Operations on One Random Variables

operations on one Random variables:-

Expectation:-

The mean or average value of a probability distribution function of a random variable 'x' is called Expectation of x or the mean value of 'x' or simply the Expected value of x.

It is denoted as $E[x]$ or $\bar{x}$.

Expected value of a Random variable:-

If 'x' be a continuous random variable with valid probability density function $f_x(x)$ then the expected value of 'x' is given by

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

Note:-

* If 'x' is a discrete random variable with set of elements $\{x_1, x_2, x_3 \ldots x_n\}$ and the corresponding probabilites $\{p(x_1), p(x_2), \ldots p(x_n)\}$ then the expected value of x is

$$E(x) = \sum_{i=1}^{n} x_i\, p(x_i)$$

* If $p(x_1) = p(x_2) = \ldots = p(x_n) = \frac{1}{n}$

$$E[x] = \sum_{i=1}^{n} x_i \frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

The Expected value of a function of a random variable:-

consider a random variable x with probability density function $f_x(x)$. If $g(x)$ is a real function of x then Expected value of $g(x)$ for a continuous random variable x is defined as

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x)\, dx$$

For discrete random variable x, it is defined as

$$E[g(x)] = \sum_{i=1}^{n} g(x_i)\, p(x_i)$$

properties of Expectation:-

1) If a random variable 'x' is a constant that is $x=k$

$$E[x] = E[k] = k$$

2) If 'a' is any constant then expectation,

$$E[ax] = aE[x]$$

3) If a and b are any two constants then

$$E[ax+b] = aE[x]+b$$

4) If x and y be the random variables such that $y \le x$

$$E[y] \le E[x]$$

5) $$|E[x]| \le E[|x|]$$

6) $$E[g_1(x)+g_2(x)] = E[g_1(x)]+E[g_2(x)]$$

7) $$\sigma_x^2 = E[x^2]-\{E[x]\}^2$$

1) what is our Mathematical expectation if we win Rs 10 if a balanced coin comes up head and losses Rs 10 if it comes up tail.

Let $x_1 = +1$, winning of Rupees 10

$x_2 = -1$, loss of Rs 10

Since the coin is balanced therefore probability of getting head

$$p(x_1) = \frac{1}{2}$$

probability of getting tail

$$p(x_2) = \frac{1}{2}$$

we know that

$$E[x] = \sum_{i=1}^{n} x_i (p(x_i))$$

$$= \sum_{i=1}^{2} x_i \, p(x_i)$$

$$= x_1 \, p(x_1) + x_2 \, p(x_2)$$

$$= 1\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right)$$

$$\therefore E[x] = \frac{1}{2} - \frac{1}{2} = 0$$

2. Let 'x' be a random variable with probabilities shown below:

$x (x_i)$	-1	1	2
$p(x_i)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

Find a) $E[x]$ b) $E[x^2]$ c) $E[(2x+1)^2]$ d) σ_x^2

Given

$x_1 = -1$ $x_2 = 1$ $x_3 = 2$

$p(x_1) = \frac{1}{6}$ $p(x_2) = \frac{1}{3}$ $p(x_3) = \frac{1}{2}$

a) To find $E[x]$:-

w.k.T

$$E[x] = \sum_{i=1}^{n} x_i \, p(x_i)$$

$$= (-1)\left(\frac{1}{6}\right) + (1)\left(\frac{1}{3}\right) + (2)\left(\frac{1}{2}\right)$$

$= -\frac{1}{6} + \frac{1}{3} + 1$

$\therefore E[x] = \frac{1}{6} + 1 = \frac{7}{6}$

b) To find $E[x^2]$:-

w.k.T $E[x^2] = \sum_{i=1}^{n} x_i^2 \, p(x_i)$

$= (-1)^2 \frac{1}{6} + (1)^2 \frac{1}{3} + (2)^2 \frac{1}{2}$

$= \frac{1}{6} + \frac{1}{3} + 2$

$= \frac{1}{2} + 2$

$\therefore E[x^2] = \frac{5}{2}$

c) To find $E[(2x+1)^2]$:-

w.k.T

$E[(ax+b)^2] = \sum_{i=1}^{n} (ax_i+b)^2 \, p(x_i)$

$E[(2x+1)^2] = \sum_{i=1}^{n} (2x_i+1)^2 \, p(x_i)$

$= (2(-1)+1)^2 \frac{1}{6} + (2(1)+1)^2 \frac{1}{3} + (2(2)+1)^2 \frac{1}{2}$

$= (-1)^2 \frac{1}{6} + \frac{9}{3} + \frac{25}{2}$

$= \frac{1}{6} + \frac{9}{3} + \frac{25}{2}$

$= \frac{47}{3} = 15.66$

d) To find σ_x^2 :-

w.k.T $\sigma_x^2 = E[x^2] - [E[x]]^2$

$\sigma_x^2 = \frac{5}{2} - \left(\frac{7}{6}\right)^2 = \frac{5}{2} - \frac{49}{36} = 1.13$

3. consider a random variable 'x' with $E[x] = 5$ and $\sigma_x^2 = 2.9$ another random variable is given as $y = -8x + 10$. find

a) $E[xy]$

b) $E[x^2]$

c) $E[y^2]$

d) σ_y^2

Given

$$E[x] = 5$$

$$\sigma_x^2 = 2.9$$

And $y = -8x + 10$

a) To find $E[xy]$:-

$$E[xy] = E[x(-8x+10)] = E[-8x^2 + 10x]$$

$$= -8E[x^2] + 10E[x]$$

$$= -8E[x^2] + 10(5)$$

$$= -8E[x^2] + 50$$

$$= -8(27.9) + 50$$

$$= -173.2$$

b) To find $E[x^2]$:-

W.K.T

$$\sigma_x^2 = E[x^2] - [E[x]]^2$$

$$2.9 = E[x^2] - (5)^2$$

$$2.9 + 25 = E[x^2]$$

$$E[x^2] = 27.9$$

c) To find $E[y^2]$:-

$$E[y^2] = E[(-8x+10)^2]$$

$$= E[64x^2 + 100 - 160x]$$

$= 64\,E[x^2] + 100 - 160\,E[x]$

$= 64(27.9) + 100 - 800$

$= 1085.6$

d) To find σ_Y^2 :-

$$\sigma_Y^2 = E[Y^2] - \{E[Y]\}^2$$

$$= 1085.6 - \{E(-8x+10)\}^2$$

$$= 1085.6 - \{-8E[x] + 10\}^2$$

$$= 1085.6 - \{-40+10\}^2$$

$$= 1085.6 - \{-30\}^2$$

$$= 1085.6 - 900$$

$$\sigma_Y^2 = 185.6$$

4. consider a pdf of a random variable 'x' is

$$f_X(x) = \begin{cases} \frac{1}{K}, & -2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

And another random variable $Y = 2X$ the find

a) value of k

b) $E[X]$

c) $E[Y]$

d) $E[XY]$

Given pdf is $f_X(x) = \begin{cases} \frac{1}{K}, & -2 \leq x \leq 3 \\ 0, & \text{otherwise} \end{cases}$

and also given $Y = 2X$

a) To find value of k:-

From the probabilities of probability density function for continuous random variable

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-2}^{3} \frac{1}{k}\,dx = 1$$

$$\frac{1}{k}\int_{-2}^{3} dx = 1$$

$$\frac{1}{k}[x]_{-2}^{3} = 1$$

$$\frac{1}{k}[3-(-2)] = 1$$

$$\frac{1}{k}[5] = 1$$

$$k = 5$$

b) To find $E[X]$:-

W.K.T $E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$

$$= \int_{-2}^{3} x\left(\frac{1}{k}\right)dx$$

$$= \int_{-2}^{3} x\left(\frac{1}{5}\right)dx$$

$$= \frac{1}{5}\int_{-2}^{3} x\,dx$$

$$= \frac{1}{5}\left[\frac{x^2}{2}\right]_{-2}^{3}$$

$$= \frac{1}{10}[9-4]$$

$$= \frac{1}{10}(5) = \frac{1}{2} = 0.5$$

c) To find $E[Y]$:-

$$E[Y] = E[2X] = 2E[X]$$

$$= 2\left(\frac{1}{2}\right) = 1$$

d) To find $E[xy]$:-

$$E[xy] = E[x(2x)] = 2E[x^2]$$

$$= 2\int_{-\infty}^{\infty} x^2 f_x(x)\, dx$$

$$= 2\int_{-2}^{3} x^2 \left(\frac{1}{5}\right) dx$$

$$= \frac{2}{5}\left[\frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{2}{15}\left[x^3\right]_{-2}^{3}$$

$$= \frac{2}{15}\left[27+8\right]$$

$$= \frac{70}{15} = 4.66$$

5. when Two unbiased dice are thrown, find the expected value of sum of the numbers shown on the dice.

we know that by throwing two dice, sample space

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Let x be a random variable i.e the number which is obtained by sum of the numbers in the dice.

$$X = \begin{Bmatrix} (2,3,4,5,6,7) & (3,4,5,6,7,8) & (4,5,6,7,8,9) \\ (5,6,7,8,9,10) & (6,7,8,9,10,11) & (7,8,9,10,11,12) \end{Bmatrix}$$

$X(x_i)$	2	3	4	5	6	7	8	9	10	11	12
$P(x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

To find $E[X]$:-

$$E[X] = \sum_{i=1}^{n} x_i \, p(x_i)$$

$$= \frac{1}{36}\left[2+6+12+20+30+42+40+36+30+22+12\right]$$

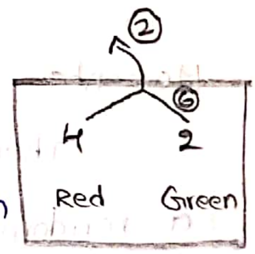
$$= \frac{1}{36}[252]$$

$$= \frac{252}{36}$$

$$\therefore E[X] = 7$$

6. A box contains 4 red, 2 Green balls. 2 balls are drawn together. Find the Expected value of no.of red balls drawn.

Let X a random variable for the event of number of red balls drawn.



Let x_1 denote no red ball and two green balls. $x_1 = 0$

Let x_2 = one red and one green ball

$x_2 = 1$

Let x_3 = two red balls and no green ball

$x_3 = 2$

The probabilities are

$$p(x_1) = \frac{4c_0 \times {}^2c_2}{6c_2} = \frac{1 \times 1}{\frac{6!}{4! \times 2!}} = \frac{1}{15}$$

$$p(x_2) = \frac{4c_1 \times {}^2c_1}{6c_2} = \frac{4 \times 2}{6c_2} = \frac{8}{15}$$

$$p(x_3) = \frac{{}^4C_2 \times {}^2C_0}{{}^6C_2}$$

$$= \frac{6 \times 1}{15}$$

$$p(x_3) = \frac{6}{15} = \frac{2}{5}$$

To find $E[x]$:-

$$E[x] = \sum_{i=1}^{n} x_i \, p(x_i)$$

$$= x_1 p(x_1) + x_2 p(x_2) + x_3 p(x_3)$$

$$= 0\left(\frac{1}{15}\right) + 1\left(\frac{8}{15}\right) + 2\left(\frac{2}{5}\right)$$

$$= 0 + \frac{8}{15} + \frac{4}{5}$$

$$\therefore E[x] = \frac{4}{5}\left(\frac{2}{3}+1\right) = \frac{4}{\cancel{5}}\left(\frac{\cancel{5}}{3}\right) = \frac{4}{3}$$

Moments:-

There are two types of moments. for a function of a random variable x.

i) Moments about the origin:-

Let $g(x)$ be a real function of the random variable x $g(x) = x^n$ for $n = 0, 1, 2 \ldots$ then the expected value of the function $g(x)$ is called the Moments about the origin of a random variable x. It is denoted as m_n where n indicates the order of the moments.

In general m^{th} moment is defined as

$$m_n = E[x^n] = \int_{-\infty}^{\infty} x^n f_x(x)\, dx$$

for discrete random variable $E[x^n] = \sum x_i^n \, p(x_i)$

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* If $n=0$, $E[x^0] = \int_{-\infty}^{\infty} x^0 f_X(x)\,dx = \int_{-\infty}^{\infty} f_X(x)\,dx = 1$

* If $n=1$, $E[x^1] = \int_{-\infty}^{\infty} x f_X(x)\,dx = E[x] = \bar{x}$

* If $n=2$, $E[x^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = E[x^2]$

2) Moments about the mean (or) central moments:-

Let $g(x)$ be a real function of a random variable x such that $g(x) = (x-\bar{x})^n$ for $n = 0,1,2,3 \ldots$ where $\bar{x}$ is the mean of x then the expected value of the function $g(x)$ is called moments about the mean of the random variable x. It is denoted by μ_n. where n indicates the order of the moment is also called central moments of the random variable x.

n^{th} central moment is given by

$$\mu_n = E[(x-\bar{x})^n] = \int_{-\infty}^{\infty} (x-\bar{x})^n f_X(x)\,dx$$

For discrete random variable $\mu_n = \Sigma (x_i - \bar{x})^n p(x_i)$

Note:-

If $n=0$, $\mu_0 = \Sigma (x_i - \bar{x})^0 p(x_i) = \Sigma p(x_i) = 1$

$\therefore \mu_0 = m_0 = 1$

Variance:-

The variance of the density function $f_X(x)$ for a random variable x is defined as the second central moment μ_2 of x. It is also denoted as σ_X^2 (or) $Var(x)$

$$\mu_2 = \sigma_X^2 = E[(x-\bar{x})^2] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\,dx$$

for discrete random variable $\sigma_X^2 = \Sigma (x_i - \bar{x})^2 p(x_i)$

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skew:-

The skew of the density function $f_X(x)$ of a random variable x is defined as the third central moment μ_3 of x. It is given by

$$\therefore \mu_3 = E\left[(x-\bar{x})^3\right] = \int_{-\infty}^{\infty} (x-\bar{x})^3 f_X(x)\, dx$$

For discrete random variable $\mu_3 = \Sigma (x_i - \bar{x})^3 p(x_i)$

properties of variance:-

* If k is a constant then variance of k is equal to 0

$$\therefore Var(k) = 0$$

* $Var(kx) = k^2 . Var(x)$

* For a given random variable x the relationship between the moments is given by

$$\sigma_x^2 = m_2 - m_1^2$$

* If x is a random variable and a, b are real constants then $Var(ax+b) = a^2 Var(x)$

Functions for moments:-

To calculate the n^{th} moment of a random variable x the following two functions are generally used.

1) characteristic function

2) Moment Generation function

characteristic function:-

The characteristic function of a random variable x is defined by $\phi_x(\omega) = E[e^{j\omega x}]$, where $j = \sqrt{-1}$

* It is a function of real variable $-\infty < \omega < \infty$

$$\therefore \phi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

For discrete random variable $\phi_X(\omega) = \sum_i e^{j\omega x} p(x_i)$

* the characteristic function the transforms into another random variable ω.

Note:-

1) The characteristic function is a F.T of $f_X(x)$ with the sign of ω is reversed.

2) The I.F.T $\phi_X(\omega)$ is the density function with the sign of x is reversed.

$$f_X(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

3) If $\phi_X(\omega)$ a characteristic function of a random variable X then n^{th} moment of X is given by

$$m_n = (-j)^n \left.\frac{d^n}{d\omega^n}[\phi_X(\omega)]\right|_{\omega=0}$$

Properties of characteristic function:-

1) The C.F is unity at $\omega = 0$ is given by

$$\left.\phi_X(\omega)\right|_{\omega=0} = \phi_X(0) = 1$$

2) The maximum Amplitude of the C.F is unity at $\omega = 0$ is given by

$$|\phi_X(\omega)| \leq \phi_X(0) \quad (\text{or}) \quad |\phi_X(\omega)| \leq 1$$

3) $\phi_X(\omega)$ is a continuous function of ω in the range $-\infty < \omega < \infty$.

4) $\phi_X(-\omega)$ and $\phi_X(\omega)$ are conjugate functions i.e

$$\phi_X(-\omega) = \phi_X^*(\omega) \quad \text{and} \quad \phi_X^*(-\omega) = \phi_X(\omega)$$

5) $\phi_{X_1+X_2}(\omega) = \phi_{X_1}(\omega) \cdot \phi_{X_2}(\omega)$

1) Find the characteristic function for a random variable with density function $f_X(x) = x$ for $0 \le x \le 1$.

Given

pdf is $f_X(x) = x$ for $0 \le x \le 1$

From the definition of C.F

$$\phi_X(\omega) = E[e^{j\omega X}]$$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3$

$$= \int_{x=0}^{1} e^{j\omega x}\, x\, dx$$

$$= \left[(x)\left(\frac{e^{j\omega x}}{j\omega}\right) - (1)\left(\frac{e^{j\omega x}}{(j\omega)^2}\right) \right]_0^1$$

$$= \frac{1}{j\omega}\left[e^{j\omega x}\left(x - \frac{1}{j\omega}\right) \right]_{x=0}^{1}$$

$$= \frac{1}{j\omega}\left[e^{j\omega}\left(1 - \frac{1}{j\omega}\right) - 1\left(0 - \frac{1}{j\omega}\right) \right]$$

$$= \frac{1}{j\omega}\left[e^{j\omega}\left(1 - \frac{1}{j\omega}\right) + \frac{1}{j\omega} \right]$$

2) The density function of a random variable is given by $f_X(x) = ae^{-bx}$, $x \ge 0$, find the characteristic function and first two moments.

Given pdf is $f_X(x) = ae^{-bx}$, $x \ge 0$

1) To find C.F:-

W.K.T $CF = \phi_X(\omega) = E(e^{j\omega x})$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_{x=0}^{\infty} e^{j\omega x} (ae^{-bx})\, dx$$

$$= a \int_0^{\infty} e^{-(-j\omega+b)x}\, dx$$

$$= a \left[\frac{e^{-(-j\omega+b)x}}{j\omega-b} \right]_{x=0}^{\infty}$$

$$= \frac{a}{j\omega-b} [0-1]$$

$$\phi_X(\omega) = \frac{-a}{j\omega-b}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

2) To find m_n:-

W.K.T

$$m_n = (-j)^n \frac{d^n}{d\omega^n} \phi_X(\omega) \Big|_{\omega=0}$$

put $n = 1$

$$m_1 = (-j)^1 \frac{d}{d\omega}(\phi_X(\omega)) \Big|_{\omega=0}$$

$$= (-j) \frac{d}{d\omega} \left(\frac{-a}{j\omega-b} \right) \Big|_{\omega=0}$$

$$= (-j) \left[(-a)(-1)(j\omega-b)^{-2}(j) \right] \Big|_{\omega=0}$$

$$= (-j) \left[\frac{aj}{(j\omega-b)^2} \right] \Big|_{\omega=0}$$

$$= (-j) \left[\frac{aj}{(-b)^2} \right]$$

$$m_1 = \frac{a}{b^2}$$

put $n = 2$

$$m_2 = (-j)^2 \frac{d^2}{d\omega^2} \phi_X(\omega) \Big|_{\omega=0}$$

$$= (-j) \frac{d}{d\omega} \left[\frac{d}{d\omega} (\phi_X(\omega)) \right] \Bigg|_{\omega=0}$$

$$= (-j) \frac{d}{d\omega} \left[\frac{aj}{(j\omega-b)^2} \right] \Bigg|_{\omega=0}$$

$$= (-aj) \left[(-2)(j\omega-b)^{-3} (j) \right] \Bigg|_{\omega=0}$$

$$= \left[\frac{2a(-1)}{(j\omega-b)^3} \right] \Bigg|_{\omega=0}$$

$$m_2 = \frac{-2a}{(-b)^3}$$

$$m_2 = \frac{-2a}{-b^3} = \frac{2a}{b^3}$$

3) Find the C.F for $f_X(x) = e^{-|x|}$

Given pdf is $f_X(x) = e^{-|x|}$

$\therefore$ CF is $\phi_X(\omega) = E[e^{j\omega x}]$

$$= \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_{-\infty}^{0} e^{j\omega x} e^{-(-x)} dx \int_{0}^{\infty} e^{j\omega x} e^{-(x)} dx$$

$$= \int_{-\infty}^{0} e^{(j\omega+1)x}\, dx + \int_{0}^{\infty} e^{(j\omega-1)x}\, dx$$

$e^{-\infty} = 0$

$e^{\infty} = \infty$

$$= \left[\frac{e^{(j\omega+1)x}}{j\omega+1} \right]_{-\infty}^{0} + \left[\frac{e^{(j\omega-1)x}}{j\omega-1} \right]_{0}^{\infty}$$

$$= \frac{1}{j\omega+1}\left[1-0\right] + \left[\frac{e^{-(-j\omega+1)x}}{-(-j\omega+1)} \right]_{0}^{\infty}$$

$$= \frac{1}{j\omega+1} + \frac{1}{-(-j\omega+1)}\left[0-1\right]$$

$$\times \left[= \frac{1}{j\omega+1} + \frac{1}{j\omega-1} \right.$$

$$= \left. \frac{2j\omega}{(j\omega)^2-1} = \frac{2j\omega}{-\omega^2-1} = \frac{2j\omega}{+(1+} \right] \times$$

$$= \frac{1}{j\omega+1} - \frac{1}{j\omega-1}$$

$$= \frac{-2}{(j\omega)^2-1} = \frac{-2}{-\omega^2-1} = \frac{-2}{-(1+\omega^2)} = \frac{2}{1+\omega^2}$$

4) Show that the distribution function for which the CF $e^{-|\omega|}$ has the density function $f_X(x) = \frac{1}{\pi(1+x^2)}$, $-\infty<x<\infty$

Given

CF is $\phi_X(\omega) = e^{-|\omega|}$

W.K.T

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{\omega}\, e^{-j\omega x}\, d\omega + \int_{0}^{\infty} e^{-\omega}\, e^{-j\omega x}\, d\omega \right]$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{(1-jx)\omega}\, d\omega + \int_{0}^{\infty} e^{-(1+jx)\omega}\, d\omega\right]$$

$$= \frac{1}{2\pi}\left\{\left[\frac{e^{(1-jx)\omega}}{1-jx}\right]_{-\infty}^{0} + \left[\frac{e^{-(1+jx)\omega}}{-(1+jx)}\right]_{0}^{\infty}\right\}$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-jx}\{1-0\} + \frac{1}{-(1+jx)}\{0-1\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-jx} - \frac{1}{(1+jx)}(-1)\right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-jx} + \frac{1}{1+jx}\right]$$

$$f_X(x) = \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{2}}{1-(jx)^2}\right] = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

5) Find the C.F of a binomial of a distribution.

consider

the binomial distribution of a random variable x with probability $P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$$p+q=1 \quad \text{also} \quad \Sigma\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= (p+q)^n = 1^n = 1$$

From the definition of C.F for discrete random Variable x.

$$\phi_X(\omega) = E\left[e^{j\omega x}\right]$$

$$= \sum_{x=0}^{n} e^{j\omega x} p(x)$$

$$= \sum_{x=0}^{n} e^{j\omega x}\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

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$$= \sum_{x=0}^{n} {}^{n}C_x \,(pe^{j\omega})^x q^{n-x}$$

6) Find the density function of the random variable X. If the C.F is $\phi_X(\omega) = \begin{cases} 1-|\omega|, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$

Given

C.F is $\phi_X(\omega) = \begin{cases} 1-|\omega|, & |\omega| \le 1 \\ 0, & \text{otherwise} \end{cases}$

$$-\infty \quad\quad -1 \quad\quad 0 \quad\quad +1 \quad\quad \infty$$

$$\underbrace{\quad\quad}_{\phi_X(\omega)=0} \quad \underbrace{\quad\quad}_{1+\omega} \quad \underbrace{\quad\quad}_{1-\omega} \quad \underbrace{\quad\quad}_{0}$$

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$= \frac{1}{2\pi}\left[\int_{-1}^{0} (1+\omega)\, e^{-j\omega x} d\omega + \int_{0}^{1} (1-\omega)\, e^{-j\omega x} d\omega\right]$$

$$= \frac{1}{2\pi}\left[\left[\left\{(1+\omega)\left(\frac{e^{-j\omega x}}{-jx}\right) - (0+1)\left(\frac{e^{-j\omega x}}{-x^2}\right)\right\}_{-1}^{0}\right] + \right.$$

$$\left.\left[\left\{(1-\omega)\left(\frac{e^{-j\omega x}}{-jx}\right) - (0-1)\left(\frac{e^{-j\omega x}}{-x^2}\right)\right\}_{0}^{1}\right]\right]$$

$$= \frac{1}{2\pi}\left[\left\{e^{-j\omega x}\left(-\frac{(1+\omega)}{jx} + \frac{1}{x^2}\right)\right\}_{-1}^{0} + \left\{e^{-j\omega x}\left(-\frac{(1-\omega)}{jx} - \frac{1}{x^2}\right)\right\}_{0}^{1}\right]$$

$$= \frac{1}{2\pi}\left[\left\{1\left(\cancel{\frac{-1}{jx}} + \frac{1}{x^2}\right) - e^{jx}\left(0 + \frac{1}{x^2}\right)\right\} + \left\{e^{-jx}\left(0 - \frac{1}{x^2}\right) - 1\left(\cancel{\frac{-1}{jx}} - \frac{1}{x^2}\right)\right\}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2}{x^2} - \frac{1}{x^2}\left(e^{jx} + e^{-jx}\right)\right]$$

$$= \frac{1}{\cancel{2}\pi}\left[\frac{\cancel{2}}{x^2} - \frac{\cancel{2}}{x^2}\cos x\right]$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$

$$2\cos x = e^{jx}+e^{-jx}$$

$$= \frac{1}{\pi x^2}\left[1-\cos x\right]$$

Moment generating function (mgf):-

The moment generating function of a random variable is also used to generate the n^{th} moments about the origin.

consider a random variable x with probability density function $f_X(x)$ then mgf x is defined as the expected value of the function e^{vx} i.e

$$M_X(v) = E[e^{vx}] \quad \text{where} \quad -\infty < v < \infty$$

$$\therefore \quad M_X(v) = \int_{-\infty}^{\infty} e^{vx} f_X(x)\,dx$$

For continuous random variable

$$M_X(v) = \sum_i e^{vx_i} p(x_i)$$

For Discrete random variable.

Properties of m.g.f:-

* If $M_X(v)$ is a m.g.f of the random variable x then the n^{th} moment of x is given by

$$m_n = \left.\frac{d^n}{dv^n} M_X(v)\right|_{v=0}$$

* The moment generating function at v=0 is unity i.e

$$\left. M_X(v)\right|_{v=0} = M_X(0) = 1$$

* If x be a random variable with m.g.f $M_x(v)$ then m.g.f for $y = ax+b$.

$$M_y(v) = e^{bv} M_x(av)$$

* If $M_x(v)$ is a m.g.f of a random variable x then

$$M_x(cv) = e^{vb} M_{cx}(v)$$

* If x_1 and x_2 are two independent random variable with m.g.f $M_{x_1}(v)$ and $M_{x_2}(v)$ then

$$M_{x_1+x_2}(v) = M_{x_1}(v) \cdot M_{x_2}(v)$$

1) The pdf of a random variable x $p(x) = \frac{1}{10^x}$ for $x = 0,1,2 \cdots \infty$ find 1) m.g.f 2) $E[x]$

Given

probability distribution

$$p(x) = \frac{1}{10^x} \quad \text{for } x = 0,1,2,3 \cdots \infty$$

1) To find m.g.f :-

$$\text{m.g.f} = M_x(v) = E[e^{vx}]$$

$$= \sum_x e^{vx} p(x)$$

$$\left[\because \sum_{x=0}^{\infty} a^x = \frac{1}{1-a} \right.$$

$$= \sum_{x=0}^{\infty} e^{vx} \frac{1}{10^x}$$

$$= \sum_{x=0}^{\infty} \left(\frac{e^v}{10} \right)^x$$

$$= \frac{1}{1 - \frac{e^v}{10}} = \frac{10}{10 - e^v}$$

2) To find $E[x]$:-

$$E[x] = E[x^1] = m_1 = \frac{d}{dv} M_x(v) \Big|_{v=0}$$

$$E[x] = E[x'] = \frac{d}{dv}\left[\frac{10}{10-e^v}\right]\Bigg|_{v=0}$$

$$= 10\frac{d}{dv}\left[(10-e^v)^{-1}\right]\Bigg|_{v=0}$$

$$= 10\left[(-1)(10-e^v)^{-2}(0-e^v)\right]\Bigg|_{v=0}$$

$$= \left[\frac{10e^v}{(10-e^v)^2}\right]\Bigg|_{v=0}$$

$$= \frac{10}{81}$$

8) The pdf of a random variable x is given as $f_x(x) = ae^{-bx}$, $x \geq 0$. find

a) $M_x(v)$

b) $E[x]$

c) $E[x^2]$

Given

pdf is $f_x(x) = ae^{-bx}$, $x \geq 0$

a) To find $M_x(v)$:-

$$M_x(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx} f_x(x)\,dx$$

$$= \int_{x=0}^{\infty} e^{vx}(ae^{-bx})\,dx$$

$$= a\int_{x=0}^{\infty} e^{-(b-v)x}\,dx$$

$$= a\left[\frac{e^{-(b-v)x}}{-(b-v)}\right]_0^{\infty}$$

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$$= \frac{a}{-(b-v)}[0-1]$$

$$\therefore M_x(v) = \frac{a}{b-v}$$

b) To find $E[x]$:-

$$E[x] = E[x^1] = m_1 = \frac{d}{dv} M_x(v)\Big|_{v=0}$$

$$= \frac{d}{dv}\left[\frac{a}{b-v}\right]\Big|_{v=0}$$

$$= a\frac{d}{dv}\left[(b-v)^{-1}\right]\Big|_{v=0}$$

$$= a\left[(-1)(b-v)^{-2}(0-1)\right]\Big|_{v=0}$$

$$= \left[\frac{a}{(b-v)^2}\right]\Big|_{v=0}$$

$$E[x] = \frac{a}{b^2}$$

c) To find $E[x^2]$:-

$$E[x^2] = m_2 = \frac{d^2}{dv^2} M_x(v)\Big|_{v=0}$$

$$= \frac{d}{dv}\left[\frac{d}{dv} M_x(v)\right]\Big|_{v=0}$$

$$= \frac{d}{dv}\left[\frac{a}{(b-v)^2}\right]\Big|_{v=0}$$

$$= \left(a(-2)(b-v)^{-3}(0-1)\right)\Big|_{v=0}$$

$$= \left[\frac{2a}{(b-v)^3}\right]\Big|_{v=0}$$

$$E[x^2] = \frac{2a}{b^3}$$

3) The probability distribution of a random variable x is

$P(x) = \frac{2}{3}\left(\frac{1}{3}\right)^x, \quad x = 0,1,2 \dots \infty$

Find out m.g.f. Also find out 1^{st} and 2^{nd} moments.

Given

$$PD \text{ is } p(x) = \frac{2}{3}\left(\frac{1}{3}\right)^x, \quad x = 0,1,2 \dots \infty$$

1) To find m.g.f :-

$$\therefore m.g.f = M_x(v) = E[e^{vx}]$$

$$= \sum_x e^{vx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{vx} \frac{2}{3}\left(\frac{1}{3}\right)^x$$

$$= \frac{2}{3} \sum_{x=0}^{\infty} e^{vx}\left(\frac{1}{3}\right)^x$$

$$= \frac{2}{3} \sum_{x=0}^{\infty} \left(\frac{e^v}{3}\right)^x$$

$$= \frac{2}{3} \frac{1}{\left(1-\frac{e^v}{3}\right)} = \frac{2}{\cancel{3}} \frac{\cancel{3}}{(3-e^v)} = \frac{2}{3-e^v}$$

2) To find m_1 :-

$$E[x] = E[x^1] = m_1 = \frac{d}{dv} M_x(v) \Big|_{v=0}$$

$$= \frac{d}{dv}\left[\frac{2}{3-e^v}\right]\Big|_{v=0}$$

$$= 2 \frac{d}{dv}(3-e^v)^{-1} \Big|_{v=0}$$

$$= 2\left[(-1)(3-e^v)^{-2}(0-e^v)\right]\Big|_{v=0}$$

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$$= \left[\frac{2e^v}{(3-e^v)^2}\right]\Bigg|_{v=0}$$

$$= \frac{2}{4} = \frac{1}{2}$$

3) To find m_2:-

$$m_2 = \frac{d^2}{dv^2} M_x(v)\Bigg|_{v=0}$$

$$= \frac{d}{dv}\left[\frac{d}{dv} M_x(v)\right]\Bigg|_{v=0} \qquad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}\right]$$

$$= \frac{d}{dv}\left[\frac{2e^v}{(3-e^v)^2}\right]\Bigg|_{v=0}$$

$$= 2\left[\frac{(3-e^v)^2 e^v - e^v\{2(3-e^v)^3(0-e^v)\}}{(3-e^v)^4}\right]\Bigg|_{v=0}$$

$$= \cancel{2}\left[\frac{4 - 1\{2(8)(-1)\}}{\cancel{16}\ 8}\right]$$

$$= \frac{4+16}{8}$$

$$\therefore m_2 = \frac{20}{8} = \frac{5}{2}$$

) Find the m.g.f and C.F of a poisson distribution.

Given that

we know that

PD of a poisson distibution is given by

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x \geq 0$$

1) To find m.g.f:-

$$m.g.f = M_x(v) = E[e^{vx}]$$

$$= \sum_x e^{vx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{vx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} e^{vx} \frac{\lambda^x}{x!} \quad \left[\because e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}\right.$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^v\lambda)^x}{x!}$$

$$= e^{-\lambda} \cdot e^{(e^v\lambda)}$$

$$= e^{-\lambda + e^v\lambda}$$

$$\therefore M_x(v) = e^{\lambda(e^v - 1)}$$

2) To find C.F :-

$$C.F = \phi_x(\omega) = E[e^{j\omega x}]$$

$$= \sum_{x=0}^{\infty} e^{j\omega x} p(x)$$

$$= \sum_{x=0}^{\infty} e^{j\omega x} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{j\omega}\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{(e^{j\omega}\lambda)}$$

$$= e^{-\lambda + e^{j\omega}\lambda}$$

$$\therefore C.F = e^{\lambda(e^{j\omega} - 1)}$$

Multiple Random variables:-

Joint probability distribution function

consider two random variables x and y with elements $\{x\}$ and $\{y\}$ in xy-plane the order pair of numbers (x,y) is called a random vector in the two dimensional product space or joint sample space.

Let the two events $A = \{X \le x\}$ and $B = \{Y \le y\}$ then the joint probability ~~density~~ distribution function is defined as

$F_{XY}(x,y) = P\{X \le x, Y \le y\} = P(A \cap B)$.

The general For discrete random variable, joint probability distribution is given by

* $$F_{XY}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{N} P(x_n, y_m)\, u(x-x_n)\, u(y-y_m)$$

Similarly for n random variables, joint distribution function is given by

$$F_{X X_2 \ldots X_n}(x_1, x_2, \ldots x_n) = P\{X_1 \le x_1, X_2 \le x_2, \ldots X_n \le x_n\}$$

properties of joint distribution function:-

* $F_{XY}(-\infty, -\infty) = 0$
* $F_{XY}(x, -\infty) = 0$
* $F_{XY}(-\infty, y) = 0$
* $F_{XY}(\infty, \infty) = 1$
* $0 \le F_{XY}(x,y) \le 1$
* $F_{XY}(x,y)$ is a monotonic and non-decreasing function of both x and y.
* The marginal distribution function are given by

$F_{XY}(x,\infty) = F_X(x)$

$$F_{XY}(\infty, y) = F_Y(y)$$

Joint probability density function:-

The joint probability density function of two random variables X and Y is defined as

$$f_{XY}(x,y) = \frac{\partial^2}{\partial x \, \partial y} F_{XY}(x,y)$$

It is simply called joint density function.

For discrete random variables, joint density function is given by

$$f_{XY}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{N} p(x_n, y_m)\, \delta(x-x_n)\, \delta(y-y_m)$$

By direct Integration, the joint distribution function can be obtained as

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy$$

properties of the joint density function:-

* $f_{XY}(x,y) \geq 0$

* $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$

* $p\{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\} = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x,y)\, dx\, dy$

* $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$

* $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$

Marginal density function:-

The functions $f_X(x)$ and $f_Y(y)$ in property number 4 are called Marginal density function or joint Marginal density function. They are the density function of the single variables x and y are also defined as

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$f_Y(y) = \frac{d}{dy} F_Y(y)$$

1) A Joint sample space for two random variables x and y has four elements (1,1), (2,2) (3,3) and (4,4) probabilities of these elements are 0.1, 0.35, 0.05, 0.5 respectively

a) Determine through logic and sketch the distribution function $F_{XY}(x,y)$

b) Find the probability of the event $\{X \leq 2.5, Y \leq 6\}$

c) Find the probability of the event $\{X \leq 3\}$

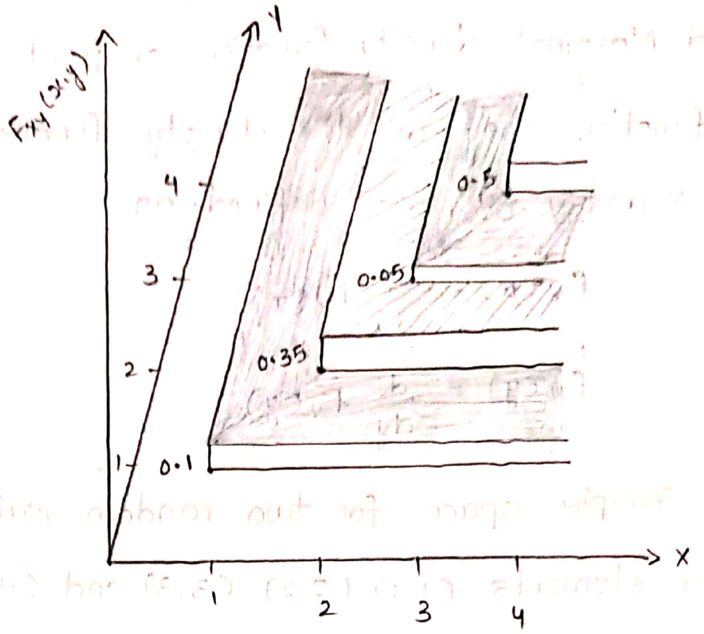
Given data

(x,y)	(1,1)	(2,2)	(3,3)	(4,4)
P(x,y)	0.1	0.35	0.05	0.5

a) To find $F_{XY}(x,y)$:-

$$F_{XY}(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{N} P(x_n, y_m)\, u(x-x_n)\, u(y-y_m)$$

$$F_{XY}(x,y) = (0.1)\, u(x-1)\, u(y-1) + (0.35)\, u(x-2)\, u(y-2) + (0.05)\, u(x-3)\, u(y-3) + (0.5)\, u(x-4)\, u(y-4)$$



b) To find $p\{X \le 2.5, Y \le 6\}$:-

$$p\{X \le 2.5, Y \le 6\} = F_{XY}(2,2) = 0.1 + 0.35 = 0.45$$

c) To find $p\{X \le 3\} = F_{XY}(3,\infty) = 0.1 + 0.35 + 0.05 = 0.50$

3) The joint distribution for two random variables x and y is

$F_{XY}(x,y) = u(x)\,u(y)\,[1 - e^{-ax} - e^{-ay} + e^{-a(x+y)}]$

i) where $a>0$, sketch $F_{XY}(x,y)$.

ii) If $a = 0.5$ in each case, find the probabilities

a) $p\{X \le 1, Y \le 2\}$ (b) $p\{0.5 < X < 1.5\}$

c) $p\{-1.5 < X \le 2, 1 < Y \le 3\}$

Given Joint probability distibution function is

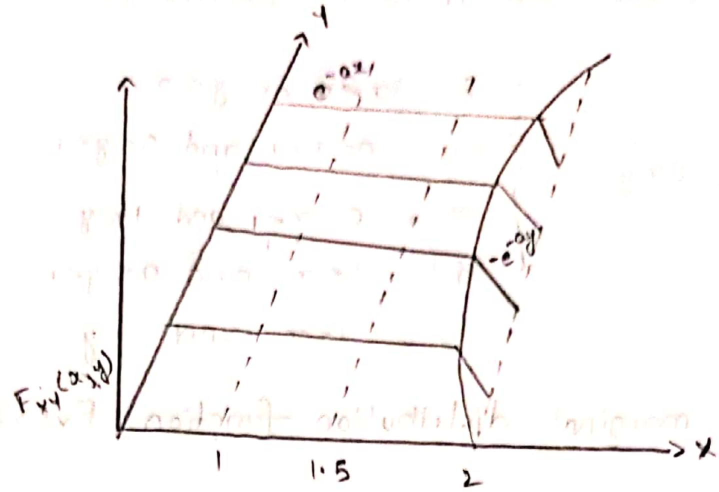
$$F_{XY}(x,y) = u(x)\,u(y)\,[1 - e^{-ax} - e^{-ay} + e^{-a(x+y)}]$$

$$= [(1-e^{-ax}) - e^{-ay} + e^{-ax}\,e^{-ay}]\,u(x)\,u(y)$$

$$= [(1-e^{-ax}) - e^{-ay}(1-e^{-ax})]\,u(x)\,u(y)$$

$$F_{XY}(x,y) = [(1-e^{-ax})(1-e^{-ay})]\,u(x)\,u(y)$$

i) sketch $F_{xy}(x,y)$:-



ii) Given $a = 0.5 = \frac{1}{2}$

$$F_{xy}(x,y) = (1-e^{-x/2})(1-e^{-y/2})$$

a) To find $p\{x \le 1, y \le 2\}$:-

$$\therefore p\{x \le 1, y \le 2\} = F_{xy}(1,2)$$

$$= (1-e^{-\frac{1}{2}})(1-e^{-1}) = 0.249$$

b) To find $p\{0.5 < x < 1.5\}$:-

$$\therefore p\{0.5 < x < 1.5\} = F_x(1.5) - F_x(0.5)$$

$$= F_{xy}(1.5, \infty) - F_{xy}(0.5, \infty)$$

$$= (1-e^{-\frac{1.5}{2}})(1-e^{-\infty}) - (1-e^{-\frac{0.5}{2}})(1-e^{-\infty})$$

$$= (1-e^{-\frac{1.5}{2}}) - (1-e^{-\frac{0.5}{2}})$$

$$= 0.306$$

c) To find $p\{-1.5 < x \le 2, 1 < y \le 3\}$:-

$$p\{-1.5 < x \le 2, 1 < y \le 3\} = p(-1.5,1) + p(2,3) - p(-1.5,3) - p(2,1)$$

$$= (1-e^{-\frac{1.5}{2}})(1-e^{-1/2}) + (1-e^{-1})(1-e^{-3/2}) - (1-e^{-\frac{1.5}{2}})(1-e^{-\frac{3}{2}}) - (1-e^{-1})(1-e^{-\frac{1}{2}})$$

$$= 0.0408$$

3) Random variables X and Y are component of two dimen-
random variables has a joint distribution.

$$F_{XY}(x,y) = \begin{cases} 0, & x<0 \text{ or } y<0 \\ xy, & 0\le x<1 \text{ and } 0<y<1 \\ x, & 0\le x<1 \text{ and } 1\le y \\ y, & 1\le x \text{ and } 0\le y<1 \\ 1, & 1\le x \text{ and } 1\le y \end{cases}$$

Find the marginal distribution function $F_X(x)$ and $F_Y(y)$

$$F_X(x) = F_{XY}(x,\infty) = \begin{cases} 0, & x<0 \\ x, & 0\le x\le 1 \\ 1, & 1\le x \end{cases}$$

$$F_Y(y) = F_{XY}(\infty,y) = \begin{cases} 0, & y<0 \\ y, & 0\le y<1 \\ 1, & 1\le y \end{cases}$$

4) The joint probability density function of a random variable X and Y is given by

$$f_{XY}(x,y) = \begin{cases} 4xy, & 0<x<1, 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Find $P(X+Y<1)$

Given

Joint probability density function is

$$f_{XY}(x,y) = \begin{cases} 4xy, & 0<x<1, 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

Also given $x+y<1 \Rightarrow x<1-y$

$$\therefore P(x+Y<1) = \iint f_{XY}(x,y)\,dx\,dy$$

$$= \int_{y=0}^{1}\int_{x=0}^{1-y} 4xy \, dx \, dy$$

$$= 4 \int_{y=0}^{1} y \left[\int_{x=0}^{1-y} x \, dx \right] dy$$

$$= 4 \int_{y=0}^{1} y \left(\frac{x^2}{2} \right)_{x=0}^{1-y} dy$$

$$= 2 \int_{y=0}^{1} y \left[(1-y)^2 - 0 \right] dy$$

$$= 2 \int_{y=0}^{1} y \left[1 + y^2 - 2y \right] dy$$

$$= 2 \int_{y=0}^{1} \left[y + y^3 - 2y^2 \right] dy$$

$$= 2 \left[\frac{y^2}{2} + \frac{y^4}{4} - 2\frac{y^3}{3} \right]_{y=0}^{1}$$

$$= 2 \left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right]$$

$$= 2 \left[\frac{3}{4} - \frac{2}{3} \right] = \cancel{2} \left[\frac{9-8}{\cancel{12}_{6}} \right] = \frac{1}{6}$$

3) Given Joint probability density function of x and y is

$$f_{xy}(x,y) = \begin{cases} ab\, e^{-(ax+by)} & , x>0, y>0 \\ 0 & , \text{otherwise} \end{cases}$$

find $P(x>y)$.

Given Joint probability density function is

$$f_{xy}(x,y) = \begin{cases} ab\, e^{-(ax+by)} & , x>0, y>1 \\ 0 & , \text{otherwise} \end{cases}$$

Also given $x>y$

$$\therefore\ p(x>y) = \iint f_{xy}(x,y)\,dx\,dy$$

$$= \int_{x=0}^{\infty}\left[\int_{y=0}^{x} ab\,e^{-(ax+by)}\,dy\right]dx$$

$$= ab\int_{x=0}^{\infty} e^{-ax}\left[\int_{y=0}^{x} e^{-by}\,dy\right]dx$$

$$= ab\int_{x=0}^{\infty} e^{-ax}\left[\frac{e^{-by}}{-b}\right]_{y=0}^{x} dx$$

$$= -a\int_{x=0}^{\infty} e^{-ax}\left[e^{-bx}-1\right]dx$$

$$= -a\int_{x=0}^{\infty}\left(e^{-(a+b)x} - e^{-ax}\right)dx$$

$$= -a\left[\frac{e^{-(a+b)x}}{-(a+b)} - \frac{e^{-ax}}{-a}\right]_{x=0}^{\infty}$$

$$= a\left[(0-0) - \left\{\frac{1}{a+b} - \frac{1}{a}\right\}\right]$$

$$= \frac{a}{\cancel{b}}\left[\frac{1}{a} - \frac{1}{a+b}\right]$$

$$= \frac{a}{\cancel{b}}\left[\frac{b}{a+b}\right]$$

$$= \frac{ab}{a+b}$$

6) The Joint probability density function of x and y

$$f_{xy}(x,y) = \begin{cases} K\,e^{-(x+2y)}, & x>0,\ y>0 \\ 0, & \text{otherwise} \end{cases}$$

a) find the value of k (b) find $p(x>1, y<1)$, $p(x<y)$ and $p(x \le 2)$

Given Jpdf is a valid density function then

$$f_{xy}(x,y) = \begin{cases} k e^{-(x+2y)}, & x>0, y>0 \\ 0, & \text{otherwise} \end{cases}$$

To find k value:

a) Given Jpdf is a valid density function then

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\, dx\, dy = 1$$

$$\int_{-\infty}^{\infty}\int_{0}^{\infty} k e^{-(x+2y)} dx.\, dy = 1$$

$$k \int_{0}^{\infty} e^{-x} dx \int_{0}^{\infty} e^{-2y} dy = 1$$

$$k \left(\frac{e^{-x}}{-1}\right)_0^{\infty} \left(\frac{e^{-2y}}{-2}\right)_0^{\infty} = 1$$

$$\frac{k}{2}\left[e^{-\infty}-1\right]\left[e^{-\infty}-1\right] = 1$$

$1 \times \frac{1}{2}$

$\frac{k}{2}$

$$2k(0-1)(0-1) = 1$$

$$k = \frac{1}{2}$$

b) find $p(x>1, y<1)$:

$$p(x>1, y<1) = \int_{x=1}^{\infty}\int_{y=0}^{1} f_{xy}(x,y)\, dy\, dx$$

$$= \int_{x=1}^{\infty}\int_{y=0}^{1} \frac{1}{2} e^{-(x+2y)} dx\, dy$$

$$= \frac{1}{2}\int_{x=1}^{\infty} e^{-x}\,dx \int_{y=0}^{1} e^{-2y}\,dy$$

$$= \frac{1}{\cancel{2}}\left[\frac{e^{-x}}{-1}\right]_{x=1}^{\infty}\cdot\left[\frac{e^{-2y}}{-\cancel{2}}\right]_{y=0}^{1}$$

$$= \left[0-e^{-1}\right]\left[e^{-2}-1\right]$$

$$= e^{-1}(1-e^{-2})$$

$$= 0.318$$

b) To find $p(x<y)$:-

$$p(x<y) = \int_{x=0}^{\infty}\int_{y=x}^{\infty} f_{xy}(x,y)\,dy\,dx$$

$$= \int_{x=0}^{\infty}\left[\int_{y=x}^{\infty}\frac{1}{2}\,e^{-(x+2y)}\,dy\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{\infty} e^{-x}\left[\int_{y=x}^{\infty} e^{-2y}\,dy\right]dx$$

$$= \frac{1}{2}\int_{x=0}^{\infty} e^{-x}\left[\frac{e^{-2y}}{-2}\right]_{y=x}^{\infty}dx$$

$$= -\frac{1}{4}\int_{x=0}^{\infty} e^{-x}\left[0-e^{-2x}\right]dx$$

$$= \frac{1}{4}\int_{x=0}^{\infty} e^{-3x}\,dx = \frac{1}{4}\left[\frac{e^{-3x}}{-3}\right]_{x=0}^{\infty}$$

$$= \frac{1}{12}\left[0-1\right] = \frac{1}{12}$$

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To find $P(X \leq 2)$:

$$P(X \leq 2) = \int_{y=0}^{\infty} \int_{x=0}^{2} \frac{1}{2} e^{-(x+2y)} \, dx \, dy$$

$$= \frac{1}{2} \int_{y=0}^{\infty} e^{-2y} dy \int_{x=0}^{2} e^{-x} dx$$

$$= \frac{1}{2} \left[\frac{e^{-2y}}{-2} \right]_{y=0}^{\infty} \left[\frac{e^{-x}}{-1} \right]_{x=0}^{2}$$

$$= \frac{1}{4} [0-1] [e^{-2} - 1]$$

$$= \frac{1}{4} [1 - e^{-2}]$$

$$P(X \leq 2) = 0.216$$

Given Joint probability density function,

$$f_{xy}(x,y) = \begin{cases} b(x+y)^2, & -2 < x < 2, \; -3 < y < 3 \\ 0, & \text{otherwise} \end{cases}$$

a) find the constant 'b' such that these is a valid joint density function.

b) find Marginal density function $f_X(x)$ and $f_Y(y)$.

Given

$$f_{xy}(x,y) = \begin{cases} b(x+y)^2, & -2 < x < 2, \; -3 < y < 3 \\ 0, & \text{otherwise} \end{cases}$$

a) To find 'b' value:

From the properties of joint probability density function.

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y) \, dx \, dy = 1$$

$$\int_{x=-2}^{2} \int_{y=-3}^{3} b(x+y)^2 \, dy \, dx = 1$$

$$b \int_{x=-2}^{2} \left[\int_{y=-3}^{3} (x^2+y^2+2xy) \, dy \right] dx = 1$$

$$b \int_{x=-2}^{2} \left[x^2 y + \frac{y^3}{3} + 2x \frac{y^2}{2} \right]_{y=-3}^{3} dx = 1$$

$$b \int_{x=-2}^{2} \left[\{ 3x^2 + 9 + \cancel{9x} \} - \{ -3x^2 - 9 + \cancel{9x} \} \right] dx = 1$$

$$b \int_{x=-2}^{2} [3x^2 + 3x^2 + 18] \, dx = 1$$

$$6b \int_{x=-2}^{2} (x^2+3) \, dx = 1$$

$$6b \left[\frac{x^3}{3} + 3x \right]_{x=-2}^{2} = 1$$

$$6b \left[\left\{ \frac{8}{3} + 6 \right\} - \left\{ -\frac{8}{3} - 6 \right\} \right] = 1$$

$$6b \left[\frac{16}{3} + 12 \right] = 1$$

$$24b \left[\frac{4}{3} + 3 \right] = 1$$

$$b \left[\frac{13}{\cancel{3}} \right] = \frac{1}{\frac{24}{8}} \Rightarrow b\,[13] = \frac{1}{8} \rightarrow b = \frac{1}{104}$$

b) To find $f_X(x)$ and $f_Y(y)$:-

$$\therefore f_X(x) = \int_{y=-3}^{3} f_{XY}(x,y)\, dy$$

$$= \int_{y=-3}^{3} \frac{1}{104} (x+y)^2\, dy$$

$$= \frac{1}{104} \int_{y=-3}^{3} (x+y)^2\, dy$$

$$= \frac{1}{104} \int_{y=-3}^{3} \left[x^2+y^2+2xy\right] dy$$

$$= \frac{1}{104} \left[x^2y + \frac{y^3}{3} + 2x\frac{y^2}{2} \right]_{y=-3}^{3}$$

$$= \frac{1}{104} \left[\{3x^2+9+9x\} - \{-3x^2-9+9x\} \right]$$

$$= \frac{1}{104} \left[6x^2+18 \right]$$

$$f_X(x) = \frac{6}{104} \left[x^2+3 \right]$$

$$\therefore f_X(x) = \begin{cases} \frac{6}{104}(x^2+3), & -2 < x < 2 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore f_Y(y) = \int_{x=-2}^{2} f_{XY}(x,y)\, dx$$

$$= \int_{x=-2}^{2} \frac{1}{104} (x+y)^2\, dx$$

$$= \frac{1}{104} \int_{-2}^{2} (x^2+y^2+2xy)\, dx$$

$$= \frac{1}{104}\left[\frac{x^3}{3} + y^2x + 2\frac{x^2}{2}y\right]_{-2}^{2}$$

$$= \frac{1}{104}\left[\left[\frac{8}{3} + 2y^2 + 4y\right] - \left[-\frac{8}{3} - 2y^2 + 4y\right]\right]$$

$$= \frac{1}{104}\left[\frac{8}{3} + 2y^2 + \frac{8}{3} + 2y^2\right]$$

$$= \frac{1}{104}\left[\frac{16}{3} + 4y^2\right]$$

$$= \frac{4}{104}\left[\frac{4}{3} + y^2\right]$$

Let $x + y = t$

$x = t - y$

$dx = dt$

$x = -2 \rightarrow -2 + y = t$

$t = -2 + y$

$x = 2 \rightarrow 2 + y = t$

$= \int_{t=-2+y}^{2+y} \frac{1}{104}[t]^2 dt$

$$\therefore f_Y(y) = \begin{cases} \frac{1}{312}\left\{(2+y)^3 - (-2+y)^3\right\}, & -3 < y < 3 \\ 0, & \text{otherwise} \end{cases}$$

Conditional distribution and density function.

1) point conditional:-

consider two random variables x and y the distribution of one random variable x at a given condition the the distribution function of random variable y is known at some value of y is defined as conditional distribution function. It is given as $F_X(x/y=y)$

$$F_X(x/y=y) = \frac{\int_{-\infty}^{x} f_{XY}(x,y)\,dx}{f_Y(y)}$$

And the conditional density function is

$$f_X(x/y=y) = \frac{d}{dx} F_X(x/y=y)$$

$$= \frac{d}{dx}\left[\frac{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}{F_Y(y)}\right]$$

$$f_x(x/y=y) = \frac{f_{xy}(x,y)}{f_y(y)}$$

i.e

$$f_x(x/y) = \frac{f_{xy}(x,y)}{f_y(y)}$$

$$f_y(y/x) = \frac{f_{xy}(x,y)}{f_x(x)}$$

For Discrete Random variable

consider both x and y are discrete random variable with elements $X = \{x_1, x_2, \ldots, x_N\}$ $Y = \{y_1, y_2, \ldots y_M\}$ the corresponding probabilities are

$$P(x_i), \quad i = 1, 2, \ldots N$$

$$P(y_j), \quad j = 1, 2, \ldots M$$

the probability distribution function is given by

$$F_x(x/y=y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} u(x-x_i)$$

The density function

$$f_x(x/y=y_k) = \sum_{i=1}^{N} \frac{P(x_i, y_k)}{P(y_k)} \delta(x-x_i)$$

similarly

$$F_y(y/x=x_k) = \sum_{j=1}^{M} \frac{P(x_i, y_j)}{P(x_k)} u(y-y_j)$$

$$f_y(y/x=x_k) = \sum_{j=1}^{M} \frac{P(x_k, y_j)}{P(x_k)} \delta(y-y_j)$$

2) Interval condition :-

consider interval condition

consider the events B's define $y_1 \le Y \le y_2$ for the random

Variable y is

$$B = \{ y_1 \leq Y \leq y_2 \}$$

And Assume that $p(B) = p\{ y_1 \leq Y \leq y_2 \} \neq 0$ then the conditional distribution function of x is given by

$$F_x(x / y_1 \leq Y \leq y_2) = \frac{\int_{y_1}^{y_2} \int_{-\infty}^{x} f_{xy}(x, y)\, dx\, dy}{\int_{y_1}^{y_2} f_y(y)\, dy}$$

$$= \frac{F_{xy}(x, y_2) - F_{xy}(x, y_1)}{F_y(y_2) - F_y(y_1)}$$

For differentiating, we get the conditional density function i.e

$$f_x(x / y_1 \leq y \leq y_2) = \frac{\int_{y_1}^{y_2} f_{xy}(x, y)\, dy}{\int_{y_1}^{y_2} \int_{-\infty}^{\infty} f_{xy}(x, y)\, dx\, dy}$$

Definition:-

stastical independence of random variable :-

Consider two random variables x and y with events $A = \{ X \leq x \}$; $B = \{ Y \leq y \}$ for two real numbers x and y.

The two random variables are said to be statically independent if and only if $p\{ X \leq x, Y \leq y \} = p\{ X \leq x \} . p\{ Y \leq y \}$

The distribution function is

$$F_{xy}(x, y) = F_x(x) \cdot F_y(y).$$

The density function is $f_{xy}(x, y) = f_x(x) \cdot f_y(y)$

Note:-

The conditional distribution function for independent random variables.

$$F_X(x/y) = F_X(x)$$

$$F_Y(y/x) = F_Y(y)$$

similarly conditional density functions are

$$f_X(x/y) = f_X(x)$$

$$f_Y(y/x) = f_Y(y)$$

if they are statiscally independent.

1) The joint probability density function of a random variables X, Y is given by $f_{XY}(x,y) = \begin{cases} K e^{-(ax+by)}, & x>0,\ y>0 \\ 0, & \text{otherwise} \end{cases}$

i) Find the value of K.

ii) Are X and Y independent?

Given

JPdf is $f_{XY}(x,y) = \begin{cases} K e^{-(ax+by)}, & x>0,\ y>0 \\ 0, & \text{otherwise} \end{cases}$

i) To find 'K' value:-

From the properties of Joint density function.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\int_{0}^{\infty}\int_{0}^{\infty} K e^{-(ax+by)}\,dx\,dy = 1$$

$$K\int_{0}^{\infty} e^{-ax}\,dx \int_{0}^{\infty} e^{-by}\,dy = 1$$

$$K\left[\frac{e^{-ax}}{-a}\right]_0^{\infty}\left[\frac{e^{-by}}{-b}\right]_0^{\infty} = 1$$

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$$\frac{k}{ab}[0-1][0-1]=1$$

$$\therefore k = ab$$

ii) To show that x and y are independent :-

w.k.T

Marginal density function for the random variable, is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_{-\infty}^{\infty} ab\, e^{-(ax+by)}\,dy$$

$$= ab e^{-ax}\int_{0}^{\infty} e^{-by}\,dy$$

$$= ab e^{-ax}\left[\frac{e^{-by}}{-b}\right]_0^{\infty}$$

$$= \frac{a\cancel{b}\, e^{-ax}}{-\cancel{b}}[0-1]$$

$$= -a e^{-ax}(0-1)$$

$$f_X(x) = a e^{-ax}$$

$$\therefore f_Y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx$$

$$= \int_{0}^{\infty} ab\, e^{-(ax+by)}\,dx$$

$$= ab e^{-by}\int_{0}^{\infty} e^{-ax}\,dx$$

$$= ab e^{-by}\left[\frac{e^{-ax}}{-a}\right]_0^{\infty}$$

$$= -\frac{\cancel{a} b e^{-by}}{\cancel{a}}[0-1]$$

$$= -b e^{-by}[0-1]$$

$$\therefore f_Y(y) = b e^{-by}$$

$$\therefore f_X(x) \cdot f_Y(y) = (a e^{-ax})(b e^{-by})$$

$$= ab e^{-ax-by}$$

$$= ab e^{-(ax+by)}$$

$$\therefore f_X(x)\cdot f_Y(y) = f_{XY}(x,y)$$

$\therefore$ x and y are independent events.

8) The Jpdf for a random variable x and y is given by

$$f_{xy}(x,y) = \begin{cases} k(x+y), & 0<x<2,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

a) Find the value of k.

b) Find the marginal probability density functions of x and y.

c) Are x and y are independent?

Given

Jpdf is $f_{xy}(x,y) = \begin{cases} k(x+y), & 0<x<2,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

a) To find 'k' value:-

From the properties of Jpdf.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x,y)\,dx\,dy = 1$$

$$\int_{x=0}^{2}\int_{y=0}^{1} k(x+y)\,dx\,dy = 1$$

$$\int_{x=0}^{2}\left[\int_{y=0}^{1} k(x+y)\,dy\right]dx = 1$$

$$k\int_{x=0}^{2}\left[xy+\frac{y^2}{2}\right]_0^1 dx = 1$$

$$k\int_{x=0}^{2}\left[\left\{x+\frac{1}{2}\right\}-\left\{0+0\right\}\right]dx = 1$$

$$k\int_{x=0}^{2}\left(x+\frac{1}{2}\right)dx = 1$$

$$k\left[\frac{x^2}{2}+\frac{x}{2}\right]_0^2 = 1$$

$$\frac{k}{2}\left[\{4+2\}-0\right] = 1$$

$$\frac{k}{\cancel{2}}\left[\overset{3}{\cancel{6}}\right] = 1$$

$$\boxed{k = \frac{1}{3}}$$

b) To find Marginal density function $f_X(x)$ and $f_Y(y)$:

W.K.T,

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$$

$$= \int_0^1 \frac{1}{3}(x+y)\,dy$$

$$= \frac{1}{3}\left[xy+\frac{y^2}{2}\right]_0^1 = \frac{1}{3}\left[\left\{x+\frac{1}{2}\right\}-0\right] = \frac{1}{3}\left[x+\frac{1}{2}\right.$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$$

$$= \int_0^2 \frac{1}{3}(x+y)\,dx$$

$$= \frac{1}{3}\int_0^2 (x+y)\,dx$$

$$= \frac{1}{3}\left[\frac{x^2}{2}+yx\right]_0^2$$

$$= \frac{1}{3}\left[\{2y+2\}-0\right]$$

$$= \frac{1}{3}\left[2y+2\right]$$

$$\therefore f_Y(y) = \frac{2}{3}\left[y+1\right]$$

c) To verify x and y are independent :-

$$\therefore \quad f_X(x)\,f_Y(y) = \frac{1}{3}\left(x+\frac{1}{2}\right)\frac{2}{3}(1+y)$$

$$= \frac{2}{9}\left(x+\frac{1}{2}\right)(1+y)$$

$$= \frac{2}{9}\left[x+xy+\frac{1}{2}+\frac{y}{2}\right]$$

$$\neq \frac{1}{3}(x+y)$$

$$f_{XY}(x,y) \neq f_X(x)\,f_Y(y).$$

$\therefore$ X and Y are not independent random variables.