SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES :: CHITTOOR

(Autonomous)

Department of Mathematics

II B Tech- I Semester

23BSC231	COMPLEX VARIABLES AND NUMERICAL METHODS	L	T	P	C
	(EEE)	3	1	-	4

PRE-REQUISITES: Algebra, Calculus, Differential equations, Real Analysis

COURSE EDUCATIONAL OBJECTIVES:

1. To analyze the functions of complex variable with a review of elementary complex Functions and to learn continuity, differentiability and analyticity of a complex function
2. To understand the Taylor and Laurent expansion with their use in finding out the residue and improper integral.
3. To develop skill to analyze appropriate method to find the root of the Algebraic and Transcendental Equations and to develop skill to apply the concept of interpolation
4. To learn the method of evaluation of numerical derivative, numerical integration
5. To learn the method of evaluation of solve ordinary differential equations numerically using numerical methods.

UNIT I : Complex Variable – Differentiation 09

Introduction to functions of complex variable-concept of Limit & continuity- Differentiation, Cauchy-Riemann equations, analytic functions harmonic functions, finding harmonic conjugate-construction of analytic function by Milne Thomson method.

UNIT II: Complex Variable – Integration 09

Line integral, Cauchy's integral theorem(Simple Case), Cauchy Integral formula, Power series expansions: Taylor's series,Laurent's series, zeros of analytic functions, singularities, Residues, Cauchy Residue theorem (without proof).

UNIT III:Solution of Algebraic & Transcendental Equations and Interpolation 09

Introduction-Bisection Method-Iterative method, Regula-falsi method and Newton Raphson method Finite differences-Newton's forward and backward interpolation formulae – Lagrange's formulae.

UNIT IV:Numerical Differentiation, Numerical Integration and Curve Fitting 09

Numerical differentiation: Newton's forward and backward formulae
Numerical integration: Trapezoidal rule - Simpson's 1/3 Rule - Simpson's 3/8 Rule
Curve fitting: Fitting of straight line, second-degree and Exponential curve by method of least squares.

UNIT V:Solution of Initial value problems to Ordinary differential equations 09

Numerical solution of Ordinary Differential equations: Solution by Taylor's series-Picard's Method of successive Approximations-Euler's and modified Euler's methods-Runge-Kutta methods (second and fourth order).

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COURSE OUTCOMES:

On successful completion of the course,students will be able to		POsrelatedtoCOs
CO1	Analyze limit, continuity and differentiation of functions of complex variables and Understand Cauchy-Riemann equations, analytic functions and various properties of analytic functions.	PO1,PO2,PO3
CO2	Understand Cauchy theorem, Cauchy integral formulas and apply these to evaluate complex integrals. Classify singularities and poles; find residues and evaluate complex integrals using the residue theorem.	PO1,PO2,PO3
CO3	Apply numerical methods to solve algebraic and transcendental equations and to Derive interpolating polynomials using interpolation formulae.	PO1,PO2,PO3
CO4	Demonstrate knowledge in finding the numerical values to derivatives, integrals through different mathematical methods and constructing a curve, or mathematical function, that has the best fit to a series of data points	PO1,PO2,PO3
CO5	Demonstrate knowledge in solving ordinary differential equations numerically through various methods	PO1,PO2,PO3

TEXT BOOKS:

1. B.S.Grewal,Higher Engineering Mathematics, KhannaPublishers,2017, 44th Edition
2. S S Sastry, Introductory Methods of Numerical Analysis, PHI Learning Private Limited.

REFERENCE BOOKS:

1. Erwin Kreyszig, Advanced Engineering Mathematics, John Wiley & Sons, 2018, 10th Edition.
2. B.V.Ramana, Higher Engineering Mathematics, by Mc Graw Hill publishers
3. R.K.JainandS.R.K.Iyengar,Advanced Engineering Mathematics, Alpha Science International Ltd.,2021 5th Edition(9th reprint).

ONLINE LEARNING RESOURCES:

1. https://onlinecourses.nptel.ac.in/noc17_ma14/preview
2. https://onlinecourses.nptel.ac.in/noc20_ma50/preview
3. http://nptel.ac.in/courses/111105090

CO\PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO.1	3	3	3	-	-	-	-	-	-	-	-	-
CO.2	3	3	3	-	-	-	-	-	-	-	-	-
CO.3	3	3	3	-	-	-	-	-	-	-	-	-
CO.4	3	3	3	-	-	-	-	-	-	-	-	-
CO.5	3	3	3	-	-	-	-	-	-	-	-	-
CO*	**3**	**3**	**3**	**-**	**-**	**-**	**-**	**-**	**-**	**-**	**-**	-

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Complex variable - Differentiation

- **Function of a complex variable**
- **Limit, continuity and differentiability of a function**
- **Analytic functions**
- **Harmonic functions**
- **Procedure of finding derivative of a complex function**
- **Construction of analytic function given real or imaginary part – Milne Thomson's method**

Pre-requisites:

Consider the quadratic equation, $m^2+1=0$. This equation does not possess solution in reals. This difficulty was overcome by introducing the imaginary unit i defined as $i^2=-1$ or $i=\sqrt{-1}$ Thus the system of real number is extended to the system of complex number. Here i is called imaginary number.

- The numbers of the form $z=x+iy$ where x , y are real numbers and $i=\sqrt{-1}$,known as imaginary unit are called complex numbers
- The real number x is called the real part and the real number y is called the imaginary part of the complex number z. These are denoted by Re(z)and Im (z).
  $z=x+iy \Rightarrow \text{Re}(z)=x, \text{Im}(z)=y$
- The complex number is purely real if $y=0$ and it is purely imaginary if real part is 0.
- The set of complex numbers is denoted by C. All real numbers are complex but all complex numbers need not be reals.
- Two complex numbers $z_1=a+ib$ and $z_2=c+id$ are said to be equal if their real and imaginary parts are separately equal, ie $z_1=z_2 \Rightarrow a+ib=c+id \Rightarrow a=c \ \ and \ \ b=d$.
- The sum of two complex numbers $z_1=a+ib$ and $z_2=c+id$ is the complex number $(a+c)+i(b+d)$
- The product of two complex numbers $z_1=a+ib$ and $z_2=c+id$ is the complex number $z_1z_2=(ac-bd)+i(ad+bc)$
- If $z=x+iy$ is any complex number, then the complex number $\bar{z}=x-iy$ is called the conjugate of the complex number z. Further $z\bar{z}=(x+iy)(x-iy)=x^2+y^2$.
- A complex number and its conjugate satisfy the following properties:

i) $\overline{z_1z_2}=\overline{z_1}\bullet\overline{z_2}$ ii) $\overline{z_1 \pm z_2}=\overline{z_1}\pm\overline{z_2}$

iii) $z+\bar{z}=2\operatorname{Re}(z)$. iv) $z-\bar{z}=2i\operatorname{Im}(z)$.

- $e^{i\theta}=\cos\theta+i\sin\theta$
- $\cos\theta=\dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta=\dfrac{e^{i\theta}-e^{-i\theta}}{2i}$.
- $\cos(i\theta)=\cosh\theta$, $\sin(i\theta)=i\sinh\theta$.
- If n is any rational number then $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$.

1.1 Complex plane:

The complex numbers can be represented by means of points on the plane. The plan on which the representation is carried out is called complex **plane** or **Argand plane.** We can represent the complex number of the form $z=x+iy$ as a point $P(x,y)$ as shown in the Cartesian plane, called complex plane or $z-$plane. The x-axis is called the real axis and y-axis the imaginary axis. If θ is the angle made by OP with the x-axis and OP$=r$, then we have

$$x=r\cos\theta, y=r\sin\theta \Rightarrow |z|=\sqrt{x^2+y^2}=r \quad and \quad \theta=\tan^{-1}\left(\frac{y}{x}\right)=amp\, z.$$

Here r is called radius vector or the modulus of the complex number and θ is called amplitude of z or the argument of z denoted as amp(z) or arg(z).

- $z\bar{z}=x^2+y^2=|z|^2$ *and* $|z|=\sqrt{z\bar{z}}$
- $|z_1\bullet z_2|=|z_1|\bullet|z_2|$, $\dfrac{|z_1|}{|z_2|}=\left|\dfrac{z_1}{z_2}\right|$.
- $amp(z_1z_2)=amp(z_1)+amp(z_2)$, $amp\left(\dfrac{z_1}{z_2}\right)=amp(z_1)-amp(z_2)$.
- Complex number in i) Cartesian form: $z=x+iy$

  ii) Polar form: $z=r(\cos\theta+i\sin\theta)$

  iii) Exponential form: $z=re^{i\theta}$.

1.2 Function of a complex variable:

Let D be the set of complex numbers and z and w be two complex variables. The complex variable w is said to be a function of the complex variable z, if to every value of z in the domain D, there correspond one or more values of w. This is denoted by $w = f(z)$. Ex: $w = e^z, w = \frac{1}{z}, w = \sin z$ *etc.* are functions of complex variable.

1.3 Limit, continuity and differentiability of a complex function:

1.3.1 Limit of a complex function:

A function $w = f(z)$ is said to tend to limit l as z tends to z_0 if for every $\varepsilon > 0$, there exists as $\delta > 0$ such that $|f(z) - l| < \varepsilon$ *for all* $|z - z_0| < \delta$.

The above definition implies that the value of $f(z)$ can be made arbitrarily close to l for all z in the neighborhood of z_0, except perhaps at $z = z_0$. Symbolically, we can write $\lim_{z \to z_0} f(z) = l$.

1.3.2 Continuity of a function of a complex variable:

A function $f(z)$ of complex variable z,is said to be continuous at a point z_0,if

i. $f(z)$ is defined at z = z_0.
ii. $\lim_{z \to z_0} f(z)$ exists.
iii. $\lim_{z \to z_0} f(z) = f(z_0)$.

Note that the sum, difference and product of continuous functions of a complex variable are continuous. And the function $f(z)$ is said to be continuous in a domain D if it is continuous at every point in D.

1.3.3 Differentiability of a complex function:

A function $f(z)$ is said to be differentiable at a point z = z_0 if $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exists. This limit is called the derivative of $f(z)$ at z = z_0 and is denoted by $f'(z_0)$. Let $z = z_0 + \delta z$, $f'(z_0) =$ $\lim_{z \to z_0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$.

Note:

Neighbourhood of a complex number z_0:

Let r be a finite number, neighbourhood of a complex number z_0 is defined as $|z - z_0| = r$ or $|z - z_0| < r$

Consider $z = x + iy \therefore z_0 = x_0 + iy_0$

$$|z - z_0| = |x + iy - (x_0 + iy_0)| = r \Rightarrow \sqrt{(x - x_0)^2 + (y - y_0)^2} = r \Rightarrow (x - x_0)^2 + (y - y_0)^2 = r$$

Hence neighbourhood of any point z_0 in the complex plane is the circle centered at z_0 and having radius r and $|z - z_0| < r$ represents only the interior points of the circle centered at z_0 and radius r.

1.3.4 Analytic function:

A function $f(z)$ of a complex variable is said to be analytic at a point z_0 if it is differentiable at z_0 and also at every point of some neighbourhood of z_0. A function $f(z)$ is said to be analytic in a domain D, if it is analytic at every point in D. It is also called as **regular** function or **holomorphic** function.

1.4 Cauchy-Riemann Equations:

Theorem:

A necessary condition that $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain D is that the first order derivatives of u and v with respect to x and y must exist and satisfy the equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. These equations are called Cauchy-Riemann equations in Cartesian form.

Proof:

Given that $f(z)$ is analytic in a domain D. Therefore $f(z)$ has a derivative

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z + \delta z) - f(z)}{\delta z}$$

We have $f(z) = u(x, y) + iv(x, y)$

$$\therefore f(z + \delta z) = u(x + \delta x, y + \delta y) + iv(x + \delta x, y + \delta y).$$

$\delta z = \delta x + i\delta y$

$$\therefore \quad f'(z) = \lim_{\delta z \to 0} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y) + iv(x,y)]}{\delta z}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta z} + i \lim_{\delta z \to 0} \frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta z} \qquad (1)$$

Case 1: Let δz be purely real. Thus $\delta y = 0, \delta z = \delta x$.

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{\delta x} + i \lim_{\delta z \to 0} \frac{v(x+\delta x, y+\delta y) - v(x,y)}{\delta x}$$

$$f'(z) = \lim_{\delta z \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta z \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad (2)$$

Case 2: Let δz be purely imaginary. Thus $\delta x = 0, \delta z = i\delta y$

$$\therefore f'(z) = \lim_{\delta y \to 0} \frac{u(x+\delta x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x+\delta x, y+\delta y) - v(x,y)}{i\delta y}$$

$$\Rightarrow f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\delta y} + i \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{i\delta y}$$

$$\Rightarrow f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\Rightarrow f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad (3)$$

Since $f'(z)$ is unique, the right sides of (2) *and* (3) are equal.

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating real and imaginary parts we get,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note:

i) $f(z)$ *analytic* $\Rightarrow v_x$ ii) $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. need not imply $f(z)$ analytic.

Sufficient Conditions:

The sufficient conditions for the function $f(z)=u+iv,$ to be analytic at any point in the domain are

i) u,v and their partial derivatives $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous at that point.

ii) C-R equations are satisfied at that point.

1.5 Cauchy-Riemann equations in polar form:

Statement:

A necessary condition that $f(z)=u(r,\theta)+iv(r,\theta)$ be analytic in a domain D is that the first order derivatives of u and v w.r.t $r \text{ and } \theta$ must exist and must satisfy the equations

$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta},\ \frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}.$ These equations are called C-R equations in polar form.

Proof:

Given that $f(z)$ is analytic in the domain D.

$$\therefore f'(z)=\lim_{\delta z\to 0}\frac{f(z+\delta z)-f(z)}{\delta z} \text{ exists.}$$

And we have $f(z)=u(r,\theta)+iv(r,\theta)$ and $f(z+\delta z)=u(r+\delta r,\theta+\delta\theta)+iv(r+\delta r,\theta+\delta\theta)$

$$\therefore f'(z)=\lim_{\delta z\to 0}\frac{[u(r+\delta r,\theta+\delta\theta)+iv(r+\delta r,\theta+\delta\theta)]-[u(r,\theta)+iv(r,\theta)]}{\delta z}$$

$$f'(z)=\lim_{\delta z\to 0}\frac{[u(r+\delta r,\theta+\delta\theta)-u(r,\theta)]-i[v(r+\delta r,\theta+\delta\theta)-v(r,\theta)]}{\delta z} \qquad (1)$$

$$We\ \ have\ z=re^{i\theta}$$
$$\delta z=\frac{\delta z}{\delta r}\delta r+\frac{\delta z}{\delta\theta}\delta\theta$$
$$\delta z=e^{i\theta}\delta r+ire^{i\theta}\delta\theta$$

Equation (1) becomes $f'(z) = \lim_{\delta z \to 0} \frac{[u(r+\delta r, \theta+\delta\theta) - u(r,\theta)]}{\delta z} + i \lim_{\delta z \to 0} \frac{[v(r+\delta r, \theta+\delta\theta) - v(r,\theta)]}{\delta z}$ (2)

Case 1: When $\delta z \to 0, \delta r \to 0$ and put $\delta\theta = 0$

Equation (1) becomes,

$$f'(z) = \lim_{\delta r \to 0} \frac{[u(r+\delta r, \theta) - u(r,\theta)]}{e^{i\theta}\delta r} + i \lim_{\delta z \to 0} \frac{[v(r+\delta r, \theta+\delta\theta) - v(r,\theta)]}{e^{i\theta}\delta r}$$

$$f'(z) = e^{-i\theta}\frac{\partial u}{\partial r} + ie^{-i\theta}\frac{\partial v}{\partial r} \qquad (3)$$

Case 2: When $\delta z \to 0, \delta\theta \to 0$ and put $\delta r = 0$

Equation (1) becomes,

$$f'(z) = \lim_{\delta\theta \to 0} \frac{[u(r, \theta+\delta\theta) - u(r,\theta)]}{ire^{i\theta}\delta\theta} + i \lim_{\delta\theta \to 0} \frac{[v(r, \theta+\delta\theta) - v(r,\theta)]}{ire^{i\theta}\delta\theta}$$

$$f'(z) = \frac{-ie^{-i\theta}}{r}\frac{\partial u}{\partial \theta} + \frac{e^{-i\theta}}{r}\frac{\partial v}{\partial r} \qquad (4)$$

Comparing (3) *and* (4),

$$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}, \; r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta}$$

1.6 Harmonic Functions: (Laplace equation)

The equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called Laplace equation in two dimensions.

The Laplace equation is of great practical importance and occurs frequently in the study of fluid flow, heat conduction, gravitation and electrostatic

Any function $\phi(x, y)$ which possesses continuous partial derivatives of the first and second order and satisfy the Laplace equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ is called a **harmonic function.**

1.6.1 Theorem: (Cartesian form)

Statement:

The real and imaginary parts of an analytic function $f(z)=u(x,y)+iv(x,y)$ are harmonic functions.

Proof:

Given $f(z)=u(x,y)+iv(x,y)$ is analytic function.

We are supposed to prove that u and v are harmonic functions.

Since $f(z)$ is analytic, C-R equations are satisfied.

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad (1)$$

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}. \qquad (2)$$

Differentiate (1) w.r.t x and (2) w.r.t y, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial y\partial x} \quad \text{and} \quad \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial x\partial y}$$

$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial y\partial x}-\frac{\partial^2 v}{\partial x\partial y}=0 \Rightarrow$ u is harmonic function.

Differentiate (1) w.r.t y and (2) w.r.t x and adding the equations we get,

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$$

1.6.2 Theorem: (Polar form)

Statement:

The real and imaginary parts of an analytic function $f(z)=u(r,\theta)+iv(r,\theta)$ are harmonic functions.

Proof:

Given $f(z)=u(r,\theta)+iv(r,\theta)$ is analytic.

We are supposed to prove that u and v are harmonic.

Since $f(z)$ is analytic C-R equations are satisfied.

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta} \qquad (1)$$

$$\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}. \qquad (2)$$

Differentiate (1) w.r.t r and (2) w.r.t θ we get,

$$r\frac{\partial^2 u}{\partial r^2}+\frac{\partial u}{\partial r}=\frac{\partial^2 v}{\partial r\partial \theta}$$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial^2 v}{\partial r\partial \theta} \qquad (3)$$

$$-\frac{\partial^2 u}{\partial \theta^2}=r\frac{\partial^2 v}{\partial \theta\partial r}$$

$$\Rightarrow \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=-\frac{1}{r}\frac{\partial^2 v}{\partial \theta\partial r} \qquad (4)$$

Adding (3) *and* (4) *we get,*

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0. \Rightarrow u \text{ is harmonic.}$$

Differentiate (1) w.r.t θ and (2) w.r.t r and adding we get,

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}=0. \Rightarrow v \text{ is harmonic.}$$

1.6.3 Theorem:

Statement:

If $f(z)=u(x,y)+iv(x,y)$ is analytic function in the domain D of the complex plane, then $u=c_1$ *and* $v=c_2$ where c_1 and c_2 are constants, represent orthogonal family of curves.

Proof:

Given $f(z)=u+iv$ is analytic function

And $u(x,y)=c_1$ (1)

$v(x, y) = c_2$ (2)

Differentiate (1) w.r.t x we get,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right) = m_1$$

Again differentiate (2) w.r.t y we get,

$$\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\left(\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right) = m_2$$

Since $f(z) = u(x, y) + iv(x, y)$ is analytic function, we have C-R equations: $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

$$\therefore m_1 m_2 = \left(\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right) = \left(\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}\right)\left(\frac{-\frac{\partial u}{\partial y}}{\frac{\partial u}{\partial x}}\right) = -1$$

This shows that the curves $u(x, y) = c_1$ and $v(x, y) = c_2$ cut orthogonally.

Note:

If $f(z) = u + iv$ is analytic function then v is an harmonic conjugate of u and vice-versa.

Procedure of finding derivative of a complex function:

1. Given $f(z)$, replace z by $x + iy$ or $re^{i\theta}$. And find the real part of $f(z)$ i.e. u and imaginary part v.

2. Verify whether $f(z)$ is analytic or not. If so, find $f'(z)$ using the relevant formula

$f'(z) = u_x + iv_x$ or $f'(z) = e^{-i\theta}(u_r + iv_r)$ where $u_x = \frac{\partial u}{\partial x}$, $v_x = \frac{\partial v}{\partial x}$, $u_r = \frac{\partial u}{\partial r}$ and $v_r = \frac{\partial v}{\partial r}$.

Problems:

1. Find the complex derivative of $f(z)=z$

Solution:

Given $f(z)=z$

$\Rightarrow f(z)=x+iy$
$\Rightarrow u=x, v=y$

Consider $u_x=1,\ v_x=0,\ v_y=1$ and $u_y=0$

Clearly $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=1$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}.=0$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z)=u_x+iv_x$

$f'(z)=1+i0=1$.

2. Find $f'(z)$ if $f(z)=e^z$

Solution:

Given $f(z)=e^z$

$\Rightarrow f(z)=e^{x+iy}=e^x e^{iy}=e^x(\cos y+i\sin y)$

$\Rightarrow u=e^x\cos y, v=e^x\sin y$

Now $u_x=\mathrm{e}^x\cos y, u_y=-\mathrm{e}^x\sin y, v_x=\mathrm{e}^x\sin y$ *and* $v_y=\mathrm{e}^x\cos y.$

Clearly $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=\mathrm{e}^x\cos y$, $\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$. C-R equations are satisfied.

And all the parial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z)=u_x+iv_x$

$f'(z)=\mathrm{e}^x\cos y+i\ \mathrm{e}^x\sin y=e^x(e^{iy})=e^z$.

3. Find $f'(z)$ if $f(z) = e^z + z$.

Solution:

Given $f(z) = e^z + z$.

$\Rightarrow f(z) = e^{x+iy} + (x+iy) = e^x e^{iy} + (x+iy)$

$= e^x(\cos y + i\sin y) + x + iy$

$\Rightarrow u = e^x \cos y + x, v = e^x \sin y + y$

$Now\ u_x = e^x \cos y + 1, u_y = -e^x \sin y$

$v_x = e^x \sin y\ and\ v_y = e^x \cos y + 1.$

Clearly $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = e^x \cos y + 1$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z) = u_x + iv_x$

$f'(z) = e^x \cos y + 1 + i(e^x \sin y)$

$f'(z) = e^z + 1.$

4. Find $f'(z)$ if $f(z) = \cos z$.

Solution:

Given $f(z) = \cos z$.

$\Rightarrow f(z) = \cos(x+iy) = \cos x \cos(iy) - \sin x \sin(iy)$

$= \cos x \cosh y - i \sin x \sinh y$

$\Rightarrow u = \cos x \cosh y, v = -\sin x \sinh y$

$Now\ u_x = -\sin x \cosh y, u_y = \cos x \sinh y, v_x = -\cos x \sinh y\ and\ v_y = -\sin x \cosh y$

Clearly $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z) = u_x + iv_x$

$$f'(z) = -\sin x \cosh y - i\cos x \sinh y$$

$$f'(z) = -\sin x \cos(iy) - \cos x \sin(iy) = -\sin(x+iy) = -\sin z.$$

5. Find $f'(z)$ if $f(z) = \cosh z$.

Solution:

Given $f(z) = \cosh z$

$\Rightarrow f(z) = \cos(iz) = \cos i(x+iy) = \cos(xi - y)$

$\Rightarrow \cos(ix)\cos y + \sin(xi)\sin y = \cosh x \cos y + i \sinh x \sin y$

$\Rightarrow u = \cosh x \cos y, v = \sinh x \sin y$

Now $u_x = \sinh x \cos y, u_y = -\cosh x \sin y,.$

$v_x = \cosh x \sin y$ *and* $v_y = \sinh x \cos y$

Clearly $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. C-R equations are satisfied

And all the partial derivatives are continuous. $\therefore f(z)$ is analytic.

We have $f'(z) = u_x + iv_x$

$$f'(z) = \sinh x \cos y + i\cosh x \sin y$$

$$f'(z) = \cos y \sin(ix) + i\cos(ix)\sin y$$

$$f'(z) = \frac{\sin(ix)}{i}\cos y + i\cos(ix)\sin y$$

$$= -i[\sin(ix)\cos y + \cos(ix)\sin y] = -i\sin[i(x+iy)]$$

$$= -i\sin(iz) = -i^2 \sinh z = \sinh z .$$

6. Find $f'(z)$ if $f(z) = \log z$.

Solution:

Given $f(z) = \log z$

$$\Rightarrow f(z) = \log\left(re^{i\theta}\right) = \log r + i\theta$$

$$\Rightarrow u = \log r, v = \theta$$

$Now\ u_r = \frac{1}{r}, u_\theta = 0, v_r = 0\ and\ v_\theta = 1.$

$Consider\ ru_r = 1 = v_\theta\ \ and\ rv_r = 0 = u_\theta$

$\therefore ru_r = v_\theta\ \ and\ rv_r = u_\theta$. C - R equations are satisfied

And the partial derivatives are continuous. $\therefore\ f(z)$ is analytic.

$$f'(z) = e^{-i\theta}\frac{\partial u}{\partial r} + ie^{-i\theta}\frac{\partial v}{\partial r}$$

$$f'(z) = u_r e^{-i\theta} + ie^{-i\theta}v_r = e^{-i\theta}(\frac{1}{r}+0) = \frac{1}{re^{i\theta}} = \frac{1}{z}.$$

7. Find the derivative of complex valued function $f(z) = z^n$

Solution:

$Given\ f(z) = z^n$

$\Rightarrow (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i\sin n\theta).$

$\therefore u = r^n \cos n\theta, v = r^n \sin n\theta$

$u_r = nr^{n-1}\cos n\theta, u_\theta = -nr^n \sin n\theta, v_r = nr^{n-1}\sin n\theta, v_\theta = nr^n \cos n\theta.$

$Clearly\ \ ru_r = v_\theta\ \ and\ \ rv_r = -u_\theta$

C-R equations are satisfied and partial derivatives are continuous.

Thus $f(z)$ is analytic.

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}(nr^{n-1}\cos n\theta + inr^{n-1}\cos n\theta) = nr^{n-1}.e^{-i\theta}e^{in\theta}$$

$$= nr^n \frac{1}{re^{i\theta}} e^{in\theta} = nz^{n-1}$$

1.7 Construction of analytic function using Milne's Thomson's method:

This method can be applied to construct the analytic function when only real or imaginary part is given.

1.7.1 Procedure:

1. If u or v of analytic function is given in Cartesian form:

Suppose the real part u is given,

Since the function is analytic we have $f'(z) = u_x + iv_x$.

By C-R equations we know that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. $\therefore\ f'(z) = u_x - iu_y$

Then put x=z and y=0 in the above equation. After substitution, integrate w.r.t z, so that the analytic function $f(z)$ is obtained. The procedure is vice-versa if v is given.

2. If u or v of analytic function is given in polar form:

Suppose the real part u is given,

Since the function is analytic we have $f'(z) = e^{-i\theta}(u_r + iv_r)$.

By C-R equations we have, $ru_r = v_\theta$ *and* $rv_r = -u_\theta \therefore f'(z) = e^{-i\theta}(u_r - i\frac{1}{r}u_\theta)$.

Then put r=z and θ=0 so that the function $f'(z)$ will be in terms of z. Now on integration, we get the required analytic function.

1.7.2 Problems:

1. Find the analytic function whose real part is $x^3 - 3xy^2$.

Solution:

Let $f(z) = u + iv$ be the required analytic function.

Given $u = x^3 - 3xy^2$.

$$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 + 3y^2, \frac{\partial u}{\partial y} = -6xy.$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = u_x - iu_y = (3x^2 + 3y^2) + 6ixy.$$

Apply Milne's Thomson's method, Put $x = z$ and $y = 0$ in the above equation,

We get, $f'(z) = 3z^2$

On integration, we get $f(z) = z^3 + c$ which is required analytic function.

2. Find the analytic function whose imaginary part is $v = e^x(x\sin y + y\cos y)$

Solution:

Let $f(z) = u + iv$ be the required analytic function.

Given $v = e^x(x\sin y + y\cos y)$

$$\Rightarrow \frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y), \frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y).$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z) = u_x + iv_x$

$$\Rightarrow f'(z) = v_y + iv_x = e^x(x\cos y + \cos y - y\sin y) + \mathrm{i}\left[e^x(x\sin y + y\cos y) + e^x(\sin y)\right]$$

Apply Milne's Thomson's method, Put $x = z$ and $y = 0$ in the above equation,

We get, $f'(z) = e^z(z+1) + i(0)$

On integration, we get $f(z) = ze^z + c$ which is required analytic function.

3. Find the analytic function whose real part is $\left(r+\frac{1}{r}\right)\cos\theta$.

Solution:

Let $f(z) = u + iv$ be the required analytic function.

Given $u = \left(r+\frac{1}{r}\right)\cos\theta$.

$$\Rightarrow \frac{\partial u}{\partial r} = \left(1-\frac{1}{r^2}\right)\cos\theta, \frac{\partial u}{\partial \theta} = -\left(r+\frac{1}{r}\right)\sin\theta.$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$\Rightarrow f'(z) = e^{-i\theta}(u_r - i\frac{1}{r}u_\theta). = e^{-i\theta}\left[\left(1-\frac{1}{r^2}\right)\cos\theta + i\left(r+\frac{1}{r}\right)\sin\theta\right]$$

Apply Milne's Thomson's method, Put $r = z$ and $\theta = 0$ in the above equation,

We get, $f'(z) = \left(1-\frac{1}{z^2}\right)$

On integration, we get $f(z) = z + \frac{1}{z} + c$ which is required analytic function.

4. Find the analytic function whose imaginary part is $r\sin\theta + \dfrac{\cos\theta}{r}$.

Solution:

Let $f(z) = u + iv$ be the required analytic function.

Given $v = r\sin\theta + \dfrac{\cos\theta}{r}$.

$$\Rightarrow \frac{\partial v}{\partial r} = \sin\theta - \frac{\cos\theta}{r^2}, \frac{\partial v}{\partial\theta} = r\cos\theta - \frac{\sin\theta}{r}.$$

We know $f(z)$ is analytic $\Rightarrow f'(z)$ exists.

And $f'(z) = e^{-i\theta}(u_r + iv_r)$

$$\Rightarrow f'(z) = e^{-i\theta}(\frac{1}{r}v_\theta + iv_r). = e^{-i\theta}\left[\frac{1}{r}\left(r\cos\theta - \frac{\sin\theta}{r}\right) + i\left(\sin\theta - \frac{\cos\theta}{r^2}\right)\right]$$

Apply Milne's Thomson's method, Put $r = z$ and θ =0 in the above equation,

We get, $f'(z) = \left(1 - i\dfrac{1}{z^2}\right)$

On integration, we get $f(z) = z + \dfrac{i}{z} + c$ which is required analytic function.

5. Find the analytic function $f(z) = u + iv$ given $u - v = e^x(\cos y - \sin y)$.

Solution:

Given: $u - v = e^x(\cos y - \sin y)$. (1)

Differentiate (1) partially w.r.t x, $u_x - v_x = e^x(\cos y - \sin y)$ (2)

Differentiate (1) partially w.r.t y, $u_y - v_y = e^x(-\sin y - \cos y)$ (3)

Since the function is analytic, we have $u_y = -v_x, u_x = v_y$

$\therefore$ (3) becomes, $-v_x - u_x = -e^x(\sin y + \cos y) \Rightarrow u_x + v_x = e^x(\sin y + \cos y)$

Add (2) and the above equations we get, $u_x = e^x\cos y \Rightarrow v_x = e^x \sin y$

$f'(z) = u_x + iv_x = e^x(\cos y + i\sin y) = e^z$

6. Find the analytic function $f(z)=u+iv$ given $2u+v=e^x(\cos y-\sin y)$.

Solution:

Given: $2u+v=e^x(\cos y-\sin y)$. (1)

Differentiate (1) partially w.r.t x, $2u_x+v_x=e^x(\cos y-\sin y)$ (2)

Differentiate (1) partially w.r.t y, $2u_y+v_y=-e^x(\sin y+\cos y)$ (3)

Since the function is analytic, we have $u_y=-v_x, u_x=v_y$

$\therefore$ (3) becomes, $-2v_x+u_x=-e^x(\sin y+\cos y)$

multiply the above equation by 2,we get $-4v_x+2u_x=-2e^x(\sin y+\cos y)$

Solve (2) and the above equation we get, $v_x=\frac{1}{5}e^x[3\cos y+\sin y]$

Substitute in (2) we get, $u_x=\frac{1}{5}e^x(\cos y-3\sin y)$

$f'(z)=u_x+iv_x=\frac{1}{5}e^x(\cos y-3\sin y)+i\frac{1}{5}e^x(3\cos y+\sin y)$

Put $x=z, y=0$ we get, $f'(z)=\frac{1}{5}e^z(1+3i)$

$f(z)=\frac{1}{5}e^z(1+3i)+c$ is required analytic function

7. Find the analytic function $f(z)=u+iv$ given $u+v=x^3-y^3+3x^2y-3xy^2$

Solution:

Given: $u+v=x^3-y^3+3x^2y-3xy^2$ (1)

Differentiate (1) partially w.r.t x, $u_x+v_x=3x^2+6xy-3y^2$ (2)

Differentiate (1) partially w.r.t y, $u_y+v_y=-3y^2+3x^2-6xy$ (3)

Since the function is analytic, we have $u_y=-v_x, u_x=v_y$

$\therefore$ (3) becomes, $u_x-v_x=3x^2-3y^2-6xy$ (4)

Add (2) and the above equation we get, $u_x = 3(x^2 - y^2)$

Substitute in equation (2), we get $v_x = 6xy$

$f'(z) = u_x + iv_x = 3(x^2 - y^2) + i6xy$

Put $x = z, y = 0$

$f'(z) = 3z^2 \Rightarrow f(z) = z^3 + c$ is the required analytic function .

8. Find the analytic function $f(z) = u + iv$ given $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$

Solution:

Given: $u + v = \frac{1}{r^2}(\cos 2\theta - \sin 2\theta)$ (1)

Differentiate (1) partially w.r.t r, $u_r + v_r = -\frac{2}{r^3}(\cos 2\theta - \sin 2\theta)$ (2)

Differentiate (1) partially w.r.t θ, $u_\theta + v_\theta = \frac{1}{r^2}(-2\sin 2\theta - 2\cos 2\theta)$ (3)

Since the function is analytic, $ru_r = v_\theta$ *and* $rv_r = -u_\theta$

$\therefore$ (3) becomes, $v_r - u_r = \frac{2}{r^3}(\sin 2\theta + \cos 2\theta)$

Add (2) and the above equation we get, $v_r = \frac{2}{r^3}\sin 2\theta$

Substitute in equation (2), we get $u_r = -\frac{2}{r^3}\cos 2\theta$

$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(-\frac{2}{r^3}\cos 2\theta + i\frac{2}{r^3}\sin 2\theta\right) = \frac{-2}{r^3 e^{3i\theta}}$

Put $r = z, \theta = 0$ we get, $f'(z) = -\frac{2}{z^3}$

On integration, $f(z) = \frac{1}{z^2} + c$ is the required analytic function.

9. If $\phi + i\psi$ represents the complex potential of an electrostatic field where $\psi = x^2 - y^2 + \frac{x}{x^2 + y^2}$. Find the complex potential as a function of complex variable z and determine ϕ.

Solution:

Given $\psi = x^2 - y^2 + \frac{x}{x^2 + y^2}$.

Differentiate partially w.r.t x we get, $\psi_x = 2x + \dfrac{-x^2+y^2}{\left(x^2+y^2\right)^2}$

Again differentiate partially w.r.t y we get, $\psi_y = -2y + \dfrac{-2xy}{x^2+y^2}$.

We have $f'(z) = \phi_x + i\psi_x$. But $\phi_x = \psi_y$.

$\therefore$ $f'(z) = \phi_x + i\psi_x$. Put x=z,y=0

$$f'(z) = 0 + i\left(2z - \frac{1}{z^2}\right)$$

On integration, $f(z) = i\left(z^2 + \dfrac{1}{z}\right) + c$ is required complex potential.

To find ϕ

$$f(z) = \phi + i\psi = i\left[(x+iy)^2 + \frac{1}{x+iy}\right] + c = \left(-2xy + \frac{y}{x^2+y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2+y^2}\right) + c.$$

Equating real part, $\phi = \left(-2xy + \dfrac{y}{x^2+y^2}\right)$

10. If $f(z)$ is analytic function of z then show that $\left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial x^2}\right]|ref(z)|^2 = 2|f'(z)|^2$.

Solution:

Let $f(z) = u + iv$ and $\operatorname{Re} f(z) = u$ and $\therefore |ref(z)|^2 = u^2$

L.H.S= $\left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial x^2}\right]|ref(z)|^2 = \dfrac{\partial^2\left(u^2\right)}{\partial x^2} + \dfrac{\partial^2\left(u^2\right)}{\partial y^2} = \dfrac{\partial}{\partial x}\left[2u\dfrac{\partial u}{\partial x}\right] + \dfrac{\partial}{\partial y}\left[2u\dfrac{\partial u}{\partial y}\right]$

$= 2(u_x)^2 + 2u_{xx} + 2(u_y)^2 + 2u_{yy} = 2\left[u_x^{\,2} + u_y^{\,2}\right] + 2u(0) \qquad \because u_{xx} + u_{yy} = 0$

$= 2|f'(z)|^2$ =R.H.S

11. If $f(z)$ is analytic function of z then show that $\left[\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial x^2}\right]|f(z)|^2 = 4|f'(z)|^2$.

Solution:

Let $f(z) = u + iv$ be analytic. And $|f(z)| = \sqrt{u^2+v^2}$

$\therefore |f(z)|^2 = u^2 + v^2$, let $\phi = u^2 + v^2$

To prove that: $\phi_{xx} + \phi_{yy} = 4(u_x^{\,2} + v_x^{\,2})$

Proof: Consider $\phi = u^2 + v^2$

Differentiate partially w.r.t x twice,

$\phi_x = 2uu_x + 2vv_x$; $\phi_{xx} = 2\left[uu_{xx} + u_x^{\ 2} + vv_{xx} + v_x^{\ 2}\right]$

Similarly $\phi_{yy} = 2\left[uu_{yy} + u_y^{\ 2} + vv_{yy} + v_y^{\ 2}\right]$

Consider $\phi_{xx} + \phi_{yy} =$

$2\left[u\left(u_{xx} + u_{yy}\right) + u\left(u_{xx} + u_{yy}\right) + u_x^{\ 2} + u_y^{\ 2} + v_x^{\ 2} + v_y^{\ 2}\right] = 4\left(u_x^{\ 2} + v_x^{\ 2}\right)$.

12. If $f(z)$ is regular function of z then show that $\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2$

Solution:

Let $f(z) = u + iv$ be analytic. And $|f(z)| = \sqrt{u^2 + v^2}$

Let $|f(z)| = \sqrt{u^2 + v^2} = \phi$

We have to prove $\phi_x^{\ 2} + \phi_y^{\ 2} = |f'(z)|^2$

Proof: Consider $\phi^2 = u^2 + v^2$

Differentiate $2\phi\phi_x = 2uu_x + 2vv_x \Rightarrow \phi\phi_x = uu_x + vv_x$

Similarly $\phi\phi_y = 2uu_y + 2vv_y$

Squaring and adding the above equations we get,

$\phi^2\left(\phi_x^{\ 2} + \phi_y^{\ 2}\right) = \left(uu_x + vv_x\right)^2 + \left(uu_y + vv_y\right)^2 = \left(u^2 + v^2\right)\left(u_x^{\ 2} + v_x^{\ 2}\right)$

$\Rightarrow \left(\phi_x^{\ 2} + \phi_y^{\ 2}\right) = \left(u_x^{\ 2} + v_x^{\ 2}\right) = |f'(z)|^2$

EXERCISE:

1. Show that $f(z) = \sin z$ is analytic and hence find $f(z)'$. (Ans: $f(z)' = \cos z$)

2. Show that $f(z) = \sinh z$ is analytic and hence find $f(z)'$. (Ans: coshz)

3. Show that $f(z) = e^x(\sin y + i\sin y)$ is holomorphic and hence find $f(z)'$. (Ans: e^z)

4. Show that $f(z) = z^2 + 2z$ is analytic and hence find $f(z)'$. (Ans : $2z + 2$)

5. Show that $f(z) = z^2$ is analytic and hence find $f(z)'$. (Ans : $z^2(1 + \log z)$)

6. Construct the analytic function whose imaginary part is $-\frac{\sin\theta}{r}$. (Ans: $\frac{1}{r}\cos\theta$).

Hence find its real part.

7. Construct the analytic function whose imaginary part is $\left(r-\frac{1}{r}\right)\sin\theta$. (Ans :

$u=\left(r+\frac{1}{r}\right)\cos\theta; \quad z+\frac{1}{z}$)

8. Construct the analytic function whose real part is $\frac{x^4-y^4-2x}{x^2+y^2}$. Hence find its imaginary

part.(Ans: $f(z)=z^2-\frac{2}{z}+c; \quad v=\frac{2xy^3+2x^3y+2y}{x^2+y^2}$)

9. Construct the analytic function whose real part is $e^{-x}\{(x^2-y^2)\cos y+2xy\sin y\}$. (Ans:

$f(z)=z^2e^{-z}+c$)

10. Construct the analytic function whose real part is $r^2\cos 2\theta$. (Ans: $f(z)=z^2+c$)

11. Show that the function $u=\frac{1}{r}\cos\theta$ is harmonic and find its harmonic conjugate. Also find the

analytic function. (Ans: $\frac{-\sin\theta}{r}$; $f(z)=\frac{1}{z}$).

12. Show that the function $v=\cos x\sinh y$ is harmonic and find its harmonic conjugate. Also find the analytic function. (Ans: $f(z)=\sin z,\ u=\sin x\cosh y$)

13. Show that the function $u=(x-1)^3-3xy^2+3y^2$ is harmonic and find its harmonic conjugate. Also find the analytic function. (Ans: $(z-1)^3; \quad v=x^2-y^2+2(2x+y)$)

14. Show that the function $u=\frac{1}{2}\log(x^2+y^2),\ x\neq 0, y\neq 0$ is harmonic and find its harmonic

conjugate. Also find the analytic function. (Ans: $\tan^{-1}\left(\frac{y}{x}\right)+c$)

15. Find the analytic function $f(z)=u+iv$ given $u-v=(x-y)+e^x(\cos y+\sin y)$. (Ans: $z+e^z$)

16. Find the analytic function $f(z)=u+iv$ given $u+v=\frac{2\sin 2x}{e^{2y}-e^{-2y}-2\cos 2x}$. (Ans : $\cot z$)

Complex Integration

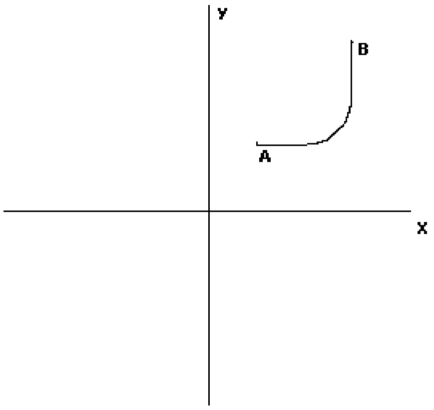
- **Complex line integral**
- **Cauchy's Theorem**
- **Cauchy's Integral Theorem**
- **Taylor's series**
- **Laurent's series**
- **Zero's, Poles and Singularities**
- **Residues**
- **Cauchy's Residue Theorem**

2.1 Introduction:

Complex variable techniques have been used in a wide variety of areas of engineering. This has been particularly true in areas such as electromagnetic field theory, fluid dynamics, aerodynamics and elasticity. With the rapid developments in computer technology and the consequential use of sophisticated algorithms for analysis and design in engineering there has been, in recent years, less emphasis on the use of complex variable techniques and a shift towards numerical techniques applied directly to the underlying full partial differential equations which model the situation. However it is useful to have an analytical solution, possibly for an idealized model in order to develop a better understanding of the solution and to develop confidence in numerical estimates for the solution of more sophisticated models. The design of aerofoil sections for aircraft is an area where the theory was developed using complex variable techniques. Throughout engineering, transforms defined as complex integrals in one form or another play a major role in analysis and design. The use of complex variable techniques allows us to develop criteria for the stability of systems.

2.2 Complex line integral:

Suppose $f(z)$ is a function defined at every point on a given continuous curve C on the argand diagram.



We define $\int_C f(z)dz$ as the limit of a sum as follows,

Introduce a very large number of points $z_1, z_2, z_3......z_{n-1}$ along the curve C. Suppose A and B represent complex numbers z_0, and z_n, respectively. Put $z_1 - z_0 = \delta z_1,\ \ z_2 - z_1 = \delta z_2\ \ldots$
$z_n - z_{n-1} = \delta z_n$

Consider the sum

$$f(z_1)\delta z_1 + f(z_2)\delta z_2 + f(z_3)\delta z_3 + f(z_4)\delta z_4 ....+ f(z_n)\delta z_n = \sum_{i=1}^{n} f(z_i)\delta z_i.$$

Let $n \to \infty$ such that each $\delta z_i \to 0$ then we define $\int_C f(z)dz = \lim_{\delta z_i \to 0} \sum_{i=1}^{n} f(z_i)\delta z_i$

This integral is called as the line integral of v the complex function f(z).

2.2.1 Properties of line integrals:

1. $\int_A^B f(z)dz = -\int_B^A f(z)dz$.
2. $\int_A^B f(z)dz = \int_A^C f(z)dz + \int_C^B f(z)dz$.
3. $\int_C k\ f(z)dz = k\int_C f(z)dz$.
4. $\int_C [f(z) \pm g(z)]dz = \int_C f(z)dz \pm \int_C g(z)dz$.

Note: Evaluation of a line integral of a complex valued function:

$$\int_C f(z)dz ,= \int [u(x,y) + iv(x,y)](dx + idy) = \int_c (udx - vdy) + i\int_c (vdx + udy).$$

Problems:

1. **Evaluate** $\int_c z^2 dz$ **where C is**

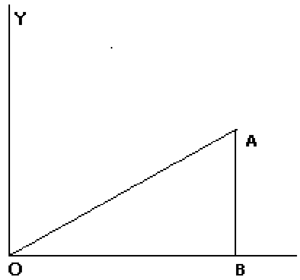
(i) The line joining the points 0 and 3 + i ,

(ii) The path OBA consisting of two line segments OB and OA, where A and B are z=3+i and z=3 respectively.

Solution:

$z = x + iy \Rightarrow z^2 = x^2 - y^2 + 2ixy$ and $dz = dx + idy$

$$\therefore z^2 dz = (x^2 - y^2 + 2ixy)(dx + idy) = \left[\left(x^2 - y^2\right)dx - 2xydy\right] + i\left[2xydx + \left(x^2 - y^2\right)dy\right]$$



(i) C is the line segment joining O(0,0) and A(3,1)
Equation of OA is $x - 3y = 0 \Rightarrow x = 3y \Rightarrow dx = 3dy$
Further, y varies from y =0 to y =1.

$$\int_C z^2 dz = \int_0^1 \left[\left(9y^2 - y^2\right)3dy - 6y^2\right] + i\left[6y^2.3dy + 8y^2 dy\right]$$

$$= \int_0^1 18y^2 dy + i\int_0^1 26y^2 dy = 6 + \frac{26}{3}i.$$

(ii) C is the path OAB. Thus, we can write,

$$\int_C z^2 dz = \int_{OBA} z^2 dz = \int_{OB} z^2 dz + \int_{BA} z^2 dz$$

Along OB: $y = 0, dy = 0$ and x varies from $x = 0$ to $x = 3$.

Along BA: $x = 0, dx = 0$ and y varies from $y = 0$ to $y = 1$.

Thus we have, $\int_{OBA} z^2 dz = \int_0^3 x^2 dx + \int_0^1 -6ydy + i\left(9 - y^2\right)dy = 6 + \frac{26}{3}i$.

2. **Evaluate** $\int_0^{3+i} z^2 dz$ **along the line** $3y = x$.

Solution:

$$\int_0^{3+i} z^2 dz = \int_0^{3+i} (x + iy)^2 (dx + idy)$$

Along the line, $3y = x.$,we have $dx = 3dy$ and y varies from 0 to 1.

$$\int_0^1 (x+iy)^2(dx+idy) = \int_0^1 (3y+iy)^2(3dy+idy)$$

$$= (3+i)^3\int_0^1 y^2 dy = \frac{(3+i)^3}{3}.$$

4. **Evaluate** $\int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)dy$ **along**

(i) The parabola $x = 2t,\ y = t^2 + 3$

(ii) The straight line from $(0,\ 3)$ **to** $(2,\ 4)$

Solution:

(i). x varies from 0 to 2 and hence

$$\left.\begin{array}{l} if\ x=0 \Rightarrow t=0, \\ if\ x=2, \Rightarrow t=1, \end{array}\right. \Rightarrow t \text{ Varies from 0 to 1}$$

$$I = \int_{(0,3)}^{(2,4)} (2y+x^2)dx + (3x-y)dy$$

$$= \int_{t=0}^{1} \left\{2(3+t^2)+4t^2\right\}2dt + \left\{3(2t)-(3+t^2)\right\}2tdt$$

$$I = \int_{t=0}^{1} (24t^2 - 2t^3 - 6t + 12)dt = 33/2.$$

(ii). Equation of the straight line joining (0,3) and (2,4) is given by $\frac{y-3}{x-0} = \frac{4-3}{2-0}$.

$\Rightarrow x = 2y - 6.$ $Hence\ dx = 2dy.$

Hence the given integral I become,

$$I = \int_{y=3}^{4} \left\{2y + (2y-6)^2\right\}2ydy + \{3(2y-6)-y\}dy$$

$$= \int_{y=3}^{4} \left(8y^2 - 39y + 54\right)dy = 97/6.$$

5. **Evaluate** $\int_c (\bar{z})^2 dz$ **around the circle (i)** $|z| = 1$ **, (ii)** $|z-1| = 1$.

Solution:

(i) $|z| = 1$, is the circle at (0,0) and radius equal to 1.

On the circle we have $z = re^{i\theta}, dz = ie^{i\theta} d\theta$ further θ varies from 0 to 2π.

$$\bar{z} = e^{-i\theta} \text{ and } (\bar{z})^2 = e^{-2i\theta}$$

Now,

$$\int_C (\bar{z})^2 dz = \int_0^{2\pi} e^{-2i\theta} ie^{i\theta} d\theta = i\int_0^{2\pi} e^{-i\theta} d\theta = 0.$$

(ii) $|z-1| = 1$ is the circle with centre at (1,0) and radius equal to 1.

On this circle $z - 1 = e^{i\theta} \quad 0 \le \theta \le 2\pi$

$\Rightarrow z = 1 + e^{i\theta}, dz = ie^{i\theta} d\theta$

Also, $\bar{z} = 1 + e^{-i\theta}$ and $(\bar{z})^2 = 1 + 2e^{-i\theta} + e^{-2i\theta}$

$$\therefore \int_c (\bar{z})^2 dz = \int_0^{2\pi} (1 + 2e^{-i\theta} + e^{-2i\theta}) ie^{i\theta} d\theta = 4\pi i.$$

6. **Show that (i)** $\int_C \frac{dz}{z-a} = 2\pi i$ **(ii)** $\int_c \frac{dz}{(z-a)^n} = 0$, **n=2,3,4…, where C is the circle** $|z-a| = r.$

Solution:

$|z-a| = r$, is the circle with centre at (a,0) and radius r.

On this circle we have $z - a = re^{i\theta}$; $0 \le \theta \le 2\pi \Rightarrow dz = ire^{i\theta} d\theta.$

$$\int_c \frac{dz}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i.$$

$$\text{Again} \int_c \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{i\theta}}{r^n e^n i\theta} d\theta$$

$$=\frac{1}{r^{n-1}}\int_0^{2\pi} e^{i(1-n)\theta}d\theta=\frac{i}{r^{n-1}}\left[\frac{e^{i(1-n)\theta}}{i(1-n)}\right]_0^{2\pi}=\frac{1}{(1-n)r^{n-1}}\left[e^{i(1-n)2\pi}-1\right]=0.$$

Because $e^{i(1-n)2\pi}=1.$

2.3 Cauchy's Theorem:

Statement: If $f(z)$ is analytic function and $f'(z)$ is continuous at each point within and on a closed curve C, then $\int_c f(z)dz=0.$

Proof: Let $f(z)=u(x,y)+iv(x,y)$ *and* $dz=dx+idy$

$$\int_C f(z)dz=\int_C(udx-vdy)+i\int_C(vdx+udy)\ldots(1)$$

Since $f'(z)$ is continuous, therefore $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}$ are also continuous in the region D enclosed by C. Hence by the Greeen's theorem, terms of equation (1) can be as follows.

$$\int_C f(z)dz=-\int\int_D\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right]dxdy+i\int\int_D\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right]dxdy \qquad \ldots(2)$$

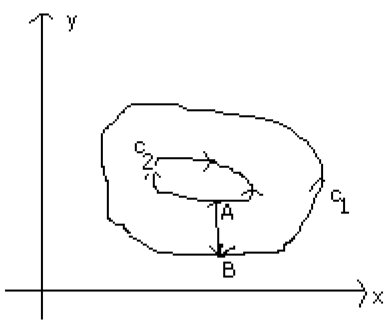
Now $f(z)$ being analytic, u and v are necessarily satisfy the C-R equations and thus the integrands of the two double integrals in (2) vanish identically.

Hence $\int_C f(z)dz=0.$

Corollary 1: If $f(z)$ is analytic in the domain D and A & B are two points in D, then $\int_A^B f(z)dz$ is independent of the path joining A & B.

Proof: Let $C_1=ATB$ & $C_2=ASB$ be two simple curves joining the points A & B such that c_1 & c_2 both lies in the domain D.

Clearly ATBSA is a closed contour in D. Thus by Cauchy's theorem we have



$$\int_{ATBSA} f(z)dz \quad =0$$

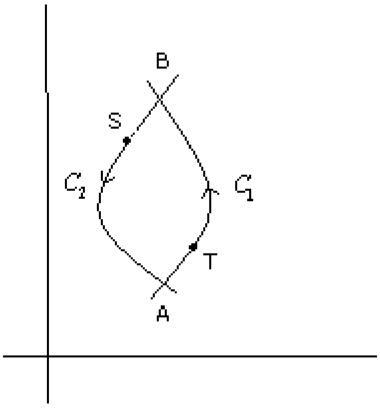
$$\Rightarrow\int_{ATB} f(z)\,dz+\int_{BSA} f(z)\,dz=0\Rightarrow\int_{ATB} f(z)\,dz-\int_{ASB} f(z)\,dz=0$$

$$\Rightarrow\int_{ATB_1} f(z)\,dz=\int_{BSA} f(z)\,dz\Rightarrow\int_{C_1} f(z)\,dz=\int_{C_2} 2\,f(z)\,dz$$

$\Rightarrow \int_A^B f(z)dz$ is independent of the path.

Corollary 2:If C_1 and C_2 are two simple curves, with C_1 enclosing C_2 and f(z) be analytic inside and on the boundary of the annular region between C_1 and C_2 then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$.

Proof: Let the curve C_1 and C_2 both be described in the positive direction
We shall introduce a cross cut AB from a point A on C_2 to a point B on C_1.



i) line AB
ii) arc BB along C_1
iii) line BA
iv) arc AA along C_2 in clock wise direction

Thus by Cauchy's theorem, we have.

$$\int_C f(z)\,dz = 0$$
$$\int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz + \int_{BA} f(z)\,dz + \int_{-C_2} f(z)\,dz = 0$$
$$\int_{AB} f(z)\,dz + \int_{C_1} f(z)\,dz - \int_{AB} f(z)\,dz - \int_{C_2} f(z)\,dz = 0$$
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

Hence the proof.

Corollary 3: If C is a simple closed curve enclosing non over lapping simple closed curves $C_1, C_2, C3,.....C_n$ and if $f(z)$ is analytic in the annular region between C and these curves then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz,..., \int_{C_n} f(z)dz.$$

2.4 Cauchy's integral Formula:

Statement: If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then $f(a) = \dfrac{1}{2\pi i}\int_C \dfrac{f(z)}{z-a}dz.$

Proof: Consider the function $\dfrac{f(z)}{z-a}$ which is analytic at all points with in C except at z=a.

With the point α as centre and radius r , draw a small circle C_1 lying entirely within C.

Now $\frac{f(z)}{z-a}$ being analytic in the region enclosed by C and C_1, we have Cauchy's theorem,

$$\int_C \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(z)}{z-a}dz = \int_{C_1} \frac{f(a+re^{i\theta})}{re^{i\theta}} irre^{i\theta}d\theta = i\int_{C_1} f(a+re^{i\theta})d\theta$$

In the limiting form, as the circle C_1, shrinks to the point 'a' , i.e as 'r' tends to '0' then the above integral approach to

$$= i\int_{C_1} f(a)d\theta \quad = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz \text{ .}$$

2.5 Generalized Cauchy's integral Formula:

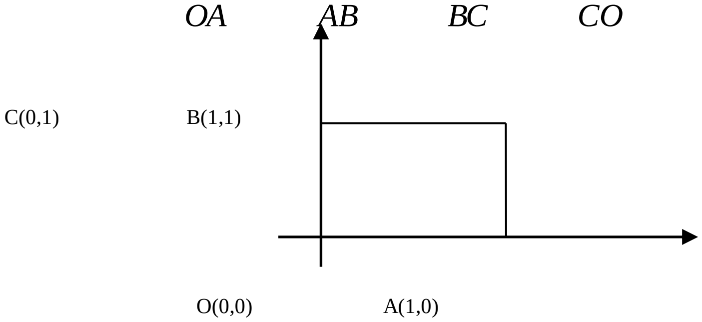
Statement: If $f(z)$ is analytic within and on a closed curve C and if 'a' is any point within C, then $f^{n}(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}dz.$

Problems:

1. Verify Cauchy's theorem for the function $f(z) = z^2$ where C is the square having vertices (0,0) (1,0) (1,1) (0,1).

Solution:

C is the square OABC and we have by Cauchy's theorem $\int_C f(z)dz = 0$. Therefore we have to show that, $\int_{OA} z^2 dz + \int_{AB} z^2 dz + \int_{BC} z^2 dz + \int_{CO} z^2 dz = 0$



Along OA, $y = 0 \,\&\, dy = 0;\ 0 \le x \le 1$

$$z^2 dz = (x+iy)^2 (dx + idy) = x^2 dx.$$

$$\int_{OA} z^2 dz = \int_{x=0}^{1} x^2 dx = 1/3. \qquad \text{....(1)}$$

Along AB, $x = 1 \,\&\, dx = 0;\ 0 \le y \le 1$

$$z^2 dz = (x+iy)^2 (dx + idy) = (1+iy)^2 idy.$$

$$\int_{AB} z^2 dz = i \int_{y=0}^{1} (1+iy)^2 dy = -1 + 2i/3. \qquad \text{...(2)}$$

Along BC, $y = 1 \,\&\, dy = 0;\ 1 \le x \le 0$

$$z^2 dz = (x+iy)^2 (dx + idy) = (x+i)^2 dx.$$

$$\int_{bc} z^2 dz = \int_{x=1}^{0} (x^2 + 2ix - 1) dx = 2/3 - i. \qquad \text{...(3)}$$

Along CO, $x = 0 \,\&\, dx = 0;\ 1 \le y \le 0$

$$z^2 dz = (x+iy)^2 (dx + idy) = -iy^2 idy.$$

$$\int_{CO} z^2 dz = i \int_{y=1}^{0} -iy^2 dy = i/3. \qquad \text{...(4)}$$

By adding (1),(2),(3) &(4), we get

$$\int_{OA} z^2 dz + \int_{AB} z^2 dz + \int_{BC} z^2 dz + \int_{CO} z^2 dz = 1/3 - 1 + 2i/3 + 2/3 - i + i/3 = 0.$$

Hence Cauchy's theorem is verified.

2. **Evaluate** $\int_c \frac{1}{z(z-1)} dz$ **where C is the circle** $|z| = 3$.

Solution:

Given C: $|z| = 3$ is the circle with centre at the origin and radius 3.

$\phi(z) = \frac{1}{z(z-1)}$ is not analytic at $z = 0$ and at $z = 1$.

These points $z = 0$ and at $z = 1$ lie inside the circle $|z| = 3$.

Now $\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}$

$$\int_C \frac{1}{z(z-1)}dz = \int_C \frac{1}{(z-1)}dz - \int_C \frac{1}{z}dz$$

Taking $f(z)=1$ which is analytic in and on C, we have

$$\int_C \frac{1}{z(z-1)}dz = \int_C \frac{f(z)}{(z-1)}dz - \int_C \frac{f(z)}{z}dz$$

By applying Cauchy's integral formula by taking z_0=1 in first integral z_0=0 in the second integral, we have

$$\int_C \frac{1}{z(z-1)}dz = 2\pi i f(1) - 2\pi i f(0) = 0.$$

3. **Evaluate** $\int_C \frac{z+4}{z^2+2z+5}dz$ **where C is the circle** $|z+1+i|=2$**.**

Solution:

Given $|z+1+i|=2 \Rightarrow |z-(-1-i)|=2$

This is a circle with centre at $z=-1-i$ and radius equal to 2.

Now $\phi(z)=\frac{z+4}{z^2+2z+5}$ will cease to be analytic where z^2+2z+5=0.

Now z^2+2z+5=0.$\Rightarrow z=-1\pm 2i$

i.e. $z^2+2z+5=(z+1-2i)(z+1+2i)$

Thus $\phi(z)$ is not analytic at $z=-1-2i$ but $z=-1+2i$ lies outside C.

Thus $\int_C \frac{z+4}{z^2+2z+5}dz = \int_C \frac{z+4}{(\mathrm{z}+1-2\mathrm{i})(\mathrm{z}\ -1-2\mathrm{i})}dz$

$$= \int_C \frac{\left[\frac{z+4}{(\mathrm{z}+1-2\mathrm{i})}\right]}{\mathrm{z}-(-1-2\mathrm{i})}dz$$

Taking $f(z)=\frac{z+4}{z-(-1+2i)}$ and $z_0=-1-2i$, applying Cauchy's integral formula, we get

$$\int_C \frac{z+4}{z^2+2z+5}dz = 2\pi i f(-1-2i) = 2\pi i\left(\frac{3-2i}{-4i}\right) = \frac{\pi}{2}(2i-3).$$

4. Evaluate $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is the circle $|z| = 3$.

Solution:

The circle C: $|z| = 3$ is the circle with centre at $z = 0$ and radius 3.

$\phi(z) = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ is analytic every where except at z=1 and at z=2. These points are inside the circle C.

Now $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$$\therefore \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \int_C \frac{f(z)}{(z-1)} dz - \int_C \frac{f(z)}{(z-2)} dz \ldots(1)$$

Taking $f(z) = \sin \pi z^2 + \cos \pi z^2$ and $z_0 = 1$, applying Cauchy's integral formula we formula,

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i f(1) = 2\pi i(-1) = -2\pi i$$

Again by taking $f(z) = \sin \pi z^2 + \cos \pi z^2$ and $z = 2\pi$, we have

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i f(2) = 2\pi i$$

(1) Becomes $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -4\pi i$.

<u>EXERCISES:</u>

1. Evaluate $\int_C (z^2 + z) dz$ along the line joining $(1-i)$ & $(2+3i)$.

2. Evaluate $\int_C (\bar{z})^2 dz$ where (i) C is the circle $|z-1| = 1$. (ii) C is the circle $|z-2| = 1$.

3. Verify Cauchy's theorem for the function $f(z) = z^2$ over the square formed by the points (0,0), (2,0), (2,2) and (0,2).

4. Evaluate $\int_C \frac{(z-1)}{(z+1)^2(z-2)} dz$ where C is the circle $|z-i| = 2$.

5. Evaluate $\int_C \frac{e^z}{(z^2+\pi^2)^2}dz$ where C is the circle $|z| = 4$.

ANSWERS:

1. $\frac{-1}{6}(103-64i)$
2. (i). $4\pi i$ (ii). $8\pi i$
4. $\frac{-2\pi i}{9}$
5. $\frac{i}{\pi}$

2.6 Taylor's series:

Statement:If $f(z)$ is analytic inside a circle C with centre at a, then for z inside C,

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + ... + \frac{f^n(a)}{n!}(z-a)^n + ...$$

Note: If $a = 0$ in the Taylor's series we obtain the series in the form

$$f(z) = f(0) + z f'(0) + z^2 \frac{f''(a)}{2!} + ...$$ This is known as Maclaurin's series for $f(z)$.

2.7 Laurent's Series:

Statement:If $f(z)$ is analytic in the ring shaped region R bounded by two concentric circles C_1 and C_2 of radii r and r_1 and with centre at 'a', then for all z in R.

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ...a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + ...$$

Where $a_n = \frac{1}{2\pi i}\int_{C_1}\frac{f(t)}{(t-a)^{n+1}}dt$ ***and*** $a_{-n} = \frac{1}{2\pi i}\int_{C_2}\frac{f(t)}{(t-a)^{-n+1}}dt$.

Examples:

Problems:

1. Find the Taylor's expansion of $f(z) = \frac{1}{(z+1)^2}$ about the point $z = -i$.

Solution:

To expand $f(z)$ about $z = -i$ i,e in powers of $z + i$, put $z + i$ =t , then

$$f(z)=\frac{1}{(t-i+1)^2}=(1-i)^{-2}[1+t/(1-i)]^{-2}=\frac{i}{2}\left[1-\frac{2t}{1-i}+\frac{3t^2}{(1-i)^2}......\right]$$

$$=\frac{i}{2}\left[1+\sum_{n=1}^{\infty}(-1)^n\frac{(n+1)(z+i)^n}{(1-i)^n}\right].$$

2. Expand $f(z)=\dfrac{z-1}{(z-2)(z-3)}$ in non-negative powers of z.

Solution:

We shall resolve into partial fractions.

$$\frac{z-1}{(z-2)(z-3)}=\frac{A}{z-2}+\frac{B}{z-3}=\frac{-1}{z-2}+\frac{2}{z-3}$$

$$f(z)=\frac{-1}{z-2}+\frac{2}{z-3}=\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}-\frac{2}{3}\left(1-\frac{z}{3}\right)^{-1} \quad ....(1)$$

The expansion is valid for $\left|\frac{z}{2}\right|<1$ and $\left|\frac{z}{3}\right|<1$. i.e., $|z|<2$ and $|z|<3$. Since $|z|<2$ implies $|z|<3$ also, we can say that the expansion is valid when $|z|<2$.

Using binomial expansion:

$$f(z)=\frac{1}{2}\left\{1+\frac{z}{2}+\frac{z^2}{4}+....\right\}-\frac{2}{3}\left\{1+\frac{z}{3}+\frac{z^2}{9}+....\right\}.$$

3. Find the Laurent's expansion of $f(z)=\dfrac{3z^2-6z+2}{z^3-3z^2+2z}$ in the region

$i)$ $1<|z|<2$

$ii)$ $|z|>2$

Solution:

By resolving into partial fraction,

$$f(z)=\frac{3z^2-6z+2}{z^3-3z^2+2z}=\frac{1}{z}+\frac{1}{z-1}+\frac{1}{z-2}. \quad ...(1)$$

Case i: $1<|z|<2\Rightarrow\dfrac{1}{|z|}<1$ and $\dfrac{|z|}{2}<1.$

Hence we have from (i)

$$f(z)=\frac{1}{z}+\frac{1}{z\left(1-\frac{1}{z}\right)}+\frac{1}{-2\left(1-\frac{z}{2}\right)}=\frac{1}{z}+\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}-\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$=\frac{1}{z}+\left\{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+......\right\}-\frac{1}{2}\left\{1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+.......\right\}$$

$$f(z)=\frac{1}{2}+\frac{2}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}.....$$

Case ii: $|z|>2\Rightarrow|z|>1\ \therefore\ \frac{2}{|z|}<1,\ \ \frac{1}{|z|}<1$

Hence from equation (1) we have

$$f(z)=\frac{1}{z}\left[1+\left(1-\frac{1}{z}\right)^{-1}+\left(1-\frac{2}{z}\right)^{-1}\right]$$

$$=\frac{1}{z}\left[1+\left\{1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+.....\right\}+\left\{1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+....\right\}\right]$$

$$=\frac{1}{z}\left[3+\frac{3}{z}+\frac{5}{z^2}+\frac{9}{z^3}+.......\right].$$

4. **Find the Laurent's expansion of** $f(z)=\frac{7z-2}{(z+1)z(z-2)}$ **in the region** $1<z+1<3$.

Solution:

put $u=z+1$, we have

$$f(z)=\frac{7(u-1)-2}{u(u-1)(u-1-2)}=\frac{7u-9}{u(u-1)(u-3)}$$

$$=-\frac{3}{u}+\frac{1}{u-1}+\frac{2}{u-3}=-\frac{3}{u}+\frac{1}{u}\left(1-\frac{1}{u}\right)^{-1}-\frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$f(z)=\frac{-3}{u}+\frac{1}{u}\left(1+\frac{1}{u}+\frac{1}{u^2}+....\right)-\frac{2}{3}\left(1+\frac{u}{3}+\frac{u^2}{3^2}+....\right)$$

$$f(z)=\frac{-2}{z+1}+\frac{1}{(z+1)^2}-\frac{1}{(z+1)^3}+....-\frac{2}{3}\left[1+\frac{z+1}{3}+\frac{(z+1)^2}{3^2}+......\right].$$

5. Expand $f(z)=\dfrac{z}{(z-1)(2-z)}$ **in Laurent's series valid for** $|z-1|<1.$

Solution:

put $u=z+1 or\, z-1=u$, we have

$$f(z)=\frac{(u+1)}{u(2-u-1)}=\frac{u+1}{u(1-u)}=g(u).$$

$$Let\,\frac{(u+1)}{u(1-u)}=\frac{A}{u}+\frac{B}{1-u}=\frac{1}{u}+\frac{2}{1-u}. \qquad ......(i)$$

$$Now\,|z-1|<1 \,is\, equivalent\, to\,|u|<1.$$

Hence (i) can be written in the form

$$g(u)=\frac{1}{u}+2(1-u)^{-1}$$

$$=\frac{1}{u}+2(1+u+u^2+u^3+...)$$

$$i.e., F(z)=\frac{1}{z-1}+2+2(z-1)+2(z-1)^2+...$$

6. Expand $f(z)=\dfrac{z-1}{(z-2)(z-3)^2}$ **as a Laurent's series valid for**

$(i)|Z|>3, \quad (ii)2<|Z|<3.$

Solution:

we shall first resolve f(z) in to partial fractions,

$$\frac{z-1}{(z-2)(z-3)^2}=\frac{A}{(z-2)}+\frac{B}{(z-3)}+\frac{C}{(z-3)^2}$$

$$f(z) \quad =\frac{1}{(z-2)}-\frac{1}{(z-3)}+\frac{2}{(z-3)^2} \qquad ...(i)$$

$$case(i):|z|>3. This\, implies\,|z|>2\, also.$$

$$i.e., \frac{3}{|z|}<1, \ \frac{2}{|z|}<1$$

We have to write (i) in the form

$$f(z)=\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{z\left(1-\frac{3}{z}\right)}+\frac{2}{\left\{z\left(1-\frac{3}{z}\right)\right\}^2}$$

$$=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{3}{z}\right)^{-1}+\frac{2}{z^2}\left(1-\frac{3}{z}\right)^{-2}$$

$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots\right)-\frac{1}{z}\left(1+\frac{3}{z}+\frac{9}{z^2}+\frac{27}{z^3}+\dots\right)+\frac{2}{z^2}\left(1+\frac{6}{z}+\frac{27}{z^2}+\frac{108}{z^3}+\dots\right)$$

$$f(z)=\frac{1}{z^2}+\frac{7}{z^3}+\frac{35}{z^4}+\dots.$$

$case(ii): 2<|z|<3.$

$i.e., 2<|z| \Rightarrow \frac{2}{|z|}<1;\ |z|<3 \Rightarrow \frac{|z|}{3}<1$

$\therefore$ $f(z)$ as given by (i) has to be written in the form

$$f(z)=\frac{1}{z\left(1-\frac{2}{z}\right)}-\frac{1}{(-3)\left(1-\frac{z}{3}\right)}+\frac{2}{\left\{(-3)\left(1-\frac{z}{3}\right)\right\}^2}$$

$$=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}+\frac{1}{3}\left(1-\frac{z}{3}\right)^{-1}+\frac{2}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$=\frac{1}{z}\left(1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\dots\right)+\frac{1}{3}\left(1+\frac{z}{3}+\frac{z^2}{9}+\frac{z^3}{27}+\dots\right)+\frac{2}{9}\left(1+\frac{2z}{3}+\frac{z^2}{3}+\frac{4z^3}{27}+\dots\right)$$

$$f(z)=\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\frac{8}{z^4}+\dots.+\frac{5}{9}+\frac{7}{27}z+\frac{1}{9}z^2+\frac{11}{243}z^3+\dots$$

EXERCISES:

1. Obtain the Taylor's Series expansion for the following functions of

   a. $\frac{z-1}{z^2}$ in powers of $z-1$

   b. $\frac{z-1}{z+1}$ in powers of $z=1$

   c. $\cos z$ about the point $z=\pi/2$

2. Find the Laurent's expansion for the following functions

   d. $\frac{e^z}{(z-1)^2}$ about $z=1$

   e. $\frac{e^{2z}}{(z-1)^3}$ about the singularity $z=1$

   f. $\frac{z-1}{z^2}$ for $|z-1|>1$

   g. $\frac{1-\cos z}{z^3}$ about z=0

   h. $\frac{z^2-1}{z^2+5z+6}$ about $z=0$ in the region $2<|z|<3$

ANSWERS:

I.

a) $\sum_{n=1}^{\infty}(-1)^{n+1}n(z-1)^n$

b) $\frac{1}{2}(z-1)-\frac{1}{2^2}(z-1)^2+\frac{1}{2^3}(z-1)^3-.....$

c) $-\left(z-\frac{\pi}{2}\right)+\frac{\left(z-\frac{\pi}{2}\right)^3}{3!}-\frac{\left(z-\frac{\pi}{2}\right)^5}{5!}+.....$

II.

a) $e\left[(z-1)^{-2}+(z-1)^{-1}+\frac{1}{2!}+\frac{1}{3!}(z-1)+\frac{1}{4!}(z-1)^2+...\right].$

b) $e^2(z-1)^{-3}+2e^2(z-1)^{-2}+2e^2(z-1)^{-1}+\frac{4e^2}{3}+\frac{2e^2}{3}(z-1)+...$

c) $\sum_{n=1}^{\infty}(-1)^{n-1}n(z-1)^{-n}\ \ for\ |z-1|>1$

d) $-\sum_{n=2}^{\infty}\frac{z^{2n-5}}{2(n-1)!}$

e) $1+\frac{3}{z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-\frac{2^3}{z^3}+......\right)-\frac{8}{3}\left(1-\frac{z}{3}+\frac{z^2}{3^2}-\frac{z^3}{3^3}+.......\right)$

2.8 Zeros of an analytic function:

A zero of an analytic function $f(z)$ is that value of z for which $\mathrm{f}(\mathrm{z})=0.$

2.8.1 Singularity: Singular point of a function is a point at which the function ceases to be analytic.

Types of singularities:

1. **Isolated Singularity:** If $\mathrm{z}=\mathrm{a}$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighborhood then $z=a$ is called an isolated singularity.
2. **Removable Singularity:** If all the negative powers of $z-a$ in Laurent's series are zero, then such singularity is said to be removable singularity.
3. **Poles:** If all the negative powers of $z-a$ in Laurent's series after the n^{th} term are missing, then the $z=a$ is called a pole of order n. Pole of order one is called **Simple pole.**
4. **Essential Singularity:** If the number of negative powers of $z-a$ in Laurent'sseries is infinite then $z=a$ is called an Essential Singularity.

Example:Find the nature and location of singularity of the function $\dfrac{z-\sin z}{z^2}$

Solution: Here $z=0$ is a singularity.

$$\frac{z-\sin z}{z^2}=\frac{1}{z^2}\left\{z-\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-.....\right)\right\}=\frac{z}{3!}-\frac{z^3}{5!}+\frac{z^5}{7!}-...$$

Therefore $z=0$ is removable singularity.

2.8.2 Residues:

The co-efficient of $(z-a)^{-1}$ in the expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z=a$.
If $z=a$ is a pole of order m of $f(z)$then the residue of $f(z)$at $z=a$ is denoted by $R[m,a]$ and is given by $R[m,a]=\dfrac{1}{(m-1)!}\lim_{z\to a}\dfrac{d^{m-1}}{dz^{m-1}}\{(z-a)^m f(z)\}$.

Note:

1. If $f(z)$ has a simple pole at $z=a$ then $\operatorname{Re}s\ f(a)=\underset{z\to a}{lt}\left[(z-a)f(z)\right]$
2. If $f(z)$ has a pole of order at $z=a$ then

$$\operatorname{Re}s\ f(a)=\frac{1}{(n-1)!}\underset{z\to a}{lt}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}.$$

2.9 Cauchy's Residue Theorem:

Statement: If $f(z)$ is analytic inside and on the boundary of a simple closed curve C except for a finite number of poles $a,\ b,\ c.....,$ then the integral of $f(z)$over C is equal to $2\pi i$ times the sum of the residues at the poles inside C.

$$\int_C f(z)dz=2\pi i(a_{-1}+b_{-1}+c_{-1}+...)$$

i.e.,

$$\int_C f(z)dz=2\pi i\sum R,\ where \sum R\ denote the sum of\ the\ residues\ at\ the\ poles\ lying\ in\ C.$$

Problems:

1. Find the residues of the function $f(z)=\dfrac{z}{(z+1)(z-2)^2}\ at\ i).z=-1\ \&\ ii)z=2$

Solution:

$z=-1$ is a pole of order 1(simple pole) and $z=2$ is a pole of order 2.

The residue of $f(z)$ for a pole of order m at $z = a$ is given by

$R[m,a] = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left\{(z-a)^m f(z)\right\}$ and in particular for a simple pole (m=1),

$$R[1,a] = \lim_{z \to a} \left\{(z-a) \quad f(z)\right\}$$

Case (i): Residue at $z = a = -1$ is given by

$$\lim_{z \to -1} (z+1) \frac{z}{(z+1)(z-2)^2} = im_{z \to -1} \frac{z}{(z-2)^2} = \frac{-1}{9}.$$

Case (ii): Residue at $z = a = 2$ where m=2 is given by

$$\lim_{z \to 2} \frac{1}{1!} \frac{d}{dz} \left\{(z-2)^2 \frac{z}{(z+1)(z-2)^2}\right\} = \lim_{z \to 2} \frac{d}{dz} \left\{\frac{z}{(z+1)}\right\}$$

$$= \lim_{z \to 2} \left\{\frac{(z+1)^2 - z}{(z+1)^2}\right\} = \lim_{z \to 2} \frac{1}{(z+!)^2} = \frac{1}{9}.$$

Thus the required residues are -1/9 and 1/9.

2. **Evaluate** $\int_C \frac{(z^2+5)}{(z-2)\ (z-3)} dz$ using residue theorem, $C : |z| = 4$

Solution:

The poles of the function $f(z) = \frac{(z^2+5)}{(z-2)\ (z-3)}$ are $z = 2,\ z = 3$ and both the poles lie within the circle $C : |z| = 4$.

Therefore, residue at $z = a = 2,$ which is a simple pole is given by

$$\lim_{z\to 2}(z-2)f(z)=\lim_{z\to 2}(z-2)\frac{(z^2+5)}{(z-2)\ (z-3)}$$

$$=\lim_{z\to 2}\left\{\frac{z^2+5}{z-3}\right\}=-9=R_1.$$

Similarly , residue at $z=a=3,$ which is given by

$$\lim_{z\to 3}(z-3)f(z)=\lim_{z\to 3}(z-3)\frac{(z^2+5)}{(z-2)\ (z-3)}$$

$$=\lim_{z\to 3}\left\{\frac{z^2+5}{z-2}\right\}=14=R_2.$$

We have by Cauchy's theorem,

$$\int_C f(z)dz = 2\pi i(R_1+R_2)$$

$$Thus,\ \int_C \frac{(z^2+5)}{(z-2)\ (z-3)}dz = 2\pi i(-9+14)=10\pi i.$$

2. **Evaluate** $\displaystyle\int_C \frac{3z^3+2}{(z-1)\ \left(z^2+9\right)}dz$ where C is the circle $i)\ |z-2|=2$ $ii)\ |z|=4$

Solution:

$$\text{let } f(z)=\frac{3z^3+2}{(z-1)\ \left(z^2+9\right)}=\frac{3z^3+2}{(z-1)\ (z+3i)(z-3i)}$$

The poles of $f(z),\ z=1,\ z=3i,\ z=-3i$ (simple poles) are represented by the points (1,0) (0,3) (0,-3) respectively.

Case (i): $C:|z-2|=2.$ this is a circle with center (2,0) and radius 2.

The point (1,0) i.e., $z=1$ only lies within C.

The residue of $f(z)$ at $z=1$ (simple pole) is given by

$$\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}(z-1)\frac{3z^3+2}{(z-1)\left(z^2+9\right)}$$

$$=\lim_{z\to 3}\left\{\frac{3z^3+2}{z^2+9}\right\}=1/2.$$

By Cauchy's theorem,

$$\int_C f(z)dz = 2\pi i(R_1)=2\pi i\frac{1}{2}=\pi i.$$

$$Thus,\ \int_C\frac{3z^3+2}{(z-1)\left(z^2+9\right)}dz=\pi i.$$

Case (ii): $C:|z|=4.$ this is a circle with center (0,0) and radius 4. All the three poles $z=1,\ z=3i,\ z=-3i$ lies within C. Let $R_1,\ R_2\ R_3,$ denotes the residues at the simple poles $z=1,\ z=3i,\ z=-3i$ respectively.

$$Now\ R_1=\lim_{z\to 1}(z-1)f(z)=\lim_{z\to 1}(z-1)\frac{3z^3+2}{(z-1)\left(z^2+9\right)}$$

$$=\lim_{z\to 3}\left\{\frac{3z^3+2}{z^2+9}\right\}=1/2.$$

$$R_2=\lim_{z\to 3i}(z-3i)f(z)=\lim_{z\to 3i}(z-3i)\frac{3z^3+2}{(z-1)(z+3i)(z-3i)}$$

$$=\lim_{z\to 3i}\frac{3z^3+2}{(z-1)(z+3i)}=\frac{15+49i}{12}.$$

$$R_3 = \lim_{z \to -3i} (z+3i) f(z) = \lim_{z \to -3i} (z+3i) \frac{3z^3+2}{(z-1)(z+3i)(z-3i)}$$

$$= \lim_{z \to -3i} \frac{3z^3+2}{(z-1)(z-3i)} = \frac{15-49i}{12}.$$

By Cauchy's theorem,

$$\int_C f(z)dz = 2\pi i(R_1+R_2+R_3) = 2\pi i\left(\frac{1}{2}+\frac{15+49i}{12}+\frac{15-49i}{12}\right) = 6\pi i.$$

$$Thus, \int_C \frac{3z^3+2}{(z-1)\left(z^2+9\right)} dz = 6\pi i.$$

3. **Evaluate** $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)} dz$ **, where C is the circle** $|z| = 3$

Solution:

$f(z) = \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2(z-2)}$ is analytic within the circle $|z| = 3$ excepting the poles $z = 1$ and $z = 2$

$$\operatorname{Re}s\ f(1) = \frac{1}{(1)!} \mathop{lt}_{z \to 1} \left\{\frac{d}{dz}\left[(z-1)^2 f(z)\right]\right\} = 2\pi + 1$$

$$\operatorname{Re}s\ f(2) = \mathop{lt}_{z \to 2} \{[(z-2)f(z)]\} = \mathop{lt}_{z \to 2} \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)^2} = 1$$

By residue theorem,

$$\int_C f(z)dz = 2\pi i[\operatorname{Re}s\, f(1) + \operatorname{Re}s\, f(2)] = 2\pi i(2\pi+1+1) = 4\pi(\pi+1)i$$

EXERCISES:

I. Find the nature and location of the singularities of the following functions

a. $\frac{1}{z(2-z)}$

b. $\sin\left(\frac{1}{z}\right)$

c. $\tan\left(\frac{1}{z}\right)$

d. $\frac{z^2-1}{(z-1)^3}$

e. $\frac{e^z}{(z-1)^4}$

f. $\frac{\cot \pi z}{(z-a)^2}$

II. Determine the poles of the following and the residue at the each pole

a. $\dfrac{z^2+1}{z^2-2z}$

c. $\dfrac{2z+4}{(z^2+1)(z+1)}$

b. $\dfrac{z^2-2z}{(z+1)^2(z^2+1)}$

III. Find the residue of the following functions at each pole

a. $\dfrac{1-e^{2z}}{z^4}$

b. $\dfrac{ze^{iz}}{(z^2+1)}$

c. $\cot z$

IV. Evaluate the following integrals

a. $\displaystyle\int_C \frac{1-2z}{z(z-1)(z-2)}\,dz \qquad C:|z|=1.5$

b. $\displaystyle\int_C \frac{4z^2-4z+1}{(z-2)(z^2+4)}\,dz \qquad C:|z|=1$

c. $\displaystyle\int_C \frac{3z^2+z+1}{(z+3)(z^2-1)}\,dz \qquad C:|z|=2$

d. $\displaystyle\int_C \frac{1+2z}{(2z-1)^2}\,dz \qquad C:|z|=1$

e. $\displaystyle\int_C \frac{z+4}{z^2+2z+5}\,dz \qquad C:|z+1-i|=2$

f. $\displaystyle\int_C \frac{z}{(z-1)(z-2)^2}\,dz \qquad C:|z-2|=0.5$

g. $\displaystyle\int_C \frac{3z^2+2}{(z-1)(z^2+9)}\,dz \qquad C:|z-2|=2$

Answers:

I.

a) $z = 0,\ z = 2$ are the isolated singularities.
b) $z = 0$ are the isolated essential singularities.
c) $z = 0$ are the non isolated essential singularities.
d) $z = 1$ is a pole of order 2.
e) $z = 1$ is a pole of order 4.
f) $z = a$ is a double pole and $z = 0, \pm 1, \pm 2, \ldots$ are simple poles.

II.

a) $\operatorname{Re} sf(0) = -1/2,\ \operatorname{Re} sf(2) = 2\frac{1}{2}$.

b) $\operatorname{Re} sf(-1) = 0,\ \operatorname{Re} sf(i) = \frac{1+2i}{2(1-i)},\ \operatorname{Re} sf(-i) = \frac{-1+2i}{2(-1+i)},$

c) $\operatorname{Re} sf(-1) = 1,\ \operatorname{Re} sf(i) = \frac{i+2}{(-1+i)},\ \operatorname{Re} sf(-i) = \frac{i-2}{(1+i)}$.

III.

a) $\operatorname{Re} sf(0) = \frac{-4}{3}$.

b) $\operatorname{Re} sf(i) = \frac{1}{2}e^{-1},\ \operatorname{Re} sf(-i) = \frac{1}{2}e.$

c) $\frac{-i\pi}{4}$, $\operatorname{Re} sf(n\pi) = 1$, n an integer.

IV.

a) $3\pi i$
b) 0
c) $\frac{\pi i}{4}$
d) πi
e) 0
f) $-2\pi i$
g) πi
h) $\frac{\pi}{16}$.

Numerical Integration

Numerical Integration:

- Numerical integration is the approximate computation of an integral using numerical techniques.
- The numerical computation of an integral is sometimes called quadrature, which mean numerical computation of a **univariate** integral.

Definite Integral

A **definite integral** is a formal calculation of area under the curve/ function, using **infinitesimal stripes** of the region. Mathematically, it is defined as follows: given a real valued function f and real numbers $a < b$, then the definite integral is written as

$$\int_a^b f(x)\,dx \qquad (1)$$

This value represents the signed area between the function f, the x-axis, and the linesx=a and x=b; regions above the x-axis have positive area, while regions below the x-axis have negative area.

Graphically, equation (1) is represented as follows:

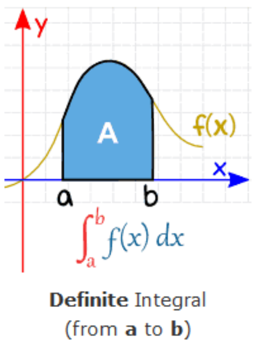


Fig. 1. Definite Integral

Indefinite Integral

An *indefinite integral* is an integral written without terminals. Mathematically, it is defined as follows: given a real valued function f, then the indefinite integral is written as

$$\int f(x)\,dx \qquad (2)$$

Graphically, equation (2) is represented as follows:

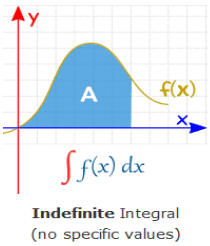


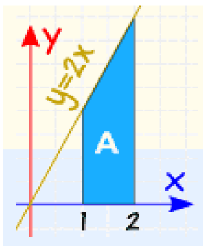
Fig. 2. Indefinite Integral

Examples

1. Evaluate $\int_1^2 2x\,dx$

Sol.

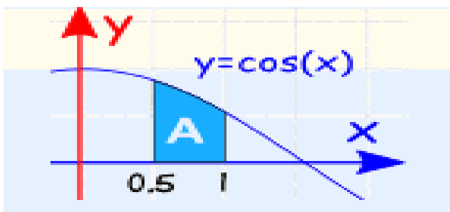
A good way to solve



$$\int_1^2 2x\,dx = \left[x^2\right]_1^2$$
$$= 2^2 - 1^1$$
$$= 3$$

2. Evaluate $\int_{0.5}^{1} \cos x\,dx$

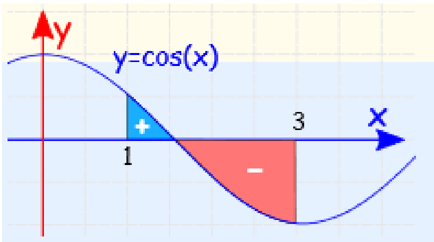
A good way to solve



$$\int_{0.5}^{1} \cos x\, dx = [\sin x]_{0.5}^{1}$$
$$= \sin(1) - \sin(0.5)$$
$$= 0.841 - 0.479$$
$$= 0.362$$

3. Evaluate $\int_{1}^{3} \cos x\, dx$

A good way to solve



$$\int_{1}^{3} \cos x\, dx = [\sin x]_{1}^{3}$$
$$= \sin(1) - \sin(3)$$
$$= 0.141 - 0.841$$
$$= -0.700$$

However, not all definite integrals are directly evaluated. For instance $\int_{0}^{1} \sqrt{x^2+1}\, dx$, $\int_{1}^{3} \frac{(x-2)^3}{\sin x + \log x} dx$, $\int_{0.5}^{2} \frac{e^x}{\sin^2 x + 2x^2} dx$ can't solve it using any of the integration methods. Hence, Numerical approaches will be used to evaluate the definite integrals. To evaluate the above kind of integrals, we have certain numerical techniques namely

1. Trapezoidal Rule
2. Simpson's 1/3 Rule
3. Simpson's 3/8 Rule

The general form of numerical integration may be stated as follows.

Given a set of data points $(x_0, y_0), (x_1, y_1), (x_2, y_2), ..., (x_n, y_n)$ of afunction $y = f(x)$, where $f(x)$ is not known explicitly, it is required to compute the value of the definite integral

$$\int_{a}^{b} f(x)\, dx \qquad (3)$$

Newton's Cotes Formula

Let $y_0, y_1, y_2, ..., y_n$ be the values of $y = f(x)$ corresponding to $a = x_0, x_1, x_2, ..., x_n = b$ which are equally spaced with step size, then $x_n = x_0 + nh$, where

$h =$ Interval size
$n =$ no.of subintervals
$a \,\&\, b =$ limits of the integration

Then the Newton's Cotes Formula is given by

$$\int_{x_0}^{x_n} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + ...\right] \quad (4)$$

Trapezoidal Rule

Trapezoidal Rule is a rule that evaluates the area under the curves by dividing the total area into smaller trapezoids rather than using rectangles. This integration works by approximating the region under the graph of a function as a trapezoid, and it calculates the area.

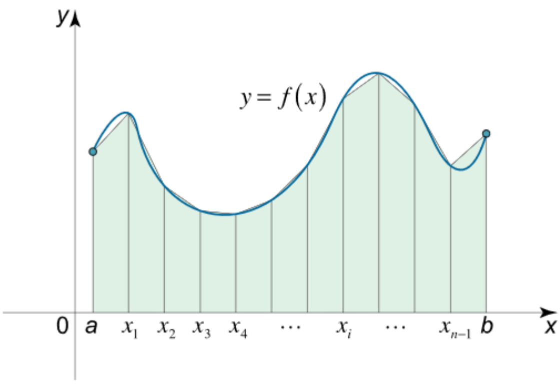


Fig. 3. Trapezoidal Rule

Put n=1, in the formula (4) , we get

$$\int_{x_0}^{x_n} f(x)\,dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right] \text{ (As higher order differences do not exist when n=1)}$$

$$= \frac{h}{2}\left[y_0 + y_1\right]$$

Similarly,

$$\int_{x_1}^{x_2} f(x)\,dx = \frac{h}{2}\left[y_1 + y_2\right]$$

$$\int_{x_2}^{x_3} f(x)\,dx = \frac{h}{2}\left[y_2 + y_3\right]$$

.

.

$$\int_{x_{0+(n-1)h}}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[y_{n-1} + y_n\right]$$

Hence,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

$$= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \ldots + \frac{h}{2}(y_{n-1} + y_n)$$

$$= \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right] \quad \text{(or)}$$

$$= \frac{h}{2}\left[(\text{Sum of first and last ordinates}) + 2(\text{Sum of remaining ordinates})\right] \qquad (5)$$

Where $h = \dfrac{b-a}{n}$ (or) $h = \dfrac{x_n - x_0}{n}$. This is known as Trapezoidal Rule

Note: This method is very simple for calculation purposes of numerical integration and error in this case is significant. **The accuracy of the result can be improved by increasing the number of subintervals or decreasing the value of *h*.**

Simpson's 1/3 Rule

This is another popular method. On putting n=2 in (4), we get

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\ldots+y_{n-1})+2(y_2+y_4+y_6+\ldots+y_{n-2})\right] \quad \text{(or)}$$

$$= \frac{h}{3}\left[(\text{Sum of first and last ordinates})+4(\text{Sum of odd ordinates})+2(\text{Sum of even ordinates})\right] \quad (5)$$

Where $h=\dfrac{b-a}{n}$ (or) $h=\dfrac{x_n-x_0}{n}$.

This formula is known as Simpson's 1/3 rule. It is also known as Simpson's rule.It is applicable only when n is a multiple of 2.

Note

This rule requires the given interval must be divided into an even number of equal subintervals of length *h*.

Simpson's 3/8 Rule

This is another popular method. On putting n=3 in (4), we get

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\ldots+y_{n-1})+2(y_3+y_6+y_9+\ldots+y_n)\right] \quad (6)$$

Where $h=\dfrac{b-a}{n}$ (or) $h=\dfrac{x_n-x_0}{n}$.

This formula is known as Simpson's 3/8 rule. It is applicable only when n is a multiple of 3. This rule is not so accurate as Simpson's 1/3 rule.

Note

There is no restriction for the number of intervals in Trapezoidal rule, whereas for Simpson's 1/3 rule must be multiple of 2, and for Simpson's 3/8 rule must be multiple of 3.

Problems

1. Evaluate $\int_0^1 x^3 dx$ with five sub-intervals by Trapezoidal Rule and compare it with actual value.

Solution:

Given $\int_0^1 x^3 dx$. Here, $a=0,\ b=1,\ n=5$ and $y=f(x)=x^3$

$$h=\frac{b-a}{n}=\frac{1-0}{5}=0.2$$

The values of *x* and *y* are given as follows:

x	0 x_0	0.2 x_1	0.4 x_2	0.6 x_3	0.8 x_4	1 x_5
$y=f(x)=x^3$	0 y_0	0.008 y_1	0.064 y_2	0.216 y_3	0.512 y_5	1 y_5

By Trapezoidal Rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\right]$$

Since

$$f(x)=x^3$$

$$\int_0^1 x^3\,dx = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$=\frac{0.2}{2}\left[(0+1)+2(0.08+0.064+0.216+0.512\right]$$

$$=0.26$$

Validation

$$\int_0^1 x^3\,dx = \left[\frac{x^4}{4}\right]_0^1$$

$$=\frac{1}{4}$$

$$=0.25$$

Hence

$$\int_0^1 x^3\,dx = 0.26$$

2. Use Trapezoidal rule with $n = 4$ to estimate $\int_0^1 \frac{1}{1+x^2}\,dx$

Solution

Given $\int_0^1 \frac{1}{1+x^2}\,dx$. Here, $a = 0,\ b = 1,\ n = 4$ and $y = f(x) = \frac{1}{1+x^2}$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

The values of x and y are given as follows:

x	0 x_0	0.25 x_1	0.5 x_2	0.75 x_3	1 x_4
$y = f(x) = \frac{1}{1+x^2}$	1 y_0	0.9412 y_1	0.8 y_2	0.6400 y_3	0.5 y_4

By Trapezoidal Rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

Since

$$y = f(x) = \frac{1}{1+x^2}$$

$$\begin{aligned}\int_0^1 \frac{1}{1+x^2}\,dx &= \frac{h}{2}\left[(y_0 + y_4) + 2(y_1 + y_2 + y_3)\right]\\ &= \frac{0.25}{2}\left[(1+0.5) + 2(0.9412 + 0.8 + 0.64)\right]\\ &= 0.7828\end{aligned}$$

Hence

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.7828$$

3. Evaluate $\int_0^{1.2} e^x\,dx$, taking six intervals by using Trapezoidal rule up to three significant figures.

Solution

Given $\int_0^{1.2} e^x\,dx$. Here, $a=0,\ b=1.2,\ n=6$ and $y=f(x)=e^x$

$$h=\frac{b-a}{n}=\frac{1.2-0}{6}=0.2$$

The values of x and y are given as follows:

x	0	0.2	0.4	0.6	0.8	1	1.2
$y=f(x)=e^x$	0	1.221	1.492	1.822	2.226	2.718	3.320

By Trapezoidal Rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\right]$$

Since

$$y=f(x)=e^x$$

$$\int_0^{1.2} e^x\,dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$=\frac{0.2}{2}\left[(0+3.320)+2(1.221+1.492+1.822+2.226+2.718)\right]$$

$$=2.3278 \approx 2.327$$

Hence $\int_0^{1.2} e^x \, dx = 2.327$

4. Evaluate $\int_2^6 \log_{10}^x \, dx$ by using Trapezoidal rule, taking $n=8$, correct to four decimal places.

Solution

Given $\int_2^6 \log_{10}^x \, dx$. Here, $a=2,\ b=6,\ n=8$ and $y=f(x)=\log_{10}^x$

$$h=\frac{b-a}{n}=\frac{6-2}{8}=0.5$$

The values of x and y are given as follows:

x	2	2.5	3	3.5	4	4.5	5	5.5	6
$y=f(x)=\log_{10}^x$	0.3010	0.3979	0.4971	0.5440	0.6020	0.6532	0.6989	0.7403	0.7781

By Trapezoidal Rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\right]$$

Since

$$y=f(x)=\log_{10}^x$$

$$\int_2^6 \log_{10}^x \, dx = \frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$=\frac{0.5}{2}\left[(0.3010+0.7781)+2(0.3979+0.4771+0.5440+0.6020+0.6532+0.6989+0.7403)\right]$$

$$=2.3266$$

Hence

$$\int_2^6 \log_{10}^x \, dx = 2.3266$$

5. Evaluate $\int_0^{\pi} t\sin t\, dt$ using the Trapezoidal rule.

Solution

Given $\int_0^{\pi} t\sin t\, dt$. Here, $a=0,\ b=\pi,\ n=6$ and $y=f(t)=t\sin t$

$$h=\frac{b-a}{n}=\frac{\pi-0}{6}=\frac{\pi}{6}$$

The values of t and y are given as follows:

t	0	$\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{3\pi}{6}$	$\frac{4\pi}{6}$	$\frac{5\pi}{6}$	π
$y=f(\mathrm{t})=t\sin t$	0	0.2618	0.9069	1.5708	1.8138	1.309	0

By Trapezoidal Rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+...+y_{n-1})\right]$$

Since

$y=f(t)=t\sin t$

$$\int_0^{\pi} t\sin t\, dt = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$=\frac{\pi}{12}\left[(0+0)+2(0.2618+0.9069+1.5708+1.8138+1.309)\right]$$

$$=3.0695$$

Hence $\int_0^{\pi} t\sin t\, dt = 3.0695$

6. Evaluate $\int_0^2 e^{-x^2}dx$ using Simpson's rule taking $h = 0.25$.

Solution

Given $\int_0^2 e^{-x^2}dx$. Here, $a = 0,\ b = 2$ and $y = f(x) = e^{-x^2}$

Since $h = 0.25$

$$h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} \Rightarrow n = \frac{2-0}{0.25} = 8$$

The values of x and y are given as follows:

x	0	0.25	0.5	0.75	1	1.25	1.5	1.75	2
$y = f(x) = e^{-x^2}$	1	0.9394	0.7788	0.5697	0.3678	0.2096	0.1054	0.0467	0.0183

By Simpson's rule (or) Simpson's $\frac{1}{3}$ rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 + ... + y_{n-2})\right] \quad \text{(or)}$$

$$= \frac{h}{3}\left[(\text{Sum of first and last ordinates}) + 4(\text{Sum of odd ordinates}) + 2(\text{Sum of even ordinates})\right]$$

Since $y = f(x) = e^{-x^2}$

$$\int_0^2 e^{-x^2}dx = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{0.25}{8}\left[(1 + 0.0183) + 4(0.9394 + 0.5697 + 0.2096 + 0.0467) + 2(0.7788 + 0.3678 + 0.1054)\right]$$

$$= 0.63486$$

Hence

$$\int_0^2 e^{-x^2}dx = 0.63486$$

7. Evaluate $\int_0^1 \frac{1}{1+x}dx$ using Trapezoidal rule, Simpson's rule and Simpson's $\frac{3}{8}$ rule.

Solution

Since Simpson's rule is applicable when n is amultiple of 2 and Simpson's $\frac{3}{8}$ rule is applicable only when n is amultiple of 3 and there is no restriction over n for Trapezoidal rule.

Therefore, on taking $n=6$ satisfies all the rules.

Given $\int_0^1 \frac{1}{1+x}dx$. Here, $a=0,\ b=1, n=6$ and $y=f(x)=\frac{1}{1+x}$

$$h=\frac{b-a}{n} \Rightarrow h=\frac{1-0}{6}=\frac{1}{6}$$

The values of *x* and *y* are given as follows:

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$y=f(x)=\frac{1}{1+x}$	1	0.8571	0.75	0.6666	0.6	0.5454	0.5

By Trapezoidal Rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx \ = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+...+y_{n-1})\right]$$

Since

$$y=f(x)=\frac{1}{1+x}$$

$$\int_0^1 \frac{1}{1+x}dx = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$=\frac{1}{12}\left[(1+0.5)+2(0.8571+0.75+0.6666+0.6+0.5454)\right]$$

$$=0.6948$$

By Simpson's rule (or) Simpson's $\frac{1}{3}$ rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0+y_n)+4(y_1+y_3+y_5+\ldots+y_{n-1})+2(y_2+y_4+y_6+\ldots+y_{n-2})\right] \quad \text{(or)}$$

$$=\frac{h}{3}\left[(\text{Sum of first and last ordinates})+4(\text{Sum of odd ordinates})+2(\text{Sum of even ordinates})\right]$$

$$\int_0^1 \frac{1}{1+x}dx = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=0.6931$$

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\ldots+y_{n-1})+2(y_3+y_6+y_9+\ldots+y_n)\right]$$

$$\int_0^1 \frac{1}{1+x}dx = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$=0.6932$$

8. Given that

x	4	4.2	4.4	4.6	4.8	5	5.2
$y=\log x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate $\int_4^{5.2} \log x\,dx$ using Simpson's $\frac{3}{8}$ rule.

Solution

Here, $a=4,\ b=5.2,\ \text{h}=0.2$ and $y=\log x$

$$h=\frac{b-a}{n}\Rightarrow n=\frac{5.2-4}{0.2}=6$$

From the given table

$$y_0=1.3863, y_1=1.4351, y_2=1.4816, y_3=1.5261, y_4=1.5686, y_5=1.6094, y_6=1.6487$$

By Simpson's $\frac{3}{8}$ rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx=\frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+...+y_{n-1})+2(y_3+y_6+y_9+...+y_n)\right]$$

$$\begin{aligned}\int_4^{5.2}\log x\,dx &=\frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]\\ &=\frac{3(0.2)}{8}\left[(1.3863+1.6487)+3(1.4351+1.4816+1.5686+1.6094)+2(1.5261)\right]\\ &=1.827847\end{aligned}$$

9. When a train is moving at 30 m/sec, steam is shut off and brakes are applied. The speed of the train per second after t seconds is given by

$Time(t)$	0	5	10	15	20	25	30	35	40
$Speed(v)$	30	24	19.5	16	13.6	11.7	10	8.5	7

Using Simpson's rule, determine the distance travelled by the train in 40 seconds.

Solution

We know that

$$\text{Speed}=\frac{\text{distance}}{\text{time}}\quad (or)$$
$$=\text{change in distance per unit time}$$

$$\text{i.e.}\frac{ds}{dt}=v \qquad (1)$$

Integrating (1) on both sides

Therefore,

$$s = \int v\,dt$$

To get s, we have to integrate v

$$s = \int_0^{40} v\,dt$$

By Simpson's rule (or) Simpson's $\frac{1}{3}$ rule, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + ... + y_{n-1}) + 2(y_2 + y_4 + y_6 + ... + y_{n-2})\right] \quad \text{(or)}$$

$$= \frac{h}{3}\left[(\text{Sum of first and last ordinates}) + 4(\text{Sum of odd ordinates}) + 2(\text{Sum of even ordinates})\right]$$

$$\int_0^{40} v\,dt = \frac{5}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{5}{3}\left[(30+7) + 4(24+16+11.7+8.5) + 2(19.5+13.6+10)\right]$$

$$= 573.4367 \textit{ meters}$$

10. A rocket is launched from the ground station. Its acceleration measured every five seconds is tabulated given below. Find the velocity and position at $t = 40$ seconds using Trapezoidal rule and Simpson's $\frac{1}{3}$ rule.

$Time(t)$	0	5	10	15	20	25	30	35	40
$Acceleration(a(t))$	40	45.25	48.50	51.25	54.35	59.48	61.50	64.30	68.70

Solution

If s is the distance travelled in time t and v is the velocity at time t, then

$$\frac{dv}{dt} = a \qquad (1)$$

Integrating (1) on both sides

Therefore,

$$(\mathrm{v})_{t=0}^{40} = \int_0^{40} a\, dt$$

Here

$\mathrm{h}=5, \mathrm{a}_0 = 40, \mathrm{a}_1 = 45.25, \mathrm{a}_2 = 48.5, \mathrm{a}_3 = 51.25, \mathrm{a}_4 = 54.35, \mathrm{a}_5 = 59.48, \mathrm{a}_6 = 61.5, \mathrm{a}_7 = 64.3, \text{and } \mathrm{a}_8 = 68.7$

By Trapezoidal Rule, we have

$$\begin{aligned}\text{The required velocity} &= \frac{h}{2}\left[(a_0 + a_8) + 2(a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7)\right] \\ &= \frac{5}{2}\left[(40 + 68.7) + 2(45.25 + 48.50 + 51.25 + 54.35 + 59.48 + 61.5 + 64.3)\right] \\ &= 2194.9\end{aligned}$$

We know that

Distance=speed * time

Position of the Rocket at $t = 40$ seconds= (2194.9*40)=87796

By Simpson's rule (or) Simpson's $\frac{1}{3}$ rule, we have

$$\begin{aligned}\text{The required velocity} &= \frac{h}{3}\left[(a_0 + a_8) + 2(a_2 + a_4 + a_6) + 4(a_1 + a_3 + a_5 + a_7)\right] \\ &= \frac{5}{2}\left[(40 + 68.7) + 2(48.50 + 54.35 + 61.5) + 2(45.25 + 51.25 + 59.48 + 64.3)\right] \\ &= 2197.5\end{aligned}$$

Position of the Rocket at $t = 40$ seconds= (2197.5*40)=87900

11. The velocity v of a particle at distance s from a point on its linear path is given by the following table:

$s(m)$	0	2.5	5	7.5	10	12.5	15	17.5	20
$v(m/\sec)$	16	19	21	22	20	17	13	11	9

Estimate the time taken by the particle to traverse the distance of 20 meter, using Trapezoidal rule.

Solution

If t seconds be the time taken to traverse a distance $s(m)$, then

$$\frac{ds}{dt} = v \quad \Rightarrow \frac{dt}{ds} = \frac{1}{v} = y(\text{say}) \text{ ....(1)}$$

Integrating (1) on both sides

Therefore,

$$t = \int_{s=0}^{20} y\,ds \text{ ....(2)}$$

Here

$$\text{h=5}, y_0 = \frac{1}{16}, y_1 = \frac{1}{19}, y_2 = \frac{1}{4}, y_3 = \frac{1}{22}, y_4 = \frac{1}{20}, y_5 = \frac{1}{17}, y_6 = \frac{1}{13}, y_7 = \frac{1}{11}, \text{and } y_8 = \frac{1}{9}$$

By Trapezoidal Rule, we have

$$\begin{aligned}\text{The required time} &= \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right] \\ &= \frac{5}{2}\left[\left(\frac{1}{16} + \frac{1}{9}\right) + 2\left(\frac{1}{19} + \frac{1}{4} + \frac{1}{22} + \frac{1}{20} + \frac{1}{17} + \frac{1}{13} + \frac{1}{11}\right)\right] \\ &= 1.45 \text{ seconds}\end{aligned}$$

Exercise Problems

1. Evaluate $\int_0^1 \frac{1}{1+x^2}\,dx$ using Simpson's $\frac{1}{3}$ ruleby taking $h=\frac{1}{6}$.

2. Evaluate $\int_0^{\pi} \frac{\sin x}{x^2}\,dx$ using Trapezoidal rule and Simpson's $\frac{3}{8}$ rule

3. Divide the range into 10 equal parts; find the approximate value of $\int_0^{\pi} \sin x\,dx$ by usingTrapezoidal rule and Simpson's rule.

Numerical Solution of Ordinary Differential Equations

Introduction

Many problems in science and engineering can be formulated into ordinary differential equations (ODE's). The analytical/exact methods of solving ODE's are applicable only to a selected class of differential equations. Quite often equations appearing in physical problems do not belong to any of these familiar types and one need to use numerical methods for solving such differential equations. We will use the following numerical methods to solve ODE's

1. Taylor's Series Method
2. Picard's Method/ Picard's Method of Successive Approximation
3. Euler's Method
4. Runge-Kutta Methods
5. Predictor-Corrector Methods
6. Milne's Method

Taylor's Series Method

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y), \text{given} \quad y(x_0) = y_0 \qquad (1)$$

Procedure

On differentiating (1) again and again, we get

$$\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \ldots\ldots$$

We know that

$$\frac{dy}{dx} = y^1 \qquad (2)$$

$$\frac{d^2y}{dx^2} = y^{11} \qquad (3)$$

$$\frac{d^3y}{dx^3} = y^{111} \qquad (4)$$

.

.

On putting $x = x_0$, and $y = y_0$ in the above equations, we get

$y_0^{1}, y_0^{11}, y_0^{111}, y_0^{1111}, \ldots y_0^{n}$

Substitute the above values in Taylor's series formula, we get

$$y = y_0 + \frac{(x-x_0)}{1!} y^1(x_0) + \frac{(x-x_0)^2}{2!} y^{11}(x_0) + \frac{(x-x_0)^3}{3!} y^{111}(x_0) + \ldots + \frac{(x-x_0)^n}{n!} y^n(x_0) \qquad (5)$$

Where $y^n(x_0)$ is the n^{th} derivative of $y(x)$ at $x = x_0$.

Thus, we can obtain a power series for $y(x)$ in powers of $(x - x_0)$.

Hence, the Taylor's series (5) gives the values of y for any value of x.

Note

$y^1(x_0) = y^1{}_0, y^{11}(x_0) = y^{11}{}_0, y^{111}(x_0) = y^{111}{}_0, \ldots..$

Merits & Demerits

Merits

It is a single step method and works well for finding y values in an easy manner.

Demerits

If $f(x, y)$ issomewhatcompleted, then evaluation of higher order derivatives may become tedious.

Problems

1. Solve $\frac{dy}{dx} = xy + 1$ and $y(0) = 1$ using Taylor's series method and compute $y(0.1)$.

Solution

Given

$$\frac{dy}{dx} = y^1 = xy + 1 \qquad (1) \;\; \& \;\; y(0) = 1 \;\; i.e. x_0 = 0, y_0 = 1$$

Differentiating (1) successively w.r.t *x*, we get

$$y^{11} = xy^{1} + y \qquad (2)$$
$$y^{111} = xy^{11} + y^{1} + y^{1}$$
$$y^{111} = xy^{11} + 2y^{1} \qquad (3)$$
$$y^{1111} = xy^{111} + y^{11} + 2y^{11}$$
$$y^{1111} = xy^{111} + 3y^{11} \qquad (4),\ \text{so on}$$

We have $x_0 = 0, y_0 = 1$

Substitute these values in equations (1-4), we get

$$y^{1}{}_{0} = x_0 y_0 + 1$$
$$= 0\times1+1$$
$$=1$$

$$y_0^{11} = x_0 y^{1}{}_{0} + y_0$$
$$= 0\times1+1$$
$$=1$$
$$y_0^{111} = x_0 y_0^{11} + 2y_0^{1}$$
$$= 0\times1+2\times1$$
$$=2$$
$$y_0^{1V} = x_0 y_0^{111} + 3y_0^{11}$$
$$= 0\times2+3\times1$$
$$=3$$

By Taylor's series method, we have

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y^{1}(x_0) + \frac{(x-x_0)^2}{2!} y^{11}(x_0) + \frac{(x-x_0)^3}{3!} y^{111}(x_0) + \ldots \quad (5)$$
$$Since\ \ x_0 = 0\ \&\ \mathrm{y}_0 = 1$$
$$\therefore (5) \Rightarrow$$
$$y(x) = 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(2) + \frac{(x-0)^4}{4!}(3) + \ldots$$
$$y(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!} + \ldots \qquad (6)$$
$$\text{When } x = 0.1$$
$$(6) \Rightarrow \mathrm{y}(0.1) = 1 + \frac{(0.1)}{1!} + \frac{(0.1)^2}{2!} + \frac{2(0.1)}{3!} + \frac{3(0.1)}{4!} + \ldots$$
$$= 1.105342$$
$$= 1.1053 \text{ (correct to four decimals)}$$

2. Using Taylor's series method, obtain the solution of $\frac{dy}{dx}=3x+y^2$ and $y=1$ when $x=0$. Find the value of y for $x=0.1$ *i.e.* $y(0.1)$ correct to four places of decimals.

Solution

Given

$$\frac{dy}{dx}=y^1=3x+y^2 \qquad (1) \;\; \& \;\; y(0)=1 \;\; i.e. x_0=0, y_0=1$$

Differentiating (1) successively w.r.t x, we get

$$y^{11}=3+2yy^1 \qquad (2)$$
$$y^{111}=2yy^{11}+2(y^1)^2 \qquad (3)$$
$$y^{1V}=2yy^{111}+2y^{11}y^1+4y^1y^{11}$$
$$=2yy^{111}+6y^1y^{11} \qquad (4), \text{ so on}$$

We have $x_0=0, y_0=1$

Substitute these values in equations (1-4), we get

$$(1) \Rightarrow y^1{}_0=3x_0+y_0^2$$
$$=3(0)+(1)^2$$
$$=1$$
$$(2) \Rightarrow y_0^{11}=3+2y_0y_0^1$$
$$=3+2(1)(1)$$
$$=5$$
$$(3) \Rightarrow y_0^{111}=2y_0y_0^{11}+2(y_0^1)^2$$
$$=2(1)(5)+2(1)^2$$
$$=12$$
$$(4) \Rightarrow y_0^{1V}=2y_0y_0^{111}+6y_0^1y_0^{11}$$
$$=2(1)(12)+6(1)(5)$$
$$=54$$

By Taylor's series method, we have

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0^1 + \frac{(x-x_0)^2}{2!}y_0^{11} + \frac{(x-x_0)^3}{3!}y_0^{111} + \frac{(x-x_0)^4}{4!}y_0^{1V} \dots \quad (5)$$

$Since\ \ x_0 = 0\ \&\ \mathrm{y}_0 = 1$

$\therefore (5) \Rightarrow$

$$y(x) = 1 + \frac{(x-0)}{1!}(1) + \frac{(x-0)^2}{2!}(5) + \frac{(x-0)^3}{3!}(12) + \frac{(x-0)^4}{4!}(54) + \dots$$

$$y(x) = 1 + \frac{x}{1!} + \frac{5x^2}{2!} + \frac{12x^3}{3!} + \frac{54x^4}{4!} + \dots$$

$$= 1 + \frac{x}{1!} + \frac{5}{2}x^2 + 2x^3 + \frac{9}{4}x^4 \qquad (6)$$

When $x = 0.1$

$$(6) \Rightarrow \mathrm{y}(0.1) = 1 + \frac{(0.1)}{1!} + \frac{5(0.1)^2}{2} + 2(0.1)^3 + \frac{9}{4}(0.1)^4 + \dots$$

$$= 1.127225$$

$$= 1.1272 \text{ (correct to four decimals)}$$

3. Using Taylor's series method, solve $\frac{dy}{dx} = x^2 + y^2$ for $x = 0.4$ given $y = 0$ when $x = 0$.

Solution

Given

$$\frac{dy}{dx} = y^1 = x^2 + y^2 \qquad (1)\ \ \&\ \ y(0) = 0\ \ i.e.\ x_0 = 0, y_0 = 0$$

Differentiating (1) successively w.r.t x, we get

$$y^{11} = 2x + 2yy^1 \qquad (2)$$

$$y^{111} = 2 + 2(\mathrm{yy}^{11} + (\mathrm{y}^1)^2) \qquad (3)$$

$$y^{1V} = 2[yy^{111} + y^{11}y^1] + 4y^1y^{11}$$

$$= 2yy^{111} + 6y^1y^{11} \qquad (4),\ \text{so on}$$

We have $x_0 = 0, y_0 = 0$

Substitute these values in equations (1-4), we get

$$(1) \Rightarrow y^{1}{}_{0} = x_0^2 + y_0^2$$
$$= (0)^2 + (0)^2$$
$$= 0$$
$$(2) \Rightarrow y_0^{11} = 2x_0 + 2y_0 y_0^1$$
$$= 2(0) + 2(0)(0)$$
$$= 0$$
$$(3) \Rightarrow y_0^{111} = 2 + 2y_0 y_0^{11} + 2(y_0^1)^2$$
$$= 2 + 2(0)(0) + 2(0)^2$$
$$= 2$$
$$(4) \Rightarrow y_0^{1V} = 2y_0 y_0^{111} + 6y_0^1 y_0^{11}$$
$$= 2(0)(2) + 6(0)(0)$$
$$= 0$$

By Taylor's series method, we have

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^1 + \frac{(x-x_0)^2}{2!} y_0^{11} + \frac{(x-x_0)^3}{3!} y_0^{111} + \frac{(x-x_0)^4}{4!} y_0^{1V} \ldots \quad (5)$$

$Since\ \ x_0 = 0\ \&\ y_0 = 0$

$\therefore (5) \Rightarrow$

$$y(x) = 0 + \frac{(x-0)}{1!}(0) + \frac{(x-0)^2}{2!}(0) + \frac{(x-0)^3}{3!}(2) + \frac{(x-0)^4}{4!}(0) + \ldots$$

$$y(x) = \frac{x^3}{3} \qquad (6)$$

When $x = 0.4$

$$(6) \Rightarrow y(0.4) = \frac{(0.4)^3}{3}$$
$$= 0.02133$$
$$= 1.1272 \text{ (correct to four decimals)}$$

4. Solve $y^1 = x^2 - y, y(0) = 1$ using Taylor's series method and compute $y(0.1), y(0.2), y(0.3)$ and $y(0.4)$.

Solution

Given

$y^1 = x^2 - y \quad (1)\ \&\ y(0) = 1$

Differentiating (1) successively w.r.t x, we get

$y^{11} = 2x - y^{1}$ (2)

$y^{111} = 2 - y^{11}$ (3)

$y^{1V} = -y^{111}$ (4), so on

We have $x_0 = 0, y_0 = 1$

Substitute these values in equations (1-4), we get

$$(1) \Rightarrow y^{1}{}_{0} = x_0^2 - y_0$$
$$= (0)^2 - (1)$$
$$= -1$$

$$(2) \Rightarrow y_0^{11} = 2x_0 - y_0^{1}$$
$$= 2(0) - (-1)$$
$$= 1$$

$$(3) \Rightarrow y_0^{111} = 2 - y_0^{11}$$
$$= 2 - 1$$
$$= 1$$

$$(4) \Rightarrow y_0^{1V} = -y_0^{111}$$
$$= -1$$

By Taylor's series method, we have

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^{1} + \frac{(x-x_0)^2}{2!} y_0^{11} + \frac{(x-x_0)^3}{3!} y_0^{111} + \frac{(x-x_0)^4}{4!} y_0^{1V} \ldots \quad (5)$$

Since $x_0 = 0$ & $y_0 = 1$

$\therefore (5) \Rightarrow$

$$y(x) = 1 + \frac{(x-0)}{1!}(-1) + \frac{(x-0)^2}{2!}(1) + \frac{(x-0)^3}{3!}(1) + \frac{(x-0)^4}{4!}(-1) + \ldots$$

$$y(x) = 1 - x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} + \ldots \quad (6)$$

Hence

When $x = 0.1$

$(6) \Rightarrow y(0.1) = 0.9051$

When $x = 0.2$

$(6) \Rightarrow y(0.2) = 0.8212$

When $x = 0.3$

$(6) \Rightarrow y(0.3) = 0.7492$

When $x = 0.4$

$(6) \Rightarrow y(0.4) = 0.6897$

5. Using Taylor's series method, solve $\frac{dy}{dx} = \log(xy)$ for $y(1.1), y(1.2)$ given $y(1) = 2$.

Solution

Given

$$\frac{dy}{dx} = \log(xy) = \log x + \log y \qquad (1) \;\; \& \;\; y(1) = 2$$

Differentiating (1) successively w.r.t x, we get

$$y^{11} = \frac{1}{x} + \frac{1}{y} y^{1} \qquad (2)$$

$$y^{111} = \frac{-1}{x^2} + \frac{1}{y} y^{11} + y^{1}\left(\frac{-1}{y^2}\right) y^{1}$$

$$= \frac{-1}{x^2} + \frac{1}{y} y^{11} + \left(\frac{-1}{y^2}\right)\left(y^{1}\right)^2 \qquad (3)$$

We have $x_0 = 1, y_0 = 2$

Substitute these values in equations (1-3), we get

$$(1) \Rightarrow y^{1}{}_{0} = \log(x_0) + \log(y_0)$$

$$= \log(1) + \log(2)$$

$$= \log 2$$

$$(2) \Rightarrow y_0^{11} = \frac{1}{x_0} + \frac{1}{y_0} y_0^{1}$$

$$= \frac{1}{1} + \frac{1}{2}(\log 2)$$

$$= 1 + \frac{1}{2}\log 2$$

$$(3) \Rightarrow y_0^{111} = \frac{-1}{x_0^2} + \frac{1}{y_0} y_0^{11} - \frac{1}{y_0^{11}}(y_0^{1})^2$$

$$= \frac{-1}{1} + \frac{1}{2}(1 + \frac{1}{2}\log 2) - \frac{1}{4}(\log 2)^2$$

$$= -1 + \frac{1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2$$

By Taylor's series method, we have

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0^1 + \frac{(x-x_0)^2}{2!} y_0^{11} + \frac{(x-x_0)^3}{3!} y_0^{111} + \frac{(x-x_0)^4}{4!} y_0^{1V} \dots \quad (4)$$

$Since\ \ x_0 = 1 \ \& \ y_0 = 2$

$\therefore (4) \Rightarrow$

$$y(x) = 2 + \frac{(x-1)}{1!}(\log 2) + \frac{(x-1)^2}{2!}\left(1 + \frac{1}{2}\log 2\right) + \frac{(x-1)^3}{3!}\left(\frac{-1}{2} + \frac{1}{4}\log 2 - \frac{1}{4}(\log 2)^2\right) + \dots \qquad (5)$$

Hence

When $x = 1.1$

$(5) \Rightarrow y(1.1) = 2.036$

When $x = 1.2$

$(5) \Rightarrow y(1.2) = 2.082$

Exercise Problems

1. Using Taylor's series method, solve $\frac{dy}{dx} = 2y + 3e^x, y(0) = 0$ for $x = 0.2$.
2. Using Taylor's series method, solve $y^1 = x + y, y(0) = 1$ for $x = 1.1 \ \& 1.2$.
3. Using Taylor's series method, solve $y^1 = x^2 - y, y(0) = 1$ for $x = 0.2 \ \& 0.4$.

Picard's Method (or) Picard's Method of Successive Approximation

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y), \text{given } \ y(x_0) = y_0 \qquad (1)$$

Procedure

Consider equation (1)

$$\frac{dy}{dx} = f(x, y)$$

$$dy = f(x, y)\, dx \qquad (2)$$

Integrating (2) between the limits x_0 & x, we get

$$\int_{x=x_0}^{x} dy = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow [y]_{x_0}^{x} = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y(x) - y(x_0) = \int_{x_0}^{x} f(x,y)\,dx$$

$$\Rightarrow y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx \qquad (3)$$

Equation (3) is a solution of equation (1). But, (3) contains the unknown y under the integral sign on the right hand side.

On putting $y = y_0$ in R.H.S of equation (4), we get a first approximation y_1.

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)\,dx \qquad (4)$$

For second approximation y_2 ,put $y = y_1$ in R.H.S of equation (4), we get

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)\,dx \qquad (5)$$

Similarly

$$y_3 = y_0 + \int_{x_0}^{x} f(x,y_2)\,dx$$

$$y_4 = y_0 + \int_{x_0}^{x} f(x,y_3)\,dx$$

.

.

In general

$$y_n = y_0 + \int_{x_0}^{x} f(x,y_{n-1})\,dx \quad ,n = 1,2,3,\ldots \qquad (6)$$

Equation (6) is known as Picard's iteration formula. It gives a sequence of approximations $y_1, y_2, ..., y_n$ each giving a better result than the preceding one.

Note

Since this method involves actual integration, sometimes it may not be possible to carry out the integration.

Problems

1. Using Picard's method, find a solution of $\frac{dy}{dx} = 1 + xy$ upto third approximation, when $x_0 = 0, y_0 = 0$

Solution

Given

$$\frac{dy}{dx} = 1 + xy \qquad (1)$$

$given \;\; x_0 = 0, y_0 = 0$

$here$

$f(x, y) = 1 + xy$

We know that Picard's Iteration formula

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \;\;, n = 1, 2, 3, ... \qquad (2)$$

First Approximation

Put $n = 1$ in equation (2), we get

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx \qquad (3)$$

since $f(x, y) = 1 + xy$

$$\therefore \; f(x, y_0) = 1 + xy_0$$
$$= 1 + x(0)$$
$$= 1$$

$(3) \Rightarrow$

$$y_1 = 0 + \int_{x_0}^{x} 1\, dx$$
$$= [x]_0^x$$
$$= x - 0$$
$$= x$$

Second Approximation

Put $n = 2$ in equation (2), we get

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx \qquad (4)$$

since $f(x, y) = 1 + xy$

$$\therefore \; f(x, y_1) = 1 + xy_1$$
$$= 1 + x(x)$$
$$= 1 + x^2$$

$(4) \Rightarrow$

$$y_2 = 0 + \int_{x_0}^{x} (1 + x^2)\, dx$$
$$= \left[x + \frac{x^3}{3} \right]_0^x$$
$$= x + \frac{x^3}{3}$$

Third Approximation

Put $n=3$ in equation (2), we get

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx \qquad (5)$$

since $f(x,y)=1+xy$

$\therefore\ f(x,y_2)=1+xy_2$

$$=1+x\left(x+\frac{x^3}{3}\right)$$

$$=1+x^2+\frac{x^4}{3}$$

$(5)\Rightarrow$

$$y_3 = 0+\int_{x_0}^{x}\left(1+x^2+\frac{x^4}{3}\right)dx$$

$$=\left[x+\frac{x^3}{3}+\frac{x^5}{15}\right]_0^x$$

$$=x+\frac{x^3}{3}+\frac{x^5}{15}$$

2. Find the value of y for $x=0.4$ by Picard's Method, given that $\frac{dy}{dx}=x^2+y^2, y(0)=0$

Solution

Given

$$\frac{dy}{dx}=x^2+y^2 \qquad (1),\ y(0)=0$$

$here\ x_0=0, y_0=0$

$\&\ f(x,y)=x^2+y^2$

We know that Picard's Iteration formula

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx\ \ , n=1,2,3,\ldots \qquad (2)$$

First Approximation

Put $n=1$ in equation (2), we get

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx \qquad (3)$$

since $f(x,y) = x^2 + y^2$

$$\therefore f(x, y_0) = x^2 + (y_0)^2 = x^2 + (0)^2 = x^2$$

$(3) \Rightarrow$

$$y_1 = 0 + \int_{x_0}^{x} x^2\,dx = \left[\frac{x^3}{3}\right]_0^x = \frac{x^3}{3}$$

Second Approximation

Put $n=2$ in equation (2), we get

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \qquad (4)$$

since $f(x,y) = x^2 + y^2$

$$\therefore f(x, y_1) = x^2 + y_1^2 = x^2 + \left(\frac{x^3}{3}\right)^2 = x^2 + \frac{x^6}{9}$$

$(4) \Rightarrow$

$$y_2 = 0 + \int_{x_0}^{x} \left(x^2 + \frac{x^6}{9}\right)dx = \left[\frac{x^3}{3} + \frac{x^7}{63}\right]_0^x = \frac{x^3}{3} + \frac{x^7}{63}$$

Since calculation of y_3 is tedious and hence approximate value is y_2

Now

When $x = 0.4$

$$y(0.4) = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63} = 0.02136 = 0.0214$$

3. Obtain $y(0.1)$ given $y^1 = \dfrac{y-x}{y+x}, y(0)=1$ by using Picard's Method

Solution

Given

$$y^1 = \frac{y-x}{y+x} \qquad (1),\ y(0)=1$$

here $x_0 = 0, y_0 = 1$

$$\& \ f(x,y) = \frac{y-x}{y+x}$$

We know that Picard's Iteration formula $y_n = y_0 + \displaystyle\int_{x_0}^{x} f(x, y_{n-1})\,dx \ , n = 1,2,3,\ldots \qquad (2)$

First Approximation

Put $n=1$ in equation (2), we get

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx \qquad (3)$$

since $f(x,y) = \dfrac{y-x}{y+x}$

$$\therefore\ f(x,y_0) = \frac{y_0 - x}{y_0 + x}$$

$$= \frac{1-x}{1+x} = \frac{-1-x+2}{1+x} = \frac{-(1+x)+2}{1+x} = -1 + \frac{2}{1+x}$$

$(3) \Rightarrow$

$$y_1 = 1 + \int_{x_0}^{x}\left(-1 + \frac{2}{1+x}\right)dx$$

$$= 1 + \left[-x + 2\log(1+x)\right]_0^x$$

$$= 1 - x + 2\log(1+x)$$

Second Approximation

Put $n=2$ in equation (2), we get

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \qquad (4)$$

since $f(x,y) = \dfrac{y-x}{y+x}$

$$\therefore\ f(x,y_1) = \frac{y_1 - x}{y_1 + x}$$

$$= \frac{1-x+2\log(1+x)-x}{1-x+2\log(1+x)+x}$$

$$= \frac{1-2x+2\log(1+x)}{1+2\log(1+x)}$$

$$= 1 - \frac{2x}{1+2\log(1+x)}$$

$(4) \Rightarrow$

$$y_2 = 1 + \int_0^x \left(1 - \frac{2x}{1+2\log(1+x)}\right)dx \qquad (5)$$

Evaluation of (5) is very difficult. Hence we use y_1 as the approximate solution of y

$\therefore$

$y(x) = y_1 = 1 - x + 2\log(1+x)$

When $x = 0.1$

$y(0.1) = 1 - 0.1 + 2\log(1+0.1)$

$= 1.0906$

4. Find the value of y for $x = 0.25, 0.5 \& 1$, by using Picard's method given that

$$\frac{dy}{dx} = \frac{x^2}{y^2+1},\ y(0)=0$$

Solution

$$\frac{dy}{dx} = \frac{x^2}{y^2+1} \qquad (1),\ y(0)=0$$

here $x_0 = 0, y_0 = 0$

$\&\ f(x,y) = \dfrac{x^2}{y^2+1}$

We know that Picard's Iteration formula

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx \ ,n = 1,2,3,\ldots \qquad (2)$$

First Approximation

Put $n=1$ in equation (2), we get

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx \qquad (3)$$

since $f(x,y) = \dfrac{x^2}{y^2+1}$

$$\therefore\ f(x, y_0) = \frac{x^2}{y_0^2+1} = \frac{x^2}{0+1} = x^2$$

$(3) \Rightarrow$

$$y_1 = 0 + \int_0^x \left(x^2\right) dx$$

$$= 0 + \left[\frac{x^3}{3}\right]_0^x$$

$$= \frac{x^3}{3}$$

Second Approximation

Put $n=2$ in equation (2), we get

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx \qquad (4)$$

since $f(x,y) = \dfrac{x^2}{y^2+1}$

$$\therefore\ f(x, y_1) = \frac{x^2}{y_1^2+1} = \frac{x^2}{\left(\frac{x^3}{3}\right)^2+1} = \frac{9x^2}{\left(x^3\right)^2+9}$$

$(4) \Rightarrow$

$$y_2 = 0 + 3\int_0^x \left(\frac{3x^2}{\left(x^3\right)^2+3^2}\right) dx$$

$$= 3\int_0^x \frac{d(x^3)}{\left(x^3\right)^2+3^2} \qquad \therefore \int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\, tan^{-1}\left(\frac{x}{a}\right)$$

$$= 3.\frac{1}{3}\tan^{-1}\left(\frac{x^3}{3}\right)$$

$$y(x) = y_2 = \tan^{-1}\left(\frac{x^3}{3}\right) \qquad (5)$$

Hence

when $x = 0.25$

$$(5) \Rightarrow y(0.25) = \tan^{-1}\left(\frac{(0.25)^3}{3}\right) = 0.005$$

when $x = 0.5$

$$(5) \Rightarrow y(0.5) = \tan^{-1}\left(\frac{(0.5)^3}{3}\right) = 0.042$$

when $x = 1$

$$(5) \Rightarrow y(1) = \tan^{-1}\left(\frac{(1)^3}{3}\right) = 0.321$$

Exercise Problems

1. Given that $y^1 + y = e^x$, $y(0) = 0$. Compute $y(0.1)$ & $y(0.2)$ using Picard's method.
2. Perform two iterations of Picard's method to find an approximate solution of the initial value problem $y^1 = x + y^2$, $y(0) = 1$.

Euler's Method

We have so far discussed the methods which yield the solution of a differential equation in the form of a function. We will now describe the methods which gives the solution in the form of a set of tabulated values.

Suppose we wish to solve the equation $\frac{dy}{dx} = f(x, y)$, given $y(x_0) = y_0$. In this method starting from x_0 we take equally spaced values of x, *say* $x_0, x_1, x_2, \ldots, x_n$ with step length *h*.

$$\therefore\ x_n = x_{n-1} + h,\ n = 1, 2, 3, \ldots$$

The successive approximation values of y are given by

$$\begin{aligned} y(x_1) &= y_1 = y_0 + h\, f(x_0, y_0) \\ y(x_2) &= y_2 = y_1 + h\, f(x_1, y_1) \\ y(x_3) &= y_3 = y_2 + h\, f(x_2, y_2) \\ &. \\ &. \\ &. \\ y(x_{n+1}) &= y_{n+1} = y_n + h\, f(x_n, y_n),\ \ n = 0, 1, 2, \ldots \end{aligned}$$

This is known as Euler's algorithm and can be used to evaluate
$y_1, y_2, ..., y_n$ *i.e.* $y(x_1), y(x_2), ..., y(x_n)$ respectively starting from the initial condition.

we wish to sol

Note

To obtain reasonable accuracy with Euler's method, we have to take a smaller value of h

Solved Problems

1. Using Euler's method, solve for y at $x = 2$ from $\frac{dy}{dx} = 3x^2 + 1$, given $y(1) = 2$ taking step size $h = 0.5$ & $h = 0.25$

Solution

Given

$$\frac{dy}{dx} = 3x^2 + 1 \qquad (1) \qquad y(1) = 2$$

Here $f(x, y) = 3x^2 + 1,\ x_0 = 1, y_0 = 2\ \&\ h = 0.5$

We know that

$$y(x_{n+1}) = y_{n+1} = y_n + h\, f(x_n, y_n),\ \ \mathrm{n} = 0, 1, 2, ... \qquad (2)$$

Since $x_n = x_{n-1} + h,\ \ n = 1, 2, 3, ...$

$\therefore\ x_0 = 1, x_1 = 1.5, x_2 = 2$

$(2) \Rightarrow$

$$\begin{aligned} \mathrm{y}_{n+1} &= y_n + h\, f(x_n, y_n) \\ &= y_n + 0.5(3x_n^2 + 1) \qquad (3) \end{aligned}$$

Take $n = 0$ in (3), we get

$(3) \Rightarrow$

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 2 + 0.5(3x_0^2 + 1)$$
$$= 2 + 0.5(3(1)^2 + 1)$$
$$= 4$$

i.e. $y(x_1) = y(1.5) = y_1 = 4$

Take $n = 1$ in (3), we get

$(3) \Rightarrow$

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 4 + 0.5(3x_1^2 + 1)$$
$$= 4 + 0.5(3(1.5)^2 + 1)$$
$$= 7.875$$

i.e. $y(x_2) = y(2) = y_2 = 7.875$

(ii) when $h = 0.25$

$\therefore\ x_0 = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75, x_4 = 2$

Take $n = 0$ in (3), we get

$(3) \Rightarrow$

$$y_1 = y_0 + h\, f(x_0, y_0)$$
$$= 2 + 0.25(3x_0^2 + 1)$$
$$= 2 + 0.25(3(1)^2 + 1)$$
$$= 3$$

i.e. $y(x_1) = y(1.25) = y_1 = 3$

Take $n = 1$ in (3), we get

$(3) \Rightarrow$

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$= 3 + 0.25(3x_1^2 + 1)$$
$$= 3 + 0.25(3(1.25)^2 + 1)$$
$$= 4.42188$$

i.e. $y(x_2) = y(1.5) = y_2 = 4.42188$

Take $n = 2$ in (3), we get

$(3) \Rightarrow$

$$y_3 = y_2 + h\, f(x_2, y_2)$$
$$= 4.42188 + 0.25(3x_2^2 + 1)$$
$$= 4.42188 + 0.25(3(1.5)^2 + 1)$$
$$= 6.35938$$

i.e. $y(x_3) = y(1.75) = y_3 = 6.35938$

Take $n = 3$ in (3), we get

$(3) \Rightarrow$

$$y_4 = y_3 + h\, f(x_3, y_3)$$
$$= 6.35938 + 0.25(3x_3^2 + 1)$$
$$= 6.35938 + 0.25(3(1.75)^2 + 1)$$
$$= 8.90626$$

i.e. $y(x_4) = y(2) = y_4 = 8.90626$

2. Use Euler's method to find $y(0.1), y(0.2)$ given $\frac{dy}{dx} = (x^3 + xy^2)e^{-x}, y(0) = 1$.

Solution

Given

$$\frac{dy}{dx} = (x^3 + xy^2)e^{-x} \qquad (1), y(0) = 1$$

Here $f(x, y) = (x^3 + xy^2)e^{-x}$, $x_0 = 0, y_0 = 1$ & $h = 0.1$

We know that

$$y(x_{n+1}) = y_{n+1} = y_n + h\,f(x_n, y_n),\ \ \mathrm{n} = 0,1,2,\ldots \qquad (2)$$

Since $x_n = x_{n-1} + h,\ \ n = 1,2,3,\ldots$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1,$$
$$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$$
$$(2) \Rightarrow$$
$$y_{n+1} = y_n + h\,f(x_n, y_n)$$
$$= y_n + 0.1(x_n^3 + x_n\,y_n^2)\,e^{-x_n} \qquad (3)$$

Take $n = 0$ in (3), we get

$$(3) \Rightarrow$$
$$y_1 = y_0 + h\,f(x_0, y_0)$$
$$= 1 + 0.1(x_0^3 + x_0 y_0^2)\,e^{-x_0}$$
$$= 1 + 0.25((0)^3 + (0)(1)^2)\,e^{-0}$$
$$= 1$$
$$i.e.\ y(x_1) = y(0.1) = y_1 = 1$$

Take $n = 1$ in (3), we get

$$(3) \Rightarrow$$
$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$= 1 + 0.1(x_1^3 + x_1 y_1^2)\,e^{-x_1}$$
$$= 1 + 0.25((0.1)^3 + (0.1)(1)^2)\,e^{-0.1}$$
$$= 1.0091$$
$$i.e.\ y(x_2) = y(0.2) = y_2 = 1.0091$$

3. Compute $y\ at\ x = 0.25$ by Euler's method,given $\frac{dy}{dx} = 2xy,\ y(0) = 1.$

Solution

Given

$$\frac{dy}{dx} = 2xy \qquad (1),\ y(0) = 1$$

Here $f(x,y)=2xy,\ x_0=0, y_0=1\ \&\ h=0.25$

We know that

$$y(x_{n+1})=y_{n+1}=y_n+h\,f(x_n,y_n),\ \ \mathrm{n}=0,1,2,... \qquad (2)$$

$(2)\Rightarrow$

$$\begin{aligned} \mathrm{y}_{n+1} &= y_n+h\,f(x_n,y_n) \\ &= y_n+0.25(2x_n\,\mathrm{y}_n) \qquad (3) \end{aligned}$$

Take $n=0$ in (3), we get

$(3)\Rightarrow$

$$\begin{aligned} \mathrm{y}_1 &= y_0+h\,f(x_0,y_0) \\ &= 1+0.25(2x_0y_0) \\ &= 1+0.25(2(0)(1)) \\ &= 1 \end{aligned}$$

$i.e.\ y(x_1)=y(0.25)=y_1=1$

Exact Solution

We have

$$\frac{dy}{dx}=2xy$$

$$\Rightarrow \frac{1}{y}dy=2x\,dx$$

Integrating on both sides

$\Rightarrow \mathrm{logy{=}x}^2+c$

$\Rightarrow y=e^{x^2}+c \qquad (3)$

Since $\mathrm{y}(0)=1$

$(3)\Rightarrow$

$\mathrm{y}(0)=\mathrm{e}^0+c\Rightarrow 1=1+c$

$c=0$

$(3)\Rightarrow\ y=e^{x^2}$

$\therefore the\ solution\ of\ (1)\ is\ y=e^{x^2}$

Hence,

$y(0.25)=e^{(0.25)^2}\ =1.0645$

We notice that y value deviates from the exact value. Hence it is required to use modified Euler's method

Modified Euler's Method

Suppose we wish to solve the equation $\frac{dy}{dx} = f(x,y)$, given $y(x_0) = y_0$.

To findthe first approximation y_1 at x_1 where $x_1 = x_0 + h$

Compute

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$
$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

.

.

$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right],\ n = 0,1,\ldots$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1

If two successive values of $y_1^{(n)}, y_1^{(n+1)}$ are almost equal, we stop there and take $y_1 = y_1^{(n)}$.

Now we have

$\frac{dy}{dx} = f(x,y)$, given $y(x_1) = y_1$

To get $y_2 = y(x_2)$, we use the above procedure again.

Solved Problems

1. Solve $y^1 = y + e^x$, $y(0) = 0$ for $x = 0.2, 0.4$ by modified Euler's method.

Solution

Given

$y^1 = y + e^x \qquad (1),\ y(0) = 0$

Here $f(x,y)=y+e^{x}$, $x_0=0, y_0=0$

Take $h=0.2$

$\therefore\ x_1=0.2,\ x_2=0.4$

To find $y_1=y(x_1)=y(0.2)$

Now

$$\begin{aligned}
y_1^{(0)} &= y_0+h\,f(x_0,y_0)\\
&= y_0+h\left[y_0+e^{x_0}\right]\\
&= 0+0.2\left[0+e^{0}\right]\\
&= 0.2
\end{aligned}$$

We know that

$$y_1^{(n+1)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(n)})\right],\ n=0,1,2,.. \qquad (2)$$

Take $n=0$ in (2), we get

$(2)\Rightarrow$

$$\begin{aligned}
y_1^{(1)} &= y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(0)})\right]\\
&= 0+\frac{0.2}{2}\left[y_0+e^{x_0}+y_1^{(0)}+e^{x_1}\right]\\
&= 0.1\left[0+e^{0}+0.2+e^{0.2}\right]\\
&= 0.24214
\end{aligned}$$

Similarly

Take $n=1$ in (2), we get

$(2)\Rightarrow$

$$\begin{aligned}
y_1^{(2)} &= y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(1)})\right]\\
&= 0+\frac{0.2}{2}\left[y_0+e^{x_0}+y_1^{(1)}+e^{x_1}\right]\\
&= 0.1\left[0+e^{0}+0.24214+e^{0.2}\right]\\
&= 0.2463
\end{aligned}$$

Take $n = 2$ in (2), we get

$(2) \Rightarrow$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[\ f(x_0, y_0) + f(x_1, y_1^{(2)})\right]$$
$$= 0 + \frac{0.2}{2}\left[y_0 + e^{x_0} + y_1^{(2)} + e^{x_1}\right]$$
$$= 0.1\left[0 + e^0 + 0.2463 + e^{0.2}\right]$$
$$= 0.2468$$

Take $n = 3$ in (2), we get

$(2) \Rightarrow$

$$y_1^{(4)} = y_0 + \frac{h}{2}\left[\ f(x_0, y_0) + f(x_1, y_1^{(3)})\right]$$
$$= 0 + \frac{0.2}{2}\left[y_0 + e^{x_0} + y_1^{(3)} + e^{x_1}\right]$$
$$= 0.1\left[0 + e^0 + 0.2468 + e^{0.2}\right]$$
$$= 0.2468$$

Since

$y_1^{(4)} = y_1^{(3)}$

$\therefore$ *first approximation is* $y_1 = y(x_1) = y(0.2) = 0.2468$

To find $y_2 = y(x_2) = y(0.4)$

We take

$x_1 = 0.2, y_1 = 0.2468$ *and* $x_2 = 0.4,\ h = 0.2$

Now

$$y_2^{(0)} = y_1 + h\ f(x_1, y_1)$$
$$= y_1 + h\left[y_1 + e^{x_1}\right]$$
$$= 0.2468 + 0.2\left[0.2468 + e^{0.2}\right]$$
$$= 0.5404$$

Take $n=0$ in (2), we get

$$(2) \Rightarrow$$

$$\begin{aligned} y_2^{(1)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right] \\ &= 0.2468 + \frac{0.2}{2}\left[y_1 + e^{x_1} + y_2^{(0)} + e^{x_2}\right] \\ &= 0.2468 + 0.1\left[0.2468 + e^{0.2} + 0.5404 + e^{0.4}\right] \\ &= 0.5968 \end{aligned}$$

Similarly

Take $n=1$ in (2), we get

$$(2) \Rightarrow$$

$$\begin{aligned} y_2^{(2)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right] \\ &= 0.2468 + \frac{0.2}{2}\left[y_1 + e^{x_1} + y_2^{(1)} + e^{x_2}\right] \\ &= 0.2468 + 0.1\left[0.2468 + e^{0.2} + 0.5968 + e^{0.4}\right] \\ &= 0.6025 \end{aligned}$$

Take $n=2$ in (2), we get

$$(2) \Rightarrow$$

$$\begin{aligned} y_2^{(3)} &= y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right] \\ &= 0.2468 + \frac{0.2}{2}\left[y_1 + e^{x_1} + y_2^{(2)} + e^{x_2}\right] \\ &= 0.2468 + 0.1\left[0.2468 + e^{0.2} + 0.6025 + e^{0.4}\right] \\ &= 0.603 \end{aligned}$$

Take $n=3$ in (2), we get

$(2) \Rightarrow$

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[\, f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$

$$= 0.2468 + \frac{0.2}{2}\left[y_1 + e^{x_1} + y_2^{(3)} + e^{x_2}\right]$$

$$= 0.2468 + 0.1\left[0.2468 + e^{0.2} + 0.6030 + e^{0.4}\right]$$

$$= 0.6031$$

Take $n = 4$ in (2), we get

$(2) \Rightarrow$

$$y_2^{(5)} = y_1 + \frac{h}{2}\left[\, f(x_1, y_1) + f(x_2, y_2^{(4)})\right]$$

$$= 0.2468 + \frac{0.2}{2}\left[y_1 + e^{x_1} + y_2^{(4)} + e^{x_2}\right]$$

$$= 0.2468 + 0.1\left[0.2468 + e^{0.2} + 0.6031 + e^{0.4}\right]$$

$$= 0.6031$$

Since

$y_2^{(5)} = y_2^{(4)} = 0.6031$

$\therefore$ *Second approximation is* $y_2 = y(x_2) = y(0.4) = 0.6031$

Exercise Problems

1. Using modified Euler's method, find an approximate value of y when $x = 0.3$, given that $\frac{dy}{dx} = x + y,\ y(0) = 1$ .Sol. $y(0.3) = 1.4004$
2. Solve $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$ by modified Euler's method and compute $y(0.02)\ \&\ y(0.04)$ Sol. $y(0.02) = 1.0202,\ y(0.04) = 1.0408$

Runge-Kutta Methods

The previous methods used for numerical solution of initial value problems are restricted due to either slow convergence or due to labour involved in finding higher order derivatives, especially in Taylor's series method. But in Runge-Kutta (R-K) method, it does not require the determination of higher order derivatives and moreover, it gives greater accuracy.

Merits and demerits of Runge-Kutta Methods

- The principal advantage of R-K method is the self-starting feature and consequently the ease of programming.
- One disadvantage of R-K method is the requirement that the function *f(x,y)*must be evaluated for several slightly different values of *x* and *y* in every step of the function.

To solve $\frac{dy}{dx} = f(x,y),\ \ y(x_0) = y_0$, we have

First order Runge-Kutta Method

We know that, by Euler's method

$$y_1 = y_0 + h\, f(x_0, y_0)$$

$$y_1 = y_0 + h\, y_0^1 \qquad (1) \qquad \left(\text{since } \frac{dy}{dx} = y^1 = f(x,y) \quad y^1(x_0) = f(x_0, y_0) \right)$$

Expanding L.H.S by Taylor's Series, we get

Since

$$y_1 = y(x_1) = y(x_0 + h) = y_0 + hy_0^1 + \frac{h}{2} y_0^{11} + ...$$

$$(1) \Rightarrow$$

$$y_0 + hy_0^1 + \frac{h}{2} y_0^{11} + ... = y_0 + hy_0^1 \qquad (2)$$

Equation (2) follows that the Euler's method agrees with Taylor's series solution upto the term in h

Hence, Euler's method is the R-K method of first order

Second order Runge-Kutta Method

We know that

By modified R-K method

$$
\begin{aligned}
&\textit{Since}\\
&y_1^{(1)} = y_0 + h\, f(x_0, y_0)\\
&y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]\\
&\quad = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0 + h, y_0 + h\, f(x_0, y_0))\right] \qquad \textit{where } f_0 = f(x_0, y_0)\\
&\quad = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h, y_0 + h\, f_0)\right] \qquad (1)
\end{aligned}
$$

If we now set

$$
\begin{aligned}
&k_1 = hf_0;\;\; k_2 = h\, f(x_0 + h, y_0 + k_1)\\
&\textit{Then } (1) \textit{ becomes}\\
&y_1 = y_0 + \frac{1}{2}(k_1 + k_2), \text{which is second order R-K formula}
\end{aligned}
$$

Hence,

Second-order R-K formula is

$$
\begin{aligned}
&y_1 = y_0 + \frac{1}{2}(k_1 + k_2)\\
&\textit{where}\\
&k_1 = h\, f(x_0, y_0)\\
&k_2 = h\, f(x_0 + h, y_0 + k_1)
\end{aligned}
$$

Third order Runge-Kutta Method

The third order R-K formula is defined as

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where

$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = h\,f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \;\&$$

$$k_3 = h\,f(x_0 + h, y_0 + 2k_2 - k_1)$$

Fourth order Runge-Kutta Method

The fourth order R-K formula is given by

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = h\,f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$$

$$k_3 = h\,f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) \;\&$$

$$k_4 = h\,f(x_0 + h, y_0 + k_3)$$

Note

The R-K method of fourth order is the most commonly used in practice and it is often referred to as Runge-Kutta method or R-K method only without any reference to the order.

Solved Problems

1. Apply R-K method of second order, find the approximate value of y when $x = 1.1$, given $\frac{dy}{dx} = 3x + y^2, y(1) = 1.2$

Solution

Given

$$\frac{dy}{dx} = 3x + y^2 \qquad (1) \qquad y(1) = 1.2$$

Here $f(x, y) = 3x + y^2,\ x_0 = 1, y_0 = 1.2,\ h = 0.1$

$\therefore\ x_1 = 1.1$

By R-K method of second order, we have

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) \qquad (2)$$

where

$$k_1 = h\,f(x_0, y_0)$$
$$k_2 = h\,f(x_0 + h, y_0 + k_1)$$

Now

$$\begin{aligned} k_1 &= h\,f(x_0, y_0) \\ &= 0.1\left[3x_0 + y_0^2\right] \\ &= 0.1\left[3(1) + (1.2)^2\right] \\ &= 0.4444 \end{aligned}$$

$$\begin{aligned} k_2 &= h\,f(x_0 + h, y_0 + k_1) \\ &= h\,f\left(x_0 + 0.1, y_0 + 0.4444\right) \\ &= 0.1\,f\left(1.1, 1.644\right) \\ &= 0.600 \end{aligned}$$

$(2) \Rightarrow$

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \\ &= 1.2 + \frac{1}{2}(0.4444 + 0.600) \end{aligned}$$

$$y_1 = y(1.1) = 1.722$$

2. Apply R-K method of second order, compute $y(2.5)$ from $\frac{dy}{dx} = \frac{x+y}{2}$, $y(2) = 2$, taking $h = 0.25$.

Solution

Given

$$\frac{dy}{dx} = \frac{x+y}{2} \qquad (1) \qquad y(2) = 2$$

Here $f(x,y)=\dfrac{x+y}{2},\ x_0=2, y_0=2,\ h=0.25$

$\therefore\ x_1=2.25, x_2=2.5$

By R-K method of second order, we have

$$y_1=y_0+\frac{1}{2}(k_1+k_2) \qquad (2)$$

where

$$k_1=h\,f(x_0,y_0)$$
$$k_2=h\,f(x_0+h, y_0+k_1)$$

First, we have to evaluate y_1

Now

$$\begin{aligned}
k_1&=h\,f(x_0,y_0)\\
&=0.25\left[\frac{x_0+y_0}{2}\right]\\
&=0.25\left[\frac{2+2}{2}\right]\\
&=0.5
\end{aligned}$$

$$\begin{aligned}
k_2&=h\,f(x_0+h, y_0+k_1)\\
&=h\,f\left(2+0.25, 2+0.5\right)\\
&=0.25\,f\left(2.25, 2.5\right)\\
&=0.528
\end{aligned}$$

$(2)\Rightarrow$

$$\begin{aligned}
y_1&=y_0+\frac{1}{2}(k_1+k_2)\\
&=2+\frac{1}{2}(0.5+0.528)\\
y_1&=y(2.25)=2.514
\end{aligned}$$

Next, we have to evaluate y_2

Now

$$\begin{aligned}
k_1 &= h\, f(x_1, y_1) \\
&= 0.25\left[\frac{x_1 + y_1}{2}\right] \\
&= 0.25\left[\frac{2.25 + 2.514}{2}\right] \\
&= 0.5293 \\
k_2 &= h\, f(x_1 + h, y_1 + k_1) \\
&= h\, f\left(2.25 + 0.25, 2.514 + 0.5293\right) \\
&= 0.5543
\end{aligned}$$

$(2) \Rightarrow$

$$\begin{aligned}
y_2 &= y_1 + \frac{1}{2}\left(k_1 + k_2\right) \\
&= 2.514 + \frac{1}{2}\left(0.5293 + 0.5543\right) \\
y_2 &= y(2.5) = 3.0558
\end{aligned}$$

3. Apply R-K method of third order, compute $y(0.1)$ from $y^1 + y = 0, y(0) = 1$

Solution

Given

$$\begin{aligned}
&y^1 + y = 0 \\
&y^1 = -y \qquad (1) \qquad y(0) = 1
\end{aligned}$$

Here $f(x, y) = -y,\ x_0 = 0, y_0 = 1,\ h = 0.1$

$\therefore\ x_1 = 0.1, x_2 = 0.2$

By R-K method of third order, we have

$$y_1 = y_0 + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right) \qquad (2)$$

where

$$k_1 = h\, f(x_0, y_0)$$

$$k_2 = h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \ \&$$

$$k_3 = h\, f(x_0 + h, y_0 + 2k_2 - k_1)$$

Now

$$\begin{aligned} k_1 &= h\, f(x_0, y_0) \\ &= 0.1\left[-y_0\right] \\ &= 0.1\left[-1\right] \\ &= -0.1 \end{aligned}$$

$$\begin{aligned} k_2 &= h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \\ &= h\, f\left(0 + \frac{1}{2}(0.1), 1 + \frac{1}{2}(-0.1)\right) \\ &= 0.1\, f\left(0.05, 0.095\right) \\ &= 0.1(-0.095)528 \\ &= -0.095 \end{aligned}$$

$$\begin{aligned} k_3 &= h\, f(x_0 + h, y_0 + 2k_2 - k_1) \\ &= 0.1\, f(0 + 0.1, 1 + 2(-0.095) + 0.1 \\ &= -0.09 \end{aligned}$$

$(2) \Rightarrow$

$$\begin{aligned} y_1 &= y_0 + \frac{1}{6}\left(k_1 + 4k_2 + k_3\right) \\ &= 1 + \frac{1}{6}\left(-0.1 + 4(-0.095) - 0.09\right) \end{aligned}$$

$$y_1 = y(0.1) = 0.905$$

4. Apply R-K method, compute $y(0.2)$ from $y^1 = x + y, y(0) = 1$

Solution

Given

$y^1 = x + y \qquad (1) \qquad y(0) = 1$

Here $f(x, y) = x + y,\ x_0 = 0, y_0 = 1,\ h = 0.2$

$\therefore\ x_1 = 0.2, x_2 = 0.4$

By fourth order R-K formula, we have

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad (2)$$

where

$k_1 = h\ f(x_0, y_0)$

$k_2 = h\ f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1)$

$k_3 = h\ f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2)$ &

$k_4 = h\ f(x_0 + h, y_0 + k_3)$

Now

$$\begin{aligned} k_1 &= h\ f(x_0, y_0) \\ &= 0.2[x_0 + y_0] \\ &= 0.2[0 + 1] \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} k_2 &= h\ f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \\ &= h\ f\left(0 + \frac{1}{2}(0.2), 1 + \frac{1}{2}(0.2)\right) \\ &= 0.2\ f(0.1, 1.1) \\ &= 0.2(0.1 + 1.1) \\ &= 0.24 \end{aligned}$$

$$
\begin{aligned}
k_3 &= h\, f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2) \\
&= 0.2\, f(0 + \frac{1}{2}(0.2), 1 + \frac{1}{2}(0.24)) \\
&= 0.2440 \\
k_4 &= h\, f(x_0 + h, y_0 + k_3) \\
&= 0.2\, f(0 + 0.2, 1 + 0.2440) \\
&= 0.2888
\end{aligned}
$$

$(2) \Rightarrow$

$$
\begin{aligned}
y_1 &= y_0 + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\
&= 1 + \frac{1}{6}\left(0.20 + 2(0.24) + 2(0.2440) + 0.2888\right) \\
&= 1.2428
\end{aligned}
$$

Hence

$y_1 = y(x_1) = y(0.2) = 1.2428$

Exercise Problems

1. Apply R-K method, compute $y(0.2)\,\&\,\mathrm{y}(0.4)$ from $\frac{dy}{dx} = \mathrm{x} + \mathrm{y}^2, y(0) = 1$
2. Using R-K method of fourth order, solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ *with* $y(0) = 1 \text{at } x = 0.2, 0.4$

Predictor-Corrector Methods

So far we have discussed many numerical methods for obtaining numerical solution of

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \qquad (1)$$

All the earlier methods require information only from the last computed point (x_i, y_i) to estimate the next point (x_{i+1}, y_{i+1}). Therefore, all these methods are single-step methods. They do not make use of the information available at the earlier steps $y_{i-1}, y_{i-2}, \ldots$ even when they are available. It is possible to improve the efficiency of estimation by using the information at several previous steps.

Methods that use information from more than one previous point to compute the next point are called multi-step methods. Sometimes, a pair of multistep methods are used in conjunction with each other, one for predicting the value of the estimate *i.e.* y_{i+1} and other for correcting the predicted value of y_{i+1}. Such methods are called predictor-corrector methods.

For instance, for solving (1), we used Euler's formula

$$y_{n+1} = y_n + h\, f(x_n, y_n), \;\; n = 0,1,2,... \qquad (2)$$

We improved this value by modified Euler's method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(1)}) \right] \qquad (3)$$

Here, we obtained initially a crude estimate of y_{i+1} and subsequently refined it by means of more accurate formula. This method is called predictor-corrector method.

As name suggests, we first predict a value of y_{n+1}(here y_{n+1}^{1}) by using certain formula and then correct this value by using a different formula. Hence (2) is used as the predictor and (3) is used as the corrector formula. A predictor formula is used to predict the value of $y_{n+1}\; at\; x_{n+1}$ and a corrector formula is used to correct the error and to improve the value of y_{n+1}

In the predictor-corrector methods, four previous values are needed i.e. $y_0, y_1, y_2, \& \, y_3$ for finding the values of $y\; at\; x_n$. These values are known as starting values, which can be obtained by using any one of the previous methods discussed earlier.

We have two popular predictor-corrector methods, namely, Adams-Bashforth- Moulton method and Milne's method.

Note

Only one value is needed for Taylor's , Picard's, Euler's & R-K methods. But, to get better approximation, predictor-corrector methods need four approximations.

Milne's Predictor-Corrector Method

To solve the equation $\frac{dy}{dx} = f(x,y), y(x_0) = y_0$ (1) numerically, starting from y_0, we have to estimate $y_1, y_2, \& y_3$ successively by using either Taylor's or Picard's or Euler's method.

The Milne's predictor formula is given by

$$y_{n+1}^{p} = y_{n-3} + \frac{4h}{3}\left(2y_{n-2}^{1} - y_{n-1}^{1} + 2y_{n}^{1}\right) \qquad (2)$$

Here, the superscript 'p' indicates that y_{n+1}^{p} is a predicted value.

The Milne's corrector formula is given by

$$y_{n+1}^{c} = y_{n-1} + \frac{h}{3}\left(y_{n-1}^{1} + 4y_{n}^{1} + y_{n+1}^{1p}\right) \qquad (3)$$

Here, the superscript 'c' indicates that y_{n+1}^{c} is a corrected value.

Note:

If in any problem, the necessary estimates $y_0, y_1, y_2, \& y_3$ are not given, then we can find them using Picard's, Taylor's, Euler's, or R-K Methods.

Solved Problems

1. Use Milne's predictor-corrector method to obtain the solution of $y^1 = x - y^2$ at $x = 0.8$, given that $y(0) = 0, y(0.2) = 0.02, y(0.4) = 0.0795, y(0.6) = 0.1762$.

Solution

Given

$y^1 = x - y^2$ (1)

$\therefore f(x,y) = x - y^2$

$x_0 = 0, y_0 = 0, h = 0.2$

$x_1 = 0.2, y_1 = 0.02$

$x_2 = 0.4, y_2 = 0.0795$

$x_3 = 0.6, y_3 = 0.1762$

$x_4 = 0.8, \; y_4 = ?.$

By Milne's predictor formula, we have

$$y_{n+1}^{p} = y_{n-3} + \frac{4h}{3}\left(2y_{n-2}^{1} - y_{n-1}^{1} + 2y_{n}^{1}\right) \qquad (2)$$

Take $n = 3 \;\; in \;\; (2)$

$$y_4^p = y_0 + \frac{4h}{3}\left(2y_1^1 - y_2^1 + 2y_3^1\right) \qquad (3)$$

From (1), we have

$$y^1 = x - y^2$$
$$y_1^1 = x_1 - y_1^2 = 0.2 - (0.02)^2 = 0.1996$$
$$y_2^1 = x_2 - y_2^2 = 0.4 - (0.0795)^2 = 0.3937$$
$$y_3^1 = x_3 - y_3^2 = 0.6 - (0.1762)^2 = 0.5689$$

$(3) \Rightarrow$

$$y_4^p = 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.5689)\right]$$
$$= 0.30488$$

Now

$$y_4^{1^p} = x_4 - y_4^2 = 0.8 - (0.30488)^2 = 0.7070$$

By Milne's corrector formula, we have

$$y_{n+1}^{c} = y_{n-1} + \frac{h}{3}\left(y_{n-1}^{1} + 4y_{n}^{1} + y_{n+1}^{1^p}\right) \qquad (4)$$

Take $n = 3$ in (4), we get

$$y_4^c = y_2 + \frac{h}{3}\left(y_2^1 + 4y_3^1 + y_4^{1^p}\right)$$
$$= 0.0795 + \frac{0.2}{3}\left(0.3937 + 4(0.5089) + 0.7070\right)$$
$$= 0.3046$$

Hence, the approximate value of $y_4 = y(0.8) = 0.3046$

2. Use Euler's method to find $y(0.1), y(0.2), \& \, y(0.3)$, given that $y^1 = (x^3 + xy^2)e^{-x}, y(0) = 1$. Compute $y(0.4) = 1$ using Milne's method.

Solution

Given

$$\frac{dy}{dx} = (x^3 + xy^2)e^{-x} \qquad (1), \mathrm{y}(0) = 1$$

Here $f(x, y) = (x^3 + xy^2)\mathrm{e}^{-x}$, $x_0 = 0, y_0 = 1 \;\&\; h = 0.1$

We know that by Euler's method

$$y(x_{n+1}) = y_{n+1} = y_n + h\, f(x_n, y_n), \;\; \mathrm{n} = 0,1,2,... \qquad (2)$$

Since $x_n = x_{n-1} + h, \;\; n = 1, 2, 3, ...$

$$\begin{aligned}
x_1 &= x_0 + h = 0 + 0.1 = 0.1,\\
x_2 &= x_1 + h = 0.1 + 0.1 = 0.2\\
x_3 &= x_2 + h = 0.2 + 0.1 = 0.3\\
x_4 &= x_3 + h = 0.3 + 0.1 = 0.4
\end{aligned}$$

$(2) \Rightarrow$

$$\begin{aligned}
\mathrm{y}_{n+1} &= y_n + h\, f(x_n, y_n)\\
&= y_n + 0.1(x_n^3 + x_n\, \mathrm{y}_n^2)\,\mathrm{e}^{-x_n} \qquad (3)
\end{aligned}$$

Take $n = 0$ in (3), we get

$(3) \Rightarrow$

$$\begin{aligned}
\mathrm{y}_1 &= y_0 + h\, f(x_0, y_0)\\
&= 1 + 0.1(x_0^3 + x_0 y_0^2)\,\mathrm{e}^{-x_0}\\
&= 1 + 0.1((0)^3 + (0)(1)^2)\,\mathrm{e}^{-0}\\
&= 1
\end{aligned}$$

$i.e.\; y(x_1) = y(0.1) = y_1 = 1$

Take $n = 1$ in (3), we get

$(3) \Rightarrow$

$$\begin{aligned} y_2 &= y_1 + h\, f(x_1, y_1) \\ &= 1 + 0.1(x_1^3 + x_1 y_1^2)\, e^{-x_1} \\ &= 1 + 0.1((0.1)^3 + (0.1)(1)^2)\, e^{-0.1} \\ &= 1.0091 \end{aligned}$$

i.e. $y(x_2) = y(0.2) = y_2 = 1.0091$

Take $n = 2$ in (3), we get

$(3) \Rightarrow$

$$\begin{aligned} y_3 &= y_2 + h\, f(x_2, y_2) \\ &= 1 + 0.1(x_2^3 + x_2 y_2^2)\, e^{-x_2} \\ &= 1 + 0.1((0.2)^3 + (0.2)(1.0091)^2)\, e^{-0.2} \\ &= 1.0173 \end{aligned}$$

i.e. $y(x_3) = y(0.3) = y_3 = 1.0173$

Hence

$x_0 = 0, y_0 = 1, h = 0.1$
$x_1 = 0.1, y_1 = 1$
$x_2 = 0.2, y_2 = 1.0091$
$x_3 = 0.3, y_3 = 1.0173$
$x_4 = 0.4,\ y_4 = ?.$

By Milne's predictor formula, we have

$$y_{n+1}^p = y_{n-3} + \frac{4h}{3}\left(2y_{n-2}^1 - y_{n-1}^1 + 2y_n^1\right) \qquad (2)$$

Take $n = 3$ in (2)

$$y_4^p = y_0 + \frac{4h}{3}\left(2y_1^1 - y_2^1 + 2y_3^1\right) \qquad (3)$$

From (1), we have

$$y^1 = (x^3 - xy^2)e^{-x}$$
$$y_1^1 = (x_1^{\ 3} - x_1 y_1^{\ 2})e^{-x_1} = ((0.1)^3 - (0.1)(1)^2)e^{-0.1} = -0.0894$$
$$y_2^1 = (x_2^{\ 3} - x_2 y_2^{\ 2})e^{-x_2} = ((0.2)^3 - (0.2)(1.0091)^2)e^{-0.2} = -0.1601$$
$$y_3^1 = (x_3^{\ 3} - x_3 y_3^{\ 2})e^{-x_3} = ((0.3)^3 - (0.3)(1.0173)^2)e^{-0.3} = -0.2099$$

$(3) \Rightarrow$

$$y_4^p = 1 + \frac{4(0.1)}{3}\left[2(-0.0894) + 0.1601 + 2(-0.2099)\right]$$
$$= 0.9415$$

(Rest of the process is left to the student)

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