UNIT - IV
Complex Variables - Differentiation

Introduction to Complex Numbers:-

Complex Number:-

An expression is of the form $x+iy$ where x and y real numbers and $i = \sqrt{-1}$, is said to be a complex number. And it is denoted by 'z'. Here x is called real part of 'z' i.e $Re(z)$ & y is called imaginary part of z, i.e $Im(z)$.

conjugate of a complex Number:-

consider a complex Number $z = x + i(-y)$ is said to be conjugate of 'z' and it is denoted by $\bar{z}$.

Thus $\bar{z} = x + i(-y)$

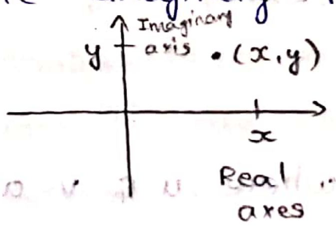
Note :-

1) $x = z + \frac{\bar{z}}{2}$

2) $y = \frac{z - \bar{z}}{2i}$

Geometrical Representation of a complex Number:-

consider a complex number $z = x+iy$. Geometrically a complex number is denoted by a point in two dimensional plane whose first co-ordinate is the real part & second coordinate is the real part & second coordinate is the Imaginary part i.e (x,y)



Note:-

1) Horizontal axis is called as Real axis.

2) vertical axis is called as Imaginary axis.

3) The plane in which complex number represented geometrically is called as complex plane. (or) Argand plane.

Polar form of a complex Number:-

If $z = x+iy$ be any complex Number then it is represented in polar form i.e (r, θ) is given by

$$z = re^{i\theta}$$

where $r = \sqrt{x^2+y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

complex variable:-

A Quantity which is varying with respect to complex numbers, is said to be complex variable and it is denoted by 'z'. Thus $z = x+iy$ where x & y are real numbers. variables.

Complex Function:-

Consider two complex variables 'z' and 'w'. If for each value of 'z', there is a value for 'w', then w is said to be function in 'z' and it is denoted by $w = f(z)$.

Note:-

Let $w = u+iv$, where u & v are real variables.

Types of Complex functions :-

1) Single valued function:-

Consider a complex function $\omega = f(z)$, If for each value of 'z', there is only single value for 'ω', then $w = f(z)$ is said to be single valued function

Ex:- $W = z$

2) Multiple valued function:-

Consider a complex function $\omega = f(z)$. If for any one value of 'z', there are multiple values for 'ω', the $\omega = f(z)$, is said to be multiple valued function.

Ex:- $W = \sqrt{z}$

Limit of a function neares to a point :-

Consider a function $f(z)$ and a point 'A'. If $f(z)$ values are nearer to a finite value for nearer value of the point 'A', then finite value is said to be limit of the point function $f(z)$ for the nearer values of point and it is denoted by $\lim f(z) = [\text{finite}]$

Method [for evaluation of $\lim\limits_{z \to a} f(z)$]

1) Evaluate $\lim f(z)$ along all 5 paths [i) rectangular path (PLQ)

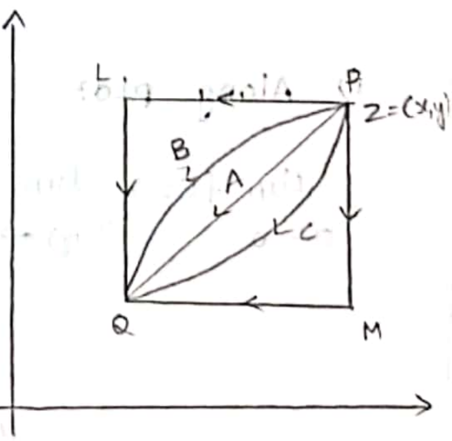
ii) rectangular path (PMQ)

iii) straight line (PQ)

iv) parabola path (PBQ)

v) parabola path (PCQ)]

from z to a



Process:-

1) convert $f(z)$ into two real variables x and y by substituting $z = x+iy$ and then convert two variables into single variable x and y depending on the mathematical form of the path.

2) If $\lim_{z \to a} f(z)$ is unique (or) same along all 5 paths, then

$\lim_{z \to a} f(z)$ exists, unique value is the limit value.

Continuity of a function at a point:-

consider a point function $f(z)$ and point 'A' if

1) $\lim_{z \to a} f(z)$ exists

2) $\lim_{z \to a} f(z) = f(a)$, then f is continuous at $z = a$

Continous of function over a region:-

consider a function $f(z)$ and a region R. If $f(z)$ is continuous at every point over R, then $f(z)$ is continuous over R.

Discuss the continuity of $f(z) = \begin{cases} \dfrac{x^3-y^3}{x^3+y^3}; & z \neq 0 \\ 0 & ; z=0 \end{cases}$ at $z=0$

i) Along PL0:-

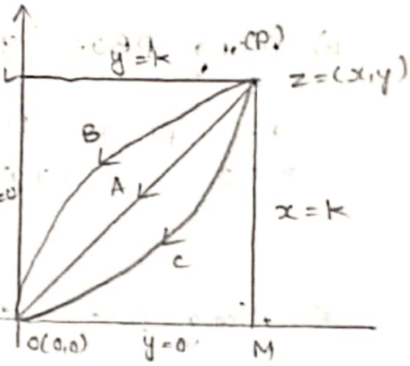
$$\lim_{z \to 0} f(z) = \lim_{(x,y) \to (0,0)} \frac{x^3-y^3}{x^3+y^3} = \lim_{x \to 0} \frac{x^3-k^3}{x^3+k^3}$$

$$= \lim_{x \to 0} \left(\frac{-k^3}{k^3}\right) = -1$$

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3} = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^3+y^3}$$

$$= \lim_{y\to 0} \left(\frac{-y^3}{y^3}\right)^{x=0}$$

$$= -1$$



Along PMO:-

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3} = \lim_{y\to 0} \frac{k^3-y^3}{k^3+y^3} = \lim_{y\to 0} \left(\frac{+k^3}{k^3}\right) = 1$$

(or)

$$\lim_{z\to 0} f(z) = \lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^3+y^3} = \lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3-y^3}{x^3+y^3} = \lim_{x\to 0} \left(\frac{x^3}{x^3}\right) = 1$$

Discuss the continuity of $f(z) = \begin{cases} \dfrac{x^3(1+i)-y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

i) Along PLO:-

$$\underset{z\to 0}{Lt} f(z) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} = \underset{\substack{x\to 0\\ y\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \underset{y\to 0}{Lt} \frac{-y^3(1+i)}{y^2} = 0$$

ii) Along PMO:-

$$\underset{z\to 0}{Lt} f(z) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} = \underset{\substack{y\to 0\\ x\to 0}}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}$$

$$= \underset{x\to 0}{Lt} \frac{x^3(1+i)}{x^2} = 0$$

3) Along PAO:- $(y = mx)$

$$\underset{z\to 0}{Lt} f(z) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} = \underset{x\to 0}{Lt} \frac{x^3(1+i)-(mx)^3(1-i)}{x^2+(mx)^2}$$

$$= 0$$

4) Along PBO:- ($y^2 = mx$)

$$\underset{z \to 0}{Lt} f(z) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \underset{y \to 0}{Lt} \frac{\left(\frac{y^2}{m}\right)^3(1+i) - y^3(1-i)}{\left(\frac{y^2}{m}\right)^2 + y^2}$$

$$= 0$$

5) Along PCO:- ($x^2 = my$)

$$\underset{z \to 0}{Lt} f(z) = \underset{(x,y)\to(0,0)}{Lt} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$$

$$= \underset{x \to 0}{Lt} \frac{x^3(1+i) - \left(\frac{x^2}{m}\right)^3(1-i)}{x^2 + \left(\frac{x^2}{m}\right)^2} = 0$$

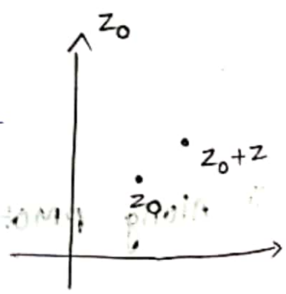
$\underset{z \to 0}{Lim} f(z)$ exists & $\underset{z \to 0}{lim} f(z) = 0 = f(0)$

$\therefore$ f is continuous at $z = 0$

Derivative of a function of a point :-

consider a function $f(z)$ and a point z_0.

If $\underset{\substack{(z_0+h)\to z_0 \\ h \to 0}}{Lt} \frac{f(z_0+h) - f(z_0)}{(z_0+h) - (z_0)}$ exists, then derivative of $f(z)$ at z_0 exists and the limit value is the derivative of f at z_0 and it is denoted by $\left.\frac{df}{dz}\right|_{z_0}$ (or) $f'(z_0)$



Derivative of a function over a region :-

consider a function $f(z)$ and region (R). If the derivative of $f(z)$ at each point exists over a region (R) then derivative of 'f' exists over the region R

Analyticity of a function at a point:-

consider a function $f(z)$ and a point 'a'. If

1) $f(z)$ is derivable at 'a'.

2) $f(z)$ is derivable in the neighbourhood of 'a' [circular region with centre as 'a' & radius as very small quantity].

then $f(z)$ is analytic at the point a.

Analyticity of a function over a region:-

Consider a function $f(z)$ and a region 'R'. If $f(z)$ is analytic at every point over the region 'R', then $f(z)$ is analytic over the region 'R'.

Result:- 1) [Necessary condition for Analyticity]

If $f(z) = w = u+iv$ is analytic over a region R, then

1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

2) $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$

2) Sufficient condition for Analyticity

If 1) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ [cauchy-Riemann eqn's (or) CR-eqn's]

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

then $f(z) = u+iv$ is Analytic everywhere.

Note:- [sufficient condition is useful for easily saying $f(z)$ is analytic over a region] process:-

1) write Real part (u) & Imaginary part (v) of $f(z)$ by substituting $z = x+iy$.

2) Evaluate $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

3) check the CR equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

4) check the continuity of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

1) show that $f(z) = z^2$, is analytic everywhere and hence find its derivative.

$$f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy) = u+iv$$

$$u = x^2-y^2 \quad \& \quad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

$\therefore$ $f(z)$ is analytic everywhere.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 2x + i(2y) = 2(x+iy) = 2z$$

Note:-

1) n^{th} degree polynomial in x & y is continuous every where.

2) Exponential function in x & y is continuous everywhere.

3) sine and cosine functions in x & y are continuous every where.

4) If f & g are continuous then $f+g$, $f-g$, $f \cdot g$, f/g $(g \neq 0)$ are continuous.

Q) Show that $f(z) = z^3$, is analytic every where and hence find its derivative.

$f(z) = z^3$

$= (x+iy)^3$

$= x^3 + i^3y^3 + 3x^2iy + 3xi^2y^2$

$= x^3 - iy^3 + 3x^2iy - 3xy^2$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$= u + iv$

$u = x^3 - 3xy^2$ $\quad\quad$ $v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$ $\quad\quad$ $\frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy$ $\quad\quad$ $\frac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous

$\therefore f(z)$ is analytic everywhere

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= (3x^2 - 3y^2) + i(6xy)$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$= 3(x+iy)^2$$

$$= 3z^2$$

Note:-

If $f(z)$ is analytic function then the rules (or) methods for evaluation of derivatives of real function are applicable to complex function.

3) show that $f(z) = z + 2\bar{z}$ is not analytic everywhere.

$$f(z) = z + 2\bar{z}$$

$$= x+iy+2(x-iy)$$

$$= x+iy+2x-2iy$$

$$= 3x - iy$$

$u = 3x$, $v = -y$

$$\frac{\partial u}{\partial x} = 3 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \ \& \ \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$f(z)$ is not analytic every where

4) show that $f(z) = e^z$ is analytic everywhere & find its derivative.

$f(z) = e^z = e^{x+iy} = e^x(\cos y + i\sin y)$

$u = e^x \cos y$ $\qquad$ $v = e^x \sin y$ $\qquad$ $[e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{\partial u}{\partial x} = e^x \cos y$ $\qquad$ $\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$ $\qquad$ $\frac{\partial v}{\partial y} = e^x \cos y$

Here $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}, \frac{\partial v}{\partial x}$ are continuous.

$\therefore$ $f(z)$ is analytic everywhere

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x(\cos y + i\sin y)$$

$$f'(z) = e^z$$

5) If $f(z) = \sin z$ is analytic everywhere & find its derivative.

$f(z) = \sin z = \sin(x+iy)$

$= \sin x \cdot \cosh y + i\cos x \sinh y$

$u = \sin x \cdot \cosh y$ $\qquad$ $v = \cos x \sinh y$

$\frac{\partial u}{\partial x} = \cos x \cdot \cosh y$ $\qquad$ $\frac{\partial v}{\partial x} = -\sin x \cdot \sinh y$

$\frac{\partial u}{\partial y} = \sin x \cdot \sinh y$ $\qquad$ $\frac{\partial v}{\partial y} = \cos x \cdot \cosh y$

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

$\therefore f(z)$ is analytic everywhere

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \cos x \cdot \cosh y + i(-\sin x \cdot \sinh y)$$

$$= \cos x \cdot \cosh y - i\sin x \cdot \sinh y$$

$$= \cos(x+iy)$$

$$= \cos z$$

show that $f(z) = \log z$ is analytic everywhere except origin.

$$f(z) = \log z = \log(x+iy)$$

$$= \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}(y/x)$$

$u = \frac{1}{2}\log(x^2+y^2)$, $v = \tan^{-1}(y/x)$

$$\frac{\partial u}{\partial x} = \frac{\cancel{2}x}{\cancel{2}(x^2+y^2)} = \frac{x}{x^2+y^2}$$

$$\frac{\partial v}{\partial x} = \frac{1}{1+(y/x)^2} \times \left(\frac{-y}{x^2}\right)$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial u}{\partial y} = \frac{\cancel{2}y}{\cancel{2}(x^2+y^2)}$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2+y^2}$$

$$\frac{\partial v}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \times \left(\frac{1}{x}\right)$$

$$\frac{\partial v}{\partial y} = \frac{x}{x^2+y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ \& } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

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$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous

$\therefore$ f(z) is analytic everywhere

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

$$= \frac{1}{x^2+y^2}(x-iy)$$

f(z) is analytic every where except at origin.

CR equations in polar coordinates:-

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

1) show that $f(z) = z^n$; $n \in z^+$ is analytic everywhere & hence find its derivative.

$$f(z) = z^n$$

$$= (x+iy)^n = (r.e^{i\theta})^n$$

$$= r^n . e^{in\theta}$$

$$= r^n(\cos n\theta + i\sin n\theta)$$

$u = r^n \cos n\theta$ $\qquad$ $v = r^n \sin n\theta$

$\frac{\partial u}{\partial r} = n r^{n-1}(\cos n\theta)$ $\qquad$ $\frac{\partial v}{\partial r} = n r^{n-1}(\sin n\theta)$

$\frac{\partial u}{\partial \theta} = r^n(-n\sin n\theta)$ $\qquad$ $\frac{\partial v}{\partial \theta} = r^n(n\cos n\theta)$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \& \quad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, \frac{\partial v}{\partial r}, \frac{\partial v}{\partial \theta}$ is continuous

$\therefore$ $f(z) = z^n$, $n \in z^+$ is analytic everywhere

$$f'(z) = n z^{n-1}$$

2) show that $f(z) = \sin x \cdot \sin y - i \cos x \cos y$ is not analytic everywhere.

3) show that $f(z) = z + 3\bar{z}$ is not analytic

4) Show that $f(z) = \cos z$ is analytic

5) show that $f(z) = |z|^2$ is not analytic.

2) Given

$$f(z) = \sin x \cdot \sin y - i \cos x \cos y$$

$$= u + iv$$

$u = \sin x \cdot \sin y$ $\quad$ $v = -\cos x \cos y$

$\frac{\partial u}{\partial x} = \cos x \cdot \sin y$ $\quad$ $\frac{\partial v}{\partial x} = \sin x \cos y$

$\frac{\partial u}{\partial y} = \sin x \cdot \cos y$ $\quad$ $\frac{\partial v}{\partial y} = +\cos x \sin y$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$f(z)$ is not analytic everywhere.

3) Given

$$f(z) = z + 3\bar{z}$$

$$= x + iy + 3(x - iy)$$

$$= x + iy + 3x - i3y$$

$$= 4x - i2y$$

$u = 4x$ $\qquad\qquad$ $v = -2y$

$\frac{\partial u}{\partial x} = 4$ $\qquad\qquad$ $\frac{\partial v}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ $\qquad\qquad$ $\frac{\partial v}{\partial y} = -2$

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

$f(z)$ is not analytic every where.

4) Given

$$f(z) = \cos z$$

$$= \cos(x+iy)$$

$$= \cos x \cos hy - i \sin x \cdot \sin hy$$

$u = \cos x \cos hy$ $\qquad\qquad$ $v = -\sin x \sinh y$

$\frac{\partial u}{\partial x} = -\sin x \cosh y$ $\qquad\qquad$ $\frac{\partial v}{\partial x} = -\cos x \cdot \sinh y$

$\frac{\partial u}{\partial y} = \cos x \cdot \sinh y$ $\qquad\qquad$ $\frac{\partial v}{\partial y} = -[\sin x \cosh y]$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous.

$f(z)$ is analytic everywhere

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= -\sin x \cosh y - i \cos x . \sinh y$$

$$= -[\sin x \cosh y + i \cos x \sinh y]$$

$$= -\sin(x+iy)$$

$$= -\sin z$$

Laplace Equation:-

An equation is of the form $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$, where f is a function in two variables x, y, is said to be a Laplace equation.

Harmonic function:-

An function satisfies laplace equation is said to be Harmonic function.

Properties of Analytic function:-

1) If $f(z) = w = u+iv$ is an analytic function then u and v are Harmonic functions.

(or)

Real and Imaginary parts of analytic function are Harmonic functions.

2) If $f(z) = w = u+iv$ is an analytic function then $u(x,y) = c_1$ & $v(x,y) = c_2$ forms orthogonal curves

(or)

Real and Imaginary parts of analytic function forms orthogonal curves.

3) If $f(z) = w = u+iv$ is an analytic function then and $u = c$ then $f(z)$ is a constant function.

(or)

An analytic function whose real part is constant, is a constant function.

4) If an analytic function whose Imaginary part is constant is a constant function.

5) If $f(z) = u+iv$ is analytic function, & $|f(z)| = c$, then $f(z)$ is a constant function.

1) If $\cosh(u+iv) = x+iy$, prove that

i) $$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = 1$$

we know that

$$\cosh z = \cosh x \cos y + i(\sinh x \cdot \sin y)$$

$$\cosh(u+iv) = x+iy$$

$$\cosh(u+iv) = \cosh u \cos v + i(\sinh u \cdot \sin v) = x+iy$$

Here $x = \cosh u \cos v$ $\quad y = \sinh u \sin v$

i) $$\frac{x^2}{\cosh^2 u} + \frac{y^2}{\sinh^2 u} = \frac{\cancel{\cosh^2 u} \cos^2 v}{\cancel{\cosh^2 u}} + \frac{\cancel{\sinh^2 u} \sin^2 v}{\cancel{\sinh^2 u}}.$$

$$= \cos^2 v + \sin^2 v$$

$$= 1$$

ii) $\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v}$

$$\frac{x^2}{\cos^2 v} - \frac{y^2}{\sin^2 v} = \frac{\cosh^2 u \cancel{\cos^2 v}}{\cancel{\cos^2 v}} - \frac{\sinh^2 u \cancel{\sin^2 v}}{\cancel{\sin^2 v}}$$

$$= \cosh^2 u - \sinh^2 u = 1$$

2) solve $\sinh z = i$

Given $\sinh z = i$

$$\frac{e^z - e^{-z}}{2} = i \qquad \left[\sinh\theta = \frac{e^\theta - e^{-\theta}}{2} \right.$$

$$e^z - e^{-z} = 2i$$

$$e^z - \frac{1}{e^z} = 2i$$

$$(e^z)^2 - 1 = 2ie^z$$

$$(e^z)^2 - 2ie^z = 1$$

$$[e^z - i]^2 - i^2 = 1$$

$$[e^z - i]^2 = 1 - 1 = 0$$

$$[e^z - i]^2 = 0$$

$$e^z - i = 0$$

$$e^z = i$$

$z = \ln(i)$. Here $x = 0$ $y = 1$

$$z = \frac{1}{2}\ln[0+1] + i\tan^{-1}\left(\frac{1}{0}\right) + 2n\pi i \ , n \in z$$

$$z = i\left(\frac{\pi}{2}\right) + 2n\pi i \ , n \in z$$

3) Test whether (or) not $f(z) = \sin x \sin y - i\cos x \cos y$ analytic.

Given

$$f(z) = \sin x \sin y - i\cos x \cos y$$

Here $u = \sin x \sin y$ $\quad v = -\cos x \cos y$

$$\frac{\partial u}{\partial x} = \cos x \sin y \qquad \frac{\partial v}{\partial x} = \sin x \cdot \cos y$$

$$\frac{\partial u}{\partial y} = \sin x \cos y \qquad \frac{\partial v}{\partial y} = \cos x \sin y$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ but } \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

Hence $f(z)$ is not analytic.

4) Seperate real and Imaginary part of $\tanh z$

$$\tanh z = \frac{\sinh z}{\cosh z}$$

$$= \frac{\sinh x \cos y + i\cosh x \sin y}{\cosh x \cos y + i\sinh x \sin y} \times \frac{\cosh x \cos y - i\sinh x \sin y}{\cosh x \cos y - i\sinh x \sin y}$$

$$= \frac{\sinh x \cos y \cosh x \cos y - i\sinh^2 x \sin y \cos y + i\cosh^2 x \sin y \cos y + \cosh x \sinh x \sin^2 y}{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}$$

$$= \frac{\cosh x \sinh x(\cos^2 y + \sin^2 y) + i(\cosh^2 x - \sinh^2 x)\sin y \cos y}{\left(\frac{\cosh 2x + 1}{2}\right)\left(\frac{1+\cos 2y}{2}\right) + \left(\frac{\cosh 2x - 1}{2}\right)\left(\frac{1-\cos 2y}{2}\right)}$$

$$= \frac{\cosh x \sinh x + i \sin y \cos y}{\frac{1}{4}\left[\cosh 2x + \cosh 2x \cancel{\cos 2y} + \cancel{1} + \cos 2y + \cosh 2x - \cosh 2x \cancel{\cos 2y} - \cancel{1} + \cos 2y\right]}$$

$$= \frac{\cosh x \sinh x + i \sin y \cos y}{\frac{1}{4}\left[2\cosh 2x + 2\cos 2y\right]}$$

$$= \frac{2\cosh x \sinh x}{\cosh 2x + \cos 2y} + i\,\frac{2\sin y \cos y}{\cosh 2x + \cos 2y}$$

$$= \frac{\sinh 2x}{\cosh 2x + \cos 2y} + i\,\frac{\sin 2y}{\cosh 2x + \cos 2y}$$

5) show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |f(z)|^2 = 4\,|f'(z)|^2$, where $f(z)$ is analytic.

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |f(z)|^2 = \left[4\,\frac{\partial^2}{\partial z \partial \bar{z}}\right] |f(z)|^2$$

$$= 4\left[\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\left[|f(z)|^2\right]\right]\right]$$

$$= 4\left[\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\left[f(z) \cdot f(\bar{z})\right]\right]\right]$$

$$= 4\left[\frac{\partial}{\partial z}\left[f(z) \times f'(\bar{z})\right]\right]$$

$$= 4 f'(\bar{z}) \cdot f'(z)$$

$$= 4\,|f'(z)|^2$$

$$\left[\begin{array}{l} |z|^2 = z\cdot\bar{z} \\ \frac{\partial^2}{x^2} + \frac{\partial^2}{y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}} \end{array}\right.$$

6) show that $\left[\frac{\partial^2}{x^2}+\frac{\partial^2}{y^2}\right]|f(z)|^p = p^2 |f(z)|^{p-2} |f(z)|^2,$

where $f(z)$ is analytic.

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]|f(z)|^p = \left[4\frac{\partial^2}{\partial z\partial \bar{z}}\right]|f(z)|^p$$

$$= 4\left[\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}[f(z)\cdot f(\bar{z})]^{p/2}\right]\right]$$

$$= 4\left[\frac{\partial}{\partial z}\left(\frac{\partial}{\partial \bar{z}}[f(z)]^{p/2}\cdot[f(\bar{z})]^{p/2}\right]\right]$$

$$= 4\left[\frac{\partial}{\partial z}[f(z)]^{p/2}\times\frac{p}{2}\times[f(\bar{z})^{p/2-1}\times f'(\bar{z})]\right]$$

$$= 4\cdot\frac{p}{2}\times[f(\bar{z})]^{p/2-1}\times f'(\bar{z})\times\frac{p}{2}[f(z)]^{p/2-1}$$

$$\times f'(z)$$

$$= p^2[f(z)\cdot f(\bar{z})]^{\frac{2p-2}{2}}\times[f'(z)\cdot f'(\bar{z})]$$

$$= p^2[|f(z)|^2]^{\frac{p-2}{2}}|f'(z)|^2$$

$$= p^2|f(z)|^{p-2}|f'(z)|^2$$

7) show that $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\log|f'(z)| = 0$, where

$f(z)$ is analytic.

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right]\log|f'(z)|$$

$$= \left[4\frac{\partial^2}{\partial z\partial\bar{z}}\right]\log|f'(z)|$$

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$$= 4\left[\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\left[\log\left[f'(z)\, f'(\bar{z})\right]^{1/2}\right]\right]\right]$$

$$= 4\left[\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\; \frac{1}{2}\left[\log(f'(z)) + \log f'(\bar{z})\right]\right]\right]$$

$$= 4\left[\frac{\partial}{\partial z}\left[\frac{1}{2}\left(\frac{f''(\bar{z})}{f'(\bar{z})}\right)\right]\right]$$

$$= 4(0)$$

$$= 0$$

$$\left[|\operatorname{Re} f(z)|^2 = \frac{1}{4}\left[f(z) + f(\bar{z})\right]^2\right.$$

8) show that $\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] |\operatorname{Re} f(z)|^2 = 2|f'(z)|^2$,

where $f(z)$ is analytic.

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \left[4 \frac{\partial^2}{\partial z\, \partial \bar{z}}\right] |\operatorname{Re} f(z)|^2 =$$

$$= 4\left[\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\left[\frac{1}{4}\left(f(z) + f(\bar{z})\right)^2\right]\right]\right]$$

$$\left[|\operatorname{Re} f(z)|^2 = \frac{1}{4}\left[f(z) + f(\bar{z})\right]^2\right.$$

$$= 4\left[\frac{\partial}{\partial z}\left[\frac{1}{4} \times 2\left(f(z) + f(\bar{z})\right) \times f'(z)\right]\right]$$

$$= 4 \times \frac{2}{4}\, f'(\bar{z})\left[f'(z) + 0\right]$$

$$= 2\,|f'(\bar{z})|^2$$

construction of an Analytic function when we know real (or) Imaginary parts :-

consider an analytic function $f(z) = u + iv$

Method - I :-

case - i :- Let u is known

$$v = \int \Big[\underbrace{-\frac{\partial u}{\partial y}}_{M} dx + \underbrace{\frac{\partial u}{\partial x}}_{N} dy \Big],$$ which is to be evaluated

by using the formula $\underbrace{\int M\, dx}_{y \text{ constants}} + \underbrace{\int N_1\, dy}_{\text{Terms of } N \text{ not containing } 'x'} + c$

case - ii :- Let v is known

$$u = \int \Big[\underbrace{\frac{\partial v}{\partial y}}_{M} dx - \underbrace{\frac{\partial v}{\partial x}}_{N} dy \Big],$$ which is to be evaluate

by using the formula $\underbrace{\int M\, dx}_{y \text{ constant}} + \underbrace{\int N_1\, dy}_{\text{terms of } N \text{ not containing } 'x'} + c$

Method - 2 :- Milon - Thomson Method :-

case - i :- Let u is known

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\, dz + c,$$ where

$$\phi_1(x,y) = \frac{\partial u}{\partial x} \quad \text{and} \quad \phi_2(x,y) = \frac{\partial u}{\partial y}$$

case - ii :- Let v is known

$$f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz + c,$$ where

$$\phi_1(x,y) = \frac{\partial v}{\partial y} \quad \& \quad \phi_2(x,y) = \frac{\partial v}{\partial x}$$

Find the analytic function whose real part is

$x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

Given

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

By milon thomson method

$$f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\, dz + c \longrightarrow ①$$

where $\phi_1(x,y) = \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = -6xy - 6y$$

$$f(z) = \int [(3z^2 + 6z) - i(0)]\, dz + c$$

$$= \frac{3z^3}{3} + \frac{6z^2}{2} + c$$

$$= z^3 + 3z^2 + c$$

2) Find the Analytic function whose Imaginary part is $3x^2y - y^3$.

Given

$$v = 3x^2y - y^3.$$

By Milon thomson Method:

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\, dz + c \longrightarrow ①$$

where $\psi_1(x,y) = \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

$$\psi_2(x,y) = \frac{\partial v}{\partial x} = 6xy$$

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$① \Rightarrow f(z) = \int [3z^2 + i(0)]\,dz + c$

$$= 3\frac{z^3}{3} + c$$

$$f(z) = z^3 + c$$

3) find the Analytic function whose Imaginary part is $e^{-x}(x\cos y + y\sin y)$.

Given

$$v = e^{-x}(x\cos y + y\sin y)$$

By Milton thomson Method :

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz + c \longrightarrow ①$$

where $\psi_1(x,y) = \frac{\partial v}{\partial y} = e^{-x}[-x\sin y + \sin y + y\cos y]$

$$\psi_2(x,y) = \frac{\partial v}{\partial x} = -e^{-x}[x\cos y + y\sin y] + e^{-x}[\cos y - x\sin y]$$

$① \Rightarrow f(z) = \int [0 + i[-e^{-z}\{z\} + e^{-z}]\,dz + c$

$$= i\int e^{-z}[1-z]\,dz + c$$

$$= i\left[(1-z)\frac{e^{-z}}{(-1)} - (-1)\times\frac{e^{-z}}{1}\right] + c$$

$$= i[ze^{-z}] + c$$

chain :-

$$\int f \times g = f g_1 - f' g_2 + f'' g_3 - f''' g_4 + \cdots$$

4) If $u = \frac{1}{2}\log(x^2+y^2)$ then show that u is harmonic and hence evaluate v & then f(z).

Given

$$u = \frac{1}{2}\log(x^2+y^2)$$

i) To show that u is harmonic:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\frac{1}{2}\,\frac{2x}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[\frac{1}{2}\times\frac{2y}{x^2+y^2}\right]$$

$$= \frac{\partial}{\partial x}\left[\frac{x}{x^2+y^2}\right] + \frac{\partial}{\partial y}\left[\frac{y}{x^2+y^2}\right]$$

$$= \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2 + x^2 - y^2}{(x^2+y^2)^2} \qquad \left[\frac{u}{v} = \frac{vu' - uv'}{v^2}\right.$$

$$= 0$$

∴ 'u' satisfies Laplace equation & u is Harmonic

ii) To find 'v':-

$$v = \int\left[-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy\right] \qquad \underset{y=\text{constant}}{\int M\,dx} + \int N_1\,dy + c$$

Terms of N not containing 'x'

$$= \int\left[\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right]$$

$$= \underset{y \text{ constant}}{\int} \frac{-y}{x^2+y^2}\,dx + \int 0\,dy + c \quad ; \text{ here } N_1 = 0$$

$$= -y \times \frac{1}{y} \tan^{-1}\left(\frac{x}{y}\right) + c$$

$$= -\tan^{-1}\left(\frac{x}{y}\right) + c$$

iii) To find $f(z)$:-

$$f(z) = u + iv$$

$$= \frac{1}{2}\log(x^2+y^2) + i\left(-\tan^{-1}(x/y)\right) + c$$

5) If $v = \cos x \sinh y$, then show that v is Harmonic & hence evaluate u & then $f(z)$.

Given

$$v = \cos x \sinh y$$

i) To show that v is harmonic:

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left[\sinh y(-\sin x)\right] + \frac{\partial}{\partial y}\left[\cos x \cosh y\right]$$

$$= -\sinh y \cos x + \cos x \sinh y$$

$$= 0$$

$\therefore$ v is Harmonic

ii) To find 'u':-

$$u = \int\left[\frac{\partial v}{\partial y}dx - \frac{\partial v}{\partial x}dy\right]$$

$$= \int\left[\cos x \cdot \cosh y \, dx + \sin x \sinh y \, dy\right]$$

$$= \int_{y=\text{constant}} (\cos x \cdot \cosh y)\, dx + \int 0 \, dy + c \text{ , here } N_1 = 0$$

$$u = \cosh y \cdot \sin x + c$$

iii) To find $f(z)$:-

$$f(z) = u + iv$$

$$= (\cosh y \sin x + c) + i(\cos x \sinh y)$$

Note:- $f(z) = \cosh y \sin x + i(\cos x \sinh y) + c$

If $u+v$ (or) $u-v$ are given, then construct $(1+i) f(z)$.

1) Find $f(z)$ where $u+v = \frac{x}{x^2+y^2}$

Given

$$u+v = \frac{x}{x^2+y^2}$$

we know that

$$f(z) = u+iv$$

$$i f(z) = iu - v$$

$$(1+i) f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

Given $u+v = V = \frac{x}{x^2+y^2}$

By Milon Thomson Method

$$(1+i) f(z) = \int [\varphi_1(z,0) + i\varphi_2(z,0)]\, dz + c \longrightarrow ①$$

$$\varphi_1(x,y) = \frac{\partial V}{\partial y} = \frac{(x^2+y^2)(0) - x(0+2y)}{(x^2+y^2)^2}$$

$$= \frac{-2xy}{(x^2+y^2)^2}$$

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$$\varphi_2(x,y) = \frac{\partial v}{\partial x} = \frac{x^2+y^2(1) - x(2x)}{(x^2+y^2)^2}$$

$$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$$

$$① \Rightarrow (1+i) f(z) = \int \left[0 + i\left(\frac{-1}{z^2}\right)\right] dz$$

$$(1+i) f(z) = -i \frac{z^{-1}}{-1} + c$$

$$f(z) = \frac{i}{1+i} z^{-1} + \frac{c}{1+i}$$

2) find $f(z)$, where $u-v = e^x(\cos y - \sin y)$

Given

$$u-v = e^x(\cos y - \sin y)$$

we know that

$$f(z) = u+iv$$

$$if(z) = iu - v$$

$$(1+i) f(z) = (u-v) + i(u+v)$$

$$F(z) = U + iV$$

Given $u-v = U = e^x(\cos y - \sin y)$

By Milon Thomson Method

$$(1+i) f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)] dz + c \rightarrow ①$$

$$\phi_1(x,y) = \frac{\partial u}{\partial x} = (\cos y - \sin y) \cdot e^x$$

$$\phi_2(x,y) = \frac{\partial u}{\partial y} = e^x[-\sin y - \cos y]$$

(i) $\Rightarrow (1+i)\, f(z) = \int e^{z} - i(-e^{z})\, dz + c$

$$= \int (1+i)\, e^{z}\, dz + c$$

$$(1+i)\, f(z) = (1+i)\, e^{z} + c$$

$$f(z) = e^{z} + \frac{c}{1+i}$$

Example 41 : Show that the function $u(x, y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate $v(x, y)$ and the analytic function $f(z) = u + iv$. **[JNTU 1996]**

Solution : Given $u(x, y) = e^x \cos y$

Differentiating with respect to x and y, we get

$$\frac{\partial u}{\partial x} = e^x \cos y \quad \text{and} \quad \frac{\partial u}{\partial y} = -e^x \sin y$$

$$\frac{\partial^2 u}{\partial x^2} = e^x \cos y \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Hence, $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Thus, u is a harmonic function. Let v be the harmonic conjugate of u. Then, by Cauchy–Riemann equations

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = e^x \sin y$$

Integrating, $v = e^x \sin y + f(y)$ ...(1)

$$\therefore \quad \frac{\partial v}{\partial y} = e^x \cos y + f'(y) \qquad ...(2)$$

Again $$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = e^x \cos y \qquad ...(3)$$

From (2) and (3), we get

$$e^x \cos y = e^x \cos y + f'(y)$$

or $$f'(y) = 0 \Rightarrow f(y) = c$$

Hence, from (1), we get

$$v = e^x \sin y + c$$

$$\therefore \quad f(z) = u + iv = e^x \cos y + ie^x \sin y + ic$$
$$= e^x (\cos y + i \sin y) + ic = e^x e^{iy} + ic = e^z + ic$$

Example 44 : Show that $u(x, y) = x^3 - 3x\,y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

Solution : Given $u = x^3 - 3x\,y^2$

$$\therefore \quad \frac{\partial u}{\partial x} = 3x^2 - 3y^2, \; \frac{\partial u}{\partial y} = -6x\,y$$

and $$\frac{\partial^2 u}{\partial x^2} = 6x, \; \frac{\partial^2 u}{\partial y^2} = -6x$$

Clearly. $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence u is harmonic.

Let v be the conjugate of u. Then

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \quad \text{(Using } C - R \text{ equations)}$$

or $\quad dv = 6\,x\,y\,dx + (3x^2 - 3y^2)\,dy$

Taking $M = 6xy$ and $N = 3x^2 - 3y^2$, we get

$$dv = M\,dx + N\,dy \qquad \ldots (1)$$

Now $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x$$

Hence (1) is exact differential equation

Integrating, $v = \int M\,dx \; + \; \int N\,dy + c$

(y constant) (only those terms which do not contain x.)

$$= \int 6\,x\,y\,dx + \int (-3\,y^2)\,dy + c$$

y constant

or $\quad v = 6y\left(\frac{x^2}{2}\right) - 3\left(\frac{y^3}{3}\right) + c = 3x^2y - y^3 + c$

$\therefore \quad f(z) = u + iv = x^3 - 3xy^2 + i(3x^2y - y^3) + ic$

$\qquad = x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + ic$

or $\quad f(z) = (x + iy)^3 + k$ where $k = ic$.

Note : The above result may be obtained by using Milne – Thomson's method.

Example 45 : Find k such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic and find its conjugate. **[JNTU 2004, 2004S, (K) Nov.2010 (Set No.1)]**

Solution : We have $f(x,y) = x^3 + 3kxy^2$

$$\therefore \quad \frac{\partial f}{\partial x} = 3x^2 + 3ky^2, \frac{\partial f}{\partial y} = 6kxy$$

and $\quad \frac{\partial^2 f}{\partial x^2} = 6x, \frac{\partial^2 f}{\partial y^2} = 6kx$

Since $f(x,y)$ is harmonic, therefore $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$

i.e. $\quad 6x + 6kx = 0 \qquad$ *i.e.* $\quad x(1+k) = 0$

i.e. $\quad 1 + k = 0 \quad (\because x \neq 0)$ or $k = -1$

Hence $\quad f(x,y) = x^3 - 3xy^2$

Let $g(x,y)$ be the conjugate of $f(x,y)$. Then

$$dg = \frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = -\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy \quad \text{(Using } C - R \text{ equations)}$$

$$= -6kxy\,dx + (3x^2 + 3ky^2)\,dy$$

or $\quad dg = 6xy\,dx + (3x^2 - 3y^2)\,dy \qquad (\because k = -1)$

This is exact differential equation.

Integrating, $\quad g = \int 6xy\,dx + \int -3y^2\,dy + c$

$\qquad$ y constant $\qquad$ only those terms which do not contain x.

$$= 6y\left(\frac{x^2}{2}\right) - 3\left(\frac{y^3}{3}\right) + c = 3x^2y - y^3 + c$$

Example 46 : If u is a harmonic function, show that $w = u^2$ is not ... unless

Example 14 : Find the conjugate harmonic function of the harmonic function $u = x^2 - y^2$.

[JNTU 1998S]

Solution : Given $u = x^2 - y^2$ ...(1)

Differentiating (1) partially w.r.t. x, $\frac{\partial u}{\partial x} = 2x$

Again differentiating $\frac{\partial^2 u}{\partial x^2} = 2$ ...(2)

Differentiating (1) partially w.r.t. y, $\frac{\partial u}{\partial y} = -2y$

Again differentiating, $\frac{\partial^2 u}{\partial y^2} = -2$ ...(3)

(2) + (3) gives, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$\therefore$ u is harmonic.

Let v be its harmonic conjugate. $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = 2x + i2y$

Using Milne–Thomson method, $f'(z) = 2z$

Integrating, we get $f(z) = z^2 + c$

$\therefore$ $u + iv = (x+iy) + ik$ where $c = ik$

$= x^2 - y^2 + i\,2xy + ik = x^2 - y^2 + i\,(2xy + k)$

Equating imaginary parts, $v = 2xy + k$ is the required form.

Example 15 : Show that $u(x, y) = e^{2x}\,(x\cos 2y - y\sin 2y)$ is harmonic and find its harmonic conjugate.

[JNTU 1998]

(or) Find the analytic function whose real part is $u = e^{2x}(x\cos 2y - y\sin 2y)$.

[JNTU (H) Nov. 2009 (Set No. 3), JNTU (A) Nov. 2010 (Set No.2)]

Solution : Given $u(x, y) = e^{2x}\,(x\cos 2y - y\sin 2y)$ ...(1)

Differentiating (1) partially w.r.t. x, $\frac{\partial u}{\partial x} = e^{2x}\left[(2x+1)\cos 2y - 2y\sin 2y\right]$

Again differentiating, $\frac{\partial^2 u}{\partial x^2} = e^{2x}\left[(4x+4)\cos 2y - 4y\sin 2y\right]$ ...(2)

Differentiating (1) partially w.r.t. y, $\frac{\partial u}{\partial y} = e^{2x}\left[-2x\sin 2y - 2y\cos 2y - \sin 2y\right]$

Again differentiating,

$$\frac{\partial^2 u}{\partial y^2} = e^{2x}\left[-(4x+4)\cos 2y + 4y \sin 2y\right] = -\frac{\partial^2 u}{\partial x^2} \quad \text{[using (2)]}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

$\therefore$ u is harmonic.

Let v be its harmonic conjugate.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}, \quad \text{using Cauchy-Riemann equations}$$

$$= e^{2x}\left[(2x+1)\cos 2y - 2y\sin 2y\right] - i\,e^{2x}\left[-2x\sin 2y - 2y\cos 2y - \sin 2y\right]$$

Using Milne–Thomson method

$$f'(z) = e^{2z}\left[(2z+1)\cos 0 - 0\right] - i\,e^{2z}(0) = e^{2z}(2z+1) = e^{2z}\cdot 2z + e^{2z}$$

Integrating, we get $f(z) = z\cdot e^{2z} + c$

or $\quad u + iv = (x+iy)\cdot e^{2x+2iy} + c$

$$= (x+iy)\,e^{2x}(\cos 2y + i\sin 2y) + c$$

$$= e^{2x}(x\cos 2y - y\sin 2y) + i\left[e^{2x}(x\sin 2y + y\cos 2y) + k\right] \text{ when } c = ik$$

Equating imaginary parts, we get $v = e^{2x}(2\sin 2y + y\cos 2y) + k$.

Example 17 : Find the conjugate harmonic of $u = e^{x^2-y^2} \cos 2xy$. Hence find $f(z)$ in terms of z.

[JNTU 2003S (Set No. 1)]

Solution : We have

$$u = e^{x^2-y^2} \cos 2xy$$

$$\therefore \quad \frac{\partial u}{\partial x} = e^{x^2-y^2}(-\sin 2xy)(2y) + \cos 2xy\, e^{x^2-y^2}(2x)$$

$$= 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$$

and
$$\frac{\partial u}{\partial y} = e^{x^2-y^2}(-\sin 2xy)(2x) + \cos 2xy\, e^{x^2-y^2}(-2y)$$

$$= -2e^{x^2-y^2}(x \sin 2xy + y \cos 2xy)$$

Let v be the conjugate of u. Then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \qquad \text{(Using Cauchy – Riemann equations)}$$

$$= 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy) + i2e^{x^2-y^2}(x \sin 2xy + y \cos 2xy)$$

By Milne – Thomson's method, $f'(z)$ is expressed in terms of z by replacing x by z and y by 0.

Hence $f'(z) = 2e^{z^2}(z-0) + i\,2e^{z^2}(0+0) = 2ze^{z^2}$

Integrating, $f(z) = \int 2ze^{z^2}dz + c = \int e^t dt + c$ (putting $z^2 = t$)

$$= e^t + c = e^{z^2} + c$$

or
$$u + iv = e^{(x+iy)^2} + c = e^{(x^2-y^2)+i2xy} + c$$

$$= e^{x^2-y^2}(\cos 2xy + i \sin 2xy) + c$$

$$= e^{x^2-y^2}\cos 2xy + i(e^{x^2-y^2}\sin 2xy + k), \text{ where } c = ik$$

Equating imaginary parts, $v = e^{x^2-y^2}\sin 2xy + k$