UNIT-II

MAGNETOSTATICS

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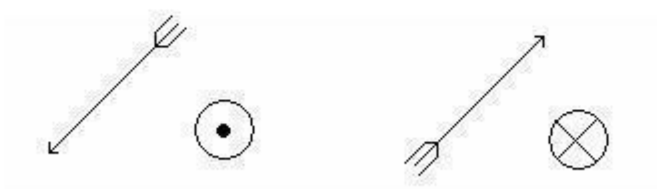
Introduction:

The source of steady magnetic field may be a permanent magnet, a direct current or an electric field changing with time. In this chapter we shall mainly consider the magnetic field produced by a direct current. The magnetic field produced due to time varying electric field will be discussed later.

There are two major laws governing the magneto static fields are:

- Biot-Savart Law
- Ampere's Law

Usually, the magnetic field intensity is represented by the vector $\vec{H}$. It is customary to represent the direction of the magnetic field intensity (or current) by a small circle with a dot or cross sign depending on whether the field (or current) is out of or into the page as shown in Fig. 2.1.

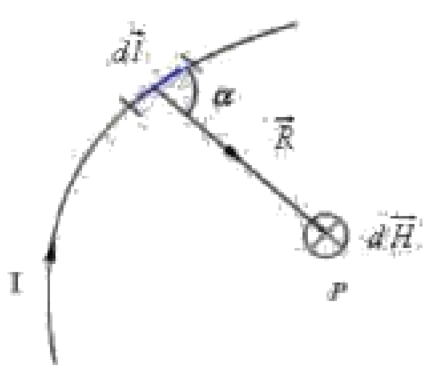


$\vec{H}$ (or l) out of the page $\vec{H}$ (or l) into the page

Fig. Representation of magnetic field (or current)

Biot- Savart's Law:

This law relates the magnetic field intensity dH produced at a point due to a differential current element $Id\vec{l}$ as shown in Fig.



The magnetic field intensity $d\vec{H}$ at P can be written as,

$$d\vec{H} = \frac{Id\vec{l} \times \hat{a}_R}{4\pi R^2} = \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$

$$dH = \frac{IdlSin\alpha}{4\pi R^2}$$

where $R = |\vec{R}|$ is the distance of the current element from the point P.
The value of the constant of proportionality 'K' depends upon a property called permeability of the medium around the conductor. Permeability is represented by symbol 'm' and the constant 'K' is expressed in terms of 'm' as

Thus

$$dB = \frac{\mu}{4\pi} \frac{I\,dl\ \sin\theta}{r^2}$$

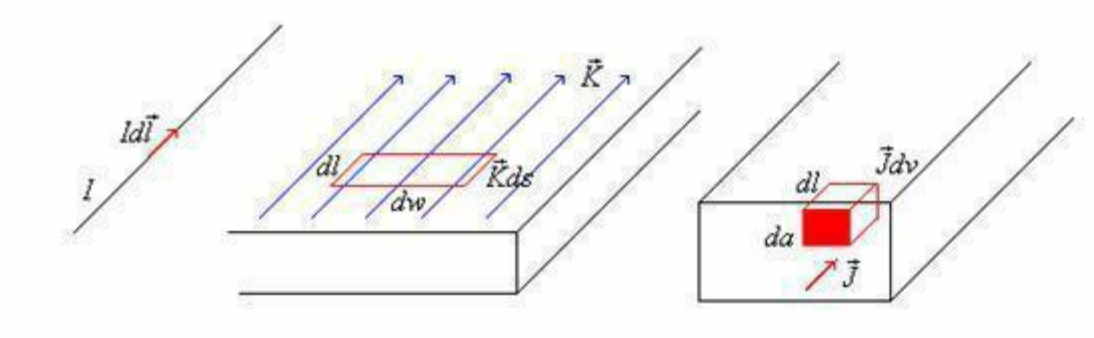
Magnetic field 'B' is a vector and unless we give the direction of 'dB', its description is not complete. Its direction is found to be perpendicular to the plane of 'r' and 'dl'.

If we assign the direction of the current 'I' to the length element 'dl', the vector product dl x r has magnitude r dl sinq and direction perpendicular to 'r' and 'dl'.

Hence, Biot–Savart law can be stated in vector form to give both the magnitude as well as direction of magnetic field due to a current element as

$$\vec{dB} = \frac{\mu}{4\pi} \frac{I\,(\vec{dl} \times \vec{r})}{r^3}$$

Similar to different charge distributions, we can have different current distribution such as line current, surface current and volume current. These different types of current densities are shown in Fig. 2.3.



Line Current Surface Current Volume Current

Fig. 2.3: Different types of current distributions

By denoting the surface current density as K (in amp/m) and volume current density as J (in amp/m2) we can write:

$$Id\vec{l} = \vec{K}ds = \vec{J}dv$$

(It may be noted that $I = Kdw = Jda$)

Employing Biot -Savart Law, we can now express the magnetic field intensity H. In terms of these current distributions as

$$\vec{H} = \int_l \frac{Id\vec{l} \times \vec{R}}{4\pi R^3}$$ .... for line current...........................

$$\vec{H} = \int_s \frac{Kd\vec{s} \times \vec{R}}{4\pi R^3}$$ .... for surface current ....

$$\vec{H} = \int_v \frac{\vec{J}dv \times \vec{R}}{4\pi R^3}$$ .... for volume current.....................

$\bar{H}$ Due to infinitely long straight conductor:

We consider a finite length of a conductor carrying a current $\vec{I}$ placed along z-axis as shown in the Fig 2.4. We determine the magnetic field at point P due to this current carrying conductor.

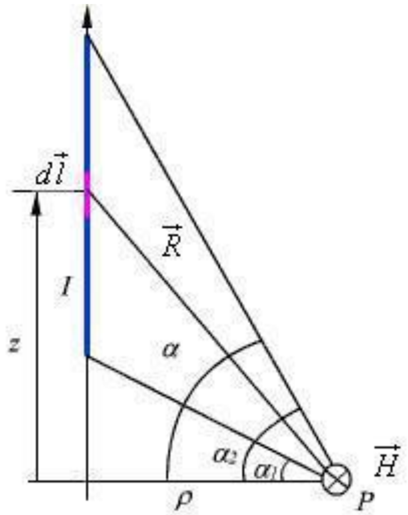


Fig. 2.4: Field at a point P due to a finite length current carrying conductor

With reference to Fig. 2.4, we find that

$$d\vec{l} = dz\,\hat{a}_z \quad and \quad \vec{R} = \rho\,\hat{a}_\rho - z\,\hat{a}_z$$

Applying Biot - Savart's law for the current element $\vec{I}\,d\vec{l}$ We can write,

$$\vec{dH} = \frac{I d\vec{l} \times \vec{R}}{4\pi R^3} = \frac{\rho dz \hat{a}_\phi}{4\pi[\rho^2 + z^2]^{3/2}}$$

Substituting $\frac{z}{\rho} = \tan\alpha$ we can write,

$$\vec{H} = \int_{\alpha_1}^{\alpha_2} \frac{I}{4\pi} \frac{\rho^2 \sec^2 \alpha d\alpha}{\rho^3 \sec^3 \alpha} \hat{a}_\phi = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\hat{a}_\phi$$

We find that, for an infinitely long conductor carrying a current I , $\alpha_2 = 90^0$ and $\alpha_1 = -90^0$ Therefore

$$\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$$

Ampere's Circuital Law:

Ampere's circuital law states that the line integral of the magnetic field $\vec{H}$ (circulation of H) around a closed path is the net current enclosed by this path. Mathematically,

$$\oint \vec{H}.d\vec{l} = I_{enc}$$

The total current I enc can be written as,

$$I_{enc} = \int_S \vec{J}.d\vec{s}$$

By applying Stoke's theorem, we can write

$$\oint \vec{H}.d\vec{l} = \int_S \nabla \times \vec{H}.d\vec{s}$$

$$\therefore \int_S \nabla \times \vec{H}.d\vec{s} = \int_S \vec{J}.d\vec{s}$$

$$\therefore \nabla \times \vec{H} = \vec{J}$$

Which is the Ampere's circuital law in the point form and Maxwell’s equation for magneto static fields.

Applications of Ampere's circuital law:

1. It is used to find $\bar{H}$ and $\bar{B}$ due to any type of current distribution.
2. If $\bar{H}$ or $\bar{B}$ is known then it is also used to find current enclosed by any closed path.

We illustrate the application of Ampere's Law with some examples.

$\bar{H}$ Due to infinitely long straight conductor :(using Ampere's circuital law)

We compute magnetic field due to an infinitely long thin current carrying conductor as shown in Fig. 2.5. Using Ampere's Law, we consider the close path to be a circle of radius ρ as shown in the Fig. 4.5.

If we consider a small current element $Id\vec{l}(=Idz\hat{a}_z)$, $d\vec{H}$ is perpendicular to the plane containing both $d\vec{l}$ and $\vec{R}(=\rho\hat{a}_\rho)$. Therefore only component of $\vec{H}$ that will be present is H_ϕ,i.e., $\vec{H}=H_\phi\hat{a}_\phi$.

By applying Ampere's law we can write,

$$\vec{H}=\frac{I}{2\pi\rho}\hat{a}_\phi \quad \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho 2\pi = I$$

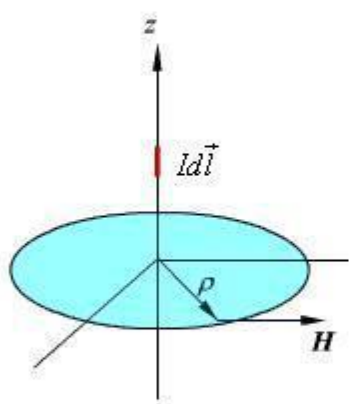


Fig. Magnetic field due to an infinite thin current carrying conductor

$\bar{H}$ Due to infinitely long coaxial conductor :(using Ampere's circuital law)

We consider the cross section of an infinitely long coaxial conductor, the inner conductor carrying a current I and outer conductor carrying current - I as shown in figure 2.6. We compute the magnetic field as a function of ρ as follows:

In the region $0 \le \rho \le R_1$

$$I_{enc} = I\frac{\rho^2}{R_1^2}$$

$$H_\phi = \frac{I_{enc}}{2\pi\rho} = \frac{I\rho}{2\pi a^2}$$

In the region $R_1 \le \rho \le R_2$

$$I_{enc} = I$$

$$H_\phi = \frac{I}{2\pi\rho}$$

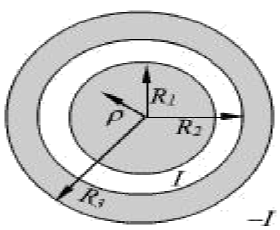


Fig. 2.6: Coaxial conductor carrying equal and opposite currents in the region

$R_2 \le \rho \le R_3$

$$H_\phi = \frac{I}{2\pi\rho}\frac{R_3^2 - \rho^2}{R_3^2 - R_2^2}$$

In the region $\rho > R_3$

$$I_{nec} = 0 \qquad H_\phi = 0$$

Magnetic Flux Density:

In simple matter, the magnetic flux density $\vec{B}$ related to the magnetic field intensity $\vec{H}$ as $\vec{B} = \mu\vec{H}$ where μ called the permeability. In particular when we consider the free space $\vec{B} = \mu_0\vec{H}$ where $\mu_0 = 4\pi\times10^{-7}$ H/m is the permeability of the free space. Magnetic flux density is measured in terms of Wb/m 2 .

The magnetic flux density through a surface is given by:

$$\psi = \int_s \vec{B}.d\vec{s} \quad \text{Wb}$$

In the case of electrostatic field, we have seen that if the surface is a closed surface, the net flux passing through the surface is equal to the charge enclosed by the surface. In case of magnetic field isolated magnetic charge (i. e. pole) does not exist. Magnetic poles always occur in pair (as N-S). For example, if we desire to have an isolated magnetic pole by dividing the magnetic bar successively into two, we end up with pieces each having north (N) and south (S) pole as shown in Fig. 6 (a). This process could be continued until the magnets are of atomic dimensions; still we will have N-S pair occurring together. This means that the magnetic poles cannot be isolated.

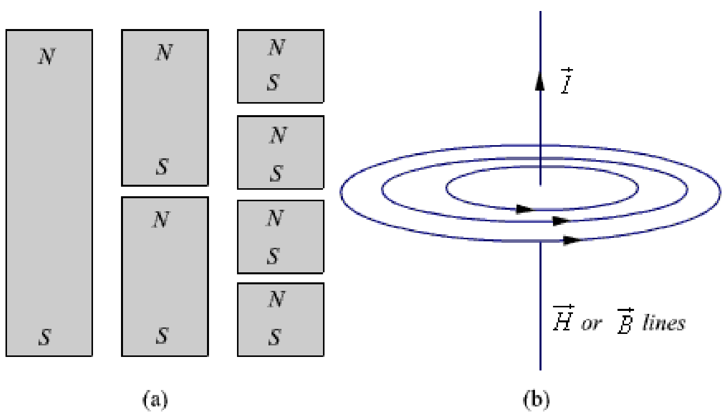


Fig. 6: (a) Subdivision of a magnet (b) Magnetic field/ flux lines of a straight current carrying conductor

Maxwell's 2nd equation for static magnetic fields:

Similarly if we consider the field/flux lines of a current carrying conductor as shown in Fig. 6 (b), we find that these lines are closed lines, that is, if we consider a closed surface, the number of flux lines that would leave the surface would be same as the number of flux lines that would enter the surface.

From our discussions above, it is evident that for magnetic field,

$\oint_S \vec{B}.d\vec{s} = 0$ ....in integral form

which is the Gauss's law for the magnetic field.

By applying divergence theorem, we can write:

$$\oint_S \vec{B}.d\vec{s} = \int_V \nabla.\vec{B}dv = 0$$

Hence, $\nabla.\vec{B} = 0$ .... in point/differential form

which is the Gauss's law for the magnetic field in point form.

Magnetic Scalar and Vector Potentials:

In studying electric field problems, we introduced the concept of electric potential that simplified the computation of electric fields for certain types of problems. In the same manner let us relate the magnetic field intensity to a scalar magnetic potential and write:

$$\vec{H} = -\nabla V_m$$

From Ampere's law , we know that

$$\nabla \times \vec{H} = \vec{J}$$

Therefore, $\nabla \times (-\nabla V_m) = \vec{J}$

But using vector identity, $\nabla \times (\nabla V) = 0$ we find that $\vec{H} = -\nabla V_m$ is valid only where $\vec{J} = 0$.

Thus the scalar magnetic potential is defined only in the region where $\vec{J} = 0$. Moreover, Vm in general is not a single valued function of position. This point can be illustrated as follows. Let us consider the cross section of a coaxial line as shown in fig 7.

In the region $a < \rho < b$, $\vec{J} = 0$ and $\vec{H} = \frac{I}{2\pi\rho}\hat{a}_\phi$

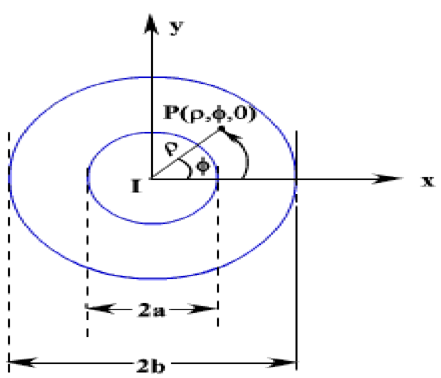


Fig. 7: Cross Section of a Coaxial Line

If Vm is the magnetic potential then,

$$-\nabla V_m = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}$$
$$= \frac{I}{2\pi\rho}$$

If we set Vm = 0 at $\phi = 0$ then c=0 and $V_m = -\frac{I}{2\pi}\phi$

$$\therefore \text{At } \phi = \phi_0 \quad V_m = -\frac{I}{2\pi}\phi_0$$

We observe that as we make a complete lap around the current carrying conductor , we reach ϕ_0 again but Vm this time becomes

$$V_m = -\frac{I}{2\pi}(\phi_0 + 2\pi)$$

We observe that value of Vm keeps changing as we complete additional laps to pass through the same point. We introduced Vm analogous to electostatic potential V.

But for static electric fields,

$\nabla \times \vec{E} = 0$ and $\oint \vec{E} \cdot d\vec{l} = 0$

whereas for steady magnetic field $\nabla \times \vec{H} = 0$ wherever $\vec{J} = 0$ but $\oint \vec{H} \cdot d\vec{l} = I$ even if $\vec{J} = 0$ along the path of integration.

We now introduce the vector magnetic potential which can be used in regions where current density may be zero or nonzero and the same can be easily extended to time varying cases. The use of vector magnetic potential provides elegant ways of solving EM field problems.

Since $\nabla . \vec{B} = 0$ and we have the vector identity that for any vector $\vec{A}$, $\nabla . (\nabla \times \vec{A}) = 0$, we can write $\vec{B} = \nabla \times \vec{A}$.

Here, the vector field $\vec{A}$ is called the vector magnetic potential. Its SI unit is Wb/m. Thus if can find $\vec{A}$ of a given current distribution, $\vec{B}$ can be found from $\vec{A}$ through a curl operation. We have introduced the vector function $\vec{B}$ and $\vec{A}$ related its curl to $\vec{B}$. A vector function is defined fully in terms of its curl as well as divergence. The choice of $\nabla . \vec{A}$ is made as follows.

$$\nabla \times \nabla \times \vec{A} = \mu \nabla \times \vec{H} = \mu \vec{J}$$

By using vector identity, $\nabla\times\nabla\times\vec{A} = \nabla(\nabla.\vec{A}) - \nabla^2\vec{A}$

$$\nabla(\nabla.\vec{A}) - \nabla^2\vec{A} = \mu\vec{J}$$

Great deal of simplification can be achieved if we choose $\nabla.\vec{A} = 0$.

Putting $\nabla.\vec{A} = 0$, we get $\nabla^2\vec{A} = -\mu\vec{J}$ which is vector poisson equation.

In Cartesian coordinates, the above equation can be written in terms of the components as

$$\nabla^2 A_x = -\mu J_x .$$

$$\nabla^2 A_y = -\mu J_y$$

$$\nabla^2 A_z = -\mu J_z$$

The form of all the above equation is same as that of

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

for which the solution is

$$V = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho}{R}dv', \qquad R = \left|\vec{r} - \vec{r}'\right|$$

$$\nabla.\vec{A} = \mu\varepsilon\frac{\partial V}{\partial t}$$

In case of time varying fields we shall see that , which is known as Lorentz condition, V being the electric potential. Here we are dealing with static magnetic field, so $\nabla.\vec{A} = 0$.

By comparison, we can write the solution for Ax as

$$A_x = \frac{\mu}{4\pi}\int_{V'}\frac{J_x}{R}dv'$$

Computing similar solutions for other two components of the vector potential, the vector potential can be written as

$$\vec{A} = \frac{\mu}{4\pi}\int_{V'}\frac{\vec{J}}{R}dv'$$

This equation enables us to find the vector potential at a given point because of a volume current density $\vec{J}$.

Similarly for line or surface current density we can write

$$\vec{A} = \frac{\mu}{4\pi}\int_{L'}\frac{I}{R}d\vec{l}'$$

$$\vec{A} = \frac{\mu}{4\pi}\int_{s'} \frac{\vec{K}}{R} ds'$$

.

The magnetic flux ψ through a given area S is given by

$$\psi = \int_S \vec{B}.d\vec{s}$$

Substituting $\vec{B} = \nabla \times \vec{A}$

$$\psi = \int_S \nabla \times \vec{A}.d\vec{s} = \oint_C \vec{A}.d\vec{l}$$

Vector potential thus have the physical significance that its integral around any closed path is equal to the magnetic flux passing through that path.

Forces due to magnetic fields

There are at least three ways in which force due to magnetic fields can be experienced. The force can be (a) due to a moving charged particle in a (B) field,
(b) on a current element in an external (B) field, or (c) between two currentelements.

Force on a Charged Particle

The electric force (Fe) on a moving electric charge (Q) in an electric field is given by Coulomb's experimental law and is related to the electric field intensity (E) as

$$F_e = Q\,E \text{--------(1)}$$

This shows that if Q is positive, F_e and E have the same direction.
A magnetic field can exert force only on a moving charge. From experiments, itis found that the magnetic force (Fm) experienced by a charge (Q) moving with avelocity (u) in a magnetic field (B) is

$$F_m = Q\,u \times B \text{-------(2)}$$

This clearly shows that (Fm) is perpendicular to both (u) and (B).
From eqs. (1) and (2), a comparison between the electric force F_e and the magnetic force $\mathbf{F_m}$ can be made. $\mathbf{F_e}$ is independent of the velocity of the charge and can perform work on the charge and change its kinetic energy. Unlike $\mathbf{F_e}$, $\mathbf{F_m}$ depends on the charge velocity and is normal to it. $\mathbf{F_m}$ cannot perform work because its at right angles to the direction of motion of the charge (Fm**.** **dl** = 0); it does not cause an increase in kinetic energy of the charge. The magnitudeof $\mathbf{F_m}$ is generally small compared to $\mathbf{F_e}$ except at high velocities.
For a moving charge **Q** in the presence of both electric and magnetic fields, the total force on the charge is given by

$$F = F_e + F_m$$

Or

$$\boxed{F = Q\,(E + u \times B)}$$

This is known as the (Lorentz ***force equation)***. It relates mechanical force to electrical force. If the mass of the charged particle moving in **E** and **B** fields is m
, by Newton's second law of motion.

$$F = m\frac{du}{dt} = Q\,(E + u \times B)$$

Force on a Current Element

To determine the force on a current element (I **dl)** of a current – carrying conductor due to the magnetic field (B**)**, we modify eq. (2) using the fact that for convection current

$$J = \rho_V u$$

To recall the relationship between current elements:

$$\boldsymbol{I}\, dl = \mathbf{K}\, ds = \mathbf{J}\, dv$$

Combining eqs. (5 – 5) and (5 – 6) yields

$$I\, dl = \rho_V\, u\, dv = d\, Q\, u$$

Alternatively , $I\, dl = \frac{d\,Q}{dt}\, dl = d\,Q\, \frac{dl}{dt} = dQ\, u$

Hence ,

$$\boldsymbol{I}\, \mathbf{dl} = dQ\, \mathbf{u}$$

This shows that an elemental charge (d Q) moving with velocity **u** (thereby producing convection current element d Q **u)** is equivalent to a conduction current element ***I*** **dl** . Thus, the force on current element ***I*** **dl** in a magnetic field **B** is found from eq. (2) by merely replacing Q **u** by ***I*** **dl;** that is,

$$d\, F = I\, dl \times B$$

If the current ***I*** is through a closed path **L** or circuit, the force on the circuit is given by

$$F = \oint_L I\, dl \times B$$

Also have surface current elements (K d **S**) or a volume current element(**J** d **v**)

$$F = \int_S K\, d\, S \times B \quad , \quad F = \int_V J\, d\, v \times B$$

Force between Two Current Elements

The force between two elements $I_1\, dl_1$ and $I_2\, dl_2$. According to Biot – Savart's law , both current element produce magnetic fields. So may find the force d (d F_1) on element $I_1\, dl_1$ due to the field d B_2 produced by element $I_2\, dl_2$ as shown in figure (5 – 1) . From eq. (5 – 8),

$$d\,(d\,F_1) = I_1\,d\,l_1 \times d\,B_2$$

But from Biot – Savart's law,

$$d\,B_2 = \frac{\mu_0 I_2\ d\,l_2 \times a_{R_{21}}}{4hR_{21}^2}$$

Hence

$$d\,(d\,F_1) = \frac{\mu_0 I_1\ d\,l_1 \times (I_2\ d\,l_2 \times a_{R_{21}})}{4hR_{21}^2}$$

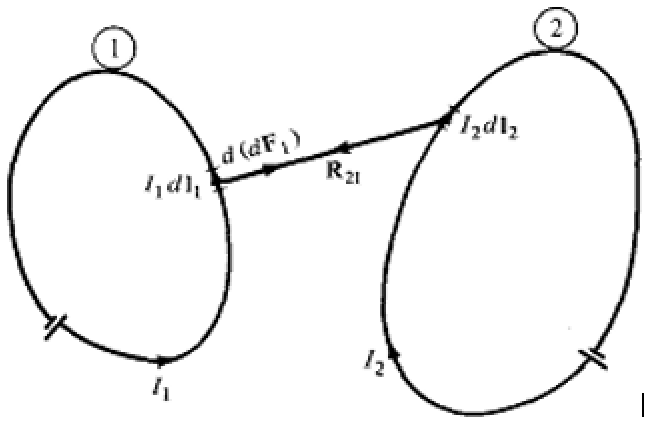


Fig. Force between Two Current Elements

This equation is essentially the law of force between two current element and is analogous to Coulomb's law, which expresses the force between two stationary charges. From eq. (5 – 12) , can to obtain the total force **F1** on current loop (1) due to current loop (2) shown in figure (5 – 1) as

$$F_1 = \frac{\mu_0 I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\,l_1 \times (d\,l_2 \times a_{R_{21}})}{R_{21}^2}$$

The force **F2** on loop (2) due to the magnetic field **B1** from loop (1) is obtained from above eq. by interchanging subscripts **1** and **2** . It can be shown that **F2** = **- F1;**

Maxwell's Equations (Time Varying Fields)

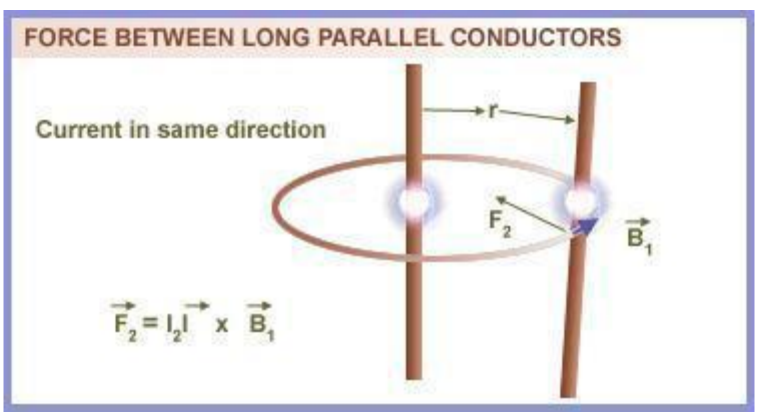
Faraday's Law:

Michael Faraday, in 1831 discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In terms of fields, we can say that a time varying magnetic field produces an electromotive force (emf) which causes a current in a closed circuit. The quantitative relation between the induced emf (the voltage that arises from conductors moving in a magnetic field or from changing magnetic fields) and the rate of change of flux linkage developed based on experimental observation is known as Faraday's law.

Any change in the magnetic environment of a coil of wire will cause a voltage (emf) to be "induced" in the coil. No matter how the change is produced, the voltage will be generated. The change could be produced by changing the magnetic field strength, moving a magnet toward or away from the coil, moving the coil into or out of the magnetic field, rotating the coil relative to the magnet, etc.

Faraday's law is a fundamental relationship which comes from Maxwell's equations. It serves as a succinct summary of the ways a voltage (or emf) may be generated by a changing magnetic environment. The induced emf in a coil is equal to the negative of the rate of change of magnetic flux times the number of turns in the coil. It involves the interaction of charge with magnetic field.

When two current carrying conductors are placed next to each other, we notice that each induces a force on the other. Each conductor produces a magnetic field around itself (Biot– Savart law) and the second experiences a force that is given by the Lorentz force.



Mathematically, the induced emf can be written as

$$\text{Emf} = -\frac{d\phi}{dt} \quad \text{Volts}$$

where ϕ is the flux linkage over the closed path. A non zero $\frac{d\phi}{dt}$ may result due to any of the following:

(a) time changing flux linkage a stationary closed path.

(b) relative motion between a steady flux a closed path.

(c) a combination of the above two cases.

The negative sign in equation (7) was introduced by Lenz in order to comply with the polarity of the induced emf. The negative sign implies that the induced emf will cause a current flow in the closed loop in such a direction so as to oppose the change in the linking magnetic flux which produces it. (It may be noted that as far as the induced emf is concerned, the closed path forming a loop does not necessarily have to be conductive).

If the closed path is in the form of N tightly wound turns of a coil, the change in the magnetic flux linking the coil induces an emf in each turn of the coil and total emf is the sum of the induced emfs of the individual turns, i.e.,

Emf = $-N\frac{d\phi}{dt}$ Volts

By defining the total flux linkage as

$$\lambda = N\phi$$

The emf can be written as

Emf = $-\frac{d\lambda}{dt}$

Continuing with equation (3), over a closed contour 'C' we can write

Emf = $\oint_C \vec{E}.d\vec{l}$

where $\vec{E}$ is the induced electric field on the conductor to sustain the current.

Further, total flux enclosed by the contour 'C ' is given by

$$\phi = \int_S \vec{B}.d\vec{s}$$

Where S is the surface for which 'C' is the contour.

From (11) and using (12) in (3) we can write

$$\oint_C \vec{E}.d\vec{l} = -\frac{\partial}{\partial t}\oint_S \vec{B}.d\vec{s}$$

By applying stokes theorem

$$\int_S \nabla \times \vec{E}.d\vec{s} = -\int_S \frac{\partial \vec{B}}{\partial t}.d\vec{s}$$

Therefore, we can write

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

which is the Faraday's law in the point form

We have said that non zero $\frac{d\phi}{dt}$ can be produced in a several ways. One particular case is when a time varying flux linking a stationary closed path induces an emf. The emf induced in a stationary closed path by a time varying magnetic field is called a transformer emf .

Displacement Current Density:

The equation

$\Delta \times H = J$ For static EM fields is modified to Modified to

$\Delta \times H = J + J_d$ ________________ (3.19)

To make the Ampere's law compatible for varying fields.

Now, applying divergence, we get

$$\Delta.(\Delta \times H) = 0 = \Delta.J + \Delta J_d$$

$$\Delta.J_d = -\Delta.J = \frac{de_v}{dt}$$

From Gauss Law, we have

$$e_v = \Delta.D$$

Therefore,

$$\Delta.J_d = \frac{d(\Delta.D)}{dt} = \Delta.\frac{dD}{dt}$$

$\Rightarrow J_d = \frac{dD}{dt}$ ___________ (3.20)

MAXWELL'S EQUATIONS (Time varying Fields)

Introduction:

In our study of static fields so far, we have observed that static electric fields are produced by electric charges, static magnetic fields are produced by charges in motion or by steady current. Further, static electric field is a conservative field and has no curl, the static magnetic field is continuous and its divergence is zero. The fundamental relationships for static electric fields among the field quantities can be summarized as:

$$\nabla \times \vec{E} = 0 \quad (1)$$

$$\nabla \cdot \vec{D} = \rho_v \quad (2)$$

For a linear and isotropic medium,

$$\vec{D} = \varepsilon \vec{E} \quad (3)$$

Similarly for the magnetostatic case

$$\nabla . \vec{B} = 0 \quad (4)$$

$$\nabla \times \vec{H} = \vec{J} \quad (5)$$

$$\nabla \times \vec{H} = \vec{J} \quad (6)$$

It can be seen that for static case, the electric field vectors $\vec{E}$ and $\vec{D}$ and magnetic field vectors $\vec{B}$ and $\vec{H}$ form separate pairs.

Maxwell's equations represent one of the most elegant and concise ways to state the fundamentals of electricity and magnetism. From them one can develop most of the working relationships in the field. Because of their concise statement, they embody a high level of mathematical sophistication and are therefore not generally introduced in an introductory treatment of the subject, except perhaps as summary relationships.

These basic equations of electricity and magnetism can be used as a starting point for advanced courses, but are usually first encountered as unifying equations after the study of electrical and magnetic phenomena.

Symbols Used

E = Electric field	ρ = charge density	i = electric current
B = Magnetic field	ε0 = permittivity	J = current density
D = Electric displacement	μ0 = permeability	c = speed of light
H = Magnetic field strength	M = Magnetization	P = Polarization

Integral form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_0}$

Gauss' law for magnetism $\oint \vec{B} \cdot d\vec{A} = 0$

III. Faraday's law of induction $\oint \vec{E} \cdot d\vec{s} = -\frac{d\Phi_B}{dt}$

IV. Ampere's law

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i + \frac{1}{c^2} \frac{\partial}{\partial t} \int \vec{E} \cdot d\vec{A}$$

Differential form in the absence of magnetic or polarizable media:

I. Gauss' law for electricity $\nabla \cdot E = \frac{\rho}{\varepsilon_0} = 4\pi k \rho$

Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\frac{\partial B}{\partial t}$

IV. Ampere's law

$$\nabla\, x\, B = \frac{4\pi k}{c^2} J + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$= \frac{J}{\varepsilon_0 c^2} + \frac{1}{c^2}\frac{\partial E}{\partial t}$$

$$k = \frac{1}{4\pi\varepsilon_0} = \text{Coulomb's constant} \qquad c^2 = \frac{1}{\mu_0 \varepsilon_0}$$

Differential form with magnetic and/or polarizable media:

I. Gauss' law for electricity $\nabla \cdot D = \rho$

$D = \varepsilon_0 E + P$ *General case*

$D = \varepsilon_0 E$ *Free space*

$D = \varepsilon E$ *Isotropic linear*

II. Gauss' law for magnetism $\nabla \cdot B = 0$

III. Faraday's law of induction $\nabla x\, E = -\frac{\partial B}{\partial t}$

IV. Ampere's law $\nabla\, x\, H = J + \frac{\partial D}{\partial t}$

$B = \mu_0 (H + M)$ *General case*

$B = \mu_0 H$ *Free space*

$B = \mu H$ *Isotropic linear magnetic medium*

Gauss's Law **(Magnetic fields)**	**Integral form:** $\underbrace{\mu_o \oint \vec{H} \cdot d\vec{S}}_{Left} = \underbrace{0}_{Right}$	**Left side:** The number of magnetic field lines – perpendicularly passing through a closed surface. **Right side:** Identically zero.	The total magnetic flux passing through any closed surface is zero. Flux enter the closed surface is same with the flux come out from the surface. The divergence of the magnetic field at any point is zero.
	Differential form: $\underbrace{\mu_o \vec{\nabla} \cdot \vec{H}}_{Left} = \underbrace{0}_{Right}$	**Left side:** Divergence of the magnetic field – the tendency of the field to "flow" away from a point than toward it. **Right side:** Identically zero.	

Faraday's Law	**Integral form:** $\underbrace{\oint_C \vec{E} \cdot d\vec{l}}_{Left} = \underbrace{-\mu_o \int_S \frac{\partial \vec{H}}{\partial t} \cdot d\vec{S}}_{Right}$	**Left side:** The circulation of the vector electric field, $\vec{E}$ around a closed path, *C*. **Right side:** The rate of change with time (*d*/*dt*) of magnetic field, through any surface, $\vec{S}$.	Changing magnetic flux through a surface induces an emf in any boundary path, *C* of that surface, and a changing magnetic field, $\vec{H}$ induces a circulating electric field.
	Differential form: $\underbrace{\vec{\nabla} \times \vec{E}}_{Left} = \underbrace{-\mu_o \frac{\partial \vec{H}}{\partial t}}_{Right}$	**Left side:** Curl of the electric field, – the tendency of the field lines to circulate around a point. **Right side:** The rate of change of the magnetic field, $\vec{H}$ over time (*d*/*dt*)	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time.

Ampere's Law	**Integral form:** $$\underbrace{\oint_C \vec{H}\cdot d\vec{l}}_{Left} = \underbrace{\int_S \left(\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}\right)\cdot d\vec{S}}_{Right}$$	**Left side:** The circulation of the magnetic field, $\vec{H}$ around a closed path, C. **Right side:** Two sources for the magnetic field, H; a steady conduction current, $\vec{J}_c$ and a changing electric field, $\vec{E}$ through any surface, bounded by closed path, C.	An electric current or a changing electric flux through a surface produces a circulating magnetic field around any path, C that bounds that surface.
	Differential form: $$\underbrace{\vec{\nabla}\times\vec{H}}_{Left} = \underbrace{\vec{J}_c + \varepsilon_o \frac{\partial \vec{E}}{\partial t}}_{Right}$$	**Left side:** Curl of the magnetic field, – the tendency of the field lines to circulate around a point. **Right side:** Two terms represent the electric current density, $\vec{J}_c$ and the time rate of change of the electric field, $\vec{E}$.	A circulating electric field, is produced by a magnetic field, $\vec{H}$ that changes with time. An electric current, or a changing electric field, through a surface produces a circulating magnetic field, $\vec{H}$ around any path that bounds that surface.

Inconsistency of amperes l

Ampere's circuit law states that the line integral of tangential component of H around a closed path is same as the net current Ienc enclosed by the path.

i.e.

$$\int H.dl = I_{enc}$$

By applying stoke's theorem,

$$\int H.dl \; becomes \int_s J.ds$$

$\therefore$ Therefore, $\Delta \times H = J$ ______ (3.14)

This is true in case of static EM fields.

But in case of time-varying fields, the above Ampere's law shows same inconsistency.

The inconsistency of ampere law for time varying fields is shown in two cases:

1. For static EM fields, we have

$$\Delta \times H = J$$

Applying divergence on both sides, we get,

$$\Delta.(\Delta \times H) = \Delta.J$$

But divergence of curl of a vector field is always zero.

Therefore,

$$\Delta.(\Delta \times H) = 0 = \Delta.J$$

The continuity of current equation is given by

$$\Delta.J = \frac{-dp_v}{dt}$$

Where J = Current density
e_v = Charge density

For static fields, no current is produced, therefore, $e_v = 0 \Rightarrow \Delta.J = 0$

Implies eq. 3.15 is satisfied but for time varying fields, current is produced and therefore,

$$\Delta..J = \frac{-de_v}{dt} \# 0 \text{ \_\_\_\_\_\_ \_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (3.16)$$

Eq. (3.15) and eq. (3.16) are contradicting each other.

This is an inconsistency of ampere's law and the Ampere's law must be modified for time varying fields.

2. Consider the typical example of where the surface passes between the capacitor plates.

The figure is shown below.

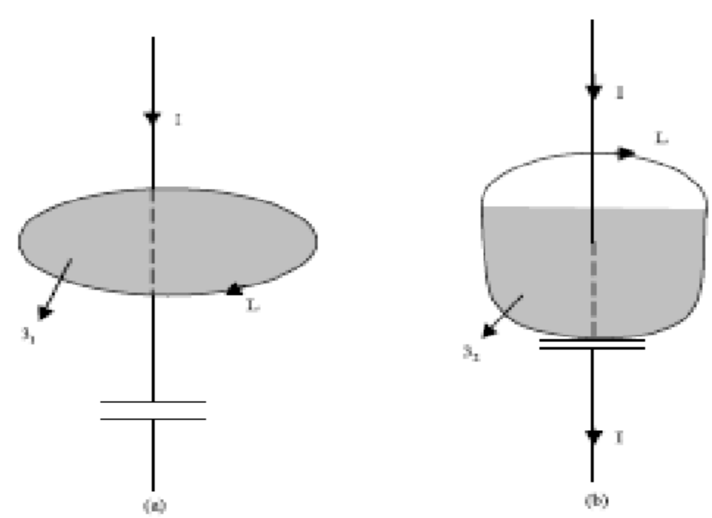


Fig 3.3 (a): Two surfaces of integration which explain the inconsistency of Ampere's law

In fig 3.3(a),

Based on Ampere's circuit law we get figure

$$\int_L H.dl = \int_{S_1} J.ds = I_{enc} = I \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ } (3.17)$$

In fig 3.3(b), based the ampere's circuit law, we get,

$$\int_L H.dl = \int_{3_2} J.ds = I_{enc} = 0 \quad \_\_\_\_\_\_\_\_\_\_\_\_ (3.18)$$

Because no conduction current flows through 3_2

i.e. J=0

in both (a) and (b), same closed path is used, but equations 3.17 and 3.18 are different.

This is an inconsistency of Ampere's circuit law.

This inconsistency of Ampere's circuit law in both cases (1) and (2) can be resolved by including displacement current in Ampere's circuit law.

Substituting in (3.19), we get,

$$\Delta \times H = J + \frac{dD}{dt} \quad \_\_\_\_\_\_\_\_\_\_\_\_ (3.21)$$

This is the Maxwell equation (based on ampere's circuit Law) for tiem varying fields.

In equation (3.21),

J_d = Displacement current density

J = Conduction current density,

The conduction current density J involves flow of charges. The displacement current density J_d does not involve flow of charges. Displacement current,

$$I_d = \int Jd.ds = \int \frac{do}{dt}.ds \quad \_\_\_\_\_\_\_\_\_\_\_\_ (3.22)$$

Solved problems:

Problem1:

(a) In a cylindrical conductor to the region $0.01 \le r \le 0.02$, $0 < z < 1$ m and the current density is given by,

$$\vec{J} = 10e^{-100r}\hat{a}_\phi \text{ A/m}^2$$

Find the total current crossing the extential of this region with φ = constant plane.

(b) Find the total current in a circular conductor of 4 mm radius if the current density varies according to $J = \frac{10^4}{r}$ A/m^2.

Solution

(a) Total current in the wire is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{r=0.01}^{0.02} \int_{z=0}^{1} \left[10e^{-100r}\hat{a}_\phi\right].\left[rdrdz\hat{a}_\phi\right]$$

$$= \int_{r=0.01}^{0.02} \int_{z=0}^{1} 10re^{-100r}drdz$$

$$= 10 \int_{r=0.01}^{0.02} re^{-100r}dr$$

$$I = 10\left[\frac{re^{-100r}}{-100}\Bigg|_{0.01}^{0.02} - \int_{r=0.01}^{0.02} \frac{e^{-100r}}{-100}dr\right]$$

$$= 10\left[-\frac{1}{100}(0.02e^{-2} - 0.01e^{-1}) + \frac{e^{-100r}}{-100\times 100}\Bigg|_{0.01}^{0.02}\right]$$

$$= 2\times 10^{-3}e^{-1}$$

$$= 310^{-3}e^{-2}$$

(b) Total current is given as,

$$I = \int_S \vec{J}.d\vec{S} = \int_{\phi=0}^{2\pi} \int_{r=0}^{0.004} \frac{10^4}{r} rdrd\phi = 2\pi \times 10^4 \int_{r=0}^{0.004} dr = 2\pi \times 10^4 \times 0.004 = 80\pi \text{ A}$$

Problem2:

If $\vec{J} = \frac{1}{r^3}(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta)$ A/m^2, calculate the current passing through

(a) A hemispherical shell of 20 cm radius

(b) A spherical shell of 10 cm radius

Solution

Total current is given as $I = \int \vec{J}.d\vec{S}$

Here, $d\vec{S} = r^2 \sin\theta\, d\phi\, d\theta\, \hat{a}_r$

(a) Total current passing through a hemispherical shell of 20 cm radius is,

$$I = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta).(r^2 \sin\theta\, d\phi\, d\theta\, \hat{a}_r)\Bigg|_{r=0.2}$$

$$= \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \frac{1}{r^3} 2\cos\theta r^2 \sin\theta\, d\phi\, d\theta\Bigg|_{r=0.2}$$

$$= 2\pi \times \frac{2}{r} \int_{\theta=0}^{\pi/2} \sin\theta d(\sin\theta)\Bigg|_{r=0.2}$$

$$= \frac{4\pi}{0.2}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 10\pi = 31.42 \text{ A}$$

(b) Total current passing through a spherical shell of 10 cm radius is,

$$I = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{r^3}(2\cos\theta\, \hat{a}_r + \sin\theta\, \hat{a}_\theta).(r^2 \sin\theta\, d\phi\, d\theta\, \hat{a}_r)\Bigg|_{r=0.1}$$

$$= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{1}{r^3} 2\cos\theta r^2 \sin\theta\, d\phi\, d\theta\Bigg|_{r=0.1}$$

$$= 2\pi \times \frac{2}{r} \int_{\theta=0}^{\pi} \sin\theta d(\sin\theta)\Bigg|_{r=0.1}$$

$$= \frac{4\pi}{0.1}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi}$$

$$= 0$$

Problem3:

For the current density, $\vec{J} = 10z\sin^2\phi\,\hat{a}_r$ A/m^2, find the current through the cylindrical surface of $r = 2$, $1 \le z \le 5$ m.

Solution

Total current passing through the cylindrical surface is,

$$I = \int \vec{J}.d\vec{S} = \int_{z=1}^{5}\int_{\phi=0}^{2\pi} (10z\sin^2\phi\,\hat{a}_r).(rd\phi dz\hat{a}_r)\Big|_{r=2} = 10r\left[\frac{z^2}{2}\right]_1^5 \int_{\phi=0}^{2\pi}\sin^2\phi d\phi\Big|_{r=2}$$

$$= 10\times 2\times\frac{24}{2}\times\frac{2\pi}{2} = 240\pi = 754\text{ A}$$

Problem4:

Determine the current density function $\vec{J}$ associated with the magnetic field defined by

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$ A/m (Cartesian)

(b) $\vec{H} = 6r\hat{a}_r + 2r\hat{a}_\phi + 5\hat{a}_z$ A/m (Cylindrical)

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$ A/m (Spherical)

(a) $\vec{H} = 3\hat{i} + 7\hat{j} + 2x\hat{k}$

By Ampere's law in Cartesian coordinates,

$$\vec{J} = \nabla\times\vec{H} = \begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3 & 7 & 2x \end{vmatrix} = -2\hat{a}_y\text{ A/m}^2$$

(b) By Ampere's law in cylindrical coordinates,

$$\vec{J} = \nabla\times\vec{H} = \begin{vmatrix} \frac{1}{r}\hat{a}_r & \hat{a}_\phi & \hat{a}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_r & rH_\phi & H_z \end{vmatrix}$$

$$= \left[\frac{1}{r}\frac{\partial H_z}{\partial\phi} - \frac{\partial H_\phi}{\partial z}\right]\hat{a}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\hat{a}_\phi + \frac{1}{r}\left[\frac{\partial(rH_\phi)}{\partial r} - \frac{\partial H_r}{\partial\phi}\right]\hat{a}_z$$

$$= \left[\frac{1}{r}\frac{\partial}{\partial\phi}(5) - \frac{\partial}{\partial z}(2r)\right]\hat{a}_r + \left[\frac{\partial}{\partial z}(6r) - \frac{\partial}{\partial r}(5)\right]\hat{a}_\phi + \left(\frac{1}{r}\right)\left[\frac{\partial}{\partial r}(r2r) - \frac{\partial}{\partial\phi}(6r)\right]\hat{a}_z$$

$$= \left(\frac{1}{r}\right)\times 4r\hat{a}_z$$

$$= 4\hat{a}_z\text{ A/m}^2$$

(c) $\vec{H} = 2\rho\hat{a}_\rho + 3\hat{a}_\theta + \cos\theta\,\hat{a}_\phi$

By Ampere's law in spherical coordinates,

$$\vec{J} = \nabla \times \vec{H} = \frac{1}{\rho^2 \sin\theta} \begin{vmatrix} \hat{a}_\rho & \hat{a}_\theta & \hat{a}_\phi \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ H_\rho & \rho H_\theta & \rho \sin\theta H_\phi \end{vmatrix}$$

$$= \frac{1}{\rho \sin\theta}\left[\frac{\partial}{\partial \theta}(H_\phi \sin\theta) - \frac{\partial H_\theta}{\partial \phi}\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial H_\rho}{\partial \phi} - \frac{\partial}{\partial \rho}(\rho H_\phi)\right]\hat{a}_\theta + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho H_\theta) - \frac{\partial H_\rho}{\partial \theta}\right]\hat{a}_\phi$$

$$= \frac{1}{\rho \sin\theta}\left[\frac{\partial}{\partial \theta}(\cos\theta \sin\theta) - \frac{\partial}{\partial \phi}(3)\right]\hat{a}_\rho + \left(\frac{1}{\rho}\right)\left[\frac{1}{\sin\theta}\frac{\partial}{\partial \phi}(2\rho) - \frac{\partial}{\partial \rho}(\rho \cos\theta)\right]\hat{a}_\theta + \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho 3) - \frac{\partial}{\partial \theta}(2\rho)\right]\hat{a}_\phi$$

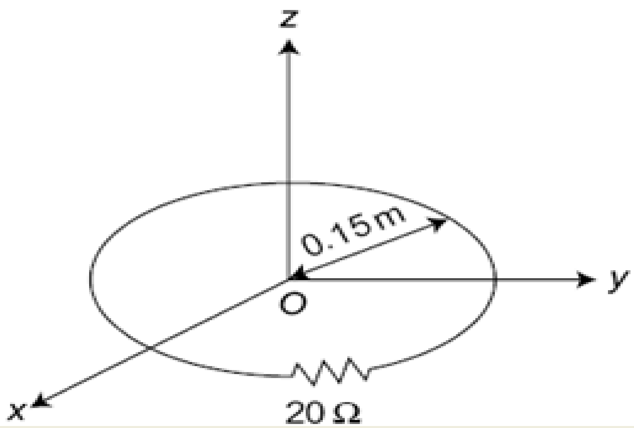
$$= \frac{1}{\rho}\left(\frac{\cos 2\theta}{\sin\theta}\right)\hat{a}_\rho - \frac{1}{\rho}\cos\theta\,\hat{a}_\theta + \frac{3}{\rho}\hat{a}_\phi \text{ A/m}^2$$

Problem7:

The circular loop conductor having a radius of 0.15 m is placed in the *xy* plane. This loop consists of a resistance of 20 Ω as shown in Fig. If the magnetic flux density is

$\vec{B} = 0.5 \sin 10^3 t\, \hat{a}_z$ **T**

Find the current flowing through the loop.



Circular loop conductor

Solution

Here since the loop is stationary and the magnetic field is time varying, only the transformer emf is induced.

Transformer emf induced is,

$$V_s = -\iint_S \frac{\partial \vec{B}}{\partial t}.d\vec{S} = -\iint_S \frac{\partial}{\partial t}(0.5\sin 10^3 t\hat{a}_z).(rdr\,d\phi\,\hat{a}_z)$$

$$= -0.5\times 10^3 \cos 10^3 t \int_{r=0}^{0.15}\int_{\phi=0}^{2\pi} rdr\,d\phi$$

$$= -0.5\times 2\pi\times 10^3 \cos 10^3 t\left[\frac{r^2}{2}\right]_0^{0.15}$$

$$= -10^3\pi \cos 10^3 t\times 0.01125$$

$$= -35.34 \cos 10^3 t \text{ V}$$