DEPARTMENT OF COMPUTER SCIENCE AND ENGINEERING-DATA SCIENCE

II B. Tech III Sem

		L	T	P	C
23BSC232T	**DISCRETE MATHEMATICS & GRAPH THEORY**				
	(Common to All Engineering Branches)	**2**	**1**	**0**	**3**

PRE-REQUISITES: Nil

COURSE EDUCATIONAL OBJECTIVES:

1. To gain the knowledge on connectives and relate the laws of logic to find the disjunctive normal form and conjunctive normal form of compound proposition.
2. To learn the various concepts related to predicate logic.
3. To understand the concept of groups, Abelian groups and group homomorphism and isomorphism.
4. To study the fundamentals of graphs, sub graphs, planar graphs, Hamiltonian graphs, Euler graphs, Spanning trees and graph traversals

UNIT-I: MATHEMATICAL LOGIC (9)

Introduction, Statements and Notation, Connectives, Well-formed formulas, Tautology, Duality law, Equivalence, Implication, Normal Forms, Functionally complete set of connectives, Inference Theory of Statement Calculus, Predicate Calculus, Inference theory of Predicate Calculus.

UNIT-II: SET THEORY (9)

The Principle of Inclusion- Exclusion, Pigeon hole principle and its application, Functions composition of functions, Inverse Functions, Recursive Functions, Lattices and its properties. Algebraic structures: Algebraic systems-Examples and General Properties, Semi groups and Monoids, groups, sub groups, homomorphism, Isomorphism

UNIT-III: ELEMENTARY COMBINATORICS (9)

Combinations and Permutations, Enumeration of Combinations and Permutations, Enumerating Combinations and Permutations with Repetitions, Enumerating Permutations with Constrained Repetitions, Binomial Coefficients, The Binomial and Multinomial Theorems.

UNIT-IV: RECURRENCE RELATIONS (9)

Generating Functions of Sequences, Calculating Coefficients of Generating Functions, Recurrence relations, Solving Recurrence Relations by Substitution and Generating functions, The Method of Characteristic roots, Solutions of Inhomogeneous, Recurrence Relations

UNIT-V: GRAPHS (9)

Basic Concepts, Isomorphism and Subgraphs, Trees and their Properties, Spanning Trees, Directed Trees, Binary Trees, Planar Graphs, Euler's Formula, Multigraphs and Euler Circuits, Hamiltonian Graphs.

TOTAL HOURS: 45

COURSE OUTCOMES:

on successful completion of the course, students able to		Pos
CO1	Apply mathematical logic to solve problems.	**PO1,PO2,PO3**
CO2	Understand the concepts and perform the operations related to sets, relations andfunctions. Gain the conceptual background needed and identify structures of algebraic nature.	**PO1,PO2,PO3**
CO3	Apply basic counting techniques to solve combinatorial problems.	**PO1,PO2,PO3, PO4**
CO4	Formulate problems and solve recurrence relations.	**PO1,PO2,PO3**
CO5	Apply Graph Theory in solving computer science problems	**PO1,PO2,PO3**

TEXT BOOKS:

1. J.P. Tremblay and R. Manohar, Discrete Mathematical Structures with Applications to Computer Science, Tata McGraw Hill, 2002.
2. Kenneth H. Rosen, Discrete Mathematics and its Applications with Combinatorics and Graph Theory, 7th Edition, McGraw Hill Education (India) Private Limited.

REFERENCE BOOKS:

1. Joe L. Mott, Abraham Kandel and Theodore P. Baker, Discrete Mathematics for Computer Scientists & Mathematicians, 2nd Edition, Pearson Education.
2. Narsingh Deo, Graph Theory with Applications to Engineering and Computer Science.

REFERENCE WEBSITE:

1. http://www.cs.yale.edu/homes/aspnes/classes/202/notes.pdf

CO-PO MAPPING

CO-PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO1	3	3	3	--	-	-	-	-	-	-	-	-
CO2	3	3		--	-	-	-	-	-	-	-	-
CO3	3	3	3	2	-	-	-	-	-	-	-	-
CO4	3	3	3	--	-	-	-	-	-	-	-	-
CO5	3	3	3		-	-	-	-	-	-	-	-
CO*	3	3	3	2	**-**	**-**	**-**	**-**	**-**	**-**	**-**	**-**

DISCRETE MATHEMATICS AND GRAPH THEORY.

UNIT-1

MATHEMATICAL LOGIC

INTRODUCTION

Logic is the science dealing with the methods of reasoning. Reasoning plays a very important role in every area of knowledge, particularly mathematics. logic expressed in such a language has come to be called 'symbolic logic' or 'mathematical logic'.

STATEMENTS AND NOTATION.

Statements ~~and~~ or Prepositions: A declarative sentence that is either true or false, but not both is called a statement or proposition.

Ex: The following sentences are statements.

① $3+4=5$ (F)

② 3 is a prime number (T)

③ New Delhi is in India (T).

The following are not statements.

① $x+y=4$

② Let me go.

③ What is your name?

Statements are of two types:

① Simple statement ② Compound statement.

Simple statement: Simple statement is a simple sentence and it can not contain any connectives.

Ex: ① $3+4=5$

② Delhi is in India.

Compound statement: The composition of two or more statements is called a compound statement.

Ex: ① $2+3=5$ and $4+5=1$.

② If $2+3=2$ then $4+5=10$.

Truth value: The truthfulness or falsity of a statement is called its truth value denoted by T(True) and F(False) respectively. If statement is true we will indicate its truth value by 1 and if it is false by the symbol 0.

CONNECTIVES

Connectives are used to form a compound statement from two or more sentences.

The connectives are negation ($\sim$ or $\neg$), conjunction ($\wedge$), disjunction ($\vee$), conditional or implication ($\rightarrow$), biconditional or bi-implication ($\leftrightarrow$).

① **Negation:** A statement which is obtained by inserting the word 'not' in appropriate place is called negation of the statement. It is denoted by '$\sim$ or $\neg$'.

If P is a statement, then negation P is the statement 'not P', denoted by $\sim P$ or $\neg P$.

Ex: P: 3 is a prime number

$\sim P$ or $\neg P$: 3 is not a prime number.

Truth table:

P	$\sim P$
T	F
F	T

P	$\sim P$
1	0
0	1

② **Conjunction:** If P and Q are any two statements then the conjunction of P and Q is denoted by $P \wedge Q$, which can be read as 'P and Q'. The truth value of $P \wedge Q$ is true only when P and Q both are true, otherwise it has the truth value F.

Truth table:

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

③ **Disjunction:** If P and Q are any two statements then the disjunction of P and Q is denoted by $P \vee Q$, which can be read as 'P or Q'. The truth value of $P \vee Q$ is F only when both P and Q have the truth value F, otherwise it has the truth value T.

Truth table:

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

④ **Conditional or Implication:** If P and Q are any two statements then conditional of P and Q is denoted by $P \rightarrow Q$, which can be read as 'P implies Q'. The truth value of $P \rightarrow Q$ is F only when P has the truth value T and Q has the truth value F, otherwise it has the truth value T.

Truth table:

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

⑤ Biconditional or Bi-implication: If P and Q are any two statements then the biconditional & bi-implication of P and Q is denoted by $P \leftrightarrow Q$ & $P \rightleftarrows Q$ which can be read as 'p if and only if Q'. The truth value of $P \leftrightarrow Q$ is T when both P and Q have the same truth value, otherwise it has truth value F.

Truth table:

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	F

Problems:

Q: Construct the truth table for the following statements.

① $P \wedge \sim P$.

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

② $(P \vee Q) \vee \sim P$.

P	Q	$P \vee Q$	$\sim P$	$(P \vee Q) \vee \sim P$
T	T	T	F	T
T	F	T	F	T
F	T	T	T	T
F	F	F	T	T

③ $P \wedge (\neg Q \wedge Q)$

P	Q	$\neg Q$	$\neg Q \wedge Q$	$P \wedge (\neg Q \wedge Q)$
T	T	F	F	F
T	F	T	F	F
F	T	F	F	F
F	F	T	F	F

④ $\neg(\neg P \vee \neg Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(\neg P \vee \neg Q)$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

⑤ ~(P∨(Q∧R)) ⇄ [(P∨Q)∧(P∨R)]

P	Q	R	Q∧R	P∨Q	P∨R	P∨(Q∧R)	~(P∨(Q∧R))	(P∨Q)∧(P∨R)	~(P∨(Q∧R)) ⇔ [(P∨Q)∧(P∨R)]
T	T	T	T	T	T	T	F	T	F
T	T	F	F	T	T	T	F	T	F
T	F	T	F	T	T	T	F	T	F
T	F	F	F	T	T	T	F	T	F
F	T	T	T	T	T	T	F	T	F
F	T	F	F	T	F	F	T	F	F
F	F	T	F	F	T	F	T	F	F
F	F	F	F	F	F	F	T	F	F.

H/W ① ¬(P∨Q)∨(¬P∧¬Q) ② ¬P∨(Q∨¬R)

③ (P→Q)⟺(~P∨Q) ④ (P⟺Q) ⇄ [(P∧Q)∨(¬P∧¬Q)]

⑤ ¬(¬P∨¬Q)∨(P→Q).

Answers of H/W:

① ¬(P∨Q)∨(¬P∧¬Q).

P	Q	¬P	¬Q	P∨Q	¬(P∨Q)	¬P∧¬Q	¬(P∨Q)∨(¬P∧¬Q)
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

② ¬P∨(Q∨¬R)

P	Q	R	¬P	¬R	Q∨¬R	¬P∨(Q∨¬R)
T	T	T	F	F	T	T
T	T	F	F	T	T	T
T	F	T	F	F	F	F
T	F	F	F	T	T	T
F	T	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	F	F	T
F	F	F	T	T	T	T

③ $(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$.

P	Q	$\sim P$	$P \rightarrow Q$	$\sim P \vee Q$	$(P \rightarrow Q) \leftrightarrow (\sim P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

$(P \leftrightarrow Q) \leftrightarrow (\sim P \vee Q)$
T
T
F
T

$\neg(\neg P \vee \neg Q) \vee (P \leftrightarrow Q)$
T
F
F
T

④ $(P \leftrightarrow Q) \rightleftarrows [(P \wedge Q) \vee (\neg P \wedge \neg Q)]$

P	Q	$\neg P$	$\neg Q$	$P \leftrightarrow Q$	$P \wedge Q$	$\neg P \wedge \neg Q$	$(P \wedge Q) \vee (\neg P \wedge \neg Q)$	$(P \leftrightarrow Q) \rightleftarrows [(P \wedge Q) \vee (\neg P \wedge \neg Q)]$
T	T	F	F	T	T	F	T	T
T	F	F	T	F	F	F	F	T
F	T	T	F	F	F	F	F	T
F	F	T	T	T	F	T	T	T

⑤ $\neg(\neg P \vee \neg Q) \vee (P \rightarrow Q)$

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$\neg(\neg P \vee \neg Q)$	$P \rightarrow Q$	$\neg(\neg P \vee \neg Q) \vee (P \rightarrow Q)$
T	T	F	F	F	T	T	T
T	F	F	T	T	F	F	F
F	T	T	F	T	F	T	T
F	F	T	T	T	F	T	T

Converse: let P and Q be any two statements. If $P \rightarrow Q$ is conditional statement then the converse of $P \rightarrow Q$ is $Q \rightarrow P$.

Inverse: Let P and Q be any two statements. If $P \rightarrow Q$ is conditional statement then the inverse of $P \rightarrow Q$ is $\neg P \rightarrow \neg Q$.

Contra-positive: Let P and Q be any two statements. If $P \rightarrow Q$ is conditional statement then the contra-positive of $P \rightarrow Q$ is $\neg Q \rightarrow \neg P$.

Ex: $P \rightarrow Q$: If the calculator is working then the battery is ~~not~~ good.

let P: The calculator is working.
Q: The battery is good.

Converse $(Q \rightarrow P)$: If the battery is good then the calculator is working.

Inverse $(\neg P \rightarrow \neg Q)$: If the calculator is not working then the battery is not good.

Contrapositive $(\neg Q \rightarrow \neg P)$: If the battery is not good then the calculator is not working.

Other connectives:

① XOR (or) Exclusive OR: let P and Q be two statements. The XOR of P and Q denoted by $P \oplus Q$, is the statement that is true when exactly one of P and Q is true but not both and is false otherwise.

Truth table:

P	Q	$P \oplus Q$
T	T	F
T	F	T
F	T	T
F	F	F

② NAND: NAND is a combination of not and AND. It is denoted by the symbol $\uparrow$.

let P and Q be two statements. The truth values of $P \uparrow Q$ is false when both P and Q have the truth value T, otherwise it has the truth value T.

Truth table:

P	Q	$P \uparrow Q$
T	T	F
T	F	T
F	T	T
F	F	T

③ NOR: NOR is the combination of Not and OR. It is denoted by $\downarrow$.

Let P and Q be two statements. The truth value of $P \downarrow Q$ is T when both P and Q have the truth value F, otherwise it has the truth value F.

Truth table:

P	Q	$P \downarrow Q$
T	T	F
T	F	F
F	T	F
F	F	T

WELL-FORMED FORMULAE

A statement involving connectives in symbolic form and consisting of variables, paranthesis in a structured and meaningful way is called a well-formed formulae (w.f.f.).

A w.f.f. can be generated by the following rules:

1. All variables and constants are w.f.f.
2. If P is a w.f.f., then $\neg P$ is also a w.f.f.
3. If P and Q are w.f.f then $P \wedge Q$, $P \vee Q$, $P \rightarrow Q$, $P \leftrightarrow Q$ are all w.f.f

Ex: $(P \rightarrow Q) \vee (Q \vee R)$, $\sim P \rightarrow (Q \vee R)$.

TAUTOLOGY

A compound statement which is true for all possible truth values of its components is called a tautology. A tautology is denoted by T_0.

Ex: $P \vee \sim P$ is a tautology.

P	$\sim P$	$P \vee \sim P$
T	F	T
F	T	T

Contradiction: A compound statement which is false for all possible truth values of its components is called a contradiction. A contradiction is denoted by F_0.

Ex: $P \wedge \sim P$ is a contradiction.

P	$\sim P$	$P \wedge \sim P$
T	F	F
F	T	F

DUALITY LAW

Let U be the compound statement that contains the connectives $\wedge$ and $\vee$. Suppose we replace each occurence of $\wedge$ by $\vee$ and $\vee$ by $\wedge$. Also, if U contains T_0 and F_0 as its components, suppose we replace each occurence of T_0 by F_0 and F_0 by T_0. Then the resulting compound statement is called the 'dual' of U and is denoted by U^d.

Ex: $U = P \wedge (Q \vee \neg R) \vee (S \wedge T_0)$

$U^d = P \vee (Q \wedge \neg R) \wedge (S \vee F_0)$

Q: Write the duals of the following propositions.

1. $\neg(P \vee Q) \wedge [P \vee \neg(Q \wedge \neg S)]$

   $\neg(P \wedge Q) \vee [P \wedge \neg(Q \vee \neg S)]$

2. $[(P \wedge F_0) \vee (Q \wedge T_0)] \wedge [(R \vee S) \vee F_0]$

   $[(P \vee T_0) \wedge (Q \vee F_0)] \vee [(R \wedge S) \vee T_0]$

EQUIVALENCE, IMPLICATION

Equivalence Implications: The compound statement P and Q are said to be logically equivalent whenever P and Q have same truth value and it is denoted by $P \Leftrightarrow Q$ (or) $P \equiv Q$.

Ex: $P \rightarrow Q$ is logically equivalent to $\sim P \vee Q$.

P	Q	$P \to Q$	$\sim P$	$\sim P \vee Q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

└ both are same ┘

$\therefore P \to Q \Leftrightarrow \sim P \vee Q$.

Laws of logic:

1. $P \wedge P \Leftrightarrow P$, $P \vee P \Leftrightarrow P$ – Idempotent law.
2. $P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$, $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$ – Associative law.
3. $P \wedge T \Leftrightarrow P$, $P \vee F \Leftrightarrow P$ – Identity law.
4. $P \wedge \sim P \Leftrightarrow F$, $P \vee \sim P \Leftrightarrow T$. – Inverse law.
5. $P \wedge Q \Leftrightarrow Q \wedge P$, $P \vee Q \Leftrightarrow Q \vee P$ – commutative law.
6. $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$, $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$ – Distributive law.
7. $\sim (P \wedge Q) \Leftrightarrow \sim P \vee \sim Q$, $\sim (P \vee Q) \Leftrightarrow \sim P \wedge \sim Q$ – De-morgan's law.
8. $P \vee (P \wedge Q) \Leftrightarrow P$, $P \wedge (P \vee Q) \Leftrightarrow P$ – Absorption law
9. $P \vee T \Leftrightarrow T$, $P \wedge F \Leftrightarrow F$
10. $\sim (\sim P) \Leftrightarrow P$.

connective $\left\{ \begin{array}{l} \to \text{ implies} \\ \leftrightarrow \text{ if and only if} \end{array} \right.$

not connective $\left\{ \begin{array}{l} \Rightarrow \text{ tautologically implies} \\ \Leftrightarrow \text{ logically equivalent} \end{array} \right.$

Tautological Implications:

A statement A is said to be tautologically imply a statement B if and only if $A \to B$ is a tautology. It is denoted by $A \Rightarrow B$, which is read as A tautologically implies B. Here $\Rightarrow$ is not a connective.

1. $P \wedge Q \Rightarrow P$
2. $P \wedge Q \Rightarrow Q$
3. $P \Rightarrow P \vee Q$
4. $\neg P \Rightarrow P \to Q$
5. $Q \Rightarrow P \to Q$
6. $\neg (P \to Q) \Rightarrow P$
7. $\neg (P \to Q) \Rightarrow \neg Q$
8. $P \wedge (P \to Q) \Rightarrow Q$
9. $\neg Q \wedge (P \to Q) \Rightarrow \neg P$
10. $\neg P \wedge (P \wedge Q) \Rightarrow Q$
11. $(P \to Q) \wedge (Q \to R) \Rightarrow P \to R$
12. $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$.

Problems:

1. Write the dual of $P \to Q$.

W.K.T., $P \to Q \Leftrightarrow \sim P \vee Q$

$U : P \to Q \Leftrightarrow U : \sim P \vee Q$

$U^d : \sim P \wedge Q$.

② Show that $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are logically equivalent by using truth tables.

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

both are same

$\therefore$ $P \vee (Q \wedge R)$ and $(P \vee Q) \wedge (P \vee R)$ are logically equivalent.

(or)

$P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$.

③ Show that $\neg(P \vee (\neg P \wedge Q))$ and $\neg P \wedge \neg Q$ are logically equivalent without using truth tables.

Sol: $\neg(P \vee (\neg P \wedge Q)) \Leftrightarrow \neg[(P \vee \neg P) \wedge (P \vee Q)]$ (By distributive law)

$\Leftrightarrow \neg[T \wedge (P \vee Q)]$ (By Inverse law)

$\Leftrightarrow \neg[P \vee Q]$ (By Identity law)

$\therefore \neg(P \vee (\neg P \wedge Q)) \Leftrightarrow \neg P \wedge \neg Q$. (By De-morgan's law.

④ Prove that $[(P \to R) \wedge (Q \to R)] \to [(P \vee Q) \to R]$ is a tautology.

P	Q	R	$P \to R$	$Q \to R$	$(P \to R) \wedge (Q \to R)$	$P \vee Q$	$(P \vee Q) \to R$	$[(P \to R) \wedge (Q \to R)] \to [(P \vee Q) \to R]$
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	T	F	T
T	F	T	T	T	T	T	T	T
T	F	F	F	T	F	T	F	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	F	T	F	T
F	F	T	T	T	T	F	T	T
F	F	F	T	T	T	F	T	T

$\therefore [(P \to R) \wedge (Q \to R)] \to [(P \vee Q) \to R]$ is a tautology.

H/W

① Prove that $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$ is a tautology.

② " " $[(P \to Q) \to R] \to [(\neg P \vee Q) \to R]$ is a tautology.

③ Prove the following logical equivalences using truth tables.

(i) $P \vee [P \wedge (P \vee Q)] \Leftrightarrow P$

(ii) $[(P \vee Q) \wedge (P \vee \neg Q)] \vee Q \Leftrightarrow P \vee Q$.

Answers of H/w:

① $[(P\rightarrow Q)\wedge(Q\rightarrow R)]\rightarrow(P\rightarrow R)$

P	Q	R	$P\rightarrow Q$	$Q\rightarrow R$	$(P\rightarrow Q)\wedge(Q\rightarrow R)$	$P\rightarrow R$	$[(P\rightarrow Q)\wedge(Q\rightarrow R)]\rightarrow(P\rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

∴ $[(P\rightarrow Q)\wedge(Q\rightarrow R)]\rightarrow(P\rightarrow R)$ is a tautology.

② $[(P\rightarrow Q)\rightarrow R]\rightarrow[(\sim P\vee Q)\rightarrow R]$

P	Q	R	$P\rightarrow Q$	$(P\rightarrow Q)\rightarrow R$	$\sim P$	$\sim P\vee Q$	$(\sim P\vee Q)\rightarrow R$	$[(P\rightarrow Q)\rightarrow R]\rightarrow[(\sim P\vee Q)\rightarrow R]$
T	T	T	T	T	F	T	T	T
T	T	F	T	F	F	T	F	T
T	F	T	F	T	F	F	T	T
T	F	F	F	T	F	F	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	F	T	T	F	T
F	F	T	T	T	T	T	T	T
F	F	F	T	F	T	T	F	T

∴ $[(P\rightarrow Q)\rightarrow R]\rightarrow[(\sim P\vee Q)\rightarrow R]$ is a tautology.

③ (i) $P\vee[P\wedge(P\vee Q)]\Leftrightarrow P$

P	Q	$P\vee Q$	$P\wedge(P\vee Q)$	$P\vee(P\wedge(P\vee Q))$
T	T	T	T	T
T	F	T	T	T
F	T	T	F	F
F	F	F	F	F

∴ $P\vee[P\wedge(P\vee Q)]\Leftrightarrow P$

④ (ii) $[(P\vee Q)\wedge(P\vee\neg Q)]\vee Q\Leftrightarrow P\vee Q$

P	Q	$P\vee Q$	$\neg Q$	$P\vee\neg Q$	$(P\vee Q)\wedge(P\vee\neg Q)$	$[(P\vee Q)\wedge(P\vee\neg Q)]\vee Q$
T	T	T	F	T	T	T
T	F	T	T	T	T	T
F	T	T	F	F	F	T
F	F	F	T	T	F	F

∴ $[(P\vee Q)\wedge(P\vee\neg Q)]\vee Q\Leftrightarrow P\vee Q$.

NORMAL FORMS

If there are more variables it is practically not possible, which is why we consider normal forms

Elementary product: A product of the variables and their negations in a formula is called an elementary product.

here sum – $\vee$
product – $\wedge$

Ex: $P, \neg P \wedge Q, \neg Q \wedge P \wedge \neg P, Q \wedge \neg P$, etc.

Elementary sum: A sum of the variables and their negations is called an elementary sum.

Ex: $P, \neg P \vee Q, \neg Q \vee P \vee \neg P, Q \vee \neg P$, etc.

→ A necessary and sufficient condition for an elementary product to be identically false is that it contain at least one pair of factors in which one is the negation of the other.

Ex: $\neg Q \wedge P \wedge \neg P$, $(\neg Q \wedge P) \wedge Q$.

→ A necessary and sufficient condition for an elementary sum to be identically true is that it contain at least one pair of factors in which one is the negation of the other.

Ex: $\neg Q \vee P \vee \neg P$, $\neg Q \vee P \vee Q$. not required

Disjunctive normal form: A formula which is equivalent to the given formula and which consists of sum of elementary products is called a disjunctive normal form (DNF) of the given formula.

~~Ex:~~ Ex: $(P \wedge R) \vee (\neg P \wedge Q) \vee (Q \wedge R \wedge \neg P)$

Q: obtain DNFs of a) $P \wedge (P \rightarrow Q)$ b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$.

Sol: a) $P \wedge (P \rightarrow Q)$ ∵ $A \rightarrow B \Leftrightarrow \neg A \vee B$

$\Leftrightarrow P \wedge (\neg P \vee Q)$ $x(y+z) = xy + xz$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$ (By distributive law).

b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$ ∵ $A \rightleftarrows B \Leftrightarrow (A \wedge B) \vee (\neg A \wedge \neg B)$

$\Leftrightarrow [\neg(P \vee Q) \wedge (P \wedge Q)] \vee [\neg(\neg(P \vee Q)) \wedge \neg(P \wedge Q)]$

Brackets can be removed bcoz $\wedge$ is associative

$\Leftrightarrow [\neg P \wedge \neg Q \wedge P \wedge Q] \vee [(P \vee Q) \wedge (\neg P \vee \neg Q)]$ (By De-morgan's law)

$\Leftrightarrow [\neg P \wedge \neg Q \wedge P \wedge Q] \vee$ ~~$(P \wedge \neg P) \vee (P \wedge \neg Q) \vee$~~ $[P \wedge (\neg P \vee \neg Q)] \vee [Q \wedge (\neg P \vee \neg Q)]$.

$\Leftrightarrow (\neg P \wedge \neg Q \wedge P \wedge Q) \vee (P \wedge \neg P) \vee (P \wedge \neg Q) \vee (Q \wedge \neg P) \vee (Q \wedge \neg Q)$. (By distributive law)

which is the required disjunctive normal form.

Conjunctive Normal Forms: A formula which is equivalent to a given formula and which consists of a product of elementary sums is called conjunctive normal form (CNF) of the given formula.

Ex: $(Q \vee P \vee R \vee \neg P) \wedge (Q \vee P) \wedge (R \vee \neg Q)$.

Q: obtain the CNFs of the following:

a) $P \wedge (P \to Q)$ b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$.

a) $P \wedge (P \to Q)$ $\because A \to B \Leftrightarrow \neg A \vee B$

$\Leftrightarrow P \wedge (\neg P \vee Q)$

b) $\neg(P \vee Q) \rightleftarrows (P \wedge Q)$ $\because A \rightleftarrows B \Leftrightarrow (A \to B) \wedge (B \to A)$

$\Leftrightarrow [\neg(P \vee Q) \to (P \wedge Q)] \wedge [(P \wedge Q) \to \neg(P \vee Q)]$ $\because A \to B \Leftrightarrow \neg A \vee B$

$\Leftrightarrow [\neg(\neg(P \vee Q)) \vee (P \wedge Q)] \wedge [\neg(P \wedge Q) \vee \neg(P \vee Q)]$ (By De-morgan law)

$\Leftrightarrow [(P \vee Q) \vee (P \wedge Q)] \wedge [(\neg P \vee \neg Q) \vee (\neg P \wedge \neg Q)]$ (By distributive law)

Brackets can be removed bcoz $\vee$ is associative.

$\Leftrightarrow (P \vee Q \vee P) \wedge (P \vee Q \vee Q) \wedge (\neg P \vee \neg Q \vee \neg P) \wedge (\neg P \vee \neg Q \vee \neg Q)$

is the required Conjunctive Normal Form.

Q: Find the DNF and CNF of $(\neg P \to R) \wedge (P \to Q)$

To find DNF:

$(\neg P \to R) \wedge (P \to Q) \Leftrightarrow [\neg(\neg P) \vee R] \wedge [\neg P \vee Q]$ $\because A \to B \Leftrightarrow \neg A \vee B$

$\Leftrightarrow [P \vee R] \wedge [\neg P \vee Q]$

$\Leftrightarrow [P \wedge (\neg P \vee Q)] \vee [R \wedge (\neg P \vee Q)]$

(By distributive law) $\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q) \vee (R \wedge \neg P) \vee (R \wedge Q)$.

To find CNF:

$(\neg P \to R) \wedge (P \to Q) \Leftrightarrow [\neg(\neg P) \vee R] \wedge [\neg P \vee Q]$

$\Leftrightarrow [P \vee R] \wedge [\neg P \vee Q]$.

~~Principal~~

~~Minterms: let P and Q be two statements~~. ~~variables~~.

Minterms: A minterm is a product of all the variables in the function, where each variable appears exactly once either in its true or negated form. For n variables there are 2^n minterms.

For two variables P and Q, there are 2^2 such formulas given by $P\wedge Q$, $P\wedge\neg Q$, $\neg P\wedge Q$ and $\neg P\wedge\neg Q$. These formulas are called minterms or Boolean conjunctions of P and Q.

Truth table:

P	Q	$P\wedge Q$	$P\wedge\neg Q$	$\neg P\wedge Q$	$\neg P\wedge\neg Q$
T	T	T	F	F	F
T	F	F	T	F	F
F	T	F	F	T	F
F	F	F	F	F	T

Principal Disjunctive Normal Forms:

For a given formula, an equivalent formula consisting of disjunctions of minterms (sum of minterms) only is known as its principal disjunctive normal form.

Such a normal form is also called as the sum of products canonical form.

Note: Min terms for 3 variables P, Q and R are $P\wedge Q\wedge R$, $P\wedge Q\wedge\neg R$, $P\wedge\neg Q\wedge R$, $P\wedge\neg Q\wedge\neg R$, $\neg P\wedge Q\wedge R$, $\neg P\wedge Q\wedge\neg R$, $\neg P\wedge\neg Q\wedge\neg R$, $\neg P\wedge\neg Q\wedge\neg R$.

Q: Obtain the PDNF of the following.

① $P\to Q$

P	Q	$P\to Q$
T	T	T
T	F	F
F	T	T
F	F	T

From truth table: $(P\wedge Q)\vee(\neg P\wedge Q)\vee(\neg P\wedge\neg Q)$

Without truth table:

$P\to Q \Leftrightarrow \neg P\vee Q$

$\Leftrightarrow (\neg P\wedge T)\vee(Q\wedge T)$

$\Leftrightarrow (\neg P\wedge(Q\vee\neg Q))\vee(Q\wedge(P\vee\neg P))$

$\Leftrightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(Q\wedge P)\vee(Q\wedge\neg P)$

$\Leftrightarrow (\neg P\wedge Q)\vee(\neg P\wedge\neg Q)\vee(P\wedge Q)$

→ If a formula is a tautology, then obviously all the minterms appear in its PDNF.

→ Equivalent forms have identical PDNFs.

→ Every formula which is not a contradiction has an equivalent PDNF which is unique except for the rearrangement of the factor in minterms as well as their disjunctions.

(2) $P \vee Q$.

$P \vee Q \Leftrightarrow (P \wedge T) \vee (Q \wedge T)$

$\Leftrightarrow [P \wedge (Q \vee \neg Q)] \vee [Q \wedge (P \vee \neg P)]$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (Q \wedge P) \vee (Q \wedge \neg P)$

$\Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q) \vee (\neg P \vee Q)$.

(3) $\neg (P \wedge Q)$

$\neg (P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

$\Leftrightarrow [\neg P \wedge (Q \vee \neg Q)] \vee [\neg Q \wedge (P \vee \neg P)]$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (\neg Q \wedge P) \vee (\neg Q \wedge \neg P)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (\neg Q \wedge P)$.

(4) $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge T) \vee (\neg P \wedge R \wedge T) \vee (Q \wedge R \wedge T)$

$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge R \wedge (Q \vee \neg Q)) \vee (Q \wedge R \wedge (P \vee \neg P))$.

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge R \wedge Q) \vee (\neg P \wedge R \wedge \neg Q)$

$\vee (Q \wedge R \wedge P) \vee (Q \wedge R \wedge \neg P)$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R)$

(5) $P \rightarrow [(P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P)]$

$\Leftrightarrow \neg P \vee [(P \rightarrow Q) \wedge \neg (\neg Q \vee \neg P)]$

$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge \neg (\neg Q \vee \neg P)]$

$\Leftrightarrow \neg P \vee [(\neg P \vee Q) \wedge (Q \wedge P)]$

$\Leftrightarrow \neg P \vee (\underbrace{\neg P \wedge (Q \wedge P)}_{F}) \vee (Q \wedge (Q \wedge P))$

$\Leftrightarrow \neg P \vee (Q \wedge P)$

$\Leftrightarrow [\neg P \wedge (Q \vee \neg Q)] \vee (P \wedge Q)$

$\Leftrightarrow (\neg P \wedge Q) \vee (\neg P \wedge \neg Q) \vee (P \wedge Q)$.

PDNF	PCNF
$\wedge T$	$\vee F$
$P \vee \neg P$	$P \wedge \neg P$

Max terms: A maxterm is a sum of all the variables in the function, where each variable appears exactly once either in its true or negated form. For n variables there are 2^n maxterms.

For two variables P and Q, there are 2^2 such formulas given by $P \vee Q$, $P \vee \neg Q$, $\neg P \vee Q$ and $\neg P \vee \neg Q$. These formulas are called maxterms or Boolean disjunctions of P and Q.

Truth table:

P	Q	$P \vee Q$	$P \vee \neg Q$	$\neg P \vee Q$	$\neg P \vee \neg Q$
T	T	T	T	T	F
T	F	T	T	F	T
F	T	T	F	T	T
F	F	F	T	T	T

Note: Max terms for 3 variables P, Q and R are
$P \vee Q \vee R$, $P \vee Q \vee \neg R$, $P \vee \neg Q \vee R$, $P \vee \neg Q \vee \neg R$, $\neg P \vee Q \vee R$, $\neg P \vee Q \vee \neg R$,
$\neg P \vee \neg Q \vee R$, $\neg P \vee \neg Q \vee \neg R$.

Principal Conjunctive Normal Form:

For a given formula, an equivalent formula consisting of conjunctions of the maxterms only is known as its principal disjunctive normal form. This normal form is also called the product of sums canonical form.

-> If a formula is contradiction, then obviously all the maxterms appear in its PCNF.

-> Equivalent forms have identical PDNFs

-> Every formula which is not a tautology has an equivalent PCNF which is unique except for the rearrangement of the factors in the maxterms as well as in their conjunctions.

Q: Obtain the PCNF of the following.

① $P \wedge Q$

$P \wedge Q \Leftrightarrow (P \vee F) \wedge (Q \vee F)$

$\Leftrightarrow (P \vee (Q \wedge \neg Q)) \wedge [Q \vee (P \wedge \neg P)]$

$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (Q \vee P) \wedge (Q \vee \neg P)$

$\Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q) \wedge (\neg P \vee Q)$

② $Q \wedge (P \vee \neg Q)$

$Q \wedge (P \vee \neg Q) \Leftrightarrow (Q \vee F) \wedge (P \vee \neg Q)$

$\Leftrightarrow [Q \vee (P \wedge \neg P)] \wedge [P \vee \neg Q]$

$\Leftrightarrow (Q \vee P) \wedge (Q \vee \neg P) \wedge (P \vee \neg Q)$

$\Leftrightarrow (P \vee Q) \wedge (\neg P \vee Q) \wedge (P \vee \neg Q)$

Equivalent

③ $(\neg P \to R) \wedge (Q \leftrightarrow P)$

$(\neg P \to R) \wedge (Q \leftrightarrow P)$

$\Leftrightarrow [\neg(\neg P) \vee R] \wedge [(Q \to P) \wedge (P \to Q)]$

$\Leftrightarrow (P \vee R) \wedge (\neg Q \vee P) \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \vee R \vee F) \wedge (\neg Q \vee P \vee F) \wedge (\neg P \vee Q \vee F)$

$\Leftrightarrow (P \vee R \vee (Q \wedge \neg Q)) \wedge (\neg Q \vee P \vee (R \wedge \neg R)) \wedge (\neg P \vee Q \vee (R \wedge \neg R))$

$\Leftrightarrow (P \vee R \vee Q) \wedge (P \vee R \vee \neg Q) \wedge (\neg Q \vee P \vee R) \wedge (\neg Q \vee P \vee \neg R)$
$\wedge (\neg P \vee Q \vee R) \wedge (\neg P \vee Q \vee \neg R)$.

$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R) \wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$\wedge (\neg P \vee Q \vee \neg R)$.

Suppose $S : (\neg P \to R) \wedge (Q \leftrightarrow P)$.

Note: The CNF of $\neg S$ can be easily obtained by writing the conjunction of remaining max terms, thus $\neg S$ has the PCNF

$\neg S \Leftrightarrow (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$.

By considering $\neg(\neg S)$ we obtain.

$\neg(\neg S) \Leftrightarrow \neg(P \vee Q \vee \neg R) \vee \neg(\neg P \vee \neg Q \vee R) \vee \neg(\neg P \vee \neg Q \vee \neg R)$

$S \Leftrightarrow (\neg P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (P \wedge Q \wedge R)$

which is the PDNF of S.

Q④ Obtain PCNF of $A = (P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$ by constructing PDNF.

Sol: $A \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (Q \wedge R)$

$\Leftrightarrow (P \wedge Q \wedge T) \vee (\neg P \wedge Q \wedge T) \vee (Q \wedge R \wedge T)$

$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge Q \wedge (R \vee \neg R)) \vee (Q \wedge R \wedge (P \vee \neg P))$

$\Leftrightarrow (P \wedge Q \wedge (R \vee \neg R)) \vee (\neg P \wedge Q \wedge (R \vee \neg R))$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$
$\wedge (Q \wedge R \wedge P) \wedge (Q \wedge R \wedge \neg P)$

$\Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge \neg R)$.

is the PDNF of A.

The PDNF of $\neg A$ is given by

$\neg A \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R)$
$\vee (\neg P \wedge \neg Q \wedge \neg R)$.

Taking negation on both sides.

$\neg(\neg A) \Leftrightarrow \neg(P \wedge \neg Q \wedge R) \wedge \neg(\neg P \wedge \neg Q \wedge R) \wedge \neg(P \wedge \neg Q \wedge \neg R)$
$\wedge \neg(\neg P \wedge \neg Q \wedge \neg R)$

$A \Leftrightarrow (\neg P \vee Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$\wedge (P \vee Q \vee R)$.

which is the required PCNF.

⑤ Obtain the PCNF of $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

Sol: Let $S = (\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

$S \Leftrightarrow \neg(\neg P \vee \neg Q) \vee (P \leftrightarrow \neg Q)$

$\Leftrightarrow (P \wedge Q) \vee [(P \rightarrow \neg Q) \wedge (\neg Q \rightarrow P)]$

$S \Leftrightarrow (P \wedge Q) \vee [(\neg P \vee \neg Q) \wedge (Q \vee P)]$

Let $A = P \wedge Q$, $B = \neg P \vee \neg Q$, $C = Q \vee P$.

$\therefore S \Leftrightarrow A \vee (B \wedge C)$

$\Leftrightarrow (A \vee B) \wedge (A \vee C)$

1st term : $A \vee B = (P \wedge Q) \vee (\neg P \vee \neg Q)$

$\Leftrightarrow [(P \wedge Q) \vee \neg P] \vee [(P \wedge Q) \vee \neg Q]$

$\Leftrightarrow [(P \vee \neg P) \wedge (Q \vee \neg P)] \vee [(P \vee \neg Q) \wedge (Q \vee \neg Q)]$

$\Leftrightarrow [T \wedge (\neg P \vee Q)] \vee [(P \vee \neg Q) \wedge T]$

$\Leftrightarrow (\neg P \vee Q) \vee (P \vee \neg Q)$

$\Leftrightarrow (\neg P \vee P) \vee (Q \vee \neg Q)$

$\Leftrightarrow T \vee T$

$\Leftrightarrow T.$

2nd term : $A \vee C = (P \wedge Q) \vee (Q \vee P)$

$\Leftrightarrow [(P \wedge Q) \vee Q] \vee [(P \wedge Q) \vee P]$ [Absorption law].

$\Leftrightarrow Q \vee P$

$\Leftrightarrow P \vee Q.$

$\therefore S \Leftrightarrow T \wedge (P \vee Q)$

$\Leftrightarrow P \vee Q.$

which is the required PCNF.

PDNF without using truth tables.

Min terms of P, Q and R.

P	Q	R	$P\wedge Q\wedge R$	$P\wedge Q\wedge\neg R$	$P\wedge\neg Q\wedge R$	$P\wedge\neg Q\wedge\neg R$	$\neg P\wedge Q\wedge R$	$\neg P\wedge Q\wedge\neg R$	$\neg P\wedge\neg Q\wedge R$	$\neg P\wedge\neg Q\wedge\neg R$
T	T	T	T	F	F	F	F	F	F	F
T	T	F	F	T	F	F	F	F	F	F
T	F	T	F	F	T	F	F	F	F	F
T	F	F	F	F	F	T	F	F	F	F
F	T	T	F	F	F	F	T	F	F	F
F	T	F	F	F	F	F	F	T	F	F
F	F	T	F	F	F	F	F	F	T	F
F	F	F	F	F	F	F	F	F	F	T

② $P \vee Q$.

$P \vee Q \Leftrightarrow (P \wedge Q) \vee (P \wedge \neg Q)$
$\vee (\neg P \wedge Q)$
is the required PDNF.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F.

③ $\neg (P \wedge Q)$

$\neg(P \wedge Q) \Leftrightarrow (P \wedge \neg Q) \vee (\neg P \wedge Q)$
$\vee (\neg P \wedge \neg Q)$.
is the required PDNF.

P	Q	$P \wedge Q$	$\neg(P \wedge Q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T.

④ $(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R)$.

P	Q	R	$\neg P$	$P \wedge Q$	$\neg P \wedge R$	$Q \wedge R$	$(P \wedge Q) \vee (\neg P \wedge R) \vee ($
T	T	T	F	T	F	T	T
T	T	F	F	T	F	F	T
T	F	T	F	F	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	F	T	T	T
F	T	F	T	F	F	F	F
F	F	T	T	F	T	F	T
F	F	F	T	F	F	F	F.

$(P \wedge Q) \vee (\neg P \wedge R) \vee (Q \wedge R) \Leftrightarrow (P \wedge Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q$
is the required PDNF.

⑤ $P \rightarrow [(P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)]$

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \vee \neg P$	$\neg(\neg Q \vee \neg P)$	$(P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)$	$P \rightarrow \{(P \rightarrow Q) \wedge \neg(\neg Q \vee \neg$
T	T	F	F	T	F	T	T	T
T	F	F	T	F	T	F	F	F
F	T	T	F	T	T	F	F	T
F	F	T	T	T	T	F	F	T

$P \rightarrow [(P \rightarrow Q) \wedge \neg(\neg Q \vee \neg P)] \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge Q) \vee (\neg P \wedge \neg Q)$
is the required PDNF.

To remember the order of min terms / max terms

$P \vee Q \vee R$	T	T	T	$P \wedge Q \wedge R$
$P \vee Q \vee \neg R$	T	T	F	$P \wedge Q \wedge \neg R$
$P \vee \neg Q \vee R$	T	F	T	$P \wedge \neg Q \wedge R$
"	"	"	"	"

~~The thing~~ The variables initially considered are atomic variables

PCNF using truth tables:

Maxterms of P, Q and R.

P	Q	R	$P \vee Q \vee R$	$P \vee Q \vee \neg R$	$P \vee \neg Q \vee R$	$P \vee \neg Q \vee \neg R$	$\neg P \vee Q \vee R$	$\neg P \vee Q \vee \neg R$	$\neg P \vee \neg Q \vee R$	$\neg P \vee \neg Q \vee \neg R$
T	T	T	T	T	T	T	T	T	T	F
T	T	F	T	T	T	T	T	T	F	T
T	F	T	T	T	T	T	T	F	T	T
T	F	F	T	T	T	T	F	T	T	T
F	T	T	T	T	T	F	T	T	T	T
F	T	F	T	T	F	T	T	T	T	T
F	F	T	T	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T	T	T	T

① $P \wedge Q$.

$P \wedge Q \Leftrightarrow (P \vee Q) \wedge (P \vee \neg Q)$
$\wedge (\neg P \vee Q)$

is the required PCNF

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

② $Q \wedge (P \vee \neg Q)$

$Q \wedge (P \vee \neg Q) \Leftrightarrow (P \vee Q)$
$\wedge (P \vee \neg Q) \wedge (\neg P \vee Q)$

is the required PCNF.

P	Q	$\neg Q$	$P \vee \neg Q$	$Q \wedge (P \vee \neg Q)$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	F
F	F	T	T	F

③ $(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$.

$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$
$\Leftrightarrow (P \vee Q \vee R) \wedge (P \vee \neg Q \vee R)$
$\wedge (P \vee \neg Q \vee \neg R) \wedge (\neg P \vee Q \vee R)$
$\wedge (\neg P \vee Q \vee \neg R)$.

is the required PCNF.

P	Q	R	$\neg P$	$\neg P \rightarrow R$	$Q \leftrightarrow P$	$(\neg P \rightarrow R) \wedge (Q \leftrightarrow P)$
T	T	T	F	T	T	T
T	T	F	F	T	T	T
T	F	T	F	T	F	F
T	F	F	F	T	F	F
F	T	T	T	T	F	F
F	T	F	T	F	F	F
F	F	T	T	T	T	T
F	F	F	T	F	T	F

④ in PDNF 4

⑤ $(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$.

$(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$

$\Leftrightarrow P \vee Q$

is the required PCNF.

P	Q	$\neg P$	$\neg Q$	$\neg P \vee \neg Q$	$P \leftrightarrow \neg Q$	$(\neg P \vee \neg Q) \rightarrow (P \leftrightarrow \neg Q)$
T	T	F	F	F	F	T
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	T	F	F

FUNCTIONALLY COMPLETE SET OF CONNECTIVES

Any set of connectives in which every formula can be expressed in terms of an equivalent formula containing the connectives from this set is called a functionally complete set of connectives.

Ex:

(OR)

The set of logical connectives from which any Boolean expression can be constructed.

Ex: 1. $\{\neg, \vee\}$

$$P \wedge Q \Leftrightarrow \neg(\neg P \vee \neg Q)$$

2. $\{\neg, \wedge\}$

$$P \vee Q \Leftrightarrow \neg(\neg P \wedge \neg Q)$$

3. $\{\neg, \oplus$ (XOR)$\}$

4. $\{$(NAND) $\uparrow\}$ alone

Also known as Sheffer stroke.

$$P \uparrow P \Leftrightarrow \neg P$$

5. $\{$(NOR) $\downarrow\}$ alone

Also called the Peirce arrow.

$$P \downarrow P \Leftrightarrow \neg P$$

Q ① Write an equivalent formula for $P \wedge [(Q \rightleftarrows R) \vee (R \rightleftarrows P)]$ which does not contain the biconditional.

$$P \wedge [(Q \rightleftarrows R) \vee (R \rightleftarrows P)]$$

$$\Leftrightarrow P \wedge [[(Q \to R) \wedge (R \to Q)] \vee [(R \to P) \wedge (P \to R)]]$$

② Write an equivalent formula for $P \wedge (Q \rightleftarrows R)$ which contains neither the biconditional nor the conditional.

$$P \wedge (Q \rightleftarrows R) \Leftrightarrow P \wedge [(Q \to R) \wedge (R \to Q)]$$

$$\Leftrightarrow P \wedge [(\neg Q \vee R) \wedge (\neg R \vee Q)]$$

INFERENCE THEORY OF STATEMENT CALCULUS

Argument: The argument is a finite sequence of propositions with the final proposition (statement) being the conclusion, and the preceding statements being the premises.

Premises: Statements or propositions that are assumed to be true.

Conclusion: A statement that logically follows from the premises.

Ex:

P1 : If it rains, the ground will be wet.

P2 : It is raining.

C : The ground will be wet.

Symolic form

P1 : $P \to Q$.

P2 : P

C : Q.

$\}\to$ Modus Ponens

Validity of the argument:

An argument is said to be valid iff (if and only if) the ~~conclusion of premises~~ conjunction of premises implies the conclusion.

For a set of Premises, $H_1, H_2, \ldots, H_n$ follows $H_1 \wedge H_2 \wedge H_3 \wedge \ldots \wedge H_n \rightarrow C$ is a tautology then the given argument is valid otherwise it is invalid.

The rules of inference is a criteria for determining the validity of an argument. (Validity is also proved using truth tables).

Rules of inference:

Any conclusion which is arrived by following these rules is called a valid conclusion and the argument is valid argument.

Rule P: A premises may be introduced at any point in the derivation.

Rule T: A formula may be introduced at any point in the derivation.

Rule CP: Rule CP is also called the deduction theorem and is generally used if the conclusion is of the form $R \rightarrow S$. If R is included as an additional premise and S is derived from the premises from $P \wedge R$, then $R \rightarrow S$ can be derived from the premises P alone.

Implications:

I_1 $P \wedge Q \Rightarrow P$ } (simplification)
I_2 $P \wedge Q \Rightarrow Q$

I_3 $P \Rightarrow P \vee Q$ } (addition)
I_4 $Q \Rightarrow P \vee Q$

I_5 $\neg P \Rightarrow P \rightarrow Q$

I_6 $Q \Rightarrow P \rightarrow Q$

I_7 $\neg(P \rightarrow Q) \Rightarrow P$

I_8 $\neg(P \rightarrow Q) \Rightarrow \neg Q$

I_9 $P, Q \Rightarrow P \wedge Q$ $\qquad P \wedge Q \Rightarrow P \wedge Q$

I_{10} $\neg P, P \vee Q \Rightarrow Q$ (disjunctive syllogism) $\qquad \neg P \wedge (P \vee Q) \Rightarrow Q$

I_{11} $P, P \rightarrow Q \Rightarrow Q$ (modus ponens).

I_{12} $\neg Q, P \rightarrow Q \Rightarrow \neg P$ (modus tollens)

I_{13} $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$ (hypothetical syllogism)

I_{14} $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$ (dilemma).

Equivalences:

E_1 $\neg\neg P \Leftrightarrow P$. (double negation)

E_2 $P \wedge Q \Leftrightarrow Q \wedge P$ } (commutative laws).

E_3 $P \vee Q \Leftrightarrow Q \vee P$

E_4 $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$ } (Associative laws)

E_5 $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

E_6 $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ } (Distributive laws)

E_7 $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

E_8 $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ } (De-morgan's laws)

E_9 $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$

E_{10} $P \vee P \Leftrightarrow P$

E_{11} $P \wedge P \Leftrightarrow P$

E_{12} $R \vee (P \wedge \neg P) \Leftrightarrow R$

E_{13} $R \wedge (P \vee \neg P) \Leftrightarrow R$

E_{14} $R \vee (P \vee \neg P) \Leftrightarrow T$

E_{15} $R \wedge (P \wedge \neg P) \Leftrightarrow F$

E_{16} $P \rightarrow Q \Leftrightarrow \neg P \vee Q$.

E_{17} $\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$.

E_{18} $P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$

E_{19} $P \rightarrow (Q \rightarrow R) \Leftrightarrow (P \wedge Q) \rightarrow R$

E_{20} $\neg(P \leftrightarrow Q) \Leftrightarrow P \rightleftarrows \neg Q$.

E_{21} $P \leftrightarrow Q \Leftrightarrow (P \rightarrow Q) \wedge (Q \rightarrow P)$

E_{22} $P \leftrightarrow Q \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$.

Q 1 show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \rightarrow R) \wedge (Q \rightarrow S)$.

Step		Derivation	Rule/Formula.
{1}	(1)	$P \vee Q$	Rule P.
{1}	(2)	$\neg P \rightarrow Q$	Rule T, (1), E_1, E_{16} $E_1: \neg\neg P \Leftrightarrow P$ $E_{16}: P \rightarrow Q \Leftrightarrow \neg P \vee Q$.
{3}	(3)	$Q \rightarrow S$	Rule P.
{1, 3}	(4)	$\neg P \rightarrow S$	Rule T, (2), (3), I_{13} $I_{13}: P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$.
{1, 3}	(5)	$\neg S \rightarrow P$	Rule T, (4), E_{18}, E_1 $E_{18}: P \rightarrow Q \Leftrightarrow \neg Q \rightarrow \neg P$ $E_1: \neg\neg P \Leftrightarrow P$.
{6}	(6)	$P \rightarrow R$	Rule P.

$\{1,3,6\}$	(7)	$\neg S \to R$	Rule T, (5), (6), I_{13} I_{13}: $P \to Q, Q \to R \Rightarrow P \to R$.
$\{1,3,6\}$	(8)	$S \vee R$	Rule T, (7), E_{16}, E_1 E_{16}: $P \to Q \Leftrightarrow \neg P \vee Q$ E_1: $\neg\neg P \Leftrightarrow P$.

② Show that $R \wedge (P \vee Q)$ is a valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\neg M$.

Step		Derivation	Rule / Formula.
$\{1\}$	(1)	$P \to M$	Rule P.
$\{2\}$	(2)	$\neg M$	Rule P.
$\{1,2\}$	(3)	$\neg P$	Rule T; (1), (2), I_{12} I_{12} is Modus Tollens. $\neg Q, P \to Q \Rightarrow \neg P$.
$\{4\}$	(4)	$P \vee Q$	Rule P.
$\{1,2,4\}$	(5)	Q	Rule T, (3), (4), I_{10} I_{10} is disjunctive syllogism $\neg P, P \vee Q \Rightarrow Q$.
$\{6\}$	(6)	$Q \to R$	Rule P.
$\{1,2,4,6\}$	(7)	R	Rule T, (5), (6), I_{11} I_{11} is modus ponens $P, P \to Q \Rightarrow Q$.
$\{1,2,4,6\}$	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7), I_9 I_9: $P, Q \Rightarrow P \wedge Q$.

(OR)

Step		Derivation	Rule / Formula
$\{1\}$	(1)	$P \vee Q$	Rule P.
$\{1\}$	(2)	$\neg P \to Q$	Rule T, (1), E_{16} E_{16}: $P \to Q \Leftrightarrow \neg P \vee Q$.
$\{3\}$	(3)	$Q \to R$	Rule P.
$\{1,3\}$	(4)	$\neg P \to R$	Rule T, (2), (3), I_{13} I_{13}: $P \to Q, Q \to R \Rightarrow P \to R$.
$\{1,3\}$	(5)	$\neg R \to P$	Rule T, (4), E_1, E_{18} E_1: $\neg\neg P \Leftrightarrow P$ E_{18}: $P \to Q \Leftrightarrow \neg Q \to \neg P$.
$\{6\}$	(6)	$P \to M$	Rule P
$\{1,3,6\}$	(7)	$\neg R \to M$	Rule T, (5), (6), I_{13} I_{13}: $P \to Q, Q \to R \Rightarrow P \to R$.

$\{1,3,6\}$	(8)	$\neg M \to R$	Rule T, (7), E_1, E_{18} $E_1: \neg\neg P \Leftrightarrow P$. $E_{18}: P \to Q \Leftrightarrow \neg Q \to \neg P$.
$\{9\}$	(9)	$\neg M$	Rule P
$\{1,3,6,9\}$	(10)	R	Rule T, (8), (9), I_{11} $I_{11}: P, P \to Q \Rightarrow Q$.
$\{1,3,6,9\}$	(11)	$R \wedge (P \vee Q)$	Rule T, (1), (10), I_{9} $I_9: P, Q \Rightarrow P \wedge Q$.

PREDICATE CALCULUS

Predicative logic: The logic based upon the analysis of predicates in any statement is called predicate logic.

Predicate: A predicate tells the nature of the objects.

In general, predicates are represented by capital letters and objects are represented by small letters

Ex: Dog is an animal
Cat is an animal
Rat is an animal

Here 'is an animal' is a predicate, which is denoted by A. Dog, Cat, Rat are objects which are denoted by d, c, r respectively, then the statements can be written as A(d), A(c), A(r).

n-placed predicate: A predicate which requires n objects to represent a statement function is called n-placed predicate (n>

Ex: 'is a man' is a predicate that needs only one object to represent a statement function, so it is called as one-placed predicate.

A predicate 'taller than' needs two objects, ~~so it is~~ to represent a statement function, so it is called a 2-placed predicate

Simple statement function: A simple statement function of one variable is defined to be an expression consisting of a predicate symbol and an individual variable.

Ex: $M(x)$: x is a man, is a simple statement function.

Compound statement function: The statement functions which are obtained from combining ~~two~~ one or more simple statement functions and the logical connectives is called compound statement function.

Ex: Let $M(x)$: x is a man
$H(x)$: x is a mortal.

$M(x) \wedge H(x)$: x is a man and x is a mortal.

$\neg M(x)$: x is not a man.

$M(x) \rightarrow \neg H(x)$: If x is a man then x is not a mortal.

$M(x) \vee \neg H(x)$: x is a man or x is not a mortal.

Free and bound variables:

Given a formula containing a part of the form $(x)\,P(x)$ or $(\exists x)P(x)$, such a ~~formula~~ part is called an x-bound part of the formula. Any occurence of x in an x-bound part of a formula is called bound ocurence of x, while any occurence of x or of any variable that is not a bound occurence is called free ocurence.

Ex: $(\exists x)(P(x) \wedge Q(x))$

The scope of $\underbrace{\exists(x)}_{\text{Quantifier}}$ is $P(x) \wedge Q(x)$.

$(\exists x)\,P(x) \wedge Q(x)$

The scope of $(\exists x)$ is $P(x)$, the last occurence of x in $Q(x)$ is free.

Quantifiers: Quantifiers are words reffer to quantities such as some or all, and indicate how frequently a certain statement is true.

Quantifiers are two types.

Universal quantifiers: The universal quantifiers of $P(x)$ is the statement, $P(x)$ is true for all values of x in the universe of discourse.

The notation is $(\forall x)P(x)$ (or) $(x)\,P(x)$.

Here $\forall$ is called universal quantifiers.

Example:

① All men are mortal

let $M(x)$: x is a man
$H(x)$: x is mortal

for all x, if x is a man then x is mortal.

$(\forall x)\,[M(x) \rightarrow H(x)]$

② Every apple is red

$A(x)$: x is apple
$R(x)$: x is red

for all x, if x is apple then x is red

$(\forall x)\,[A(x) \rightarrow R(x)]$.

③ Any integer is either positive or negative.

for all x, if x is an integer then x is either positive or negative.

$(\forall x)\,[I(x) \rightarrow (P(x) \vee N(x))]$.

~~Existenc~~

Existential Quantifiers: The existential quantification of $P(x)$ is the statement, there exist an element x in the universe of discourse such that $P(x)$ is true. We use the notation $(\exists x)\, P(x)$

Here $\exists x$ is called existential quantifier.

Ex: ① Some men are clever.

For some x, x is a man and x is clever.

$M(x)$: x is a man

$C(x)$: x is clever.

$\exists(x)\ [M(x) \wedge C(x)]$

② Some cats are black

$\exists(x)\ [C(x) \wedge B(x)]$

③ Some real numbers are rational

$\exists(x)\ [R_1(x) \wedge R_2(x)]$.

Formulas:

E_{23}: $(\exists x)\,[A(x) \vee B(x)] \Leftrightarrow (\exists x)\,A(x) \vee (\exists x)\,B(x)$

E_{24}: $(x)\,[A(x) \wedge B(x)] \Leftrightarrow (x)\,A(x) \wedge (x)\,B(x)$

E_{25}: $\neg(\exists x)\,A(x) \Leftrightarrow (\forall x)\,\neg A(x)$

E_{26}: $\neg(\forall x)\,A(x) \Leftrightarrow (\exists x)\,\neg A(x)$

E_{27}: $(x)\,[A \vee B(x)] \Leftrightarrow A \vee (x)\,B(x)$

E_{28}: $(\exists x)\,[A \wedge B(x)] \Leftrightarrow A \wedge (\exists x)\,B(x)$

E_{29}: $(x)\,A(x) \rightarrow B \Leftrightarrow (\exists x)\,(A(x) \rightarrow B)$

E_{30}: $(\exists x)\,A(x) \rightarrow B \Leftrightarrow (x)\,(A(x) \rightarrow B)$

E_{31}: $A \rightarrow (x)\,B(x) \Leftrightarrow (x)\,(A \rightarrow B(x))$

E_{32}: $A \rightarrow (\exists x)\,B(x) \Leftrightarrow (\exists x)\,(A \rightarrow B(x))$.

I_{15}: $(x)\,A(x) \vee (x)\,B(x) \Rightarrow (x)\,(A(x) \vee B(x))$

I_{16}: $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)\,A(x) \wedge (\exists x)\,B(x)$.

INFERENCE THEORY OF PREDICATE CALCULUS.

Rule US (Universal specification):

From $(\forall x)\,A(x)$ one can conclude $A(y)$.

Rule ES (Existential specification):

From $(\exists x)\,A(x)$ one can conclude $A(y)$.

Rule UG (Universal generalisation):

From $A(x)$ one can conclude $(\forall y)A(y)$.

Rule EG (Existential generalization):

From $A(x)$ one can conclude $(\exists y)A(y)$.

Q: ① Show that: All men are mortal
Socrates is a man
Therefore, Socrates is mortal.

Sol: Let $M(x)$: x is a man.
$H(x)$: x is a mortal.
S: Socrates.

Given, $\forall(x)[M(x) \to H(x)]$, $M(S) \Rightarrow H(S)$.

Step		Derivation	Rule/Formula
{1}	(1)	$(\forall x)[M(x) \to H(x)]$	Rule P
{1}	(2)	$M(S) \to H(S)$	Rule US, (1)
{3}	(3)	$M(S)$	Rule P
{1,3}	(4)	$H(S)$	Rule T, (2), (3), I_{11}

I_{11}: Modus ponens
$P, P \to Q \Rightarrow Q$.

② Show that $(x)(P(x) \to Q(x)) \wedge (\forall x)(Q(x) \to R(x)) \Rightarrow (\forall x)(P(x) \to R(x))$

Step		Derivation	Rule/Formula
{1}	(1)	$(x)(P(x) \to Q(x))$	Rule P.
{1}	(2)	$P(y) \to Q(y)$	Rule US, (1)
{3}	(3)	$(x)(Q(x) \to R(x))$	Rule P.
{3}	(4)	$Q(y) \to R(y)$	Rule US, (3).
{1,3}	(5)	$P(y) \to R(y)$	Rule T, (2), (4), $P \to Q, Q \to R \Rightarrow P \to R$.
{1,3}	(6)	$(x)(P(x) \to R(x))$	Rule UG, (5).

③ Show that $(\exists x) M(x)$ follows logically from the premises $(x)(H(x) \to M(x))$ and $(\exists x) H(x)$.

Step		Derivation	Rule, Formula
{1}	(1)	$(\exists x) H(x)$	Rule P.
{1}	(2)	$H(y)$	Rule ES, (1).
{3}	(3)	$(x)[H(x) \to M(x)]$	Rule P.
{3}	(4)	$H(y) \to M(y)$	Rule US, (3).
{1,3}	(5)	$M(y)$	Rule T, (2), (4), $P, P \to Q \Rightarrow Q$
{1,3}	(6)	$(\exists x) M(x)$	Rule EG, (5).

(6) Verify the validity of the argument:

All integers are rational numbers.

Some integers are powers of 3.

Therefore, some rational numbers are powers of 3.

Sol: let $I(x)$: x is an integer.

$R(x)$: x is a rational number.

$P(x)$: x is a power of 3.

Given, $(\forall x)[I(x) \to R(x)], (\exists x)[I(x) \land P(x)] \Rightarrow (\exists x).[R(x) \land P(x)]$

Step		Derivation	Rule / Formula
{1}	(1)	$(\forall x)[I(x) \to R(x)]$	Rule P
{1}	(2)	$I(y) \to R(y)$	Rule US, (1).
{3}	(3)	$(\exists x)[I(x) \land P(x)]$	Rule P.
{3}	(4)	$I(y) \land P(y)$	Rule ES, (3)
{3}	(5)	$I(y)$	Rule T, (4), I_1 $I_1: P \land Q \Rightarrow P$.
{3}	(6)	$P(y)$	Rule T, (4), I_2 $I_2: P \land Q \Rightarrow Q$.
{1,3}	(7)	$R(y)$	Rule T, (1), (5), I_{11} Modus ponens $P, P \to Q \Rightarrow Q$.
{1,3}.	(8)	$R(y) \land P(y)$	Rule T, (6), (7), $P, Q \Rightarrow P \land Q$.
{1,3}	(9)	$(\exists x)[R(x) \land P(x)]$	Rule EG, (8).

Inconsistency of statements:

- A set of formulas $H_1, H_2, ---, H_m$ is inconsistent if their ~~contradiction~~ conjunction implies a contradiction, i.e., $H_1 \land H_2 \land H_3 ----- \land H_m \Rightarrow \underbrace{R \land \neg R}_{F}$,

where R is any formula.

Q (1) Show that the set of premises $P \to Q, P \to R, Q \to \neg R$ and P are inconsistent.

Step		Derivation	Rule / Formula
{1}	(1)	$P \to Q$	Rule P.
{2}	(2)	$Q \to \neg R$	Rule P.
{1,2}	(3)	$P \to \neg R$	Rule T, (1), (2), $P \to Q, Q \to R \Rightarrow P \to R$.
{4}	(4)	P	Rule P.
{1,2,4}	(5)	$\neg R$	Rule T, (3), (4) $P, P \to Q \Rightarrow Q$
{6}	(6)	$P \to R$	Rule P.

{4,6}	(7)	R	Rule T, (4), (6), Modus ponens: $P, P \to Q \Rightarrow Q$.
{1,2,4,6}	(8)	$R \wedge \neg R$	Rule T, (7), (5) $P, Q \Rightarrow P \wedge Q$.
{1,2,4,6}	(9)	F	Rule T (8), $P \wedge \neg P \Leftrightarrow F$.

(2) Show that the following premises are inconsistent:

If Jack misses many classes through illness, then he fails high school.

If Jack fails high school, then he is uneducated.

If Jack reads a lot of books, then he is not uneducated.

Jack misses many classes through illness and reads a lots of books.

Sol: let C : Jack misses many classes through illness.

F : Jack fails high school.

B : Jack reads a lot of books.

E : Jack is ~~undeca~~ uneducated.

The given premises are, $C \to F$, $F \to E$, $B \to \neg E$, $C \wedge B$.

Step		Derivation	Rule / Formula
{1}	(1)	$C \to F$	Rule P.
{2}	(2)	$F \to E$	Rule P.
{1,2}	(3)	$C \to E$	Rule T, (1), (2), $P \to Q, Q \to R \Rightarrow P \to R$.
{4}	(4)	$B \to \neg E$	Rule P.
{4}	(5)	$E \to \neg B$	Rule T, (4), $P \to Q \Leftrightarrow \neg Q \to \neg P$, $\neg(\neg P) \Leftrightarrow P$.
{1,2,4}	(6)	$C \to \neg B$	Rule T, (3), (5), $P \to Q, Q \to R \Rightarrow P \to R$.
{1,2,4}	(7)	$\neg C \vee \neg B$	Rule T, (6), $P \to Q \Leftrightarrow \neg P \vee Q$
{1,2,4}	(8)	$\neg(C \wedge B)$	Rule T, (7), De-morgan's law: $\neg P \vee \neg Q \Leftrightarrow \neg(P \wedge Q)$.
{9}	(9)	$C \wedge B$	Rule P.
{1,2,4,9}	(10)	$(C \wedge B) \wedge \neg(C \wedge B)$	Rule T, (8), (9), $P, Q \Rightarrow P \wedge Q$.
{1,2,4,9}	(11)	F	Rule T, (10), $P \wedge \neg P \Leftrightarrow F$.

$\therefore (C \wedge B) \wedge \neg (C \wedge B) \Rightarrow F$

$\therefore$ The given statements are inconsistent.

UNIT-2

SET THEORY

THE PRINCIPLE OF INCLUSION - EXCLUSION

Let A and B are subsets of some universal set U then the principle of inclusion- exclusion for two sets A and B is

$|A \cup B| = |A| + |B| - |A \cap B|$. $|A|$ - mod A, cardinality

where $|A|$ = No. of elements in set A.

$|B|$ = No. of elements in set B.

$|A \cup B|$ = No. of elements in set $A \cup B$.

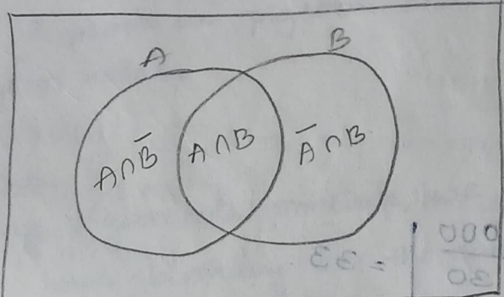
$|A \cap B|$ = No. of elements in set $A \cap B$.

Note:

1. The principle of inclusion-exclusion for three sets A, B and C is

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$.

2. Venn diagram for two sets A and B is



3. $|\bar{A}| = N - |A|$, where N is the number of elements in the universal set.

4. $\lfloor x \rfloor$ = ~~The least positive integer~~ The largest integer that is less that or equal to it called a greatest integer function or floor function.

Q: ① Out of 80 students in a class, 60 play football, 53 play hockey and 35 play both the games. How many students (i) do not play any of these games.
(ii) play only hockey but not football.

Sol: Given, number of students in a class, N = 80.

Let F : The students play foot ball,

H : The students play hockey.

then, $|F| = 60$, $|H| = 53$, $|F \cap H| = 35$.

(i) the no. of students do not play football and hockey is

$|\bar{F} \cap \bar{H}| = \overline{F \cup H} = N - |F \cup H|$

$= N - [|F| + |H| - |F \cap H|]$

$= 80 - [60 + 53 - 35]$

$|\bar{F} \cap \bar{H}| = 2$

(ii) The no. of student play only hockey but not football is,

$$|\bar{F} \cap H| = |H| - |F \cap H|$$
$$= 53 - 35$$
$$|\bar{F} \cap H| = 18.$$

~~② Find the minimum number of students~~

② Find the number of integers between 1 and 1000 inclusive that are divisible by none 5, 6 and 8.

Sol: Total numbers from 1 to 1000 = 1000 (N).

let, A : Integers divisible by 5 b/w 1 to 1000

B : 6

C : 8.

$$|A| = \left\lfloor \frac{1000}{5} \right\rfloor = 200$$

$$|B| = \left\lfloor \frac{1000}{6} \right\rfloor = 166$$

$$|C| = \left\lfloor \frac{1000}{8} \right\rfloor = 125$$

$$|A \cap B| = \left\lfloor \frac{1000}{LCM(5,6)} \right\rfloor = \left\lfloor \frac{1000}{30} \right\rfloor = 33.$$

$$|A \cap C| = \left\lfloor \frac{1000}{LCM(5,8)} \right\rfloor = \left\lfloor \frac{1000}{40} \right\rfloor = 25$$

$$|B \cap C| = \left\lfloor \frac{1000}{LCM(6,8)} \right\rfloor = \left\lfloor \frac{1000}{24} \right\rfloor = 41.$$

$$|A \cap B \cap C| = \left\lfloor \frac{1000}{LCM(5,6,8)} \right\rfloor = \left\lfloor \frac{1000}{120} \right\rfloor = 8.$$

The no. of integers b/w 1 and 1000 that are divisible by none 5, 6 and 8 are.

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = |\overline{A \cup B \cup C}| = N - |A \cup B \cup C|$$
$$= N - [|A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A|] + |A \cap B \cap C|$$
$$= 1000 - [200 + 166 + 125 - 33 - 25 - 41 + 8]$$
$$= 1000 - 400$$
$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 600$$

PIGEON HOLE PRINCIPLE AND ITS APPLICATION

Statement: If m pigeons are assigned to n pigeon holes and $m>n$ then two or more pigeon occupies the same pigeon hole.

(or)

In other words,

If m pigeons occupies n pigeon holes and if $m>n$ then atleast one pigeon hole must contain two or more pigeons in it.

The extended pigeon hole principle:

If m pigeons are assigned to n pigeon holes then one of the pigeon hole must contain $\left\lfloor \frac{m-1}{n} \right\rfloor + 1$ pigeons.

Note: To apply the principle, we must decide which objects will play the role of pigeon and which objects will play the role of pigeon hole.

Q ① Show that if 8 people are in a room atleast two of them have birthdays that occur on the same day of the week.

Sol: let $m=8$ and $n=7$.

i.e., Treat 8 people as pigeons and 7 days of the week as the pigeon holes.

Here $8>7$

i.e., $m>n$.

Then by using pigeon hole principle, we can say that atleast two people have birthday on the same day of the week.

② Applying pigeon hole principle show that of any 14 integers that are selected from the set $S=\{1,2,3,-\,-\,-,25\}$, there are at least two whose sum is 26. Also write a statement that generalizes their result.

Sol: Given, $S=\{1,2,3,---,25\}$.

let the integer sets having the sum 26 are

$S_1=\{1,25\}$, $S_2=\{2,24\}$, $S_3=\{3,23\}$, $S_4=\{4,22\}$, $S_5=\{5,21\}$, $S_6=\{6,20\}$, $S_7=\{7,19\}$, $S_8=\{8,18\}$, $S_9=\{9,17\}$, $S_{10}=\{10,16\}$, $S_{11}=\{11,15\}$, $S_{12}=\{12,14\}$, $S_{13}=\{13,13\}$.

Treat 14 integers as pigeons and the sets as pigeon holes.

$m=14$, $n=13$

$m>n$.

According to pigeon hole principle, at least one ~~group~~ set must contain more than one selected number ~~at least one pair~~.

$\therefore$ Among any 14 integers chosen from S, ~~must st~~ there are atleast two whose sum is 26.

Generalisation: In any set $\{1, 2, \ldots, 2n-1\}$ selecting $n+1$ numbers guarantees that at least two of them sum to $2n$.

③ Find the minimum number of students in ~~the~~ a class to be sure that 4 out of them are born on the same month.

Sol: Here we treat months as pigeon holes, $n = 12$. and students as pigeons (m).

To be sure that 4 out of them are born on the same month, AT, applied pigeon hole principle

Here $\left\lfloor \frac{m-1}{n} \right\rfloor + 1 = 4$

$$\left\lfloor \frac{m-1}{12} \right\rfloor + 1 = 4$$

$$\frac{m-1}{12} = 3$$

$$m - 1 = 36$$

$$m = 37.$$

$\therefore$ Minimum no. of students required $= 37$.

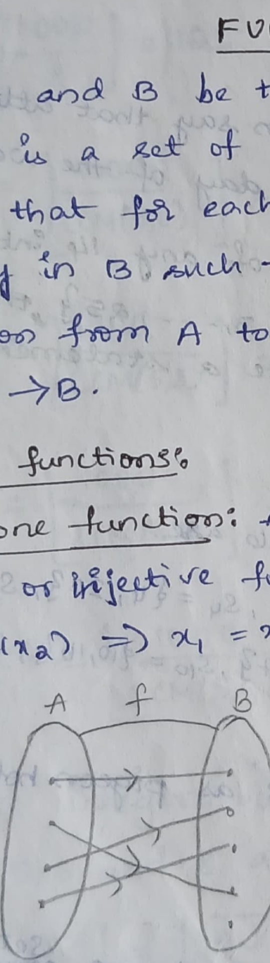
FUNCTIONS

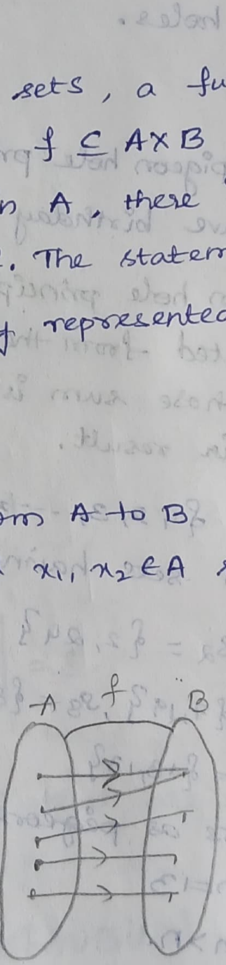
Let A and B be two non-empty sets, a function f from A to B is a set of ordered pairs $f \subseteq A \times B$ with the property that for each element x in A, there is a unique element y in B such that $(x, y) \in f$. The statement f is a function from A to B is usually represented symbolically by $f: A \to B$.

Types of functions:

① One-one function: A function f from A to B is said to be one-one or injective function for all $x_1, x_2 \in A$ such that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Ex:



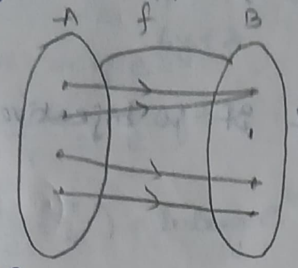


f is one-one $\quad$ f is not one-one.

$f(x) = 2x$, $f: \mathbb{R} \to \mathbb{R}$

② **Many to one function:** A function f from A to B is said to be many to one function iff two or more elements of A have the same image in B.

Ex: $f(x) = x^2$, $f: R \to R$.

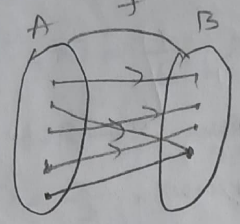


f is many to one.

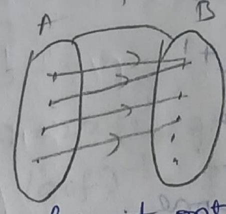
③ **Onto function:** A function $f: A \to B$ is said to be onto function or surjective if every element of B has a pre-image in A.

i.e, Co domain of B = Range of f.

Ex: $f(x) = x^3$, $f: \mathbb{R} \to \mathbb{R}$.



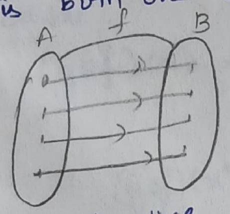
f is onto



f is not onto

④ **Bijective function:** A function $f: A \to B$ is said to be bijective if it is both one-one and onto.

Ex:



f is bijective

$f(x) = x + 1$, $f: \mathbb{R} \to \mathbb{R}$

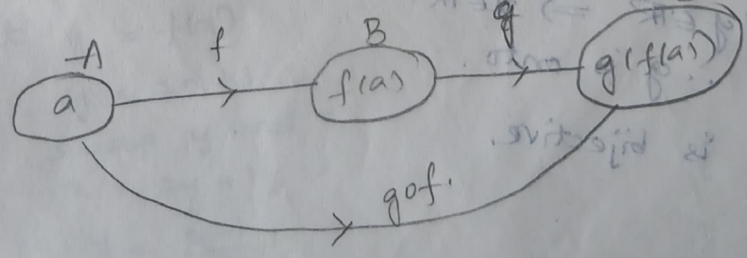
⑤ **Identity function:** A mapping $I_x: X \to X$ is called an identity function if and only if $I_x = \{(x, x) \mid x \in X\}$.

Thus, $f: A \to A$ is said to be identity function iff $f(x) = x \; \forall x \in X$.

COMPOSITION OF FUNCTIONS

Let $f: A \to B$ and $g: B \to C$. The composition of f and g is a new function from $A \to C$. denoted by $g \circ f$

$(g \circ f)(x) = g[f(x)]$, $\forall x \in A$.



INVERSE FUNCTIONS

If $f: X \to Y$ is a bijective function, then the ~~inverse~~ function $f^{-1}: Y \to X$ is called inverse function which satisfies $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

Thus $f(x) = y \Leftrightarrow x = f^{-1}(y)$.

A function $f: X \to Y$ is invertable iff it is bijective.

Q:

~~Let $f: R \to R$~~

① verify $f(x) = 2x+1 \ \forall x \in R$ is bijective from $\mathbb{R} \to \mathbb{R}$.

Sol: Given, $f(x) = 2x+1$

one-one: Let $x_1, x_2 \in R$

$$f(x_1) = f(x_2)$$

$$2x_1 + 1 = 2x_2 + 1$$

$$2x_1 = 2x_2$$

$$x_1 = x_2$$

$\therefore f$ is one-one

onto: Let $y = f(x)$

$$y = 2x+1$$

$$y - 1 = 2x$$

$$x = \frac{y-1}{2} \in \mathbb{R}$$

$\forall y \in R \Rightarrow x \in \mathbb{R}$.

$\therefore f$ is onto.

Hence, f is bijective.

② H/W verify $g(x) = x \ \forall x \in R$, is bijective from $\mathbb{R} \to \mathbb{R}$.

Sol: Given, $g(x) = x$.

one-one: Let $x_1, x_2 \in \mathbb{R}$

$$g(x_1) = g(x_2)$$

$$x_1 = x_2$$

$\therefore g$ is one-one

onto: Let $y = g(x)$

$$y = x$$

$$x = y \in \mathbb{R}$$

$\forall y \in \mathbb{R} \Rightarrow x \in \mathbb{R}$.

$\therefore g$ is onto.

Hence g is bijective.

③ Find the inverse of $f(x) = 4e^{6x+2}$.

Sol: Given, $f(x) = 4e^{6x+2}$.

let $y = 4e^{6x+2}$.

We now swap x and y.

$$x = 4e^{6y+2}.$$

$$\frac{x}{4} = e^{6y+2}$$

$$\ln\cancel{\log}\left(\frac{x}{4}\right) = \ln\cancel{\log}\, e^{6y+2}$$

$$\ln\cancel{\log}\left(\frac{x}{4}\right) = 6y+2$$

$$\ln\left(\frac{x}{4}\right) - 2 = 6y$$

$$y = \frac{1}{6}\left[\ln\left(\frac{x}{4}\right) - 2\right]$$

$$\therefore f^{-1}(x) = \frac{1}{6}\left[\ln\left(\frac{x}{4}\right) - 2\right].$$

$\log e^{x} = x \log e = x$

$e^{\log x} = x$

$y = 4e^{6x+2}$

$\frac{y}{4} = e^{6x+2} \Rightarrow \log\frac{y}{4} = 6x+2$

$\ln\left(\frac{y}{4}\right) - 2 = 6x$

$x = \frac{1}{6}\left[\ln\left(\frac{y}{4}\right) - 2\right]$

$\therefore f^{-1}(x) = \frac{1}{6}\left[\ln\left(\frac{x}{4}\right) - 2\right].$

④ Let $f: R \to R$ such that $f(x) = 2x+1$, $g: R \to R$ such that $g(x) = \frac{x}{3}$, then verify that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Sol: Given, $f(x) = 2x+1$, $g(x) = \frac{x}{3}$.

WKT f and g are bijective from $\mathbb{R} \to \mathbb{R}$. Therefore f and g are invertible.

Now, $(g \circ f)(x) = g[f(x)]$

$= g[2x+1]$

$(g \circ f)(x) = \frac{2x+1}{3}$

let $y = (g \circ f)(x)$

$y = \frac{2x+1}{3}$

swap x and y

$x = \frac{2y+1}{3}$

$3x = 2y+1$

$3x - 1 = 2y$

$y = \frac{3x-1}{2}$

$\therefore (g \circ f)^{-1}(x) = \frac{3x-1}{2}$ —— ①

Again, let $y = f(x)$

$y = 2x+1$

swap y and x.

$x = 2y+1$

$y = \frac{x-1}{2}$.

$\therefore f^{-1}(x) = \frac{x-1}{2}$ —— ②

Now, let $y = g(x)$

$y = x/3$.

Swap x and y.

$x = y/3$.

$y = 3x$.

$\therefore g^{-1}(x) = 3x$. — (3)

Now, $(f^{-1} o g^{-1})(x) = f^{-1}[g^{-1}(x)]$

$= f^{-1}[3x]$

$(f^{-1} o g^{-1})(x) = \frac{3x-1}{2}$ — (4)

From (1) & (4)

$(gof)^{-1}(x) = (f^{-1} o g^{-1})(x)$

i.e. $(gof)^{-1} = f^{-1} o g^{-1}$.

(5) Let $f: A \to B$, $g: B \to C$ and $h: C \to D$ then show that $ho(gof) = (hog)of$.

Sol: ~~Let~~ Given,

$f: A \to B$, $g: B \to C$ and $h: C \to D$.

By composition of function we have,

$gof: A \to C$, $hog: B \to D$

and $ho[gof]: A \to D$, $[hog]of: A \to D$.

So, domain of $ho(gof)$ = domain of $(hog)of$

Let x be a element in the domain.

Now, $[ho(gof)](x) = h[(gof)(x)]$

$= h[g(f(x))]$

$[(hog)of](x) = (hog)[f(x)]$

$= h[g(f(x))]$

$\therefore ho(gof) = (hog)of$.

RECURSIVE FUNCTIONS

Recursive Function: A recursive function is a function which is defined in terms of itself recursively. A recursive definition has two parts.

1. Definition of the smallest argument i.e., $f(0)$ or $f(1)$.
2. Definition of $f(n)$, given $f(n-1)$, $f(n-2)$ etc....

Ex: let $f(x) = n!$

The explicit method of describing this function is
$f(0) = 1$ and $f(n) = 1.2.3 \ldots n$ for $n \in Z^+$.

The recursive method of describing this function is
$f(0) = 1$ and $f(n) = n f(n-1)$ for $n \in Z^+$. First $f(0)$ is specified, then $f(n)$ is said in the aid of $f(0)$.

Q: A function $f(n) = a_n$ is defined recursively by $a_0 = 4$ and ~~$a_n = a_{n-1} + n$~~ $a_n = a_{n-1} + n$ for $n \geq 1$. Find $f(n)$ in explicit form.

Sol: Given, $a_0 = 4$ and $a_n = a_{n-1} + n$ for $n \geq 1$.

Using the given recursive formula repeatedly, we find that

$$a_n = a_{n-1} + n$$
$$= [a_{n-2} + (n-1)] + n$$
$$= [a_{n-3} + (n-2)] + (n-1) + n$$
$$= \ldots\ldots$$
$$\ldots\ldots$$
$$= (a_1 + 2) + 3 + \cdots + n$$
$$= (a_0 + 1) + 2 + 3 + \cdots + n$$
$$= 4 + 1 + 2 + 3 + \cdots + n$$
$$a_n = 4 + \frac{n(n+1)}{2}$$

which is the explicit formula for $f(n) = a_n$.

Relation (or) Binary relation: A binary relation

let A and B are two non-empty sets. A binary relation or simply a relation from A to B is a subset of $A \times B$.

Given, $x \in A$ and $y \in B$, we write xRy if $(x, y) \in R$
and $x \not R y$ if $(x, y) \notin R$.

If R is a relation from A to A, then R is said to be relation on A.

Properties of relations:

Reflexive: $(a,a) \in R$ i.e., aRa $\forall a \in A$.

Symmetry: $(a,b) \in R \Rightarrow (b,a) \in R$ $\forall a,b \in A$.

Anti-symmetry: $(a,b) \in R$ & $(b,a) \in R \Rightarrow a=b$ $\forall a,b \in A$

Transitivity: $(a,b) \in R$ & $(b,c) \in R \Rightarrow (a,c) \in R$ $\forall a,b,c \in A$.

Partially ordered set:

A relation R on a set S is called partial order, if it is reflexive, antisymmetric and transitive.

i.e., ① aRa $\forall a \in S$ (reflexive)

② aRb and $bRa \Rightarrow a=b$, $\forall a,b \in S$ (antisymmetric)

③ aRb and $bRc \Rightarrow aRc$, $\forall a,b,c \in S$ (transitivity).

A set together with a partial order R is called a partially ordered set or Poset. It is denoted by (S,R)

Note: In a poset $(P, \leq)$, the relation is usually denoted by $\leq$, which is read as less than or equal to and $a \leq b$ is read as a preceeds b, $b \geq a$ is read as b succeeds a.

Upper bound and lower bound:

Let $(A, \leq)$ be a poset and B is a subset of A.

An element $u \in A$ is called an upper bound of B if u succeeds every element of B i.e., $b \leq u$ $\forall b \in B$.

An element $l \in A$ is called an lower bound of B if l preceeds every element of B i.e., $l \leq b$ $\forall b \in B$.

If $B \subseteq A$ and $x \in A$, then 'x' is called the least upper bound of B iff

upper bound of B $(b \leq x)$

① x is a upper bound of B

② $x \leq u$ for every upper bound u of B.

It is also called as supremum of B and is denoted by sup B (or) lub B.

If $B \subseteq A$ and $y \in B$, then 'y' is called the greatest lower bound of B if

① y is a lower bound of B $(y \leq b)$

② $l \leq y$ for every lower bound l of B.

It is also called as infimum of B and is denoted by glb B or inf B.

LATTICES AND ITS PROPERTIES

a element subset of this poset

A partially ordered set or poset in which every ~~pair of~~ elements has both a least upper bound (lub) and greatest lower bound (glb) is called a lattice.

Let (L, R) be a lattice.

i.e., $lub\{x,y\}$ and $glb\{x,y\}$ exists $\forall x,y \in L$.

$x \vee y = lub\{x,y\}$, read as join of x and y or x join y

$x \wedge y = glb\{x,y\}$, read as meet of x and y or x meet y.

Properties of lattices:

Let (L,R) be a lattice & let $a,b,c \in L$

① Idempotent property: (i) $a \vee a = a$ (ii) $a \wedge a = a$

② Commutative property: (i) $a \vee b = b \vee a$ (ii) $a \wedge b = b \wedge a$

③ Associative property: (i) $a \vee (b \vee c) = (a \vee b) \vee c$
(ii) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

④ Absorption property: (i) $a \vee (a \wedge b) = a$
(ii) $a \wedge (a \vee b) = a$

Note: A lattice is distributive lattice iff it satisfies the following conditions.

① $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

② $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Q: If $A = \{1, 2, 3, 5, 30\}$ and R is the divisibility relation. Prove that (A,R) is a lattice but not distributive lattice.

Sol: Given $A = \{1, 2, 3, 5, 30\}$.

R is the divisibility relation.

We have to show that (A,R) is a lattice but not a distributive lattice

Least upper bound table:

→ The least no. x which is divisible by both a and b in the set

$\vee$	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

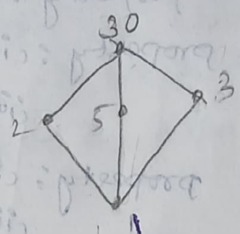
By observing the LUB table, for every pair of elements on set A, LUB ~~is~~ exist.

Greatest lower bound table: → the greatest no. y, where both a & b are divisible by y.

$\wedge$	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

By observing the GLB table, for every pair of elements on set A, GLB exist.

Hasse Diagram



Since for every ~~pair~~ 2 element subset of A, both LUB and GLB exists, ∴ (A, R) is a lattice.

To check whether (A, R) is a distributive lattice;

we consider, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

$2 \vee (3 \wedge 5) = (2 \vee 3) \wedge (2 \wedge 5)$

$2 \vee 1 = 30 \wedge 30$

$2 \neq 30$

∴ (A, R) is not a distributive lattice.

ALGEBRAIC STRUCTURES

ALGEBRAIC SYSTEMS - EXAMPLES AND GENERAL PROPERTIES

Binary and n-ary operation:

Let X be a set and f be a mapping $f: X \times X \to X$ then f is said to be a binary operation on X.

In general, a mapping $f: X^n \to X$ is called a n-ary operation and n is called the order of the operation.

For n=1, $f: X \to X$ is called a uniary operation.

For n=2, $f: X \times X \to X$ is called a binary operation.

The binary operations are generally denoted by the symbols $*, +, \times, \div, -, \otimes, \oplus, \Delta$, etc..

ALGEBRAIC SYSTEMS: A system consisting of a set and one or more n-ary operations on the set will be called an algebraic systems or algebraic structures.

We shall denote an algebraic structure by $(S, f_1, f_2, \ldots)$ where S is a non-empty set and $f_1, f_2 \ldots$ are operations on S.

Ex: ① $(\mathbb{Z}, +, \times)$ is a algebraic structure, since $+$ and $\times$ are binary operations on $\mathbb{Z}$

② $(\mathbb{R}, +)$ is a algebraic structure, since $+$ is a binary operation on $\mathbb{R}$.

General Properties:

Let G be a set and $(G, *, \Delta)$ is an algebraic system then the properties of the binary operation $*$ is given by:

① Closure law: The set G is said to be closed under $*$ if $a * b \in G$, $\forall a, b \in G$.

② Associativity: The set G is said to be associative under $*$ if $a*(b*c) = (a*b)*c$, $\forall a, b, c \in G$.

③ Identity: The set G is said to satisfy identity property under $*$ if there exists an element e in G such that $a*e = e*a = a$, $\forall a \in G$.

④ Inverse: The set G is said to be invertible under $*$ if there exists an element x in G such that $a*x = e$, $\forall a \in G$

⑤ Commutative: The set G is said to be commutative under $*$ if $a*b = b*a$, $\forall a, b \in G$.

⑥ Distributive: The set G is said to be distributive under $*$ over Δ if $a*(b \Delta c) = (a*b) \Delta (a*c)$
$(b \Delta c)*a = (b*a) \Delta (c*a)$, $\forall a, b, c \in G$.

⑦ Cancellation law: For any $a, b, c \in G$ and $a \neq 0$,

$a*b = a*c \Rightarrow b = c$ (Left-cancellation law)

$b*a = c*a \Rightarrow b = c$ (Right-cancellation law).

SEMI GROUP

An algebraic system $(S, *)$ is called a semi group, if $*$ is closed and associative under $*$.

(or)

$(S, *)$ is said to be a semi group if,

1. $a*b \in S \quad \forall a,b \in S$
2. $(a*b)*c = a*(b*c)$, $a,b,c \in S$.

Ex: 1. $(\mathbb{Z}, +)$ is a semi group, since $\mathbb{Z}$ is closed and associative under addition.

2. $(\mathbb{Q}, \times)$ is a semigroup, since $\mathbb{Q}$ is closed and associative under multiplication.

MONOID

A semi group $(M, *)$ ~~with~~ containing the identity element is said to be a monoid.

In other words, an algebraic system $(M, *)$ is called a monoid if

1. $a*b \in M \quad \forall a,b \in M$ (closure)
2. $a*(b*c) = (a*b)*c$, $\forall a,b,c \in M$ (associative)
3. $\exists e \in M$ such that $a*e = e*a = a$, $\forall a \in M$ (identity).

Ex: 1. $(\mathbb{N}, \times)$ is a monoid with identity element 1.

2. $(\mathbb{Z}, +)$ is a monoid with identity element 0.

3. If E denotes the set of positive even numbers then $(E, +)$, $(E, \times)$ are semigroups but not monoids

4. $(\mathbb{N}, +)$ is not a monoid.

Note: A monoid $(M, *)$ is called a commutative monoid if $*$ is commutative.

GROUPS

A non-empty set G is said to form a group with respect to binary operation $*$, if

1. $a*b \in G \quad \forall a,b \in G$ (closure)
2. $(a*b)*c = a*(b*c)$, $\forall a,b,c \in G$ (associative)
3. $\exists e \in G$ such that $a*e = e*a = a$, $\forall a \in G$ (identity)
4. $\exists a^{-1} \in G$ such that $a^{-1}*a = a*a^{-1} = e$, for any $a \in G$ (inverse).

In other words, an algebraic system $(G, *)$ is said to be a group if $*$ satisfies closure, associative, identity and inverse properties.

Ex: ① $(\mathbb{Z}, +)$ is a group. ② $(\mathbb{Q}, +)$ is a group. ③ $(\mathbb{Z}, \times)$ is not a group $\because 0$ - no inverse ④ $(\mathbb{R}, \times)$ is not a group

Abelian group: A group $(G, *)$ is said to be abelian group if G is commutative under the binary operation $*$.

Ex: ① $(\mathbb{Z}, +)$ is an abelian group

② $(\mathbb{Q}', \times)$ is an abelian group, where $\mathbb{Q}' = \mathbb{Q} - \{0\}$.

Order of a group:

The order of a group is the total number of elements in the group.

$|G|$ = number of elements in G.

Ex: ① $(\mathbb{Z}, +)$, $(\mathbb{R}, +)$ have infinite no. of elements.

② $(\mathbb{Z}_4, +_4)$ is a addition modulo 4 group which consists of 4 element $\mathbb{Z}_4 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}\}$.

$\therefore |\mathbb{Z}_4| = 4$.

Finite group: A group $(G, *)$ is called finite if the set G has a finite number of elements.

Ex: $(\mathbb{Z}_6, +_6)$, the integers modulo 6 under addition.

$\mathbb{Z}_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

$|\mathbb{Z}_6| = 6$, since $(\mathbb{Z}_6, +_6)$ have finite number of elements it is a finite group.

Infinite group: A group $(G, *)$ is called infinite if the set G has infinitely many elements.

Ex: $(\mathbb{Z}, +)$ have infinite number of elements, so it is an infinite group.

Order of an element:

The order of an element $g \in (G, *)$ is the smallest positive integer n such that

$$g * g * \cdots (n \text{ times}) = e.$$

where e is the identity of the group.

If no such n exists, the element has infinite order.

Ex: (1) In $(Z_6, +_6)$, the element 2 has order 3

because $2+2+2 \equiv 0 \pmod 6$

(2) In $(Z, +)$, the element 1 has infinite order

(never returns to identity element 0 in finite steps).

Q (1) On the set Q of all rational numbers, the operation $*$ is defined by $a*b = a+b-ab$. Show that this operation and Q forms a commutative monoid.

Sol: Given Q is the set of all rational numbers.

and $a*b = a+b-ab$.

Closure: let $a, b \in Q$

$a*b = a+b-ab \in Q$.

$\therefore (Q, *)$ is closed.

Associative: let $a, b, c \in Q$, then

$$a*(b*c) = a*(b+c-bc)$$
$$= a+(b+c-bc) - a(b+c-bc)$$
$$= a+b+c-bc-ab-ac+abc$$

$$(a*b)*c = (a+b-ab)*c$$
$$= (a+b-ab)+c - (a+b-ab)c$$
$$= a+b-ab+c-ac-bc+abc$$

$\therefore a*(b*c) = (a*b)*c$

$\therefore (Q, *)$ is associative.

Identity: let $e \in Q$.

let $a*e = a \quad \forall a \in Q$

$a+e-ae = a$

$e(1-a) = 0$

$e = 0$

$\therefore e = 0 \in Q$ is the identity element.

$\therefore (Q, *)$ satisfies the identity property.

Commutative: let $a, b \in Q$ then

$a * b = a + b - ab$

$= b + a - ba$

$= b * a$.

$\therefore (Q, *)$ is commutative.

Thus $(Q, *)$ is commutative monoid.

(2) Prove that the set Z of all integers with the binary operation $*$ defined as $a*b = a+b+1 \ \forall\ a, b \in Z$ is an abelian group.

Sol: Given, Z is set of all integers

and $a*b = a+b+1, \forall\ a, b \in Z$

let a, b, c be any 3 elements in Z

Closure:

$a*b = a+b+1 \in Z$

$\therefore (Z, *)$ is closed.

Associative: $(a*b)*c = (a+b+1)*c$

$= (a+b+1)+c+1$

$= a+b+c+2$

$a*(b*c) = a*(b+c+1)$

$= a+(b+c+1)+1$

$= a+b+c+2$

$\therefore (a*b)*c = a*(b*c)$

$\therefore (Z, *)$ is associative.

Identity: let $e \in Z$ is a identity element

then $\forall a \in Z$, $a*e = a$

$a+e+1 = a$

$e+1 = 0$

$e = -1 \in Z$

$\therefore e = -1$ is the identity element for $(Z, *)$.

Inverse: let the integer a have inverse x, then

$a * x = e$.

$a + x + 1 = e = -1$

$x = -2 - a \in Z$

$\therefore$ The inverse element of a is $-2-a, \forall a$.

Commutative: $a*b = a+b+1$

$= b+a+1$

$a*b = b*a$.

$\therefore (Z, *)$ is commutative.

Thus, $(Z, *)$ is an abelian group.

(3) Show that the set $G = \{1, \omega, \omega^2\}$, where $1, \omega, \omega^2$ are cube roots of unity, form an abelian group under the operation of ordinary multiplication.

Sol: Given, $G = \{1, \omega, \omega^2\}$.

The composition table of multiplication is as follows. $\omega^3 = 1$

$\times$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

Closure: Since all entries of the composition table are also elements of the set G, $(G, \times)$ is closed.

Associative: $1 \times (\omega \times \omega^2) = (1 \times \omega) \times \omega^2$

$(G, \times)$ is associative.

Identity: From above table we see that

$1 \times 1 = 1$

$1 \times \omega = \omega \times 1 = \omega$

$1 \times \omega^2 = \omega^2 \times 1 = \omega^2$.

$\therefore$ 1 is the identity element.

Inverse: From the above table, we see that

$1 \times 1 = 1$, $\omega^2 \times \omega = \omega \times \omega^2 = 1$.

$\therefore$ The inverse of $1, \omega, \omega^2$ are $1, \omega^2, \omega$ respectively.

Commutative: Since the composition is symmetric along the principal diagonal, the operation $\times$ is commutative,

(or)

We have, $1 \times \omega = \omega = \omega \times 1$

$\omega \times \omega^2 = 1 = \omega^2 \times \omega$

$\omega^2 \times 1 = \omega^2 = 1 \times \omega^2$.

$\therefore (G, \times)$ is commutative.

Thus, $G = \{1, \omega, \omega^2\}$, form an abelian group under the operation of ordinary multiplication.

(4) Prove that the four roots of unity $1, -1, i, -i$, where $i = \sqrt{-1}$, form an abelian multiplicative group.

Sol: Given, let $G = \{1, -1, i, -i\}$

The composition table of multiplication is as follows.

$\times$	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Closure: Since all entries of the composition table are also elements of the set G, $(G,\times)$ is closed.

Associative: WKT, ordinary multiplication is already associative.

$\therefore (G,\times)$ is associative.

Identity: From above table we see that

$1\times 1 = 1$

$1\times -1 = -1\times 1 = -1$

$1\times i = i\times 1 = i$

$1\times -i = -i\times 1 = -i$.

$\therefore 1$ is the identity element, since $1\times a = a\times 1 = a, \forall a \in G$.

~~Inverse: Since the composition is symmetric along the principal diagonal of the composition table;~~

Inverse: From the above table, we see that

$1\times 1 = 1$, $-1\times -1 = 1$, $+i\times -i = 1$, $-i\times i = 1$

$\therefore$ The inverse of $1, -1, i, -i$ are $1, -1, -i$ and i respectively.

Commutative: Since the composition is symmetric along the principal diagonal of the composition table, $(G,*)$ is commutative.

Thus, $G = \{1, -1, i, -i\}$, the four roots of unity form an abelian multiplicative group.

SUBGROUP

Let $(G,*)$ be a group. A subset $H \subseteq G$ is called a subgroup of G if H itself forms a group under the same operation $*$.

Ex: Let $(\mathbb{Z},+)$ be a group, where

$\mathbb{Z} = \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$.

let $2\mathbb{Z} = \{ \ldots, -4, -2, 0, 2, 4, \ldots \} \subseteq \mathbb{Z}$

Closure: WKT, sum of two even integers is even

$\therefore a+b \in 2\mathbb{Z}, \forall a, b \in 2\mathbb{Z}$

$\therefore (2\mathbb{Z}, +)$ is closed.

Associative: Since, addition is associative on integers, $(2\mathbb{Z}, +)$ is associative.

Identity: We observe that, $\exists\, 0 \in 2\mathbb{Z}$ such that $a+0 = 0+a = a, \forall a \in 2\mathbb{Z}$

$\therefore 0$ is the identity element of $2\mathbb{Z}$.

Inverse: ~~For any $a \in 2\mathbb{Z}$, the inverse $-a \in 2\mathbb{Z}$ where $a+(-a) = 0$.~~

For every $a \in 2\mathbb{Z}$, $\exists\, -a \in 2\mathbb{Z}$ such that $a+(-a) = 0$.

$\therefore (2\mathbb{Z}, +)$ is a group.

Hence $(2\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Z}, +)$.

Here Note on next page.

Theorem: The necessary and sufficient condition for a non-empty subset H of a group $(G, *)$ to be a subgroup is $a \in H, b \in H \Rightarrow a * b^{-1} \in H$.

Proof:

Necessary: Given $(H, *)$ is a subgroup of $(G, *)$ we have to prove $a \in H, b \in H \Rightarrow a * b^{-1} \in H$.

let $a \in H, b \in H$

we have $b^{-1} \in H$, since $(H, *)$ is a group and inverse element exists.

$a \in H, b^{-1} \in H \Rightarrow a * b^{-1} \in H$ $\because H$ is closed.

$\therefore a \in H, b \in H \Rightarrow a * b^{-1} \in H$.

Sufficient: Given, H is a non-empty subset of a group $(G, *)$ such that $a \in H, b \in H \Rightarrow a * b^{-1} \in H$.

We have to prove $(H, *)$ is a subgroup of $(G, *)$.

let $a \in H$, then $a \in H, a \in H \Rightarrow a * a^{-1} \in H$.

i.e., $e \in H$.

$\therefore H$ contains the identity element

$e \in H, a \in H \Rightarrow e * a^{-1} \in H$
i.e., $a^{-1} \in H$.

$\therefore$ Inverse of each element in H exists in H.

Let $b \in H$, then $b^{-1} \in H$
$a \in H, b^{-1} \in H \Rightarrow a * (b^{-1})^{-1} \in H$.
i.e., $a * b \in H$.

$\therefore H$ is closed under $*$.

Since $H \subseteq G$ and G is associative
$\therefore (H, *)$ is associative.

Thus $(H, *)$ is a group.
and hence $(H, *)$ is a subgroup of $(G, *)$.

Hence proved.

HOMOMORPHISM

Let $(G, *)$ and $(H, \oplus)$ be two groups. A mapping $\phi : G \to H$ is said to be homomorphism if
$\phi(a * b) = \phi(a) \oplus \phi(b)$, $\forall a, b \in G$.

Monomorphism: A one-one homomorphism is said to be monomorphism.

Epimorphism: A onto homomorphism is called epimorphism.

~~Isomorphism~~.

ISOMORPHISM

A homomorphism is said to be an isomorphism if it is both monomorphism (one-one) and epimorphism (onto).

Endomorphism: A homomorphism from a group G onto itself is said to be endomorphism.

Automorphism: An isomorphism from group G onto itself is said to be an automorphism.
(or)
A homomorphism from a group G onto itself is said to be automorphism if it is one-one and onto.

—— Note from for before page

Note: All other subgroup of $(G, *)$ are called proper subgroups except $(\{e\}, *)$ and $(G, *)$, which are called as trivial subgroups of $(G, *)$.

Q: Let $S = \{a, b, c\}$ and let $*$ denote a binary operation on S given by following table. Also let $P = \{1, 2, 3\}$ and $\oplus$ be a binary operation on P given by following table. Show that $(S, *)$ and $(P, \oplus)$ are isomorphic.

$*$	a	b	c
a	a	b	c
b	b	b	c
c	c	b	c

$\oplus$	1	2	3
1	1	2	1
2	1	2	2
3	1	2	3

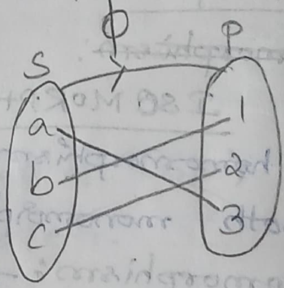
Sol: Given, $S = \{a, b, c\}$, $P = \{1, 2, 3\}$. The binary operations $*$ and $\oplus$ are given by.

$*$	a	b	c
a	a	b	c
b	b	b	c
c	c	b	c

$\oplus$	1	2	3
1	1	2	1
2	1	2	2
3	1	2	3

Let $\phi : S \to P$ given by

$\phi(a) = 3$, $\phi(b) = 1$ and $\phi(c) = 2$.



Now for $a, b \in S$

$$\phi(a * b) = \phi(a) \oplus \phi(b)$$
$$\phi(b) = 3 \oplus 1$$
$$1 = 1$$

for $b, c \in S$

$$\phi(b * c) = \phi(b) \oplus \phi(c)$$
$$\phi(c) = 1 \oplus 2$$
$$2 = 2$$

Similary $\forall\, x, y \in S$, $\phi(x * y) = \phi(x) \oplus \phi(y)$.

$\therefore \phi$ is a homomorphism.

Clearly we see that ϕ is one-one and there is not element in P without a pre-image in S.

Hence ϕ is onto.

$\therefore$ Hence ϕ is a isomorphism

Thus $(S, *)$ and $(P, \oplus)$ are isomorphic.

Example of isomorphism:

Let $(G, +)$ and $(G', \cdot)$ are two groups with binary operations $+$ and $\cdot$.

f is defined from G to G' as $f(x) = e^x$.

We prove that $f: G \to G'$ is a ~~homo~~ isomorphism.

(i) homomorphism:

let $x_1, x_2 \in G$

$$f(x_1 + x_2) = e^{x_1 + x_2}$$

$$= e^{x_1} \cdot e^{x_2}$$

$$\therefore f(x_1 + x_2) = f(x_1) \cdot f(x_2)$$

$\therefore f$ is a homomorphism.

(ii) one-one:

let $f(x_1) = f(x_2)$

$$e^{x_1} = e^{x_2}$$

$$\Rightarrow x_1 = x_2$$

$\therefore f$ is one-one

(iii) onto:

$$f(x) = e^x$$

let $f(x) = y$

$$\Rightarrow y = e^x.$$

$$\Rightarrow x = \log_e y = \ln y.$$

$\therefore f$ is onto.

$\therefore f$ is isomorphism.

UNIT - III

ELEMENTARY COMBINATORICS

Basis of counting: There are two fundamental principles of counting called

① Sum rule (or) Addition principle (or) Principle of disjunctive counting.

② Product rule (or) Multiplication principle (or) Principle of sequential counting.

① Sum rule: If a set X is the union of disjoint non-empty subsets $S_1, S_2, \ldots, S_n$ then $|X| = |S_1| + |S_2| \ldots + |S_n|$ which can also be stated as if $E_1, E_2, E_3, \ldots, E_n$ are mutually exclusive events and E_1 can happen in e_1 ways, E_2 can happen in e_2 ways, ..., E_n can happen in e_n ways then E_1 or E_2 or E_3 ..... or E_n can happen $e_1 + e_2 + \cdots + e_n$ ways.

Q: ① If there are 14 boys and 12 girls in a class, find the number of ways of selecting one student.

Sol: For a selection of one student either of the two tasks is to be performed.

① Selecting a boy among 14 boys (or)

② Selecting a girl among 12 girls.

Selecting a boy among 14 boys can happen in 14 ways.

Selecting a girl among 12 girls can happen in 12 ways.

∴ By sum rule, either of the two tasks can be performed in $14 + 12 = 26$ ways.

② In how many ways can we draw a heart or a spade card from an ordinary deck of playing cards? A heart or an ace? An ace or a king? A card numbered 2 through 10? A numbered card or a king?

Sol: ① The no. of ways of selecting a heart = 13 ways.

② The no. of ways of selecting a spade card = 13 ways.

∴ By sum rule, the no. of ways of selecting a heart card or a spade card $= 13 + 13 = 26$ ways.

(2). The number of ways of selecting a heart card = 13 ways
The number of ways of selecting an ace = 4 - 1 = 3 ways.
∴ By sum rule, the no. of ways of selecting a heart & an ace card = 13 + 3 = 16 ways.

(3) The no. of ways of selecting an ace = 4 ways.
The no. of ways of selecting a king = 4 ways.
∴ By sum rule, the no. of ways of selecting an ace & a king = 4 + 4 = 8 ways.

(4) The no. of ways of selecting numbered cards 2 through 10 = 9+9+9+9 = 36 ways.

(5) The no. of ways of selecting numbered card = 36 ways.
The no. of ways of selecting a king = 4 ways.
∴ By sum rule, the no. of ways of selecting a numbered card & a king = 36 + 4 = 40 ways.

Product rule: If an event can occur in 'm' ways and a second event in 'n' ways and if the no. of ways the second event occurs does not depend upon the occurence of first event then the two events can occur simultaneously in 'mn' ways. In general if events $E_1, E_2, ----, E_n$ can happen in ~~e_1, e_2~~ $n_1, n_2, ---, n_n$ ways, then the sequence of events E_1 first follows by E_2 and so on follows by E_n can happen in $n_1 \times n_2 \times ---\times n_n$ ways.

Q: (1) Three persons enter into a car, there are 5 seats in a car. In how many ways can they take up their seats.

Sol: The first person has a choice of 5 seats and can sit in any of those 5 seats.
So there are 5 ways for first person
The second person has choice of 4 seats.
The third person has choice of 3 seats.
Hence the required no. of ways in which all three persons can take up their seats is $5 \times 4 \times 3 = 60$ ways.

(2) In how many ways can one select two books ~~from~~ of different subjects ~~from~~ among six distinct computer science books, 3 distinct maths books and two distinct chemistry books.

Sol: Using product rule, one can select two books from different subjects as follows.

One CS and one maths in 6×3 ways $= 18$ ways.
One CS and one chem in 6×2 ways $= 12$ ways.
One maths and one chem in 3×2 ways $= 6$ ways.

These sets are pairwise disjoint one can use sum rule to get the required no. of ways which is $18+12+6 = 36$ ways

Factorial: The product of first n natural numbers $1, 2, 3, ---, n$ is denoted by $n!$

Thus $n! = n(n-1)(n-2) ------ 3.2.1$

$n! = n.(n-1)!$.

$\to 0! = 1$
$\to 1! = 1$

COMBINATION: An unordered selection of objects taken some or all at a time is called Combination.

Thus any unordered selection of r objects from a set of n objects is called an r-combination of n objects and is denoted by $C(n, r)$ or ^nC_r ~~given by~~ (without repetition)

$\cancel{C(n, r) \text{ or } {}^nC_r = \frac{n!}{r!(n-r)!}}$ $C(n, r)$ or $^nC_r = \frac{n!}{r!(n-r)!}$

Ex: Selection of 2 batsmen from a set of 3 batsmen {Sachin, Dravid, Virat}.

Here $n = 3$, $r = 2$

$$^3C_2 = \frac{3!}{2!(3-2)!} = 3$$

i.e., {S, D}, {D, V}, {V, S}.

Results:

1. $^nC_0 = 1$, $^nC_1 = n$.
2. $^nC_n = 1$
3. $^nC_r = {}^nC_{n-r}$
4. $^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$.
5. The no. of combinations of r objects among n objects if the repetitions is allowed is given by

$$C(n+r-1, n-1) = \frac{(n+r-1)!}{r!(n-1)!}.$$

⑥ The no. of ways of distributing n object to r people so that each one get atleast one object is ~~$C(n-1)$~~ ${}^{n-1}C_{n-r}$.

⑦ The no. of non-negative integer solutions of $x_1 + x_2 + \cdots + x_r = n$ such that $x_i > 0$ is ${}^{n-1}C_{n-r}$.

⑧ The no. of non-negative integer solutions of $x_1 + x_2 + \cdots + x_r = n$ such that $x_i \geq 0$ is ${}^{n+r-1}C_n$.

PERMUTATION: An arrangement of a given set of objects taken some or all of them at a time is called a ~~permutation~~ permutation.

Thus any arrangement of r objects from a set of n-objects is called r-permutation of n objects, and without repet is denoted by $P(n, r)$ or ${}^nP_r$ given by

$$P(n,r) = {}^nP_r = \frac{n!}{(n-r)!}$$

Ex: No. of ways of arranging the letters in ~~MATHS~~ HAT taken 2 at a time is

Here $n = 3, r = 2$

$${}^nP_r = {}^3P_2 = \frac{3!}{(3-2)!} = 6.$$

i.e., HA, AH, AT, TA, HT, TH.

Results:

① ${}^nP_0 = 1$

② ${}^nP_n = n!$

③ ${}^nP_1 = n$.

④ ${}^nP_{n-1} = {}^nP_n$

⑤ The no. of permutation of n objects in which r_1 objects are of 1st type, r_2 objects are of 2nd type, ----, r_m objects are of m^{th} type and the rest are distinct is $\frac{n!}{r_1! r_2! \cdots r_m!}$.

⑥ The circular permutation of n objects is $(n-1)!$.

PROBLEMS

① In how many ways five students can be selected from 12 student (i) without replacement (ii) If two students not to be included.

Sol: (i) Total no. of students = 12.
5 students can be selected from 12 students without replacement in ${}^{12}C_5 = 792$ ways.
(ii) If two students not to be included, then we have to select 5 students from 10 students, which can be done in ${}^{10}C_5 = 252$ ways

② How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5. (i) If repetition of digits is allowed. (ii) If repetition of digits is not allowed.

Sol: (ii) To form a 4-digit no. zero should not be in thousands place.
So, 1st place can be filled in 5 ways (1, 2, 3, 4, 5).
2nd place can be filled in 5 ways including 0.
3rd " " " " " 4 ways.
4th " " " " " 3 ways.
Hence by product rule, total no. of 4-digit nos = $5 \times 5 \times 4 \times 3$
= 300 ways.

(i) If repetition is allowed, the first place can be filled in 5 ways (excluding 0) because 0 should not be in thousands place. And each of the remaing places can be filled in 6 ways.
Hence by product rule, total no. of 4-digit no.s = $5 \times 6 \times 6 \times 6$
= 1080 ways.

③ How many different license plates are there that involve 1, 2 or 3 letters followed by 4 digits?

Sol: One letter followed by 4 digits can be filled in

L	D	i	git	s
□	□	□	□	□
26	10	10	10	10

$= 26 \times 10^4$ ways.

Two letters followed by 4 digits can be filled in

L	L	D	D	D	D
□	□	□	□	□	□
26	26	10	10	10	10

$= 26^2 \times 10^4$ ways.

Three letters followed by 4 digits can be filled in

L	L	L	D	D	D	D
□	□	□	□	□	□	□
26	26	26	10	10	10	10

$= 26^3 \times 10^4$ ways.

∴ By sum rule, total no. of license plates $= (26 + 26^2 + 26^3) \times 10^4$
$= 18278 \times 10^4$ ways

④ How many ways are there to seat 10 boys and 10 girls around a circular table?

Sol: Total no. of people = 10 boys + 10 girls
= 20.

The no. of circular permutations = ~~20~~ $(n-1)!$
$= (20-1)!$
$= 19!$

⑤ Out of 5 men and 2 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included.

Sol: There are 5 men and 3 women, committee of 3 is to be formed with atleast one women included.

Total - men
$({}^{8}C_3 - {}^{5}C_3)$

(5) Men	(2) Women	No. of ways
2	1	${}^{5}C_2 \times {}^{2}C_1 = 20$
1	2	${}^{5}C_1 \times {}^{2}C_2 = 5$

∴ Total no. of ways = 20+5 = 25 ways.

⑥ Find the no. of arrangements of the letters in the word ACCOUNTANT.

Sol: In the word ~~account~~ ACCOUNTANT.

there are
A - 2
C - 2
O - 1
U - 1
N - 2
T - 2

Total no. of letters = 10.

∴ No. of arrangements $= \dfrac{10!}{2!\,2!\,2!\,2!} = 226800$

⑦ In how many ways can the letters of the word "COMPUTER" be arranged? How many of them begin with C and end with R? How many of them do not begin with C but end with R?

Sol: Total no. of letters in COMPUTER = 8.

No. of ways it can be arranged $= 8! = {}^{8}P_8 = 40320$ arr

If C occupies 1st place and R occupies last place, the remaining places should be arranged by 6 letters

(C, M, P, U, T, E) which can be done in ${}^6P_6 = 6! = 720$ ways

If C does not occupy the first place and R occupies the last place.

☐ ☐ ☐ ☐ ☐ ☐ ☐ $\boxed{R}$ $= 6 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

6 6 5 4 3 2 1 $= 4320$ ways.

(without C) (including C)

⑧ A student is to answer 12 out of 15 questions in an examination. How many choices does the students have

(i) In all?

(ii) If he must answer the first two questions?

(iii) If he must answer the first or the second question but not both?

(iv) If he must answer exactly 3 of the first 5 questions?

(v) If he must answer at least 3 of the first 5 questions.

Sol: (i) Total no. of questions = 15.

No. of Questions to be answered = 12.

No. of choices the student have = ${}^{15}C_{12} = 455$

(ii) If he must answer the first two questions, then he must answer the remaining 10 questions out of other 13 questions.

No. of choices the student have = ${}^{13}C_{10} = 286$

(iii) One question from the first or second question can be answered in ${}^2C_1$ ways.

11 question from the remaining 13 questions can be answered in ${}^{13}C_{11}$ ways.

No. of choices the student have in this case = ${}^2C_1 \times {}^{13}C_{11}$ = 156.

(iv) Exactly 3 of the first 5 questions can be answered in ${}^5C_3$ ways.

And 9 out of remaining 10 questions can be answered in ${}^{10}C_9$ ways.

No. of choices the student have = ${}^5C_3 \times {}^{10}C_9 = 100$.

(v) If he must answer at least 3 of the first 5 questions,

1st 5 Q	Remaining 10 Q	No. of ways
3	9	${}^{5}C_3 \times {}^{10}C_9 = 100$
4	8	${}^{5}C_4 \times {}^{10}C_8 = 225$
5	7	${}^{5}C_5 \times {}^{10}C_7 = 120$

∴ By sum rule total no. of choices $= 100 + 225 + 120$
$= 445$.

(9) Find the no. of ~~word~~ arrangements of the letters of the word MATHEMATICS.

Sol: In the word MATHEMATICS, no. of letters = 11,
i.e., 2-M's, 2-A's, 2-T's, 1-H, 1-E, 1-I, 1-C, 1-S.

∴ No. of arrangements $= \dfrac{11!}{2!\,2!\,2!} = 4989600$

(10) There are 21 consonants and 5 vowels in the English alphabet. Consider only 10-letter words with 4 different vowels and 6 consonants.

(i) How many such words can be formed?
(ii) How many contain the letter 'a'?
(iii) How many begin with 'b' and end with 'c'?

Sol:

(i) 4 different vowels from 5 vowels can be selected in ${}^{5}C_4$ ways.
6 consonants from 21 consonants can be selected in ${}^{21}C_6$ ways.
These 10 letters can be arranged in ${}^{10}P_{10} = 10!$ ways.

∴ By product rule, no. of such words $= {}^{5}C_4 \times {}^{21}C_6 \times 10!$

(ii) If the word should contain letter 'a', then 3 vowels are selected from remaining 4 vowels in ${}^{4}C_3$ ways.
6 consonants are selected from 21 consonants in ${}^{21}C_6$ ways
These 10 letters can be arranged in $10!$ ways.

∴ No. of words containing 'a' $= {}^{4}C_3 \times {}^{21}C_6 \times 10!$.

(iii) Since the words begin with 'b' and end with 'c', 4 consonants are selected from remaining 19 consonants in $^{19}C_4$ ways.

4 vowels out of 5 vowels are selected in $^5C_4$ ways.

These letters can be arranged in $8!$ ways.

$\therefore$ No. of words begin with 'b' and end with 'c' $= {}^{19}C_4 \times {}^5C_4 \times 8!$.

(11) Consider the word "TALLAHASSEE". How many arrangements are there

(i) Altogether?

(ii) Where two letters A appear together?

(iii) where the letters S are taken together and the letters E are together?

(iv) If 4 of the letters are taken?

Sol: (i) No. of letters in TALLAHASSEE = 11

we have 3-A, 1-T, 2-L, 1-H, 2-S, 2-E.

$\therefore$ No. of arrangements altogether $= \dfrac{11!}{3!\,2!\,2!\,2!} = 831600.$

(ii) Let us consider 2-A's as one letter. Then we have

total no. of letters = 10.

i.e., 2-A, 1-T, 2-L, 1-H, 2-S and 2-E

$\therefore$ No. of arrangements $= \dfrac{10!}{2!\,2!\,2!\,2!} = 226800$

(iii) Let us consider 2-S's as one letter and 2-E's as one letter. Then we have,

total no. of letter = 9.

i.e., 3-A's, 1-T, 2-L, 1-H, 1-S, 1-E.

$\therefore$ No. of arrangements $= \dfrac{9!}{3!\,2!} = 30240.$

(iv) The possible ways to arrange four letters is given by.

Types of letters	No. of arrangements.
4-different	$^6C_4 \times 4!$ or $^6P_4 = 360$
2-alike, 2-different	$^4C_1 \times {}^5C_2 \times \dfrac{4!}{2!} = 480$
2-alike, 2-alike	$^4C_2 \times \dfrac{4!}{2!\,2!} = 36$
3-alike, 1-different.	$1 \times {}^5C_1 \times \dfrac{4!}{3!} = 20.$

$\therefore$ By sum rule, total no. of arrangements $= 360+480+36+20$

$= 896.$

(12) A man has 15 close friends of whom 6 are women.
(i) In how many ways can he invite 3 or more of his friends to a party?
(ii) In how many ways can he invite 3 or more of his friends if he wants to invite same number of men and women?

Sol: (i) No. of ways he can invite 3 or more of his friends is $= {}^{15}C_3 + {}^{15}C_4 + \dots + {}^{15}C_{15} = 32647$

(ii) The possible ways to select same number of men and women if he can invite 3 or more friends is given below.
No. of men = Total friends - No. of women $= 15 - 6 = 9$

Men	Women	No. of ways.
2	2	${}^9C_2 \times {}^6C_2 = 540$
3	3	${}^9C_3 \times {}^6C_3 = 1680$
4	4	${}^9C_4 \times {}^6C_4 = 1890$
5	5	${}^9C_5 \times {}^6C_5 = 756$
6	6	${}^9C_6 \times {}^6C_6 = 84.$

∴ Total no. of ways $= 540 + 1680 + 1890 + 756 + 84 = 4950.$

(13) A committee of 5 men and 3 women is to be formed out of 7 men and 6 women. If two particular women are not to be together in the committee, then how many such committees can be formed?

Sol: There are two possible cases to select such committees.
Case (i): If two women are excluded.
Then 5 men out of 7 men are selected in ${}^7C_5$ ways, 3 women out of remaining 4 women can be selected in ${}^4C_3$ ways.
∴ By product rule, no. of ways $= {}^7C_5 \times {}^4C_3 = 84.$
Case (ii): If one of the two women is selected.
One of the two women are selected in ${}^2C_1$ ways and the other other are selected from the remaining 4 women in ${}^4C_2$ ways, while the men are selected in ${}^7C_5$ ways.
∴ By product rule, no. of ways $= {}^7C_5 \times {}^2C_1 \times {}^4C_2 = 252.$

∴ By sum rule total no. of required ways $= 84 + 252 = 336.$

(14) Enumerate the number of non-negative integral solutions to the inequality $x_1+x_2+x_3+x_4+x_5 \le 19$.

Since we want non negative solutions, $x_i \ge 0$

Sol: If $x_1+x_2+x_3+x_4+x_5=0$ then $n=0, r=5$.

$^{n+r-1}C_n = {}^{0+5-1}C_0 = {}^4C_0$

If $x_1+x_2+x_3+x_4+x_5=1$ then $n=1, r=5$

$^{n+r-1}C_n = {}^{1+5-1}C_1 = {}^5C_1$

If $x_1+x_2+x_3+x_4+x_5=2$, then $n=2, r=5$

$^{n+r-1}C_n = {}^{2+5-1}C_2 = {}^6C_2$.

If $x_1+x_2+x_3+x_4+x_5=3$, then $n=3, r=5$.

$^{n+r-1}C_n = {}^{3+5-1}C_3 = {}^7C_3$

If $x_1+x_2+x_3+x_4+x_5=4$, then $n=4, r=5$.

$^{n+r-1}C_n = {}^{4+5-1}C_4 = {}^8C_4$

If $x_1+x_2+x_3+x_4+x_5=5$, then $n=5, r=5$.

$^{n+r-1}C_n = {}^{5+5-1}C_5 = {}^9C_5$

If $x_1+x_2+x_3+x_4+x_5=6$, then $n=6, r=5$

$^{n+r-1}C_n = {}^{6+5-1}C_6 = {}^{10}C_6$.

Similarly for $n=7,8,9,10,11,12,13,14,15,16,17,18,19$

we get $^{11}C_7$, $^{12}C_8$, $^{13}C_9$, $^{14}C_{10}$, $^{15}C_{11}$, $^{16}C_{12}$, $^{17}C_{13}$, $^{18}C_{14}$, $^{19}C_{15}$, $^{20}C_{16}$, $^{21}C_{17}$, $^{22}C_{18}$, $^{23}C_{19}$.

$\therefore$ The no. of non-negative integral solutions

$= {}^4C_0 + {}^5C_1 + {}^6C_2 + {}^7C_3 + {}^8C_4 + {}^9C_5 + {}^{10}C_6 + {}^{11}C_7 + {}^{12}C_8 + {}^{13}C_9 + {}^{14}C_{10} + {}^{15}C_{11} + {}^{16}C_{12} + {}^{17}C_{13} + {}^{18}C_{14} + {}^{19}C_{15} + {}^{20}C_{16} + {}^{21}C_{17} + {}^{22}C_{18} + {}^{23}C_{19}$

$= 1+5+15+35+70+126+210+330+495+715+1001+1365+1820+2380+3060+3876+4845+5985+7315+8855$

$= 42504$

(15) How many integral solutions are there to the equation $x_1+x_2+x_3+x_4+x_5=20$ where $x_1 \geq -3$, $x_2 \geq 0$, $x_3 \geq 4$, $x_4 \geq 2$ and $x_5 \geq 2$.

Sol: Given, $x_1+x_2+x_3+x_4+x_5=20$ — (1), where

$x_1 \geq -3$, $x_2 \geq 0$, $x_3 \geq 4$, $x_4 \geq 2$ and $x_5 \geq 2$.

~~let us consider~~

$\Rightarrow x_1+3 \geq 0,\ x_2 \geq 0,\ x_3-4 \geq 0,\ x_4-2 \geq 0,\ x_5-2 \geq 0$

Let us consider,

$y_1 = x_1+3,\ y_2 = x_2,\ y_3 = x_3-4,\ y_4 = x_4-2,\ y_5 = x_5-2.$

$\Rightarrow x_1 = y_1-3,\ x_2 = y_2,\ x_3 = y_3+4,\ x_4 = y_4+2,\ x_5 = y_5+2.$

Substituting these x_1, x_2, x_3, x_4 and x_5 values in eqn (1)

we get:

$y_1-3+y_2+y_3+4+y_4+2+y_5+2=20.$

$y_1+y_2+y_3+y_4+y_5+5=20$

$y_1+y_2+y_3+y_4+y_5=20-5$

$y_1+y_2+y_3+y_4+y_5=15$, where $y_i \geq 0$.

Here $n=15$, $r=5$.

$\therefore$ The no. of integral solutions $= {}^{n+r-1}C_n$

$= {}^{15+5-1}C_{15}$

$= {}^{19}C_{15}$

$= 3876.$

(16) Compute $P(8,5)$ and $C(6,3)$.

Sol: $P(8,5) = {}^8P_5 = \dfrac{n!}{(n-r)!} = \dfrac{8!}{(8-5)!}$

$= \dfrac{8\times7\times6\times5\times4\times\cancel{3!}}{\cancel{3!}}$

$\therefore P(8,5) = 6720.$

$C(6,3) = {}^6C_3 = \dfrac{n!}{r!(n-r)!} = \dfrac{6!}{3!(6-3)!} = \dfrac{\cancel{6}\times5\times4\times\cancel{3!}}{\cancel{3!}\times\cancel{3}\times\cancel{2}\times1}$

$\therefore C(6,3) = 20$

(17) How many integers are between 1 and 10,00,000 having the sum of the digits as 18?.

Note: It is not possible for two digits to be ≥ 10 simultaneously as their sum at least would be 20.

Sol: Let x_1 = digit in one's place

x_2 = " " ten's place

x_3 = " " hundredth's place

x_4 = " " thousands "

x_5 = " " ten thousands place

x_6 = " " lakhs place.

where $0 \leq x_i \leq 9$.

Then we have to calculate the non-negative integral solutions to the equation $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$ —(1)

where $0 \leq x_i \leq 9$.

Firstly, the total no. of non-negative integral solutions to $x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 18$, where $x_i \geq 0$ is given by,

Here $n = 18$, $r = 6$.

No. of solutions $= {}^{n+r-1}C_n = {}^{18+6-1}C_{18} = {}^{23}C_{18} = 33649$.

Now for $x_i \leq 9$.

solutions where at least one digit is 10 or greater must be ~~ect~~ excluded.

i.e., Let $x_1 \geq 10$, and $x_i \geq 0$ for $i = 2, 3, 4, 5, 6$.

$x_1 - 10 \geq 0$

~~Let $y_1 \geq 0$~~

Let $y_1 = x_1 - 10 \geq 0$, $y_2 = x_2$, $y_3 = x_3$, $y_4 = x_4$, $y_5 = x_5$, $y_6 = x_6$.

$\Rightarrow y_1 + 10 = x_1$

Eqn (1) $\Rightarrow y_1 + 10 + y_2 + y_3 + y_4 + y_5 + y_6 = 18$

$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 + y_6 = 18 - 10 = 8$

Here $n = 8$, $r = 6$.

No. of solutions $= {}^{n+r-1}C_n = {}^{8+6-1}C_8 = {}^{13}C_8 = 1287$

Since any of the 6 digits can be greater than or equal to 10, the no. of solutions to be subtracted $= 6 \times 1287$
$= 7722$.

$\therefore$ Total no. of integers b/w 1 and 1000000 having sum of digits as 18 $= 33649 - 7722 = 25927$.

(18) How many integral solutions are there to the equation $x_1+x_2+x_3+x_4+x_5=30$ where for each i,

i) $x_i \geq 1$

ii) $x_1 \geq 2, x_2 \geq 2, x_3 \geq 4, x_4 \geq 2$ and $x_5 \geq 0$.

iii) $x_i \geq i+1$.

Sol: Given, $x_1+x_2+x_3+x_4+x_5=30$ — ①.

(i) Given, $x_i \geq 1 \Rightarrow x_i - 1 \geq 0$.

Let us consider $y_i = x_i - 1 \geq 0$

$\Rightarrow y_i + 1 = x_i$.

Now, ① $\Rightarrow y_1+1+y_2+1+y_3+1+y_4+1+y_5+1 = 30$

$\Rightarrow y_1+y_2+y_3+y_4+y_5 = 30-5 = 25$

Here $n=25, r=5$.

No. of integral solutions of ① $= {}^{n+r-1}C_n = {}^{25+5-1}C_{25}$

$= {}^{29}C_{25} = 23751$.

(ii) Given, $x_1 \geq 2, x_2 \geq 2, x_3 \geq 4, x_4 \geq 2$ and $x_5 \geq 0$,

Let us consider $y_1 = x_1-2, y_2 = x_2-2$ $y_3 = x_3-4$, $y_4 = x_4-2, y_5 = x_5-0$.

$\Rightarrow y_1+2 = x_1, y_2+2 = x_2, y_3+4 = x_3, y_4+2 = x_4, y_5 = x_5$.

Now ① $\Rightarrow y_1+2+y_2+2+y_3+4+y_4+2+y_5 = 30$.

$\Rightarrow y_1+y_2+y_3+y_4+y_5 = 30 - \cancel{11}\ 10 = \cancel{20}\ \cancel{19}\ 20$

Here $n = \cancel{19}\ 20, r=5$.

No. of integral solutions of ① $= {}^{n+r-1}C_n = {}^{\cancel{19}\ 20+5-1}C_{\cancel{19}\ 20} = \cancel{{}^{23}C_{19}} = \cancel{10626} = \cancel{8855}$

$= {}^{24}C_{20} = 10626$

(iii) Given $x_i \geq i+1$

i.e., $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 5$ and $x_5 \geq 6$.

Let us consider, $y_1 = x_1-2, y_2 = x_2-3, y_3 = x_3-4$, $y_4 = x_4-5, y_5 = x_5-6$, where $y_i \geq 0$.

$\Rightarrow y_1+2 = x_1, y_2+3 = x_2, y_3+4 = x_3, y_4+5 = x_4, y_5+6 = x_5$.

Now ① $\Rightarrow y_1+2+y_2+3+y_3+4+y_4+5+y_5+6 = 30$.

$\Rightarrow y_1+y_2+y_3+y_4+y_5 = 30-20 = 10$

Here $n=10, r=5$.

No. of integral solutions $= {}^{n+r-1}C_n = {}^{10+5-1}C_{10} = {}^{14}C_{10} = 1001$

BINOMIAL THEOREM

If $n \in \mathbb{N}$ be any non-negative integer, then the binomial theorem is given by

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r y^{n-r}.$$

Note:

1. Binomial theorem is also stated as

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r.$$

2. ${}^nC_r$ is the binomial coefficient
3. The expansion has $(n+1)$ terms.

Ex: $(x+y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3 y + {}^4C_2 x^2y^2 + {}^4C_3 xy^3 + {}^4C_4 y^4$

$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$

MULTINOMIAL THEOREM

The multinomial theorem generalizes the binomial theorem to more than two terms.

It is stated as, for any positive integer n,

$$(x_1 + x_2 + \cdots + x_t)^n = \sum_{r_1+r_2+\cdots+r_t = n} \frac{n!}{r_1!\, r_2! \cdots r_t!} x_1^{r_1} x_2^{r_2} \cdots x_t^{r_t}.$$

where $r_1 + r_2 + \cdots + r_t = n$, $r_i \geq 0$.

~~Here~~

Note:

1. $\dfrac{n!}{r_1!\, r_2! \cdots r_t!}$ is the multinomial coefficient which is also written as $\binom{n}{r_1, r_2, \ldots, r_t}$

2. The general term of the multinomial theorem is

$$= \frac{n!}{r_1!\, r_2! \cdots r_t!} x_1^{r_1} x_2^{r_2} \cdots x_t^{r_t}.$$

where $r_1 + r_2 + \cdots + r_t = n$; $r_i \geq 0$.

PROBLEMS :

① Find the binomial expansion of $(2a-3b)^4$.

Sol: Here $x = 2a,\ y = -3b,\ n = 4$.

Binomial theorem is given by

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r.$$

$$\Rightarrow \cancel{(2a-3b)^4 = \sum_{r=0}^{4} {}^4C_r (2a)^{4-r} y^r}.$$

$$= \cancel{{}^4C_0 (2a)^{4-0} (3b)^0} + \cancel{{}^4C_1 (2a)^{4-1} (-3b)^1}$$
$$+ \cancel{{}^4C_2 (2a)^{4-2} (3b)^2} + \cancel{{}^4C_3 (2a)^{4-3} (3b)^3}.$$
$$+ \cancel{{}^4C_4 (2a)^{4-4} (3b)^4}.$$

$$= \cancel{16a^4 + 3}$$

$$\Rightarrow (2a-3b)^4 = \sum_{r=0}^{4} {}^4C_r (2a)^{4-r} (-3b)^r$$

$$= {}^4C_0 (2a)^{4-0} (-3b)^0 + {}^4C_1 (2a)^{4-1} (-3b)^1$$
$$+ {}^4C_2 (2a)^{4-2} (-3b)^2 + {}^4C_3 (2a)^{4-3} (-3b)^3$$
$$+ {}^4C_4 (2a)^{4-4} (-3b)^4.$$

$$= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4.$$

② Find the coefficient of x^3y^7 in $(x+y)^{10}$.

Sol: Here $n = 10$

Binomial theorem is given by

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r} y^r.$$

$$(x+y)^{10} = \sum_{r=0}^{10} {}^{10}C_r x^{10-r} y^r.$$

Consider $r = 7$ term in the above expansion.

$$= {}^{10}C_7 x^{10-7} y^7 = 120 x^3 y^7.$$

∴ The coefficient of x^3y^7 in $(x+y)^{10}$ is 120.

③ Find the coefficient of x^2y^4 in $(x-2y)^6$.

Sol: Here $n=6$, $x=x$, $y=-2y$.

Binomial theorem is

$$(x+y)^n = \sum_{r=0}^{n} {}^nc_r\, x^{n-r} y^r$$

$$(x-2y)^6 = \sum_{r=0}^{6} {}^6c_r\, x^{6-r} (-2y)^r.$$

Consider $r=4$ term in the above expansion.

$$= {}^6c_4\, x^{6-4} (-2y)^4 = 15 \times x^2 \times 16y^4$$
$$= 240x^2y^4.$$

∴ The coefficient of x^2y^4 is 240.

④ Find the coefficient of x^6y^3 in $(x-3y)^9$.

Sol: Here $n=9$, $x=x$, $y=-3y$.

Binomial theorem is

$$(x+y)^n = \sum_{r=0}^{n} {}^nc_r\, x^{n-r} y^r.$$

$$(x-3y)^9 = \sum_{r=0}^{9} {}^9c_r\, x^{9-r} (-3y)^r.$$

Consider $r=3$ term in the above expansion.

$$= {}^9c_3\, x^{9-3} (-3y)^3 = 84 \times x^6 \times (-27) \times y^3.$$
$$= -2268x^6y^3.$$

∴ The coefficient of x^6y^3 in $(x-3y)^9$ is -2268.

⑤ Find the coefficient of x^9y^3 in $(2x-3y)^{12}$.

Sol: Here $n=12$, $x=2x$, $y=-3y$.

Binomial theorem is

$$(x+y)^n = \sum_{r=0}^{n} {}^nc_r\, x^{n-r} y^r,$$

$$(2x-3y)^{12} = \sum_{r=0}^{12} {}^{12}c_r\, (2x)^{12-r} (-3y)^r.$$

Consider $r=3$ term in the above expansion.

$$= {}^{12}c_3\, (2x)^{12-3} (-3y)^3 = {}^{12}c_3 \times 2^9 \times (-3)^3 \times x^9 y^3$$
$$= 3041280\, x^9y^3.$$

∴ The coefficient of x^9y^3 in $(2x-3y)^{12}$ is 3041280

(or)

$27 \times 2^9 \times {}^{12}c_3$.

⑥ Find the coefficient of $x^3y^2z^2$ in $(2x-y+z)^7$.

Sol: ~~Here~~

The general term of multinomial theorem is

$$= \frac{n!}{r_1!\, r_2! \ldots r_t!} x_1^{r_1} x_2^{r_2} \ldots x_t^{r_t}.$$

Here $n=7$, $x_1 = 2x$, $x_2 = -y$, $x_3 = z$.

$$\text{General term} = \frac{7!}{r_1!\, r_2!\, r_3!} (2x)^{r_1} (-y)^{r_2} (z)^{r_3}.$$

Consider $r_1 = 3$, $r_2 = 2$, $r_3 = 2$, then we ~~get~~ have

$r_1 + r_2 + r_3 = n$

$3+2+2 = 7$.

$$\text{Term} = \frac{7!}{3!\,2!\,2!} 2^3 x^3 (-y)^2 z^2 = 1680 x^3y^2z^2.$$

∴ Coefficient of $x^3y^2z^2$ in $(2x-y+z)^7$ is 1680.

⑦ Find the coefficient of xyz^2 in $[2x-y-z]^4$.

Sol: The general term of multinomial theorem is

$$= \frac{n!}{r_1!\, r_2! \ldots r_t!} x_1^{r_1} x_2^{r_2} \ldots x_t^{r_t}.$$

Here $n=4$, $x_1 = 2x$, $x_2 = -y$, $x_3 = -z$.

$$\text{General term} = \frac{4!}{r_1!\, r_2!\, r_3!} (2x)^{r_1} (-y)^{r_2} (-z)^{r_3}.$$

Consider $r_1 = 1$, $r_2 = 1$, $r_3 = 2$, then

$r_1 + r_2 + r_3 = n$

$1+1+2 = 4$

$$\text{Term} = \frac{4!}{1!\,1!\,2!} (2x)^1 (-y)^1 (-z)^2.$$

$$= -24\,xyz^2.$$

∴ Coefficient of xyz^2 in $(2x-y-z)^4$ is -24.

⑧ Find the coefficient of $ab^2c^3d^4$ in $[a+2b-3c+2d+5]^{13}$.

Sol: The general term of multinomial theorem is

$$= \frac{n!}{r_1!\, r_2! \ldots r_t!} x_1^{r_1} x_2^{r_2} \ldots x_t^{r_t}.$$

Here $n = 13$, $x_1 = a$, $x_2 = 2b$, $x_3 = -3c$, $x_4 = 2d$, $x_5 = 5$.

General term $= \dfrac{13!}{r_1! r_2! r_3! r_4! r_5!} a^{r_1} (2b)^{r_2} (-3c)^{r_3} (2d)^{r_4} (5)^{r_5}$

~~We have~~
Consider $r_1 = 1, r_2 = 2, r_3 = 3, r_4 = 4$.

WKT, $r_1 + r_2 + r_3 + r_4 + r_5 = n$

$1 + 2 + 3 + 4 + r_5 = 13$

$r_5 = 13 - 10$

$r_5 = 3$.

Term $= \dfrac{13!}{1!\, 2!\, 3!\, 4!\, 3!} a^1 (2b)^2 (-3c)^3 (2d)^4 (5)^3$.

$= -13! \times 125\ ab^2c^3d^4$. $= -7.783776 \times 10^{11}$

$\therefore$ Coefficient of $ab^2c^3d^4$ in $[a+2b-3c+2d+5]^{13}$ is $-13! \times 125$

(9) Find the coefficient of x^2yz in $[2x-y+z+1]^7$.

Sol: The general term of multinomial theorem is

$$= \frac{n!}{r_1! r_2! --- r_t!} x_1^{r_1} x_2^{r_2} \cdots\cdots x_t^{r_t}.$$

Here $n = 7$, $x_1 = 2x$, $x_2 = -y$, $x_3 = z$, $x_4 = 1$.

General term $= \dfrac{7!}{r_1! r_2! r_3! r_4!} (2x)^{r_1} (-y)^{r_2} (z)^{r_3} (+1)^{r_4}$.

Consider $r_1 = 2, r_2 = 1, r_3 = 1$

WKT, $r_1 + r_2 + r_3 + r_4 = n$

$2 + 1 + 1 + r_4 = 7$

$r_4 = 7 - 4$

$r_4 = 3$.

Term $= \dfrac{7!}{2!\, 1!\, 1!\, 3!} (2x)^2 (-y)^1 \cdot (z)^1 (1)^3$.

$= -1680\, x^2yz$

$\therefore$ Coefficient of x^2yz in $[2x-y+z+1]^7$ is -1680

UNIT - IV

RECURRENCE RELATIONS

GENERATING FUNCTIONS OF SEQUENCES:

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of real numbers. The function $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

is called the generating function.

Here $\{a_0, a_1, a_2, \ldots, a_n, \ldots\}$ is called the numeric function of a_n.

Ex: The generating function for the numeric function $\{3^0, 3^1, 3^2, \ldots, 3^n, \ldots\}$ is $f(x) = \sum_{n=0}^{\infty} 3^n x^n$.

① Find the generating function for the sequence. $1, 1, 0, 1, 1, 1, \ldots$

Sol: The given sequence is $1, 1, 0, 1, 1, 1, \ldots$

Let $a_0 = 1, a_1 = 1, a_2 = 0, a_3 = 1, a_4 = 1, a_5 = 1, \ldots$

The generating function is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + a_1 x + a_2 x^2 + \ldots$$

$$= 1 + x + 0 \cdot x^2 + x^3 + \ldots$$

$$= (1 + x + x^2 + x^3 + \ldots) - x^2$$

$$f(x) = (1-x)^{-1} - x^2.$$

② Find the generating function for the sequence $1, 1, 1, 3, 1, 1, \ldots$

Sol: Given sequence is $1, 1, 1, 3, 1, 1, \ldots$

Let $a_0 = 1, a_1 = 1, a_2 = 1, a_3 = 3, a_4 = 1, a_5 = 1, \ldots$

The generating function is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$= 1 + 1 \cdot x + 1 \cdot x^2 + 3 \cdot x^3 + 1 \cdot x^4 + 1 \cdot x^5 + \ldots$$

$$= (1 + x + x^2 + x^3 + x^4 + \ldots) + 2x^3$$

$$f(x) = (1-x)^{-1} + 2x^3.$$

Note:

① $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n.$

② $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-x)^n.$

③ $\frac{1}{(1-x)^2} = (1-x)^{-2} = 1+2x+3x^2+4x^3+ \cdots = \sum_{n=0}^{\infty} (n+1)x^n.$

④ $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1-2x+3x^2-4x^3+ \cdots = \sum_{n=0}^{\infty} (n+1)(-x)^n.$

⑤ $e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+ \cdots.$

Q: Determine the sequence generated by the following generating functions.

③ $f(x) = 2e^x + 3x^2.$

Sol: Given, $f(x) = 2e^x + 3x^2$

$$= 2\left[1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+ \cdots\right]+3x^2$$

$$= 2+2x+\frac{2x^2}{2!}+\frac{2x^3}{3!}+\frac{2x^4}{4!}+ \cdots +3x^2$$

$$f(x) = 2+2x+4x^2+\frac{2x^3}{3!}+\frac{2x^4}{4!}+ \cdots$$

∴ The sequence is $\{2, 2, 4, \frac{2}{3!}, \frac{2}{4!}, \frac{2}{5!}, \cdots\}$

④ $f(x) = 7e^{8x} - 4e^{3x}.$

Sol: Given $f(x) = 7e^{8x} - 4e^{3x}.$

$$= 7\left[1+8x+\frac{(8x)^2}{2!}+\frac{(8x)^3}{3!}+ \cdots\right] - 4\left[1+3x+\frac{(3x)^2}{2!}+\frac{(3x)^3}{3!}+ \cdots\right]$$

$$= 7+56x+224x^2+ \cdots -4x-12x-18x^2- \cdots$$

$$f(x) = 3+44x+206x^2+ \cdots$$

∴ The sequence is $\{3, 44, 206, \cdots\}$.

⑤ $f(x) = (2x-3)^3$

Sol: Given $f(x) = (2x-3)^3$

By binomial theorem,

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r} y^r$$

Here $x = 2x$, $y = -3$, $n = 3$.

$$(2x-3)^3 = \sum_{r=0}^{3} {}^3C_r\, (2x)^{3-r} (-3)^r$$

$$= {}^3C_0 (2x)^{3-0}(-3)^0 + {}^3C_1 (2x)^{3-1}(-3)^1 + {}^3C_2 (2x)^{3-2}(-3)^2 + {}^3C_3 (2x)^{3-3}(-3)^3.$$

$$= 8x^3 - 36x^2 + 54x - 27.$$

$$= -27 + 54x - 36x^2 + 8x^3.$$

∴ The sequence is $\{-27, 54, -36, 8\}$.

⑥ $\frac{x^4}{1-x}$

Sol: $\frac{x^4}{1-x} = x^4(1-x)^{-1}$

$= x^4(1+x+x^2+x^3+\cdots)$

$= x^4+x^5+x^6+x^7+\cdots$

∴ The sequence is $\{0,0,0,0,1,1,1,\cdots\}$

⑦ $\frac{1}{1-ax}$

Sol: $\frac{1}{1-ax} = (1-ax)^{-1}$

$= 1+ax+(ax)^2+(ax)^3+(ax)^4+\cdots$

$= 1+ax+a^2x^2+a^3x^3+a^4x^4+\cdots$

∴ The sequence is $\{1,a,a^2,a^3,a^4,\cdots\}$.

CALCULATING COEFFICIENTS OF GENERATING FUNCTIONS:

① $c(n,r) = \binom{n}{r} = {}^nC_r = \frac{n!}{r!(n-r)!}$

② $1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x}$.

③ The coefficient of x^r in $(1+x+x^2+\cdots)^n$ is ${}^{n+r-1}C_r$.

i.e., $(1+x+x^2+\cdots)^n = [(1-x)^{-1}]^n = (1-x)^{-n}$

$= \sum_{r=0}^{n} {}^{n+r-1}C_r\, x^r$.

PROBLEMS:

① Find the coefficient of x^{20} in $(x^3+x^4+x^5+\cdots)^5$

Sol: Given $(x^3+x^4+x^5+\cdots)^5 = (x^3)^5(1+x+x^2+\cdots)^5$

$= x^{15}[(1-x)^{-1}]^5$

$= x^{15}(1-x)^{-5}$

$\because (1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r x^r$.

$= x^{15}\sum_{r=0}^{\infty} {}^{5+r-1}C_r\, x^r$

$= x^{15}\sum_{r=0}^{\infty} {}^{4+r}C_r\, x^r$.

For $r=5$, we get x^{20} term

$= x^{15}\cdot {}^{4+5}C_5\, x^5 = {}^9C_5\, x^{20} = 126\, x^{20}$.

∴ The coefficient of $x^{20} = 126$.

② Find the coefficient of x^5 in $(1-2x)^{-7}$.

Sol: WKT, $(1-x)^{-n} = \sum_{r=0}^{\infty} {}^{n+r-1}C_r \, x^r$.

$\Rightarrow (1-2x)^{-7} = \sum_{r=0}^{\infty} {}^{7+r-1}C_r \, (2x)^r$

$= \sum_{r=0}^{\infty} {}^{6+r}C_r \, (2x)^r$.

For $r=5$, we get x^5 term

$= {}^{6+5}C_5 \, (2x)^5$

$= {}^{11}C_5 \times 2^5 . x^5 = 14784$.

$\therefore$ The coefficient of x^5 is 14784.

③ Find the coefficient of x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$.

Sol: $(x^2+x^3+x^4+x^5+x^6)^5 = [x^2(1+x+x^2+x^3+x^4)]^5$

$= (x^2)^5 \left[\frac{1-x^{4+1}}{1-x}\right]^5 \quad \because 1+x+x^2+\dots+x^n = \frac{1-x^{n+1}}{1-x}$.

$= x^{10}(1-x^5)^5(1-x)^{-5}$.

$= x^{10}\left[\sum_{r=0}^{5} {}^5C_r (1)^{5-r}(-x^5)^r\right]\left[\sum_{r=0}^{\infty} {}^{5+r-1}C_r \, x^r\right]$.

$= x^{10}\left[{}^5C_0 - {}^5C_1 x^5 + {}^5C_2 x^{10} - {}^5C_3 x^{15} + {}^5C_4 x^{20} - {}^5C_5 x^{25}\right]$

$\left[\sum_{r=0}^{\infty} {}^{4+r}C_r \, x^r\right]$

$\therefore$ The coefficient of $x^{20} = {}^5C_0 \, {}^{14}C_{10} - {}^5C_1 \, {}^9C_5 + {}^5C_2 \, {}^4C_0$

$= 381$

RECURRENCE RELATION :

A mathematical equation that defines a sequence by relating each term to one or more previous terms in the same sequence.

One can carry out step by step computation to determine the values of the a_n from a_{n-1}, a_{n-2}, ... provided that the values of the function at one or more points are given. The given values are called initial conditions or boundary conditions of the recurrence relation.

Ex: The numeric function $(5, 8, 11, 14, \dots)$ is defined by the recurrence relation $a_n = a_{n-1} + 3, n \geq 1$ with the initial condition $a_0 = 5$.

Order of recurrence relation: The order of the recurrence relation is the difference between the largest and smallest subscript appearing in the relation.

Ex: ① $a_n = -3a_{n-1}$ is a RR with order $n-(n-1)=1$.

② $a_{n+2} = a_{n+1} - 2a_n$ is a RR with order $n+2-n=2$.

Linear recurrence relation with constant coefficients:

A recurrence relation of the form,

$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$, where c_i's are constant, is called a linear recurrence relation of k^{th} order provided both c_0 and c_k are non-zero.

* If $f(n)=0$, then the relation is known as linear homogeneous equation.
* If $f(n)\neq 0$, then the relation is known as linear non-homogeneous equation.

Solution of linear recurrence relations:

An explicit formula which satisfy the RR with initial condition is called a solution of RR.

① Iteration (or) Substitution method.

② Characteristic roots.

③ Generating functions

~~Solving recurrence relation by substitution method:~~

SOLVING RECURRENCE RELATION BY SUBSTITUTION METHOD:

In this method RR for a_n is used repeatedly to solve for a general expression for a_n in terms of n.

Problems:

Q: Solve the following RR by substitution method.

① $a_n = a_{n-1} + 2$, $n \geq 2$ subject to initial condition $a_1 = 3$.

Sol: Given, $a_n = a_{n-1} + 2$ —①, $a_1 = 3$, $n \geq 2$.

$n=2$, ① $\Rightarrow a_2 = a_{2-1} + 2 = a_1 + 2$.

$n=3$, ① $\Rightarrow a_3 = a_{3-1} + 2 = a_2 + 2 = a_1 + 2 + 2 = a_1 + 2(2)$.

$n=4$, ① $\Rightarrow a_4 = a_3 + 2 = a_1 + 2(2) + 2 = a_1 + 3(2)$.

⋮

Illy, $a_n = a_1 + (n-1)(2) = 3 + 2n - 2$

$\therefore a_n = 2n+1$.

② $a_n = a_{n-1} + 3^n$, $n \geq 1$, $a_0 = 1$.

Sol: Given, $a_n = a_{n-1} + 3^n$ —①, $a_0 = 1$, $n \geq 1$.

For $n=1$, ① $\Rightarrow a_1 = a_0 + 3$.

For $n=2$, ② $\Rightarrow a_2 = a_1 + 3^2 = a_0 + 3 + 3^2$

For $n=3$, ① $\Rightarrow a_3 = a_2 + 3^3 = a_0 + 3 + 3^2 + 3^3$.

⋮

Similarly, $a_n = a_0 + 3 + 3^2 + 3^3 + \dots + 3^n$.

$a_n = 1 + 3 + 3^2 + \dots + 3^n$.

$a_n = \frac{1 - 3^{n+1}}{1-3}$.

$\therefore a_n = \frac{3^{n+1} - 1}{2}$.

③. $a_n + 3na_{n-1} = 0$, $a_0 = 1$.

Sol: Given, $a_n + 3na_{n-1} = 0$, $a_0 = 1$ — ①

For $n=1$, ① $\Rightarrow a_1 + 3.1.a_0 = 0$

$a_1 = -3.1.a_0$.

For $n=2$, ① $\Rightarrow a_2 + 3.2.a_1 = 0$

$a_2 = -3.2.(-3.1.a_0)$

$= 3^2 . 2! \, a_0$.

For $n=3$, ① $\Rightarrow a_3 + 3.3.a_2 = 0$

$a_3 = -3.3.(3^2.2! \, a_0)$

$= -3^3 . 3! \, a_0$.

⋮

Similarly, $a_n = (-3)^n n! \, a_0 = (-3)^n n!$ $(\because a_0 = 1)$.

SOLVING RECURRENCE RELATIONS BY GENERATING FUNCTIONS METHOD:

Let $a_n + Aa_{n-1} + Ba_{n-2} = 0$ — ① for $n \geq 2$, be a 2nd order recurrence relation.

Multiply ① by x^n and take summation over $\sum_{n=2}^{\infty}$, then we get

$$① \Rightarrow \sum_{n=2}^{\infty} a_n x^n + \sum_{n\geq 2}^{\infty} A a_{n-1} x^n + \sum_{n=2}^{\infty} B a_{n-2} x^n = 0$$

$$\Rightarrow (a_2x^2 + a_3x^3 + a_4x^4 + \dots) + A(a_1x^2 + a_2x^3 + \dots) + B(a_0x^2 + a_1x^3 + \dots) = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n - a_0 - a_1 x + Ax\sum_{n=0}^{\infty} a_n x^n - Axa_0 + Bx^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow G(x) - a_0 - a_1 x + Ax\,G(x) - Axa_0 + Bx^2 G(x) = 0$$

$$\Rightarrow G(x)[1 + Ax + Bx^2] = a_0 + a_1 x + Axa_0.$$

$$\Rightarrow G(x) = \frac{a_0 + x(a_1 + Aa_0)}{1 + Ax + Bx^2}.$$

where $G(x) = \sum_{n=0}^{\infty} a_n x^n$ is the generating function.

Now substitute a_0, a_1 values in above eqn and resolve into partial fractions. The coefficient of x^n in RHS of $G(x)$ determines a_n.

Q: Solve the following recurrence relation using generating functions.

① $a_n - 2a_{n-1} - 3a_{n-2} = 0$, $n \geq 2$ with $a_0 = 3, a_1 = 1$.

Sol: Given, $a_n - 2a_{n-1} - 3a_{n-2} = 0$ —① , $n \geq 2$

with $a_0 = 3, a_1 = 1$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ —②

Multiply both sides of ① by x^n and take $\sum_{n=2}^{\infty}$, we get

$$① \Rightarrow \sum_{n=2}^{\infty} a_n x^n - 2\sum_{n=2}^{\infty} a_{n-1} x^n - 3\sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$$\Rightarrow (a_2 x^2 + a_3 x^3 + \cdots) - 2(a_1 x^2 + a_2 x^3 + \cdots) - 3(a_0 x^2 + a_1 x^3 + \cdots) = 0$$

$$\Rightarrow (a_0 + a_1 x + a_2 x^2 + \cdots) - a_0 - a_1 x - 2x(a_0 + a_1 x + a_2 x^2 + \cdots) + 2x a_0$$
$$-3x^2(a_0 + a_1 x + a_2 x^2 + \cdots) = 0$$

$$\Rightarrow G(x) - a_0 - a_1 x - 2xG(x) + 2xa_0 - 3x^2 G(x) = 0 \quad [\because \text{From } ②]$$

$$\Rightarrow G(x)[1 - 2x - 3x^2] = a_0 + a_1 x - 2x a_0$$

$$\Rightarrow G(x) = \frac{3 + x(1-6)}{1 - 3x + x - 3x^2} = \frac{3-5x}{1(1-3x) + x(1-3x)}$$

$$\Rightarrow G(x) = \frac{3-5x}{(1-3x)(1+x)} \quad —③$$

Now by partial fractions,

$$\frac{3-5x}{(1-3x)(1+x)} = \frac{A}{1-3x} + \frac{B}{1+x}$$

$$3 - 5x = A(1+x) + B(1-3x) \quad —④$$

For $x = -1$, ④ $\Rightarrow 3 - 5(-1) = A(1-1) + B(1 - 3(-1))$

$$8 = B(4)$$

$$\therefore B = 2.$$

For $x = \frac{1}{3}$, ④ $\Rightarrow 3 - 5\left(\frac{1}{3}\right) + A\left(1 + \frac{1}{3}\right) + B\left(1 - 3\left(\frac{1}{3}\right)\right)$

$$\frac{9-5}{3} = A\left(\frac{3+1}{3}\right) + B(0)$$

$$\frac{4}{3} = A\left(\frac{4}{3}\right)$$

$$\therefore A = 1.$$

Then, ③ $\Rightarrow G(x) = \frac{1}{1-3x} + \frac{2}{1+x}$

$$= 1 + 3x + (3x)^2 + (3x)^3 + \cdots + 2(1 - x + x^2 - x^3 + \cdots)$$

$$\Rightarrow G(x) = \sum_{n=0}^{\infty} 3^n x^n + 2\sum_{n=0}^{\infty} (-1)^n x^n.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} (3^n + 2(-1)^n) x^n.$$

$$\therefore a_n = 3^n + 2(-1)^n.$$

(2) $a_n - 5a_{n-1} + 6a_{n-2} = 2^n$, $n \geq 2$ with $a_0 = 1, a_1 = 1$.

Sol: Given, $a_n - 5a_{n-1} + 6a_{n-2} = 2^n$ — (1)

$n \geq 2$, with $a_0 = 1, a_1 = 1$.

let $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ — (2) be the generating function.

Multiply both sides of (1) by x^n and take summation $\sum_{n=2}^{\infty}$, we get,

$$\sum_{n=2}^{\infty} a_n x^n - 5\sum_{n=2}^{\infty} a_{n-1} x^n + 6\sum_{n=2}^{\infty} a_{n-2} x^n = \sum_{n=2}^{\infty} 2^n x^n.$$

$$(a_2 x^2 + a_3 x^3 + \cdots) - 5(a_1 x^2 + a_2 x^2 + \cdots) + 6(a_0 x^2 + a_1 x^3 + \cdots)$$
$$= 2^2 x^2 + 2^3 x^3 + \cdots$$

$$\Rightarrow (a_0 + a_1 x + a_2 x^2 + \cdots) - a_0 - a_1 x - 5x(a_0 + a_1 x + a_2 x^2 + \cdots)$$
$$+ 5x a_0 + 6x^2 (a_0 + a_1 x + a_2 x^2 + \cdots) = 1 + 2x + (2x)^2 + \cdots - 1 - 2x.$$

$$\Rightarrow G(x) - a_0 - a_1 x - 5x G(x) + 5x a_0 + 6x^2 G(x) = \frac{1}{1-2x} - 1 - 2x \quad (\because \text{From } (2))$$

$$\Rightarrow G(x)[1 - 5x + 6x^2] = \frac{1}{1-2x} - 1 - 2x + a_0 + a_1 x - 5x a_0$$
$$(\because a_0 = 1, a_1 = 1)$$

$$\Rightarrow G(x)[1 - 2x - 3x + 6x^2] = \frac{1}{1-2x} \cancel{-1} - 2x \cancel{+1} + x - 5x.$$

$$\Rightarrow G(x)(1-2x)(1-3x) = \frac{1}{1-2x} - 6x = \frac{1 - 6x + 12x^2}{1-2x}$$

$$\Rightarrow G(x) = \frac{12x^2 - 6x + 1}{(1-2x)^2(1-3x)} \quad \text{— (3)}$$

Now by partial fractions:

$$\frac{12x^2 - 6x + 1}{(1-2x)^2(1-3x)} = \frac{A}{1-2x} + \frac{B}{(1-2x)^2} + \frac{C}{1-3x}.$$

$$\Rightarrow 12x^2 - 6x + 1 = A(1-2x)(1-3x) + B(1-3x) + C(1-2x)^2 \quad \text{— (4)}$$

Put $x = 1/2$,

$$(1) \Rightarrow 12\left(\tfrac{1}{4}\right) - 6\left(\tfrac{1}{2}\right) + 1 = A(0) + B\left(1 - 3\left(\tfrac{1}{2}\right)\right) + C(0)$$
$$3 - 3 + 1 = B\left(\frac{2-3}{2}\right) = B\left(-\frac{1}{2}\right)$$
$$1 = -B/2$$
$$\therefore B = -2.$$

Put $x = \frac{1}{3}$,

$$① \Rightarrow 12\left(\frac{1}{9}\right) - 6\left(\frac{1}{3}\right) + 1 = A(0) + B(0) + C\left(1 - \frac{2}{3}\right)^2$$

$$\frac{4}{3} - \frac{6}{3} + \frac{3}{3} = C\left(\frac{1}{3}\right)^2$$

$$\frac{1}{3} = C\left(\frac{1}{9}\right)$$

$$C = \frac{9}{3}$$

$$\therefore C = 3.$$

Put $x = 0$,

$$① \Rightarrow 1 = A + B + C$$

$$1 = A - 2 + 3$$

$$1 = A + 1$$

$$\therefore A = 0$$

$$③ \Rightarrow G(x) = \frac{0}{1-2x} + \frac{(-2)}{(1-2x)^2} + \frac{3}{(1-3x)}$$

$$\Rightarrow G(x) = -2(1-2x)^{-2} + 3(1-3x)^{-1}$$

$$\Rightarrow G(x) = -2\left[1 + 2(2x) + 3(2x)^2 + \cdots\right] + 3\left[1 + 3x + (3x)^2 + \cdots\right]$$

$$\Rightarrow G(x) = -2\sum_{n=0}^{\infty}(n+1)2^n x^n + 3\sum_{n=0}^{\infty} 3^n x^n.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[-2(n+1)2^n + 3 \cdot 3^n\right] x^n$$

$$\therefore a_n = -(n+1)2^{n+1} + 3^{n+1}.$$

(3) $a_n = 8a_{n-1} + 10^{n-1}$ with $a_0 = 1$, $a_1 = 9$.

Sol: Given, $a_n = 8a_{n-1} + 10^{n-1}$ with $a_0 = 1$, $a_1 = 9$.

$\Rightarrow a_n - 8a_{n-1} = 10^{n-1}$ —— (1)

Let $G(x)$ be the generating funⁿ, $G(x) = a_0 + a_1 x + a_2 x^2 + \cdots$ —— (2)

Multiply both sides of (1) by x^n and take $\sum_{n=1}^{\infty}$, we get

$$(1) \Rightarrow \sum_{n=1}^{\infty} a_n x^n - 8\sum_{n=1}^{\infty} a_{n-1} x^n = \sum_{n=1}^{\infty} 10^{n-1} x^n.$$

$$\Rightarrow (a_1 x + a_2 x^2 + \cdots) - 8(a_0 x + a_1 x^2 + \cdots) = x + 10x^2 + 10^2 x^3 + \cdots$$

$$\Rightarrow G(x) - a_0 - 8x(a_0 + a_1 x + \cdots) = x(1 + 10x + 10^2 x^2 + \cdots)$$

$$\Rightarrow G(x) - a_0 - 8x\,G(x) = x\left(\frac{1}{1-10x}\right) \quad [\because \text{From (2)}]$$

$$\Rightarrow G(x)[1 - 8x] = \frac{x}{1-10x} + a_0 = \frac{x}{1-10x} + 1 = \frac{x + 1 - 10x}{1-10x} \quad [\because a_0 = 1]$$

$$\Rightarrow G(x) = \frac{1-9x}{(1-10x)(1-8x)} \text{ —— (3)}$$

Now by partial fractions.

$$\frac{1-9x}{(1-10x)(1-8x)} = \frac{A}{1-10x} + \frac{B}{1-8x} \Rightarrow 1-9x = A(1-8x) + B(1-10x) \text{ —— (4)}$$

For $x = \frac{1}{8}$, (4) $\Rightarrow 1 - \frac{9}{8} = A(0) + B\left(1 - \frac{10}{8}\right) \Rightarrow -\frac{1}{8} = B\left(-\frac{2}{8}\right)$

$$\therefore \boxed{B = \frac{1}{2}}$$

For $x = \frac{1}{10}$, (4) $\Rightarrow 1 - \frac{9}{10} = A\left(1 - \frac{8}{10}\right) + B(0) \Rightarrow \frac{1}{\cancel{10}} = A\left(\frac{2}{\cancel{10}}\right)$

$\therefore \boxed{A = \frac{1}{2}}$.

(3) $\Rightarrow G(x) = \frac{1}{2(1-10x)} + \frac{1}{2(1-8x)} = \frac{1}{2}\sum_{n=0}^{\infty} 8^n x^n + \frac{1}{2}\sum_{n=0}^{\infty} 10^n x^n$.

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\left[\frac{1}{2}(8^n + 10^n)\right] x^n$.

$\therefore a_n = \frac{1}{2}(8^n + 10^n)$.

④ $a_n - 9a_{n-1} + 20a_{n-2} = 0$ for $n \geq 2$ and $a_0 = -3$, $a_1 = -10$.

Sol: Given $a_n - 9a_{n-1} + 20a_{n-2} = 0$ —— ① , $n \geq 2$

and $a_0 = -3$, $a_1 = -10$.

Let $G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ —— ② be the generating function.

Multiply both sides of ① by x^n and take summation $\sum_{n=2}^{\infty}$, we get

$$\sum_{n=2}^{\infty} a_n x^n - 9\sum_{n=2}^{\infty} a_{n-1} x^n + 20\sum_{n=2}^{\infty} a_{n-2} x^n = 0.$$

$\Rightarrow (a_2x^2 + a_3x^3 + \cdots) - 9(a_1x^2 + a_2x^3 + \cdots) + 20(a_0x^2 + a_1x^3 + \cdots) = 0$

$\Rightarrow (a_0 + a_1x + a_2x^2 + \cdots) - a_0 - a_1x - 9x(a_0 + a_1x + a_2x^2 + \cdots) + 9a_0x$

$+ 20x^2(a_0 + a_1x + a_2x^2 + \cdots) = 0$

$\Rightarrow G(x) - a_0 - a_1x - 9xG(x) + 9a_0x + 20x^2G(x) = 0$ [∵ from ②]

$\Rightarrow G(x)[1 - 9x + 20x^2] = a_0 + a_1x - 9a_0x = -3 + (-10 - 9(-3))x$

$$\Rightarrow G(x) = \frac{-3 + 17x}{1 - 9x + 20x^2} = \frac{-3 + 17x}{(1-5x)(1-4x)} \text{ —— ③}$$

By partial fractions,

$$\frac{-3+17x}{(1-5x)(1-4x)} = \frac{A}{1-5x} + \frac{B}{1-4x}.$$

$$-3+17x = A(1-4x) + B(1-5x) \quad \text{--- (4)}$$

Put $x = \frac{1}{4}$

$$(4) \Rightarrow -3 + \frac{17}{4} = A(0) + B\left(1-\frac{5}{4}\right)$$

$$\frac{-12+17}{4} = B\left(\frac{4-5}{4}\right)$$

$$\frac{5}{\cancel{4}} = B\left(\frac{-1}{\cancel{4}}\right)$$

$$\therefore B = -5.$$

Put $x = \frac{1}{5}$

$$(4) \Rightarrow -3 + \frac{17}{5} = A\left(1-\frac{4}{5}\right) + B(0)$$

$$\frac{-15+17}{5} = A\left(\frac{5-4}{5}\right)$$

$$\frac{2}{\cancel{5}} = \frac{A}{\cancel{5}}$$

$$\therefore A = 2.$$

$$(3) \Rightarrow G(x) = \frac{2}{1-5x} + \frac{(-5)}{1-4x}$$

$$= 2(1+5x+(5x)^2+\cdots) - 5(1+4x+(4x)^2+\cdots).$$

$$\Rightarrow G(x) = 2\sum_{n=0}^{\infty} 5^n x^n - 5\sum_{n=0}^{\infty} 4^n x^n.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} [2(5^n) - 5(4^n)]x^n.$$

$$\therefore a_n = 2(5^n) - 5(4^n).$$

THE METHOD OF CHARACTERISTIC ROOTS:

In this method the solution of a recurrence relation is obtained as a sum of two parts.

(1) The homogeneous solution, which satisfy the recurrence relation when the RHS of the relation is zero (i.e., $f(n)=0$).

(2) The particular solution, which satisfies the relation with $f(n)$ on the right hand side.

The general solution is given by

$$\boxed{a_n = a_n^{(h)} + a_n^{(P)}}$$

where $a_n^{(h)}$ = Homogeneous solution.

$a_n^{(P)}$ = Particular solution.

Note: If RHS of the given recurrence relation is zero then the general solution is $a_n = a_n^{(h)}$.

Solution of linear homogeneous equation:

Let $a_n = r^n$ is a solution of the recurrence relation

$C_0 a_n + C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} = 0$. Then,

$C_0 r^n + C_1 r^{n-1} + C_2 r^{n-2} + \cdots + C_k r^{n-k} = 0$.

Dividing both sides by r^{n-k}, we get

$C_0 r^k + C_1 r^{k-1} + C_2 r^{k-2} + \cdots + C_k = 0$.

The above equation is called as the characteristic equation of the recurrence relation of kth order and it has k roots called characteristic roots.

① Distinct roots: If the characteristic roots are distinct roots, say $r_1, r_2, \ldots, r_k$ then the general solution for homogeneous equation is

$$a_n = a_n^{(h)} = b_1 r_1^n + b_2 r_2^n + b_3 r_3^n + \cdots + b_k r_k^n.$$

where $b_1, b_2, \ldots, b_k$ are constants which may be choosen to satisfy initial conditions.

Ex: If characteristic roots $r = 2, 3, 4$, then the solution is $a_n = a_n^{(h)} = b_1 2^n + b_2 3^n + b_3 4^n$.

② Multiple roots: If 'r' is a root of the characteristic equation of rth order with multiplicity m (repeated m times) then the general solution is

$$a_n = a_n^{(h)} = (b_1 + b_2 n + b_3 n^2 + \cdots + n^{m-1} b_m) r^n.$$

Ex: If the characteristic eqn is $(r-4)^3 = 0$, then the roots are $r = 4, 4, 4$ (repeated 3 times) then the general solution is $a_n = (b_1 + b_2 n + b_3 n^2) 4^n$.

③ Mixed roots: If some roots of a characteristic eqn are distinct and some roots are equal, then we add both the types of solution.

Ex: If the characteristic equation is $(r-2)(r-4)(r-5)^3 = 0$ then the roots are $r = 2, 4, 5, 5, 5$. and the general solution is $a_n = b_1 2^n + b_2 4^n + (b_3 + b_4 n + b_5 n^2) 5^n$.

Problems:

Q:

Solve the following RR using the method of characteristic roots.

① $a_n - a_{n-1} - 2a_{n-2} = 0$.

Sol: Given, $a_n - a_{n-1} - 2a_{n-2} = 0$ — ①

Let $a_n = r^n$ be the solution of ①, then

$r^n - r^{n-1} - 2r^{n-2} = 0$

Dividing by r^{n-2} on both sides, we get

$r^2 - r - 2 = 0$ is the characteristic eqn.

$r^2 - 2r + r - 2 = 0$

$r(r-2) + 1(r-2) = 0$

$(r-2)(r+1) = 0$

$r = 2, -1$.

∴ The general solution is $a_n = b_1(2)^n + b_2(-1)^n$.

② $a_n - 7a_{n-1} + 12a_{n-2} = 0$, $n \geq 2$ with $a_0 = 2, a_1 = 5$.

Sol: Given, $a_n - 7a_{n-1} + 12a_{n-2} = 0$ — ①, $n \geq 2$ with $a_0 = 2$, $a_1 = 5$.

Let $a_n = r^n$ be the solution of ①, then

$r^n - 7r^{n-1} + 12r^{n-2} = 0$

Dividing by r^{n-2} on both sides, we get

$r^2 - 7r + 12 = 0$ is the characteristic eqn.

$\Rightarrow r^2 - 4r - 3r + 12 = 0$

$\Rightarrow r(r-4) - 3(r-4) = 0$

$\Rightarrow (r-4)(r-3) = 0$

$\Rightarrow r = 3, 4$.

∴ The general solution is $a_n = b_1(3)^n + b_2(4)^n$ — ②

We have $a_0 = 2$

$\Rightarrow b_1(3)^0 + b_2(4)^0 = 2$ [Put $n=0$ in ②]

$\Rightarrow b_1 + b_2 = 2$ — ③

Put $n=1$ in ②, $a_1 = b_1(3)^1 + b_2(4)^1$

$5 = 3b_1 + 4b_2$ [$\because a_1 = 5$]

$\Rightarrow 3b_1 + 4b_2 = 5$ — ④

Solving ③ & ④

④ $\Rightarrow \cancel{3b_1} + 4b_2 = 5$

③ × 3 $\Rightarrow \cancel{3b_1} + 3b_2 = 6$

$b_2 = -1$

② $\Rightarrow b_1 - 1 = 2$

$\boxed{b_1 = 3}$

② $\Rightarrow a_n = 3(3)^n + (-1)(4)^n$

$\therefore$ The general solution is

$a_n = 3^{n+1} - 4^n$.

③ Solve $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n \geq 3$ with $a_0 = 3, a_1 = 1$ and $a_2 = 0$.

Sol: Given, $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$, $n \geq 2$ with $a_0 = 3, a_1 = 1, a_2 = 0$

$\Rightarrow a_n - 2a_{n-1} - a_{n-2} + 2a_{n-3} = 0$ — ①

Let $a_n = r^n$ be the solution of ①, then

$r^n - 2r^{n-1} - r^{n-2} + 2r^{n-3} = 0$

Dividing by r^{n-3} on both sides, we get

$r^3 - 2r^2 - r + 2 = 0$ is the characteristic eqn

For $r = 1$, $\Rightarrow (1)^3 - 2(1)^2 - (1) + 2 = 0$

$1 - 2 - 1 + 2 = 0$

$0 = 0$

$$\begin{array}{r|rrrr} r=1 & 1 & -2 & -1 & 2 \\ & 0 & 1 & -1 & -2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\Rightarrow (r-1)(r^2 - r - 2) = 0$

$(r-1)(r-2)(r+1) = 0$

$r = 1, 2, -1$.

$\therefore$ The general solution is

$a_n = b_1(1)^n + b_2(2)^n + b_3(-1)^n$ — ②

We have, $a_0 = 3$

$b_1 + b_2 + b_3 = 3$ — ③

and $a_1 = 1$

$\Rightarrow b_1 + 2b_2 - b_3 = 1$ — ④

and $a_2 = 0$

$\Rightarrow b_1 + 4b_2 + b_3 = 0$ — ⑤

③ + ④

$\cancel{b_1 + b_2 + b_3 = 3}$

$\cancel{b_1 + 2b_2 - b_3 = 1}$

$\cancel{2b_1 +}$

③ $\Rightarrow b_3 = 3 - b_1 - b_2$

~~④ $\Rightarrow b_1 + 2b_2 - 3 + b_1 + b_2 = 1$~~

⑤ $\Rightarrow b_1 + 4b_2 + 3 - b_1 - b_2 = 0$

$\Rightarrow 3b_2 + 3 = 0$

$\Rightarrow \boxed{b_2 = -1}$

④ $\Rightarrow b_1 + 2b_2 - 3 + b_1 + b_2 = 1$

$2b_1 + 3(-1) - 3 = 1$

$2b_1 = 1 + 6$

$\boxed{b_1 = 7/2}$.

③ $\Rightarrow b_3 = 3 - 7/2 - (-1)$

$= 4 - 7/2$

$\boxed{b_3 = 1/2}$.

(or)

③ - ⑤

$\cancel{b_1} + b_2 + \cancel{b_3} = 3$

$\cancel{b_1} + 4b_2 + \cancel{b_3} = 0$

$-3b_2 = 3$

$b_2 = -1$

③ $\Rightarrow b_1 + b_3 = 4$

④ $\Rightarrow b_1 - b_3 = 3$

$2b_1 = 7$

$b_1 = 7/2$.

③ $\Rightarrow 7/2 + (-1) + b_3 = 3$

$b_3 = 3 + 1 - 7/2$

$= \frac{8-7}{2}$

$b_3 = 1/2$.

$\therefore$ The general solution is

② $\Rightarrow a_n = 7/2 - 2^n + \frac{(-1)^n}{2}$

(or)

$$a_n = \frac{7 - 2^{n+1} + (-1)^n}{2} = \frac{1}{2}\left[7 - 2^{n+1} + (-1)^n\right].$$

④ Find the solution for $a_{n+2} - 6a_{n+1} + 9a_n = 0$.

Sol: Given, $a_{n+2} - 6a_{n+1} + 9a_n = 0$ — ①

Let $a_n = r^n$ be the solution of ①, then

$r^{n+2} - 6r^{n+1} + 9r^n = 0$

Dividing by r^n on both sides, we get

$r^2 - 6r + 9 = 0$ is the characteristic eqn.

$r^2 - 3r - 3r + 9 = 0$

$(r-3)^2 = 0$

$r = 3, 3$.

$\therefore$ The general solution is

$a_n = (b_1 + b_2 n)\, 3^n$.

⑤ Suppose that a person deposits Rs. 10,000/- in a saving account at bank yielding 12% per year with interest compounded anually. How much will be in the account after 30 years?

Sol: Let P_0 be the amount deposited = Rs. 10,000/-

$\therefore P_0 = 10{,}000$.

Let P_n be the amount in the account after n years.

ATP, $P_n = P_{n-1} + 12\%\, P_{n-1}$.

$\Rightarrow P_n = P_{n-1} + \frac{12}{100} P_{n-1}$

$\Rightarrow P_n = P_{n-1} + 0.12 P_{n-1}$

$\Rightarrow P_n = 1.12 P_{n-1}$ ——— ①

$\Rightarrow P_n - 1.12 P_{n-1} = 0$

let $P_n = r^n$ be the solution of ①,

$r^n - 1.12 r^{n-1} = 0$

Dividing by r^{n-1} on both sides,

$r - 1.12 = 0$

$r = 1.12$.

∴ The general solution is

$P_n = b_1(1.12)^n$.

we have $P_0 = 10,000$

$b_1 (1.12)^0 = 10,000$

$\Rightarrow b_1 = 10,000$.

∴ The general solution is

$P_n = 10,000 (1.12)^n$.

(or)

For $n=1$, $P_1 = 1.12 P_{1-1} = 1.12 P_0$

For $n=2$, $P_2 = 1.12 P_1 = 1.12(1.12 P_0)$

$= (1.12)^2 P_0$

For $n=3$, $P_3 = 1.12 P_2 = (1.12)^3 P_0$

⋮

Similarly,

$P_n = (1.12)^n P_0$.

$P_n = 10,000 (1.12)^n$.

Amount in the account after 30 years is

$P_{30} = 10,000 (1.12)^{30}$

∴ P_{30} = Rs. 2,99,599.221209.

SOLUTION OF INHOMOGENEOUS RECURRENCE RELATION:

The general solution of a linear non-homogeneous recurrence relation $c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n)$ where $c_0, c_1, c_2, \ldots, c_k$ are constants is $a_n = a_n^{(h)} + a_n^{(P)}$

There is no general method for finding the particular solution of a RR for every function of $f(n)$. So the solution can be obtained by the method of inspection.

In this method for finding a particular solution we use trial solution in each case as shown in the following table.

f(n)	Trial solution.
① If $f(n)$ = constant	① $a_n^{(P)} = A_0$
② If $f(n)$ = polynomial of degree m	② $a_n^{(P)} = A_0 + A_1 n + A_2 n^2 + \cdots + A_m n^m$.
③ If $f(n) = b^n$ (b is not a root)	③ $a_n^{(P)} = A_0 b^n$.
④ If $f(n) = b^n$ (b is a root of multiplicity m)	④ $a_n^{(P)} = A_0 n^m b^n$.
⑤ If $f(n) = b^n \times$ polynomial (b is a root of multiplicity m)	⑤ $a_n^{(P)} = b^n n^m (A_0 + A_1 n + \cdots)$

Problems:

① Find the particular solution $a_n = 3a_{n-1} + 7$.

Sol: Given, $a_n = 3a_{n-1} + 7$.

$\Rightarrow a_n - 3a_{n-1} = 7$ — ①.

To find $a_n^{(P)}$:

The RHS of ① is constant.

Let $a_n^{(P)} = A_0$ — ②

① $\Rightarrow A_0 - 3A_0 = 7$

$-2A_0 = 7$

$\Rightarrow A_0 = -7/2$.

$\therefore$ ② $\Rightarrow a_n^{(P)} = -7/2$.

② Solve $a_{n+2} + 4a_{n+1} + 4a_n = 7$, $n \geq 0$ with $a_0 = 1$, $a_1 = 2$.

Sol: Given $a_{n+2} + 4a_{n+1} + 4a_n = 7$ — ①, $n \geq 0$ with $a_0 = 1$, $a_1 = 2$.

① is a non-homogeneous RR.

$\therefore$ ~~let~~ ~~the~~ the general solution is

$a_n = a_n^{(h)} + a_n^{(P)}$ — ②

To find $a_n^{(h)}$: let $a_{n+2} + 4a_{n+1} + 4a_n = 0$ — ③

Let $a_n = r^n$ is a solution of ③, then

$r^{n+2} + 4r^{n+1} + 4r^n = 0$

Dividing r^n on both sides, we get:

$r^2 + 4r + 4 = 0$ is the characteristic eqn,

$r^2 + 2r + 2r + 4 = 0$

$(r+2)^2 = 0$

$r = -2, -2$.

$\therefore a_n^{(h)} = (b_1 + b_2 n)(-2)^n$ — ④

To find $a_n^{(P)}$: The RHS of the given relation, $f(n)$ = constant.

Let $a_n^{(P)} = A_0$ — ⑤

① $\Rightarrow A_0 + 4A_0 + 4A_0 = 7$.

$9A_0 = 7$

$A_0 = 7/9$.

⑤ $\Rightarrow a_n^{(P)} = 7/9$

$\therefore$ The general solution is

② $\Rightarrow a_n = (b_1 + b_2 n)(-2)^n + 7/9$ — ⑥.

Put $n = 0$ in ⑥

$a_0 = [b_1 + b_2(0)](-2)^0 + 7/9$.

$1 = b_1 + 7/9$.

$$b_1 = 1 - 7/9 = \frac{9-7}{9}$$

$$\boxed{b_1 = \frac{2}{9}}$$

Put $n = 1$ in (6)

$$a_1 = [b_1 + b_2(1)](-2)^1 + 7/9$$

$$2 = \left[\frac{2}{9} + b_2\right](-2) + 7/9$$

$$2 = -\frac{4}{9} - 2b_2 + 7/9$$

$$2b_2 = \frac{-4}{9} + 7/9 - 2 = \frac{3}{9} - 2$$

$$2b_2 = \frac{3-18}{9} = \frac{-15}{9}$$

$$\boxed{b_2 = -\frac{15}{18}} = -\frac{5}{6}$$

$$\therefore \boxed{b_2 = \frac{-5}{6}}$$

$\therefore$ The solution is

$$(6) \Rightarrow a_n = \left[\frac{2}{9} - \frac{5}{6}n\right](-2)^n + 7/9$$

(3) Solve $y_{n+2} - y_{n+1} - 2y_n = n^2$.

Sol: Given $y_{n+2} - y_{n+1} - 2y_n = n^2$ ——— (1)

(1) is a non-homogeneous relation

$\therefore$ The general solution is $y_n = y_n^{(h)} + y_n^{(p)}$ ——— (2).

To find $y_n^{(h)}$: let $y_{n+2} - y_{n+1} - 2y_n = 0$ ——— (3)

Let $y_n = r^n$ be the solution of (3), then

$$r^{n+2} - r^{n+1} - 2r^n = 0$$

Dividing by r^n on both sides, we get

$r^2 - r - 2 = 0$ is the characteristic eqn.

$$r^2 - 2r + r - 2 = 0$$

$$r(r-2) + 1(r-2) = 0$$

$$(r-2)(r+1) = 0$$

$$r = 2, -1.$$

$$\therefore y_n^{(h)} = b_1 2^n + b_2(-1)^n \text{ ——— (4)}.$$

To find $y_n^{(p)}$: The RHS of (1), $f(n)$ = polynomial $= n^2$.

Let $y_n^{(p)} = A_0 + A_1 n + A_2 n^2$ ——— (5).

(5), we have, $y_n = A_0 + A_1 n + A_2 n^2$

$$\Rightarrow y_{n+1} = A_0 + A_1(n+1) + A_2(n+1)^2$$

$$\Rightarrow y_{n+2} = A_0 + A_1(n+2) + A_2(n+2)^2.$$

$$① \Rightarrow A_0 + A_1(n+2) + A_2(n+2)^2 - [A_0 + A_1(n+1) + A_2(n+1)^2]$$
$$- 2[A_0 + A_1 n + A_2 n^2] = n^2.$$

$$\Rightarrow \cancel{A_0} + A_1 n + 2A_1 + A_2(n^2+4n+4) - A_0 - A_1 n - A_1$$
$$- A_2(n^2+1+2n) - 2A_0 - 2A_1 n - 2A_2 n^2 = n^2 \quad ⑥$$

Equating n^2 coefficients of ⑥

$$A_2 - A_2 - 2A_2 = 1$$
$$A_2 = -\frac{1}{2}.$$

Equating n coefficients of ⑥

$$\cancel{A_1} + 4A_2 - \cancel{A_1} - 2A_2 - 2A_1 = 0$$
$$2A_2 - 2A_1 = 0$$
$$2\left(-\frac{1}{2}\right) - 2A_1 = 0 \quad (\because)$$
$$-1 - 2A_1 = 0$$
$$2A_1 = -1$$
$$A_1 = -\frac{1}{2}.$$

$$2A_2 = 2A_1$$
$$A_2 = A_1 = -\frac{1}{2}.$$

Equating constant terms of ⑥

$$\cancel{A_0} + 2A_1 + 4A_2 - \cancel{A_0} - A_1 - A_2 - 2A_0 = 0$$
$$A_1 + 3A_2 - 2A_0 = 0$$
$$2A_0 = A_1 + 3A_2 = \left(-\frac{1}{2}\right) + 3\left(-\frac{1}{2}\right)$$
$$= \frac{-4}{2} = -2$$
$$2A_0 = -2$$
$$A_0 = -1$$

$$⑤ \Rightarrow y_n^{(P)} = -1 - \frac{1}{2}n - \frac{1}{2}n^2$$

$\therefore$ the general solution is

$$② \Rightarrow y_n = y_n^{(h)} + y_n^{(P)} = b_1 2^n + b_2(-1)^n - 1 - \frac{n}{2} - \frac{n^2}{2}$$

SOLVING RECURRENCE RELATION BY ITERATIVE METHOD:

In this method we use the RR iteratively to solve the expression.

Q: Solve the following recurrence relation by iterative method

① $a_n = a_{n-1} + 2$, $a_0 = 3$.

Sol: Given, $a_n = a_{n-1} + 2$ — ① and $a_0 = 3$.

Replace n by $n-1$ in ①, we get

$a_{n-1} = a_{n-2} + 2$

Substituting a_{n-2} in ①, we get

$a_n = a_{n-2} + 2 + 2 = a_{n-2} + 2(2)$ —— ②

Replace n by $n-2$ in ①, we get

$a_{n-2} = a_{n-3} + 2$.

Substituting a_{n-2} in ②, we get

$a_n = a_{n-3} + 2 + 2(2) = a_{n-3} + 3(2)$.

Similarly,

$a_n = a_{n-k} + k(2)$.

Put $k = n \Rightarrow a_n = a_{n-n} + n(2)$

$= a_0 + 2n$

$\therefore a_n = 3 + 2n$

② $a_n = a_{n-1} + 2n + 3$, $a_0 = 1$

Sol: Given, $a_n = a_{n-1} + 2n + 3$ —— ①, $a_0 = 1$.

Replace n by $n-1$ in ① we get

$a_{n-1} = a_{n-2} + 2(n-1) + 3$

Substituting a_{n-1} in ①, we get

$a_n = a_{n-2} + [2(n-1) + 3] + [2n+3]$ —— ②

Replace n by $n-2$ in ①, we get

$a_{n-2} = a_{n-3} + [2(n-2) + 3] \cancel{+ [2(n-1)+3] + [2n+3]}$

Substituting a_{n-2} in ②, we get

$a_n = a_{n-3} + [2(n-2) + 3] + [2(n-1) + 3] + [2n+3]$.

Similarly

$a_n = a_{n-k} + [2(n-(k-1)) + 3] + [2(n-(k-2)) + 3] + \dots$

$\dots + [2n+3]$.

Put $k = n \Rightarrow a_n = a_{n-n} + [2[1] + 3] + [2(2) + 3] + \dots + [2n+3]$

$\Rightarrow a_n = a_0 + \sum_{n=1}^{n} (2n+3) = a_0 + 2[1+2+\dots+n] + 3n$.

$= a_0 + \cancel{2}\frac{n(n+1)}{\cancel{2}} + 3n$

$= 1 + n^2 + n + 3n$

$\therefore a_n = n^2 + 4n + 1$

UNIT-V

GRAPH THEORY

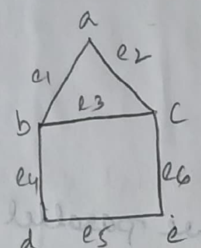
Graph: A graph G is a pair of sets (V, E) where V is the set of points which are called as vertices and E is a set of lines called edges.

Here V and E are assumed to be finite sets.

Note:

1. A vertex in a graph G is also called as node or a point or a junction.
2. A edge in G is called a line or a branch or an arc.
3. The number of vertices in G is denoted by $|V|$ or $|V(G)|$ and the number of edges in G is denoted by $|E|$ or $|E(G)|$.

Ex:



Here $|V| = 5$
$|E| = 6$.

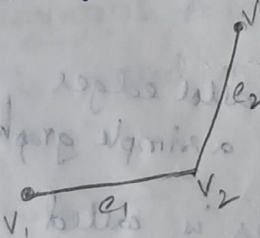
Order and size:

The number of vertices of a graph is called its order and is denoted by $|V|$.

The number of ~~edge~~ edges of a graph is called its size and is denoted by $|E|$.

Adjacent vertices: Two vertices 'u' and 'v' are said to be adjacent if there exist an edge connecting u and v.

Ex:

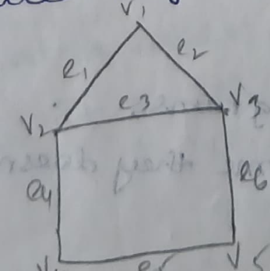


Here, v_1, v_2 are adjacent vertices.

v_1, v_3 are not adjacent vertices.

Adjacent edges: If two edges have common vertex then they are called adjacent edge.

Ex:



Here e_1, e_3 are adjacent edges.

e_1, e_6 are not adjacent edges.

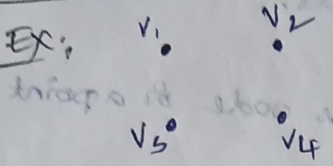
Finite graph: A graph with finite number of vertices and finite number of edges is called finite graph.

Infinite graph: A graph with infinite number of vertices is called infinite graph.

Trivial graph: A graph of 0 or 1 vertex is called trivial graph.

Null graph: A graph without edges is called null graph.

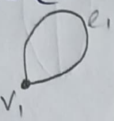
Ex:



Here $|V| = 4$, $|E| = 0$.

Loop: An edge is said to be a loop if it has same endpoints. (or) A loop is an edge that connects from a vertex to itself. It is also called as self-loop.

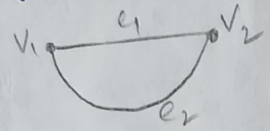
Ex:



Here e_1 is the loop.

Parallel edges: Two edges are said to be parallel, if they have same vertices.

Ex:

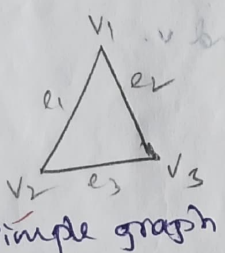


Here e_1, e_2 are parallel edges

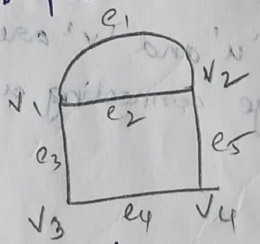
Multiple edges: If two or more edges have the same vertices then they are called multiple edges.

Simple graph: A graph is said to be simple graph if it has neither loop nor parallel edge.

EX:



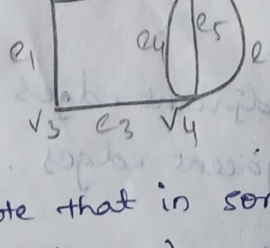
G - simple graph



Here e_1, e_2 are parallel edges. Hence it is not a simple graph.

Multi graph: A graph with multiple edges is called a multigraph or multiple graph.

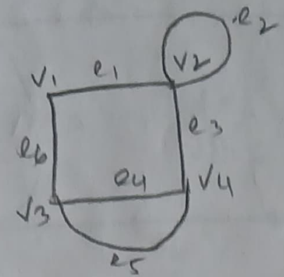
Ex:



[Note that in some networks and programs, multigraphs allow loops in some they doesn't].

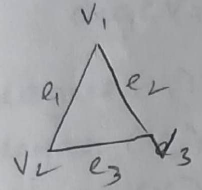
Pseudo graph: A graph in which both loops and multiple edges are allowed is called a Pseudo graph.

Ex:



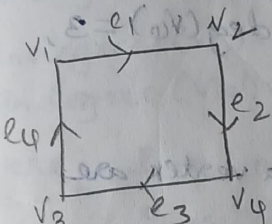
directed graph: A graph in which every edge is undirected is called undirected graph.

Ex:



Directed graph: A graph in which every edge is directed is called a digraph or diagraph or directed graph.

Ex:



Note: Suppose $e_1 = (v_1, v_2)$ is a directed edge of a digraph, then

① v_1 is called the initial vertex of e_1.

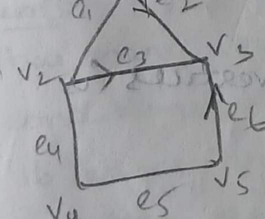
② v_2 is called the terminal vertex of e_1.

③ e_1 is called the incident from v_1 to v_2

④ v_1, v_2 are called adjacent nodes of e_1.

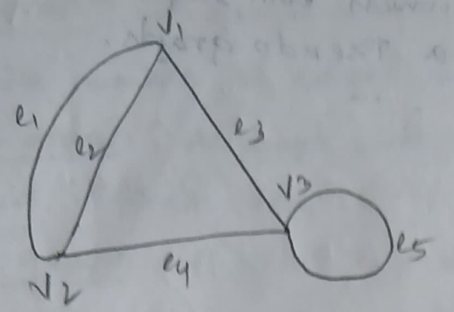
Mixed graph: A graph with both directed and undirected edges is called a mixed graph.

Ex:



Degree of a vertex: The degree of a vertex in an undirected graph is the number of edges incident with it except that a loop at a vertex contributes twice to the degree of that vertex. It is denoted by $deg_G(v)$ or $d(v)$.

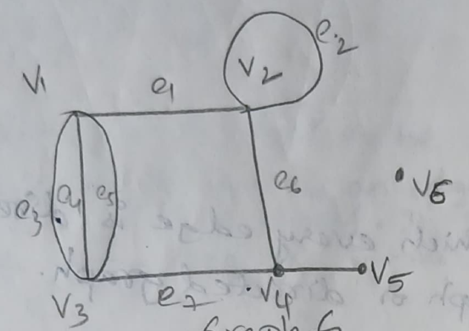
Ex: (1)



Graph G.

Here $deg_G(V_1) = 3$, $deg_G(V_2) = 3$, $deg_G(V_3) = 4$.

(2)



Graph G.

Here $deg_G(V_1) = 4$, $deg_G(V_2) = 4$, $deg_G(V_3) = 4$, $deg(V_4) = 3$, $deg_G(V_5) = 1$, $deg_G(V_6) = 0$.

Note: The minimum and maximum degrees of a vertex are denoted by $\delta(G)$ & $\Delta(G)$ respectively.

In the above ex-2 - $\delta(G) = 0$, $\Delta(G) = 4$.

Isolated vertex: An isolated vertex is a vertex in a graph with no edges connecting to it, i.e., it has a degree zero.

Ex: V_6 in above ex-(2).

Pendent vertex: If the degree of the vertex is one then the vertex is called pendent vertex.

Ex: V_5 in above ex-(2).

Note: A graph containing only isolated vertices is called a null graph.

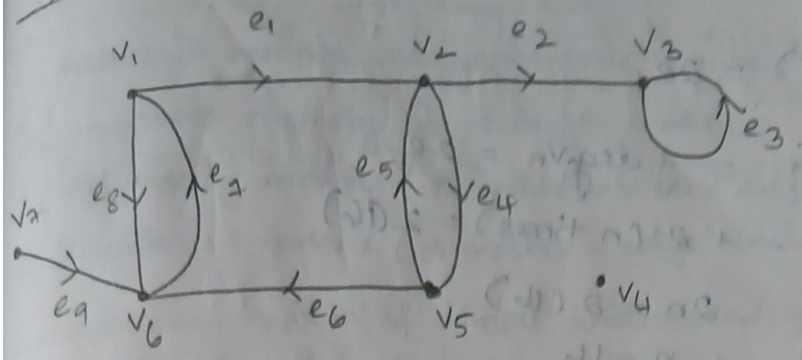
Indegree and Out degree:

For the directed graph we can found indegree and out degree.

→ The indegree of a vertex in a digraph is the number of inward direction edges to the vertex.

→ The outdegree of a vertex in a digraph is the number of outward direction edges of the vertex.

Ex :



Here $deg(V_4) = 0$, So, V_4 is isolated vertex.
$deg(V_7) = 1$, So V_7 is pendent vertex.

Indegree $(V_1) = 1$ Outdegree $(V_1) = 2$
Indegree $(V_2) = 2$ Out degree $(V_2) = 2$
Indegree $(V_3) = 2$ Out degree $(V_3) = 1$
Indegree $(V_4) = 0$ Out degree $(V_4) = 0$
Indegree $(V_5) = 1$ Out degree $(V_5) = 2$
Indegree $(V_6) = 3$ Out degree $(V_6) = 1$
Indegree $(V_7) = 0$ Out degree $(V_7) = 1$.

Here Sum of indegree $= 9$.
Sum of outdegree $= 9$.

Hand shaking theorem (or) Euler's first theorem:

The sum of the degrees of all vertices in a graph G is twice the number of edges in that graph.

i.e., $\sum_{v \in V} deg_G(v) = 2e$

(or)

$\sum_{i=1}^{n} deg(V_i) = 2e$.

where e is the number of edges and n is number of vertices.

Problems:

Q: How many vertices the following graphs will have if the conditions are

(i) 16 edges and all the vertices of degree 2.

(ii) 21 edges, three vertices of degree 4 and other vertices of degree 3.

(iii) 24 edges and all the vertices of same degree.

Sol: (i) Given, the graph G has 16 edges and all the vertices of degree 2.

By handshaking theorem,

$$\sum_{i=1}^{n} \deg_G(v_i) = 2e.$$

$\Rightarrow \deg_G(v_1) + \deg_G(v_2) + \cdots + \deg_G v_n = 2e.$

$2 + 2 + \cdots + 2 \text{ (n times)} = 2(16)$

$2n = 2(16)$

$n = 16$

$\therefore$ No. of vertices in the graph is 16.

(ii) Given the graph G has 21 edges.

By handshaking theorem,

$$\sum_{i=1}^{n} \deg_G(v_i) = 2e.$$

$\deg v_1 + \deg v_2 + \deg v_3 + \cdots + \deg v_n = 2e$

$4 + 4 + 4 + 3 + 3 + \cdots + 3 \text{ (n-3 times)} = 2(21)$

$12 + 3(n-3) = 42$

$3(n-3) = 42 - 12 = 30$

$n - 3 = \frac{30}{3} = 10$

$n = 10 + 3$

$n = 13.$

$\therefore$ The total no. of vertices in the graph G is 13.

(iii) Given, the graph G has 24 edges and all vertices have same degree say 'd'.

By handshaking theorem,

$$\sum_{i=1}^{n} \deg_G(v_i) = 2e.$$

$\deg v_1 + \deg v_2 + \cdots + \deg v_n = 2e.$

$d + d + \cdots + d \text{ (n times)} = 2(24)$

$n \times d = 48.$

WKT, For a simple graph $0 \le d \le n-1$.

So the possible values of (n, d) such that $n \times d = 48$ are.

$(1, 48), (2, 24), (3, 16), (4, 12), (6, 8),$
$(8, 6), (12, 4), (16, 3), (24, 2), (48, 1).$

Out of above values the possible values satisfying $d \le n-1$ are $(8, 6), (12, 4), (16, 3), (24, 2)$ and $(48, 1)$

$\therefore$ No of possible vertices = 8, 12, 16, 24 and 48 with degree 6, 4, 3, 2 and 1 respectively.

Matrix representation of Graphs:

Although graphs are geometric figures, their representation in computer memory is through matrices.

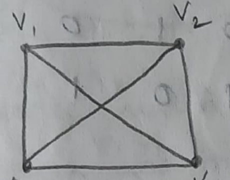
1. Adjacency matrix (or vertex - vertex adjacency matrix)
2. Incidence matrix (or vertex - edge incidence matrix).

1. Adjacency matrix of simple undirected graph:

For a graph G, which consists of n vertices, an $n \times n$ adjacency matrix $A = [a_{ij}]$ is defined as

$$a_{ij} = \begin{cases} 1, & \text{if vertex } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

Ex:



Graph G

The adjacency matrix $A_G =$

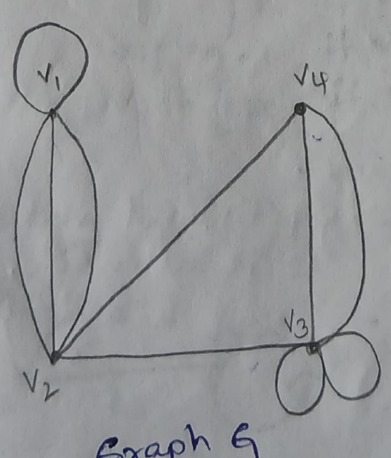
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

2. Adjacency matrix for multigraph:

For a multigraph G, consists of n vertices, a $n \times n$ adjacency matrix $A = [a_{ij}]$ is defined as

$$a_{ij} = \begin{cases} m, & m = \text{no. of edges between } v_i \text{ and } v_j, \text{ if } m \geq 1. \\ 0, & \text{otherwise.} \end{cases}$$

Ex:



Graph G

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 3 & 0 & 0 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

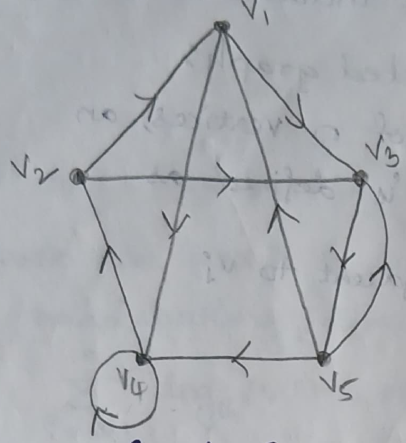
Adjacency matrix.

3. Adjacency matrix for Directed graphs:

For a directed graph G consists of n vertices, an $n \times n$ adjacency matrix $A = [a_{ij}]$ is defined as

$$a_{ij} = \begin{cases} m, & \text{no. of edges beginning at } v_i \text{ and ending at } v_j \\ 0, & \text{otherwise.} \end{cases}$$

Ex:



Graph G

Adjacency matrix

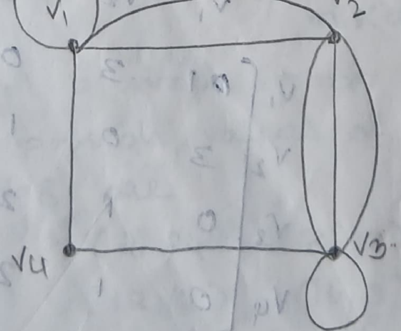
$$A_G = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & V_4 & V_5 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Q: Draw the graph represented by given adjacency matrix.

(i) $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

Sol. Given, Adjacency matrix is $A = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & V_3 & V_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$

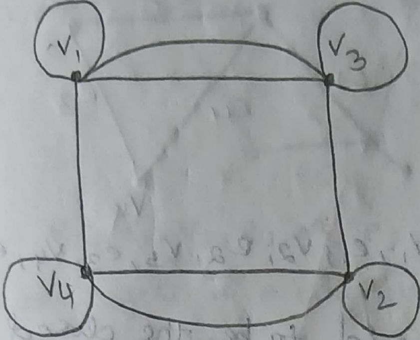
The corresponding graph is



(ii) $\begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$

Sol: Given, adjacency matrix is $A=$

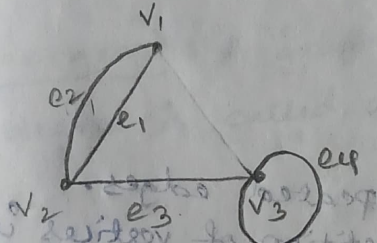
$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \end{array}$$



4. Incidence matrix: For a graph G, which consists n vertices and m edges, an $n \times m$ matrix, $X = [x_{ij}]$ is defined as

$$x_{ij} = \begin{cases} 1, & \text{if vertex } v_i \text{ is incident to edge } e_j \\ 2, & \text{if } e_j \text{ is a loop at } v_i \\ 0, & \text{otherwise} \end{cases}$$

Ex:



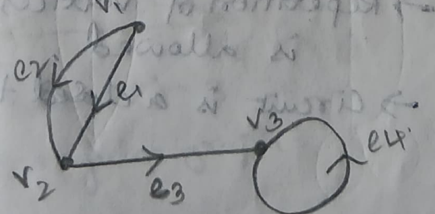
Incidence matrix $=$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array}$$

5. Incidence matrix of directed graph: For a directed graph G, which consists of n vertices and m edges, an $n \times m$ incidence matrix $X = [x_{ij}]$ is defined as

$$x_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is initial vertex of edge } e_j \\ -1 & \text{if } v_i \text{ is terminal vertex of edge } e_j \\ 0 & \text{if } v_i \text{ not incident on edge } e_j \end{cases}$$

Ex:



Incidence matrix $=$

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \end{array}$$

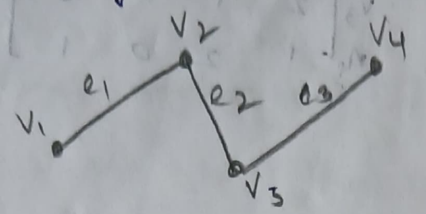
Walk: A walk in a graph is a sequence of vertices and edges in which each edge's endpoints are the preceding and succeding vertices in the sequence.

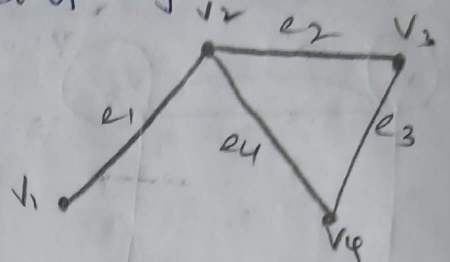
$$v_1, e_1, v_2, e_2, v_3, e_3, \ldots, e_{n-1}, v_n.$$

→ Repetition of vertices and edges is allowed.

→ The length of the walk = number of edges.

Ex:



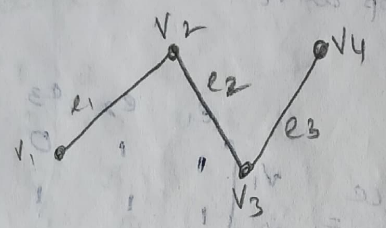


$v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_3, v_3.$ $v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_2$

Closed and open walk: A walk is said to be the closed walk if it is possible that a walk begins and end at the same vertices. Otherwise the walk is called open, i.e., terminal vertices are different.

Path: A path is a walk with no repeated vertices, and hence no repeated edges.

Ex:

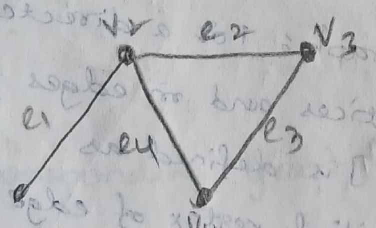


$v_1, e_1, v_2, e_2, v_3, e_3, v_4.$

Trial: A trial is a walk with no repeated edges.

→ Repetition of vertices is allowed.

Ex:



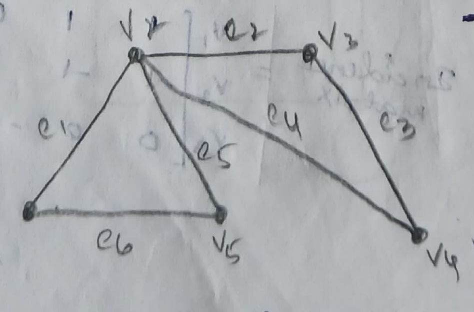
$v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_2.$

Circuit: A circuit is a closed walk with no repeated edges. i.e, start and end points are same.

→ Repetition of vertices is allowed.

→ Circuit is a closed trial.

Ex:



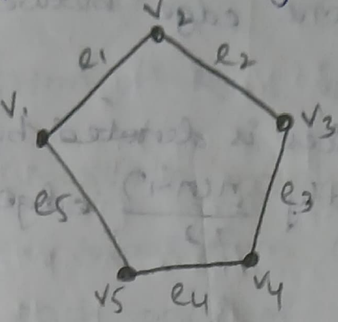
$v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_2, e_5, v_5, e_6, v_1.$

__Cycle__: A cycle is a closed walk with no repeated edges and no repeated vertices other than starting and ending vertices.

→ A ~~circuit~~ cycle is a closed path.

→ No repetition of edges and vertices.

Ex:



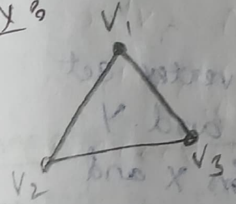
$v_1, e_1, v_2, e_2, v_3, e_3, v_4, e_4, v_5, e_5, v_1$.

→ Every cycle is a circuit but not every circuit is a cycle.

	Repeated vertices	Repeated edges	closed
Walk	✓	✓	✓✗
Path	✗	✗	✓✗
Trial	✓	✗	✓✗
circuit	✓	✗	✓
cycle	✗	✗	✓

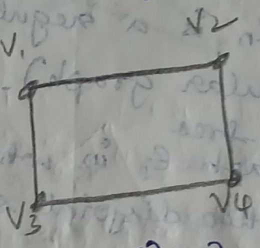
__Regular graph__: A graph in which all the vertices are of equal degree is called a regular graph.

Ex:



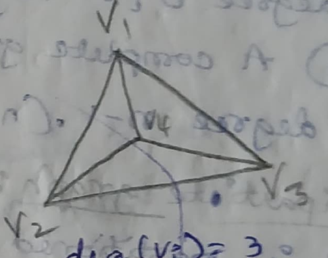
$deg(v_i) = 2$

2-regular graph



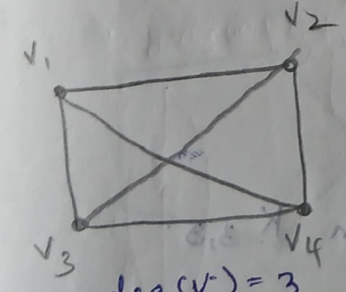
$deg(v_i) = 2$

2-regular graph.



$deg(v_i) = 3$ is

3-regular graph



$deg(v_i) = 3$

3-regular graph.

<u>Complete graph</u>: A simple graph G in which every pair of distinct vertices are adjacent is called complete graph.

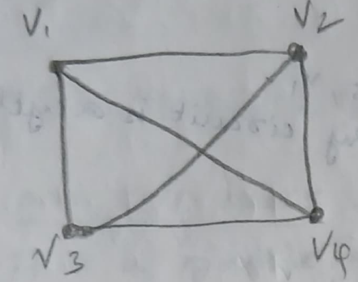
(or)

A simple graph G is said to be a complete graph if G contains one and only one edge between each distinct pair of vertices.

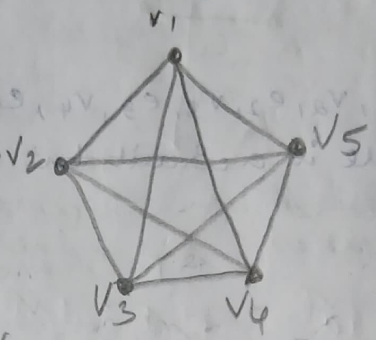
→ A complete graph of n vertices is denoted by K_n.

→ A complete graph has exactly $\frac{n(n-1)}{2}$ edges.

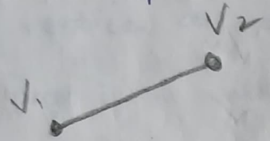
<u>Ex</u>:



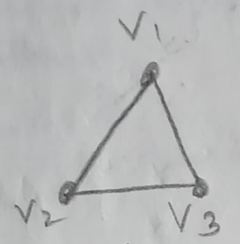
K_4



K_5.



K_2



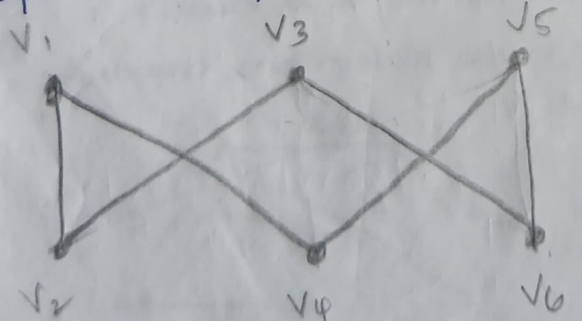
K_3

<u>Note</u>: ① Every null graph is a regular graph of degree 0 (0 - regular graph)

② A complete graph K_n is a regular graph of degree $n-1$. ($n-1$ regular graph).

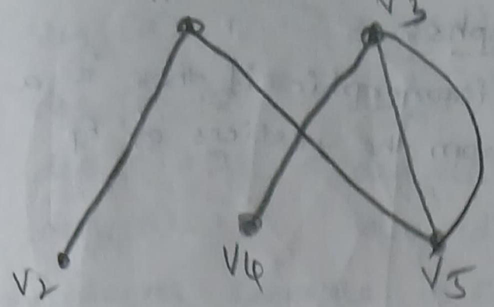
<u>Bipartiate graph</u>: A loop-free graph G in which the vertex set V is partitioned into two disjoint sets X and Y such that every edge of G has one end in X and the other end in Y is known as Bi-partiate graph. The Bi-partitioned X, Y of vertices is known as Bi-partition of G.

<u>Ex</u>:



Graph $K_{3,3}$.

Bi-partition of $K_{3,3}$ is $X = \{v_1, v_3, v_5\}$, $Y = \{v_2, v_4, v_6\}$.

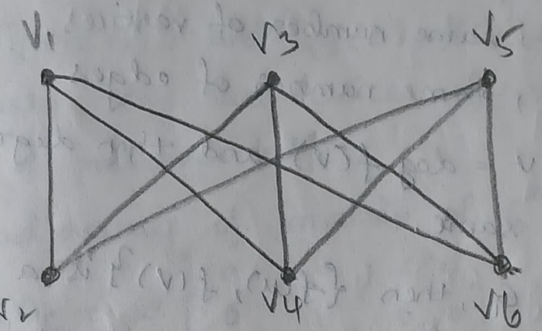


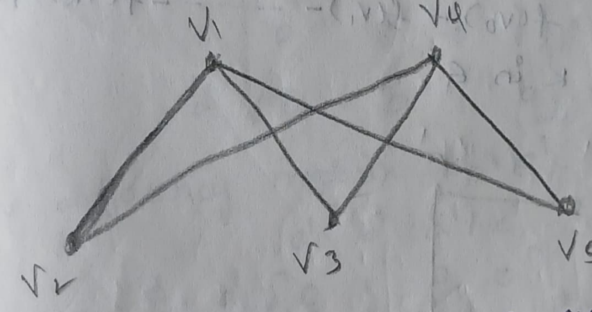
Graph $K_{2,4}$

Bipartition of $K_{2,4}$ is $X = \{v_1, v_3\}$, $Y = \{v_2, v_4, v_5\}$.

Complete Bipartiate graph: For a simple bipartiate graph with bipartition (X, Y), if there is an edge between each pair of vertices from X and Y, then such a graph is called complete bipartiate graph.

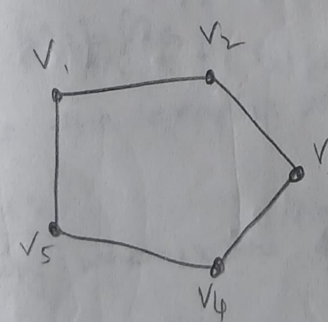
Ex:

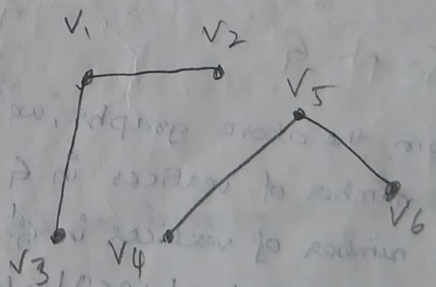




Connected graph: A graph $G = (V, E)$ is said to be connected if, for every pair of vertices $u, v \in V$, there exists a path between u and v.

Ex:





Disconnected graph.

Connected graph

Isomorphism or Isomorphic graphs:

Two graphs G and G' are isomorphic if there is a function $f: V(G) \to V(G')$ from the vertices of G to the vertices of G' such that

(i) f is one-one

(ii) f is onto

(iii) For each pair of vertices u and v of G,

$$(u,v) \in E(G) \Leftrightarrow [f(u), f(v)] \in E(G').$$

Any function f with the above three properties is called an isomorphism. ~~or isomorphic graph,~~

Note: If such an isomorphism f exists then there are several conclusions we can make.

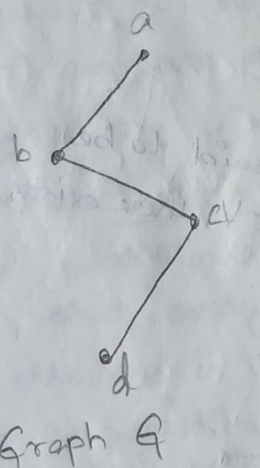
① $|V(G)| = |V(G')|$ i.e., same number of vertices.

② $|E(G)| = |E(G')|$ i.e., same number of edges.

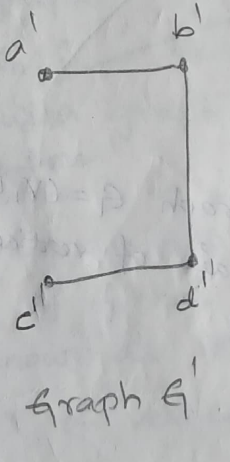
③ If $v \in V(G)$, then $\deg v = \deg f(v)$ and then degree sequence of G and G' are same.

④ If $\{u, v\}$ is a loop in G, then $\{f(u), f(v)\}$ is a loop in G' and more generally if $v_0 - v_1 - v_2 - \cdots - v_k - v_0$ is a cycle of length k then $f(v_0) - f(v_1) - \cdots - f(v_k) - f(v_0)$ is also a cycle of length k in G'.

Ex:



Graph G



Graph G'

From the above graphs, we have

The number of vertices in G, $|V(G)| = 4$.

The number of vertices in G', $|V(G')| = 4$

$$|V(G)| = |V(G')| = 4.$$

The number of edges in G, $|E(G)| = 3$.

The number of edges in G', $|E(G')| = 3$.

$$|E(G)| = |E(G')| = 3.$$

$deg_G(a) = 1$ $\quad$ $deg_{G'}(a') = 1$

$deg_G(b) = 2$ $\quad$ $deg_{G'}(b') = 2$

$deg_G(c) = 2$ $\quad$ $deg_{G'}(e') = 1$

$deg_G(d) = 1$ $\quad$ $deg_{G'}(d') = 2.$

$\therefore$ The degree sequence in $G = \{1, 1, 2, 2\}$

The degree sequence in $G' = \{1, 1, 2, 2\}$.

$\therefore$ The degree sequence is same.

We define a function $f : V(G) \to V(G')$ as

$f(a) = a'$, $f(b) = b'$, $f(c) = d'$, $f(d) = c'$.

We observe that f is one-one and onto.

Further, $\{a, b\} \in E(G)$ and $\{f(a), f(b)\} = \{a', b'\} \in E(G')$

$\{b, c\} \in E(G)$ and $\{f(b), f(c)\} = \{b', c'\} \in E(G')$.

Similarly any two vertices $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(G')$

Hence f preserves edjes.

The adjacency of matrix of G,

$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix of G'

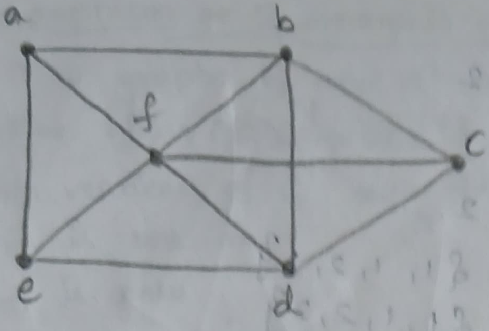
$$A_{G'} = \begin{array}{c} \\ a' \\ b' \\ d' \\ c' \end{array} \begin{array}{c} \begin{array}{cccc} a' & b' & d' & c' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore [A_G] = [A_{G'}]$

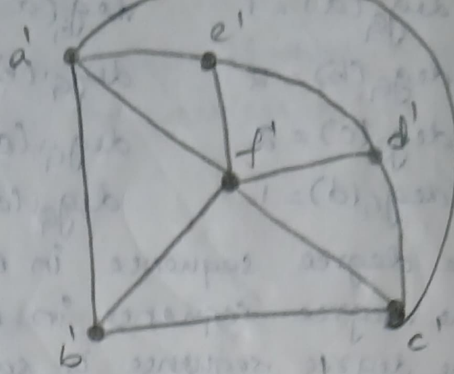
Hence the graphs are isomorphic.

Problems:

Q: Verify whether the following graphs are isomorphic or not.



Graph G



Graph G'

Sol: Order and size:

In graph G, no. of vertices, $|V(G)| = 6$.

no. of edges, $|E(G)| = 11$

In graph G', no of vertices, $|V(G')| = 6$.

no. of edges, $|E(G')| = 11$.

$\therefore |V(G)| = |V(G')|$

$|E(G)| = |E(G')|$.

Degree sequence:

$deg_G(a) = 3$ $\quad deg_{G'}(a') = 4$

$deg_G(b) = 4$ $\quad deg_{G'}(b') = 3$

$deg_G(c) = 3$ $\quad deg_{G'}(c') = 4$

$deg_G(d) = 4$ $\quad deg_{G'}(d') = 3$

$deg_G(e) = 3$ $\quad deg_{G'}(e') = 3$

$deg_G(f) = 5$. $\quad deg_{G'}(f') = 5$.

Degree sequence of G = $\{3, 3, 3, 4, 4, 5\}$

Degree sequence of G' = $\{3, 3, 3, 4, 4, 5\}$.

$\therefore$ Degree sequence of G and G' are same.

Function:

We define a function $f: V(G) \to V(G')$ as

$f(a) = d'$, $f(b) = c'$, $f(c) = b'$, $f(d) = a'$, $f(e) = e'$,

$f(f) = f'$,

We know that f is one-one and onto.

Edge preserving:

$[a, b] \in E(G) \Leftrightarrow [f(a), f(b)] = [d', c'] \in E(G')$.

$[b, c] \in E(G) \Leftrightarrow [f(b), f(c)] = [c', b'] \in E(G')$

$[c, d] \in E(G) \Leftrightarrow [f(c), f(d)] = [b', a'] \in E(G')$.

$[d,e] \in E(G) \Leftrightarrow [f(d), f(e)] = [a', e'] \in E(G')$

$[a,e] \in E(G) \Leftrightarrow [f(a), f(e)] = [d', e'] \in E(G')$

$[a,f] \in E(G) \Leftrightarrow [f(a), f(f)] = [d', f'] \in E(G')$

$[b,f] \in E(G) \Leftrightarrow [f(b), f(f)] = [c', f'] \in E(G')$.

$[d,f] \in E(G) \Leftrightarrow [f(d), f(f)] = [a', f'] \in E(G')$

$[e,f] \in E(G) \Leftrightarrow [f(e), f(f)] = [e', f'] \in E(G')$.

$[c,f] \in E(G) \Leftrightarrow [f(c), f(f)] = [b', f'] \in E(G')$

$[b,d] \in E(G) \Leftrightarrow [f(b), f(d)] = [c', a'] \in E(G']$

$\therefore$ Edge preserving is satisfied. (or). f preserves adjacency of vertices.

Adjacency matrix:

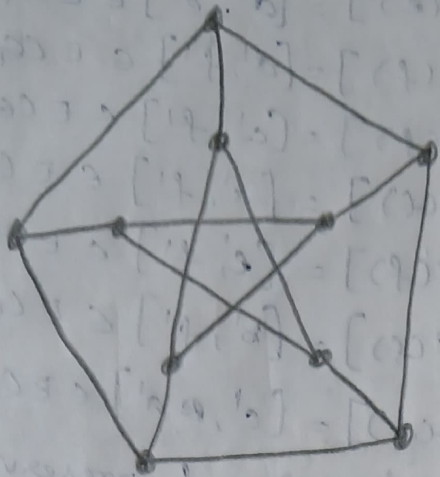
$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \\ f \end{array} \begin{array}{c} \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

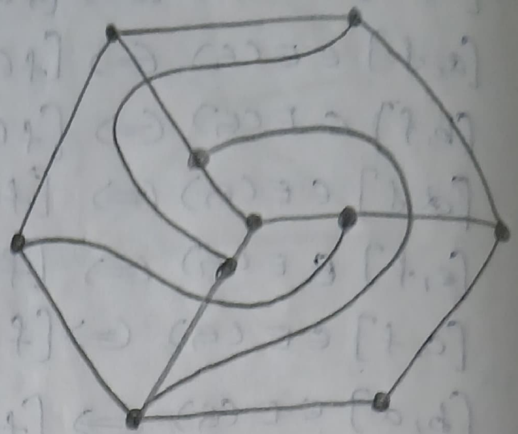
$$A_{G'} = \begin{array}{c} \\ d' \\ c' \\ b' \\ a' \\ e' \\ f' \end{array} \begin{array}{c} \begin{array}{cccccc} d' & c' & b' & a' & e' & f' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

$\therefore A_G = A_{G'}$.

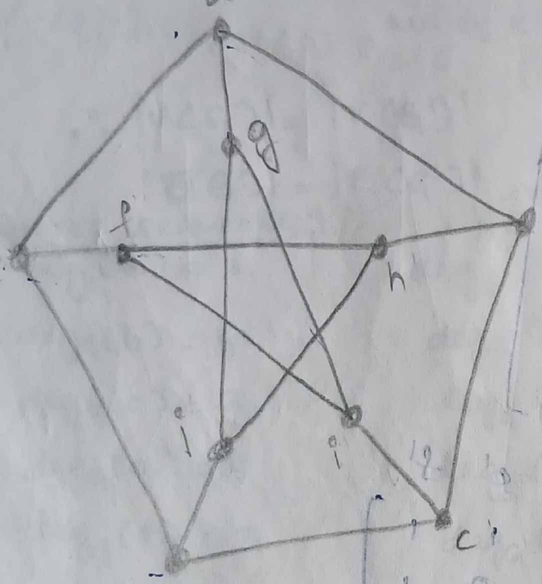
Hence the graphs are isomorphic.

②.

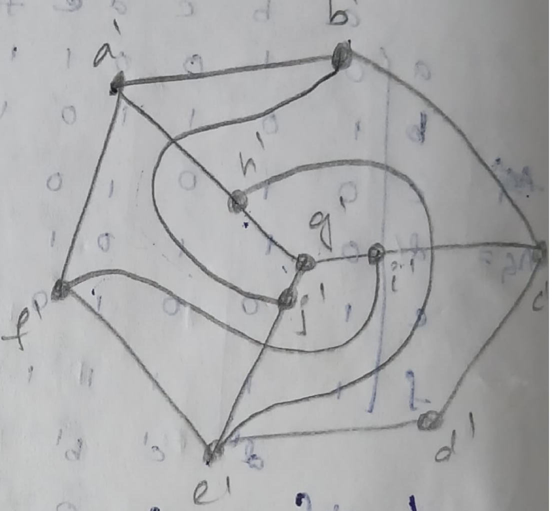




Sol:



Graph G



Graph G'.

Vertices (order):

No. of vertices in G, $|V(G)| = 10$.

No. of vertices in G', $|V(G')| = 10$.

$\therefore |V(G)| = |V(G')|$

Edges (size):

No. of edges in G, $|E(G)| = 15$

No. of edges in G', $|E(G')| = 15$.

$\therefore |E(G)| = |E(G')|$.

Degree sequence:

$deg_G(a) = 3$ $\quad$ $deg_{G'}(a') = 3$

$deg_G(b) = 3$ $\quad$ $deg_{G'}(b') = 3$

$deg_G(c) = 3$ $\quad$ $deg_{G'}(c') = 3$.

$\deg_G(d) = 3$ $\deg_{G'}(d') = 2$

$\deg_G(e) = 3$ $\deg_{G'}(e') = 4$

$\deg_G(f) = 3$ $\deg_{G'}(f') = 3$

$\deg_G(g) = 3$ $\deg_{G'}(g') = 3$

$\deg_G(h) = 3$ $\deg_{G'}(h') = 3$

$\deg_G(i) = 3$ $\deg_{G'}(i) = 3$

$\deg_G(j) = 3.$ $\deg_{G'}(j) = 3.$

Degree sequence of $G = \{3, 3, 3, 3, 3, 3, 3, 3, 3, 3\}$.

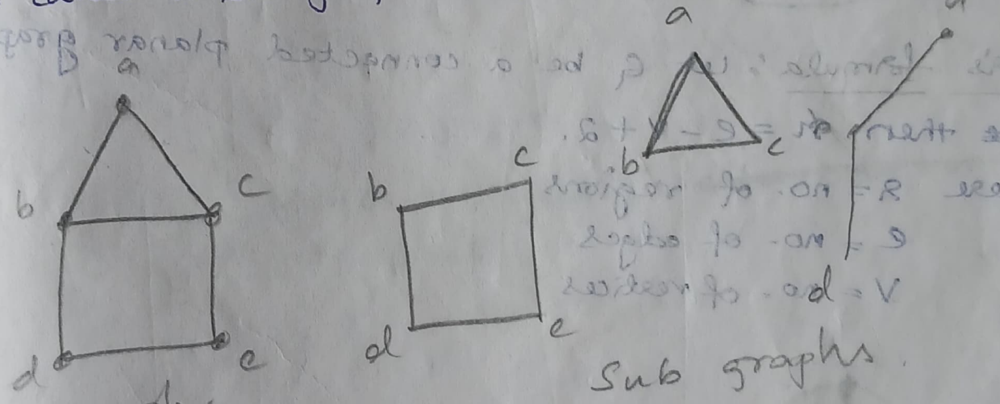
Degree sequence of $G' = \{2, 3, 3, 3, 3, 3, 3, 3, 3, 4\}$.

Here degree sequence is not same.

$\therefore$ The given graphs are not isomorphic.

Sub graphs: Let G and H are two graphs with vertex $V(H)$, $V(G)$ and edges set $E(G)$, $E(H)$. such that if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ then H is called a sub graph of G.

Ex:



Graph

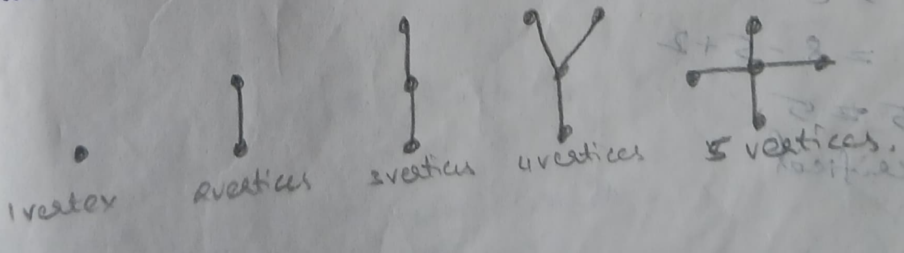
Sub graphs

Trees: A tree is connected acyclic graph.

Its edges are called branches.

A tree with only one vertex is called a trivial tree otherwise it is a non-trivial tree.

Eg:



Note:

① If in a graph G, there is one and only one path between every pair of vertices then G is a tree.

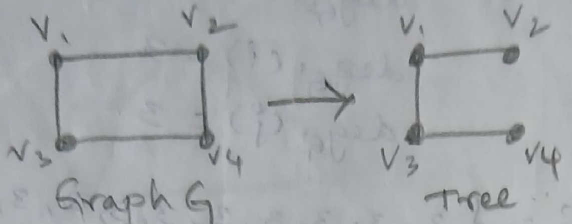
② A tree with n vertices has n-1 edges.

Spanning tree : A subgraph H of a graph G is called a spanning tree of G if

① H is a tree.

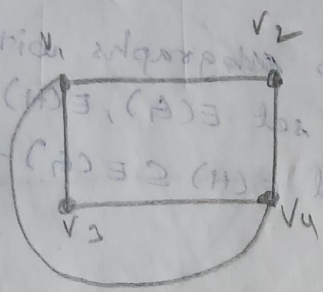
② H contains all the vertices of G.

Ex:



Planar graph : A graph is said to be planar if it can be drawn in a plane with its edges do not intersect each other except at vertices.

Ex:



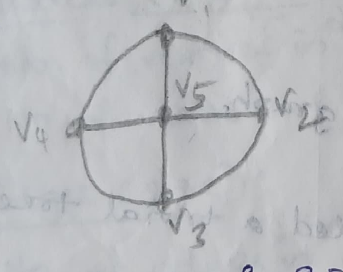
Euler's formula : Let G be a connected planar graph, ~~there~~ then $r = e - v + 2$.

where r = no. of regions

e = no. of edges

V = no. of vertices

Ex:



$v = 5,\ e = 8,\ r = 5$

$r = e - v + 2$

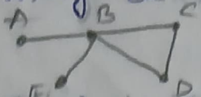
$= 8 - 5 + 2$

$5 = 5.$

Hence Verified

Euler path: A path that passes through each edge exactly once but vertices may be repeated is called Euler path.

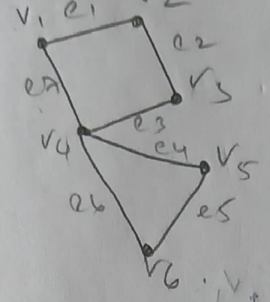
Ex:



Note: At most 2-vertices have odd degree

Euler circuit: A graph G is said to have an Euler circuit if there is a circuit in G that traverses every edge of the graph exactly once, i.e., start and end vertex are same. In a circuit all the vertices should have even degree.

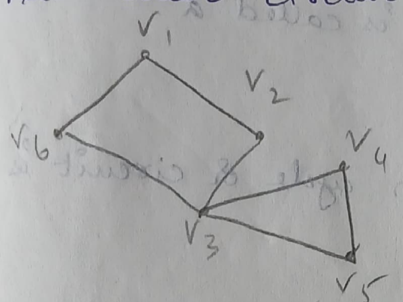
Ex:



$V_1, e_1, V_2, e_2, V_3, e_3, V_4, e_4, V_5, e_5, V_6, e_6, V_4, e_7, V_1$ - Euler circuit

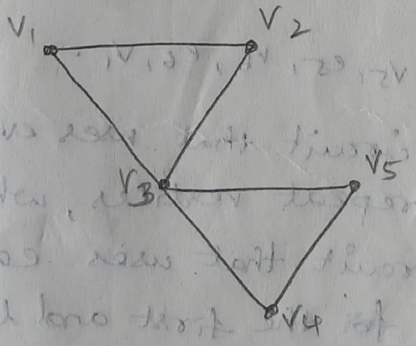
Euler graph: A graph G is called a euler graph if it have euler circuit.

Ex:



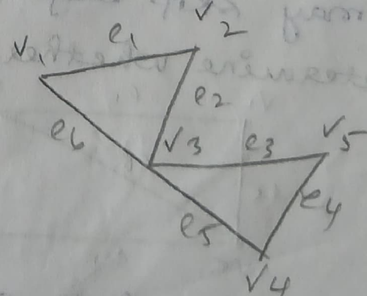
$V_1 - V_2 - V_3 - V_4 - V_5 - V_3 - V_6 - V_1$

Q: Determine whether the graph is ~~euler circuit or not~~. euler path or circuit.



Sol: In the above graph, we can write a euler circuit as

$V_1, e_1, V_2, e_2, V_3, e_3, V_5, e_4, V_4, e_5, V_3, e_6, V_1$.

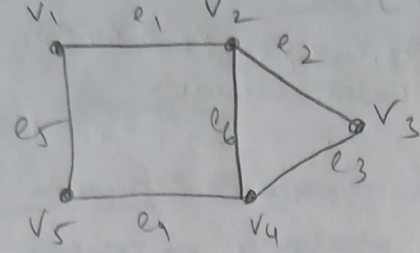


Hence the graph is Euler circuit and hence euler path.

Hamiltonian path: A path in graph G that uses each vertex of the graph exactly once is called Hamiltonian path.

Hamiltonian circuit or **Hamiltonian cycle**: A circuit that contains each vertex in G exactly once, except start and end vertices, is called Hamiltonian circuit.

Ex:



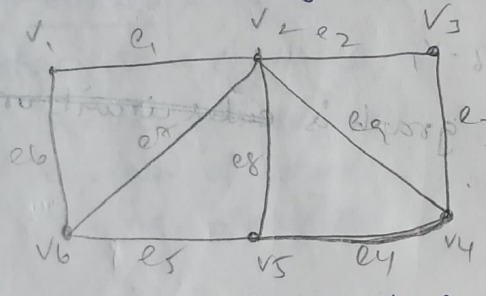
$V_1, e_1, V_2, e_2, V_3, e_3, V_4, e_4, V_5, e_5, V_1$.

Hamiltonian graph: A graph with a closed path that includes every vertex exactly once is called a Hamiltonian graph.

(or)

A graph containing Hamiltonian cycle or circuit is called Hamiltonian graph.

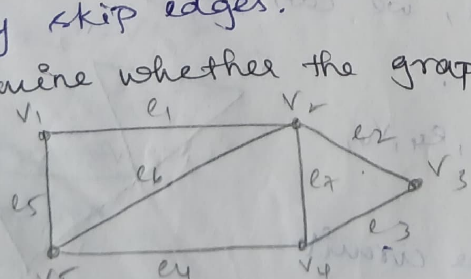
Ex:



$V_1, e_1, V_2, e_2, V_3, e_3, V_4, e_4, V_5, e_5, V_6, e_6, V_1$.

Note: Eulerian graph has a circuit that uses every edge exactly once but may repeat vertices, while a hamiltonian graph has a circuit that uses each vertex exactly once (except for the first and last) but may skip edges.

Q: Determine whether the graph is Hamiltonian path or circuit



Sol: From the above graph, $V_1, e_1, V_2, e_2, V_3, e_3, V_4, e_4, V_5$ is Hamiltonian path. and $V_1, e_1, V_2, e_2, V_3, e_3, V_4, e_4, V_5, e_5, V_1$ is a Hamiltonian circuit.

Hence the graph is both Hamiltonian path and circuit.