of fluid per second flowing through elemental strip would have been

$$= \rho \times \text{Velocity} \times \text{Area} = \rho \times U \times b \times dy \qquad ...(ii)$$

As U is more than u, hence due to the presence of the plate and consequently due to the formation of the boundary layer, there will be a reduction in mass flowing per second through the elemental strip.

This reduction in mass/sec flowing through elemental strip

= mass/sec given by equation (*ii*) – mass/sec given by equation (*i*)

$$= \rho U b dy - \rho u b dy = \rho b(U - u)dy$$

$\therefore$ Total reduction in mass of fluid/s flowing through BC due to plate

$$= \int_0^\delta \rho b(U-u)dy = \rho b \int_0^\delta (U-u)dy \qquad ...(iii)$$

{if fluid is incompressible}

Let the plate is displaced by a distance δ^* and velocity of flow for the distance δ^* is equal to the free-stream velocity (*i.e.*, U). Loss of the mass of the fluid/sec flowing through the distance δ^*

$$= \rho \times \text{Velocity} \times \text{Area}$$

$$= \rho \times U \times \delta^* \times b \qquad \{\because \text{Area} = \delta^* \times b\} \; ...(iv)$$

Equating equation (*iii*) and (*iv*), we get

$$\rho b \int_0^\delta (U-u)dy = \rho \times U \times \delta^* b$$

Cancelling ρb from both sides, we have

$$\int_0^\delta (U-u)dy = U \times \delta^*$$

or

$$\delta^* = \frac{1}{U}\int_0^\delta (U-u)dy = \int_0^\delta \frac{(U-u)dy}{U} \qquad \left\{\begin{array}{l}\because\ U \text{ is constant and can} \\ \text{be taken inside the integral}\end{array}\right\}$$

$\therefore$

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy. \qquad ...(13.2)$$

Momentum Thickness (θ). Momentum thickness is defined as the distance, measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in **momentum** of the flowing fluid on account of boundary layer formation. It is denoted by θ.

Consider the flow over a plate as shown in Fig. 13.3. Consider the section 1-1 at a distance x from leading edge. Take an elemental strip at a distance y from the plate having thickness (dy). The mass of fluid flowing per second through this elemental strip is given by equation (i) and is equal to $\rho u b dy$.

Momentum of this fluid = Mass × Velocity = $(\rho u b dy) u$

Momentum of this fluid in the absence of boundary layer = $(\rho u b dy) U$

$\therefore$ Loss of momentum through elemental strip = $(\rho u b dy) U - (\rho u b dy) \times u = \rho b u (U - u) dy$

$$\therefore \text{ Total loss of momentum/sec through } BC = \int_0^{\delta} \rho b u (U - u) dy \qquad \text{...(13.3)}$$

Let θ = distance by which plate is displaced when the fluid is flowing with a constant velocity U

$\therefore$ Loss of momentum/sec of fluid flowing through distance θ with a velocity U

$$\begin{aligned} &= \text{Mass of fluid through } \theta \times \text{velocity} \\ &= (\rho \times \text{area} \times \text{velocity}) \times \text{velocity} \\ &= [\rho \times \theta \times b \times U] \times U \qquad \{\because \text{Area} = \theta \times b\} \\ &= \rho \theta b U^2 \qquad \text{...(13.4)} \end{aligned}$$

Equating equations (13.4) and (13.3), we have

$$\rho \theta b U^2 = \int_0^{\delta} \rho b u (U - u) dy = \rho b \int_0^{\delta} u (U - u) dy \qquad \{\text{If fluid is assumed incompressible}\}$$

or
$$\theta U^2 = \int_0^{\delta} u (U - u) dy \qquad \{\text{cancelling } \rho b \text{ from both sides}\}$$

or
$$\theta = \frac{1}{U^2} \int_0^{\delta} u (U - u) dy = \int_0^{\delta} \frac{u (U - u) dy}{U^2}$$

$$\therefore \qquad \theta = \int_0^{\delta} \frac{u}{U} \left[1 - \frac{u}{U} \right] dy. \qquad \text{...(13.5)}$$

Energy Thickness (δ^{**}). It is defined as the distance measured perpendicular to the boundary of the solid body, by which the boundary should be displaced to compensate for the reduction in kinetic energy of the flowing fluid on account of boundary layer formation. It is denoted by δ^{**}.

Consider the flow over the plate as shown in Fig. 13.3 having section 1-1 at a distance x from leading edge. The mass of fluid flowing per second through the elemental strip of thickness 'dy' at a distance y from the plate as given by equation (i) = $\rho u b dy$

Kinetic energy of this fluid $= \frac{1}{2} m \times \text{velocity}^2 = \frac{1}{2} (\rho u b dy) u^2$

Kinetic energy of this fluid in the absence of boundary layer

$$= \frac{1}{2} (\rho u b dy) U^2$$

$\therefore$ Loss of K.E. through elemental strip

$$= \frac{1}{2} (\rho u b dy) U^2 - \frac{1}{2} (\rho u b dy) u^2 = \frac{1}{2} \rho u b [U^2 - u^2] dy$$

$\therefore$ Total loss of K.E. of fluid passing through BC

$$= \int_0^\delta \frac{1}{2} \rho u b \, [U^2 - u^2] \, dy = \frac{1}{2} \rho b \int_0^\delta u \, (U^2 - u^2) \, dy$$

{If fluid is considered incompressible}

Let δ^{**} = distance by which the plate is displaced to compensate for the reduction in K.E.

$\therefore$ Loss of K.E. through δ^{**} of fluid flowing with velocity U

$$= \frac{1}{2} \text{ (mass)} \times \text{velocity}^2 = \frac{1}{2} (\rho \times \text{area} \times \text{velocity}) \times \text{velocity}^2$$

$$= \frac{1}{2} (\rho \times b \times \delta^{**} \times U) U^2 \qquad \{\because \text{ Area} = b \times \delta^{**}\}$$

$$= \frac{1}{2} \rho b \delta^{**} U^3$$

Equating the two losses of K.E., we get

$$\frac{1}{2} \rho b \delta^{**} U^3 = \frac{1}{2} \rho b \int_0^\delta u \, (U^2 - u^2) \, dy$$

or

$$\delta^{**} = \frac{1}{U^3} \int_0^\delta u \, (U^2 - u^2) \, dy$$

$\therefore$

$$\delta^{**} = \int_0^\delta \frac{u}{U} \left[1 - \frac{u^2}{U^2} \right] dy. \qquad \text{...(13.6)}$$

Problem 1 *Find the displacement thickness, the momentum thickness and energy thickness for the velocity distribution in the boundary layer given by* $\frac{u}{U} = \frac{y}{\delta}$, *where u is the velocity at a distance y from the plate and* $u = U$ *at* $y = \delta$, *where* δ *= boundary layer thickness. Also calculate the value of* δ^*/θ.

Solution. Given :

Velocity distribution $\quad \frac{u}{U} = \frac{y}{\delta}$

(*i*) Displacement thickness δ^* is given by equation (13.2),

$$\delta^* = \int_0^\delta \left(1 - \frac{u}{U}\right) dy = \int_0^\delta \left(1 - \frac{y}{\delta}\right) dy \qquad \left\{\because \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \left[y - \frac{y^2}{2\delta} \right]_0^\delta \qquad \{\delta \text{ is constant across a section}\}$$

$$= \delta - \frac{\delta^2}{2\delta} = \delta - \frac{\delta}{2} = \frac{\delta}{2}. \textbf{ Ans.}$$

(*ii*) Momentum thickness, θ is given by equation (13.5),

$$\theta = \int_0^\delta \frac{u}{U} \left(1 - \frac{u}{U}\right) dy$$

Substituting the value of $\frac{u}{U} = \frac{y}{\delta}$,

$$\theta = \int_0^\delta \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right) dy = \int_0^\delta \left(\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) dy$$

$$= \left[\frac{y^2}{2\delta} - \frac{y^3}{3\delta^2}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^3}{3\delta^2} = \frac{\delta}{2} - \frac{\delta}{3} = \frac{3\delta - 2\delta}{6} = \frac{\delta}{6}. \textbf{ Ans.}$$

(*iii*) Energy thickness δ^{**} is given by equation (13.6), as

$$\delta^{**} = \int_0^\delta \frac{u}{U}\left[1 - \frac{u^2}{U^2}\right] dy = \int_0^\delta \frac{y}{\delta}\left[1 - \frac{y^2}{\delta^2}\right] dy \qquad \left\{\because \ \frac{u}{U} = \frac{y}{\delta}\right\}$$

$$= \int_0^\delta \left[\frac{y}{\delta} - \frac{y^3}{\delta^3}\right] dy = \left[\frac{y^2}{2\delta} - \frac{y^4}{4\delta^3}\right]_0^\delta = \frac{\delta^2}{2\delta} - \frac{\delta^4}{4\delta^3}$$

$$= \frac{\delta}{2} - \frac{\delta}{4} = \frac{2\delta - \delta}{4} = \frac{\delta}{4}. \textbf{ Ans.}$$

(*iv*)

$$\frac{\delta^*}{\theta} = \frac{\left(\frac{\delta}{2}\right)}{\left(\frac{\delta}{6}\right)} = \frac{\delta}{2} \times \frac{6}{\delta} = \textbf{3. Ans.}$$

SEPARATION OF BOUNDARY LAYER

When a solid body is immersed in a flowing fluid, a thin layer of fluid called the boundary layer is formed adjacent to the solid body. In this thin layer of fluid, the velocity varies from zero to free-stream velocity in the direction normal to the solid body. Along the length of the solid body, the thickness of the boundary layer increases. The fluid layer adjacent to the solid surface has to do work against surface friction at the expense of its kinetic energy. This loss of the kinetic energy is recovered from the immediate fluid layer in contact with the layer adjacent to solid surface through momentum exchange process. Thus the velocity of the layer goes on decreasing. Along the length of the solid body, at a certain point a stage may come when the boundary layer may not be able to keep sticking to the solid body if it cannot provide kinetic energy to overcome the resistance offered by the solid body. In other words, the boundary layer will be separated from the surface. This phenomenon is called the boundary layer separation. The point on the body at which the boundary layer is on the verge of separation from the surface is called point of separation.

Effect of Pressure Gradient on Boundary Layer Separation. The effect of pressure gradient $\left(\frac{dp}{dx}\right)$ on boundary layer separation can be explained by considering the flow over a curved surface *ABCSD* as shown in Fig. 13.7. In the region *ABC* of the curved surface, the area of flow decreases and hence velocity increases. This means that flow gets accelerated in this region. Due to the increase of the velocity, the pressure decreases in the direction of the flow and hence pressure gradient $\frac{dp}{dx}$ is negative in this region. As long as $\frac{dp}{dx} < 0$, the entire boundary layer moves forward as shown in Fig. 13.7.

Region *CSD* of the curved surface. The pressure is minimum at the point *C*. Along the region *CSD* of the curved surface, the area of flow increases and hence velocity of flow along the direction of fluid decreases. Due to decrease of velocity, the pressure increases in the direction of flow and hence pressure gradient $\frac{dp}{dx}$ is positive or $\frac{dp}{dx} > 0$. Thus in the region *CSD*, the pressure gradient is positive and velocity of fluid layer along the direction of flow decreases. As explained in the Art. 13.7, the velocity of the layer adjacent to the solid surface along the length of the solid surface goes on decreasing as the kinetic energy of the layer is used to overcome the frictional resistance of the surface. Thus the combined effect of positive pressure gradient and surface resistance reduce the momentum of the fluid is unable to the surface. A stage comes, when the momentum of the fluid is unable to overcome the surface resistance and the boundary layer starts separating from the surface at the point *S*. Downstream the point *S*, the flow is taking place in reverse direction and the velocity gradient becomes negative.

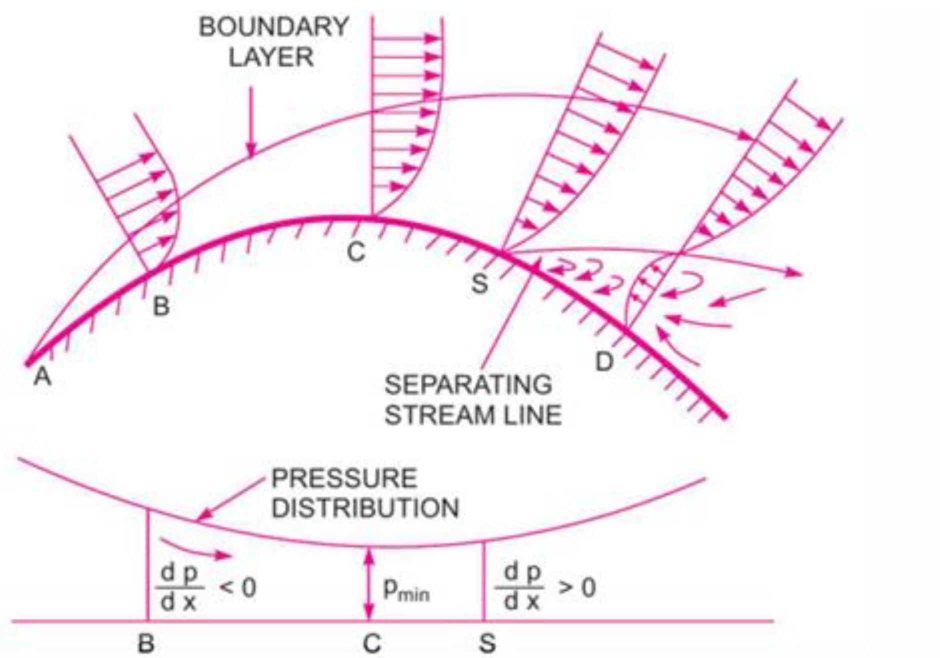


Fig. 13.7 *Effect of pressure gradient on boundary layer separation.*

Location of Separation Point. The separation point *S* is determined from the condition,

$$\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0 \qquad \text{...(13.46)}$$

For a given velocity profile, it can be determined whether the boundary layer has separated, or on the verge of separation or will not separate from the following conditions :

1. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is negative ... the flow has separated.
2. If $\left(\frac{\partial u}{\partial y}\right)_{y=0} = 0$ ... the flow is on the verge of separation.
3. If $\left(\frac{\partial u}{\partial y}\right)_{y=0}$ is positive ... the flow will not separate or flow will remain attached with the surface.

Stream-lined Body. A stream-lined body is defined as that body whose surface coincides with the stream-lines, when the body is placed in a flow. In that case the separation of flow will take place only at the trailing edge (or rearmost part of the body). Though the boundary layer will start at the leading edge, will become turbulent from laminar, yet it does not separate upto the rearmost part of the body in the case of stream-lined body. Thus behind a stream-lined body, wake formation zone will be very small and consequently the pressure drag will be very small. Then the total drag on the stream-lined body will be due to friction (shear) only. A body may be stream-lined :

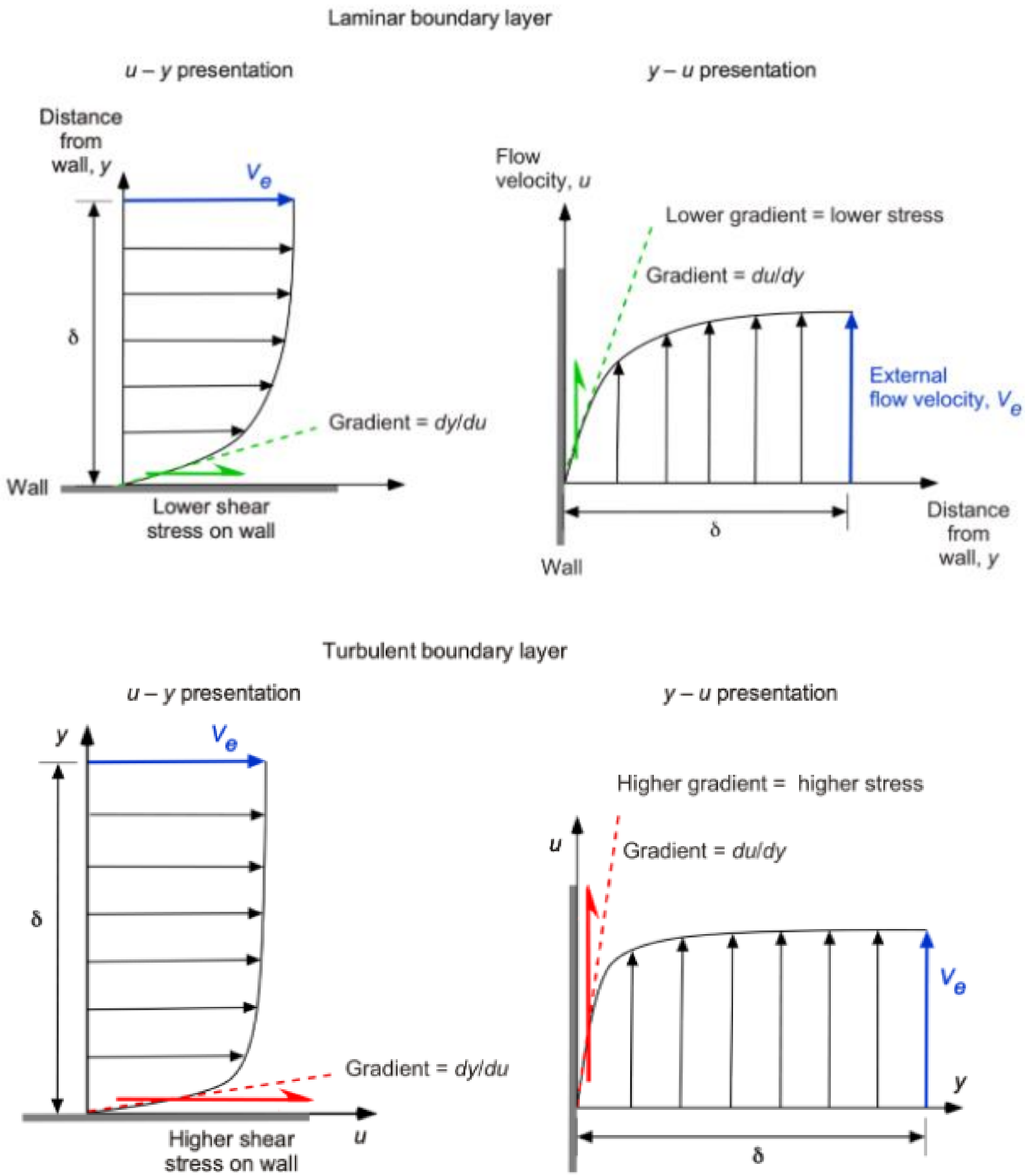
1. at low velocities but may not be so at higher velocities.
2. when placed in a particular position in the flow but may not be so when placed in another position.

Bluff Body. A bluff body is defined as that body whose surface does not coincide with the streamlines, when placed in a flow. Then the flow is separated from the surface of the body much ahead of its trailing edge with the result of a very large wake formation zone. Then the drag due to pressure will be very large as compared to the drag due to friction on the body. Thus the bodies of such a shape in which the pressure drag is very large as compared to friction drag are called bluff bodies.

Basic concepts of velocity profiles:

A velocity profile is a graphical representation of how the speed of a fluid varies across a cross-section of flow, typically showing slower speeds near boundaries (due to friction) and higher speeds in the center. It is essential for determining flow rate, pressure drop, and the nature of flow (laminar vs. turbulent).

As implied by the shape of the boundary layer velocity profiles, as shown in the figure below, the magnitude of the wall shear stress, , produced in the boundary layer (and hence on the wall) will be greater if a turbulent boundary layer flows over it compared with a laminar one.



The shear stress on a surface produced by a flowing laminar boundary layer is much lower than that produced by a turbulent boundary layer.

Dimensional Analysis:

INTRODUCTION

Dimensional analysis is a method of dimensions. It is a mathematical technique used in research work for design and for conducting model tests. It deals with the dimensions of the physical quantities involved in the phenomenon. All physical quantities are measured by comparison, which is made with respect to an arbitrarily fixed value. Length L, mass M and time T are three fixed dimensions which are of importance in Fluid Mechanics. If in any problem of fluid mechanics, heat is involved then temperature is also taken as fixed dimension. These fixed dimensions are called fundamental dimensions or fundamental quantity.

SECONDARY OR DERIVED QUANTITIES

Secondary or derived quantities are those quantities which possess more than one fundamental dimension. For example, velocity is denoted by distance per unit time (L/T), density by mass per unit volume $\left(\frac{M}{L^3}\right)$ and acceleration by distance per second square (L/T^2). Then velocity, density and acceleration become as secondary or derived quantities. The expressions (L/T), $\left(\frac{M}{L^3}\right)$ and $\left(\frac{L}{T^2}\right)$ are called the dimensions of velocity, density and acceleration respectively. The dimensions of mostly used physical quantities in Fluid Mechanics are given in Table 12.1.

Table 12.1

S. No.	*Physical Quantity*	*Symbol*	*Dimensions*
	(*a*) **Fundamental**		
1.	Length	L	L
2.	Mass	M	M
3.	Time	T	T

S.No.	Physical Quantity	Symbol	Dimensions
	(*b*) **Geometric**		
4.	Area	A	L^2
5.	Volume	$\forall$	L^3
	(*c*) **Kinematic Quantities**		
6.	Velocity	v	LT^{-1}
7.	Angular Velocity	ω	T^{-1}
8.	Acceleration	a	LT^{-2}
9.	Angular Acceleration	α	T^{-2}
10.	Discharge	Q	L^3T^{-1}
11.	Acceleration due to Gravity	g	LT^{-2}
12.	Kinematic Viscosity	ν	L^2T^{-1}
	(*d*) **Dynamic Quantities**		
13.	Force	F	MLT^{-2}
14.	Weight	W	MLT^{-2}
15.	Density	ρ	ML^{-3}
16.	Specific Weight	w	$ML^{-2}T^{-2}$
17.	Dynamic Viscosity	μ	$ML^{-1}T^{-1}$
18.	Pressure Intensity	p	$ML^{-1}T^{-2}$
19.	Modulus of Elasticity	$\begin{cases} K \\ E \end{cases}$	$ML^{-1}T^{-2}$
20.	Surface Tension	σ	MT^{-2}
21.	Shear Stress	τ	$ML^{-1}T^{-2}$
22.	Work, Energy	W or E	ML^2T^{-2}
23.	Power	P	ML^2T^{-3}
24.	Torque	T	ML^2T^{-2}
25.	Momentum	M	MLT^{-1}

DIMENSIONAL HOMOGENEITY

Dimensional homogeneity means the dimensions of each terms in an equation on both sides are equal. Thus if the dimensions of each term on both sides of an equation are the same the equation is known as dimensionally homogeneous equation. The powers of fundamental dimensions (*i.e.*, L, M, T) on both sides of the equation will be identical for a dimensionally homogeneous equation. Such equations are independent of the system of units.

Let us consider the equation, $V = \sqrt{2gH}$

Dimension of L.H.S. $= V = \frac{L}{T} = LT^{-1}$

Dimension of R.H.S. $= \sqrt{2gH} = \sqrt{\frac{L}{T^2} \times L} = \sqrt{\frac{L^2}{T^2}} = \frac{L}{T} = LT^{-1}$

Dimension of L.H.S. = Dimension of R.H.S. $= LT^{-1}$

$\therefore$ Equation $V = \sqrt{2gH}$ is dimensionally homogeneous. So it can be used in any system of units.

Method of Selecting Repeating Variables. The number of repeating variables are

Buckingham's π-Theore The Rayleigh's method of dimensional analysis becomes more laborious if the variables are more than the number of fundamental dimensions (M, L, T). This difficulty is overcame by using Buckingham's π-theorem, which states, "If there are n variables (independent and dependent variables) in a physical phenomenon and if these variables contain m fundamental dimensions (M, L, T), then the variables are arranged into $(n - m)$ dimensionless terms. Each term is called π-term".

Let $X_1, X_2, X_3, ..., X_n$ are the variables involved in a physical problem. Let X_1 be the dependent variable and $X_2, X_3, ..., X_n$ are the independent variables on which X_1 depends. Then X_1 is a function of $X_2, X_3, ..., X_n$ and mathematically it is expressed as

$$X_1 = f(X_2, X_3, ..., X_n) \qquad ...(12.1)$$

Equation (12.1) can also be written as

$$f_1(X_1, X_2, X_3, ..., X_n) = 0. \qquad ...(12.2)$$

Equation (12.2) is a dimensionally homogeneous equation. It contains n variables. If there are m fundamental dimensions then according to Buckingham's π-theorem, equation (12.2) can be written in terms of number of dimensionless groups or π-terms in which number of π-terms is equal to $(n - m)$. Hence equation (12.2) becomes as

$$f(\pi_1, \pi_2, ..., \pi_{n-m}) = 0. \qquad ...(12.3)$$

Each of π-terms is dimensionless and is independent of the system. Division or multiplication by a constant does not change the character of the π-term. Each π-term contains $m + 1$ variables, where m is the number of fundamental dimensions and is also called repeating variables. Let in the above case X_2, X_3 and X_4 are repeating variables if the fundamental dimension m (M, L, T) = 3. Then each π-term is written as

$$\left.\begin{aligned} \pi_1 &= X_2^{a_1} . X_3^{b_1} . X_4^{c_1} . X_1 \\ \pi_2 &= X_2^{a_2} . X_3^{b_2} . X_4^{c_2} . X_5 \\ &\vdots \\ \pi_{n-m} &= X_2^{a_{n-m}} . X_3^{b_{n-m}} . X_4^{c_{n-m}} . X_n \end{aligned}\right\} \qquad ...(12.4)$$

Each equation is solved by the principle of dimensional homogeneity and values of a_1, b_1, c_1 etc., are obtained. These values are substituted in equation (12.4) and values of $\pi_1, \pi_2, ..., \pi_{n-m}$ are obtained. These values of π's are substituted in equation (12.3). The final equation for the phenomenon is obtained by expressing any one of the π-terms as a function of others as

$$\pi_1 = \phi\,[\pi_2, \pi_3, ..., \pi_{n-m}]$$

or

$$\pi_2 = \phi_1\,[\pi_1, \pi_3, ..., \pi_{n-m}] \qquad ...(12.5)$$

Problem *The resisting force R of a supersonic plane during flight can be considered as dependent upon the length of the aircraft l, velocity V, air viscosity μ, air density ρ and bulk modulus of air K. Express the functional relationship between these variables and the resisting force.*

Solution. The resisting force R depends upon

(*i*) density, l, (*ii*) velocity, V,

(*iii*) viscosity, μ, (*iv*) density, ρ,

(*v*) Bulk modulus, K.

$$\therefore \quad R = Al^a \cdot V^b \cdot \mu^c \cdot \rho^d \cdot K^e \quad \text{...}(i)$$

where A is the non-dimensional constant.

Substituting the dimensions on both sides of the equation (*i*),

$$MLT^{-2} = AL^a \cdot (LT^{-1})^b \cdot (ML^{-1}T^{-1})^c \cdot (ML^{-3})^d \cdot (ML^{-1}T^{-2})^e$$

Equating the powers of M, L, T on both sides,

Power of M, $\quad 1 = c + d + e$

Power of L, $\quad 1 = a + b - c - 3d - e$

Power of T, $\quad -2 = -b - c - 2e.$

There are five unknowns but equations are only three. Expressing the three unknowns in terms of two unknowns (μ and K).

$\therefore$ Express the values of a, b and d in terms of c and e.

Solving,

$$d = 1 - c - e$$

$$b = 2 - c - 2e$$

$$a = 1 - b + c + 3d + e = 1 - (2 - c - 2e) + c + 3(1 - c - e) + e$$

$$= 1 - 2 + c + 2e + c + 3 - 3c - 3e + e = 2 - c.$$

Substituting these values in (*i*), we get

$$R = A\, l^{2-c} \cdot V^{2-c-2e} \cdot \mu^c \cdot \rho^{1-c-e} \cdot K^e$$

$$= A\, l^2 \cdot V^2 \cdot \rho(l^{-c}\, V^{-c}\, \mu^c\, \rho^{-c}) \cdot (V^{-2e} \cdot \rho^{-e} \cdot K^e)$$

$$= A\, l^2 V^2 \rho \left(\frac{\mu}{\rho V L}\right)^c \cdot \left(\frac{K}{\rho V^2}\right)^e$$

$$= \mathbf{A \rho l^2 V^2\, \phi \left[\left(\frac{\mu}{\rho VL}\right) \cdot \left(\frac{K}{\rho V^2}\right)\right]}. \textbf{ Ans.}$$