FINITE ELEMENT METHODS
23CIV471T

IV-B.Tech & I-SEM

DEPARTMENT OF CIVIL ENGINEERING

SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES (SITAMS)
(UGC - AUTONOMOUS INSTITUTION & ACCREDITED BY NBA FOR CSE, ECE, EEE & MCA)
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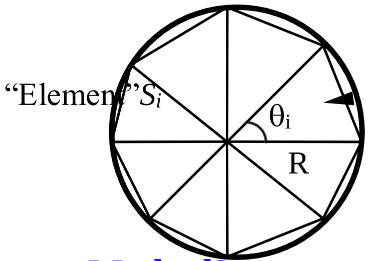
INTRODUCTION

Basic Concepts

The *finite element method* (FEM), or *finite element analysis*(FEA), is based on the idea of building a complicated object with simple blocks, or, dividing a complicated object into small and manageable pieces. Application of this simple idea can be found every where in everyday life, as well as in engineering.

Examples:

- Lego (kids’ play)
- Buildings
- Approximation of the area of a circle:



Why Finite Element Method?

- Design analysis: hand calculations, experiments, andcomputer simulations
- FEM/FEA is the most widely applied computer simulation method in engineering
- Closely integrated with CAD/CAM applications

A Brief History of the FEM

- 1943 ----- Courant (Variational methods)
- 1956 ----- Turner,Clough,MartinandTopp(Stiffness)
- 1960 ----- Clough(“FiniteElement”,planeproblems)
- 1970s ---- Applicationsonmainframecomputers
- 1980s ---- Microcomputers,pre-andpostprocessors
- 1990s ---- Analysisoflargestructuralsystems

What is FEM ?

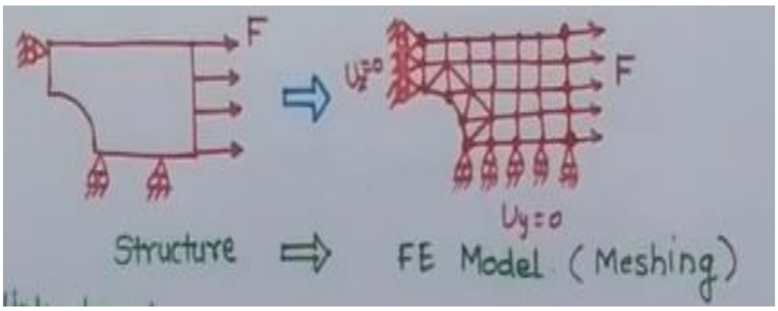
- Many physical phenomena in engineering and science can be described in terms of *partial differential equations*.
- In general, solving these equations by classical analytical methods for arbitrary shapes is almost impossible.
- The *finite element method* (FEM) is a numerical approach by which these partial differential equations can be solved approximately.
- From an engineering standpoint, the FEM is a method for solving engineering problems such as *stress analysis, heat transfer, fluid flow* and*electromagnetics* by computer simulation.
- Millions of engineers and scientists worldwide use the FEM to predict the behaviour of structural, mechanical, thermal, electrical and chemical systems for both design and performance analyses.

BASIC STEPS OF FEM

1 Discretization of the structure
2 Identify primary unknown quantity
3 Selection of Displacement function
4 Formation of the element stiffness matrix and load vector
5 Formation of Global stiffness matrix and load vector
6 Incorporation of Boundary conditions
7 Solution of Simultaneous equations
8 Calculation of element strains and stresses
9 Interpretation of the result obtained.

Step1: Discretization of or structure – (Establish the FE mesh)

- The continuum is divided into a number of elements by imaginary lines or surfaces.
- The interconnected elements may have different sizes and shapes.
- Establish the FE mesh with set coordinates, element numbers and node numbers
- The discretized FE model must be situated with a coordinate system
- Elements and nodes in the discretized FE model need to be identified by "element numbers" and "nodal numbers."
- Nodes are identified by the assigned node numbers and their corresponding coordinates

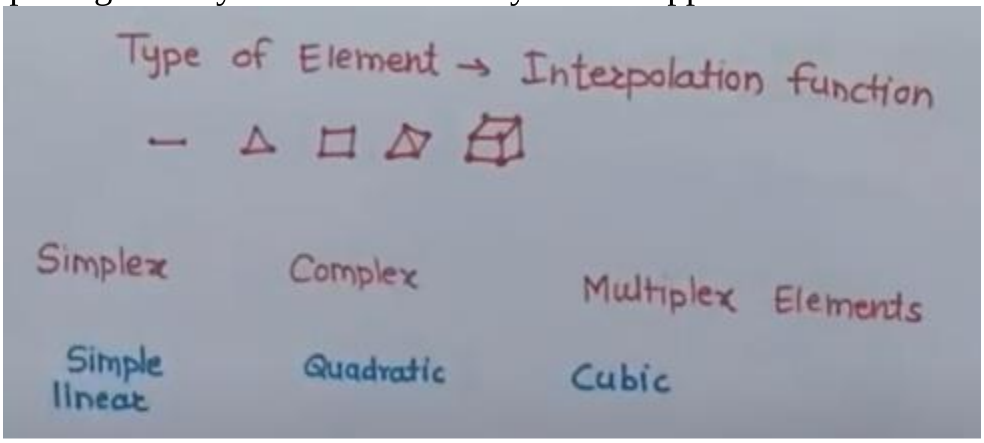


Step2: Identify primary unknown quantity

- Primary unknown quantity - The first and principal unknown quantity to be obtained by the FEM
  Eg: Stress analysis: Displacement {u} at nodes
- In stress analysis, The primary unknowns are nodal displacements, but secondary unknown quantities include: strains in elements can be obtained by the "strain-displacement relations," and the unknown stresses in the elements by the stress-strain relations (the Hooke's law).

Step 3: Choice of approximating functions

- Displacement function is the starting point of the mathematical analysis.
- This represents the variation of the displacement within the element.
- The displacement function may be approximated in the form a linear function or a higher-order function.
- A convenient way to express it is by polynomial expressions.
- Shape or geometry of the element may also be approximated.



Step 4: Formation of the element stiffness matrix & load vector

- After continuum is discretized with desired element shapes, the individual element stiffness matrix is formulated.
- Basically it is a minimization procedure whatever may be the approach adopted.
- For certain elements, the form involves a great deal of sophistication.
- The geometry of the element is defined in reference to the global frame.
- Coordinate transformation must be done for elements where it is necessary.

$$\{F\}e= \{K\}e * \{q\}e$$

Step 5: Formation of overall stiffness matrix & load vector

- After the element stiffness matrices in global coordinates are formed, they are assembled to form the overall stiffness matrix.
- The assembly is done through the nodes which are common to adjacent elements.
- The overall stiffness matrix is symmetric and banded.

$$\{F\}G= \{K\}\ G * \{q\}\ G$$

Step6 : Incorporation of boundary conditions

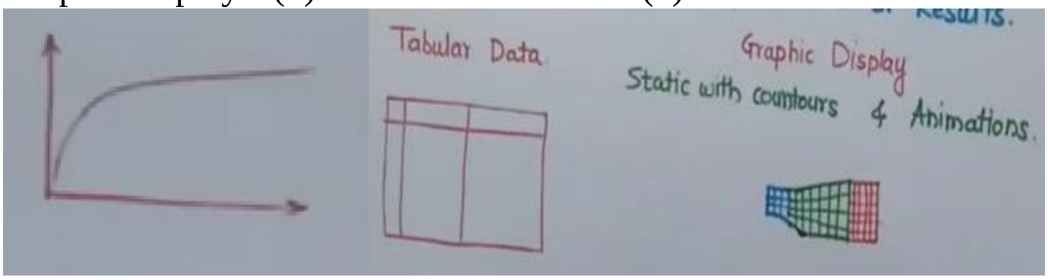
- The boundary restraint conditions are to be imposed in the stiffness matrix.
- There are various techniques available to satisfy the boundary conditions.
- One is the size of the stiffness matrix may be reduced or condensed in its final

form.
- To ease computer programming aspect and to elegantly incorporate the boundary conditions, the size of overall matrix is kept the same.

Step7: Solution of simultaneous equations
- The unknown nodal displacements are calculated by the multiplication of force vector with the inverse of stiffness matrix.
- [δ]=inverse of [k].[F]

Step8: Calculation of stresses or stress-resultants
- Nodal displacements are utilized for the calculation of stresses or stress-resultants.
- This may be done for all elements of the continuum or it may be limited to some predetermined elements.

Step9: Display and Interpretation of Results
- Results may also be obtained by graphical means.
- It may desirable to plot the contours of the deformed shape of the continuum.
- Tabulation of results
- Graphic displays: (1) Static with contours. (2) Animations



Advantages of Finite Element Method

- Modeling of complex geometries and irregular shapes are easier as varieties of finite elements are available for discretization of domain.
- Boundary conditions can be easily incorporated in FEM.
- Different types of material properties can be easily accommodated in modeling from element to element or even within an element.
- Higher order elements may be implemented.
- FEM is simple, compact and result-oriented and hence widely popular among engineering community.
- Availability of large number of computer software packages and literature makes FEM a versatile and powerful numerical method.

Disadvantages of Finite Element Method
- Large amount of data is required as input for the mesh used in terms of nodal connectivity and other parameters depending on the problem.
- It requires a digital computer and fairly extensive
- It requires longer execution time compared with FEM.
- Output result will vary considerably.

Limitations of FEA

1. Proper engineering judgment is to be exercised to interpret results.
2. It requires large computer memory and computational time to obtain intend results.
3. There are certain categories of problems where other methods are more effective, e.g., fluid problems having boundaries at infinity are better treated by the boundary element method.
4. For some problems, there may be a considerable amount of input data. Errors may creep up in their preparation and the results thus obtained may also appear to be acceptable which indicates deceptive state of affairs. It is always desirable to make a visual check of the input data.
5. In the FEM, many problems lead to round-off errors. Computer works with a limited number of digits and solving the problem with restricted number of digits may not yield the desired degree of accuracy or it may give total erroneous results insome cases. For many problems the increase in the number of digits for the purpose of calculation improves the accuracy.

Applications of FEM

1. Mechanical engineering: In mechanical engineering, FEM applications include steady and transient thermal analysis in solids and fluids, stress analysis in solids, automotive design and analysis and manufacturing process simulation.

2. Geotechnical engineering: FEM applications include stress analysis, slope stability analysis, soil structure interactions, seepage of fluids in soils and rocks, analysis of dams, tunnels, bore holes, propagation of stress waves and dynamic soil structure interaction.

3. Aerospace engineering: FEM is used for several purposes such as structural analysis for natural frequencies, modes shapes, response analysis and aerodynamics.

4. Nuclear engineering: FEM applications include steady and dynamic analysis of reactor containment structures, thermo-viscoelastic analysis of reactor components, steady and transient temperature-distribution analysis of reactors and related structures.

5. Electrical and electronics engineering: FEM applications include electrical network analysis, electromagnetics, insulation design analysis in high-voltage equipments, dynamic analysis of motors and heat analysis in electrical and electronic equipments.

6. Metallurgical, chemical engineering: In metallurgical engineering, FEM is used for the metallurgical process simulation, moulding and casting. In chemical engineering, FEM can be used in the simulation of chemical processes, transport processes and chemical reaction simulations.

7. Meteorology and bio-engineering: In the recent times, FEM is used in climate predictions, monsoon prediction and wind predictions. FEM is also used in bio-engineering for the simulation of various human organs, blood circulation prediction and even total synthesis of human body.

8 Civil Engineering Structure: Finite element analysis (FEA) is an extremely useful tool in the field of civil engineering for numerically approximating physical structures that are too complex for regular analytical solutions. Consider a concrete beam with support at both ends, facing a concentrated load on its center span. The deflection at the center span can be determined mathematically in a relatively simple way, as the initial and boundary conditions are finite and in control. However, once you transport the same beam into a practical application, such as within a bridge, the forces at play become much more difficult to analyze with simple mathematics.

finite element method vs classical method

Classical Methods	Finite Element method
1) Exact equations are formed and exact solutions are obtained.	1) Exact equations are formed but approximate solutions are formed.
2) Solutions can be obtained for few standard cases.	2) Solution can be obtained for all problems.
3) For the solution of shape, Boundary conditions and loading some assumptions are made.	3) No assumptions are made problem is treated as it is.
4) When material is not isotropic, solution for the problems becomes very difficult.	4) All type of property can handle without any difficulty.
5) If structure consist more than one material, it is difficult to analyze.	6) If structure consist more than one material then it can be analyzed without any difficulty.

POTENTIAL ENERGY AND EQUILIBRIUM; THE RAYLEIGH–RITZ METHOD

In mechanics of solids, our problem is to determine the displacement $\mathbf{u}$ of the body shown in Fig. 1.1, satisfying the equilibrium equations 1.6. Note that stresses are related to strains, which, in turn, are related to displacements. This leads to requiring solution of second-order partial differential equations. Solution of this set of equations is generally referred to as an *exact* solution. Such exact solutions are available for simple geometries and loading conditions, and one may refer to publications in theory of elasticity. For problems of complex geometries and general boundary and loading conditions, obtaining such solutions is an almost impossible task. Approximate solution methods usually employ potential energy or variational methods, which place less stringent conditions on the functions.

Potential Energy, Π

The total potential energy Π of an elastic body, is defined as the sum of total strain energy (U) and the work potential:

$$\Pi = \underset{(U)}{\text{Strain energy}} + \underset{(\mathrm{WP})}{\text{Work potential}} \tag{1.24}$$

For linear elastic materials, the strain energy per unit volume in the body is $\frac{1}{2}\boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}$. For the elastic body shown in Fig. 1.1, the total strain energy U is given by

$$U = \frac{1}{2}\int_V \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\epsilon}\, dV \tag{1.25}$$

The work potential WP is given by

$$\mathrm{WP} = -\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\, dV - \int_S \mathbf{u}^{\mathrm{T}}\mathbf{T}\, dS - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{1.26}$$

The total potential for the general elastic body shown in Fig. 1.1 is

Principle of Minimum Potential Energy

For conservative systems, of all the kinematically admissible displacement fields, those corresponding to equilibrium extremize the total potential energy. If the extremum condition is a minimum, the equilibrium state is stable.

Example 1.2

The potential energy for the linear elastic one-dimensional rod (Fig. E1.2), with body forc neglected, is

$$\Pi = \frac{1}{2}\int_0^L EA\left(\frac{du}{dx}\right)^2 dx - 2u_1$$

where $u_1 = u(x = 1)$.

Let us consider a polynomial function

$$u = a_1 + a_2x + a_3x^2$$

This must satisfy $u = 0$ at $x = 0$ and $u = 0$ at $x = 2$. Thus,

$$0 = a_1$$
$$0 = a_1 + 2a_2 + 4a_3$$

Hence,

$$a_2 = -2a_3$$
$$u = a_3(-2x + x^2) \qquad u_1 = -a_3$$

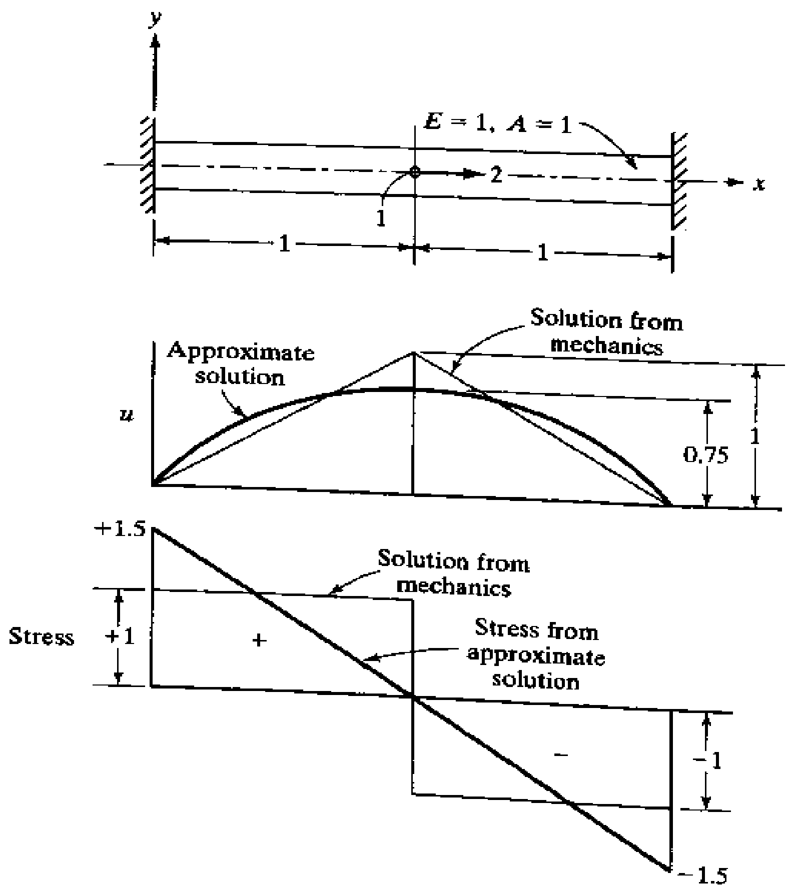


FIGURE E1.2

Then $du/dx = 2a_3(-1 + x)$ and

$$\Pi = \frac{1}{2}\int_0^2 4a_3^2(-1 + x)^2\,dx - 2(-a_3)$$
$$= 2a_3^2 \int_0^2 (1 - 2x + x^2)\,dx + 2a_3$$
$$= 2a_3^2\left(\tfrac{2}{3}\right) + 2a_3$$

We set $\partial\Pi/\partial a_3 = 4a_3\left(\frac{2}{3}\right) + 2 = 0$, resulting in

$$a_3 = -0.75 \qquad u_1 = -a_3 = 0.75$$

The stress in the bar is given by

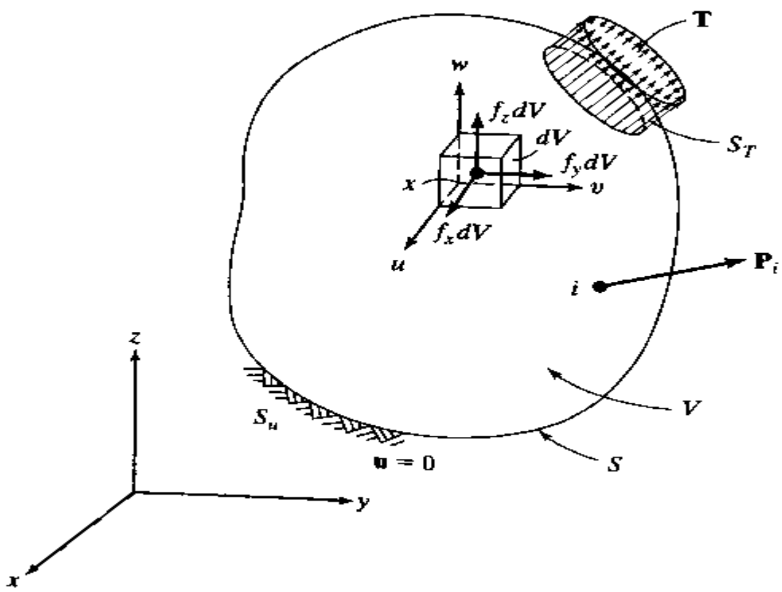
$$\sigma = E\frac{du}{dx} = 1.5(1 - x)$$ ■

We note here that an exact solution is obtained if piecewise polynomial interpolation is used in the construction of u.

The finite element method provides a systematic way of constructing the basis functions ϕ_i used in Eq. 1.30.

STRESSES AND EQUILIBRIUM

A three-dimensional body occupying a volume V and having a surface S is shown in Fig. 1.1. Points in the body are located by x, y, z coordinates. The boundary is constrained on some region, where displacement is specified. On part of the boundary, dis-



tributed force per unit area **T**, also called traction, is applied. Under the force, the body deforms. The deformation of a point **x** ($= [x, y, z]^T$) is given by the three components of its displacement:

$$\mathbf{u} = [u, v, w]^T \tag{1.1}$$

The distributed force per unit volume, for example, the weight per unit volume, is the vector **f** given by

$$\mathbf{f} = [f_x, f_y, f_z]^T \tag{1.2}$$

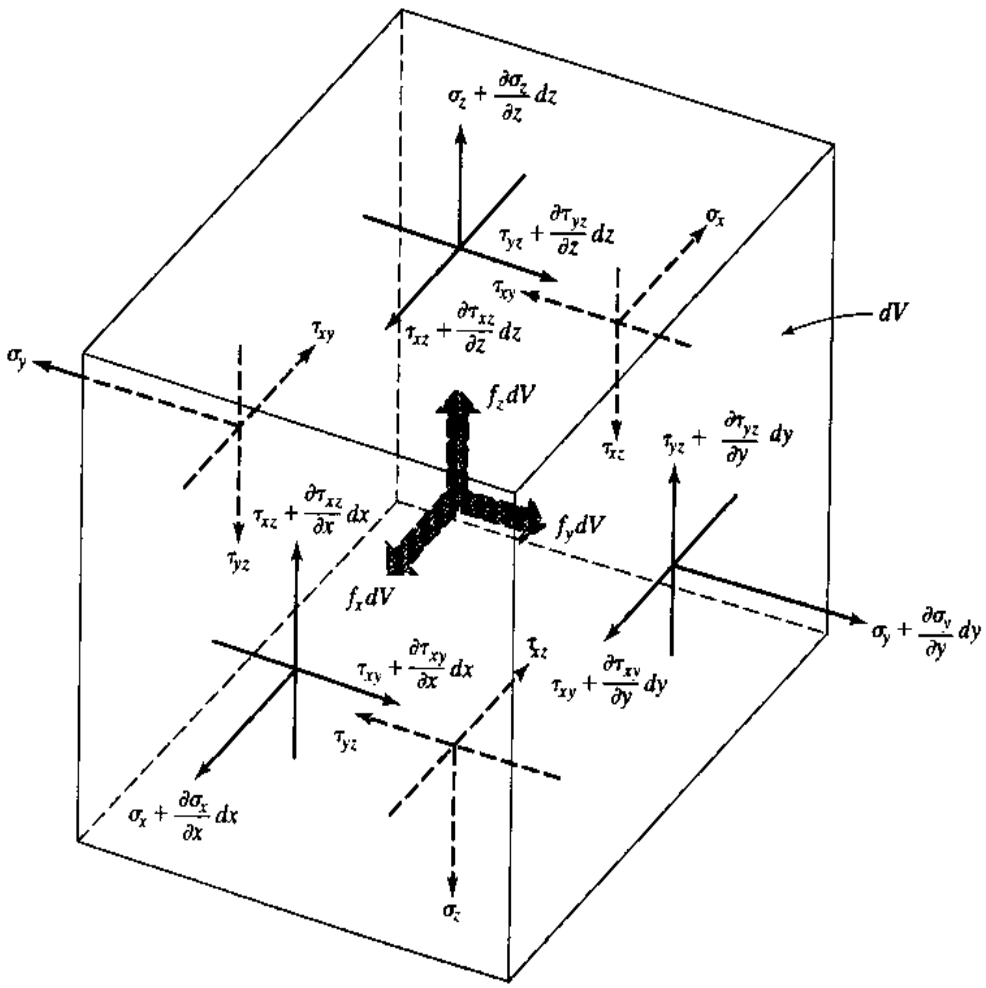
The body force acting on the elemental volume dV is shown in Fig. 1.1. The surface traction **T** may be given by its component values at points on the surface:

$$\mathbf{T} = [T_x, T_y, T_z]^T \tag{1.3}$$

Examples of traction are distributed contact force and action of pressure. A load **P** acting at a point i is represented by its three components:

$$\mathbf{P}_i = [P_x, P_y, P_z]_i^T \tag{1.4}$$

The stresses acting on the elemental volume dV are shown in Fig. 1.2. When the volume dV shrinks to a point, the stress tensor is represented by placing its components in a



(3 × 3) symmetric matrix. However, we represent stress by the six independent components as in

$$\boldsymbol{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy}]^{\mathrm{T}} \tag{1.5}$$

where $\sigma_x, \sigma_y, \sigma_z$ are normal stresses and $\tau_{yz}, \tau_{xz}, \tau_{xy}$, are shear stresses. Let us consider equilibrium of the elemental volume shown in Fig. 1.2. First we get forces on faces by multiplying the stresses by the corresponding areas. Writing $\Sigma F_x = 0$, $\Sigma F_y = 0$, and $\Sigma F_z = 0$ and recognizing $dV = dx\, dy\, dz$, we get the equilibrium equations

$$\begin{aligned}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + f_x &= 0 \\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + f_y &= 0 \\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z &= 0
\end{aligned} \tag{1.6}$$

Special Cases

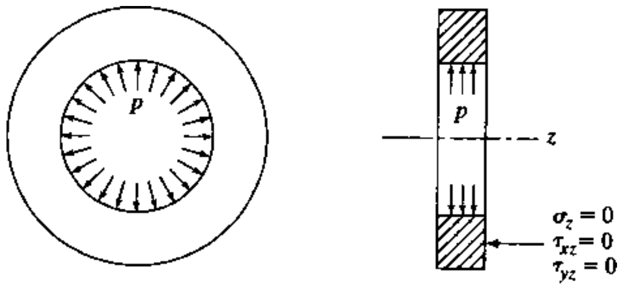
One dimension. In one dimension, we have normal stress σ along x and the corresponding normal strain ϵ. Stress–strain relations (Eq. 1.14) are simply

$$\sigma = E\epsilon \tag{1.16}$$

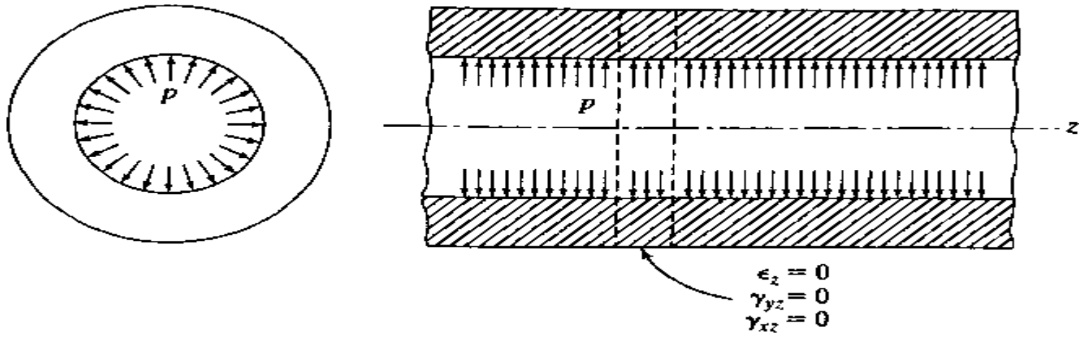
Two dimensions. In two dimensions, the problems are modeled as plane stress and plane strain.

Plane Stress. A thin planar body subjected to in-plane loading on its edge surface is said to be in plane stress. A ring press fitted on a shaft, Fig. 1.5a, is an example. Here stresses σ_z, τ_{xz}, and τ_{yz} are set as zero. The Hooke's law relations (Eq. 1.11) then give us

$$\begin{aligned}
\epsilon_x &= \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} \\
\epsilon_y &= -\nu\frac{\sigma_x}{E} + \frac{\sigma_y}{E} \\
\gamma_{xy} &= \frac{2(1+\nu)}{E}\tau_{xy} \\
\epsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y)
\end{aligned} \tag{1.17}$$



(a)



(b)

FIGURE 1.5 (a) Plane stress and (b) plane strain.

The inverse relations are given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{1.18}$$

which is used as $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\epsilon}$.

Plane Strain. If a long body of uniform cross section is subjected to transverse loading along its length, a small thickness in the loaded area, as shown in Fig. 1.5b, can be treated as subjected to plane strain. Here ϵ_z, γ_{zx}, γ_{yz} are taken as zero. Stress σ_z may not be zero in this case. The stress–strain relations can be obtained directly from Eqs. 1.14 and 1.15:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \frac{1}{2}-\nu \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \tag{1.19}$$

D here is a (3 × 3) matrix, which relates three stresses and three strains.

Anisotropic bodies, with uniform orientation, can be considered by using the appropriate **D** matrix for the material.

Strain Displacement Relationship for Axisymmentric element:

Consider an axisymmetric ring element and its cross section to represent the general state of strain for an axisymmetric problem. The displacements can be expressed for element *ABCD* in the plane of a cross-section in cylindrical coordinates. We then let *u* and *w* denote the displacements in the radial and longitudinal directions, respectively. The side *AB* of the element is displaced an amount *u,* and side *CD* is then displaced an amount *u* + in the radial direction.

The strain in the tangential direction depends on the tangential displacement v and on the radial displacement u.
However, for axisymmetric deformation behavior, recall that the tangential displacement v is equal to zero.

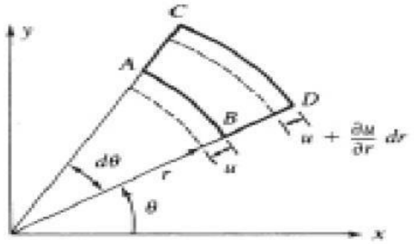
The tangential strain is due only to the radial displacement.

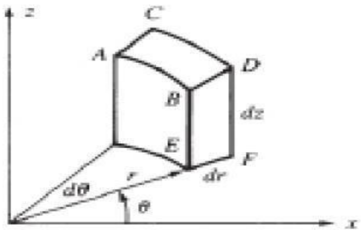
Having only radial displacement u, the new length of the arc AB is $(r+u)d\theta$, and the tangential strain is then given by:

$$\varepsilon_\theta = \frac{(r+u)d\theta - rd\theta}{rd\theta} = \frac{u}{r}$$

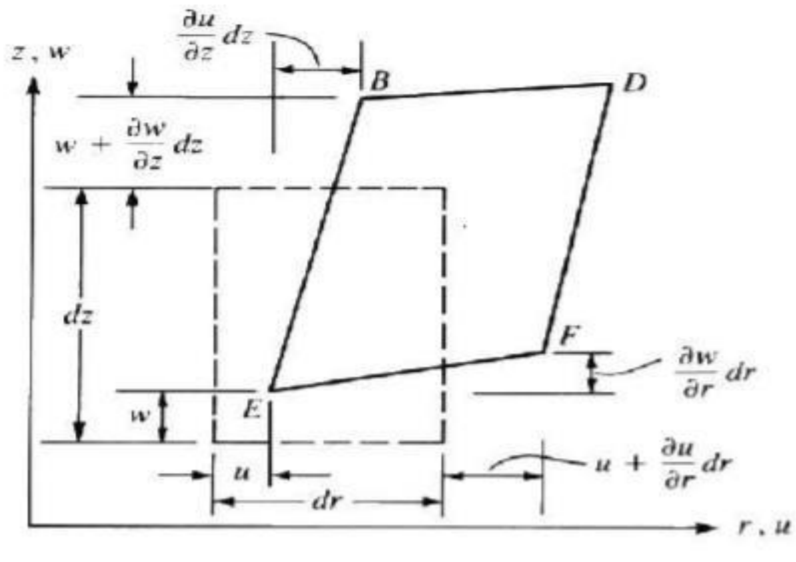
Consider the longitudinal element BDEF to obtain the longitudinal strain and the shear strain. The element displaces by amounts u and w in the radial and longitudinal directions at point E.

The element displaces additional amounts: $(\partial w/\partial z)dz$ along line BE and $(\partial u/\partial r)dr$ along line EF.





The normal strain in the radial direction is then given by: $\varepsilon_r = \frac{\partial u}{\partial r}$



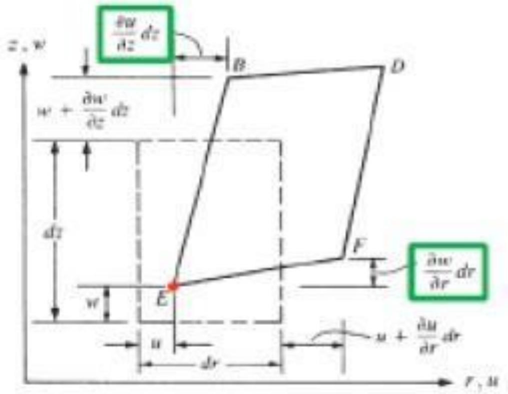
Furthermore, observing lines EF and BE, we see that point F moves upward an amount $(\partial w/\partial r)dr$ with respect to point E and point B moves to the right an amount $(\partial u/\partial z)dz$ with respect to point E.

The longitudinal normal strain is given by:

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

The shear strain in the *r-z* plane is:

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$



Summarizing the strain-displacement relationships gives:

$$\varepsilon_r = \frac{\partial u}{\partial r} \qquad \varepsilon_\theta = \frac{u}{r} \qquad \varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

UNIT –II
ONE DIMENSIONAL & TWO DIMENSIONAL ELEMENTS

COORDINATES AND SHAPE FUNCTIONS

Consider a typical finite element e in Fig. 3.5a. In the local number scheme, the first node will be numbered 1 and the second node 2. The notation x_1 = x-coordinate of node 1, x_2 = x-coordinate of node 2 is used. We define a **natural** or **intrinsic** coordinate system, denoted by ξ, as

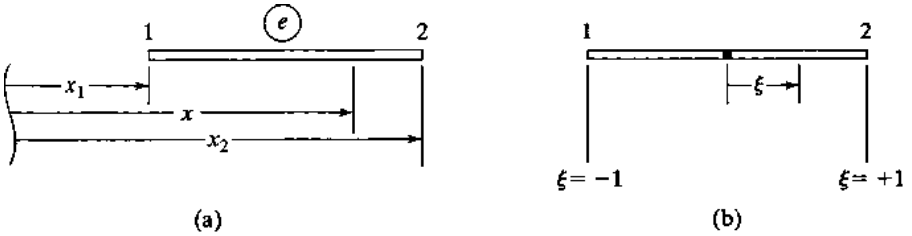


FIGURE 3.5 Typical element in x- and ξ-coordinates.

$$\xi = \frac{2}{x_2 - x_1}(x - x_1) - 1 \tag{3.4}$$

From Fig. 3.5b, we see that $\xi = -1$ at node 1 and $\xi = 1$ at node 2. The length of an element is covered when ξ changes from -1 to 1. We use this system of coordinates in defining shape functions, which are used in interpolating the displacement field.

Now *the unknown displacement field within an element will be interpolated by a linear distribution* (Fig. 3.6). This approximation becomes increasingly accurate as more elements are considered in the model. To implement this linear interpolation, linear **shape functions** will be introduced as

$$N_1(\xi) = \frac{1 - \xi}{2} \tag{3.5}$$

$$N_2(\xi) = \frac{1 + \xi}{2} \tag{3.6}$$

The shape functions N_1 and N_2 are shown in Figs. 3.7a and b, respectively. The graph of the shape function N_1 in Fig. 3.7a is obtained from Eq. 3.5 by noting that $N_1 = 1$ at $\xi = -1$, $N_1 = 0$ at $\xi = 1$, and N_1 is a straight line between the two points. Similarly, the graph of N_2 in Fig. 3.7b is obtained from Eq. 3.6. Once the shape functions are defined, the linear displacement field within the element can be written in terms of the nodal displacements q_1 and q_2 as

$$u = N_1 q_1 + N_2 q_2 \tag{3.7a}$$

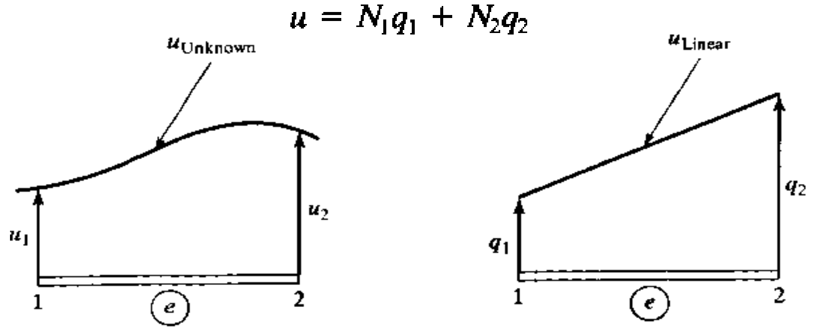


FIGURE 3.6 Linear interpolation of the displacement field within an element.

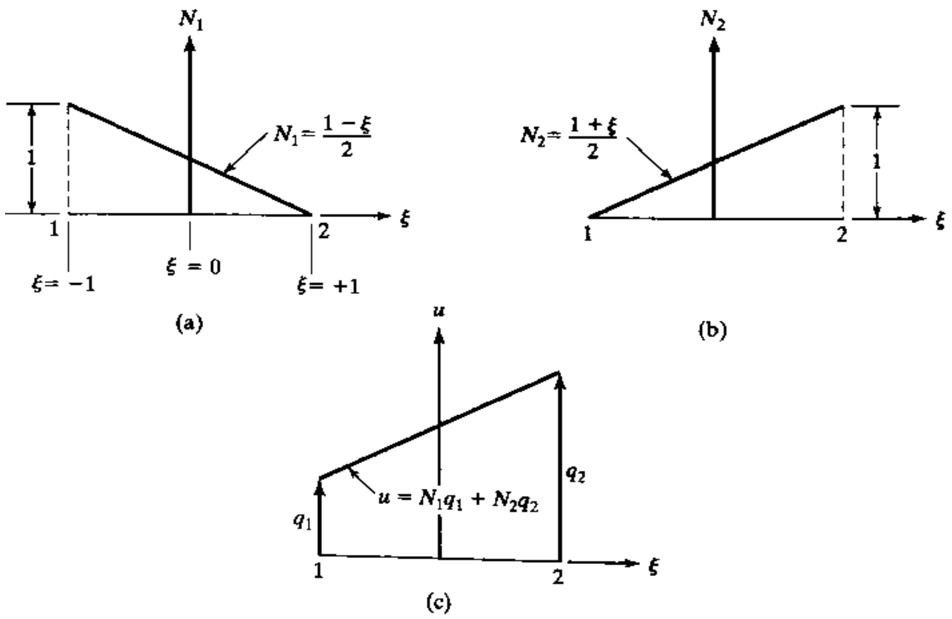


FIGURE 3.7 (a) Shape function N_1, (b) shape function N_2, and (c) linear interpolation using N_1 and N_2.

or, in matrix notation, as

$$u = \mathbf{Nq} \tag{3.7b}$$

where

$$\mathbf{N} = [N_1, N_2] \quad \text{and} \quad \mathbf{q} = [q_1, q_2]^T \tag{3.8}$$

In these equations, $\mathbf{q}$ is referred to as the *element displacement vector*. It is readily verified from Eq. 3.7a that $u = q_1$ at node 1, $u = q_2$ at node 2, and that u varies linearly (Fig. 3.7c).

It may be noted that the transformation from x to ξ in Eq. 3.4 can be written in terms of N_1 and N_2 as

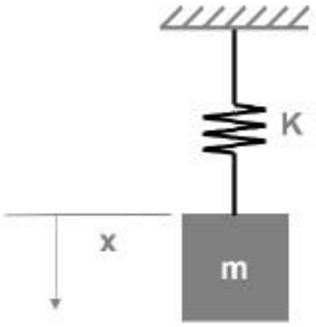
$$x = N_1 x_1 + N_2 x_2 \tag{3.9}$$

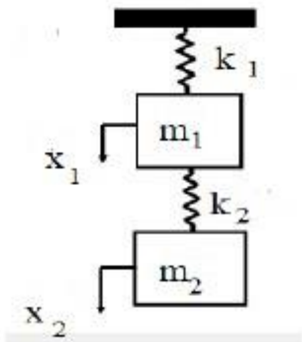
Comparing Eqs. 3.7a and 3.9, we see that both the displacement u and the coordinate x are interpolated within the element using the *same* shape functions N_1 and N_2. This is referred to as the *isoparametric* formulation in the literature.

Though linear shape functions have been used previously, other choices are possible. Quadratic shape functions are discussed in Section 3.9. In general, shape functions need to satisfy the following:

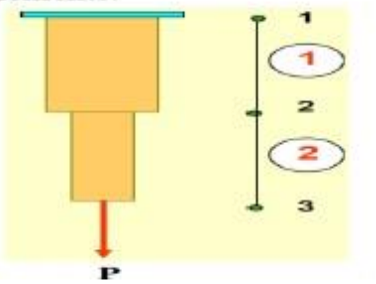
Degrees of Freedom in FEA:

- Degree of Freedom (DoF) is a “possibility” to move in a defined direction. There are 6 DoF in a 3D space: you can move or rotate along axis x, y or z. Together, those components describe a motion in 3D. DoF in FEA also do other things: they control supports, information about stresses and more.
- Degree of freedom or DOF means the number of independent coordinates a structure can move. There are 6 DOF possible for a structure. They are movement on x,y and z axis and rotation about these axis.
- Whatever be the field, degree of freedom, dof in short, represents the minimum number of independent coordinates required to specify the position of every mass in the system uniquely.
- eg. A simple spring mass system as shown in Fig. which is constrained to move only in the vertical direction requires the displacement x only to specify the position of the mass m. Hence it has one degree of freedom.
- If we attach another spring and another mass below the first mass then each mass will undergo different displacement and hence we need to specify x1 and x2 which are the displacements of masses 1 and 2 respectively. Hence thishas 2 dof.

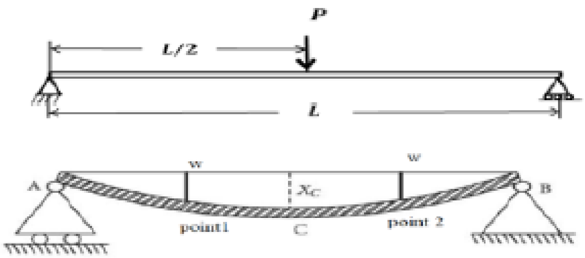




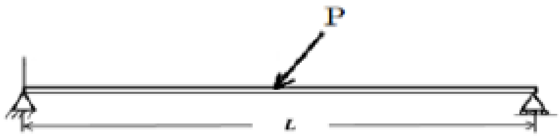
- Now coming to FEM when we want to find the stresses in a member subjected to axial loads such as the stepped bar shown below, since the bar is long and thin we can assume that the longitudinal displacements are significantly higher than the lateral displacements. So neglecting this lateral displacement we can discretise this system into two 2 noded elements.



- When we use a 2 noded element it is assumed that there is only on degree of freedom at each node namely the axial displacement of that node. So total degrees of freedom in this case for one element is 2.
- On the other hand in a beam element that is subjected to only vertical transverse loads, we require minimum 2 dofs. Why is this so?
- Take the case of a simply supported beam subjected to a central point load as shown in the figure below.



- If we specify the position of every point in the beam with only one variable namely the transverse displacement w, then if we look for two symmetrically placed points along the beam such as points 1 and 2, the displacements will be the same and equal to w. So if have to specifically refer to only one point uniquely we need one more variable that can be used to identify that point. Hence we introduce another variable namely the slope of the deflection curve.
- So a simple beam subjected to only vertical loads can be modelled using a beam element that has 2 dofs per node namely So total dof s for one two noded beam element is 2 x 2 =4.
- If the beam is subjected to a load as shown below



- Then there is an axial displacement that comes into the picture additionally. So we have to introduce one more dof namely axial displacement u at each node thus bringing the dof per node to 3 and total dof to 6.
- Similarly a 3 noded triangular element used to model a thin rectangular fin has one dof (variable) per node namely temperature so total dof is 3 x 1 =3. In a structural application there will be two dof per node namely u and v displacements. Hence total dof for a 3 noded triangular element for stress analysis will be 3 x 2 =6.
- A 4 noded tetrahedral solid element has 3 dof per node (u,v,and v displacements) when used in structural applications so total dof is 4 x 3 =12.
- So we need to understand the physical behaviour of the system and model it appropriately.

Shape Functions:

In the finite element analysis aim is to find the field variables at nodal points by rigorous analysis, assuming at any point inside the element basic variable is a function of values at nodal points of the element. This function which relates the field variable at any point within the element to the field variables of nodal points is called shape function. This is also called as interpolation function and approximating function. In two dimensional stress analysis in which basic field variable is displacement,

Shape functions are the polynomials meant to describe the variation of primary variable along the domain of element.

$$u = \Sigma N_i u_i, \; v = \Sigma N_i v_i \qquad \ldots(5.1)$$

where summation is over the number of nodes of the element. For example for three noded triangular element, displacement at $P(x, y)$ is

$$u = \Sigma N_i u_i = N_1 u_1 + N_2 u_2 + N_3 u_3$$

$$v = \Sigma N_i v_i = N_1 v_1 + N_2 v_2 + N_3 v_3$$

i.e.,

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N & 0 \\ 0 & N \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} N_1 & N_2 & N_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & N_1 & N_2 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{Bmatrix}$$

$$\underset{2\times 1}{\{\delta\}} = \underset{2\times 6}{[N]} \underset{6\times 1}{\{\delta\}_e}$$

where q is displacement at any point in the element

$[N]$ shape function

$\{\delta\}_e$ is vector of nodal displacements

Similarly in case of 6 noded triangular element

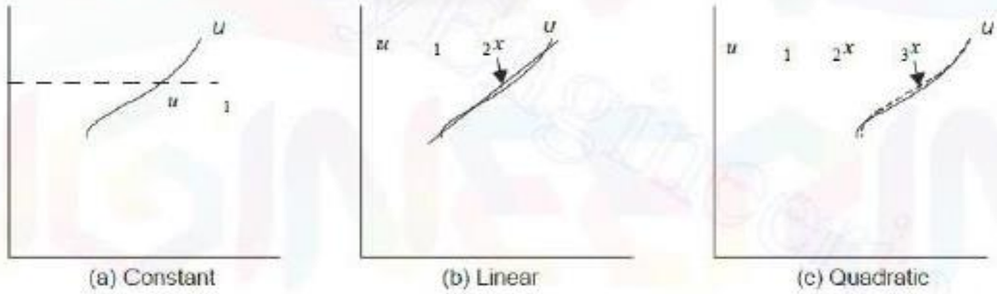
$$\underset{2\times 1}{\{\delta\}} = \underset{2\times 12}{[N]} \underset{12\times 1}{\{\delta\}_e}$$

POLYNOMIAL SHAPE FUNCTIONS

Polynomials are commonly used as shape functions. There are two reasons for using them:

(i) They are easy to handle mathematically i.e. differentiation and integration of polynomials is easy.

(ii) Using polynomial any function can be approximated reasonably well. If a function is highly nonlinear we may have to approximate with higher order polynomial. Fig. 5.1 shows approximation of a nonlinear one dimensional function by polynomials of different order.



Approximation with polynomials

One Dimensional Polynomial Shape Function

A general one dimensional polynomial shape function of nth Order is given by,

$$u(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \dots \alpha_{n+1} x^n$$

In matrix form $u = [G]\{\alpha\}$

where $$[G] = [1, x, x^2 \dots x^n]$$

and $$\{\alpha\}^T = [\alpha_1 \; \alpha_2 \; \alpha_3 \dots \alpha_{n+1}]$$

Thus in one dimensional n^{th} order complete polynomial there are $m = n + 1$ terms.

Two Dimensional Polynomial Shape Function

A general form of two dimensional polynomial model is

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 + \alpha_7 x^3 \dots + \alpha_m y^n$$
$$v(x, y) = \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \dots + \alpha_{2m} y^n \quad \dots(5.6)$$

or $$\{\delta\} = \begin{Bmatrix} u(x, y) \\ v(x, y) \end{Bmatrix} = [G]\{\alpha\} = \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix}\{\alpha\} \quad \dots(5.7)$$

where $$G_1 = [1 \; x \; y \; x^2 \; xy \; y^2 \; x^3 \dots y^n]$$

$$\{\alpha\}^T = [\alpha_1 \; \alpha_2 \; \alpha_3 \; \alpha_4 \dots \alpha_{2m}]$$

It may be observed that in two dimensional problem, total number of terms m in a complete nth degree polynomial is

$$m = \frac{(n+1)(n+2)}{2} \quad \dots(5.8)$$

For first order complete polynomial $n = 1$,

$$\therefore m = \frac{(1+1)(1+2)}{2} = 3$$

Another convenient way to remember complete two dimensional polynomial is in the form of Pascal Triangle shown in Fig. 5.2

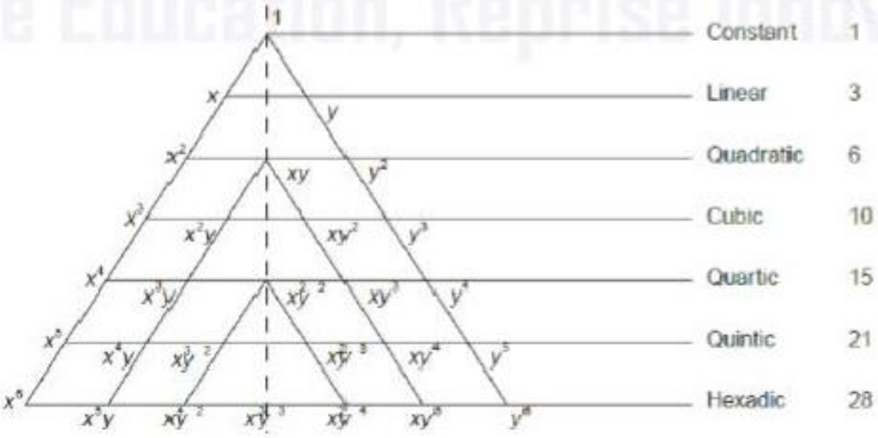


Fig. 5.2 *Pascal triangle*

Three Dimensional Polynomial Shape Function

A general three dimensional shape function of nth order complete polynomial is given by

$$\begin{aligned} u(x, y, z) &= \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z + \alpha_5 x^2 + \ldots + \alpha_m x^{n-1} z \\ v(x, y, z) &= \alpha_{m+1} + \alpha_{m+2} x + \alpha_{m+3} y + \alpha_{m+4} z + \alpha_{m+5} x^2 + \ldots + \alpha_{2m} x^{n-1} z \\ w(x, y, z) &= \alpha_{2m+1} + \alpha_{2m+2} x + \alpha_{2m+3} y + \alpha_{2m+4} z + \ldots + \alpha_{3m} x^{n-1} z \end{aligned} \qquad \ldots(5.9)$$

or $$\delta(x, y, z) = \begin{Bmatrix} u(x, y, z) \\ v(x, y, z) \\ w(x, y, z) \end{Bmatrix} = [G]\{\alpha\} = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & G_1 & 0 \\ 0 & 0 & G_1 \end{bmatrix} \{\alpha\} \qquad \ldots(5.10)$$

Where $G_1 = [1\ x\ y\ z\ x^2\ xy\ y^2\ yz\ z^2\ zx\ \ldots z^n\ z^{n-1}x\ \ldots\ zx^{n-1}]$

and $$\{\alpha\}^T = [\alpha_1\ \alpha_2\ \alpha_3\ \ldots\ \alpha_{3m}]$$

It may be observed that a complete nth order polynomial in three dimensional case is having number of terms m given by the expression

$$m = \frac{(n+1)(n+2)(n+3)}{6}$$

Thus when $n = 1$, $m = \dfrac{(1+1)(1+2)(1+3)}{6} = 4$

i.e. $$\alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 z$$

5.3 CONVERGENCE REQUIREMENTS OF SHAPE FUNCTIONS

Numerical solutions are approximate solutions. Stiffness coefficients for a displacements model have higher magnitudes compared to those for the exact solutions. In other words the displacements obtained by finite element analysis are lesser than the exact values. Thus the FEM gives lower bound values. Hence it is desirable that as the finite element analysis mesh is refined, the solution approaches the exact values. This requirement is shown graphically in Fig. 5.4. In order to ensure this convergence criteria, the shape functions should satisfy the following requirement:

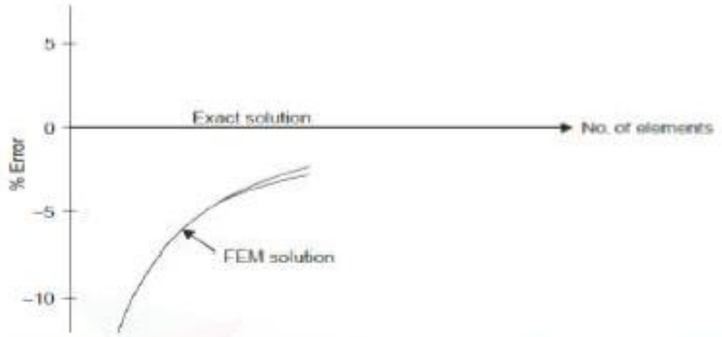


Fig. 5.4 *Convergence of FEM solution*

1. The displacement models must be continuous within the elements and the displacements must be compatible between the adjacent elements. The second part implies that the adjacent elements must deform without causing openings, overlaps or discontinuities between the elements. This requirement is called **'compatibility requirement'**.

According to Felippa and Clough this requirement is satisfied, if the displacement and its partial derivatives upto one order less than the highest order derivative appearing in strain energy function is continuous. Hence in plane stress and plane strain problems, it is enough if continuity of displacement is satisfied, since strain energy function includes only first order derivatives of the displacement (SE = ½ stress × strain). It implies, it is enough if C^0 continuity is ensured in plane stress and plane strain problems. In case of flexure problems (beams, plates, shells) the strain energy terms include second derivatives of displacements $\left(\text{like } \frac{1}{2}\frac{M^2}{EI} \text{ where } M = -EI\frac{d^2w}{dx^2}\right)$.

Hence to satisfy compatibility requirement, not only displacement continuity but slope continuity (C^1-continuity) should be satisfied. Hence in flexure problems displacements and their first derivatives are selected as nodal field variables.

2. The displacement model should include the **rigid body displacements** of the element. It means in displacement model there should be a term which permit all points on the element to experience the same displacement. It is obvious, if such term do not exists, shifting of the origin of the coordinate system will cause additional stresses and strains, which should not occur. In the displacement model,

$$u = \alpha_1 + \alpha_2 x + \alpha_3 y$$

the term α_1 provides for the rigid body displacement. Hence to satisfy the requirement of rigid body displacement, there should be constant term in the shape function selected.

3. The displacement models must include the **constant strain state** of the element. This means, there should exist combination of values of polynomial terms that cause all points in the element to experience the same strain. One such combination should occur for each possible strain. The necessity of this requirement is understood physically, if we imagine the refinement of the mesh. As these elements approach infinitesimal size, the strains within the element approach constant values. Unless the shape function term includes these constant strain terms, we cannot hope to converge to a correct solution. In the displacement model,

$$v = \alpha_{m+1} + \alpha_{m+2}\,x + \alpha_{m+3}\,y + \alpha_{m+4}\,x^2 + \ldots + \alpha_{2m}\,y^n$$

α_2 and α_{m+2} provide for uniform strain ε_x,

α_3 and α_{m+3} provide for uniform strain ε_y

An additional consideration in the selection of polynomial shape function for the displacement model is that the pattern should be independent of the orientation of the local coordinate system. This property is known as **Geometric Isotropy, Spatial Isotropy** or **Geometric Invariance.** There are two simple guidelines to construct polynomial series with the desired property of isotropy:

1. Polynomial of order n that are complete, have geometric isotropy.
2. Polynomial of order n that are not complete, yet contain appropriate terms to preserve 'symmetry' have geometric isotropy. The simple test for this property is to interchange x and y in two dimensional problems or x, y, z in cyclic order in three dimensional problems and see that the total expression do not change. However the arbitrary constants may change.

   For example, we wish to construct a cubic polynomial expression for an element that has eight nodal values assigned to it. In this situation, we have to drop two terms from the complete cubic polynomial which contains 10 terms. To maintain geometric isotropy drop only terms that occur in symmetric pairs i.e. x^3, y^3 or x^2y, xy^2. Thus the acceptable eight term cubic polynomials shape function exhibiting geometric isotropy are

$$\alpha_1 + \alpha_2\,x + \alpha_3\,y + \alpha_4\,x^2 + \alpha_5\,xy + \alpha_6\,y^2 + \alpha_7\,x^2y + \alpha_8\,xy^2$$

and $$\alpha_1 + \alpha_2\,x + \alpha_3\,y + \alpha_4\,x^2 + \alpha_5\,xy + \alpha_6\,y^2 + \alpha_7\,x^3 + \alpha_8\,y^3$$

$$\alpha_1 + \alpha_2\,x + \alpha_3\,y + \alpha_4\,x^2 + \alpha_5\,xy + \alpha_6\,y^2 + \alpha_7\,x^2y + \alpha_8\,xy^2$$

and $$\alpha_1 + \alpha_2\,x + \alpha_3\,y + \alpha_4\,x^2 + \alpha_5\,xy + \alpha_6\,y^2 + \alpha_7\,x^3 + \alpha_8\,y^3$$

In finite element analysis, the safest approach to reach correct solution is to pick the shape functions that satisfy all the requirements. For some problems, however, choosing shape functions that meet all the requirements may be difficult and may involve excessive numerical computations. For this reason some investigators have ventured to formulate shape functions for the elements that do not meet compatibility requirements. In some cases acceptable convergence has been obtained. Such elements are called '**non-conforming elements**'. The main disadvantage of using non-conforming elements is that we no longer know in advance that correct solution is reached.

Characteristic of Shape function

1. Value of shape function of particular node is one and is zero to all other nodes.
2. Sum of all shape function is one.
3. Sum of the derivative of all the shape functions for a particular primary variable is zero.

Coordinate Systems

The following terms are commonly referred in FEM
(i) Global coordinates
(ii) Local coordinates and
(iii) Natural coordinates.
However there is another term '**generalized coordinates**' used for defining a polynomial form of interpolation function. This has nothing to do with the 'coordinates' term used here to define the location of points in the element.

Global coordinates

The coordinate system used to define the points in the entire structure is called global coordinate system. Figure 4.14 shows the cartesian global coordinate system used for some of the typical cases.

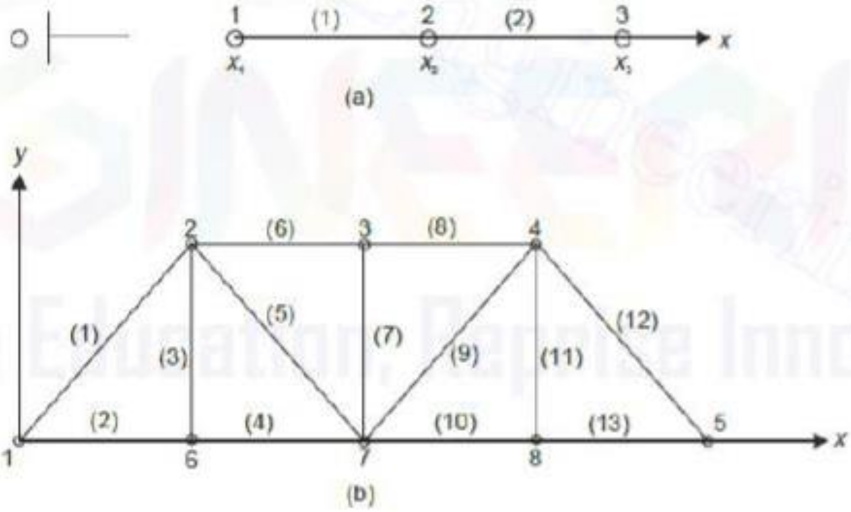


Fig. 4.14 *Global coordinate system*

Local coordinates

For the convenience of deriving element properties, in FEM many times for each element a separate coordinate system is used. For example, for typical elements shown in Fig. 4.14, the local coordinates may be as shown in Fig. 4.15. However the final equations are to be formed in the common coordinate system i.e. global coordinate system only.

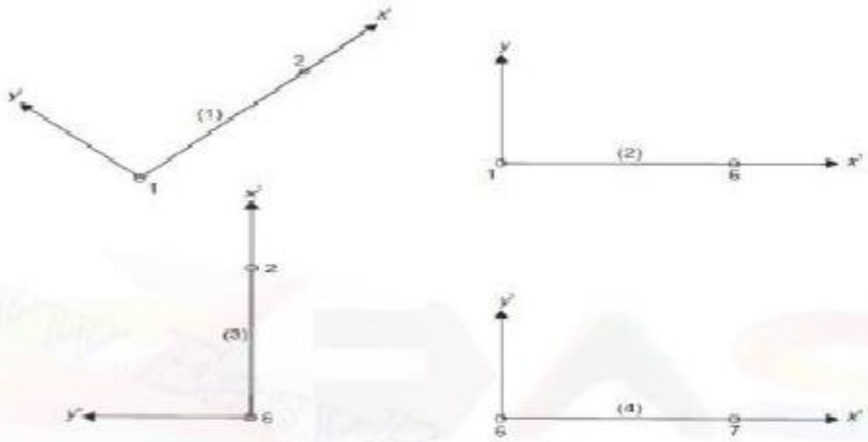


Fig. 4.15 *Local coordinate system*

Natural coordinate

A natural coordinate system is a coordinate system which permits the specification of a point within the element by a set of dimensionless numbers, whose magnitude never exceeds unity. It is obtained by assigning weightages to the nodal coordinates in defining the coordinate of any point inside the element. Hence such system has the property that ith coordinate has unit value at node i of the element and zero value at all other nodes. The use of natural coordinate system is advantages in assembling element properties (stiffness matrices), since closed form integrations formulae are available when the expressions are in natural coordinate systems.

Natural coordinate systems for one dimensional, two dimensional and three dimensional elements are discussed below:

Natural Coordinates in Two Dimensions

Natural coordinates for triangular and rectangular elements are discussed below:

1. Natural Coordinates for Triangular Elements: Consider the typical 3 noded triangular element shown in Fig. 4.20. Since there are three nodes, for any point there are three coordinates, say L_1, L_2 and L_3. From the definition of natural coordinates, we have

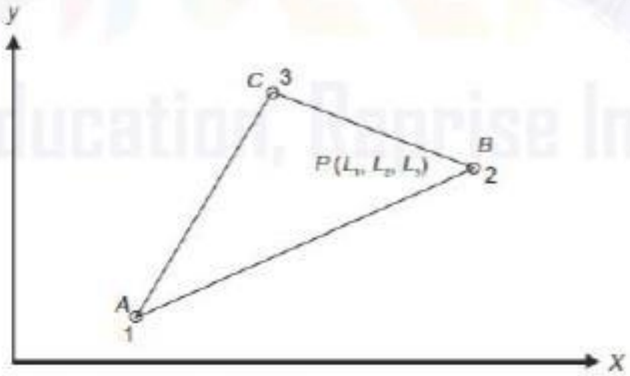


Fig. 4.20 *Typical 3 noded triangular element*

$$L_1 + L_2 + L_3 = 1 \quad \ldots(4.7a)$$
$$L_1 x_1 + L_2 x_2 + L_3 x_3 = x \quad \ldots(4.7b)$$
$$L_1 y_1 + L_2 y_2 + L_3 y_3 = y \quad \ldots(4.7c)$$

Expressing the above equations in matrix form,

$$\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1} \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

It can be shown that the determinant,

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

Proof: Now,

$$\text{Det} = \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2) - (x_1 y_3 - x_3 y_1) + (x_1 y_2 - x_2 y_1)$$

Consider the triangle *ABC* shown in Fig. 4.21. Drop perpendiculars *AD*, *BE* and *CF* on to x-axis.

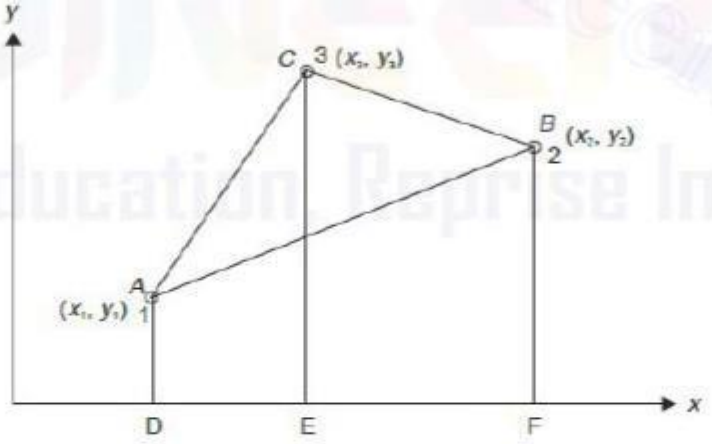


Fig. 4.21

Now, Area of triangle ABC

=Area *ADEC* + Area *CEFB* – Area *ADFB*

$$= \frac{1}{2}(AD + CE)\,DE + \frac{1}{2}(CE + BF)EF - \frac{1}{2}(AD + BF)\,DF$$

$$= \frac{1}{2}(y_1 + y_3)(x_3 - x_1) + \frac{1}{2}(y_3 + y_2)(x_2 - x_3) - \frac{1}{2}(y_1 + y_2)(x_2 - x_1)$$

$$= \frac{1}{2}\left[y_1x_3 - y_1x_1 + y_3x_3 - y_3x_1 + y_3x_2 - y_3x_3 + y_2x_2 - y_2x_3 - y_1x_2 + y_1x_1 - y_2x_2 + y_2x_1\right]$$

$$= \frac{1}{2}\left[y_1x_3 - y_3x_1 + y_3x_2 - y_2x_3 - y_1x_2 + y_2x_1\right]$$

$$= \frac{1}{2}\left[(x_2y_3 - x_3y_2) - (x_1y_3 - x_3y_1) + (x_1y_2 - x_2y_1)\right]$$

$$= \frac{1}{2}\,Det$$

$\therefore$ Det = 2 Area of triangle *ABC* = 2*A* ...(4.8)

$$\therefore \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & x_3y_1 - x_1y_3 & x_1y_2 - x_2y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix}^T \begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

$$= \frac{1}{2A}\begin{bmatrix} x_2y_3 - x_3y_2 & y_2 - y_3 & x_3 - x_2 \\ x_3y_1 - x_1y_3 & y_3 - y_1 & x_1 - x_3 \\ x_1y_2 - x_2y_1 & y_1 - y_2 & x_2 - x_1 \end{bmatrix}\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix} = \frac{1}{2A}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}\begin{Bmatrix} 1 \\ x \\ y \end{Bmatrix}$$

where $a_1 = x_2y_3 - x_3y_2 \qquad a_2 = x_3y_1 - x_1y_3 \qquad a_3 = x_1y_2 - x_2y_1$

$b_1 = y_2 - y_3 \qquad b_2 = y_3 - y_1 \qquad b_3 = y_1 - y_2$

$c_1 = x_3 - x_2 \qquad c_2 = x_1 - x_3 \qquad c_3 = x_2 - x_1$

[Note the cyclic order of subscript and absence of subscript of left hand term in right hand terms]

Thus

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{a_1 + b_1x + c_1y}{2A} \\ \dfrac{a_2 + b_2x + c_2y}{2A} \\ \dfrac{a_3 + b_3x + c_3y}{2A} \end{Bmatrix} \qquad \ldots(4.9)$$

Referring to Fig. 4.22 and applying equation 4.8, we get Area of subtraingle *CPB*

$$= 2A_1 = \begin{bmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{bmatrix}$$

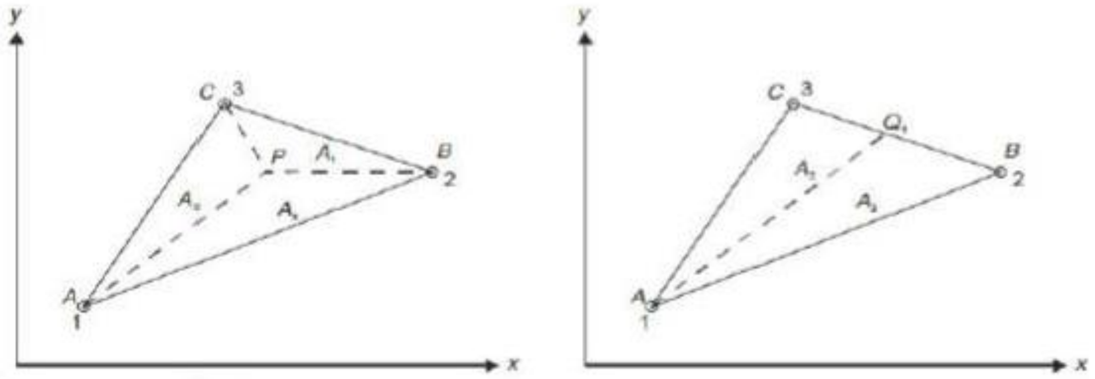


Fig. 4.22 *Area coordinates for a triangle*

i.e., $2A_1 = x_2y_3 - x_3y_2 - (xy_3 - x_3y) + xy_2 - x_2y$

$= x_2y_3 - x_3y_2 + x(y_2 - y_3) + y(x_3 - x_2)$

$= a_1 + b_1x + c_1y$

Thirdly $2A_2 = a_2 + b_2x + c_2y$

and $2A_3 = a_3 + b_3x + c_3y$

$\therefore$ Equation 4.9 reduces to

$$\begin{Bmatrix} L_1 \\ L_2 \\ L_3 \end{Bmatrix} = \begin{Bmatrix} \dfrac{2A_1}{2A} \\ \dfrac{2A_2}{2A} \\ \dfrac{2A_3}{2A} \end{Bmatrix} = \frac{1}{A}\begin{Bmatrix} A_1 \\ A_2 \\ A_3 \end{Bmatrix} \qquad \ldots(4.10)$$

where A_1, A_2 and A_3 are the areas of sub-triangles *PCB*, *PAC* and *PAB*, which are opposite to nodes 1, 2 and 3 respectively. Hence the natural coordinates in triangles are also known as **area coordinates**.

Natural Coordinates in Three Dimensions

Natural coordinates for a 4 noded tetrahedron may be derived and it results into volume coordinates. Consider the typical tetrahedron shown in Fig. 4.24.

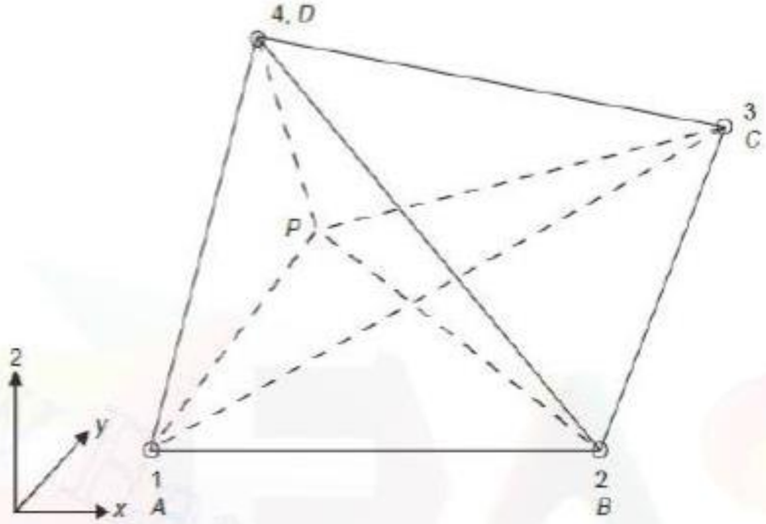


Fig. 4.24 *Tetrahedron coordinates*

The natural coordinates are related to the Cartesian coordinates as follows:

$$\begin{Bmatrix} 1 \\ x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix} \begin{Bmatrix} L_1 \\ L_2 \\ L_3 \\ L_4 \end{Bmatrix} \qquad \dots(4.12)$$

The above equation may be solved by inverting the 4 × 4 matrix. It gives

$$L_i = \frac{1}{6V}(a_i + b_i x + c_i y + d_i z), \text{ for } i = 1, 2, 3 \text{ and } 4 \qquad \dots(4.13)$$

where $6V = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$ = 6 × volume of tetrahedron defined by nodes 1, 2, 3 and 4

and $$a_1 = \begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix} \quad b_1 = \begin{vmatrix} 1 & 1 & 1 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}$$

$$c_1 = \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ z_2 & z_3 & z_4 \end{vmatrix} \quad \text{and} \quad d_1 = \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{vmatrix}$$

The other constants are obtained by cyclic changes in the subscripts. It may be noted that the above equations are valid only when the nodes are numbered so that nodes 1, 2 and 3 are ordered counter clockwise when viewed from node 4. It is also necessary that for coordinates system of right hand rule is strictly adhered to.

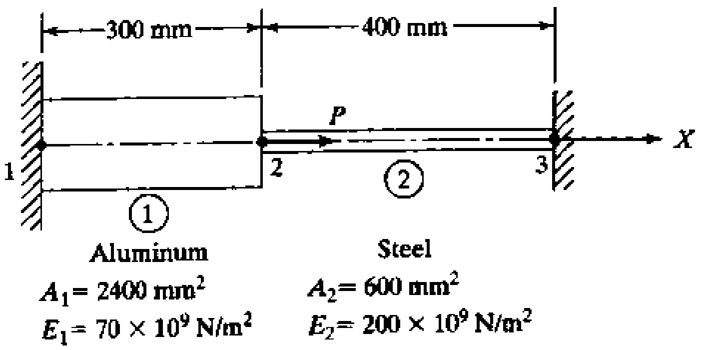
If V_i is the volume of the smaller tetrahedron which has vertices P and the three nodes other than the node i, then the tetrahedron coordinates can be considered as volume coordinates, defined as

$$L_i = \frac{V_i}{V} \quad \text{for } i=1, 2, 3 \text{ and } 4 \qquad \ldots(4.14)$$

Example 3.4

Consider the bar shown in Fig. E3.4. An axial load $P = 200 \times 10^3$ N is applied as shown. Using the penalty approach for handling boundary conditions, do the following:

(a) Determine the nodal displacements.
(b) Determine the stress in each material.
(c) Determine the reaction forces.



Solution

(a) The element stiffness matrices are

$$\begin{array}{ccc} & 1 & 2 \leftarrow \text{Global dof} \end{array}$$

$$\mathbf{k}^1 = \frac{70 \times 10^3 \times 2400}{300}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

and

$$\begin{array}{cc} 2 & 3 \end{array}$$

$$\mathbf{k}^2 = \frac{200 \times 10^3 \times 600}{400}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

The structural stiffness matrix that is assembled from $\mathbf{k}^1$ and $\mathbf{k}^2$ is

$$\begin{array}{ccc} 1 & 2 & 3 \end{array}$$

$$\mathbf{K} = 10^6\begin{bmatrix} 0.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 0.30 \end{bmatrix}$$

The global load vector is

$$\mathbf{F} = [0, \quad 200 \times 10^3, \quad 0]^{\mathrm{T}}$$

Now dofs 1 and 3 are fixed. When using the penalty approach, therefore, a large number C is added to the first and third diagonal elements of $\mathbf{K}$. Choosing C based on Eq. 3.80, we get

$$C = [0.86 \times 10^6] \times 10^4$$

Thus, the modified stiffness matrix is

$$\mathbf{K} = 10^6\begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}$$

The finite element equations are given by

$$10^6\begin{bmatrix} 8600.56 & -0.56 & 0 \\ -0.56 & 0.86 & -0.30 \\ 0 & -0.30 & 8600.30 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 200 \times 10^3 \\ 0 \end{Bmatrix}$$

which yields the solution

$$\mathbf{Q} = [15.1432 \times 10^{-6}, \quad 0.23257, \quad 8.1127 \times 10^{-6}]^{\mathrm{T}}\ \text{mm}$$

(b) The element stresses (Eq. 3.16) are

$$\sigma_1 = 70 \times 10^3 \times \frac{1}{300}[-1 \quad 1]\begin{Bmatrix} 15.1432 \times 10^{-6} \\ 0.23257 \end{Bmatrix}$$

$$= 54.27\ \text{MPa}$$

where 1 MPa = 10^6 N/m^2 = 1 N/mm^2. Also,

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{400}[-1 \quad 1]\begin{Bmatrix} 0.23257 \\ 8.1127 \times 10^{-6} \end{Bmatrix}$$

$$= -116.29 \text{ MPa}$$

(c) The reaction forces are obtained from Eq. 3.78 as

$$\begin{aligned} R_1 &= -CQ_1 \\ &= -[0.86 \times 10^{10}] \times 15.1432 \times 10^{-6} \\ &= -130.23 \times 10^3 \end{aligned}$$

Also,

$$\begin{aligned} R_3 &= -CQ_3 \\ &= -[0.86 \times 10^{10}] \times 8.1127 \times 10^{-6} \\ &= -69.77 \times 10^3 \text{ N} \end{aligned}$$

■

Example 3.5

In Fig. E3.5a, a load $P = 60 \times 10^3$ N is applied as shown. Determine the displacement field, stress, and support reactions in the body. Take $E = 20 \times 10^3$ N/mm^2.

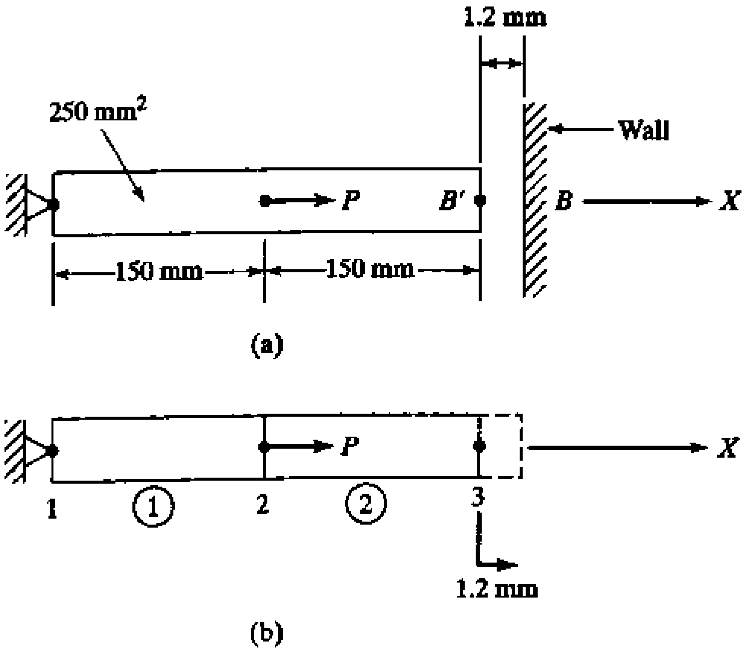


FIGURE E3.5

Solution In this problem, we should first determine whether contact occurs between the bar and the wall, B. To do this, assume that the wall does not exist. Then, the solution to the problem can be verified to be

$$Q_{B'} = 1.8 \text{ mm}$$

where $Q_{B'}$ is the displacement of point B'. From this result, we see that contact does occur. The problem has to be re-solved, since the boundary conditions are now different: The displacement at B' is specified to be 1.2 mm. Consider the two-element finite element model in Fig. 3.5b. The boundary conditions are $Q_1 = 0$ and $Q_3 = 1.2$ mm. The structural stiffness matrix **K** is

$$\mathbf{K} = \frac{20 \times 10^3 \times 250}{150}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

and the global load vector **F** is

$$\mathbf{F} = [0, \quad 60 \times 10^3, \quad 0]^T$$

In the penalty approach, the boundary conditions $Q_1 = 0$ and $Q_3 = 1.2$ imply the following modifications: A large number C chosen here as $C = (2/3) \times 10^{10}$, is added on to the 1st and 3rd diagonal elements of **K**. Also, the number $(C \times 1.2)$ gets added on to the 3rd component of **F**. Thus, the modified equations are

$$\frac{10^5}{3}\begin{bmatrix} 20001 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 20001 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 60.0 \times 10^3 \\ 80.0 \times 10^7 \end{Bmatrix}$$

The solution is

$$\mathbf{Q} = [7.49985 \times 10^{-5}, \quad 1.500045, \quad 1.200015]^T \text{ mm}$$

The element stresses are

$$\sigma_1 = 200 \times 10^3 \times \frac{1}{150}[-1 \quad 1]\begin{Bmatrix} 7.49985 \times 10^{-5} \\ 1.500045 \end{Bmatrix}$$
$$= 199.996 \text{ MPa}$$

$$\sigma_2 = 200 \times 10^3 \times \frac{1}{150}[-1 \quad 1]\begin{Bmatrix} 1.500045 \\ 1.200015 \end{Bmatrix}$$
$$= -40.004 \text{ MPa}$$

The reaction forces are

$$R_1 = -C \times 7.49985 \times 10^{-5}$$
$$= -49.999 \times 10^3 \text{ N}$$

and

$$R_3 = -C \times (1.200015 - 1.2)$$
$$= -10.001 \times 10^3 \text{ N}$$

The results obtained from the penalty approach have a small approximation error due to the flexibility of the support introduced. In fact, the reader may verify that the elimination approach for handling boundary conditions yields the exact reactions, $R_1 = -50.0 \times 10^3$ N and $R_3 = -10.0 \times 10^3$ N. ■

Truss

INTRODUCTION

The finite element analysis of truss structures is presented in this chapter. Two-dimensional trusses (or plane trusses) are treated in Section 4.2. In Section 4.3, this treatment is readily generalized to handle three-dimensional trusses. A typical plane truss is shown in Fig. 4.1. A truss structure consists only of two-force members. That is, every truss element is in direct tension or compression (Fig. 4.2). In a truss, it is required that all loads and reactions are applied only at the joints and that all members are connected together at their ends by frictionless pin joints. Every engineering student has, in a course on statics, analyzed trusses using the method of joints and the method of sections. These methods, while illustrating the fundamentals of statics, become tedious when applied to large-scale statically indeterminate truss structures. Further, joint displacements are not readily obtainable. The finite element method on the other hand is applicable to statically determinate or indeterminate structures alike. The finite element method also provides joint deflections. Effects of temperature changes and support settlements can also be routinely handled.

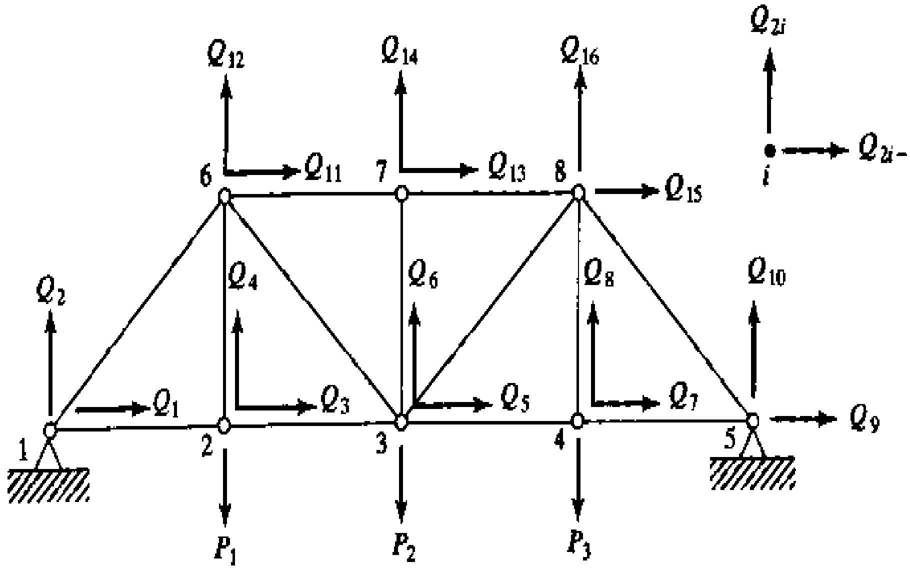


FIGURE 4.1 A two-dimensional truss.

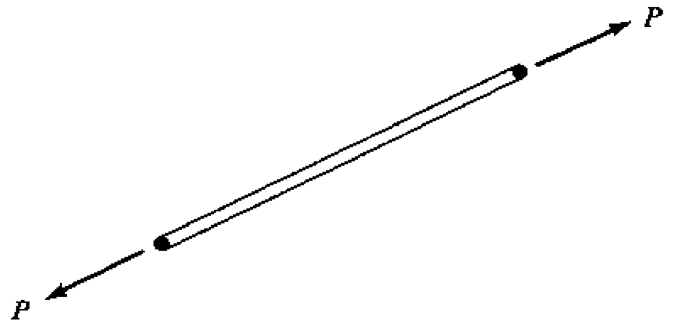


FIGURE 4.2 A two-force member.

PLANE TRUSSES

Modeling aspects discussed in Chapter 3 are now extended to the two-dimensional truss. The steps involved are discussed here.

Local and Global Coordinate Systems

The main difference between the one-dimensional structures considered in Chapter 3 and trusses is that the elements of a truss have various orientations. To account for these different orientations, **local** and **global** coordinate systems are introduced as follows:

A typical plane-truss element is shown in local and global coordinate systems in Fig. 4.3. In the local numbering scheme, the two nodes of the element are numbered 1 and 2. The local coordinate system consists of the x'-axis, which runs along the element from node 1 toward node 2. All quantities in the local coordinate system will be denoted by a prime ($'$). The global x-, y-coordinate system is fixed and does not depend on the orientation of the element. Note that x, y, and z form a right-handed coordinate system with the z-axis coming straight out of the paper. In the global coordinate system,

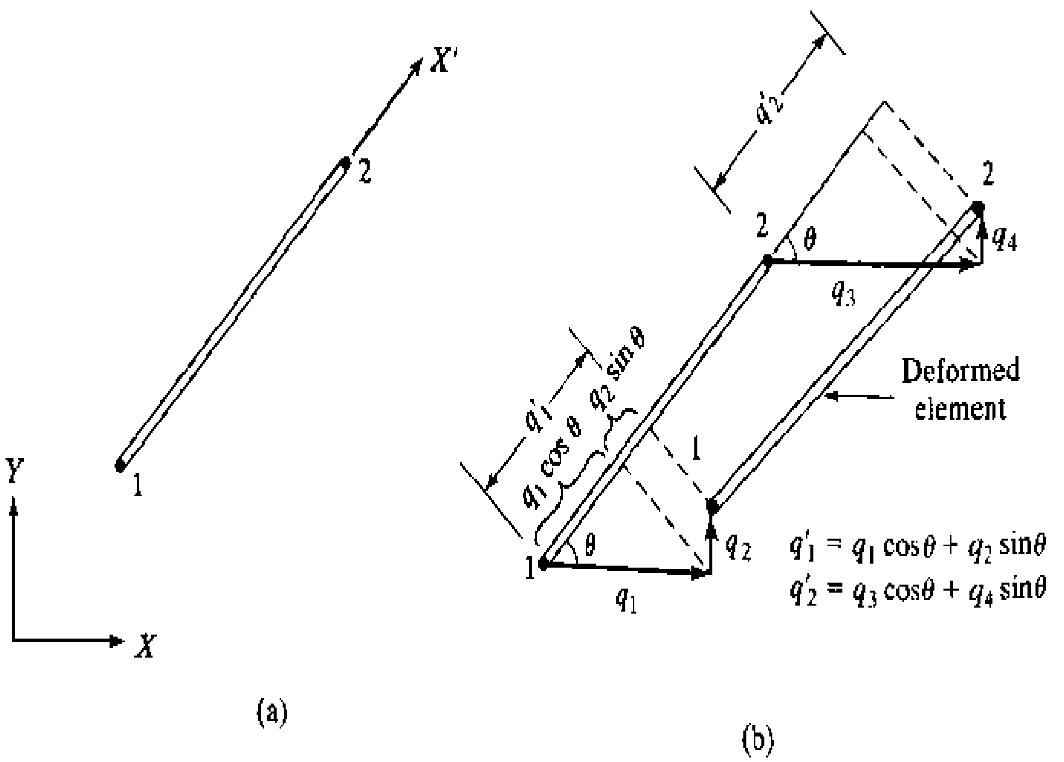


FIGURE 4.3 A two-dimensional truss element in (a) a local coordinate system and (b) a global coordinate system.

every node has two degrees of freedom (dofs). A systematic numbering scheme is adopted here: A node whose global node number is j has associated with it dofs $2j-1$ and $2j$. Further, the global displacements associated with node j are Q_{2j-1} and Q_{2j}, as shown in Fig. 4.1.

Let q'_1 and q'_2 be the displacements of nodes 1 and 2, respectively, in the local coordinate system. Thus, the element displacement vector in the local coordinate system is denoted by

$$\mathbf{q}' = [q'_1, q'_2]^{\mathrm{T}} \tag{4.1}$$

The element displacement vector in the global coordinate system is a (4×1) vector denoted by

$$\mathbf{q} = [q_1, q_2, q_3, q_4]^{\mathrm{T}} \tag{4.2}$$

The relationship between $\mathbf{q}'$ and $\mathbf{q}$ is developed as follows: In Fig. 4.3b, we see that q'_1 equals the sum of the projections of q_1 and q_2 onto the x'-axis. Thus,

$$q'_1 = q_1 \cos\theta + q_2 \sin\theta \tag{4.3a}$$

Similarly,

$$q'_2 = q_3 \cos\theta + q_4 \sin\theta \tag{4.3b}$$

At this stage, the **direction cosines** ℓ and m are introduced as $\ell = \cos\theta$ and $m = \cos\phi$ $(= \sin\theta)$. These direction cosines are the cosines of the angles that the local x'-axis makes with the global x-, y-axes, respectively. Equations 4.3a and 4.3b can now be written in matrix form as

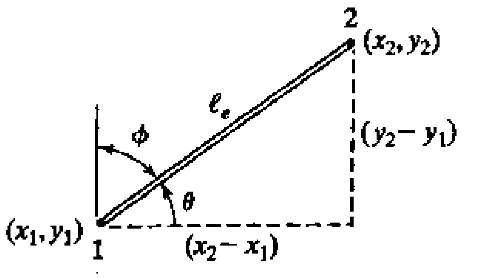
$$\mathbf{q}' = \mathbf{L}\mathbf{q} \tag{4.4}$$

where the **transformation matrix** $\mathbf{L}$ is given by

$$\mathbf{L} = \begin{bmatrix} \ell & m & 0 & 0 \\ 0 & 0 & \ell & m \end{bmatrix} \tag{4.5}$$

Formulas for Calculating ℓ and m

Simple formulas are now given for calculating the direction cosines ℓ and m from nodal coordinate data. Referring to Fig. 4.4, let (x_1, y_1) and (x_2, y_2) be the coordinates of nodes 1 and 2, respectively. We then have



$$\ell = \cos\theta = \frac{x_2 - x_1}{\ell_e}$$

$$m = \cos\phi = \frac{y_2 - y_1}{\ell_e} \; (= \sin\theta)$$

$$\ell_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

FIGURE 4.4 Direction cosines.

$$\ell = \frac{x_2 - x_1}{\ell_e} \qquad m = \frac{y_2 - y_1}{\ell_e} \tag{4.6}$$

where the length ℓ_e is obtained from

$$\ell_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{4.7}$$

The expressions in Eqs. 4.6 and 4.7 are obtained from nodal coordinate data and can readily be implemented in a computer program.

Element Stiffness Matrix

An important observation will now be made: *The truss element is a one-dimensional element when viewed in the local coordinate system.* This observation allows us to use previously developed results in Chapter 3 for one-dimensional elements. Consequently, from Eq. 3.26, the element stiffness matrix for a truss element in the local coordinate system is given by

$$\mathbf{k} = \frac{E_e A_e}{\ell_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{4.8}$$

where A_e is the element cross-sectional area and E_e is Young's modulus. The problem at hand is to develop an expression for the element stiffness matrix in the global coordinate system. This is obtainable by considering the strain energy in the element. Specifically, the element strain energy in local coordinates is given by

$$U_e = \tfrac{1}{2}\mathbf{q}'^{\mathrm{T}}\mathbf{k}'\mathbf{q}' \tag{4.9}$$

Substituting for $\mathbf{q}' = \mathbf{L}\mathbf{q}$ into Eq. 4.9, we get

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}[\mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L}]\mathbf{q} \tag{4.10}$$

The strain energy in global coordinates can be written as

$$U_e = \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}\mathbf{q} \tag{4.11}$$

where $\mathbf{k}$ is the element stiffness matrix in global coordinates. From the previous equation, we obtain the element stiffness matrix in global coordinates as

$$\mathbf{k} = \mathbf{L}^{\mathrm{T}}\mathbf{k}'\mathbf{L} \tag{4.12}$$

Substituting for $\mathbf{L}$ from Eq. 4.5 and for $\mathbf{k}'$ from Eq. 4.8, we get

$$\mathbf{k} = \frac{E_e A_e}{\ell_e}\begin{bmatrix} \ell^2 & \ell m & -\ell^2 & -\ell m \\ \ell m & m^2 & -\ell m & -m^2 \\ -\ell^2 & -\ell m & \ell^2 & \ell m \\ -\ell m & -m^2 & \ell m & m^2 \end{bmatrix} \tag{4.13}$$

The element stiffness matrices are assembled in the usual manner to obtain the structural stiffness matrix. This assembly is illustrated in Example 4.1. The computer logic for directly placing element stiffness matrices into global matrices for banded and skyline solutions is explained in Section 4.4.

The derivation of the result $\mathbf{k} = \mathbf{L}^T\mathbf{k}'\mathbf{L}$ also follows from Galerkin's variational principle. The virtual work δW as a result of virtual displacement $\boldsymbol{\psi}'$ is

$$\delta W = \boldsymbol{\psi}'^{T}(\mathbf{k}'\mathbf{q}') \tag{4.14a}$$

Since $\boldsymbol{\psi}' = \mathbf{L}\boldsymbol{\psi}$ and $\mathbf{q}' = \mathbf{L}\mathbf{q}$, we have

$$\begin{aligned} \delta W &= \boldsymbol{\psi}^{T}[\mathbf{L}^T\mathbf{k}'\mathbf{L}]\mathbf{q} \\ &= \boldsymbol{\psi}^{T}\mathbf{k}\mathbf{q} \end{aligned} \tag{4.14b}$$

Stress Calculations

Expressions for the element stresses can be obtained by noting that a truss element in local coordinates is a simple two-force member (Fig. 4.2). Thus, the stress σ in a truss element is given by

$$\sigma = E_e \epsilon \tag{4.15a}$$

Since the strain ϵ is the change in length per unit original length,

$$\begin{aligned} \sigma &= E_e \frac{q_2' - q_1'}{\ell_e} \\ &= \frac{E_e}{\ell_e}[-1 \quad 1]\begin{Bmatrix} q_1' \\ q_2' \end{Bmatrix} \end{aligned} \tag{4.15b}$$

This equation can be written in terms of the global displacements $\mathbf{q}$ using the transformation $\mathbf{q}' = \mathbf{L}\mathbf{q}$ as

$$\sigma = \frac{E_e}{\ell_e}[-1 \quad 1]\mathbf{L}\mathbf{q} \tag{4.15c}$$

Substituting for $\mathbf{L}$ from Eq. 4.5 yields

$$\sigma = \frac{E_e}{\ell_e}[-\ell \quad -m \quad \ell \quad m]\mathbf{q} \tag{4.16}$$

Once the displacements are determined by solving the finite element equations, the stresses can be recovered from Eq. 4.16 for each element. Note that a positive stress implies that the element is in tension and a negative stress implies compression.

Example 4.1

Consider the four-bar truss shown in Fig. E4.1a. It is given that $E = 29.5 \times 10^6$ psi and $A_e = 1$ in.^{2} for all elements. Complete the following:

(a) Determine the element stiffness matrix for each element.
(b) Assemble the structural stiffness matrix $\mathbf{K}$ for the entire truss.
(c) Using the elimination approach, solve for the nodal displacement.
(d) Recover the stresses in each element.
(e) Calculate the reaction forces.

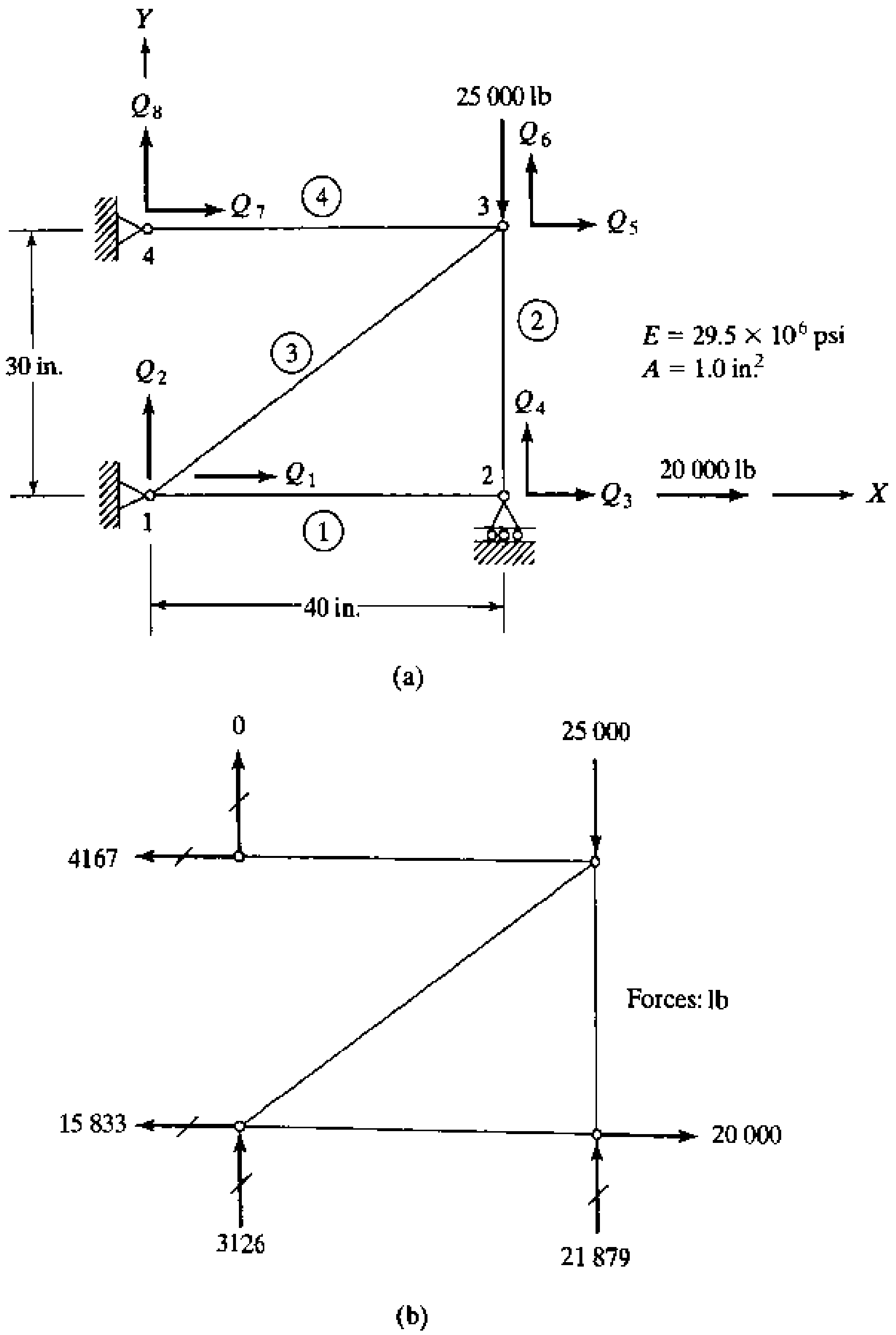


FIGURE E4.1

Solution

(a) It is recommended that a *tabular* form be used for representing nodal coordinate data and element information. The nodal coordinate data are as follows:

Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

The element connectivity table is

Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

Note that the user has a choice in defining element connectivity. For example, the connectivity of element 2 can be defined as 2–3 instead of 3–2 as in the previous table. However, calculations of the direction cosines will be consistent with the adopted connectivity scheme. Using formulas in Eqs. 4.6 and 4.7, together with the nodal coordinate data and the given element connectivity information, we obtain the direction cosines table:

Element	ℓ_e	ℓ	m
1	40	1	0
2	30	0	−1
3	50	0.8	0.6
4	40	1	0

For example, the direction cosines of elements 3 are obtained as $\ell = (x_3 - x_1)/\ell_e = (40 - 0)/50 = 0.8$ and $m = (y_3 - y_1)/\ell_e = (30 - 0)/50 = 0.6$. Now, using Eq. 4.13, the element stiffness matrices for element 1 can be written as

$$\mathbf{k}^1 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \leftarrow \text{Global dof} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} \end{array}$$

The global dofs associated with element 1, which is connected between nodes 1 and 2, are indicated in $\mathbf{k}^1$ earlier. These global dofs are shown in Fig. E4.1a and assist in assembling the various element stiffness matrices.

The element stiffness matrices of elements 2, 3, and 4 are as follows:

$$\mathbf{k}^2 = \frac{29.5 \times 10^6}{30} \begin{array}{c} \begin{array}{cccc} 5 & 6 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{array}{c} 5 \\ 6 \\ 3 \\ 4 \end{array} \end{array}$$

$$\mathbf{k}^3 = \frac{29.5 \times 10^6}{50} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 5 & 6 \end{array} \\ \begin{bmatrix} .64 & .48 & -.64 & -.48 \\ .48 & .36 & -.48 & -.36 \\ -.64 & -.48 & .64 & .48 \\ -.48 & -.36 & .48 & .36 \end{bmatrix} \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array}$$

$$\mathbf{k}^4 = \frac{29.5 \times 10^6}{40} \begin{array}{c} \begin{array}{cccc} 7 & 8 & 5 & 6 \end{array} \\ \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{c} 7 \\ 8 \\ 5 \\ 6 \end{array} \end{array}$$

(b) The structural stiffness matrix **K** is now assembled from the element stiffness matrices. By adding the element stiffness contributions, noting the element connectivity, we get

$$\mathbf{K} = \frac{29.5 \times 10^6}{600} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15.0 & 0 \\ -5.76 & -4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}$$

(c) The structural stiffness matrix **K** given above needs to be modified to account for the boundary conditions. The elimination approach discussed in Chapter 3 will be used here. The rows and columns corresponding to dofs 1, 2, 4, 7, and 8, which correspond to fixed supports, are deleted from the **K** matrix. The reduced finite element equations are given as

$$\frac{29.5 \times 10^6}{600} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix} \begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 20\,000 \\ 0 \\ -25\,000 \end{Bmatrix}$$

Solution of these equations yields the displacements

$$\begin{Bmatrix} Q_3 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{Bmatrix} 27.12 \times 10^{-3} \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \end{Bmatrix} \text{in.}$$

The nodal displacement vector for the entire structure can therefore be written as

$$\mathbf{Q} = [0, 0, 27.12 \times 10^{-3}, 0, 5.65 \times 10^{-3}, -22.25 \times 10^{-3}, 0, 0]^T \text{ in.}$$

(d) The stress in each element can now be determined from Eq. 4.16, as shown below. The connectivity of element 1 is 1 − 2. Consequently, the nodal displacement vector for element 1 is given by $\mathbf{q} = [0, 0, 27.12 \times 10^{-3}, 0]^T$, and Eq. 4.16 yields

$$\sigma_1 = \frac{29.5 \times 10^6}{40}[-1 \quad 0 \quad 1 \quad 0] \begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= 20\,000.0 \text{ psi}$$

The stress in member 2 is given by

$$\sigma_2 = \frac{29.5 \times 10^6}{30}[0 \quad 1 \quad 0 \quad -1] \begin{Bmatrix} 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ +27.12 \times 10^{-3} \\ 0 \end{Bmatrix}$$

$$= -21\,880.0 \text{ psi}$$

Following similar steps, we get

$$\sigma_3 = -5208.0 \text{ psi}$$

$$\sigma_4 = 4167.0 \text{ psi}$$

(e) The final step is to determine the support reactions. We need to determine the reaction forces along dofs 1, 2, 4, 7, and 8, which correspond to fixed supports. These are obtained by substituting for $\mathbf{Q}$ into the original finite element equation $\mathbf{R} = \mathbf{KQ} - \mathbf{F}$. In this substitution, only those rows of $\mathbf{K}$ corresponding to the support dofs are needed, and $\mathbf{F} = \mathbf{0}$ for these dofs. Thus, we have

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \frac{29.5 \times 10^6}{600} \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix}$$

which results in

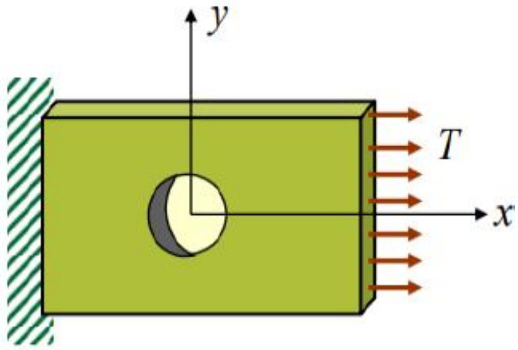
$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15833.0 \\ 3126.0 \\ 21879.0 \\ -4167.0 \\ 0 \end{Bmatrix} \text{lb}$$

A free body diagram of the truss with reaction forces and applied loads is shown in Fig. E4.1b. ■

UNIT –III
BEAM OF ELEMENT

A thin plate of thickness t, with a hole in the middle, is subjected to a uniform traction load, T as shown. This 3-D plate can be analyzed as a **two-dimensional** problem.

2-D problems generally fall into two categories: *plane stress* and *plane strain*.



A *plane stress* problem

a) Plane Stress

The thin plate can be analyzed as a *plane stress* problem, where the normal and shear stresses perpendicular to the x-y plane are *assumed* to be zero, i.e.

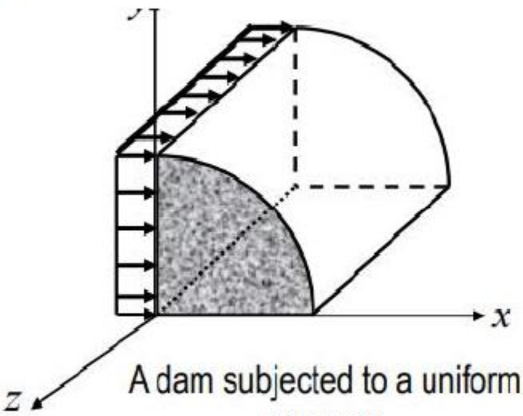
$$\sigma_z = 0; \ \tau_{xz} = 0; \ \tau_{yz} = 0$$

The *nonzero* stress components are

$$\sigma_x \neq 0; \ \sigma_y \neq 0; \ \tau_{xy} \neq 0$$

b) Plane Strain

A dam subjected to uniform pressure and a pipe under a uniform internal pressure can be analyzed in two-dimension as *plain strain* problems.



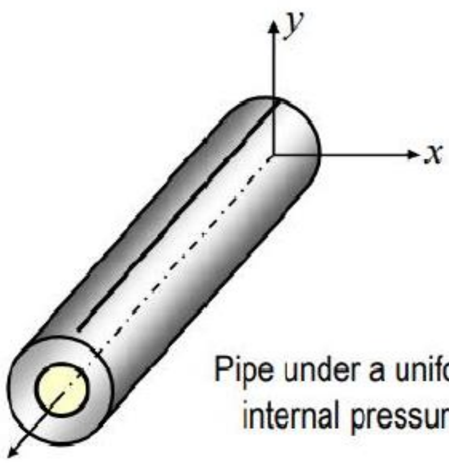
A dam subjected to a uniform pressure

The strain components perpendicular to the x-y plane are assumed to be zero, i.e.

$$\varepsilon_z = 0; \ \gamma_{xz} = 0; \ \gamma_{yz} = 0$$

Thus, the *nonzero* strain components are ε_x, ε_y, and γ_{xy}.

$$\varepsilon_x \neq 0; \ \varepsilon_y \neq 0; \ \gamma_{xy} \neq 0$$



Pipe under a uniform internal pressure

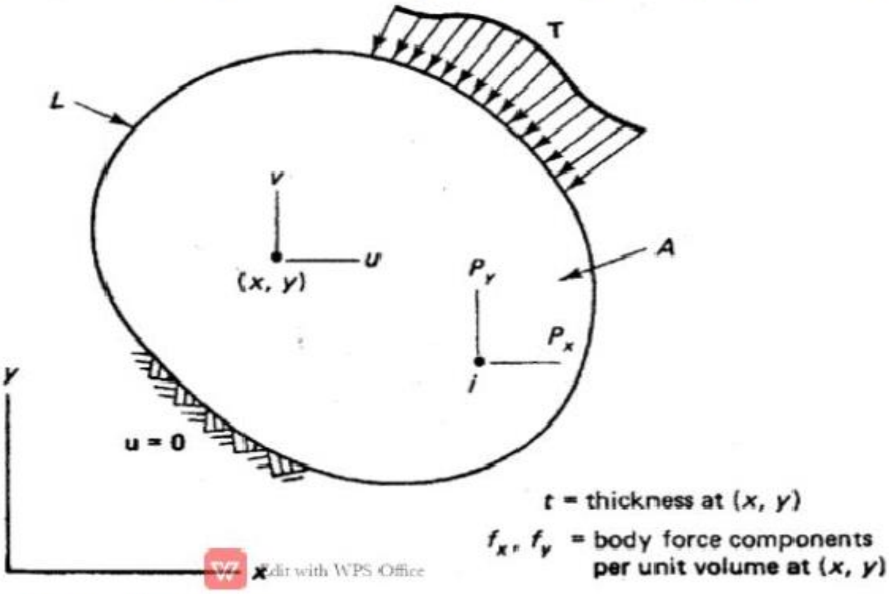
8-2 General Loading Condition

A two-dimensional body can be subjected to **three** types of forces:

a) Concentrated forces, P_x & P_y at a point, i;

b) Body forces, $f_{b,x}$ & $f_{b,y}$ acting at its *centroid*;

c) Traction force, T (i.e. force per unit length), acting along a *perimeter*

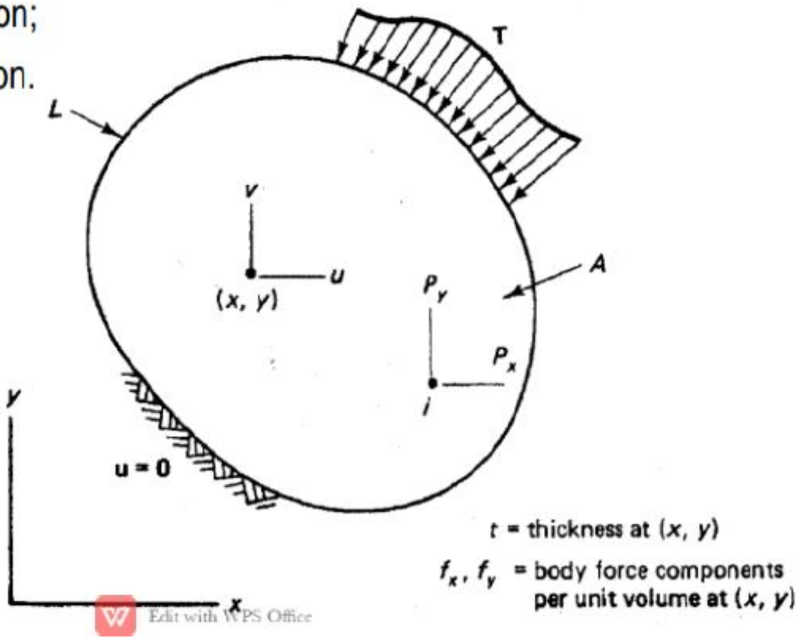


The 2-dimensional body experiences a deformation due to the applied loads.

At any point in the body, there are two components of displacement, i.e.

u = displacement in x-direction;

v = displacement in y-direction.



Stress-Strain Relation

Recall, at any point in the body, there are three components of strains, i.e.

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

The corresponding stress components at that point are

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

The stresses and strains are related through,

$$\{\sigma\} = [D]\{\varepsilon\}$$

where $[D]$ is called the *material matrix*, given by

$$[D] = \frac{E}{1-v^2} \cdot \begin{Bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{Bmatrix}$$

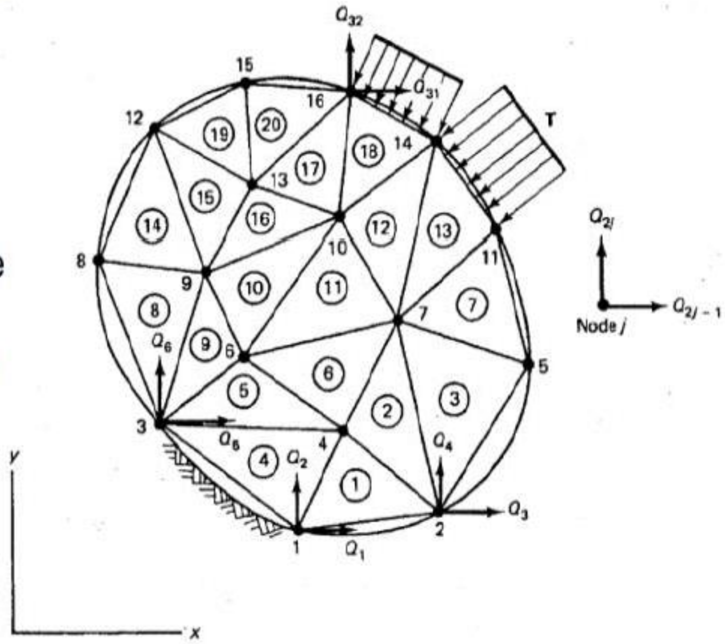
for *plane stress* problems and

$$[D] = \frac{E}{(1+v)(1-2v)} \cdot \begin{Bmatrix} 1-v & v & 0 \\ v & 1-v & 0 \\ 0 & 0 & \frac{1}{2}-v \end{Bmatrix}$$

for *plane strain* problems.

8-3 Finite Element Modeling

The two-dimensional body is transformed into finite element model by subdividing it using triangular elements.



Note:

1. *Unfilled* region exists for curved boundaries, affecting accuracy of the solution. The accuracy can be improved by using smaller elements.

2. There are **two** displacement components at a node. Thus, at a node j, the displacements are:

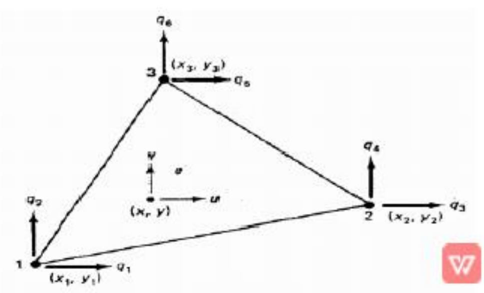
Q_{2j-1} in x-direction
Q_{2j} in y-direction

Difference B/W CST & LST elements

CST

- CST - Constant Strain Triangle
- 3 nodes per Triangle
- First order Triangle Element
- Strain in the element won't vary. Through out the element surface constant strain is observed.
- Displacement function is Linear
- Hence the displacement model is

$$\begin{cases} u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y \\ v(x,y) = \beta_1 + \beta_2 x + \beta_3 y \end{cases}$$

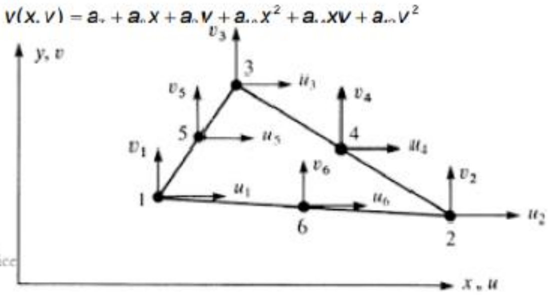


LST

- LST - Linear Strain Triangle
- 6 nodes in Triangle
- Second order Triangle Element
- Strain will vary in the element as Linear
- Displacement function is quadratic
- The variation of the displacements over the element may be expressed as:

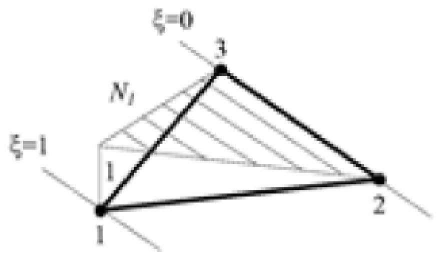
$$u(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2$$

$$v(x,y) = a_7 + a_8 x + a_9 y + a_{10} x^2 + a_{11} xy + a_{12} y^2$$

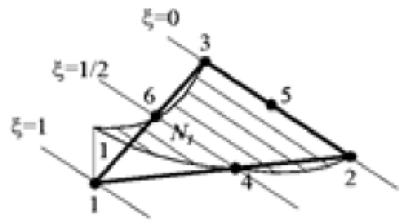


- CST elements are poor in Capturing the bending behaviour
- For same number of elements, true displacement and stresses not obtained in CST elements
- Fig below shows the variation of shape function N1 for the CST element

- LST elements are good in Capturing the bending behaviour
- For same number of elements, true displacement and stresses obtained better in LST elements
- Fig below shows the variation of shape function N1 for the LST element



Shape Function N_1 for CST



Shape Function N_1 for LST

Example 5.1

Evaluate the shape functions N_1, N_2, and N_3 at the interior point P for the triangular element shown in Fig. E5.1.

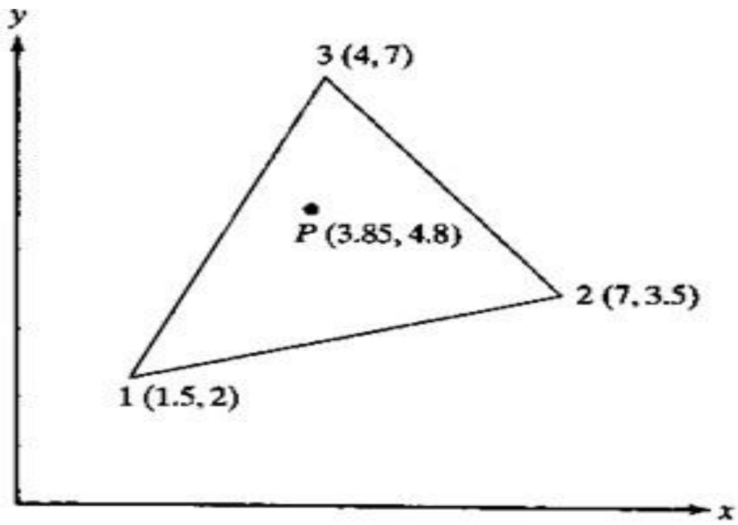


FIGURE E5.1 Examples 5.1 and 5.2.

Solution Using the isoparametric representation (Eqs. 5.15), we have

$$3.85 = 1.5N_1 + 7N_2 + 4N_3 = -2.5\xi + 3\eta + 4$$
$$4.8 = 2N_1 + 3.5N_2 + 7N_3 = -5\xi - 3.5\eta + 7$$

These two equations are rearranged in the form

$$2.5\xi - 3\eta = 0.15$$

$$5\xi + 3.5\eta = 2.2$$

Solving the equations, we obtain $\xi = 0.3$ and $\eta = 0.2$, which implies that

$$N_1 = 0.3 \qquad N_2 = 0.2 \qquad N_3 = 0.5$$ ■

In evaluating the strains, partial derivatives of u and v are to be taken with respect to x and y. From Eqs. 5.12 and 5.15, we see that u, v and x, y are functions of ξ and η. That is, $u = u(x(\xi, \eta), y(\xi, \eta))$ and similarly $v = v(x(\xi, \eta), y(\xi, \eta))$. Using the chain rule for partial derivatives of u, we have

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

which can be written in matrix notation as

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} \tag{5.16}$$

where the (2×2) square matrix is denoted as the *Jacobian* of the transformation, $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{5.17}$$

Some additional properties of the Jacobian are given in the appendix. On taking the derivative of x and y,

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} \tag{5.18}$$

Also, from Eq. 5.16,

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.19}$$

where $\mathbf{J}^{-1}$ is the inverse of the Jacobian $\mathbf{J}$, given by

$$\mathbf{J}^{-1} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & -y_{13} \\ -x_{23} & x_{13} \end{bmatrix} \tag{5.20}$$

$$\det \mathbf{J} = x_{13}y_{23} - x_{23}y_{13} \tag{5.21}$$

Example 5.2

Determine the Jacobian of the transformation **J** for the triangular element shown in Fig. E5.1.

Solution We have

$$\mathbf{J} = \begin{bmatrix} x_{13} & y_{13} \\ x_{23} & y_{23} \end{bmatrix} = \begin{bmatrix} -2.5 & -5.0 \\ 3.0 & -3.5 \end{bmatrix}$$

Thus, det $\mathbf{J}$ = 23.75 units. This is twice the area of the triangle. If 1, 2, 3 are in a clockwise order, then det $\mathbf{J}$ will be negative. ■

From Eqs. 5.19 and 5.20, it follows that

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial u}{\partial \xi} - y_{13}\dfrac{\partial u}{\partial \eta} \\ -x_{23}\dfrac{\partial u}{\partial \xi} + x_{13}\dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{5.23a}$$

Replacing u by the displacement v, we get a similar expression

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}\dfrac{\partial v}{\partial \xi} - y_{13}\dfrac{\partial v}{\partial \eta} \\ -x_{23}\dfrac{\partial v}{\partial \xi} + x_{13}\dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{5.23b}$$

Using the strain–displacement relations (5.5) and Eqs. 5.12b and 5.23, we get

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$= \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}(q_1 - q_5) - y_{13}(q_3 - q_5) \\ -x_{23}(q_2 - q_6) + x_{13}(q_4 - q_6) \\ -x_{23}(q_1 - q_5) + x_{13}(q_3 - q_5) + y_{23}(q_2 - q_6) - y_{13}(q_4 - q_6) \end{Bmatrix} \tag{5.24a}$$

From the definition of x_{ij} and y_{ij}, we can write $y_{31} = -y_{13}$ and $y_{12} = y_{13} - y_{23}$, and so on. The foregoing equation can be written in the form

$$\boldsymbol{\epsilon} = \frac{1}{\det \mathbf{J}} \begin{Bmatrix} y_{23}q_1 + y_{31}q_3 + y_{12}q_5 \\ x_{32}q_2 + x_{13}q_4 + x_{21}q_6 \\ x_{32}q_1 + y_{23}q_2 + x_{13}q_3 + y_{31}q_4 + x_{21}q_5 + y_{12}q_6 \end{Bmatrix} \tag{5.24b}$$

This equation can be written in matrix form as

$$\boldsymbol{\epsilon} = \mathbf{Bq} \tag{5.25}$$

where $\mathbf{B}$ is a (3×6) element strain–displacement matrix relating the three strains to the six nodal displacements and is given by

$$\mathbf{B} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix} \tag{5.26}$$

It may be noted that all the elements of the $\mathbf{B}$ matrix are constants expressed in terms of the nodal coordinates.

Example 5.3

Find the strain–nodal displacement matrices $\mathbf{B}^e$ for the elements shown in Fig. E5.3. Use local numbers given at the corners.

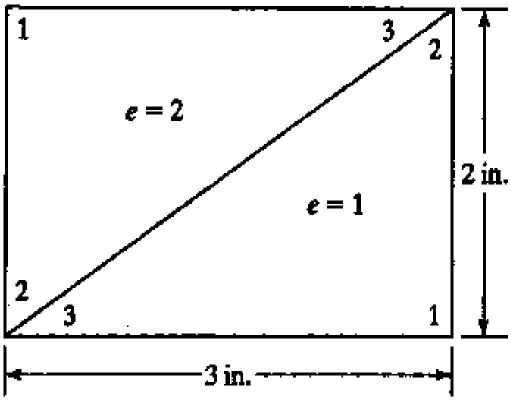


FIGURE E5.3

Solution We have

$$\mathbf{B}^1 = \frac{1}{\det \mathbf{J}}\begin{bmatrix} y_{23} & 0 & y_{31} & 0 & y_{12} & 0 \\ 0 & x_{32} & 0 & x_{13} & 0 & x_{21} \\ x_{32} & y_{23} & x_{13} & y_{31} & x_{21} & y_{12} \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & 0 & 0 & 0 & -2 & 0 \\ 0 & -3 & 0 & 3 & 0 & 0 \\ -3 & 2 & 3 & 0 & 0 & -2 \end{bmatrix}$$

where $\det \mathbf{J}$ is obtained from $x_{13}y_{23} - x_{23}y_{13} = (3)(2) - (3)(0) = 6$. Using the local numbers at the corners, $\mathbf{B}^2$ can be written using the relationship as

$$\mathbf{B}^2 = \frac{1}{6}\begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & -3 & 0 & 0 \\ 3 & -2 & -3 & 0 & 0 & 2 \end{bmatrix} \qquad \blacksquare$$

Potential-Energy Approach

The potential energy of the system, Π, is given by

$$\Pi = \frac{1}{2}\int_A \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA - \int_A \mathbf{u}^{\mathrm{T}}\mathbf{f}\, t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}\, t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{5.27}$$

In the last term in Eq. 5.27, i indicates the point of application of a point load $\mathbf{P}_i$ and $\mathbf{P}_i = [P_x, P_y]_i^{\mathrm{T}}$. The summation in i gives the potential energy due to all point loads.

Using the triangulation shown in Fig. 5.2, the total potential energy can be written in the form

$$\Pi = \sum_e \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}\, t\, dA - \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}\, t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{5.28a}$$

or

$$\Pi = \sum_e U_e - \sum_e \int_e \mathbf{u}^{\mathrm{T}}\mathbf{f}\, t\, dA - \sum \int_L \mathbf{u}^{\mathrm{T}}\mathbf{T}\, t\, d\ell - \sum_i \mathbf{u}_i^{\mathrm{T}}\mathbf{P}_i \tag{5.28b}$$

where $U_e = \frac{1}{2}\int_e \boldsymbol{\epsilon}^{\mathrm{T}}\mathbf{D}\boldsymbol{\epsilon}\, t\, dA$ is the element strain energy.

THE FOUR-NODE QUADRILATERAL

Consider the general quadrilateral element shown in Fig. 7.1. The local nodes are numbered as 1, 2, 3, and 4 in a *counterclockwise* fashion as shown, and (x_i, y_i) are the coordinates of node i. The vector $\mathbf{q} = [q_1, q_2, \ldots, q_8]^T$ denotes the element displacement vector. The displacement of an interior point P located at (x, y) is represented as $\mathbf{u} = [u(x, y), v(x, y)]^T$.

Shape Functions

Following the steps in earlier chapters, we first develop the shape functions on a master element, shown in Fig. 7.2. The master element is defined in ξ-, η-coordinates (or *natural* coordinates) and is square shaped. The Lagrange shape functions where $i = 1, 2, 3$, and 4, are defined such that N_i is equal to unity at node i and is zero at other nodes. In particular, consider the definition of N_i:

$$\begin{aligned} N_1 &= 1 \quad \text{at node 1} \\ &= 0 \quad \text{at nodes 2, 3, and 4} \end{aligned} \tag{7.1}$$

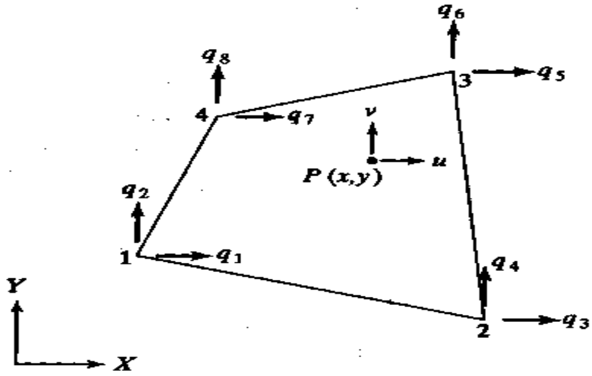


FIGURE 7.1 Four-node quadrilateral element.

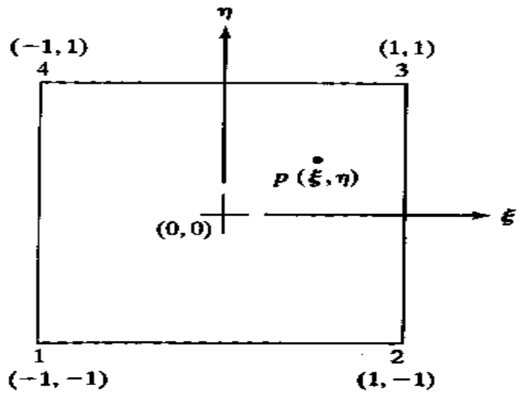


FIGURE 7.2 The quadrilateral element in ξ, η space (the *master* element).

Now, the requirement that $N_1 = 0$ at nodes 2, 3, and 4 is equivalent to requiring that $N_1 = 0$ along edges $\xi = +1$ and $\eta = +1$ (Fig. 7.2). Thus, N_1 has to be of the form

$$N_1 = c(1 - \xi)(1 - \eta) \tag{7.2}$$

where c is some constant. The constant is determined from the condition $N_1 = 1$ at node 1. Since $\xi = -1$, $\eta = -1$ at node 1, we have

$$1 = c(2)(2) \tag{7.3}$$

which yields $c = \frac{1}{4}$. Thus,

$$N_1 = \tfrac{1}{4}(1 - \xi)(1 - \eta) \tag{7.4}$$

All the four shape functions can be written as

$$\begin{aligned}
N_1 &= \tfrac{1}{4}(1-\xi)(1-\eta)\\
N_2 &= \tfrac{1}{4}(1+\xi)(1-\eta)\\
N_3 &= \tfrac{1}{4}(1+\xi)(1+\eta)\\
N_4 &= \tfrac{1}{4}(1-\xi)(1+\eta)
\end{aligned} \tag{7.5}$$

While implementing in a computer program, the compact representation of Eqs. 7.5 is useful

$$N_i = \tfrac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \tag{7.6}$$

where (ξ_i, η_i) are the coordinates of node i.

We now express the displacement field within the element in terms of the nodal values. Thus, if $\mathbf{u} = [u, v]^{\mathrm{T}}$ represents the displacement components of a point located at (ξ, η), and $\mathbf{q}$, dimension (8×1), is the element displacement vector, then

$$\begin{aligned}
u &= N_1q_1 + N_2q_3 + N_3q_5 + N_4q_7\\
v &= N_1q_2 + N_2q_4 + N_3q_6 + N_4q_8
\end{aligned} \tag{7.7a}$$

which can be written in matrix form as

$$\mathbf{u} = \mathbf{Nq} \tag{7.7b}$$

where

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \tag{7.8}$$

In the isoparametric formulation, we use the *same* shape functions N_i to also express the coordinates of a point within the element in terms of nodal coordinates. Thus,

$$\begin{aligned}
x &= N_1x_1 + N_2x_2 + N_3x_3 + N_4x_4\\
y &= N_1y_1 + N_2y_2 + N_3y_3 + N_4y_4
\end{aligned} \tag{7.9}$$

Subsequently, we will need to express the derivatives of a function in x-, y-coordinates in terms of its derivatives in ξ-, η-coordinates. This is done as follows: A function $f = f(x, y)$, in view of Eqs. 7.9, can be considered to be an implicit function of ξ and η as $f = f[x(\xi,\eta), y(\xi,\eta)]$. Using the chain rule of differentiation, we have

$$\begin{aligned}
\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}\\
\frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}
\end{aligned} \tag{7.10}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \mathbf{J} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} \tag{7.11}$$

where $\mathbf{J}$ is the Jacobian matrix

$$\mathbf{J} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \tag{7.12}$$

In view of Eqs. 7.5 and 7.9, we have

$$\mathbf{J} = \frac{1}{4}\left[\begin{array}{c|c} -(1-\eta)x_1+(1-\eta)x_2+(1+\eta)x_3-(1+\eta)x_4 & -(1-\eta)y_1+(1-\eta)y_2+(1+\eta)y_3-(1+\eta)y_4 \\ -(1-\xi)x_1-(1+\xi)x_2+(1+\xi)x_3+(1-\xi)x_4 & -(1-\xi)y_1-(1+\xi)y_2+(1+\xi)y_3+(1-\xi)y_4 \end{array}\right] \tag{7.13a}$$

$$\equiv \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{7.13b}$$

Equation 7.11 can be inverted as

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14a}$$

or

$$\begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} \tag{7.14b}$$

These expressions will be used in the derivation of the element stiffness matrix.

An additional result that will be needed is the relation

$$dx\, dy = \det \mathbf{J}\, d\xi\, d\eta \tag{7.15}$$

The proof of this result, found in many textbooks on calculus, is given in the appendix.

Element Stiffness Matrix

The stiffness matrix for the quadrilateral element can be derived from the strain energy in the body, given by

$$U = \int_V \tfrac{1}{2} \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\epsilon}\, dV \tag{7.16}$$

or

$$U = \sum_e t_e \int_e \tfrac{1}{2} \boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\epsilon}\, dA \tag{7.17}$$

where t_e is the thickness of element e.

The strain–displacement relations are

$$\boldsymbol{\epsilon} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} \tag{7.18}$$

By considering $f \equiv u$ in Eq. 7.14b, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} \tag{7.19a}$$

Similarly,

$$\begin{Bmatrix} \dfrac{\partial v}{\partial x} \\ \dfrac{\partial v}{\partial y} \end{Bmatrix} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.19b}$$

Equations 7.18 and 7.19a,b yield

$$\boldsymbol{\epsilon} = \mathbf{A} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} \tag{7.20}$$

where $\mathbf{A}$ is given by

$$\mathbf{A} = \frac{1}{\det \mathbf{J}} \begin{bmatrix} J_{22} & -J_{12} & 0 & 0 \\ 0 & 0 & -J_{21} & J_{11} \\ -J_{21} & J_{11} & J_{22} & -J_{12} \end{bmatrix} \tag{7.21}$$

Now, from the interpolation equations Eqs. 7.7a, we have

$$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial v}{\partial \xi} \\ \dfrac{\partial v}{\partial \eta} \end{Bmatrix} = \mathbf{Gq} \tag{7.22}$$

where

$$\mathbf{G} = \frac{1}{4}\begin{bmatrix} -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) & 0 \\ -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) & 0 \\ 0 & -(1-\eta) & 0 & (1-\eta) & 0 & (1+\eta) & 0 & -(1+\eta) \\ 0 & -(1-\xi) & 0 & -(1+\xi) & 0 & (1+\xi) & 0 & (1-\xi) \end{bmatrix} \tag{7.23}$$

Equations 7.20 and 7.22 now yield

$$\boxed{\boldsymbol{\epsilon} = \mathbf{Bq}} \tag{7.24}$$

where

$$\mathbf{B} = \mathbf{AG} \tag{7.25}$$

The relation $\boldsymbol{\epsilon} = \mathbf{Bq}$ is the desired result. The strain in the element is expressed in terms of its nodal displacement. The stress is now given by

$$\boxed{\boldsymbol{\sigma} = \mathbf{DBq}} \tag{7.26}$$

where $\mathbf{D}$ is a (3×3) material matrix. The strain energy in Eq. 7.17 becomes

$$U = \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\left[t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det \mathbf{J}\, d\xi\, d\eta\right]\mathbf{q} \tag{7.27a}$$

$$= \sum_e \tfrac{1}{2}\mathbf{q}^{\mathrm{T}}\mathbf{k}^e\mathbf{q} \tag{7.27b}$$

where

$$\boxed{\mathbf{k}^e = t_e \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^{\mathrm{T}}\mathbf{DB}\det \mathbf{J}\, d\xi\, d\eta} \tag{7.28}$$

is the element stiffness matrix of dimension (8×8).

We note here that quantities $\mathbf{B}$ and $\det \mathbf{J}$ in the integral in Eq. (7.28) are involved functions of ξ and η, and so the integration has to be performed numerically. Methods of numerical integration are discussed subsequently.

Element Force Vectors

Body Force A body force that is distributed force per unit volume, contributes to the global load vector $\mathbf{F}$. This contribution can be determined by considering the body force term in the potential-energy expression

$$\int_V \mathbf{u}^{\mathrm{T}}\mathbf{f}\, dV \tag{7.29}$$

Using $\mathbf{u} = \mathbf{Nq}$, and treating the body force $\mathbf{f} = [f_x, f_y]^{\mathrm{T}}$ as constant within each element, we get

UNIT –IV
ISOPARAMETRIC FORMULATION

Definition:

The term ***isoparametric*** (same parameters) is derived from the use of the same shape (interpolation) functions ***N*** to define the element's ***geometric shape*** as are used to define the ***displacements*** within the element.

Alternatively:

The basic principle of isoparametric elements is that the interpolation functions for the displacements are also used to represent the geometry of the element.

$$u = \sum_{i=1}^{4} N_i u_i \quad , \quad v = \sum_{i=1}^{4} N_i v_i$$

$$x = \sum_{i=1}^{4} N_i x_i \quad , \quad y = \sum_{i=1}^{4} N_i y_i$$

Basic Principle of Isoparametric Elements

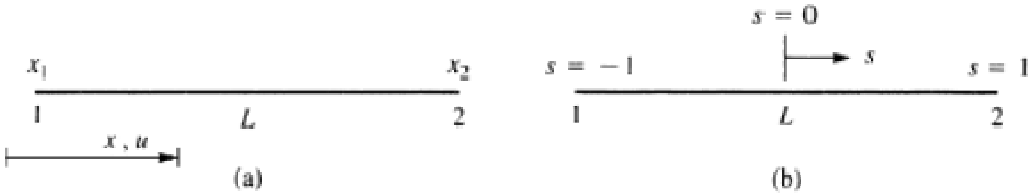
- In this formulation, displacements are expressed in terms of the ***natural*** (local) coordinates and then differentiated with respect to global coordinates. Accordingly, a transformation matrix [J], called ***Jacobian***, is produced.

- If the geometric interpolation functions are of lower order than the displacement shape functions, the element is called ***subparametric***. If the reverse is true, the element is referred to as ***superparametric***.

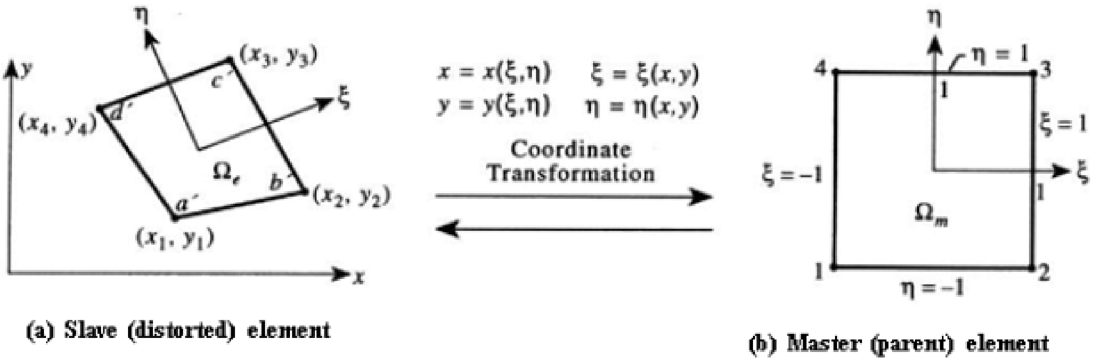
- The ***isoparametric formulation*** is generally applicable to 1-, 2- and 3- dimensional stress analysis. The isoparametric family includes elements for plane, solid, plate, and shell problems. Also, it is applicable for ***nonstructural*** problems.

- The isoparametric formulation makes it possible to generate elements that are **nonrectangular** and have **curved** sides. So it can facilitate an accurate representation of irregular elements.

- Numerous **commercial** computer programs have adopted this formulation for their various libraries of elements.



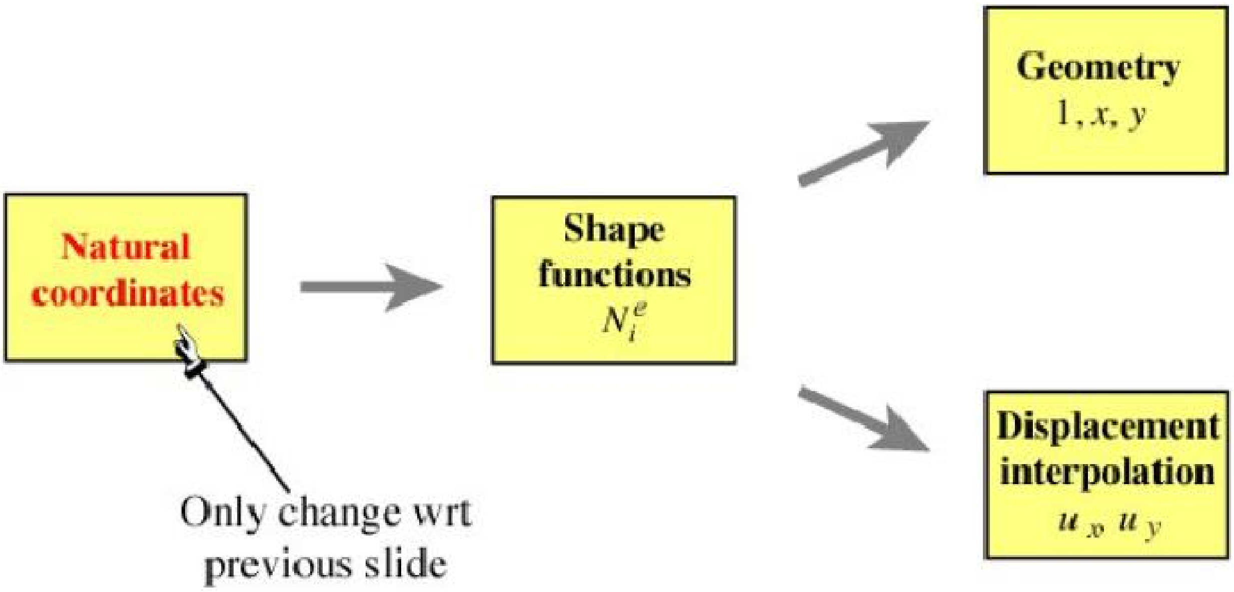
Two-Noded Bar Isoparametric Element



Isoparametric coordinate transformation.

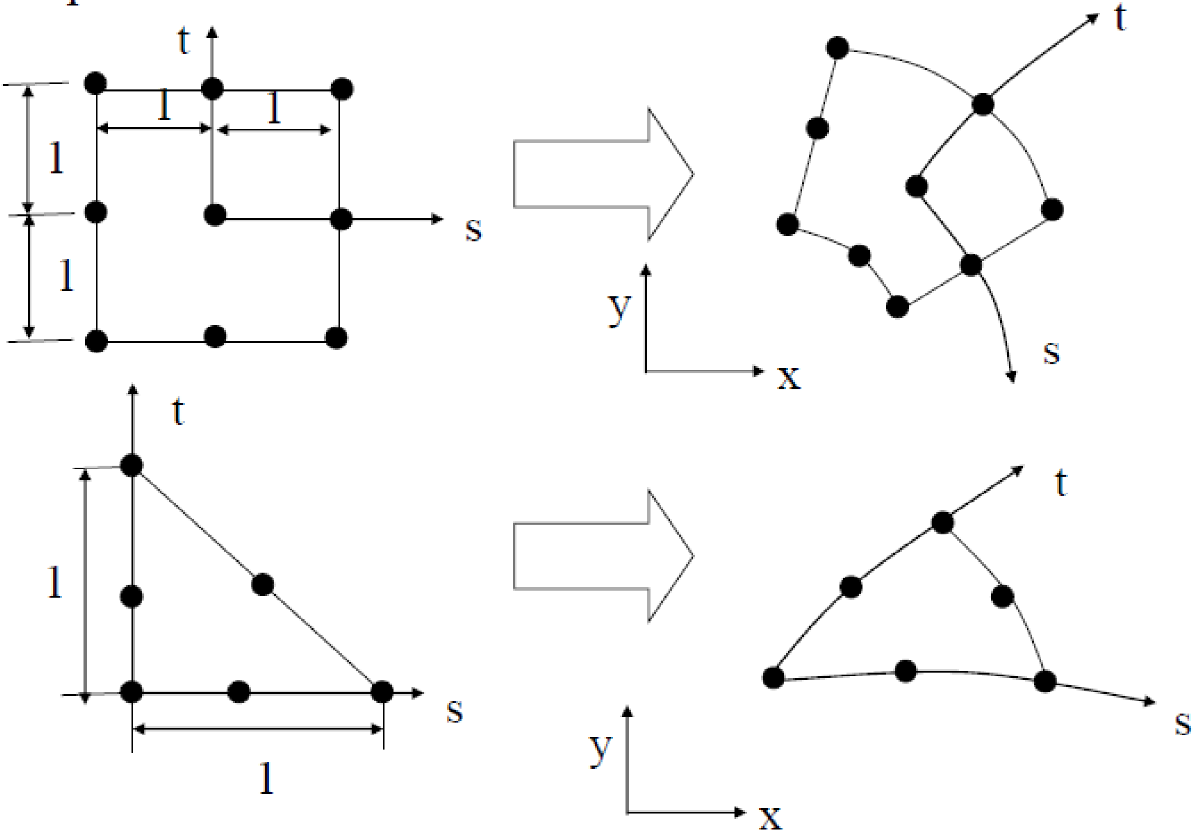
As shown in the figure, the local (natural) coordinate system (ξ, η) for the two elements have their origins at the centroids of the elements, with (ξ, η) varying form –1 to 1. The natural coordinate system needs not to be orthogonal and neither has to be parallel to the x-y axes. The coordinate transformation will map the point (ξ, η) in the master element to $x(\xi, \eta)$ and $y(\xi, \eta)$ in the slave element.

Isoparametric Representation for *any* 2D Element



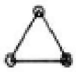
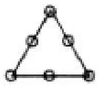
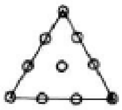
In 3D: $1, x, y$ **becomes** $1, x, y, z$ **etc**

Examples



Step 2: Select Displacement Functions

In other words, we look for shape functions that map the regular shape element in isoparametric coordinates to the quadrilateral in the x-y coordinates whose size and shape are determined by the eight nodal coordinates $x_1, y_1, x_2, y_2, \ldots, x_4, y_4$.

Terms in Pascal Triangle	*Polynomial Degree*	*Number of Terms Triangle*	
1	0 (constant)	1	
$x \quad y$	1 (linear)	3	CST 
$x^2 \quad xy \quad y^2$	2 (quadratic)	6	LST 
$x^3 \quad x^2y \quad xy^2 \quad y^3$	3 (cubic)	10	QST 

$$x(\xi,\eta) = a_1 + a_2\,\xi + a_3\,\eta + a_4\,\xi\eta$$

$$y(\xi,\eta) = a_5 + a_6\,\xi + a_7\,\eta + a_8\,\xi\eta$$

$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2\,\xi + a_3\,\eta + a_4\,\xi\eta \\ a_5 + a_6\,\xi + a_7\,\eta + a_8\,\xi\eta \end{Bmatrix}$$

$$\begin{Bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} 1 & \xi & \eta & \xi\eta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \xi & \eta & \xi\eta \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \vdots \\ a_8 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix}$$

$$x(\xi,\eta)=\begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\\a_4\end{Bmatrix}=\begin{bmatrix}1 & \xi & \eta & \xi\eta\end{bmatrix}\frac{1}{4}\begin{bmatrix}1 & 1 & 1 & 1\\-1 & 1 & 1 & -1\\-1 & -1 & 1 & 1\\1 & -1 & 1 & -1\end{bmatrix}\begin{Bmatrix}x_1\\x_2\\x_3\\x_4\end{Bmatrix}$$

$$=\frac{1}{4}[(1-\xi)(1-\eta)x_1+(1+\xi)(1-\eta)x_2+(1+\xi)(1+\eta)x_3+(1-\xi)(1+\eta)x_4]$$

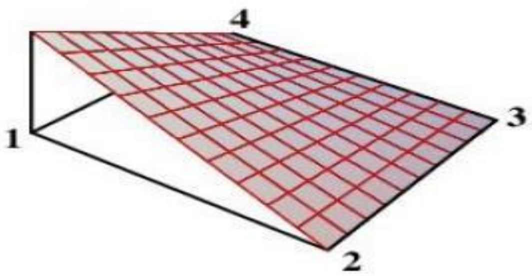
$$\begin{Bmatrix}x(\xi,\eta)\\y(\xi,\eta)\end{Bmatrix}=\begin{bmatrix}N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0\\0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4\end{bmatrix}\begin{Bmatrix}x_1\\y_1\\x_2\\y_2\\x_3\\y_3\\x_4\\y_4\end{Bmatrix}=\begin{Bmatrix}\sum_{i=1}^{4}N_i\,x_i\\\sum_{i=1}^{4}N_i\,y_i\end{Bmatrix}$$

Shape Function for 4-Nodes quadrilateral Elements

$$N_1=\frac{1}{4}(1-\xi)(1-\eta)\quad,\quad N_2=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3=\frac{1}{4}(1+\xi)(1+\eta)\quad,\quad N_4=\frac{1}{4}(1-\xi)(1+\eta)$$

These shape functions are seen to map the (ξ,η) coordinates of any point in the rectangular element in the above master element to ***x and y*** coordinates in the quadrilateral (slave) element.
For example, consider the coordinates of node 1, where:
$\xi=-1,\eta=-1$ using the above equation, we get $x=x_1$, $y=y_1$



$$N_1^e=\tfrac{1}{4}(1-\xi)(1-\eta)$$

Shape Function for 4-Nodes quadrilateral Elements

$$N_i = \begin{cases} 1 \; at \; node \, i \\ 0 \; at \; all \; other \; nodes \end{cases}$$

$$\sum_{i=1}^{n} N_i = 1 \quad , \qquad (i = 1, 2, , \cdots, n)$$

where n = the number of shape functions associated with number of nodes

$$\begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} \sum_{i=1}^{4} N_i \, u_i \\ \sum_{i=1}^{4} N_i \, v_i \end{Bmatrix}$$

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [N][d]$$

where u *and* v are displacements parallel to the global x and y coordinates

Step 3: Define the Strain/Displacement and Stress/Strain Relationships

Using Chain Rule

$$\frac{\partial f}{\partial \xi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial f}{\partial \eta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}$$

$$\begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix}$$

Can be computed

$$\begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix}$$

We want to compute these for the B matrix

This is known as the **Jacobian** matrix (J) for the mapping $(\xi, \eta) \rightarrow (x, y)$

$$\begin{Bmatrix} \dfrac{\partial N}{\partial \xi} \\ \dfrac{\partial N}{\partial \eta} \end{Bmatrix} = [J] \begin{Bmatrix} \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial y} \end{Bmatrix} \quad \Rightarrow \quad \begin{Bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial f}{\partial \xi} \\ \dfrac{\partial f}{\partial \eta} \end{Bmatrix}$$

$$[J]=\begin{bmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{bmatrix}=\begin{bmatrix}\sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}x_i & \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}y_i\\ \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}x_i & \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}y_i\end{bmatrix}$$

$$[J]^{-1}=\frac{1}{|J|}\begin{bmatrix}\frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi}\\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}\end{bmatrix}$$

where

$$|J|=\frac{\partial x}{\partial \xi}\cdot\frac{\partial y}{\partial \eta}-\frac{\partial x}{\partial \eta}\cdot\frac{\partial y}{\partial \xi}$$

Since:

$$\begin{Bmatrix}\frac{\partial N}{\partial x}\\ \frac{\partial N}{\partial y}\end{Bmatrix}=[J]^{-1}\begin{Bmatrix}\frac{\partial N}{\partial \xi}\\ \frac{\partial N}{\partial \eta}\end{Bmatrix}$$

$$\begin{Bmatrix}\frac{\partial N}{\partial x}\\ \frac{\partial N}{\partial y}\end{Bmatrix}=\frac{1}{|J|}\begin{bmatrix}\frac{\partial y}{\partial \eta} & -\frac{\partial y}{\partial \xi}\\ -\frac{\partial x}{\partial \eta} & \frac{\partial x}{\partial \xi}\end{bmatrix}\begin{Bmatrix}\frac{\partial N}{\partial \xi}\\ \frac{\partial N}{\partial \eta}\end{Bmatrix}$$

$$\frac{\partial N}{\partial x}=\frac{1}{|J|}\left[\frac{\partial y}{\partial \eta}\frac{\partial N}{\partial \xi}-\frac{\partial y}{\partial \xi}\frac{\partial N}{\partial \eta}\right]$$

$$\frac{\partial N}{\partial y}=\frac{1}{|J|}\left[\frac{\partial x}{\partial \xi}\frac{\partial N}{\partial \eta}-\frac{\partial x}{\partial \eta}\frac{\partial N}{\partial \xi}\right]$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} & 0 \\ 0 & \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

$$\{\varepsilon\} = [D'] \begin{Bmatrix} u \\ v \end{Bmatrix} = [D'][N][d]$$

$$\{\varepsilon\} = [D'][N][d]$$

$$\{\varepsilon\} = [B][d]$$

$$[D'] = \frac{1}{|J|} \begin{bmatrix} \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} & 0 \\ 0 & \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} \\ \dfrac{\partial x}{\partial \xi}\dfrac{\partial ()}{\partial \eta} - \dfrac{\partial x}{\partial \eta}\dfrac{\partial ()}{\partial \xi} & \dfrac{\partial y}{\partial \eta}\dfrac{\partial ()}{\partial \xi} - \dfrac{\partial y}{\partial \xi}\dfrac{\partial ()}{\partial \eta} \end{bmatrix}$$

$$\underset{(3\times 8)}{[B]} = \underset{(3\times 2)}{[D']}\ \underset{(2\times 8)}{[N]}$$

$$[k] = \iint_A [B]^T [D][B]\, t\, dx\, dy$$

$$\iint_A f(x,y)\, dx\, dy = \iint_A f(\xi,\eta) \left| J \right| d\xi\, d\eta$$

$$[k] = \int_{-1}^{1} \int_{-1}^{1} [B]^T [D][B]\, t \left|J\right| d\xi\, d\eta$$

The shape function are:

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) \quad , \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta) \quad , \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Their derivatives:

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta) \quad , \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta) \quad , \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta) \quad , \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta)$$

and

$$\frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi) \quad , \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi) \quad , \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi) \quad , \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi)$$

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \xi} y_i \\[2ex] \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} x_i & \sum_{i=1}^{4} \dfrac{\partial N_i}{\partial \eta} y_i \end{bmatrix}$$

$$J_{11} = \frac{\partial x}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} x_i \quad , \quad J_{12} = \frac{\partial y}{\partial \xi} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \xi} y_i$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} x_i \quad , \quad J_{22} = \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} y_i$$

$$[J] = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} N_{1,\xi} & N_{2,\xi} & N_{3,\xi} & N_{4,\xi} \\ N_{1,\eta} & N_{2,\eta} & N_{3,\eta} & N_{4,\eta} \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}$$

$$J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi) \right]$$

$$J_{12} = \frac{1}{4}\left[y_1(\eta-1) + y_2(1-\eta) + y_3(1+\eta) + y_4(-1-\eta) \right]$$

$$J_{11} = \frac{1}{4}\left[x_1(\eta-1) + x_2(1-\eta) + x_3(1+\eta) + x_4(-1-\eta) \right]$$

$$J_{21} = \frac{1}{4}\left[x_1(\xi-1) + x_2(-1-\xi) + x_3(1+\xi) + x_4(1-\xi) \right]$$

➢ Derive the Element Stiffness Matrix and Equations

$$[J]=\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

$$|J|=J_{11}\,J_{22}-J_{12}\,J_{21}$$

Explicit formulation for |J| for 4 node Element

$$|J|=\begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix}=\frac{1}{8}\{x_1 \quad x_2 \quad x_3 \quad x_4\}\begin{bmatrix} 0 & 1-\eta & \eta-\xi & \xi-1 \\ \eta-1 & 0 & \xi+1 & -\xi-\eta \\ \xi-\eta & -\xi-1 & 0 & \eta+1 \\ 1-\xi & \xi+\eta & -\eta-1 & 0 \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$$

➢ Derive the Element Stiffness Matrix and Equations

$$[B]=[D'][N]$$

$$[B]=\frac{1}{|J|}\begin{bmatrix} J_{22}\frac{\partial()}{\partial\xi}-J_{12}\frac{\partial()}{\partial\eta} & 0 \\ 0 & J_{11}\frac{\partial()}{\partial\eta}-J_{21}\frac{\partial()}{\partial\xi} \\ J_{11}\frac{\partial()}{\partial\eta}-J_{21}\frac{\partial()}{\partial\xi} & J_{22}\frac{\partial()}{\partial\xi}-J_{12}\frac{\partial()}{\partial\eta} \end{bmatrix}\begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}$$

$$[B]=\frac{1}{|J|}[\,B_1 \quad B_2 \quad B_3 \quad B_4\,]$$

$$[B_i]=\begin{bmatrix} J_{22}\,N_{i,\xi}-J_{12}\,N_{i,\eta} & 0 \\ 0 & J_{11}\,N_{i,\eta}-J_{21}\,N_{i,\xi} \\ J_{11}\,N_{i,\eta}-J_{21}\,N_{i,\xi} & J_{22}\,N_{i,\xi}-J_{12}\,N_{i,\eta} \end{bmatrix}$$

➢ Derive the Element Stiffness Matrix and Equations

$$[B_i]=\begin{bmatrix} J_{22}\,N_{i,\xi}-J_{12}\,N_{i,\eta} & 0 \\ 0 & J_{11}\,N_{i,\eta}-J_{21}\,N_{i,\xi} \\ J_{11}\,N_{i,\eta}-J_{21}\,N_{i,\xi} & J_{22}\,N_{i,\xi}-J_{12}\,N_{i,\eta} \end{bmatrix}$$

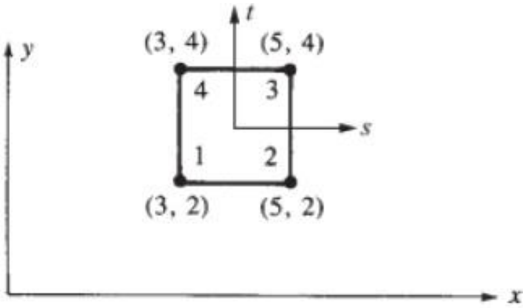
$$J_{22}=\frac{1}{4}\left[y_1(\xi-1)+y_2(-1-\xi)+y_3(1+\xi)+y_4(1-\xi)\right]$$

$$J_{12}=\frac{1}{4}\left[y_1(\eta-1)+y_2(1-\eta)+y_3(1+\eta)+y_4(-1-\eta)\right]$$

$$J_{11}=\frac{1}{4}\left[x_1(\eta-1)+x_2(1-\eta)+x_3(1+\eta)+x_4(-1-\eta)\right]$$

$$J_{21}=\frac{1}{4}\left[x_1(\xi-1)+x_2(-1-\xi)+x_3(1+\xi)+x_4(1-\xi)\right]$$

Evaluate the stiffness matrix for the quadrilateral element shown in Figure using the four-point Gaussian quadrature rule.
Let E = 30×10^6 psi, ν = 0.25 and h=1 in.

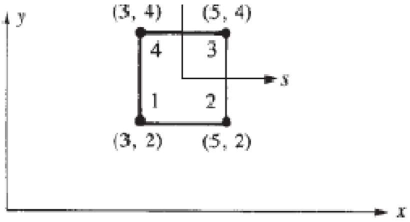


Solution

we evaluate the ***k*** matrix. Using the four-point rule, the four points are:

$$(\xi_1,\eta_1)=(-0.5773,-0.5773)$$
$$(\xi_2,\eta_2)=(-0.5773,0.5773)$$
$$(\xi_3,\eta_3)=(0.5773,-0.5773)$$
$$(\xi_4,\eta_4)=(0.5773,0.5773)$$

$$W_1=W_2=W_3=W_4=1.0$$



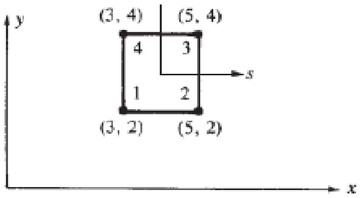
$$
\begin{aligned}
[k] = {} & [B(-0.5773,-0.5773)]^T[D][B(-0.5773,-0.5773)] \\
& |[J(-0.5773,-0.5773)]|(1)(1.000)(1.000) \\
& + [B(-0.5773,0.5773)]^T[D][B(-0.5773,0.5773)] \\
& |[J(-0.5773,0.5773)]|(1)(1.000)(1.000) \\
& + [B(0.5773,-0.5773)]^T[D][B(0.5773,-0.5773)] \\
& |[J(0.5773,-0.5773)]|(1)(1.000)(1.000) \\
& + [B(0.5773,0.5773)]^T[D][B(0.5773,0.5773)] \\
& |[J(0.5773,0.5773)]|(1)(1.000)(1.000)
\end{aligned}
$$

$$
|[J(-0.5773,-0.5773)]| = \frac{1}{8}[3 \quad 5 \quad 5 \quad 3]
\begin{bmatrix}
0 & 1-(-0.5773) & -0.5773-(-0.5773) & -0.5773-1 \\
-0.5773-1 & 0 & -0.5773+1 & -0.5773-(-0.5773) \\
-0.5773-(-0.5773) & -(-0.5773)-1 & 0 & -0.5773+1 \\
1-(-0.5773) & -0.5773+(-0.5773) & -0.5773-1 & 0
\end{bmatrix}
\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix} = 1.000
$$

Similarly,

$$
\begin{aligned}
|[J(-0.5773,0.5773)]| &= 1.000 \\
|[J(0.5773,-0.5773)]| &= 1.000 \\
|[J(0.5773,0.5773)]| &= 1.000
\end{aligned}
$$

Even though $|[J]| = 1$ in this example, in general, $|[J]| \neq 1$ and varies in space.



$$
[B(-0.5773,-0.5773)] = \frac{1}{|[J(-0.5773,-0.5773)]|}[[B_1] \quad [B_2] \quad [B_3] \quad [B_4]]
$$

$$
[B_1] = \begin{bmatrix}
J_{22} N_{1,\xi} - J_{12} N_{1,\eta} & 0 \\
0 & J_{11} N_{1,\eta} - J_{21} N_{1,\xi} \\
J_{11} N_{1,\eta} - J_{21} N_{1,\xi} & J_{22} N_{1,\xi} - J_{12} N_{1,\eta}
\end{bmatrix}
$$

$$
J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]
$$

$$
J_{22} = \frac{1}{4}\left[2(-0.5773-1) + 2(-1+0.5773) + 4(1-0.5773) + 4(1+0.5773)\right] = 1.0
$$

with similar computations used to obtain J_{12}, J_{11} and J_{21}. Also,

$$
N_{1,\xi} = -\frac{1}{4}(1-\eta) = -\frac{1}{4}(1+0.5773) = -0.3943
$$

$$
N_{1,\eta} = -\frac{1}{4}(1-\xi) = -\frac{1}{4}(1+0.5773) = -0.3943
$$

Similarly, $[B_2]$, $[B_3]$, and $[B_4]$ must be evaluated like $[B_1]$, at $(-0.5773, -0.5773)$. We then repeat the calculations to evaluate $[B]$ at the other Gauss points [Eq. (10.4.4a)].

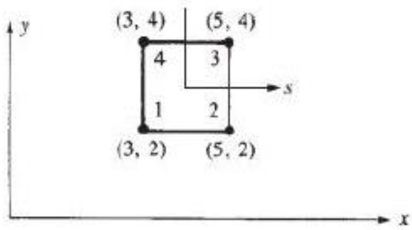
Using a computer program written specifically to evaluate $[B]$ at each Gauss point and then $[k]$, we obtain the final form of $[B(-0.5773, -0.5773)]$ as

$$[B(-0.5773, -0.5773)] =$$

$$\begin{bmatrix} -0.1057 & 0 & 0.1057 & 0 & 0 & -0.1057 & 0 & -0.3943 \\ -0.1057 & -0.1057 & -0.3943 & 0.1057 & 0.3943 & 0 & -0.3943 & 0 \\ 0 & 0.3943 & 0 & 0.1057 & 0.3943 & 0.3943 & 0.1057 & -0.3943 \end{bmatrix} \tag{10.4.4h}$$

with similar expressions for $[B(-0.5773, 0.5773)]$, and so on.

$$[D] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 32 & 8 & 0 \\ 8 & 32 & 0 \\ 0 & 0 & 12 \end{bmatrix} \quad 10^6 \text{ psi}$$



$$\begin{aligned}[k] = {} & [B(-0.5773, -0.5773)]^T[D][B(-0.5773, -0.5773)] \\ & |[J(-0.5773, -0.5773)]|(1)(1.000)(1.000) \\ & + [B(-0.5773, 0.5773)]^T[D][B(-0.5773, 0.5773)] \\ & |[J(-0.5773, 0.5773)]|(1)(1.000)(1.000) \\ & + [B(0.5773, -0.5773)]^T[D][B(0.5773, -0.5773)] \\ & |[J(0.5773, -0.5773)]|(1)(1.000)(1.000) \\ & + [B(0.5773, 0.5773)]^T[D][B(0.5773, 0.5773)] \\ & |[J(0.5773, 0.5773)]|(1)(1.000)(1.000)\end{aligned}$$

$$[k] = 10^4\begin{bmatrix} 1466 & 500 & -866 & -99 & -733 & -500 & 133 & 99 \\ 500 & 1466 & 99 & 133 & -500 & -733 & -99 & -866 \\ -866 & 99 & 1466 & -500 & 133 & -99 & -733 & 500 \\ -99 & 133 & -500 & 1466 & 99 & -866 & 500 & -733 \\ -733 & -500 & 133 & 99 & 1466 & 500 & -866 & -99 \\ -500 & -733 & -99 & -866 & 500 & 1466 & 99 & 133 \\ 133 & -99 & -733 & 500 & -866 & 99 & 1466 & -500 \\ 99 & -866 & 500 & -733 & -99 & 133 & -500 & 1466 \end{bmatrix}$$

For the rectangular element shown previous Example, assume plane stress conditions
Let E = 30×10^6 psi, $\nu = 0.3$ and displacements:

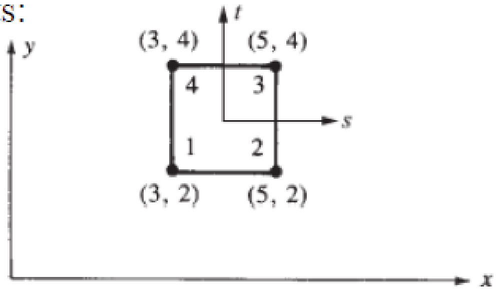
$u_1 = 0$, $v_1 = 0$

$u_2 = 0.001$, $v_2 = 0.0015$

$u_3 = 0.003$, $v_3 = 0.0016$

$u_4 = 0$, $v_4 = 0$

Evaluate the stresses at $s=0$, $t=0$



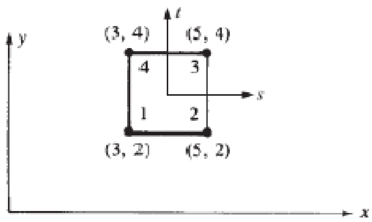
Solution

$$[B] = \frac{1}{|[J]|}[[B_1]\ \ [B_2]\ \ [B_3]\ \ [B_4]]$$

$$[B(0,0)] = \frac{1}{|[J(0,0)]|}[B_1(0,0)]\ \ [B_2(0,0)]\ \ [B_3(0,0)]\ \ [B_4(0,0)]$$

Example 3

$$|[J(0,0)]| = \frac{1}{8}[3\quad 5\quad 5\quad 3]\begin{bmatrix} 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 \end{bmatrix}\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$



$$= \frac{1}{8}[-2\quad -2\quad 2\quad 2]\begin{Bmatrix} 2 \\ 2 \\ 4 \\ 4 \end{Bmatrix}$$

$$|[J(0,0)]| = 1$$

$$[B_i] = \begin{bmatrix} J_{22}\ N_{i,\xi} - J_{12}\ N_{i,\eta} & 0 \\ 0 & J_{11}\ N_{i,\eta} - J_{21}\ N_{i,\xi} \\ J_{11}\ N_{i,\eta} - J_{21}\ N_{i,\xi} & J_{22}\ N_{i,\xi} - J_{12}\ N_{i,\eta} \end{bmatrix}$$

$$J_{22} = \frac{1}{4}\left[y_1(\xi-1) + y_2(-1-\xi) + y_3(1+\xi) + y_4(1-\xi)\right]$$

$$J_{22} = \frac{1}{4}\left[2(0-1) + 2(-1-0) + 4(1+0) + 4(1-0)\right] = 1$$

Similiarly $J_{12} = 0,\quad J_{11} = 1,\quad J_{21} = 0$

$N_{1,\xi} = -\frac{1}{4}(1-\eta) = -\frac{1}{4}(1-0) = -\frac{1}{4}$ Similarly $N_{2,\xi} = \frac{1}{4}$, $N_{3,\xi} = \frac{1}{4}$ and $N_{4,\xi} = -\frac{1}{4}$

$N_{1,\eta} = -\frac{1}{4}(1-\xi) = -\frac{1}{4}(1-0) = -\frac{1}{4}$ Similarly $N_{2,\eta} = -\frac{1}{4}$, $N_{3,\eta} = \frac{1}{4}$ and $N_{4,\eta} = \frac{1}{4}$

$$[B_1] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \qquad [B_2] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad [B_3] = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \qquad [B_4] = \begin{bmatrix} -\frac{1}{4} & 0 \\ 0 & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

$$\{\sigma\} = [D][B]\{d\} =$$

$$= (30)\frac{10^6\begin{bmatrix} 1 & 0.3 & 0 \\ 0.3 & 1 & 0 \\ 0 & 0 & 0.35 \end{bmatrix}}{1-0.09}\begin{bmatrix} -0.25 & 0 & 0.25 & 0 & 0.25 & 0 & -0.25 & 0 \\ 0 & -0.25 & 0 & -0.25 & 0 & 0.25 & 0 & 0.25 \\ -0.25 & -0.25 & -0.25 & 0.25 & 0.25 & 0.25 & 0.25 & -0.25 \end{bmatrix}\begin{Bmatrix} 0 \\ 0 \\ 0.001 \\ 0.0015 \\ 0.003 \\ 0.0016 \\ 0 \\ 0 \end{Bmatrix}$$

$$\{\sigma\} = \begin{Bmatrix} 3.321 \quad 10^4 \\ 1.071 \quad 10^4 \\ 1.471 \quad 10^4 \end{Bmatrix} \text{psi}$$

Higher-Order Shape Functions

- In general, higher-order element shape functions can be developed by adding additional nodes to the sides of the linear element.
- These elements result in higher-order strain variations within each element, and convergence to the exact solution thus occurs at a faster rate using fewer elements.
- Another advantage of the use of higher-order elements is that curved boundaries of irregularly shaped bodies can be approximated more closely than by the use of simple straight-sided linear elements.

Shape function of a quadratic isoparametric element

$$N_1 = \tfrac{1}{4}(1-s)(1-t)(-s-t-1)$$

$$N_2 = \tfrac{1}{4}(1+s)(1-t)(s-t-1)$$

$$N_3 = \tfrac{1}{4}(1+s)(1+t)(s+t-1)$$

$$N_4 = \tfrac{1}{4}(1-s)(1+t)(-s+t-1)$$

or, in compact index notation, we express

$$N_i = \tfrac{1}{4}(1+ss_i)(1+tt_i)(ss_i+tt_i-1)$$

where *i* is the number of the shape function

$$s_i = -1, 1, 1, -1 \qquad (i = 1, 2, 3, 4)$$

$$t_i = -1, -1, 1, 1 \qquad (i = 1, 2, 3, 4)$$

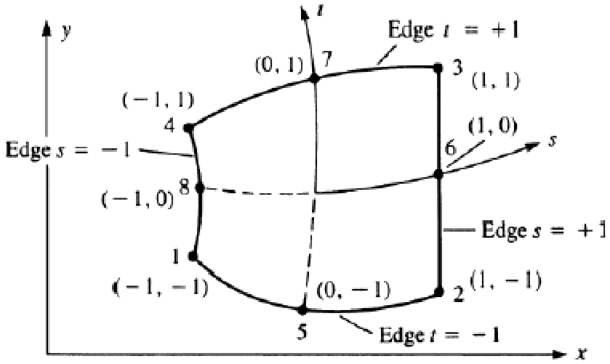


Figure 10–16 Quadratic isoparametric element

For the midside nodes ($i = 5, 6, 7, 8$),

$$N_5 = \tfrac{1}{2}(1-t)(1+s)(1-s)$$

$$N_6 = \tfrac{1}{2}(1+s)(1+t)(1-t)$$

$$N_7 = \tfrac{1}{2}(1+t)(1+s)(1-s)$$

$$N_8 = \tfrac{1}{2}(1-s)(1+t)(1-t)$$

Shape function of a cubic isoparametric element

For the corner nodes ($i = 1, 2, 3, 4$),

$$N_i = \tfrac{1}{32}(1+ss_i)(1+tt_i)[9(s^2+t^2)-10]$$

For the nodes on sides $s = \pm 1$ ($i = 7, 8, 11, 12$),

$$N_i = \tfrac{9}{32}(1+ss_i)(1+9tt_i)(1-t^2)$$

with $s_i = \pm 1$ and $t_i = \pm \frac{1}{3}$.

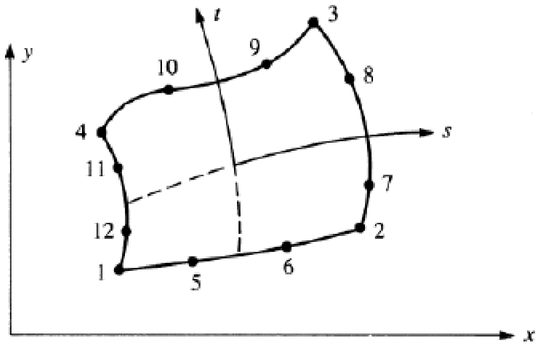


Figure 10–18 Cubic isoparametric element

For the nodes on sides $t = \pm 1$ ($i = 5, 6, 9, 10$),

$$N_i = \tfrac{9}{32}(1+tt_i)(1+9ss_i)(1-s^2)$$

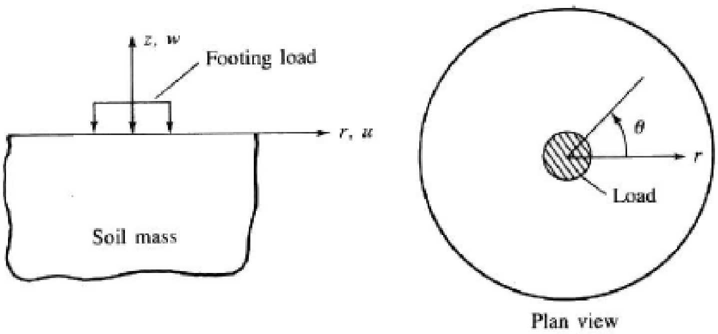
with $t_i = \pm 1$ and $s_i = \pm \frac{1}{3}$.

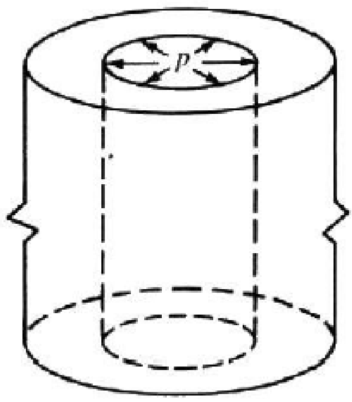
Definition of an axisymmetric solid

- An axisymmetric solid (or a thick-walled body) of revolution is defined as a 3-D body that is generated by rotating a plane and is most easily described in cylindrical coordinates. Where z is called the axis of symmetry.
- If the geometry, support conditions, loads, and material properties are all axially symmetric (all are independent of θ), then the problem can be idealized as a two-dimensional one.

Examples of an axisymmetric solid

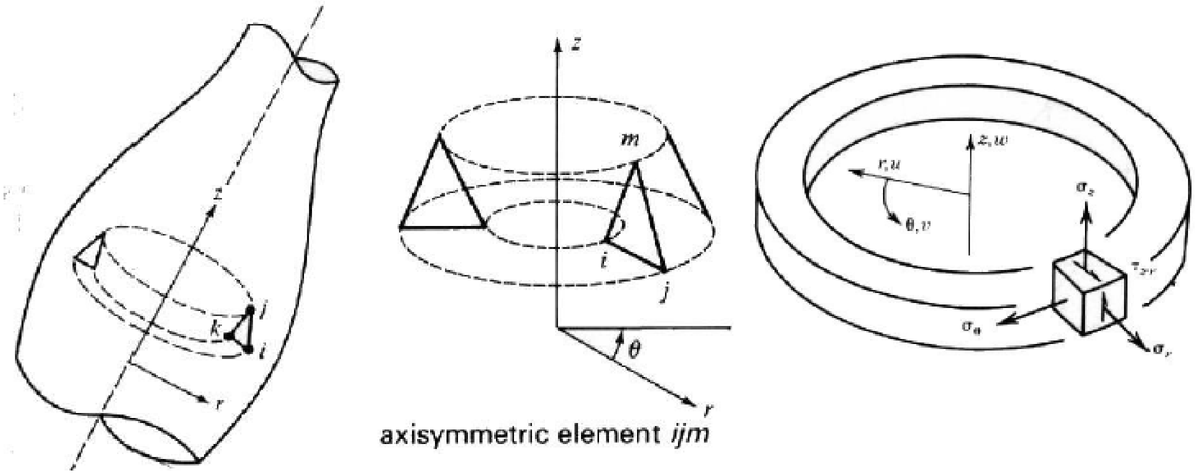
Problems such as soil masses subjected to circular footing loads, thick-walled pressure vessels, and a rocket nozzle subjected to thermal and pressure loading can often be analyzed using axisymmetric elements.





FE axisymmetric elements

axisymmetric problems can be analyzed by a finite element of revolution, called axisymmetric elements. Each element consists of a solid ring, the cross-section of which is the shape of the particular element chosen (triangular, rectangular, or quadrilateral elements). An axisymmetric element has nodal circles rather than nodal points



axisymmetric element *ijm*

Equations of Equilibrium:

The three-dimensional elasticity equations in cylindrical coordinates can be summarized as follows

$$\left.\begin{aligned}
&\frac{\partial \sigma_r}{\partial r}+\frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+\frac{\partial \tau_{rz}}{\partial z}+\frac{\sigma_r-\sigma_\theta}{r}+X_r=0\\
&\frac{\partial \tau_{r\theta}}{\partial r}+\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta}+\frac{\partial \tau_{\theta z}}{\partial z}+\frac{2}{r}\tau_{r\theta}+Y_\theta=0\\
&\frac{\partial \tau_{rz}}{\partial r}+\frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta}+\frac{\partial \sigma_z}{\partial z}+\frac{1}{r}\tau_{rz}+Z_b=0
\end{aligned}\right\}$$

The three-dimensional strain-displacement relationships of elasticity in cylindrical coordinates were ***u, v, w*** are the displacements in the ***r, θ,dz*** , respectively, are:

$$\left.\begin{aligned}
\varepsilon_r &= \frac{\partial u}{\partial r} &, \quad \gamma_{r\theta} &= \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \\
\varepsilon_\theta &= \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} &, \quad \gamma_{rz} &= \frac{\partial w}{\partial r} + \frac{\partial u}{\partial z} \\
\varepsilon_z &= \frac{\partial w}{\partial z} &, \quad \gamma_{\theta z} &= \frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}
\end{aligned}\right\}$$

The three-dimensional stress-strain relationships for isotropic elasticity are:

$$\begin{Bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \\ \tau_{r\theta} \\ \tau_{rz} \\ \tau_{\theta z} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{rz} \\ \gamma_{\theta z} \end{Bmatrix}$$

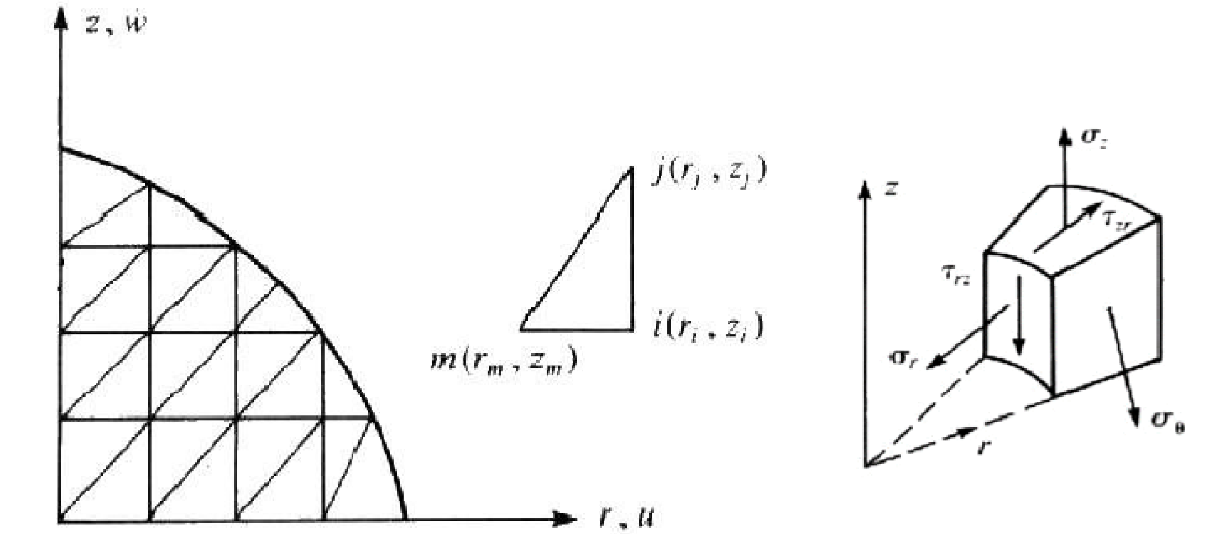
In axisymmetric problems, because of the symmetry about the z -axis, the stresses are independent of the θ coordinate. Therefore, all derivatives with respect to θ vanish and the circumferential (tangent to θ direction) displacement component is zero; therefore,

$$\gamma_{r\theta} = \gamma_{\theta z} = 0 \quad and \quad \tau_{r\theta} = \tau_{\theta z} = 0$$

$$\varepsilon_r = \frac{\partial u}{\partial r} \quad , \quad \varepsilon_\theta = \frac{u}{r} \quad , \quad \varepsilon_z = \frac{\partial w}{\partial z} \quad , \quad \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

Derivation of the Stiffness Matrix and Equations

Step 1: Discretize and Select Element Type



Typical slice through an axisymmetric solid Discretized into triangular elements

$$\{d\} = \begin{Bmatrix} \underline{d}_i \\ \underline{d}_j \\ \underline{d}_m \end{Bmatrix} = \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

(u_i, w_i) → displacement components of ***node i*** in the *r* and *z* directions, respectively.

Step 2: Select Displacement Functions

$$u(r,z) = a_1 + a_2 r + a_3 z$$
$$w(r,z) = a_4 + a_5 r + a_6 z$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} a_1 + a_2 r + a_3 z \\ a_4 + a_5 r + a_6 z \end{Bmatrix} = \begin{bmatrix} 1 & r & z & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{Bmatrix}$$

$$u_i = a_1 + a_2 r_i + a_3 z_i$$

$$u_j = a_1 + a_2 r_j + a_3 z_j$$

$$u_m = a_1 + a_2 r_m + a_3 z_m$$

$$w_i = a_4 + a_5 r_i + a_6 z_i$$

$$w_j = a_4 + a_5 r_j + a_6 z_j$$

$$w_m = a_4 + a_5 r_m + a_6 z_m$$

In Matrix Form

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \qquad \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \begin{bmatrix} 1 & r_i & z_i \\ 1 & r_j & z_j \\ 1 & r_m & z_m \end{bmatrix}^{-1} \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

Solving for the a's

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix} \qquad \begin{Bmatrix} a_4 \\ a_5 \\ a_6 \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} w_i \\ w_j \\ w_m \end{Bmatrix}$$

$$2A = \begin{vmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_m & y_m \end{vmatrix}$$

$$2A = x_i(y_j - y_m) + x_j(y_m - y_i) + x_m(y_i - y_j)$$
$$= \alpha_i + \alpha_j + \alpha_m$$

A is the area of the triangle

$\alpha_i = r_j z_m - z_j r_m$ $\qquad$ $\alpha_j = r_m z_i - z_m r_i$ $\qquad$ $\alpha_m = r_i z_j - z_i r_j$

$\beta_i = z_j - z_m$ $\qquad$ $\beta_j = z_m - z_i$ $\qquad$ $\beta_m = z_i - z_j$

$\gamma_i = r_m - r_j$ $\qquad$ $\gamma_j = r_i - r_m$ $\qquad$ $\gamma_m = r_j - r_i$

$$\{u\} = \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A} \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{bmatrix} \alpha_i & \alpha_j & \alpha_m \\ \beta_i & \beta_j & \beta_m \\ \gamma_i & \gamma_j & \gamma_m \end{bmatrix} \begin{Bmatrix} u_i \\ u_j \\ u_m \end{Bmatrix}$$

$$\{u\} = \frac{1}{2A} \begin{bmatrix} 1 & r & z \end{bmatrix} \begin{Bmatrix} \alpha_i u_i + \alpha_j u_j + \alpha_m u_m \\ \beta_i u_i + \beta_j u_j + \beta_m u_m \\ \gamma_i u_i + \gamma_j u_j + \gamma_m u_m \end{Bmatrix}$$

$$u(r,z) = \frac{1}{2A} \left\{ (\alpha_i + \beta_i r + \gamma_i z) u_i + (\alpha_j + \beta_j r + \gamma_j z) u_j + (\alpha_m + \beta_m r + \gamma_m z) u_m \right\}$$

$$w(r,z) = \frac{1}{2A} \left\{ (\alpha_i + \beta_i r + \gamma_i z) w_i + (\alpha_j + \beta_j r + \gamma_j z) w_j + (\alpha_m + \beta_m r + \gamma_m z) w_m \right\}$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

$$\{\psi\} = \begin{Bmatrix} u(r,z) \\ w(r,z) \end{Bmatrix} = \begin{Bmatrix} N_i u_i + N_j u_j + N_m u_m \\ N_i v_i + N_j v_j + N_m v_m \end{Bmatrix}$$

$$N_i = \frac{1}{2A}\left(\alpha_i + \beta_i r + \gamma_i z\right)$$

$$N_j = \frac{1}{2A}\left(\alpha_j + \beta_j r + \gamma_j z\right)$$

$$N_m = \frac{1}{2A}\left(\alpha_m + \beta_m r + \gamma_m z\right)$$

$$\{\psi\} = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

$$\{\psi\} = [N]\{d\}$$

$$[N] = \begin{bmatrix} N_i & 0 & N_j & 0 & N_m & 0 \\ 0 & N_i & 0 & N_j & 0 & N_m \end{bmatrix}$$

Step 3: **Define the Strain/Displacement and Stress/Strain Relationships**

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial r} \\ \dfrac{\partial w}{\partial z} \\ \dfrac{u}{r} \\ \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial r} \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{Bmatrix} a_2 \\ a_6 \\ \dfrac{a_1}{r} + a_2 + \dfrac{a_3 z}{r} \\ a_3 + a_5 \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} \beta_i & 0 & \beta_j & 0 & \beta_m & 0 \\ 0 & \gamma_i & 0 & \gamma_j & 0 & \gamma_m \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 & \frac{\alpha_j}{r} + \beta_j + \frac{\gamma_j z}{r} & 0 & \frac{\alpha_m}{r} + \beta_m + \frac{\gamma_m z}{r} & 0 \\ \gamma_i & \beta_i & \gamma_j & \beta_j & \gamma_m & \beta_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix}$$

$$\{\varepsilon\} = \begin{bmatrix} B_i & B_j & B_m \end{bmatrix} \begin{Bmatrix} u_i \\ w_i \\ u_j \\ w_j \\ u_m \\ w_m \end{Bmatrix} \qquad B_i = \begin{bmatrix} \beta_i & 0 \\ 0 & \gamma_i \\ \frac{\alpha_i}{r} + \beta_i + \frac{\gamma_i z}{r} & 0 \\ \gamma_i & \beta_i \end{bmatrix}$$

$$\{\varepsilon\} = [B]\,\{d\}$$

B is a function of *r* and *z*

Stress Strain Relationship

$$\{\sigma\} = [D][B]\{d\}$$

$$\begin{Bmatrix} \sigma_r \\ \sigma_z \\ \sigma_\theta \\ \tau_{rz} \end{Bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{Bmatrix}$$

Step 4 :Derive the Element Stiffness Matrix and Equations

$$[k] = \iiint_V [B]^T [D][B] \, dV$$

$$[k] = 2\pi \iint_A [B]^T [D][B] \, r \, dr \, dz$$

1) Numerical integration (Gaussian quadrature)
2) Explicit multiplication and term-by-term integration.
3) Evaluate [B] at a centroidal point of the element

$$r = \bar{r} = \frac{r_i + r_j + r_m}{3} \qquad z = \bar{z} = \frac{z_i + z_j + z_m}{3}$$

$$[B(\bar{r}, \bar{z})] = [\bar{B}]$$

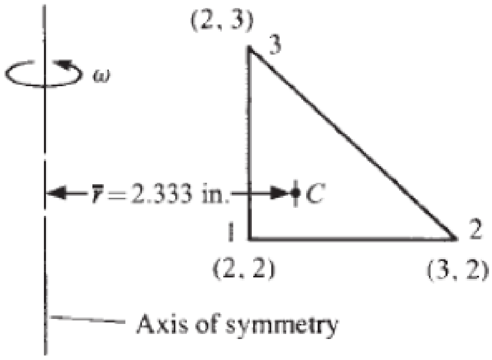
$$[k] = 2\pi \bar{r} A [\bar{B}]^T [D][\bar{B}]$$

Example 1

For the element of an axisymmetric body rotating with a constant angular velocity ω = ***100 rev/min***.
Evaluate the approximate body force matrix, include the weight of the material, where the weight density ρ_w **is 0.283** ***lb/in³***.
The coordinates of the element (in inches) are shown in the figure.

The body forces per unit volume evaluated at the centroid of the element are

Z_b =0.283 lb/in³



$$\bar{R}_b = \omega^2 \rho \bar{r} = \left[100 \frac{\text{rev}}{\text{min}} \left(2\pi \frac{\text{rad}}{\text{rev}} \right) \left(\frac{1 \text{ min}}{60 \text{ s}} \right) \right]^2 \frac{(0.283 \text{ lb/in}^3)}{(32.2 \times 12) \text{ in./s}^2} (2.333 \text{ in.})$$

$$\bar{R}_b = 0.187 \text{ lb/in}^3$$

$$\frac{2\pi \bar{r} A}{3} = \frac{2\pi(2.333)(0.5)}{3} = 2.44 \text{ in}^3$$

$$f_{b1r} = (2.44)(0.187) = 0.457 \text{ lb}$$

$$f_{b1z} = -(2.44)(0.283) = -0.691 \text{ lb} \qquad \text{(downward)}$$

All r-directed and z-directed nodal body forces are equal

$$f_{b2r} = 0.457 \text{ lb} \qquad f_{b2z} = -0.691 \text{ lb}$$

$$f_{b3r} = 0.457 \text{ lb} \qquad f_{b3z} = -0.691 \text{ lb}$$