Operations on Multiple Random Variables

Function of Joint random variable:-

If $g(x,y)$ is a function of two random variables x and y with Joint density function $f_{xy}(x,y)$ then the expected value of the function $g(x,y)$ is given as

$$\bar{g} = E[g(x,y)]$$

$$\bar{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{xy}(x,y)\, dx\, dy$$

Similarly for n random variables $x_1, x_2, \ldots x_N$ with Joint density function $f_{x_1 x_2, \ldots x_n}(x_1, x_2, \ldots x_N)$, the Expected value of the function $g(x_1, x_2, \ldots x_n)$ is given as

$$\bar{g} = E[g(x_1, x_2 \ldots x_N)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} g(x_1, x_2 \ldots x_N) f_{x_1 x_2 \ldots x_n}(x_1, x_2 \ldots x_N)\, dx_1\, dx_2 \ldots dx_N$$

Joint Moments about the origin:-

The Joint moments about the origin for two random variables x, y is the Expected Value of function $g(x,y) = x^n y^k$ it is denoted by m_{nk}.

In general $m_{nk} = E[x^n y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{xy}(x,y)\, dx\, dy$

* The sum $n+k$ is called the order of the moment.

* If $k=0$ then $m_{n0} = E[x^n y^0] = E[x^n]$ are the moments of x.

* If $n=0$ then $m_{0k} = E[x^0 y^k] = E[y^k]$ are the moments of y.

The first order moments are

$$m_{10} = E[X^1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\,dx\,dy$$

$$m_{01} = E[Y^1] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f_{XY}(x,y)\,dx\,dy$$

The second order moments are

$$m_{20} = E[X^2]$$

$$m_{02} = E[Y^2]$$

$$m_{11} = E[XY]$$

Correlation:-

Consider two random variables X and Y, the second order joint moment m_{11} is called correlation of X and Y. It is denoted as

dx_N

$$R_{XY} = m_{11} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{XY}(x,y)\,dx\,dy$$

(continuous random variable)

For discrete random variable

$$R_{XY} = \sum_m \sum_n x_m y_n P_{XY}(x_m, y_n)$$

Properties of correlation:-

If two random variables X and Y are stastically independent then X and Y are said to be uncorrelated that is

$$R_{XY} = E[XY] = E[X]\,E[Y]$$

If the random variables X and Y are orthogonal then there correlation is zero i.e $R_{XY} = 0$

Joint central moments:-

Consider the two random variables x and y then the Expected values of the function

$g(x,y) = (x-\bar{x})^n (y-\bar{y})^k$ are called Joint central moments.

In general it is denoted by μ_{nk}

$$\mu_{nk} = E[(x-\bar{x})^n (y-\bar{y})^k]$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y}) f_{xy}(x,y)\, dx\, dy$$

The order of the central moment is $n+k$.

Note:-

1) The 0^{th} order central moment is

$$\mu_{00} = E[1] = 1$$

2) The first order central moments are

$$\mu_{10} = E[(x-\bar{x})^1 (y-\bar{y})^0] = E[x-\bar{x}] = E[x] - E[\bar{x}] = 0$$

$$\mu_{10} = 0$$

$$\mu_{01} = E[(x-\bar{x})^0 (y-\bar{y})^1] = E[y-\bar{y}] = E[y] - E[\bar{y}] = 0$$

3) The second order central moments are

$$\mu_{20} = E[(x-\bar{x})^2] = \sigma_x^2$$

$$\mu_{02} = E[(y-\bar{y})^2] = \sigma_y^2$$

$$\mu_{11} = E[(x-\bar{x})(y-\bar{y})] = \sigma_{xy}^2$$

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covariance:-

consider the random variables x and y the second order joint central moments μ_{11} is called the covariance of x and y. it is expressed as C_{xy}

$$\therefore C_{xy} = \mu_{11} = E[(x-\bar{x})(y-\bar{y})]$$

$$\therefore \mu_{11} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y}) f_{xy}(x,y)\,dx\,dy$$ this is for continuous random variable.

for discrete random variable

$$C_{xy} = \sum_{n}\sum_{m} (x_n-\bar{x}_n)(y_m-\bar{y}_m)\,P(x_n, y_m)$$

correlation coefficient:-

The normalized second order central moment is called correlation coefficient. It is denoted as ρ defined as

$$\rho = \frac{\mu_{11}}{\sqrt{\mu_{20}\,\mu_{02}}} = \frac{C_{xy}}{\sqrt{\sigma_x^2\,\sigma_y^2}} = \frac{C_{xy}}{\sigma_x\sigma_y}$$

$$\therefore \rho = \frac{C_{xy}}{\sigma_x\,\sigma_y} = \frac{E[(x-\bar{x})(y-\bar{y})]}{\sigma_x\sigma_y}$$

Note:-

1) The range of correlation coefficient is $-1 \le \rho \le 1$

2) If x and y are independent then $\rho = 0$

3) If the correlation between x and y is perfect then $\rho = \pm 1$

4) If $x = y$ then $\rho = 1$

Properties of covariance:-

* If x and y are two random variables when the covariance

is $C_{XY} = R_{XY} - \bar{X} \cdot \bar{Y}$

* If two random variables x and y are independent then the covariance is zero.

$$C_{XY} = 0$$

But the converse is not true.

* $Var(X+Y) = var(X) + var(Y) + 2C_{XY}$

$Var(X-Y) = Var(X) + Var(Y) - 2C_{XY}$

* $|\sigma_{XY}| = \sigma_X \sigma_Y$

* $cov(X+a, Y+b) = cov(X,Y) = C_{XY}$

* $cov(aX, bY) = ab\,cov(X,Y) = ab\,C_{XY}$

1) Random variables x and y have the Joint density $f_{XY}(x,y) = \frac{1}{24}$, $0<x<6$, $0<y<4$. what is the Expected Value of the function $g(X,Y) = (XY)^2$.

Given

$$f_{XY}(x,y) = \frac{1}{24}, \quad 0<x<6;\ 0<y<4$$

Also given $g(X,Y) = (XY)^2$.

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{x=0}^{6}\int_{y=0}^{4} (xy)^2 \frac{1}{24}\, dy\, dx$$

$$= \frac{1}{24}\int_{x=0}^{6} x^2\, dx \int_{y=0}^{4} y^2\, dy$$

$$= \frac{1}{24}\left[\frac{x^3}{3}\right]_0^6 \left[\frac{y^3}{3}\right]_0^4$$

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$$= \frac{1}{24\times3\times3}\left[6^3\right]\left[4^3\right]$$

$$= \frac{1}{\cancel{24}\times\cancel{3}\times\cancel{3}}\left[\cancel{6}\times\cancel{6}\times\cancel{6}\right]\left[\cancel{4}\times4\times4\right]$$

$$= 64$$

2) show that two random variables X_1 and X_2 with Joint probability density function

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{16}, & |x_1| \le 4,\ 2 < x_2 < 4 \\ 0, & \text{otherwise} \end{cases}$$

are independent random variables. Also show that they are orthogonal.

Given

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{16}, & |x_1| \le 4,\ 2 < x_2 < 4 \\ 0, & \text{otherwise} \end{cases}$$

i) To show that X_1 and X_2 are independent random variables:-

i.e S.T $f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1)\, f_{X_2}(x_2)$

W.K.T

$$f_{X_1}(x_1) = \int_{x_2=-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_2$$

$$f_{X_1}(x_1) = \int_{2}^{4} \frac{1}{16}\,dx_2 = \frac{1}{16}\left[x_2\right]_2^4 = \frac{2}{16} = \frac{1}{8}$$

$$f_{X_2}(x_2) = \int_{x_1=-\infty}^{\infty} f_{X_1,X_2}(x_1,x_2)\,dx_1 = \int_{-4}^{4} \frac{1}{16}\,dx_1$$

$$f_{X_2}(x_2) = \frac{1}{16}\left[x_1\right]_{-4}^{4} = \frac{8}{16} = \frac{1}{2}$$

$\therefore f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} = f_{X_1,X_2}(x_1,x_2)$

$\therefore$ X_1 and X_2 are independent.

2) To show that X_1 & X_2 are orthogonal:-

i.e S.T $E[X_1 X_2] = 0$

$$\therefore E[X_1 X_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1,x_2)\, dx_1\, dx_2$$

$$= \int_{x_1=-4}^{4} \int_{x_2=2}^{4} (x_1 x_2) \frac{1}{16}\, dx_2\, dx_1$$

$$= \frac{1}{16} \left[\int_{x_1=-4}^{4} x_1\, dx_1 \right] \left[\int_{x_2=2}^{4} x_2\, dx_2 \right]$$

$$= \frac{1}{16} \left[\frac{x_1^2}{2} \right]_{-4}^{4} \left[\frac{x_2^2}{2} \right]_{2}^{4}$$

$$= \frac{1}{16 \times 4} [16 - 16][16 - 4]$$

$E[X_1 X_2] = 0$

$\therefore$ X_1 and X_2 are orthogonal.

3) A Joint density function is given by $f_{XY}(x,y) = \begin{cases} x(y+1.5), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

Find the first order and second order moments.

Given Jpdf is

$$f_{XY}(x,y) = \begin{cases} x(y+1.5), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$$

w.k.T

$$m_{nk} = E[x^n y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{xy}(x,y)\,dx\,dy$$

First order:-

put $n=0 \quad k=1$

$$m_{01} = E[x^0 y^1] = \int_{x=0}^{1}\int_{y=0}^{1} y\{x(y+1.5)\}\,dy\,dx$$

$$= \int_{x=0}^{1} x\,dx \int_{y=0}^{1} \left(y^2+\frac{3}{2}y\right) dy$$

$$= \left(\frac{x^2}{2}\right)_0^1 \left[\frac{y^3}{3}+\frac{3}{2}\frac{y^2}{2}\right]_0^1$$

$$= \left(\frac{1}{2}\right)\left[\frac{1}{3}+\frac{3}{2}\frac{1}{2}\right]$$

$$E[Y] = \frac{1}{2}\left[\frac{4+9}{12}\right] = \frac{13}{24}$$

put $n=1 \quad k=0$

$$m_{10} = E[x] = \int_{x=0}^{1}\int_{y=0}^{1} x\{x(y+1.5)\}\,dx\,dy$$

$$= \int_{x=0}^{1} x^2\,dx \int_{y=0}^{1} \left(y+\frac{3}{2}\right) dy$$

$$= \left(\frac{x^3}{3}\right)_0^1 \left(\frac{y^2}{2}+\frac{3}{2}y\right]_0^1$$

$$= \left(\frac{1}{3}\right)\left[\frac{1}{2}+\frac{3}{2}\right]$$

$$E[X] = \frac{1}{3}[2] = \frac{2}{3}$$

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Second order:-

put $n=2$, $k=0$

$$m_{20} = E[x^2] = \int_{x=0}^{1}\int_{y=0}^{1} x^2\{x(y+1.5)\}\,dx\,dy$$

$$= \int_{x=0}^{1} x^3\,dx \int_{y=0}^{1} \left(y+\frac{3}{2}\right) dy$$

$$= \left[\frac{x^4}{4}\right]_0^1 \left[\frac{y^2}{2}+\frac{3}{2}y\right]_0^1 = \frac{1}{4}[2] = \frac{1}{2}$$

put $n=0$ $k=2$

$$m_{02} = E[y^2] = \int_{x=0}^{1}\int_{y=0}^{1} y^2\{x(y+1.5)\}\,dx\,dy$$

$$= \int_{x=0}^{1} x\,dx \int_{y=0}^{1} (y^3+1.5y^2)\,dy$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^4}{4}+\frac{\cancel{3}}{2}\left(\frac{y^3}{\cancel{3}}\right)\right]_0^1$$

$$E[y^2] = \frac{1}{2}\left[\frac{1}{4}+\frac{1}{2}\right] = \frac{1}{2}\left(\frac{3}{4}\right) = \frac{3}{8}$$

put $n=1$ $k=1$

$$m_{11} = E[xy] = \int_{x=0}^{1}\int_{y=0}^{1} xy\{x(y+1.5)\}\,dx\,dy$$

$$= \int_{x=0}^{1} x^2\,dx \int_{y=0}^{1} (y^2+1.5y)\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 \left[\frac{y^3}{3} + \frac{3}{2}\left(\frac{y^2}{2}\right)\right]_0^1$$

$$= \frac{1}{3}\left[\frac{1}{3} + \frac{3}{4}\right]$$

$$= \frac{1}{3}\left[\frac{13}{12}\right] = \frac{13}{36}$$

4) A Joint density function is given as $f_{XY}(x,y) = \begin{cases} x(y+1.5), & 0<x<1 \\ & 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

Find all the joint moments m_{nk}, $n,k = 0,1,2,\ldots$

Given

JPdf is $f_{XY}(x,y) = \begin{cases} x(y+1.5), & 0<x<1,\ 0<y<1 \\ 0, & \text{otherwise} \end{cases}$

we know that

$$m_{nk} = E[x^n y^k] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\, dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} x^n y^k x\left(y+\frac{3}{2}\right) dy\, dx$$

$$= \int_{x=0}^{1} x^{n+1}\, dx \int_{y=0}^{1} y^k\left(y+\frac{3}{2}\right) dy$$

$$= \int_{x=0}^{1} x^{n+1}\, dx \int_{y=0}^{1} \left(y^{k+1} + \frac{3}{2} y^k\right) dy$$

$$= \left[\frac{x^{n+2}}{n+2}\right]_0^1 \left[\frac{y^{k+2}}{k+2} + \frac{3}{2}\frac{y^{k+1}}{k+1}\right]_0^1$$

$$= \frac{1}{n+2}[1-0]\left[\frac{1}{k+2} + \frac{3}{2(k+1)}\right]$$

$$= \frac{1}{n+2}\left[\frac{2(k+1)+3(k+2)}{2(k+1)(k+2)}\right]$$

$$= \frac{1}{n+2}\left[\frac{5k+8}{2(k+1)(k+2)}\right]$$

$$m_{nk} = \frac{5k+8}{2(k+1)(k+2)(n+2)}$$

put $n=1,\ k=0$

$$m_{10} = \frac{8}{4(3)} = \frac{8}{12} = \frac{2}{3}$$

put $n=0,\ k=1$

$$m_{01} = \frac{13}{12(2)} = \frac{13}{24}$$

put $n=1,\ k=1$

$$m_{11} = \frac{13}{12(3)} = \frac{13}{36} = \text{------}$$

5) The Independent random variables x and y have moments $m_{10}=2$, $m_{20}=14$, $m_{02}=12$ and $m_{11}=-6$. Find the moments μ_{22}.

Given

x and y are independent random variables

Also given $m_{10}=2$ $\quad m_{02}=12$

$m_{20}=14 \quad m_{11}=-6$

μ_{22} is a central moment, it is given by

$$\mu_{22} = E\left[(x-\bar{x})^2(y-\bar{y})^2\right]$$

$$= \left[E[x^2]-\{E[x]\}^2\right]\left[E[y^2]-\{E[y]\}^2\right]$$

$$= \left[\mu_{20}-m_{10}^2\right]\left[\mu_{02}-m_{01}^2\right] \longrightarrow (1)$$

Also w.k.T $m_{11} = m_{10} \times m_{01}$

$$m_{01} = \frac{m_{11}}{m_{10}}$$

$$= \frac{-6}{2}$$

$$m_{01} = -3$$

① $\Rightarrow$ $\mu_{22} = [14 - (2)^2][12 - (-3)^2]$

$$= [14-4][12-9]$$

$$= 10(3)$$

$$\mu_{22} = 30$$

6) Three stastically independent random variables X_1, X_2, X_3 have mean values $\overline{X_1} = 3$, $\overline{X_2} = 6$ and $\overline{X_3} = -2$. Find the mean values of the following functions.

a) $g(X_1, X_2, X_3) = X_1 + 3X_2 + 4X_3$

b) $g(X_1, X_2, X_3) = X_1 X_2 X_3$

c) $g(X_1, X_2, X_3) = -2X_1X_2 - 3X_1X_3 + 4X_2X_3$

d) $g(X_1, X_2, X_3) = X_1 + X_2 + X_3$

Given

X_1, X_2, X_3 are stastically independent random variables

Also given $\overline{X_1} = 3 = E[X_1]$

$\overline{X_2} = E[X_2] = 6$

$\overline{X_3} = E[X_3] = -2$

a) $E[g(X_1, X_2, X_3)] = E[X_1 + 3X_2 + 4X_3]$

$$= E[X_1] + 3E[X_2] + 4E[X_3]$$

$$= 3 + 3(6) + 4(-2)$$

$= 3 + 18 - 8 = 13$ because they are independent

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b) $E[g(x_1, x_2, x_3)] = E[X_1 X_2 X_3]$

$= E[X_1] \cdot E[X_2] \cdot E[X_3]$

$= (3)(6)(-2)$

$= -36$

c) $E[g(x_1, x_2, x_3)] = E[-2X_1X_2 - 3X_1X_3 + 4X_2X_3]$

$= -2E[X_1]E[X_2] - 3E[X_1]E[X_3] + 4E[X_2]E[X_3]$

$= -2(3)(6) - 3(3)(-2) + 4(6)(-2)$

$= -36 + 18 - 48$

$= -36 - 30$

$= -66$

d) $E[g(x_1, x_2, x_3)] = E[X_1 + X_2 + X_3]$

$= E[X_1] + E[X_2] + E[X_3]$

$= 3 + 6 - 2$

$= 7$

7) If x and y are two Independent random variables such that $E[x] = \lambda_1$, variance of $x = \sigma_1^2$

$E[y] = \lambda_2$, variance of $y = \sigma_2^2$

Prove that $V[xy] = \sigma_1^2\sigma_2^2 + \lambda_1^2\sigma_2^2 + \lambda_2^2\sigma_1^2$

Given

x and y are Independent random Variables

Also given $E[x] = \lambda_1$ $E[y] = \lambda_2$

variance of $x = \sigma_1^2$ variance of $y = \sigma_2^2$

$\therefore V[xy] = E[(xy)^2] - \{E[xy]\}^2$

$= E[x^2y^2] - E[xy]E[xy]$

$$= E[X^2]\,E[Y^2] - E[X]\,E[Y]\,E[X]\,E[Y]$$

$$V[XY] = E[X^2]\,E[Y^2] - \{E[X]\}^2\{E[Y]\}^2 \longrightarrow ①$$

$$V(X) = \sigma_1^2 = E[X^2] - \{E[X]\}^2$$

$$\sigma_1^2 = E[X^2] - \lambda_1^2$$

$$\sigma_1^2 + \lambda_1^2 = E[X^2]$$

$$V[Y] = \sigma_2^2 = E[Y^2] - \{E[Y]\}^2$$

$$\sigma_2^2 = E[Y^2] - \lambda_2^2$$

$$\sigma_2^2 + \lambda_2^2 = E[Y^2]$$

$$① \Rightarrow V[XY] = [\sigma_1^2 + \lambda_1^2][\sigma_2^2 + \lambda_2^2] - \lambda_1^2\lambda_2^2$$

$$= \sigma_1^2\sigma_2^2 + \sigma_1^2\lambda_2^2 + \sigma_2^2\lambda_1^2 + \cancel{\lambda_1^2\lambda_2^2} - \cancel{\lambda_1^2\lambda_2^2}$$

$$\therefore V[XY] = \sigma_1^2\sigma_2^2 + \lambda_1^2\sigma_2^2 + \lambda_2^2\sigma_1^2$$

Hence proved

8) The Joint density function of two random variables x and y is

$$f_{XY}(x,y) = \begin{cases} \frac{(x+y)^2}{40}, & -1 < x < 1 \text{ and } -3 < y < 3 \\ 0, & \text{otherwise} \end{cases}$$

Find a) the variance of x and y. (σ_X^2, σ_Y^2)

b) The correlation coefficient. (ρ_{XY})

Given

Jpdf is

$$f_{XY}(x,y) = \begin{cases} \frac{(x+y)^2}{40}, & -1 < x < 1 \text{ and } -3 < y < 3 \\ 0, & \text{otherwise} \end{cases}$$

a) To find σ_X^2 and σ_Y^2:-

we know that $\sigma_X^2 = E[X^2] - \{E[X]\}^2 \rightarrow ①$

$$\sigma_Y^2 = E[Y^2] - \{E[Y]\}^2 \rightarrow ②$$

$$m_{10} = E[x] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{x=-1}^{1}\int_{y=-3}^{3} x \frac{(x+y)^2}{40}\, dy\, dx$$

$$= \frac{1}{40}\int_{x=-1}^{1} x\left[x^2 y + \frac{y^3}{3} + 2x\frac{y^2}{2}\right]_{y=-3}^{3} dx$$

$$= \frac{1}{40}\int_{x=-1}^{1} x\left[\{3x^2+9+9x\} - \{-3x^2-9+9x\}\right] dx$$

$$= \frac{1}{40}\int_{x=-1}^{1} x\,[6x^2+18]\, dx$$

$$= \frac{1}{40}\left[\cancel{6}^{3}\frac{x^4}{\cancel{4}_{2}} + \cancel{18}^{9}\frac{x^2}{\cancel{2}}\right]_{x=-1}^{1}$$

$$= \frac{1}{40}\left[\left(\frac{3}{2}+9\right) - \left(\frac{3}{2}+9\right)\right]$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 f_{XY}(x,y)\, dx\, dy$$

$$= \int_{x=-1}^{1}\int_{y=-3}^{3} x^2 \frac{(x+y)^2}{40}\, dy\, dx$$

$$= \frac{1}{40}\int_{x=-1}^{1} x^2\,[6x^2+18]\, dx$$

$$= \frac{1}{40}\left[6\frac{x^5}{5} + \cancel{18}^{6}\frac{x^3}{\cancel{3}}\right]_{x=-1}^{1} = \frac{1}{40}\left[\left(\frac{6}{5}+6\right) - \left(-\frac{6}{5}-6\right)\right]$$

$$= \frac{1}{40}\left[\frac{12}{5}+12\right] = \frac{\cancel{12}^{3}}{\cancel{40}_{10}}\left[\frac{\cancel{6}^{3}}{5}\right] = \frac{9}{25}$$

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$$E[x^2] = \frac{9}{25}$$

$$① \rightarrow \quad \sigma_x^2 = \frac{9}{25} - 0 = 0.36$$

$$\therefore \quad m_{01} = E[Y] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y \, f_{XY}(x,y) \, dx \, dy$$

$$= \int_{x=-1}^{1} \int_{y=-3}^{3} y \, \frac{(x+y)^2}{40} \, dy \, dx$$

$$= \frac{1}{40} \int_{y=-3}^{3} y \left[\frac{x^3}{3} + y^2 x + \cancel{2} y \frac{x^2}{\cancel{2}} \right]_{x=-1}^{1} dy$$

$$= \frac{1}{40} \int_{y=-3}^{3} y \left[\left\{ \frac{1}{3} + y^2 + \cancel{y} \right\} - \left\{ -\frac{1}{3} - y^2 + \cancel{y} \right\} \right] dy$$

$$= \frac{1}{40} \int_{y=-3}^{3} y \left[\frac{2}{3} + 2y^2 \right] dy$$

$$= \frac{1}{40} \left[\frac{\cancel{2}}{3} \frac{y^2}{\cancel{2}} + \cancel{2} \frac{y^4}{\cancel{4}_2} \right]_{y=-3}^{3}$$

$$= \frac{1}{40} \left[\left\{ \frac{\cancel{9}^3}{\cancel{3}} + \frac{81}{2} \right\} - \left\{ \frac{\cancel{9}^3}{\cancel{3}} + \frac{81}{2} \right\} \right]$$

$$E[Y] = 0$$

$$E[y^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 \, f_{xy}(x,y) \, dx \, dy$$

$$= \int_{x=-1}^{1} \int_{y=-3}^{3} y^2 \, \frac{(x+y)^2}{40} \, dx \, dy$$

$$= \frac{1}{40} \int_{y=-3}^{3} y^2 \left[\frac{2}{3} + 2y^2 \right] dy$$

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$$= \frac{1}{40}\left[\frac{2}{3}\frac{y^3}{3} + \frac{2y^5}{5}\right]_{y=-3}^{3}$$

$$= \frac{1}{40}\left[\frac{2y^3}{9} + \frac{2y^5}{5}\right]_{y=-3}^{3}$$

$$= \frac{1}{40}\left[\left\{\frac{2(\cancel{27})}{\cancel{9}} + \frac{2(243)}{5}\right\} - \left\{\frac{-2(\cancel{27})}{\cancel{9}} - \frac{2(243)}{5}\right\}\right]$$

$$= \frac{1}{40}\left[\left\{6 + \frac{486}{5}\right\} - \left\{-6 - \frac{486}{5}\right\}\right]$$

$$= \frac{1}{40}\left[12 + \frac{2}{5}(486)\right]$$

$$= \frac{2}{40}\left[6 + \frac{486}{5}\right]$$

$$= \frac{1}{20}\left[6 + \frac{486}{5}\right] = \frac{1}{20}\left[\frac{30+486}{5}\right] = \frac{1}{20}\left[\frac{516}{5}\right]$$

$$= \frac{516}{100}$$

$$E[y^2] = 5.16$$

$$\sigma_y^2 = 5.16 - 0$$

$$\therefore \sigma_y^2 = 5.16$$

b) To find ρ_{xy} :-

W.K.T

$$\rho_{xy} = \frac{C_{xy}}{\sigma_x \sigma_y} \longrightarrow (3)$$

Where $C_{xy} = R_{xy} - \bar{X}\bar{Y} = E[xy] - E[x]E[y]$

(OR)

$m_{11} - m_{10} \times m_{01}$

$$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{xy}(x,y)\, dx\, dy$$

$$= \int_{x=-1}^{1} \int_{y=-3}^{3} xy \frac{(x+y)^2}{40} \, dx \, dy$$

$$= \frac{1}{40} \int_{x=-1}^{1} x \left[x^2 \frac{y^2}{2} + \frac{y^4}{4} + 2x \frac{y^3}{3} \right]_{y=-3}^{3} dx$$

$$= \frac{1}{40} \int_{x=-1}^{1} x \left[\left\{ \cancel{\frac{9x^2}{2}} + \cancel{\frac{81}{4}} + 18x \right\} - \left\{ \cancel{\frac{9x^2}{2}} + \cancel{\frac{81}{4}} - 18x \right\} \right] dx$$

$$= \frac{1}{40} \int_{x=-1}^{1} x \, [36x] \, dx$$

$$= \frac{1}{40} \int_{x=-1}^{1} x[36x] \, dx = \frac{36}{40} \left[\frac{x^3}{3} \right]_{x=-1}^{1} = \frac{36}{40} \left[1^3 - (-1)^3 \right]$$

$$\therefore E[xy] = \frac{3}{10} [2] = \frac{6}{10} = 0.6$$

$$\therefore \; ③ \Rightarrow \; \rho_{xy} = \frac{0.6}{\sqrt{0.36}\,\sqrt{5.16}} = 0.440$$

9) For Two random variables X and Y the Joint density function is $f_{XY}(x,y) = 0.15\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2)$ $+ 0.4\,\delta(x-1)\,\delta(y+2 + 0.2\,\delta(x-1)\,\delta(y-1) + 0.05\,\delta(x-1)\,\delta(y-3)$

Find a) The correlation

b) The correlation coefficient of X and Y

c) are X and Y either uncorrelated or orthogonal.

Sol:- From the given data

(x,y)	$(-1,0)$	$(0,0)$	$(0,2)$	$(1,-2)$	$(1,1)$	$(1,3)$
$P(x,y)$	0.15	0.1	0.1	0.4	0.2	0.05

a) To find correlation:-

W.K.T $R_{XY} = E[XY] = \sum x_i P(x_i, y_i)$

$= (-1)(0)(0.15) + 0 + 0 + (1)(-2)(0.4) + (1)(1)(0.2) + (1)(3)(0.05)$

$= -0.45$

b) To find ρ_{xy} :-

w.k.T $\rho_{xy} = \dfrac{C_{xy}}{\sigma_x \sigma_y} \longrightarrow ①$

$C_{xy} = R_{xy} - E[x]E[y] = E[xy] - E[x]E[y]$

$\sigma_x^2 = E[x^2] - \{E[x]\}^2 \qquad \sigma_y^2 = E[y^2] - \{E[y]\}^2$

$\therefore E[x] = \Sigma x_i \, p(x_i, y_i)$

$= (-1)(0.15) + 0 + 0 + 1(0.4) + 1(0.2) + 1(0.05)$

$= 0.5$

$E[x^2] = \Sigma x_i^2 \, p(x_i, y_i)$

$= (-1)^2(0.15) + 1^2(0.4) + 1^2(0.2) + 1^2(0.05)$

$= 0.8$

$\therefore E[y] = \Sigma y_i \, p(x_i, y_i)$

$= 0 + 0 + 2(0.1) + (-2)(0.4) + 1(0.2) + 3(0.05)$

$= -0.25$

$E[y^2] = \Sigma y_i^2 \, p(x_i, y_i)$

$= (2)^2(0.1) + (-2)^2(0.4) + 1^2(0.2) + 9(0.05)$

$E[y^2] = 2.65$

$\therefore C_{xy} = -0.45 - (0.5)(-0.25) = -0.325$

$\sigma_x^2 = 0.8 - (0.5)^2 = 0.55$

$\sigma_y^2 = (2.65) - (-0.25)^2 = 2.58$

Eq ① $\Rightarrow \rho_{xy} = \dfrac{-0.325}{\sqrt{0.55}\,\sqrt{2.58}} = -0.272$

c) since $C_{XY} \neq 0$

$\therefore$ X and Y are not uncorrelated and

$R_{XY} \neq 0$

$\therefore$ X and Y are not orthogonal

10) If the joint density function is $f_{XY}(x,y) = \begin{cases} x+y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

find out correlation coefficient.

Given

JPdf is $f_{XY}(x,y) = \begin{cases} x+y, & 0 \le x \le 1, 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

To find ρ_{XY}:

W.K.T $\rho_{XY} = \dfrac{C_{XY}}{\sigma_X \sigma_Y} \longrightarrow (1)$

$C_{XY} = R_{XY} - E[X]E[Y] = E[XY] - E[X]E[Y]$

$\sigma_X^2 = E[X^2] - (E(X))^2 \qquad \sigma_Y^2 = E[Y^2] - \{E(Y)\}^2$

$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx \qquad E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$

$\therefore f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_{y=0}^{1}(x+y)\,dy = \left[xy + \frac{y^2}{2}\right]_{y=0}^{1} = x + \frac{1}{2}$

$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{x=0}^{1}(x+y)\,dx = \left[\frac{x^2}{2} + yx\right]_{x=0}^{1} = \frac{1}{2} + y$

$f_X(x) = \begin{cases} x + \frac{1}{2}, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

$f_Y(y) = \begin{cases} \frac{1}{2} + y, & 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$

$\therefore E[X] = \int_0^1 x\left(x + \frac{1}{2}\right)dx = \left[\frac{x^3}{3} + \frac{1}{2}\left(\frac{x^2}{2}\right)\right]_{x=0}^{1} = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

$$E[Y] = \int_{y=0}^{1} y\left(\tfrac{1}{2}+y\right) dy = \left[\tfrac{1}{2}\left(\tfrac{y^2}{2}\right) + \tfrac{y^3}{3}\right]_{y=0}^{1} = \tfrac{1}{4} + \tfrac{1}{3} = \tfrac{7}{12}$$

$$\therefore E[XY] = \int_{x=0}^{1} \int_{y=0}^{1} xy\,(x+y)\, dy\, dx$$

$$= \int_{x=0}^{1} x\left[x\frac{y^2}{2} + \frac{y^3}{3}\right]_{y=0}^{1} dx$$

$$= \int_{x=0}^{1} x\left[\frac{x}{2} + \frac{1}{3}\right] dx$$

$$= \left[\frac{1}{2}\frac{x^3}{3} + \frac{1}{3}\frac{x^2}{2}\right]_{x=0}^{1} = \left[\frac{1}{6} + \frac{1}{6}\right] = \frac{1}{3}$$

$$E[X^2] = \int_{x=0}^{1} x^2\left(x+\tfrac{1}{2}\right)dx = \left[\frac{x^4}{4} + \frac{1}{2}\left(\frac{x^3}{3}\right)\right]_{x=0}^{1} = \frac{1}{4} + \frac{1}{6} = \frac{10}{24}$$

$$E[Y^2] = \int_{y=0}^{1} y^2\left(\tfrac{1}{2}+y\right)dy = \left[\frac{1}{2}\frac{y^3}{3} + \frac{y^4}{4}\right]_{y=0}^{1} = \frac{1}{6} + \frac{1}{4} = \frac{10}{24}$$

$$C_{XY} = \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = \frac{1}{3} - \frac{49}{144} = -0.006$$

$$\sigma_X^2 = \frac{10}{24} - \left(\frac{7}{12}\right)^2 = 0.076$$

$$\sigma_Y^2 = \frac{10}{24} - \left(\frac{7}{12}\right)^2 = 0.076$$

Eq ① $\Rightarrow$ $$\rho_{XY} = \frac{-0.006}{\sqrt{0.076}\,\sqrt{0.076}} = -0.078$$

11) For two random variables X and Y $f_{XY}(x,y) = 0.5\,\delta(x+1)\,\delta(y) + 0.1\,\delta(x)\,\delta(y) + 0.1\,\delta(x)\,\delta(y-2) + 0.4\,\delta(x-1)\,\delta(y+2) + 0.2\,\delta(x-1)\,\delta(y-1) + 0.5\,\delta(x-1)\,\delta(y-3)$

Find a) The correlation

b) covariance

c) The correlation coefficient of X and Y

d) Are x and y either uncorrelated or orthogonal?

sol:- From the given data

(x,y)	$(-1,0)$	$(0,0)$	$(0,2)$	$(1,-2)$	$(1,1)$	$(1,3)$
$p(x,y)$	0.5	0.1	0.1	0.4	0.2	0.5

a) To find correlation:-

w.k.T $\quad R_{xy} = E[xy] = \sum x_i \, p(x_i, y_i)$

$= (1)(-2)(0.4) + (1)(1)(0.2) + (1)(3)(0.5)$

$= 0.9$

c) To find ρ_{xy} :-

w.k.T $\quad \rho_{xy} = \dfrac{C_{xy}}{\sigma_x \sigma_y} \longrightarrow ①$

$C_{xy} = R_{xy} - E[x]E[y] = E[xy] - E[x]E[y]$

$\sigma_x^2 = E[x^2] - \{E[x]\}^2 \qquad \sigma_y^2 = E[y^2] - \{E[y]\}^2$

$\therefore E[x] = \sum x_i \, p(x_i, y_i)$

$= (-1)(0.5) + 1(0.4) + 1(0.2) + 1(0.5)$

$= 0.6$

$E[x^2] = \sum x_i^2 \, p(x_i, y_i)$

$= (-1)^2(0.5) + (0.4) + 0.2 + 0.5$

$= 1.6$

$\therefore E[y] = \sum y_i \, p(x_i, y_i)$

$= 2(0.1) + (-2)(0.4) + 1(0.2) + 3(0.5)$

$= 1.1$

$E[y^2] = \sum y_i^2 \, p(x_i, y_i)$

$= 4(0.1) + 4(0.4) + 1(0.2) + 9(0.5)$

$= 6.7$

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$\therefore C_{XY} = 0.9 - (0.6)(1.1)$

$= 0.24$

$\sigma_X^2 = 1.6 - (0.6)^2$ $\qquad$ $\sigma_Y^2 = 6.7 - (1.1)^2$

$= 1.24$ $\qquad$ $= 5.49$

Eq ① $\Rightarrow$ $\rho_{XY} = \dfrac{0.24}{\sqrt{1.24}\ \sqrt{5.49}}$

$\therefore \rho_{XY} = 0.091$

b) w.k.T

$cov(X,Y) = E[XY] - E[X]E[Y] = C_{XY}$

$cov(X,Y) = 0.24$

d) since $C_{XY} \neq 0$

$\therefore$ X and Y are not uncorrelated

$R_{XY} \neq 0$

$\therefore$ X and Y are not orthogonal

Joint characteristic function:

The JCF of two random variables X and Y is defined as the expected value of the joint function $g(x,y) = e^{j\omega_1 x} e^{j\omega_2 y}$ it is given as

$$\phi_{XY}(\omega_1, \omega_2) = E\left[e^{j\omega_1 X + j\omega_2 Y}\right]$$

$$= E\left[e^{j(\omega_1 X + \omega_2 Y)}\right]$$

where ω_1 & ω_2 are real variables

$$\phi_{XY}(\omega_1, \omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j\omega_1 x + j\omega_2 y} \cdot f_{XY}(x,y)\, dx\, dy$$

Note:-

1) Above expression is known as the two dimensional Fourier transform of Joint density function.

2) The Inverse fourier transform of the JCF gives the Jdf. i.e

$$f_{XY}(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{XY}(\omega_1,\omega_2)\, e^{-j(\omega_1 x+\omega_2 y)}\, d\omega_1 d\omega_2$$

3) If x and y are discrete random variable with probability Mass function $p(x,y)$

$$\phi_{XY}(\omega_1,\omega_2) = \sum_m \sum_n e^{j\omega_1 x_m + j\omega_2 y_n}\, p(x_m, y_n)$$

And $$p(x_m, y_n) = \sum_m \sum_n e^{-j\omega_1 x_m - j\omega_2 y_n}\, \phi_{XY}(\omega_1,\omega_2)$$

properties of Joint characteristic function:-

1) Marginal characteristic function are given by

$$\phi_X(\omega_1) = \phi_{XY}(\omega_1, 0)$$

$$\phi_Y(\omega_2) = \phi_{XY}(0, \omega_2)$$

2) If two random variable x and y are independent then the JCF is equal to the product of the individual characteristic function.

$$\phi_{XY}(\omega_1,\omega_2) = \phi_X(\omega_1)\,\phi_Y(\omega_2)$$

3) If x and y are independent random variables then

$$\phi_{X+Y}(\omega) = \phi_X(\omega)\cdot\phi_Y(\omega)$$

4) If x and y are two random variables, the joint moments can be derived from the JCF has

$$m_{nk} = (-j)^{n+k} \left.\frac{\partial^{n+k} \phi_{XY}(\omega_1,\omega_2)}{\partial\omega_1^n\, \partial\omega_2^k}\right|_{\omega_1=\omega_2=0}$$

Joint moment Generating function:-

The Joint moment Generating function of two random variables x and y is defined as the expected value of the Joint function, $g(x,y) = e^{v_1 x + v_2 y}$ it is expressed as

$$M_{xy}(v_1, v_2) = E[e^{v_1 x + v_2 y}]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{v_1 x + v_2 y} f_{xy}(x,y)\,dx\,dy$$

for discrete random variable

$$M_{xy}(v_1, v_2) = \sum_m \sum_n e^{v_1 x_m + v_2 y_n} P_{xy}(x_m, y_n)$$

properties of Joint moment Generating function:-

1) The marginal moment Generating functions are

$$M_x(v_1) = M_{xy}(v_1, 0)$$

$$M_y(v_2) = M_{xy}(0, v_2)$$

2) If two random variables x and y are independent then

$$M_{xy}(v_1, v_2) = M_x(v_1)\, M_y(v_2)$$

$$M_{x+y}(v) = M_x(v)\, M_y(v)$$

3) If x and y are two random variables then the Joint moments can be derived from the JMGF as

$$m_{nk} = \left. \frac{\partial^{n+k} M_{xy}(v_1, v_2)}{\partial v_1^n\, \partial v_2^k} \right|_{v_1 = v_2 = 0}$$

1) Two Random variables x and y have Joint characteristic function $\phi_{xy}(\omega_1, \omega_2) = \exp(-2\omega_1^2 - 8\omega_2^2)$

i) show that x and y are zero mean random variables.

ii) Are x and y correlated ?

Given

JCF is $\phi_{xy}(\omega_1, \omega_2) = \exp(-2\omega_1^2 - 8\omega_2^2)$

$$\phi_{xy}(\omega_1, \omega_2) = e^{(-2\omega_1^2 - 8\omega_2^2)}$$

W.K.T

The Joint moment of order n+k or m+k

$$m_{nk} = (-j)^{n+k} \left. \frac{\partial^{n+k} \phi_{xy}(\omega_1, \omega_2)}{\partial \omega_1^n \, \partial \omega_2^k} \right|_{\omega_1 = \omega_2 = 0} \rightarrow ①$$

i) To show that x and y are zero mean r.v:-

$$m_{10} = E[x] = 0$$

$$m_{01} = E[y] = 0$$

Put n=1 k=0 in eq①

$$m_{10} = (-j)^1 \left. \frac{\partial \left[e^{(-2\omega_1^2 - 8\omega_2^2)} \right]}{\partial \omega_1} \right|_{\omega_1 = \omega_2 = 0}$$

$$= (-j) \left[e^{-2\omega_1^2 - 8\omega_2^2} (-4\omega_1 + 0) \right] \Big|_{\omega_1 = 0}$$

$$m_{10} = 0$$

Put n=0 k=1 in eq①

$$m_{01} = (-j)^1 \left. \frac{\partial \left[e^{(-2\omega_1^2 - 8\omega_2^2)} \right]}{\partial \omega_2} \right|_{\omega_2 = \omega_1 = 0}$$

$$= (-j) \left[e^{-2\omega_1^2 - 8\omega_2^2} (-16\omega_2 + 0) \right] \Big|_{\omega_2 = 0}$$

$$m_{01} = 0$$

ii) To find correlation:-

w.k.T $\quad C_{XY} = R_{XY} - E[X]E[Y]$

$$C_{XY} = R_{XY} - 0$$

$$C_{XY} = E[XY]$$

Put $n=1$, $k=1$ in eq(1)

$$m_{11} = (-j)^2 \frac{\partial^2 \left[e^{(-2\omega_1^2 - 8\omega_2^2)}\right]}{\partial\omega_1 \, \partial\omega_2}\Bigg|_{\omega_1=\omega_2=0}$$

$$= \frac{\partial}{\partial\omega_1}\left(e^{-2\omega_1^2}\right) \frac{\partial}{\partial\omega_2}\left(e^{-8\omega_2^2}\right)\Bigg|_{\omega_1=\omega_2=0}$$

$$= \left[e^{-2\omega_1^2}(-4\omega_1) \; e^{-8\omega_2^2}(-16\omega_2)\right]\Bigg|_{\omega_1=\omega_2=0}$$

$$= 0$$

$\therefore \quad C_{XY} = 0$

X and Y are uncorrelated.

Gaussian Random Variables:-

i) Two Random variables:-

If two random variables X and Y are said to be Jointly Gaussian then Joint density function is given as

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\bar{x})^2}{\sigma_X^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_X\sigma_Y} + \frac{(y-\bar{y})^2}{\sigma_Y^2}\right]\right\}$$

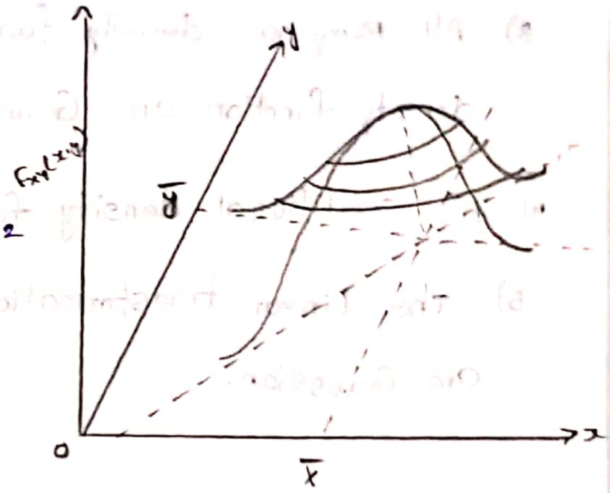
This is called bivariant Gaussian density function where $\bar{x} = E[x]$

$\bar{y} = E[y]$

$\sigma_x^2 = E[x^2] - \{E[x]\}^2$

$\sigma_y^2 = E[y^2] - (E[y])^2$

$\rho = \dfrac{C_{xy}}{\sigma_x \sigma_y}$



Maximum of $f_{xy}(x,y) = \dfrac{1}{2\pi \sigma_x \sigma_y \sqrt{1-\rho^2}}$

ii) N Random Variables:-

Consider 'N' Random Variables X_n, $n = 1, 2, 3 \cdots N$ are said to be jointly Gaussian, if there joint density function is given by

$$f_{X_1 X_2 \cdots X_N}(x_1, x_2, \cdots x_N) = \frac{1}{(-2\pi)^{\frac{N}{2}} |[C_x]|^{1/2}} \exp\left\{ \frac{-[x-\bar{x}]' [C_x]^{-1} [x-\bar{x}]}{2} \right\}$$

where $[C_x] = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1N} \\ C_{21} & C_{22} & \cdots & C_{2N} \\ \vdots & & & \\ C_{N1} & C_{N2} & \cdots & C_{NN} \end{bmatrix}$

$$[x-\bar{x}] = \begin{bmatrix} x_1 - \bar{X}_1 \\ x_2 - \bar{X}_2 \\ \vdots \\ x_N - \bar{X}_N \end{bmatrix}$$

Properties of Gaussian random variables:-

1) Gaussian random variables are completely defined by their means, variance, & covariances.

2) If the Gaussian random variable are uncorrelated then they are statistically independent.

3) All Marginal density functions derived from N - Gaussian density function are Gaussian.

4) All conditional density functions are also Gaussian.

5) The linear transformation of Gaussian random variables are Gaussian.