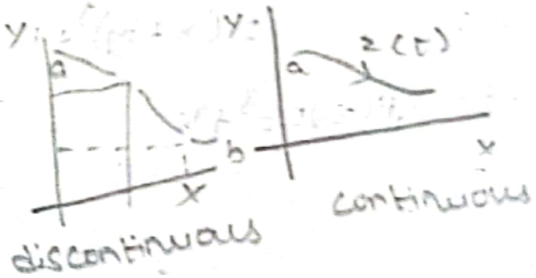
UNIT- V

complex integration

Continuous Curve (or) arc:-

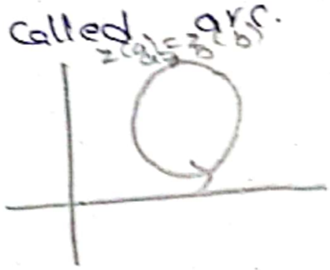
If $x(t)$ and $y(t)$ are two continuous real functions of a variable t in the interval $a \le t \le b$ the parametric equations $x = x(t)$ and $y = y(t)$ —① defineds a continuous curve (or) arc.



closed Curve:

A curve is said to be a closed if the initial point and final point coinside.

$\therefore z(a) = z(b)$. otherwise it is called arc.



multiple point of the arc:-

If the eq① are satisfied by more than one value of t in the given range then we say that the point z or (x,y) is a multiple point of the arc.

Jordan arc:-

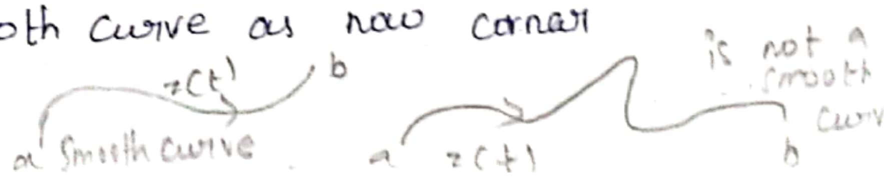
A continuous arc without multiple points is called a Jordan arc:-

Jordan Curve:-

It consist of a chain of finite no of continuous arcs. (or) a simple closed curve is defined as Jordan curve.

Regular arc:-

If $z(t)$ have continuous derivatives in $a \le t \le b$, the curve is said to be a regular arc it is also known as smooth curve. A smooth curve as now cornar



Contour:-
A continuous Jordan Curve consist of a finite no. of regular arc is called Contour.

Line integral:-
let $f(z)$ be a function which is continuous at all points on a curve c with end points are a & b Then line integral along the curve 'c' is given by

$$\int_C f(z)\,dz \text{ (or) } \int_{z=a}^{b} f(z)\,dz$$

properties of the line integrals:

(1) $\int_C f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz.$

2. $\int_a^b f(z)\,dz = -\int_a^b f(z)\,dz$

3. If $f(z)=1$ then $\int_a^b f(z)dz = b-a$

4. $\int_C [f(z) \pm g(z)]dz = \int_C f(z)dz \pm \int_C g(z)dz$

5. $\int_C [\lambda f(z)]dz = \lambda \int_C f(z)dz$

① Evaluate $\int_C f(z)\,dz$ where $f(z) = y-x-3x^2 i$ and c is the straight line segment from $z=0$ to $z=1+i$.

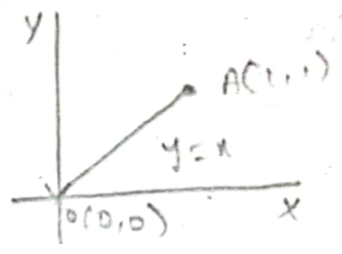
Sol:- Given
$f(z) = y-x-3x^2 i$

W.k.T $z = x+iy$
$dz = dx+idy$

Given curve 'c' is a straight line segment from $z=0$ to $z=1+i$

$x+iy = 0+i0$ to $x+iy = 1+i$.

(i.e) $(x,y) = (0,0)$ to $(x,y) = (1,1)$



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We know that

equation of a line joining the points (x_1, y_1) and (x_2, y_2)

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 0 = \frac{1-0}{1-0}(x-0)$$

$$\Rightarrow y = x$$

$$dy = dx.$$

$$\int_C f(z)dz = \int_{z=0}^{1+i} (y - x - 3x^2 i)(dx + i dy)$$

$$= \int_{x=0}^{1} (\cancel{x} - \cancel{x} - 3ix^2)(dx + i dx)$$

$$= \int_{x=0}^{1} (-3i) x^2 (1+i) dx$$

$$= (-3i)(1+i) \int_{x=0}^{1} x^2 \, dx$$

$$= (-3i - 3i^2) \left(\frac{x^3}{3}\right)_0^1$$

$$= \cancel{3}(-i + i^2)\left(\frac{1}{\cancel{3}}\right)$$

$$= 1 - i$$

2. Evaluate $\int_0^{2+i} (2x + iy + 1) dz$ along the straight line $2y = x$

Sol:- $$\int_0^{2+i} (2x + iy + 1) dz = \int_0^{2+i} (2x + iy + 1)(dx + i dy)$$

$$= \int_{(0,0)}^{(2,1)} (2x + iy + 1)(dx + i dy) \quad \text{--- (1)}$$

Given eq of a curve 'c' is

$$2y = x \Rightarrow x = 2y$$

$$\Rightarrow dx = 2 dy$$

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$$\int_{0}^{2+i} (2x+iy+1)\,dz = \int_{y=0}^{1} [2(2y)+iy+1][2\,dy+i\,dy]$$

$$= \int_{y=0}^{1} (4y+iy+1)\,(2+i)\,dy$$

$$= (2+i)\left[(4+i)\frac{y^2}{2}+y\right]_{y=0}^{1}$$

$$= (2+i)\left[(4+i)\frac{1}{2}+1\right]$$

$$= (2+i)\left[\frac{4+i+2}{2}\right]$$

$$= (2+i)\left(\frac{6+i}{2}\right)$$

$$= \frac{1}{2}(12+2i+6i-1)$$

$$= \frac{1}{2}(11+8i)$$

Evaluate $\int_{0}^{1+i} (x^2-iy)\,dz$ along the path (a) $y=x$ (b) $y=x^2$

a) $$\int_{0}^{1+i} (x^2-iy)\,dy = \int_{x=0}^{1} x^2 - i(x)\,dx$$

$$= \int_{0}^{1} x\,(x-i)\,dx$$

$$= (x-i)\int_{0}^{1} x\,dx$$

$$= (x-i)\left(\frac{x^2}{2}\right)_{0}^{1}$$

$$= \frac{x-i}{2}$$

b) $$\int_{0}^{1+i} (x^2-iy)\,dy = \int_{x=0}^{1} x^2 - i(x^2)(2x\,dx)$$

$$= \int_{x=0}^{1} x^2 - 2x^3 i\,dx$$

$$= \int_{x=0}^{1} x^2(1-i)(2x\,dx)$$

$$= (1+i)\int_{0}^{1} 2x^3\,dx$$

$$= 2(1-i)\left[\frac{x^4}{4}\right]_0^1$$

$$= \cancel{2}[1-i](1/4)_2$$

$$= \frac{1-i}{2}$$

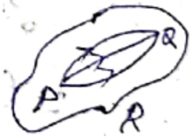
Cauchy's integral theorem:-

Statement: If $f(z)$ is an analytic function and $f'(z)$ is continuous within and on a simple closed curve 'c' then

$$\oint_c f(z)\,dz = 0$$

Note:-

1. If $f(z)$ is analytic in a region R and ~~f'(z) is~~ if P and Q be any two points in R then is independent of the path join in P & Q lying inside 'R'. $\int_P^Q f(z)\,dz$.



2. If $f(z)$ is analytic in the region bounded by two simple closed curves C_1 & C_2 then

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

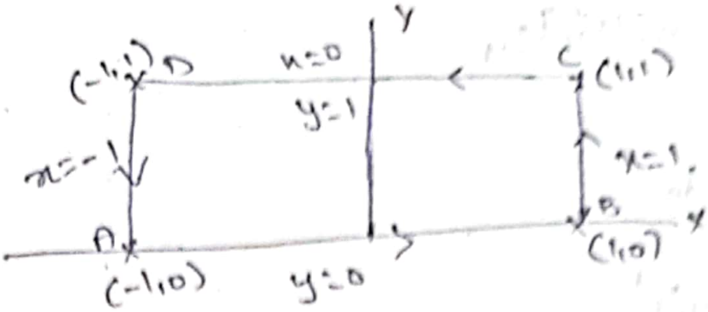
Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle with vertices $-1, 1, 1+i, -1+i$

Sol:- Given $f(z) = z^3$

Given closed curve c is a rectangle with vertices $-1, 1, 1+i, -1+i$.

(i.e) $-1+i0,\ 1+i0,\ 1+i,\ -1+i$

(i.e) $(-1,0), (1,0), (1,1)$ and $(-1,1)$



To find $\int_C f(z)dz = \int_C z^3 dz$:-

$$\int_C z^3 dz = \int_{\substack{D\,\square\,C \\ A\quad B}} z^3 dz = \left[\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}\right] z^3 dz \quad \text{--- (1)}$$

To find $\int_{AB} z^3 dz$:-

$\therefore$ eq of $\overline{AB}$ is $y=0 \Rightarrow dy=0$

$$\int_{\overline{AB}} z^3 dz = \int_{A(-1,0)}^{B(1,0)} (x+iy)^3 (dx+idy)$$

$$= \int_{x=-1}^{1} (x+i0)^3 (dx+i0)$$

$$= \int_{x=-1}^{1} x^3 dx$$

$$= \left[\frac{x^4}{4}\right]_{-1}^{1}$$

$$= \frac{1}{4}\left[(1)^4 - (-1)^4\right]$$

$$= \frac{1}{4}[1-1]$$

$$= 0$$

To find $\int_{\overline{BC}} z^3 dz$:-

eq of $\overline{BC}$ is $x=1 \Rightarrow dx=0$

$$\int_{\overline{BC}} z^3 dz = \int_{B(1,0)}^{C(1,1)} (x+iy)^3 (dx+idy)$$

$$= \int_{y=0}^{1} (1+iy)^3 (0+idy)$$

$$= i\int_{y=0}^{1}\left[1+(iy)^3+3(1)^2(iy)+3(1)(iy)^2\right]dy$$

$$= i\int_{y=0}^{1}\left[1-iy^3+3iy-3y^2\right]dy$$

$$= i\left[y-\frac{iy^4}{4}+3i\frac{y^2}{2}-3\frac{y^3}{3}\right]_{y=0}^{1}$$

$$= i\left[\cancel{1}-\frac{i}{4}+3i/2-\cancel{1}\right]$$

$$= i\left[\frac{3i}{2}-\frac{i}{4}\right]$$

$$= \frac{3}{2}(-1)+\frac{1}{4}$$

$$= \frac{1}{4}-\frac{3}{2}$$

$$= \frac{1-6}{4}$$

$$= \frac{-5}{4}$$

To find $\int_{\overline{CD}} z^3 dz$:-

$\therefore$ eq of $\overline{CD}$ is $y=1 \Rightarrow dy=0$

$$\int_{\overline{CD}} z^3 dz = \int_{C(1,1)}^{D(-1,1)} (x+iy)^3(dx+idy)$$

$$= \int_{x=1}^{-1} (x+i)^3(dx+i0)$$

$$= \int_{1}^{-1} (x+i)^3 dx$$

$$= \int_{x=1}^{-1}\left[x^3+i^3+3x^2i+3xi^2\right]dx$$

$$= \int_{x=1}^{-1}\left[x^3-i+3ix^2-3x\right]dx$$

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$$= \left[\frac{x^4}{4} - ix + 3i\frac{x^3}{3} - \frac{3x^2}{2}\right]_1^{-1}$$

$$= \left[\left\{\frac{1}{4} + i - i - \frac{3}{2}\right\} - \left\{\frac{1}{4} - i + i - \frac{3}{2}\right\}\right]$$

$$= 0$$

To find $\int_{\overline{DA}} z^3 dz$

eq of $\overline{DA}$ is $x = -1 \Rightarrow dx = 0$

$$\int_{\overline{DA}} z^3 dz = \int_{D(-1,1)}^{A(-1,0)} (x+iy)^3 (dx + idy)$$

$$= \int_{y=1}^{0} (-1+iy)^3 (0 + idy)$$

$$= i \int_{y=1}^{0} \left[-1 - iy^3 + 3(-1)^2 iy + 3(-1)(iy)^2\right] dy$$

$$= i \int_{y=1}^{0} \left[-1 - iy^3 + 3iy + 3y^2\right] dy$$

$$= i\left[-y - i\frac{y^4}{4} + 3i\, y^2/2 + \cancel{3}\, y^3/\cancel{3}\right]_{y=1}^{0}$$

$$= i\left[0 - \left\{-\cancel{1} - \frac{i}{4} + \frac{3i}{2} + \cancel{1}\right\}\right]$$

$$= i\left[\frac{i}{4} - \frac{3i}{2}\right]$$

$$= \left[-\frac{1}{4} + \frac{3}{2}\right]$$

$$= \frac{-1+6}{4} = \frac{5}{4}$$

i.e. eq ①

$$\int_C z^3 dz = 0 - \frac{5}{4} + 0 + \frac{5}{4}$$

$$= 0$$

Hence verified Cauchy's theorem.

Cauchy's integral formula:-

If $f(z)$ is analytic with in and on a closed curve 'c'. If 'a' is any point with in c then

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a).$$

Ganelaised Cauchy's integral formula.

If any function $f(z)$ is analytic at each point of a region R on a closed curve 'c'. 'a' be any point with in c.

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

problem:-

Evaluate $\int_C \frac{\sin z}{(z-\frac{\pi}{6})^2}\,dz$ where c is the circle $|z|=1$

Sol:- $\int_C \frac{\sin z}{(z-\pi/6)^2}\,dz$

$$= \int_C \frac{\sin z}{(z-\pi/6)^{1+1}}\,dz$$

$$= \int_C \frac{f(z)}{(z-a)^{n+1}}\,dz.$$

where $f(z) = \sin z$, $a = \frac{\pi}{6}$, $n=1$

Given closed curve c is $|z|=1$

$\Rightarrow x^2+y^2=1$

let $\phi(z) = |z| - 1$

put $z=a=\pi/6$

$$\therefore \phi(a) = \phi(\pi/6) = \left|\frac{\pi}{6}\right| - 1$$

$$= \frac{\pi}{6} - 1$$

$$= -0.47 < 0$$

$\therefore z = a = \pi/6$ lies inside Given curve.

according to Ganelaised Cauchy's integral formula.

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(a)$$

$$\int_C \frac{\sin z}{(z-\pi/6)^{1+1}} = \frac{2\pi i}{1!} f'(\pi/6) = 2\pi i \left[\frac{df(z)}{dr}\right]_{z=\pi/6}$$

$= 2\pi i (\cos z)_{z=\pi/6}$

$= 2\pi i (\cos \pi/6)$

$= 2\pi i \frac{\sqrt{3}}{2}$

$= \pi i \sqrt{3}$

② Evaluate $\oint_c \frac{e^z}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$

Sol:- $\oint_c \frac{e^z}{(z-1)(z-2)} dz = \oint_c \frac{f(z)}{(z-a)(z-b)} fz$

where $f(z) = e^z$, $a = 1$, $b = 2$

Given equation of a closed curve c is $|z| = 3$

(i.e) $x^2 + y^2 = 3^2$

∴ that is a circle center at the horizon radius 3 units.

let $\phi(z) = |z| - 3$

put $z = a = 1$ ∴ $\phi(a) = \phi(1) = |1| - 3 = 1 - 3 = -2 < 0$

$z = b = 2$ ∴ $\phi(b) = \phi(2) = |2| - 3 = |2| - 3 = -1 < 0$

∴ $z = a = 1$ and $z = b = 2$ lies inside the Given circle

$\therefore \oint_c \frac{e^z}{(z-1)(z-2)} dz = \oint_c e^z \left[\frac{1}{z-2} - \frac{1}{z-1} \right] dz$

$= \oint_c \frac{e^z}{z-2} dz - \oint_c \frac{e^z}{z-1} dz$

$\boxed{\therefore \int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)}$

$= 2\pi i f(2) - 2\pi i f(1)$

$= 2\pi i [f(2) - f(1)]$

$= 2\pi i [e^2 - e^1]$

③ use Cauchy's integral formula to evaluate $\oint_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where c is the circle $|z| = 3$

Sol:- $\oint_c \frac{\sin \pi z^2 + \cos \pi^2}{(z-1)(z-2)} dz = \oint_c \frac{f(z)}{(z-a)(z-b)} dz$

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where $f(z) = \sin\pi z^2 + \cos\pi z^2$, $a=1$, $b=2$

$|z| = 3$ (ie) $x^2+y^2 = 3^2$

let $\phi(z) = |z| - 3$

Put $z = a = 1 \therefore \phi(a) = \phi(1) = |1| - 3 = 1 - 3 = -2 < 0$

$z = b = 2 \therefore \phi(b) = \phi(2) = |2| - 3 = 2 - 3 = -1 < 0$

$$\oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)} dz = \oint_C (\sin\pi z^2 + \cos\pi z^2)\left[\frac{1}{z-2} - \frac{1}{z-1}\right] dz$$

$$= \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-2} dz - \oint_C \frac{\sin\pi z^2 + \cos\pi z^2}{z-1} dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i\,[f(2) - f(1)]$$

$$= 2\pi i\left[\{\sin\pi(2)^2 + \cos\pi(2)^2\} - \{\sin\pi(1)^2 + \cos\pi(1)^2\}\right]$$

$$= 2\pi i\,[\sin 4\pi + \cos 4\pi - \sin\pi - \cos\pi]$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$= 2\pi i\,[0 + 1 - 0 - (-1)]$$

$$= 2\pi i(1+1)$$

$$= 4\pi i$$

complex power series:

Sequence:

A function defined on the set of natural numbers into the set of complex numbers is called a sequence of complex numbers denoted by $[u_n]$ where $n = 1, 2, 3, \ldots$ i-e, $u_1, u_2, u_3, \ldots\ldots u_n \ldots$

Here u_n is called the nth term of the sequence

* sequence $[u_n]$ is said to be convergent if $\lim_{n\to\infty} u_n = a$, a finite quantity (any constant) otherwise is called as divergent.

Infinite Series:

let $u_1(z), u_2(z), --- u_n(z), ...$ be a sequence of functions defined on a set of complex numbers then we can form a new sequence $S_n(z)$ defined by

$S_n(z) = \sum_{n=1}^{\infty} U_n(z) = u_1(z) + U_2(z) + --- + u_n(z) + ---$ is called an infinite Series.

→ The series $\sum_{n=1}^{\infty} u_n$ is also written as Σu_n.

The series is said to be convergent on the set of complex numbers if $\lim_{n\to\infty} S_n(z) = s(z)$

Power series:

A power series is defined as the series of the form $\sum_{n=0}^{\infty} a_n (z-a)^n = a_0 + a_1 (z-a)^1 + a_2(z-a)^2 + --- + a_n(z-a)^n + ---$

where 'a' is any fixed point, ~~the complex plane~~ of the complex plane is known as centre of the power series. $a_0, a_1, a_2, --- a_n$ are called coefficients of the power series.

Taylor series:

An Analytic function can be represented by power series is called a taylor's series of the function. It is given in the following theorem.

Taylor's Theorem:

Statement: If a function $f(z)$ is analytic inside a circle 'c' with centre 'a' then for all 'z' inside 'c'

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + --- \frac{(z-a)^n}{n!} f^n(a) + -- \quad \text{--- (1)}$$

Proof:- Given $f(z)$ is analytic with in the given circle 'c' centre at 'a'.

Note:

(1) from eq (1) It is clear that a complex analytic function can always be represented by powerseries.

(2) put $a=0$ in eq (1), we get $f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + ---$ is called maclaurin's series.

Laurent's series:

If the complex function $f(z)$ has a singularity at 'a' we can't apply Taylor's theorem to get an expression of $f(z)$ in non negative powers of $(z-a)$.

We shall show that certain function can be expanded as an infinite series even though the function is not analytic at a point of the Given region by Laurent's theorem consist of positive & negative powers of $(z-a)$.

Laurent's theorem:

Statement: If $f(z)$ is analytic inside & on the boundary of the ring shaped region 'R' bounded by two concentric circles C_1 & C_2 of radii r_1 & r_2 ($r_1 > r_2$) respectively, having centre at 'a' then for all 'z' in 'R' the

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \cdots$$

where $a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw, \; n = 0, 1, 2, \ldots$

$$a_{-n} = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw, \; n = 1, 2, 3, \ldots$$

1. obtain the Taylor's series expansion of $f(z) = \frac{1}{z}$ about ~~show that~~ the point $z = 1$

Sol:- Given $f(z) = \frac{1}{z}$, $z = 1$

W.K.T Taylor's series for the function $f(z)$ about the point $z = a$ is given by,

$$f(z) = f(a) + \frac{(z-a)}{1!} f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots \longrightarrow (1)$$

if $f(z) = \frac{1}{z} \Rightarrow f(a) = f(1) = \frac{1}{1} = 1$,

$f'(z) = \frac{-1}{z^2} \Rightarrow f'(a) = f'(1) = \frac{-1}{1^2} = -1$

$f''(z) = \frac{2}{z^3} \Rightarrow f''(a) = f''(1) = \frac{2}{1^3} = 2$

$f'''(z) = \frac{-6}{z^4} \Rightarrow f'''(a) = f'''(1) = \frac{-6}{1^4} = -6$

$\therefore$ substituting all above values in eq ①

$$\frac{1}{z} = 1 + \frac{(z-1)}{1!}(-1) + \frac{(z-1)^2}{2!}(2) + \frac{(z-1)^3}{3!}(-6) + \cdots +$$

$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

(OR)

let $z-1 = \omega \Rightarrow z = 1+\omega$

$$\therefore f(z) = \frac{1}{z} = \frac{1}{1+\omega} = (1+\omega)^{-1}$$
$$= 1 - \omega + \omega^2 - \omega^3 + \cdots$$
$$= 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

2) Expand $f(z) = \frac{1}{z^2}$.

a) in powers of $z+1$

b) in powers of $z-2$

Sol: Given $f(z) = \frac{1}{z^2}$

a) To find expansion in powers of $z+1$:

let $z+1 = \omega \Rightarrow z = \omega - 1$

$$f(z) = \frac{1}{z^2} = \frac{1}{(\omega-1)^2} = \frac{1}{(1-\omega)^2}$$
$$= (1-\omega)^{-2}, \ |\omega| < 1$$
$$= 1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots$$
$$= 1 + (z+1) + 3(z+1)^2 + 4(z+1)^3 + \cdots$$

b) To find expansion in powers of $z-2$:

let $z-2 = \omega \Rightarrow z = \omega + 2$

$$z = 2 + \omega$$

$$f(z) = \frac{1}{z^2} = \frac{1}{(2+\omega)^2} = \frac{1}{2^2(1+\omega/2)^2}$$
$$= \frac{1}{4}\left[1 + \frac{\omega}{2}\right]^{-2}$$
$$= \frac{1}{4}\left[1 - 2(\omega/2) + 3(\omega/2) + 3(\omega/2)^2 - 4(\omega/2)^3 + \cdots\right], \ \left|\frac{\omega}{2}\right| < 1$$
$$= \frac{1}{4}\left[1 - (z-2) + 3/4 (z-2)^2 - \frac{1}{2}[z-2]^3 + \cdots\right]$$

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Note:

$\Rightarrow (1-x)^{-1} = 1+x+x^2+x^3+ \cdots$

$(1+x)^{-1} = 1-x+x^2-x^3 \cdots$

$(1-x)^{-2} = 1+2x+3x^2+4x^3+ \cdots$

$(1+x)^{-2} = 1-2x+3x^2-4x^3+ \cdots$

3) Represent the function $f(z) = \frac{z}{(z-1)(z-3)}$ by a series of positive & negative powers of $(z-1)$ which converges to $f(z)$ when $0<|z-1|<2$

Sol:- Given

$$f(z) = \frac{z}{(z-1)(z-3)}, \quad 0<|z-1|<2$$

$\therefore$ we expand the above function in terms of $(z-1)$

let $z-1=\omega \Rightarrow z=1+\omega$

$$z-3 = (1+\omega)-3$$
$$= \omega-2$$

$$\therefore f(z) = \frac{z}{(z-1)(z-3)} = \frac{1+\omega}{\omega(\omega-2)} \quad —① $$

Resolve $\frac{1+\omega}{\omega(\omega-2)}$ into partial fractions.

$$\frac{1+\omega}{\omega(\omega-2)} = \frac{A}{\omega} + \frac{B}{\omega-2}$$

$$\Rightarrow 1+\omega = A(\omega-2)+B(\omega) \quad —②$$

put $\omega=0$ in eq ②

$$1 = A(-2)+0$$

$$\boxed{A = \frac{-1}{2}}$$

put $\omega=2$ in eq ②

$$3 = A(0)+B(2)$$

$$3=2B$$

$$\boxed{B = \frac{3}{2}}$$

$$\therefore \frac{1+\omega}{\omega(\omega-2)} = \frac{(-1/2)}{\omega} + \frac{3/2}{\omega-2}$$

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$$f(z)=\frac{z}{(z-1)(z-2)}=\left(\frac{-1}{2}\right)\frac{1}{z-1}+\left(\frac{3}{2}\right)\left(\frac{1}{z-3}\right)$$

(OR)

$$=\left(\frac{-1}{2}\right)\frac{1}{\omega}+\frac{3}{2}\left(\frac{1}{\omega-2}\right)$$

$$f(z)=\frac{-1}{2}\frac{1}{\omega}-\frac{3}{2}\left(\frac{1}{\frac{2-\omega}{}}\right)$$

$$=\frac{-1}{2}\left(\frac{1}{\omega}\right)-\frac{3}{4}\left(\frac{1}{1-\omega/2}\right)$$

$$=\frac{-1}{2}\left(\frac{1}{\omega}\right)-\frac{3}{4}\left(1-\omega/2\right)^{-1}$$

$$=\frac{-1}{2}\left(\frac{1}{\omega}\right)-\frac{3}{4}\left(1+\left(\frac{\omega}{2}\right)+\left(\frac{\omega}{2}\right)^2+\left(\frac{\omega}{2}\right)^3+\cdots\right]$$

$$=\frac{-1}{2}\left(\frac{1}{\omega}\right)-\frac{3}{4}\left(1+\frac{1}{2}\omega+\frac{1}{4}\omega^2+\frac{1}{8}\omega^3+\cdots\right]$$

put $\omega=z-1$.

$$f(z)=\frac{-1}{2}\left(\frac{1}{z-1}\right)-\frac{3}{4}\left(1+\frac{1}{2}(z-1)+\frac{1}{4}(z-1)^2+\frac{1}{8}(z-1)^3+\cdots\right]$$

Def: zero of an Analytic function:

A zero of an Analytic function $f(z)$ is a value of z such that $f(z)=0$. particularly. a point 'a' is called a zero of an analytic function $f(z)$ is $f(a)=0$

zero of mth order: if an analytic function $f(z)$ can be expressed in the form $f(z)=(z-a)^m\phi(z)$ then $z=a$ is a zero of order m for $f(z)$.

for example if,

i) $f(z)=(z-1)^3$, Then $z=1$ in a zero of order 3 of $f(z)$.

ii) $f(z)=\frac{1}{1-z}$, then $z=\infty$ in a Simple zero of $f(z)$

iii) $f(z)=\sin z$, then $z=0,\pm\pi,\pm 2\pi,\cdots$ are simple zeros of $f(z)$.

iv) $f(z)=e^{\tan z}$ has no zeros since $e^z\neq 0$ for any z.

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Def: singular points.

A singular point of a function f(z) in the point at which the function f(z) causes to be analytic

Defferent types of singularities:-

i) Isolated singularity:- A point $z=a$ is called an isolated singularity of an analytic functions $f(z)$, if

a) $f(z)$ is not analytic at the point $z=a$

b) $f(z)$ is analytic there exist a neighbourhood of the point $z=a$ which contains no other singularity.

Ex:- i) $f(z)=\frac{e^z}{z-1}$ then $z=1$ in the isolated singular point of $f(z)$.

ii) $f(z)=\frac{e^z}{z^2+1}$ then $z=\pm i$ are two isolated singular points.

ii) poles of an analytic function:

if $z=a$ is an isolated singular point of an analytic function $f(z)$, then $f(z)$ can be expanded in Laurent's Series about the point $z=a$.

i.e, $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$ — ①

The series of negative integral powers $(z-a)$ namely $\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$ is known as the principal part of the Laurent's series of $f(z)$. If the principal part contains a finite number of terms, say m, them the singular point $z=a$ is called a pole of order m of $f(z)$.

Simple pole:

simple pole is a pole of order one

iii) essential singularity:

If the principal part of $f(z)$ contains an infinite number of terms i.e the series $\sum_{n=1}^{\infty}\frac{b_n}{(z-a)^n}$ contains an infinite no. of terms, then the point $z=a$ is called essential singularity of f(z).

(iv) Removable singularity:

If the principal part of f(z) contains no term i.e, if $b_n = 0 \ \forall n$, then the singularity z=a is called removable singularity of f(z). in this case

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n.$$

Ex:- $f(z) = \frac{1-\cos z}{z}$, then z=0 is a removable singularity.

Def: Residues.

The coefficient of $\frac{1}{z-a}$ in the laurent's expansion of f(z) about the isolated similarity z=a is called the residue of f(z) at the point

$\therefore$ Residue of f(z) at $z=a = \frac{1}{2\pi i}\int_c f(z)dz$

i.e., $\int_c f(z)dz = 2\pi i \times$ [Residue of f(z) at z=a]

$\therefore \int_c f(z)dz = 2\pi i [\text{Res} f(z)]_{z=a}$.

where 'c' is a closed curve containing the point z=a

Cauchy's Residue theorem:

Statement: If f(z) is analytic with in and on a closed curve 'c' except at a finite no. of poles $a_1, a_2, \ldots a_n$ with in 'c' & $R_1, R_2 \ldots R_n$ be the residue of the f(z) at these poles,

then $\int_c f(z)\,dz = 2\pi i [R_1 + R_2 + \cdots + R_n]$

(or)

$\int_c f(z)dz = 2\pi i$ [Sum of the residues at the poles with in 'c']

Note!

1) when $z=z_0$ is a simple pole.

In this case the laurent's expansion becomes

① $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + a_{-1}(z-z_0)^{-1}$

(2) $\therefore \lim_{z \to z_0} (z-z_0) f(z) = a_1$

(3) $\therefore [\text{Res} f(z)]_{z=z_0} = \lim_{z \to z_0} (z-z_0) f(z)$

2. If $f(z)$ is are the form $f(z) = \frac{\phi(z)}{\psi(z)}$ where $\psi(z)=0$, but $\psi'(z_0) \neq 0$

(4) $\underset{\text{at } z=z_0}{\text{Res} f(z)} = \frac{\phi(z_0)}{\psi'(z_0)} = \lim_{z \to z_0} \left[\frac{\phi(z)}{\psi'(z)} \right]$

Residue at a pole of order m:-

If $f(z)$ is analytic within a curve 'c' and have a pole of order 'm' at $z=z_0$ then the residue at $z=z_0$ is.

(5) $\lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left[(z-z_0)^m f(z) \right]$

Problems related to Poles & Residues

1) Determine the poles of the function.

i) $\frac{z}{\cos z}$ ii) $\cot z$

Sol: the poles of $f(z) = \frac{z}{\cos z}$ are given by

$\cos z = 0$

i.e., $z = (2n+1)\frac{\pi}{2}$, n being zero or an integer

i.e., $z = (2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

Hence these are simple poles of $f(z)$.

ii) the poles of $f(z) = \cot z = \frac{\cos z}{\sin z}$ are given by

$\sin z = 0$ i.e., $z = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$

which are simple poles of $f(z)$

2) find zeros and poles of $\left[\frac{z+1}{z^2+1} \right]^n$

Sol: let $f(z) = \left[\frac{z+1}{z^2+1} \right]^n = \frac{(z+1)^n}{(z^2+1)^n}$

a) zeros of the $f(z)$ are given by

$(z+1)^2 = 0$ ie, $z = -1, -1$

$\therefore z = -1$ is a zero of order '2'

b) poles of $f(z)$ are obtained by putting the denominator equal to zero

ie., $(z^2+1)^2 = 0$ ie. $(z^2 - i^2)^2 = 0$

ie., $(z-i)^2 (z+i)^2 = 0$

ie., $z = i, i$
$-i, -i$

$\therefore z = i$ & $z = -i$ both are poles of order 2

③ find the poles of the function $f(z) = \dfrac{1}{(z+1)(z+3)}$ and residues at these poles.

Sol:- The Given function $f(z)$ ~~lies~~ has two simple poles and $z=-1$ & $z=-3$

$\therefore$ Residues of $f(z)$ at $z=-1$ is $\lim_{z\to -1} \{[z-(-1)] f(z)\}$

$$= \lim_{z\to -1} [(z+1) f(z)] = \lim_{z \to -1} \left[\frac{1}{z+3}\right] = \frac{1}{2}$$

Also the residue of $f(z)$ at $z=-3$ is $\lim_{z\to -3} (z+3) f(z)$

$$= \lim_{z \to -3} \frac{1}{z+1} = \frac{-1}{2}$$

④ Determine the poles of the function $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ & the residue at each pole.

Sol:- $z=1$ & $z=-2$ are the zeros of the denominator of order 2 &,

$\therefore z=1$ is a pole of order 2 & $z=-2$ is a pole of order 1 of $f(z)$

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$\therefore \left[\operatorname{Res} f(z)\right]_{z=-2} = \lim_{z\to -2} (z+2) f(z) = \lim_{z\to -2} \cancel{(z+2)} \frac{z^2}{(z-1)^2 (z+2)}$

$$= \lim_{z\to -2} \frac{z^2}{(z-1)^2} = \frac{4}{9}$$

$\therefore \left[\operatorname{Res} f(z)\right]_{z=1}$ formula of pole of order 'm'

$$= \lim_{z\to 1} \frac{1}{1!} \frac{d}{dz}\left[(z-1)^2 f(z)\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left[\cancel{(z-1)^2} \frac{z^2}{\cancel{(z-1)^2}(z+2)}\right]$$

$$= \lim_{z\to 1} \frac{d}{dz}\left(\frac{z^2}{z+2}\right)$$

$$= \lim_{z\to 1}\left[\frac{(z+2)\,2z - z^2}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^2+4z}{(z+2)^2}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^2+4z}{(z+2)^2}\right] = \frac{5}{9}$$

find the residue of $\frac{ze^z}{(z-1)^3}$ at this pole.

let $f(z) = \frac{ze^z}{(z-1)3}$

i.e, we find that $z=1$ is a pole of the $f(z)$ is order: 3

i.e we know that if $f(z)$ has a pole of order m at $z=a$.

Then $\left[\operatorname{Res} f(z)\right]_{z=0} = \frac{1}{(m+1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$

Here $a=1, m=3$

$$\left[\operatorname{Res} f(3)\right]_{z=1} = \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2}\left[(z-1)^3 f(z)\right]$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d^2}{dz^2}(ze^z)$$

$$= \frac{1}{2} \lim_{z\to 1} \frac{d}{dz}(ze^z + e^z)$$

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$= \frac{1}{2} \lim_{z \to 1} (ze^z + e^z + e^2)$

$= \frac{1}{2} \, e(2)$

$= \frac{3e}{2}$

Evaluate $\oint_c \frac{2z-1}{z(z+2)(2z+1)} dz$ where 'c' is the circle $|z|=1$

Sol:- Here $f(z) = \frac{2z-1}{z(z+2)(2z+1)}$ has 3 simple poles at $z=0, z=-2$ & $z=\frac{-1}{2}$

But the only poles $z=0, z=-\frac{1}{2}$ lies inside the circle $|z|=1$.

Now, the residue of $f(z)$ at $z=0$ is

$\therefore Res(f; z=0) = \lim_{z \to 0} z \cdot f(z) = \lim_{z \to 0} \frac{2z-1}{(z+2)(2z+1)} = \frac{-1}{2 \cdot 1} = -\frac{1}{2}$

Also, the residue of $f(z)$ at $z=-\frac{1}{2}$ is

$Res[f; z=-\frac{1}{2}] = \lim_{z \to -\frac{1}{2}} (2z+1) f(z)$

$= \lim_{z \to -\frac{1}{2}} \frac{2z-1}{z(z+2)}$

$= \frac{-2}{-3/4}$

$= \frac{8}{3}$

$\therefore$ By Residue theorem, we have.

$\oint_c \frac{2z-1}{z(z+2)(2z+1)} = 2\pi i \left[-\frac{1}{2} + \frac{8}{3}\right]$

$= \frac{13}{3} \pi i$