UNIT – III
EM WAVE CHARACTERISTICS

Contents:

- Wave Equations for Conducting and Perfect Dielectric Media
- Uniform Plane Waves - Definition, All Relations Between E & H
- Reflection and Refraction of Plane Waves
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Wave equations:

The Maxwell's equations in the differential form are

$$\nabla\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\cdot\vec{D} = \vec{\rho}$$

$$\nabla\cdot\vec{B} = 0$$

Let us consider a source free uniform medium having dielectric constant ε, magnetic permeability μ and conductivity σ. The above set of equations can be written as

$$\nabla\times\vec{H} = \sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t} \qquad (5.29(a))$$

$$\nabla\times\vec{E} = -\mu\frac{\partial\vec{H}}{\partial t} \qquad (5.29(b))$$

$$\nabla\cdot\vec{E} = 0 \qquad (5.29(c))$$

$$\nabla\cdot\vec{H} = 0 \qquad (5.29(d))$$

Using the vector identity ,

$$\nabla\times\nabla\times\vec{A} = \nabla\cdot\left(\nabla\cdot\vec{A}\right) - \nabla^2 A$$

We can write from 2

$$\nabla\times\nabla\times\vec{E} = \nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E}$$

$$= -\nabla\times\left(\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Substituting $\nabla\times\vec{H}$ from 1

$$\nabla\cdot\left(\nabla\cdot\vec{E}\right) - \nabla^2\vec{E} = -\mu\frac{\partial}{\partial t}\left(\sigma\vec{E} + \varepsilon\frac{\partial\vec{E}}{\partial t}\right)$$

But in source free($\nabla\cdot\vec{E} = 0$) medium (eq3)

$$\nabla^2\vec{E} = \mu\sigma\frac{\partial\vec{E}}{\partial t} + \mu\varepsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

In the same manner for equation eqn 1

$$\nabla\times\nabla\times\vec{H} = \nabla\cdot\left(\nabla\cdot\vec{H}\right) - \nabla^2\vec{H}$$

$$= \sigma\left(\nabla\times\vec{E}\right) + \varepsilon\frac{\partial}{\partial t}\left(\nabla\times\vec{E}\right)$$

$$= \sigma\left(-\mu\frac{\partial\vec{H}}{\partial t}\right) + \varepsilon\frac{\partial}{\partial t}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Since $\nabla\cdot\vec{H} = 0$ from eqn 4, we can write

$$\nabla^2\vec{H} = \mu\sigma\left(\frac{\partial\vec{H}}{\partial t}\right) + \mu\varepsilon\left(\frac{\partial^2\vec{H}}{\partial t^2}\right)$$

These two equations

$$\nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t} + \mu\varepsilon \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\nabla^2 \vec{H} = \mu\sigma \left(\frac{\partial \vec{H}}{\partial t} \right) + \mu\varepsilon \left(\frac{\partial^2 \vec{H}}{\partial t^2} \right)$$

are known as wave equations.

Uniform plane waves:

A uniform plane wave is a particular solution of Maxwell's equation assuming electric field (and magnetic field) has same magnitude and phase in infinite planes perpendicular to the direction of propagation. It may be noted that in the strict sense a uniform plane wave doesn't exist in practice as creation of such waves are possible with sources of infinite extent. However, at large distances from the source, the wave front or the surface of the constant phase becomes almost spherical and a small portion of this large sphere can be considered to plane. The characteristics of plane waves are simple and useful for studying many practical scenarios

Let us consider a plane wave which has only E_x component and propagating along z . Since the plane wave will have no variation along the plane perpendicular to z

i.e., xy plane, $\frac{\partial E_x}{\partial x} = \frac{\partial E_x}{\partial y} = 0$. The Helmholtz's equation reduces to,

$$\frac{d^2 E_x}{dz^2} + k^2 E_x = 0$$

The solution to this equation can be written as

$$E_x(z) = E_x^{+}(z) + E_x^{-}(z)$$
$$= E_0^{+} e^{-jkz} + E_0^{-} e^{jkz}$$

E_0^+ & E_0^- are the amplitude constants (can be determined from boundary conditions).

In the time domain, $\varepsilon_X(z,t) = \mathrm{Re}(E_x(z) e^{j\omega t})$

$$\varepsilon_X(z,t) = E_0^{+} \cos(\omega t - kz) + E_0^{-} \cos(\omega t + kz)$$

assuming E_0^+ & E_0^- are real constants.

Here, $\varepsilon_X^{+}(z,t) = E_0^{+} \cos(\omega t - \beta z)$ represents the forward traveling wave. The plot of $\varepsilon_X^{+}(z,t)$ for several values of t is shown in the Figure below

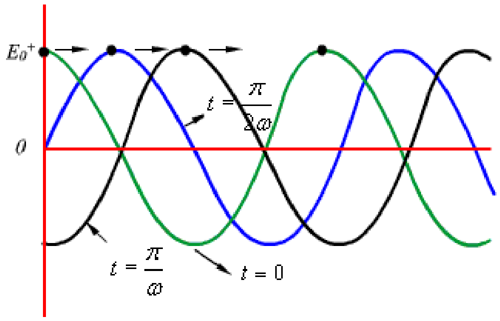


Figure : Plane wave traveling in the + *z* direction

As can be seen from the figure, at successive times, the wave travels in the +z direction.

If we fix our attention on a particular point or phase on the wave (as shown by the dot) i.e. , $\omega t - kz$ = constant

Then we see that as *t* is increased to $t + \Delta t$, z also should increase to $z + \Delta z$ so that

$\omega(t + \Delta t) - k(z + \Delta z) = \text{constant} = \omega t - \beta z$

Or, $\omega \Delta t = k \Delta z$

Or, $\frac{\Delta z}{\Delta t} = \frac{\omega}{k}$

When $\Delta t \to 0$,

we write $\lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = \frac{dz}{dt}$ = phase velocity v_p.

$\therefore v_p = \frac{\omega}{k}$

If the medium in which the wave is propagating is free space i.e., $\varepsilon = \varepsilon_0$, $\mu = \mu_0$

Then $v_p = \frac{\omega}{\omega\sqrt{\mu_0 \varepsilon_0}} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = C$

Where '*C*' is the speed of light. That is plane EM wave travels in free space with the speed of light.

The wavelength λ is defined as the distance between two successive maxima (or minima or any other reference points).

i.e., $(\omega t - kz) - \left[\omega t - k(z + \lambda)\right] = 2\pi$

or, $k\lambda = 2\pi$

or, $\lambda = \frac{2\pi}{k}$

Substituting $k = \frac{\omega}{v_p}$, $\lambda = \frac{2\pi v_p}{2\pi f} = \frac{v_p}{f}$

or, $\lambda f = v_p$

Thus wavelength λ also represents the distance covered in one oscillation of the wave. Similarly, $\varepsilon^{-}(z,t) = E_0^{-}\cos(\omega t + kz)$ represents a plane wave traveling in the -z direction.

The associated magnetic field can be found as follows:

From (6.4),

$$\vec{E}_x^{\,+}(z) = E_0^{+} e^{-jkz}\hat{a}_x$$

$$\vec{H} = -\frac{1}{j\omega\mu}\nabla\times\vec{E}$$

$$= -\frac{1}{j\omega\mu}\begin{vmatrix} \hat{a}_x & \hat{a}_y & \hat{a}_z \\ 0 & 0 & \frac{\partial}{\partial z} \\ E_0^{+}e^{-jkz} & 0 & 0 \end{vmatrix}$$

$$= \frac{k}{\omega\mu}E_0^{+}e^{-jkz}\hat{a}_y$$

$$= \frac{E_0^{+}}{\eta}e^{-jkz}\hat{a}_y = H_0^{+}e^{-jkz}\hat{a}_y$$

where $\eta = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$ is the intrinsic impedance of the medium.

When the wave travels in free space

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cong 120\pi = 377\Omega$$

is the intrinsic impedance of the free space.

In the time domain,

$$\vec{H}^{+}(z,t) = \hat{a}_y\frac{E_0^{+}}{\eta}\cos(\omega t - \beta z)$$

Which represents the magnetic field of the wave traveling in the +z direction.

For the negative traveling wave,

$$\vec{H}^{-}(z,t) = -a_y\frac{E_0^{+}}{\eta}\cos(\omega t + \beta z)$$

For the plane waves described, both the E & H fields are perpendicular to the direction of propagation, and these waves are called TEM (transverse electromagnetic) waves.

The *E* & *H* field components of a TEM wave is shown in Fig below

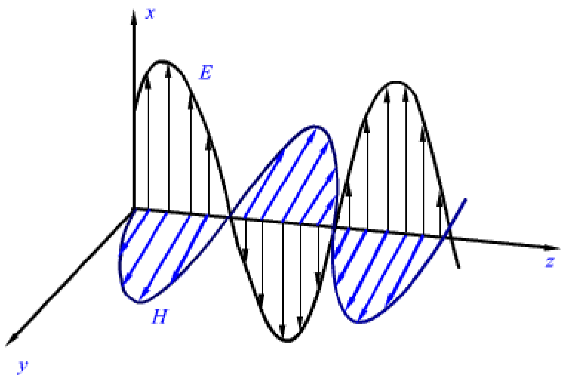


Figure : E & H fields of a particular plane wave at time t.

Solved Problems:

1. The vector amplitude of an electric field associated with a plane wave that propagates in the negative **z** direction in free space is given by $\hat{E}_m = 2a_x + 3a_y \, V/m$
   Find the magnetic field strength.

Solution:
The direction of propagation $\mathbf{n}_\beta$ is $-\mathbf{a}_z$. The vector amplitude of the magnetic field is then given by

$$\hat{H}_m = \frac{n_\beta \wedge \hat{E}}{\eta} = \frac{1}{\eta_\circ}\begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & -1 \\ 2 & 3 & 0 \end{vmatrix} = \left(\frac{1}{377} 3a_x - 2a_y\right) A/m$$

***note** $\eta_\circ = \sqrt{\frac{\mu_\circ}{\varepsilon_\circ}}$ 120π~377Ω (Appendix D – Table D.1)

2. The phasor electric field expression in a phase is given by

$$\hat{E} = \left[a_x + \hat{E}_y\, a_y + (2 + j5)\, a_z\right] e^{-j2.3(-0.6x+0.8y)}$$

Find the following:

1. $\hat{E}_y$.
2. Vector magnetic field, assuming $\mu = \mu_\circ \, and \varepsilon = \varepsilon_\circ$.
3. Frequency and wavelength of this wave.

Solution:

1. The general expression for a uniform plane wave propagating in an arbitrary direction is given by

$$\hat{E} = \hat{E}_m e^{-j\beta \cdot r}$$

where the amplitude vector $\hat{E}_m$, in general, has components in the x, y, and z directions. Comparing equation 6.3 with the general field equation for the plane wave propagating in an arbitrary direction, we obtain

$\beta \cdot r = \beta_x x + \beta_y y + \beta_z z$
$= \beta (\cos \theta_x x + \cos \theta_y y + \cos \theta_z z)$
$= 2.3(-0.6x + 0.8y + 0)$

Hence, a unit vector in the direction of propagation $\mathbf{n}_\beta$ is given by

$$\mathbf{n}_\beta = -0.6a_x + 0.8a_y.$$

Because the electric field $\hat{E}$ must be perpendicular to the direction of propagation $\mathbf{n}_\beta$, it must satisfy the following relations:

$$\mathbf{n}_\beta \cdot \hat{E} = 0$$

Therefore, $(-0.6a_x + 0.8a_y) \cdot \left[a_x + \hat{E}_y a_y + (2 + j5) a_z\right] = 0$

Or

$$-0.6 + 0.8 \hat{E}_y = 0$$

Hence, $\hat{E}_y = 0.75$. The electric field is given by

$$\hat{E} = \left[a_x + \hat{E}_y a_y + (2 + j5) a_z\right] e^{-j2.3(-0.6x+0.8y)}$$

2. The vector magnetic field $\hat{H}$ is given by

$$\hat{H} = \frac{1}{\eta} n_\beta \wedge \hat{E} = \frac{1}{377}\begin{vmatrix} a_x & a_y & a_z \\ -0.6 & 0.8 & 0 \\ 1 & 0.75 & 2+j5 \end{vmatrix}$$

so that

$$\hat{H}_x = \frac{0.8(2+j5)}{377} = (4.24 + j10.6) * 10^{-3}$$

$$\hat{H}_y = \frac{0.6(2+j5)}{377} = (3.18 - j7.95)*10^{-3}$$

$$\hat{H}_z = \frac{0.6(0.75)+0.8}{377} = -3.31*10^{-3}$$

The vector magnetic field is then given by

$$\hat{H} = \left(\hat{H}_x\, a_x + \hat{H}_y\, a_y + \hat{H}_z\, a_z\right) e^{-j2.3(-0.6x+0.8y)}$$

3. The wavelength λ is given by

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{2.3} = 2.73 \quad m$$

and the frequency

$$f = \frac{c}{\lambda} = \frac{3*10^8}{2.73} = 0.11\,GHz$$

Reflection and Refraction at Plane Interface between Two Media:

Figure 6.7 shows two media with electrical properties ε_1 and μ_1 in medium 1, and ε_2 and μ_2 in medium 2. Here a plane wave incident angle θ_i on a boundary between the two media will be partially transmitted into and partially reflected at the dielectric surface. The transmitted wave is reflected into the second medium, so its direction of propagation is different from the incidence wave. The figure also shows two rays for each the incident, reflected, and transmitted waves. A ray is a line drawn normal to the equiphase surfaces, and the line is along the direction of propagation.

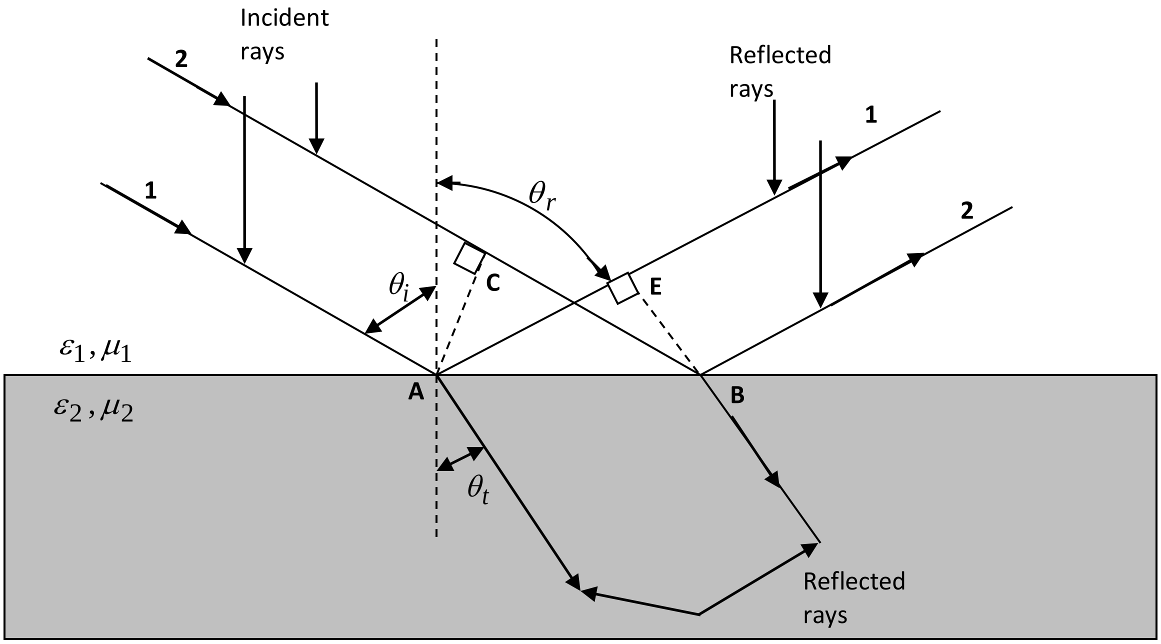


Figure 6.7

The incident ray 2 travels the distance CB, while on the contrary the reflected ray 1 travels the distance AE. For both AC and BE to be the incident and reflected wave fronts or planes of equiphase, the incident wave should take the same time to cover the distance AE. The reason being that the incident and reflected wave rays are located in the same medium, therefore their velocities will be equal,

$$\frac{CB}{V_1} = \frac{AE}{V_2}$$

OR

$$AB \sin\theta_i = AB \sin\theta_r$$

With this being the case then it follows that

$$\theta_i = \theta_r$$

What is the relationship between the angles of incidence θ_i and refraction θr ?

It takes the incident ray the equal amount of time to cover distance CB as it takes the refracted ray to cover distance AD –

$$\frac{CB}{V_1} = \frac{AD}{V_2}$$

And the magnitude of the velocity V_1 in medium 1 is:

$$V_1 = \frac{1}{\sqrt{\mu_1 * \varepsilon_1}}$$

And in medium 2:

$$V_2 = \frac{1}{\sqrt{\mu_2 * \varepsilon_2}}$$

Also,

$$CB = AB \sin\theta_i$$
$$AD = AB \sin\theta_i$$

Therefore,

$$\frac{CB}{AD} = \frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2} = \sqrt{\frac{\mu_2 * \varepsilon_2}{\mu_1 * \varepsilon_1}}$$

For most dielectrics $\mu_2 = \mu_1 = \mu_\circ$

Therefore,

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\Bigg|_{\mu_1 = \mu_2 = \mu_\circ} \qquad (6.12)$$

Equation 6.12 is known as Snell's Law of Refraction.

Behavior of Plane waves at the interface of two media:

We have considered the propagation of uniform plane waves in an unbounded homogeneous medium. In practice, the wave will propagate in bounded regions where several values of ε, μ, σ will be present. When plane wave travelling in one medium meets a different medium, it is partly reflected and partly transmitted. In this section, we consider wave reflection and transmission at planar boundary between two media.

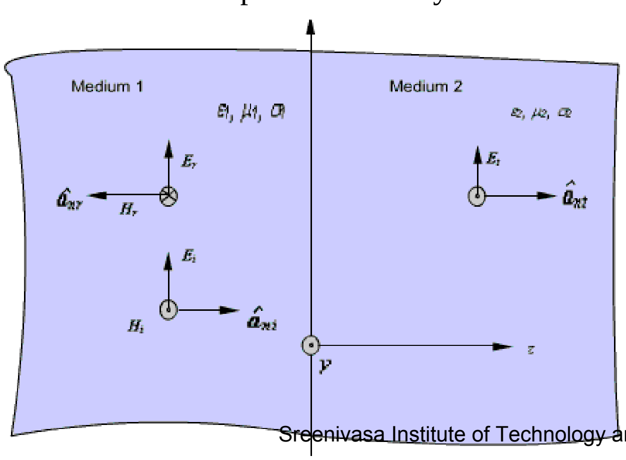


Fig 6 : Normal Incidence at a plane boundary

Case1: Let z = 0 plane represent the interface between two media. Medium 1 is characterised by $(\varepsilon_1, \mu_1, \sigma_1)$ and medium 2 is characterized by $(\varepsilon_2, \mu_2, \sigma_2)$.Let the subscripts '*i*' denotes incident, '*r*' denotes reflected and '*t*' denotes transmitted field components respectively. The incident wave is assumed to be a plane wave polarized along *x* and travelling in medium 1 along $\hat{a}_z$ direction. From equation (6.24) we can write

$$\vec{E}_i(z) = E_{io} e^{-\gamma_1 z} \hat{a}_x \quad ....(1)$$

$$\vec{H}_i(z) = \frac{1}{\eta_i} \hat{a}_z \times E_{io}(z) = \frac{E_o}{\eta_i} e^{-\gamma_1 z} \hat{a}_y \quad ....(2)$$

where $\gamma_1 = \sqrt{j\omega\mu_1(\sigma_1 + j\omega\varepsilon_1)}$ and $\eta_1 = \sqrt{\frac{j\omega\mu_1}{\sigma_1 + j\omega\varepsilon_2}}$.

Because of the presence of the second medium at *z* =0, the incident wave will undergo partial reflection and partial transmission.The reflected wave will travel along $\hat{a}_z$ in medium 1.

The reflected field components are:

$$\vec{E}_r = E_{ro} e^{\gamma_1 z} \hat{a}_x \quad ....(3)$$

$$\vec{H}_r = \frac{1}{\eta_1}\left(-\hat{a}_z\right) \times E_{ro} e^{\gamma_1 z} \hat{a}_x = -\frac{E_{ro}}{\eta_1} e^{\gamma_1 z} \hat{a}_y \quad ....(4)$$

The transmitted wave will travel in medium 2 along $\hat{a}_z$ for which the field components are

$$\vec{E}_t = E_{t0} e^{-\gamma_2 z} \hat{a}_x \quad ....(5)$$

$$\vec{H}_t = \frac{E_{t0}}{\eta_2} e^{-\gamma_2 z} \hat{a}_y \quad ....(6)$$

where $\gamma_2 = \sqrt{j\omega\mu_2(\sigma_2 + j\omega\varepsilon_2)}$ and $\eta_2 = \sqrt{\frac{j\omega\mu_2}{\sigma_2 + j\omega\varepsilon_2}}$

In medium 1,

$\vec{E}_1 = \vec{E}_i + \vec{E}_r$ and $\vec{H}_1 = \vec{H}_i + \vec{H}_r$

and in medium 2,

$\vec{E}_2 = \vec{E}_t$ and $\vec{H}_2 = \vec{H}_t$

Applying boundary conditions at the interface z = 0, i.e., continuity of tangential field components and noting that incident, reflected and transmitted field components are tangential at the boundary, we can write

$$\vec{E}_i(0) + \vec{E}_r(0) = \vec{E}_i(0)$$

& $\vec{H}_i(0) + \vec{H}_r(0) = \vec{H}_t(0)$

From equation 3to 6 we get,

$E_{io} + E_{ro} = E_{to}$ ....(7)

$\dfrac{E_{io}}{\eta_1} - \dfrac{E_{ro}}{\eta_1} = \dfrac{E_{to}}{\eta_2}$ ....(8)

Eliminating E_{to} ,

$$\frac{E_{io}}{\eta_1} - \frac{E_{ro}}{\eta_1} = \frac{1}{\eta_2}(E_{io} + E_{ro})$$

or, $E_{io}\left(\dfrac{1}{\eta_1} - \dfrac{1}{\eta_2}\right) = E_{ro}\left(\dfrac{1}{\eta_1} + \dfrac{1}{\eta_2}\right)$

or, $E_{ro} = \tau E_{io}$

$\tau = \dfrac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$ ....(8)

is called the reflection coefficient.

From equation (8), we can write

$$2E_{io} = E_{to}\left[1 + \frac{\eta_1}{\eta_2}\right]$$

or, $E_{to} = \dfrac{2\eta_2}{\eta_1 + \eta_2} E_{io} = TE_{io}$

$T = \dfrac{2\eta_2}{\eta_1 + \eta_2}$ ....(9)

is called the transmission coefficient.

We observe that,

$T = \dfrac{2\eta_2}{\eta_1 + \eta_2} = \dfrac{\eta_2 - \eta_1 + \eta_1 + \eta_2}{\eta_1 + \eta_2} = 1 + \tau$ ....(10)

The following may be noted

(i) both τ and T are dimensionless and may be complex

(ii) $0 \le |\tau| \le 1$

Let us now consider specific cases:

Case I: Normal incidence on a plane conducting boundary

The medium 1 is perfect dielectric $(\sigma_1 = 0)$ and medium 2 is perfectly conducting $(\sigma_2 = \infty)$.

$\therefore \eta_1 = \sqrt{\frac{\mu_1}{\epsilon_1}}$

$\eta_2 = 0$

$\gamma_1 = \sqrt{(j\omega\mu_1)(j\omega\epsilon_1)}$

$= j\omega\sqrt{\mu_1\epsilon_1} = j\beta_1$

From (9) and (10)

τ = -1

and T =0

Hence the wave is not transmitted to medium 2, it gets reflected entirely from the interface to the medium 1.

$\therefore \vec{E}_1(z) = E_{io}e^{-j\beta_1 z}\,\hat{a}_x - E_{io}e^{j\beta_1 z}\,\hat{a}_x = -2jE_{io}\sin\beta_1 z\,\hat{a}_x$

& $\therefore \vec{E}_1(z,t) = \mathrm{Re}\left[-2jE_{io}\sin\beta_1 z e^{j\omega t}\right]\hat{a}_x = 2E_{io}\sin\beta_1 z\sin\omega t\,\hat{a}_x$ ....(11)

Proceeding in the same manner for the magnetic field in region 1, we can show that,

$\vec{H}_1(z,t) = \hat{a}_y\frac{2E_{io}}{\eta_1}\cos\beta_1 z\cos\omega t$ ....(12)

The wave in medium 1 thus becomes a **standing wave** due to the super position of a forward travelling wave and a backward travelling wave. For a given ' t', both $\vec{E}_1$ and $\vec{H}_1$ vary sinusoidally with distance measured from z = 0. This is shown in figure 6.9.

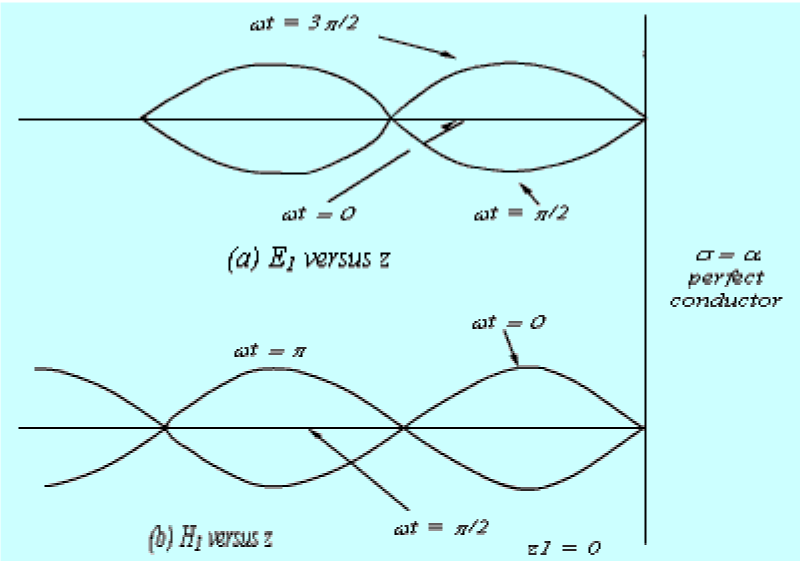


Figure 7: Generation of standing wave

Zeroes of $E_1(z,t)$ and Maxima of $H_1(z,t)$.

Maxima of $E_1(z,t)$ and zeroes of $H_1(z,t)$.

$$\left.\right\}\text{occur at } \beta_1 z = -n\pi \quad \text{or } z = -n\frac{\lambda}{2}$$

$$\left.\right\}\text{occur at } \beta_1 z = -(2n+1)\frac{\pi}{2} \quad \text{or } z = -(2n+1)\frac{\lambda}{4},\ \ n = 0,1,2,....$$

Case2: Normal incidence on a plane dielectric boundary : If the medium 2 is not a perfect conductor (i.e. $\sigma_2 \neq \infty$) partial reflection will result. There will be a reflected wave in the medium 1 and a transmitted wave in the medium 2.Because of the reflected wave, standing wave is formed in medium 1.

From equation (10) and equation (13) we can write

$$\vec{E}_1 = E_{io}\left(e^{-\gamma_1 z} + \Gamma e^{\gamma_1 z}\right)\hat{a}_x \quad ....(14)$$

Let us consider the scenario when both the media are dissipation less i.e. perfect dielectrics ($\sigma_1 = 0,\ \sigma_2 = 0$)

$$\gamma_1 = j\omega\sqrt{\mu_1\varepsilon_1} = j\beta_1 \qquad \eta_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}}$$

$$\gamma_2 = j\omega\sqrt{\mu_2\varepsilon_2} = j\beta_2 \qquad \eta_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} \quad ....(15)$$

In this case both η_1 and η_2 become real numbers.

$$\begin{aligned}\vec{E}_1 &= \hat{a}_x E_{io}\left(e^{-j\beta_1 z} + \Gamma e^{j\beta_1 z}\right) \\ &= \hat{a}_x E_{io}\left((1+T)e^{-j\beta_1 z} + \Gamma\left(e^{j\beta_1 z} - e^{-j\beta_1 z}\right)\right) \\ &= \hat{a}_x E_{io}\left(Te^{-j\beta_1 z} + \Gamma(2j\sin\beta_1 z)\right)\end{aligned} \quad ....(16)$$

From (6.61), we can see that, in medium 1 we have a traveling wave component with amplitude TE_{io} and a standing wave component with amplitude $2JE_{io}$. The location of the maximum and the minimum of the electric and magnetic field components in the medium 1from the interface can be found as follows. The electric field in medium 1 can be written as

$$\vec{E}_1 = \hat{a}_x E_{io} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right) \quad ....(17)$$

If $\eta_2 > \eta_1$ i.e. $\Gamma > 0$

The maximum value of the electric field is

$$\left|\vec{E}_1\right|_{max} = E_{io}(1+T) \quad ....(18)$$

and this occurs when

$$2\beta_1 z_{max} = -2n\pi$$

or $z_{max} = -\frac{n\pi}{\beta_1} = -\frac{n\pi}{2\pi/\lambda_1} = -\frac{n}{2}\lambda_1$, n = 0, 1, 2, 3......................(19)

The minimum value of $\left|\vec{E}_1\right|$ is

$\left|\vec{E}_1\right|_{min} = E_{io}\left(1-\Gamma\right)$ ....(20)

And this occurs when

$2\beta_1 z_{min} = -\left(2n+1\right)\pi$

or $z_{min} = -\left(2n+1\right)\frac{\lambda_1}{4}$, n = 0, 1, 2, 3............................(21)

For $\eta_2 < \eta_1$ i.e. $\Gamma < 0$

The maximum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1-\Gamma\right)$ which occurs at the z_{min} locations and the minimum value of $\left|\vec{E}_1\right|$ is $E_{io}\left(1+\Gamma\right)$ which occurs at z_{max} locations as given by the equations (6.64) and (6.66).

From our discussions so far we observe that $\frac{\left|E\right|_{max}}{\left|E\right|_{min}}$ can be written as

$S = \frac{\left|E\right|_{max}}{\left|E\right|_{min}} = \frac{1+\left|\Gamma\right|}{1-\left|\Gamma\right|}$ ....(22)

The quantity *S* is called as the standing wave ratio. As $0 \le \left|\Gamma\right| \le 1$ the range of *S* is given by $1 \le S \le \infty$

From (6.62), we can write the expression for the magnetic field in medium 1 as

$\vec{H}_1 = \hat{a}_y \frac{E_{io}}{\eta_1} e^{-j\beta_1 z}\left(1-\Gamma e^{j2\beta_1 z}\right)$ ....(23)

From (6.68) we find that $\left|\vec{H}_1\right|$ will be maximum at locations where $\left|\vec{E}_1\right|$ is minimum and vice versa.

In medium 2, the transmitted wave propagates in the + z direction.

Brewster Angle:

Brewster angle is defined as the angle of incidence at which there will be no reflected wave. It occurs when the incident wave is polarized such that the **E** field is parallel to the plane of incidence.

Brewster Angle – (from Brewster's Law), the polarizing angle of which (when light is incident) the reflected and refracted index is equal to the tangent of the polarizing angle. In other words, the angle of incidence of which there is no reflection.

From the reflection coefficient expression-

$$\hat{\Gamma}_{||} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

It can be seen that there is an angle of incidence at $\hat{\Gamma}_{||} = 0$. This angle can be obtained when

$$\eta_1 \cos\theta_i = \eta_2 \cos\theta_t$$

Or

$$\cos\theta_i = \frac{\eta_2}{\eta_1} \cos\theta_t$$

The angle of incidence θ_i, at which $\hat{\Gamma}_{||} = 0$, is known as the Brewster angle. The expression for this angle in terms of the dielectric properties of media 1 & 2, considering Snell's Law for the special case $\mu_1 = \mu_2 = \mu_\circ$ is

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\Bigg|_{\mu_1=\mu_2=\mu_\circ}$$

This condition is important, because it is usually satisfied by the materials often used in optical applications.

Equation 6.19 will take the form –

$$\cos\theta_i = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} \cos\theta_t$$

Square both sides of equation 6.20 and use Snell's Law for the special case of $\mu_1 = \mu_2 = \mu_\circ$ for the following result:

$$\cos^2\theta_i \frac{\varepsilon_1}{\varepsilon_2} = \cos^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\left(1 - \sin^2\theta_t\right)$$

$$= \frac{\varepsilon_1}{\varepsilon_2}\left(1 - \sin^2\theta_i\right)$$

The last substitution was based on Snell's Law of refraction. Therefore,

$$\left(1 - \sin^2\theta_i\right) = \frac{\varepsilon_1}{\varepsilon_2} - \frac{\varepsilon_1^2}{\varepsilon_2^2}\sin^2\theta_i$$

$$1-\frac{\varepsilon_1}{\varepsilon_2} = \sin^2\theta_i\left(1-\frac{\varepsilon_1^2}{\varepsilon_2^2}\right)$$

And

$$\sin^2\theta_i = \frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}$$

The Brewster angle of incidence is

$$\sin\ \theta_i\ = \sqrt{\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}}$$

A specific value of θ_i can be obtained from equation 6.21 -

$$1-\cos^2\theta_i = \frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}$$

Or

$$\cos^2\theta_i = 1-\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1} = \frac{\varepsilon_1}{\varepsilon_2+\varepsilon_1} =$$

$$\cos\ \theta_i = \sqrt{\frac{\varepsilon_1}{\varepsilon_2+\varepsilon_1}} \qquad (6.23)$$

From equations 6.22 & 6.23 –

$$\tan\theta_i = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

This specific angle of incidence θ_i is called the Brewster angle θ_β.

$$\theta_\beta = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

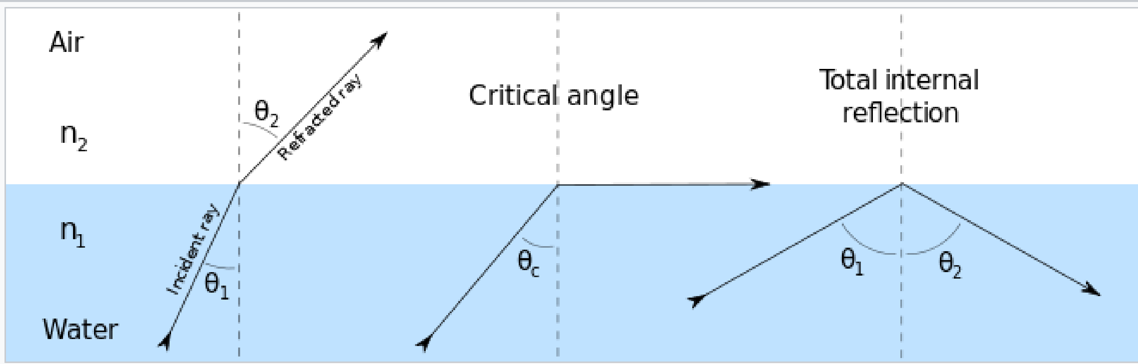
Critical angle:

In geometric optics, at a refractive boundary, the smallest angle of incidence at which total internal reflection occurs. The critical angle is given by

$$\theta_c = \sin^{-1}\left(\frac{n_1}{n_2}\right),$$

Where Θ_c is the critical angle, n_1 is the refractive index of the less dense medium, and n_2 is the refractive index of the denser medium.

Angle of incidence: The angle between an incident ray and the normal to a reflecting or refracting surface



Refraction of light at the interface between two media, including total internal reflection.

Total Reflection at Critical Angle of Incidence

In the previous section it was shown that for common dielectrics, the phenomenon of total transmission exists only where the electric field is parallel to the plane of incidence known as parallel polarization.
There is a second phenomenon existing for both polarizations:

- Total reflection occurring at the interface between two dielectric media
- A wave passing from a medium with a larger dielectric constant to a medium with smaller value of ε

Snell's Law of refraction shows –

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{or} \qquad \sin\theta_i = \frac{\sin\theta_t}{\sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}} \qquad (6.26)$$

Therefore, if $\varepsilon_1 > \varepsilon_2$, *and* $\theta_t > \theta_i$ then a wave incident at an angle θ_i will pass into medium 2 at a larger angle θ_t .

<u>**Definition:**</u>

θ_c, (critical angle of incidence) is the value of θ_i that makes $\theta_t = \pi/2$, see Figure 6.13.

Substitute $\theta_t = \pi/2$ in equation 6.26 to get –

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}, \text{ or } \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

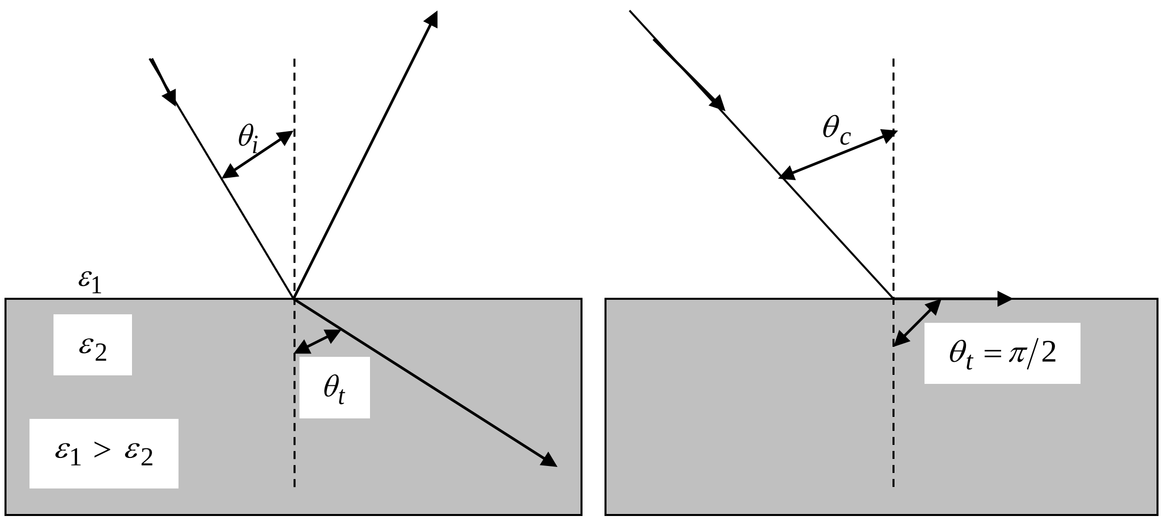


Figure 6.13 illustrates the fact that $\theta_t > \theta_i, if\ \varepsilon_1 > \varepsilon_2$. The critical angle θ_c is defined as the value of θ_i at which $\theta_t = \pi/2$.

Envision a beam of light impinging on an interface between two transparent media where $n_i < n_t$. At normal incidence ($\theta_i = 0$) most of the incoming light is transmitted into the less dense medium. As θ_i increases, more and more light is reflected back into the dense medium, while θ_t increases. When $\theta_t = 90°$, θ_i is defined to be θ_c and the transmittance becomes zero. For $\theta_i > \theta_c$ all of the light is totally internally reflected, remaining in the incident medium.

Poynting Vector and Power Flow in Electromagnetic Fields:

Electromagnetic waves can transport energy from one point to another point. The electric and magnetic field intensities asscociated with a travelling electromagnetic wave can be related to the rate of such energy transfer.

Let us consider Maxwell's Curl Equations:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

Using vector identity

$$\nabla . \left(\vec{E} \times \vec{H}\right) = \vec{H}.\nabla \times \vec{E} - \vec{E}.\nabla \times \vec{H}$$

the above curl equations we can write

$$\nabla . \left(\vec{E} \times \vec{H}\right) = -\vec{H}.\frac{\partial \vec{B}}{\partial t} - \vec{E}.\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

$$or,\ \nabla.\left(\vec{E}\times\vec{H}\right) = -\vec{H}.\frac{\partial\vec{B}}{\partial t} - \vec{E}.\vec{J} - \vec{E}.\frac{\partial\vec{D}}{\partial t}$$ ....(1)

In simple medium where $\in, \mu$ and σ are constant, we can write

$$\vec{H}.\frac{\partial\vec{B}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu H^2\right)$$

$$\vec{E}.\frac{\partial\vec{D}}{\partial t} = \frac{\partial}{\partial t}\left(\frac{1}{2}\mu E^2\right)$$ and $\vec{E}.\vec{J} = \sigma E^2$

$$\therefore \nabla.\left(\vec{E}\times\vec{H}\right) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right) - \sigma E^2$$

Applying Divergence theorem we can write,

$$\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = -\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV - \int_V \sigma E^2 dV$$ ....(2)

The term $\frac{\partial}{\partial t}\int_V\left(\frac{1}{2}\in E^2 + \frac{1}{2}\mu H^2\right)dV$ represents the rate of change of energy stored in the electric and magnetic fields and the term $\int_V \sigma E^2 dV$ represents the power dissipation within the volume. Hence right hand side of the equation (6.36) represents the total decrease in power within the volume under consideration.

The left hand side of equation (6.36) can be written as $\oint_S \left(\vec{E}\times\vec{H}\right).d\vec{S} = \oint_S \vec{P}.d\vec{S}$ where $\vec{P} = \vec{E}\times\vec{H}$ (W/mt^2) is called the Poynting vector and it represents the power density vector associated with the electromagnetic field. The integration of the Poynting vector over any closed surface gives the net power flowing out of the surface. Equation (6.36) is referred to as Poynting theorem and it states that the net power flowing out of a given volume is equal to the time rate of decrease in the energy stored within the volume minus the conduction losses.

Poynting vector for the time harmonic case:

For time harmonic case, the time variation is of the form $e^{j\omega t}$, and we have seen that instantaneous value of a quantity is the real part of the product of a phasor quantity and $e^{j\omega t}$ when $\cos\omega t$ is used as reference. For example, if we consider the phasor

$$\vec{E}(z) = \hat{a}_x E_x(z) = \hat{a}_x E_0 e^{-j\beta z}$$

then we can write the instanteneous field as

$$\vec{E}(z,t) = \mathrm{Re}\left[\vec{E}(z)e^{j\omega t}\right] = E_0\cos(\omega t - \beta z)\hat{a}_x$$ ....(1)

when E_0 is real.
Let us consider two instanteneous quantities A and B such that

$$A = \mathrm{Re}\left(Ae^{j\omega t}\right) = |A|\cos(\omega t + \alpha)$$ ....(2)

$$B = \mathrm{Re}\left(Be^{j\omega t}\right) = |B|\cos(\omega t + \beta)$$

where A and B are the phasor quantities.

i.e, $A = |A|e^{j\alpha}$

$B = |B|e^{j\beta}$

Therefore,

$$AB = |A|\cos(\omega t + \alpha)|B|\cos(\omega t + \beta)$$

$$= \frac{1}{2}|A||B|\left[\cos(\alpha - \beta) + \cos(2\omega t + \alpha + \beta)\right] \qquad \text{..............................(3)}$$

Since *A* and *B* are periodic with period $T = \frac{2\pi}{\omega}$, the time average value of the product form AB, denoted by $\overline{AB}$ can be written as

$$\overline{AB} = \frac{1}{T}\int_0^T AB\,dt$$

$$\overline{AB} = \frac{1}{2}|A||B|\cos(\alpha - \beta) \qquad \text{.....................................(4)}$$

Further, considering the phasor quantities *A* and *B*, we find that

$$AB^* = |A|e^{j\alpha}|B|e^{-j\beta} = |A||B|e^{j(\alpha-\beta)}$$

and $\mathrm{Re}\left(AB^*\right) = |A||B|\cos(\alpha - \beta)$, where * denotes complex conjugate.

$$\therefore \overline{AB} = \frac{1}{2}\mathrm{Re}\left(AB^*\right) \qquad \text{..(5)}$$

The poynting vector $\vec{P} = \vec{E} \times \vec{H}$ can be expressed as

$$\vec{P} = \hat{a}_x\left(E_y H_z - E_z H_y\right) + \hat{a}_y\left(E_z H_x - E_x H_z\right) + \hat{a}_z\left(E_x H_y - E_y H_x\right) \qquad \text{...................................(6)}$$

If we consider a plane electromagnetic wave propagating in +z direction and has only E_x component, from (6.42) we can write:

$$\vec{P}_z = E_x(z,t)H_y(z,t)\hat{a}_3$$

Using (6)

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z)H_y^*(z)\hat{a}_z\right)$$

$$\vec{P}_{zav} = \frac{1}{2}\mathrm{Re}\left(E_x(z) \times H_y(z)\right) \qquad \text{..(7)}$$

where $\vec{E}(z) = E_x(z)\hat{a}_x$ and $\vec{H}(z) = H_y(z)\hat{a}_y$, for the plane wave under consideration.

For a general case, we can write

$$\vec{P}_{av} = \frac{1}{2}\mathrm{Re}\left(\vec{E}\times\vec{H}^{*}\right)$$ ....(8)

We can define a complex Poynting vector

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^{*}$$

and time average of the instantaneous Poynting vector is given by $\vec{P}_{av} = \mathrm{Re}\left(\vec{S}\right)$.

Solved Problems:

1. Calculate the polarization angle (Brewster angle) for an air water $(\varepsilon_r = 81)$ interface at which plane waves pass from the following:
   (a) Air into water.
   (b) Water into air.

SOLUTION

1. (a) Air into water:

$$\varepsilon_{r1} = 1 \quad and \quad \varepsilon_{r2} = 81$$

The Brewster angle is then given by

$$\theta_\beta = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 6.34°$$

Therefore,

$$\theta_\beta = \tan^{-1}\sqrt{81} = 83.7°$$

(b) Water into air:

$$\varepsilon_{r1} = 81 \quad and \quad \varepsilon_{r2} = 1$$

Hence,

$$\theta_\beta = \tan^{-1}\sqrt{\frac{1}{81}} = 6.34°$$

To relate the Brewster angles in both cases, let us calculate the angle of refraction.

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Therefore, in case a,

$$\frac{\sin\theta_B}{\sin\theta_t} = \sqrt{81}$$

Therefore,

$$\sin\theta_t = \frac{\sin 83.7}{9} = 0.11$$

Or $\theta_t = 6.34^\circ$, which is the same as the Brewster angle for case b. Also, the angle of refraction in case b is given by Snell's Law as:

$$\frac{\sin\theta_B}{\sin\theta_t} = \sqrt{\frac{\varepsilon_\circ}{81\varepsilon_\circ}} = \sqrt{\frac{1}{81}}$$

Therefore,

$$\sin\theta_t = \frac{\sin 6.34^\circ}{\sqrt{\frac{1}{81}}} = 0.99$$

Or $\theta_t = 83.7^\circ$, which is the Brewster angle for case a.

2. The index of refraction of liquid is 1.9. What is the critical angle for a light ray travelling in the liquid toward a flat layer of air?

Solution

The critical angle is determined by the following expression (Snell's law, in which the angle of refraction is 90^0):

$$n_1 \sin\theta_{cr} = n_2 \sin 90^0$$

Here $n_1 = 1.9$ is the index of refraction of medium 1 (liquid), $n_2 = 1$ is the index of refraction of medium 2 (air). We substitute the known values in the above expression and find the critical angle

$$\sin\theta_{cr} = \frac{n_2}{n_1} = \frac{1}{1.9}$$

$$\theta_{cr} = \sin^{-1}\frac{1}{1.9} = 37.36^0$$

3. Find the critical angle for total internal reflection for light going from ice (index of refraction = 1.31) into air.

Solution

The critical angle is defined as the angle of incidence for which the corresponding angle of refraction is 90^0. Then the Snell's law takes the following form

$$n_1 \sin\theta_{cr} = n_2 \sin\theta_2$$

Here $n_1 = 1.31$ is the index of refraction of medium 1 (ice), θ_{cr} is the unknown critical angle, $\theta_2 = 90^0$ is the angle of refraction (angle in air), and $n_2 = 1$ is the index of refraction of medium 2 (air). We substitute these values into above expression and obtain

$$1.31 \sin \theta_{cr} = 1 \sin 90^0 = 1$$

Then

$$\theta_{cr} = \sin^{-1} \frac{1}{1.31} = 49.76^0$$