SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES, CHITTOOR
(AUTONOMOUS)
Approved by AICTE, New Delhi, Affiliated to JNTUA, Ananthapuramu.

LECTURE NOTES

Subject Name : Discrete Mathematics and Graph Theory

Year / Branch : II/CSE,CAI,CSM,CDS

Regulation : R23

Prepared By :Dr. P. Madhu Mohan Reddy
Mrs. P. Radhiga
Ms. A. Poojitha Reddy

II B Tech- III Semester

23BSC232	**DISCRETE MATHEMATICS & GRAPH THEORY**	**L**	**T**	**P**	**C**
	Common to CSE/CSD/CSM/CAI	**3**	**1**	**-**	**4**

PRE-REQUISITES: **Set Theory, Permutations and Combinations, Calculus, Linear Algebra**

COURSE EDUCATIONAL OBJECTIVES:

1. To gain the knowledge on connectives and relate the laws of logic to find the disjunctive normal form and conjunctive normal form of compound proposition and various concepts related to predicate logic.
2. To perform the operations associated with sets, functions, and relations and study the basic Properties of lattices and the concept of groups, Abelian groups and group homomorphism and isomorphism.
3. To develop skill to analyze the concepts of Permutations, Combinations, Binomial and Multinomial Theorems
4. To develop skill to apply the concept of Recurrence relations and Generating Functions in various computer programming languages.
5. To study the fundamentals of graphs, sub graphs, planar graphs, Hamiltonian graphs, Euler graphs, Spanning trees and graph traversals.

UNIT I: Mathematical Logic 09

Introduction, Statements and Notation, Connectives, Well-formed formulas, Tautology, Duality law, Equivalence, Implication, Normal Forms, Functionally complete set of connectives, Inference Theory of Statement Calculus, Predicate Calculus, Inference theory of Predicate Calculus.

UNIT II: Set theory 09

The Principle of Inclusion- Exclusion, Pigeon hole principle and its application, Functions-composition of functions, Inverse Functions, Recursive Functions, Lattices and its properties. Algebraic structures: Algebraic Systems-Examples and General Properties, Semi groups and Monoids, groups, sub groups, homomorphism, Isomorphism.

UNIT III: Elementary Combinatorics 09

Combinations and Permutations, Enumeration of Combinations and Permutations, Enumerating Combinations and Permutations with Repetitions, Enumerating Permutations with Constrained Repetitions, Binomial Coefficients, The Binomial and Multinomial Theorems.

UNIT IV: Recurrence Relations 09

Generating Functions of Sequences, Calculating Coefficients of Generating Functions, Recurrence relations, Solving Recurrence Relations by Substitution and Generating functions, The Method of Characteristic roots, Solutions of Inhomogeneous Recurrence Relations.

UNIT V: Graphs 09

Basic Concepts, Isomorphism and Subgraphs, Trees and their Properties, Spanning Trees, Directed Trees, Binary Trees, Planar Graphs, Euler's Formula, Multigraphs and Euler Circuits, Hamiltonian Graphs.

COURSE OUTCOMES:

On successful completion of the course, students will be able to		Pos
CO1	Apply mathematical logic to solve problems.	PO1, PO2, PO3
CO2	Understand the concepts and perform the operations related to sets, relations and functions. Gain the conceptual background needed and identify structures of algebraic nature.	PO1, PO2, PO3
CO3	Apply basic counting techniques to solve combinatorial problems.	PO1, PO2,PO3
CO4	Formulate problems and solve recurrence relations.	PO1, PO2, PO3
CO5	Apply Graph Theory in solving computer science problems	PO1, PO2, PO3

TEXT BOOKS:

1. J.P. Tremblay and R. Manohar, Discrete Mathematical Structures with Applications to Computer Science, Tata McGraw Hill, 2002.
2. Kenneth H. Rosen, Discrete Mathematics and its Applications with Combinatorics and Graph Theory, 7th Edition, McGraw Hill Education (India) Private Limited.

REFERENCE BOOKS:

1. Joe L. Mott, Abraham Kandel and Theodore P. Baker, Discrete Mathematics for Computer Scientists & Mathematicians, 2nd Edition, Pearson Education.
2. NarsinghDeo, Graph Theory with Applications to Engineering and Computer Science.

ONLINE LEARNING RESOURCES:

1. http://www.cs.yale.edu/homes/aspnes/classes/202/notes.pdf

CO-PO MAPPING:

CO-PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO.1	3	3	3	-	-	-	-	-	-	-	-	-
CO.2	3	3	3	-	-	-	-	-	-	-	-	-
CO.3	3	3	3	-	-	-	-	-	-	-	-	-
CO.4	3	3	3	-	-	-	-	-	-	-	-	-
CO.5	3	3	3	-	-	-	-	-	-	-	-	-
CO*	3	3	3	-	-	-	-	-	-	-	-	-

UNIT-1: Mathematical Logic

Connectives

Connectives are logical operators that are used to connect several statements or propositions into a single statement.

Types of Connectives

1. Negation
2. Conjunction
3. Disjunction
4. Implication
5. Bi-implication
6. Exclusive Disjunction

1. Negation (¬P or ~P)

Negation is the opposite truth value of a given statement or proposition.

Truth Table

P	**~P**
T	F
F	T

2. Conjunction (∧)

A **compound proposition** obtained by combining two given propositions by inserting the word **"AND"** between them is called **Conjunction**.

Truth Table

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

3. Disjunction (∨)

A **compound proposition** obtained by combining two given propositions by inserting the word **"OR"** between them is called **Disjunction**.

Truth Table

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

4. Implication (→)

A **compound proposition** is obtained by combining two given propositions by inserting the words **"If … then"** between them. This is called **Implication**.

Truth Table

P	Q	P → Q
T	T	T
T	F	F
F	T	T
F	F	T

5. Bi-Implication (⇔)

If **P** and **Q** are two propositions, then the conjunction of the conditions **P → Q** and **Q → P** is called **Bi-Implication** of **P** and **Q**.

Truth Table

P	Q	P ⇔ Q
T	T	T
T	F	F
F	T	F
F	F	T

6. Exclusive Disjunction (XOR) (⨁)

Symbols: XOR, ⨁, ⊻

If **P** and **Q** are any two propositions, then the compound proposition **P ⨁ Q** is called **Exclusive Disjunction**.

Truth Table

P	**Q**	**P ⊕ Q**
T	T	F
T	F	T
F	T	T
F	F	F

Statement

A **statement** is a sentence that says something is **true** or **false**. Generally, it is denoted by small letters like **P, Q, R, S, X, Y, …**

State which of the following is a statement (or proposition)

1. A triangle contains three lines.

Answer: True

2. X + 2 is a positive value.

Answer: Not a statement

3. Five divides X.

Answer: Not a statement

4. India had a woman Prime Minister.

Answer: True

Identify the Negation of the Following

1. 2 + 3 = 5

Negation: $2 + 3 \neq 5$

2. 5 divides 27

Negation: 5 doesn`t divide 27

3. Computer Science is a hard subject.

Negation: Computer Science is **not** a hard subject.

Consider the following statements related to a certain triangle.

Let,

- **P:** ABC is **isosceles**.
- **Q:** ABC is **equilateral**.
- **R:** ABC is **equiangular**.

Write down the following compound propositions in words.

1. $P \wedge (\sim Q)$**:**
   ABC is **isosceles AND ABC is not equilateral**.
2. $P \vee (\sim Q)$**:**
   ABC is **isosceles OR ABC is not equilateral**.
3. $\sim R \rightarrow \sim Q$**:**
   If ABC is not equiangular, then ABC is not equilateral.
4. **Indicate the truth value of each of the following.**

   1. If $\mathbf{3+4=7}$, then $\mathbf{5+3=8}$.

   Solution:

$$T \rightarrow T$$
$$= T$$

Truth Value: True (T)

Indicate the truth value of each of the following.

1) **If $\mathbf{3+4=7}$, then $\mathbf{5+3=8}$.**

Solution:

$$T \rightarrow T$$
$$= T$$

Truth Value: True (T)

From the information given, determine the required truth value.
1. $\boldsymbol{P \wedge Q}$is False and $\boldsymbol{P}$is True. Find $\boldsymbol{Q}$.
Solution:

$$P \wedge Q = T \wedge F$$
$$= F$$

∴ Q is False.
2. $\boldsymbol{P \vee Q}$is False and $\boldsymbol{Q}$is False. Find $\boldsymbol{P}$.
Solution:

$$P \vee Q = F \vee F$$
$$= F$$

∴ P is False.

3. $P \rightarrow Q$is True and $Q$is False. Find the truth value of P.
Solution:

$$P \rightarrow Q = F \rightarrow F$$
$$= T$$

∴ P is False.

Given that $P$is True and $Q$is False, find the truth value of the following.
1. $\sim P \wedge Q$

Solution:

$$=\sim T \wedge F$$
$$= F \wedge F$$
$$= F$$

Truth Value: False (F)
2. $\sim (P \rightarrow\sim Q)$
Solution:

$$=\sim [P \rightarrow\sim Q]$$
$$=\sim [T \rightarrow\sim F]$$
$$=\sim [T \rightarrow T]$$
$$=\sim [T]$$
$$= F$$

Truth Value: False (F)
3. $(P \wedge Q) \rightarrow (P \vee Q)$
Solution:

$$P \wedge Q = T \wedge F = F$$
$$P \vee Q = T \vee F = T$$
$$(P \wedge Q) \rightarrow (P \vee Q)$$
$$= F \rightarrow T$$
$$= T$$

Truth Value: True (T)

4. $\sim (P \wedge Q) \vee [\sim (Q \Leftrightarrow P)]$
Solution:

$$\sim (P \wedge Q) =\sim F = T$$
$$\sim (Q \Leftrightarrow P) =\sim (F \Leftrightarrow T)$$
$$=\sim (F) = T$$
$$\sim (P \wedge Q) \vee [\sim (Q \Leftrightarrow P)]$$
$$= T \vee T$$
$$= T$$

Tautology
A **compound proposition** that is **true for all possible truth values** of its component propositions is called **Tautology**.

Contradiction
A **compound proposition** that is **false for all possible truth values** of its component propositions is called **Contradiction**. It is denoted by **F**.
Contingency
A **compound proposition** that can be **either true or false** is called a **Contingency**.
Construct the Truth Table of the Following
1. $\sim \boldsymbol{P} \vee \sim \boldsymbol{Q}$

P	**Q**	**~P**	**~Q**	**~P ∨ ~Q**
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

iii. Construct the Truth Table for $P \vee \sim Q$

P	**Q**	**~Q**	**P ∨ ~Q**
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

iv. Construct the Truth Table for $(P \vee Q) \vee \sim P$

P	**Q**	**~P**	**P ∨ Q**	**(P ∨ Q) ∨ ~P**
T	T	F	T	T
T	F	F	T	T
F	T	T	T	T
F	F	T	F	T

v. Construct the Truth Table for $(P \rightarrow Q) \wedge (\sim P \rightarrow Q)$

P	Q	$P \rightarrow Q$	$\sim P \rightarrow Q$	$(P \rightarrow Q) \land (\sim P \rightarrow Q)$
T	T	T	F	F
T	F	F	T	F
F	T	T	F	F
F	F	T	F	F

Definitions

Converse

If P and Q are any two propositions and consider a compound proposition $P \rightarrow Q$, then $Q \rightarrow P$ is called the Converse.

Inverse

If P and Q are any two propositions and consider a compound proposition $P \rightarrow Q$, then $\neg P \rightarrow \neg Q$ is called the Inverse.

Contrapositive

If P and Q are any two propositions and consider a compound proposition $P \rightarrow Q$, then $\neg Q \rightarrow \neg P$ is called the Contrapositive.

Example

Let:

- P: n is positive.
- Q: n^3 is positive.

A. Converse

If n^3 is positive, then n is positive.

B. Inverse

If n is not positive, then n^3 is not positive.

C. Contrapositive

If n^3 is not positive, then n is not positive.

Dual

If a statement is given, then its dual is obtained by:

- Replacing ∧ (AND) with ∨ (OR).
- Replacing ∨ (OR) with ∧ (AND).
- Replacing T (True) with F (False).
- Replacing F (False) with T (True).

Example

Given statement:

U : $[(P \wedge Q) \vee R] \wedge T$

Its dual is:

U^d : $[(P \vee Q) \wedge R] \vee F$

Principle of Duality

For any two statements U and V,

If U ⇔ V

Then $U^d \Leftrightarrow V^d$

is called the Principle of Duality.

Example

$$U \Leftrightarrow V\colon P \wedge (P \vee T) \Leftrightarrow Q \wedge (P \vee F)$$
$$U^d \Leftrightarrow V^d\colon P \vee (P \wedge F) \Leftrightarrow Q \vee (P \wedge T)$$

De Morgan's Law

If P and Q are two statements, then

1. $\neg(\boldsymbol{P} \wedge \boldsymbol{Q}) = \neg\boldsymbol{P} \vee \neg\boldsymbol{Q}$
2. $\neg(\boldsymbol{P} \vee \boldsymbol{Q}) = \neg\boldsymbol{P} \wedge \neg\boldsymbol{Q}$

These are called De Morgan's Laws.

Example

Write the negation of the following statement:

Statement: I want a car and a Worthy cycle.

Let,

- **P: I want a car.**
- **Q: I want a Worthy cycle.**

Therefore,

$$P \land Q$$

Its negation is

$$\neg(P \land Q)$$

Using De Morgan's Law,

$$\neg(P \land Q) = \neg P \lor \neg Q$$

Hence, the negation is:

"I do not want a car or I do not want a Worthy cycle."

(ii) Statement:

My cat stays outside or it makes a mess.

Let,

- **P: My cat stays outside.**
- **Q: It makes a mess.**

Therefore,

$$P \lor Q$$

Its negation is

$$\neg(P \lor Q)$$

Using De Morgan's Law,

$$\neg(P \lor Q) = \neg P \land \neg Q$$

Hence, the negation is:

My cat does not stay outside and it does not make a mess.

(iii) Statement:

I have fallen and I cannot get up.

Let,

- **P: I have fallen.**
- **Q: I cannot get up.**

Therefore,

$$\boldsymbol{P \wedge Q}$$

Its negation is

$$\neg(\boldsymbol{P \wedge Q})$$

Using De Morgan's Law,

$$\neg(\boldsymbol{P \wedge Q}) = \neg\boldsymbol{P} \vee \neg\boldsymbol{Q}$$

Hence, the negation is:

I have not fallen or I can get up.

Laws of Logic

1. Idempotent Law

(i) $P \vee P \Leftrightarrow P$

(ii) $P \wedge P \Leftrightarrow P$

2. Associative Law

(i) $(P \vee Q) \vee R \Leftrightarrow P \vee (Q \vee R)$

(ii) $(P \wedge Q) \wedge R \Leftrightarrow P \wedge (Q \wedge R)$

3. Commutative Law

(i) $P \vee Q \Leftrightarrow Q \vee P$

(ii) $P \wedge Q \Leftrightarrow Q \wedge P$

4. Absorption Law

(i) $P \vee (P \wedge Q) \Leftrightarrow P$

(ii) $P \wedge (P \vee Q) \Leftrightarrow P$

5. Distributive Law

(i) $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$

(ii) $P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$

6. De Morgan's Law

(i) $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
(ii) $\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$

Some Other Equivalent Formulae

(i) $P \vee F \Leftrightarrow P$
(ii) $P \vee T \Leftrightarrow T$
(iii) $P \vee \neg P \Leftrightarrow T$
(iv) $P \wedge T \Leftrightarrow P$

(iv) $P \wedge F \Leftrightarrow F$
(v) $P \wedge \neg P \Leftrightarrow F$
(vi) $(P \rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ and $(\neg P \rightarrow Q) \Leftrightarrow (P \vee Q)$
(vii) $(P \Leftrightarrow Q) \Leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)]$

Implications

I_1: $P \wedge Q \Rightarrow P$ **[Simplification]**

I_2: $P \wedge Q \Rightarrow Q$ **[Simplification]**

I_3: $P \Rightarrow P \vee Q$ **[Addition]**

I_4: $Q \Rightarrow P \vee Q$ **[Addition]**

I_5: $\sim P \Rightarrow (P \Rightarrow Q)$

I_6: $Q \Rightarrow (P \Rightarrow Q)$

I_7: $\sim (P \Rightarrow Q) \Rightarrow P$

I_8: $\sim (P \Rightarrow Q) \Rightarrow \sim Q$

I_9: $P, Q \Rightarrow P \wedge Q$

I_{10}: $\sim P, (P \vee Q) \Rightarrow Q$

I_{11}: $P, (P \Rightarrow Q) \Rightarrow Q$ **[Modus Ponens]**

I_{12}: $\sim Q, (P \Rightarrow Q) \Rightarrow \sim P$ **[Modus Tollens]**

I_{13}: $P \Rightarrow Q, Q \Rightarrow R \Rightarrow P \Rightarrow R$ **[Hypothetical Syllogism]**

I_{14}: $P \vee Q, P \Rightarrow R, Q \Rightarrow R \Rightarrow R$ **[Dilemma]**

Equivalence

E_1: $\sim\sim P \Leftrightarrow P$ **[Double Negation]**

Normal Form

Let $A(P_1, P_2, \dots, P_n)$be the statement formula, where $P_1, P_2, \dots, P_n$are the atomic variables. The truth table of the formula Acontains 2^nrows. Construction of truth tables for such formulas may not be practical.
There are two types of normal forms:

1. **Disjunctive Normal Form (DNF)**
2. **Conjunctive Normal Form (CNF)**

DNF (Disjunctive Normal Form)
A formula which is equivalent to the given formula and which consists of the **sum of elementary products** is called the **Disjunctive Normal Form (DNF)**.
The normal form of the given formula is:

$$(\wedge) \vee (\wedge) \vee (\wedge) \vee \cdots$$

CNF (Conjunctive Normal Form)
A formula which is equivalent to a given formula and which consists of the **product of elementary sums** is called the **Conjunctive Normal Form (CNF)**.
The normal form of the given formula is:

$$(\vee) \wedge (\vee) \wedge (\vee) \wedge \cdots$$

PDNF (Principal Disjunctive Normal Form)
Given a compound proposition involving two or more simple propositions, the **Principal Disjunctive Normal Form (PDNF)** is obtained by expressing the proposition as a disjunction (OR) of minterms corresponding to the truth table rows for which the proposition is **True**.

PDNF (Principal Disjunctive Normal Form)
An equivalent compound proposition consisting of the **disjunction of the minterms** involving only the variables $P$and $Q$is known as **PDNF (Principal Disjunctive Normal Form)** or **Sum of Products (Canonical Form)**.

PCNF (Principal Conjunctive Normal Form)
Given a compound proposition Uinvolving two simple propositions $P$and Q, an equivalent compound proposition consisting of the **conjunction of the maxterms** involving only $P$and $Q$is known as **PCNF (Principal Conjunctive Normal Form)** or **Product of Sums (Canonical Form)**
Minterms and Maxterms
Given two simple propositions $P$and Q, the compound propositions

- $P \wedge Q$
- $P \wedge \sim Q$
- $\sim P \wedge Q$
- $\sim P \wedge \sim Q$

are called **minterms** involving $P$and Q.
Similarly, the compound propositions

-
- $P \vee Q$
- $P \vee \sim Q$
- $\sim P \vee Q$
- $\sim P \vee \sim Q$

are called **maxterms**.

Truth Table

P	**Q**	**~P**	**~Q**	**P∧Q**	**~P∧Q**	**P∧~Q**	**~P∧~Q**
T	T	F	F	T	F	F	F
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	F
F	F	T	T	F	F	F	T

Problems

1. Obtain Principal Disjunctive Normal Form (PDNF) of $P \rightarrow Q$

Truth Table

P	**Q**	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

The formula is **True** in the 1st, 3rd, and 4th rows.

- In the **1st row**, the minterm is: $P \wedge Q$
- In the **3rd row**, the minterm is: $\sim P \wedge Q$
- In the **4th row**, the minterm is: $\sim P \wedge \sim Q$

Hence, the **PDNF** is

$$(P \wedge Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$$

2. Find the Principal Conjunctive Normal Form (PCNF) of $P \wedge (P \rightarrow Q)$

Truth Table

P	**Q**	$P \rightarrow Q$	$P \wedge (P \rightarrow Q)$
T	T	T	T
T	F	F	F
F	T	T	F
F	F	T	F

The formula is **False** in the 2nd, 3rd, and 4th rows.

- In the **2nd row**, the maxterm is: $\sim P \vee Q$
- In the **3rd row**, the maxterm is: $P \vee \sim Q$
- In the **4th row**, the maxterm is: $P \vee Q$

Hence, the **PCNF** is

$$(\sim P \vee Q) \wedge (P \vee \sim Q) \wedge (P \vee Q)$$

Predicate Calculus (Second Half)

Predicate

A predicate is a declarative sentence that describes the properties of an object or the relation among objects and variables. A predicate is generally represented by capital letters, such as:

P(x) , A(x) , Q(x) , ……………

Example

Consider the following two statements:

Ram is a bachelor.
Arun is a bachelor.
In the above example:

Bachelor is the predicate.
Ram and Arun are the objects.

Quantifiers

Words such as all, every, each, some, whole, at least, and little, which are associated with the idea of quantity, are called quantifiers.

There are two types of quantifiers:

Existential Quantifier (∃)
Universal Quantifier (∀)

Universal Quantifiers

The Universal quantifier 'All' is called the universal quantifier and it is denoted by \(\forall \). The symbol \(\forall \) [for 'All'] represents the following phrases which have the same meaning:

i) For all
ii) For each
iii) For every
iv) Everything
v) No

Existential Quantifiers

The quantifier 'some' is called the existential quantifier and it is denoted by \(\exists \) [at least]. The symbol \(\exists \) represents the following phrases which have the same meaning:

i) There exist
ii) There is an
iii) There is at least
iv) Few
v) A little

Free Variable

An occurrence of a variable that is not bounded by a quantifier is said to be free variable.

Bound Variable

An occurrence of a variable that is bounded by either a universal (∀) or existential (∃) quantifier is said to be a bound variable.

Problems

1. Write the following arguments with quantifiers.

(i) All birds can fly.

Let,

$P(x)$: x is a bird.

$Q(x)$: x can fly.

$\therefore \forall x$, $[P(x) \rightarrow Q(x)]$

(ii) All real numbers are complex numbers.

Let,

$P(x)$: x is a real number.

$Q(x)$: x is a complex number.

$\therefore \forall x$, $[P(x) \rightarrow Q(x)]$

(iii) All cats are black.

Let,

$P(x)$: x is a cat.

$Q(x)$: x is black.

$\therefore \forall x$, $[P(x) \rightarrow Q(x)]$

(iv) Some integers are divisible by 5.

Let,

$P(x)$: x is an integer.

$Q(x)$: x is divisible by 5.

$\therefore \exists x$, $[P(x) \wedge Q(x)]$

2) Write negations of the following:

(i) If all triangles are right angle then no triangle is equiangular.

Let,

$P(x)$: x is right angle triangle.

$Q(x)$: x is not equiangular.

∴ $\forall x$, $[P(x) \rightarrow Q(x)]$

The negation is

$\sim[\forall x\ [P(x) \rightarrow Q(x)]]$

$\sim[\forall x, \sim(P(x) \rightarrow Q(x))] \Rightarrow \sim(P \rightarrow Q) \equiv P \wedge \sim Q$

∴ $\exists x$, $[P(x) \wedge \sim Q(x)]$

(ii) $\exists x$, $[P(x) \wedge Q(x)]$

The negation is

$\sim[\exists x\ [P(x) \wedge Q(x)]]$

∴ $\forall x$, $(\sim P(x) \vee \sim Q(x))$

Theory of Inference for Predicate Calculus

Along with Rule P, Rule T, and Rule CP of Statement Calculus. We use the statement calculus , we use the rules refered to the following:

Rules of Statement Calculus

We use the statement calculus. We use the rules applied to the quantifiers $\forall x$ and $\exists x$, and they are given below.

Rule US (Universal Specification):

For $\forall x\ [P(x) \rightarrow Q(x)]$,

we conclude that

$P(a) \rightarrow Q(a)$

Rule ES (Existential Specification):

For $\exists x\ [P(x) \wedge Q(x)]$,

we conclude that $P(a) \wedge Q(a)$

Definition:

1. Modus Ponens:

For any two propositions P and Q, if P is true and $P \rightarrow Q$ is true, then Q is true.

Symbolic Form:

P

P → Q

∴ Q

Example:

If Dhoni hits a century, then he gets a free car.

If Dhoni hits a century.

∴ Dhoni gets a free car.

Predicate Form:

Let

P(x): x hits a century

Q(x): x gets a free car

P(x)

P(x) → Q(x)

∴ Q(x)

∴ By Modus Ponens, the argument is valid.

2. Modus Tollens:

For any two statements P and Q, if the negation of Q is true and P → Q is true, then the negation of P is true.

Symbolic form:

~Q

P → Q

∴ ~ P

Example:

Dhoni doesn't get a free car.

If Dhoni hits a century, then he gets a free car.

Therefore, Dhoni doesn't hit a century.

Let

P(x): x gets a free car

Q(x): x hits a century

$\sim P(x)$

$Q(x) \rightarrow P(x)$

$\therefore \sim Q(x)$

3) Syllogism

Syllogism states that for any three propositions P, Q, and R, if P implies Q and Q implies R, then P implies R.

Symbolic form:

$P \rightarrow Q$

$Q \rightarrow R$

$\therefore P \rightarrow R$

Example:

If x is a triangle, then triangle is equiangular.

If x is equiangular, then it has three sides.

∴ If x is a triangle, then it has three sides.

Let:

P(x): x is a triangle

Q(x): x is equiangular

R(x): x has three sides

$P(x) \rightarrow Q(x)$

$Q(x) \rightarrow R(x)$

$\therefore P(x) \rightarrow R(x)$

∴ By syllogism, the argument is valid.

4) Rule of Conjunctive Simplification:

For any two propositions P and Q, if $P \wedge Q$ is true, then P is true.

Symbolic form:

$P \wedge Q$

$\therefore P$

5) Rule of Disjunctive Simplification

For any two statements P and Q, if P is true then $P \vee Q$ is true.

Symbolic form:

$P = P \vee Q$

P

$\therefore P \vee Q$

6) Rule of Disjunctive Syllogism

For any two propositions P and Q, if $\neg P$ is true and $P \vee Q$ is true, then Q is true. This is called Disjunctive Syllogism.

Symbolic form:

$P \vee Q, \neg P \therefore Q$

or

$P \vee Q, \neg Q \therefore P$

Example (Validity using syllogism and modus ponens):

1) Prove that $P \rightarrow Q$, $Q \rightarrow R$, $P \rightarrow R$ then R is valid

S.No	Assertion	Reason
1	$P \rightarrow Q$	Premise 1
2	$Q \rightarrow R$	Premise 2
3	$P \rightarrow R$	From (1) and (2), Syllogism
4	P	Premise 3
5	R	From (3) and (4), Modus Ponens

Therefore, the argument is valid.

2) Check the validity of: $P \rightarrow Q, P \rightarrow R, P \vee Q$

S.No	Assertion	Reason
1	$P \rightarrow Q$	Premise 1
2	P	Premise 2
3	Q	Modus ponen
4	$P \rightarrow R$	Premise 3

It is not valid.

3) Show that

$\exists x\,[P(x) \rightarrow Q(x)], \exists x\,[Q(x) \rightarrow R(x)] \vdash \exists x\,[P(x) \rightarrow R(x)]$ is valid.

Proof:

S.No	Assertion	Reason
1	$\exists x\,[P(x) \rightarrow Q(x)]$	Premise 1
2	$P(a) \rightarrow Q(a)$	ES
3	$\exists x\,[Q(x) \rightarrow R(x)]$	Premise 2
4	$Q(a) \rightarrow R(a)$	Existential Instantiation (ES)
5	$P(a) \rightarrow R(a)$	Hypothetical Syllogism (from 2 and 4)
6	$\exists x\,[P(x) \rightarrow R(x)]$	Existential Generalization (EG)

Hence,

$\exists x\,[P(x) \rightarrow Q(x)], \exists x\,[Q(x) \rightarrow R(x)] \vdash \exists x\,[P(x) \rightarrow R(x)]$ is valid.

4) Show that

$\forall x\,[P(x)], \forall x\,[P(x) \rightarrow Q(x)] \vdash \forall x\,[Q(x)]$ is valid.

Proof:

S.No	Assertion	Reason
1	$\forall x\,[P(x)]$	Premise 1
2	$P(a)$	Universal Instantiation (US)
3	$\forall x\,[P(x) \rightarrow Q(x)]$	Premise 2
4	$P(a) \rightarrow Q(a)$	Universal Instantiation (US)
5	$Q(a)$	Modus Ponens (from 2 and 4)
6	$\forall x\,[Q(x)]$	Universal Generalization (UG)

Hence , $\forall x\,[P(x)], \forall x\,[P(x) \rightarrow Q(x)] \vdash \forall x\,[Q(x)]$ is valid.

Word Problems :
Check the validity of the following:
1. All men are mortal
Socrates is a man.
∴ Socrates is mortal.
Predicate Definitions:

Proof :

P(x): x is a man
Q(x): x is mortal
Symbolic Form:
$\forall x\ [P(x) \rightarrow Q(x)]$
P(a)
∴ Q(a)

S. No	Assertion	Reason
1	$\forall x\ [P(x) \rightarrow Q(x)]$	Premise – 1
2	P(a)	Universal Instantiation (UI)
3	P(a)	Premise – 2
4	Q(a)	From (1), (3) Modus Ponens

Conclusion:
∴ The argument is valid.

2.If Jack misses many classes through illness, then he fails high school.
If Jack fails high school, then he is uneducated.
If Jack reads a lot of books, then he is not uneducated.

PROVE THAT : Jack misses many classes through illness and reads a lot of books.

Proof:
Let the propositions be:
P(x): x misses many classes through illness
Q(x): x fails high school
R(x): x is uneducated
S(x): x reads a lot of books
Symbolic Form:
$P(x) \rightarrow Q(x)$
$Q(x) \rightarrow R(x)$
$S(x) \rightarrow \neg R(x)$
$P(x) \wedge S(x)$
To Prove:

Contradiction (⊥) / Inconsistency

S. No	Statement	Reason
1	$P(x) \rightarrow Q(x)$	Premise 1
2	$Q(x) \rightarrow R(x)$	Premise 2
3	$P(x) \rightarrow R(x)$	From (1), (2) by syllogism
4	$S(x) \rightarrow \neg R(x)$	Premise 3
5	$R(x) \rightarrow \neg S(x)$	Contrapositive of (4)
6	$P(x) \rightarrow \neg S(x)$	From (3), (5) by syllogism
7	$P(x) \wedge S(x) \rightarrow (P(x) \wedge \neg S(x))$	From (6)
8	$\neg[P(x) \wedge S(x)]$	Contradiction derived
9	$P(x) \wedge S(x)$	Premise 4
10	F	From (8) and (9)

Conclusion:
The statements lead to a contradiction.
Therefore, the set of premises is inconsistent.

Unit 2: Set Theory

Set
A **set** is a well-defined collection of unique objects.
Example:

- $A = \{\text{green,black,white}\}$

Union
The **union** of two sets is the set containing all elements from both sets without repetition.
Example:

- $A = \{1,2,3\}$
- $B = \{4,5,6\}$

$$A \cup B = \{1,2,3,4,5,6\}$$

Intersection
The **intersection** of two sets is the set of common elements present in both sets.
Example:

- $A = \{1,2,3\}$
- $B = \{3,4,5\}$

$$A \cap B = \{3\}$$

Subset
A **subset** is a set whose every element belongs to another set.
Example:

- $A = \{1,2,3\}$
- $B = \{2,3\}$

Since every element of $B$is also in A,

$$B \subset A$$

Principle of Inclusion and Exclusion
Let $A$and Bbe subsets of the same universal set. Then, according to the **Principle of Inclusion and Exclusion**,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Where:

- $|A|$= Number of elements in set A
- $|B|$= Number of elements in set B
- $|A \cup B|$= Number of elements in the union of sets $A$and B
- $|A \cap B|$= Number of elements in the intersection of sets $A$and B

Note

1. Principle of Inclusion and Exclusion for Three Sets

For three sets A, B, and C,

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |A \cap C| + |A \cap B \cap C|$$

2. Venn Diagram for Two Sets

The Venn diagram for two sets $A$and Bconsists of the following regions:

- $A \cap B$— Common elements of both sets
- $A \cap \bar{B}$— Elements only in A
- $B \cap \bar{A}$— Elements only in B

3. Venn Diagram for Three Sets

The important regions are:

1. $\bar{A} \cap B \cap C$
2. $A \cap C \cap \bar{B}$
3. $A \cap B \cap \bar{C}$
4. $A \cap B \cap C$

4. Complement of a Set

$$|\bar{A}| = N - |A|$$

where,

- N= Number of elements in the universal set.

5. Greatest Integer (Floor) Function

$$[x]$$

denotes **the greatest integer less than or equal to** x(also called the **floor function**).

Problems

1. Out of **150 students**,

- 70 read the **English** paper.
- 60 read the **Hindi** paper.
- 50 read the **Telugu** paper.
- 20 read both **English and Hindi** papers.
- 15 read both **English and Telugu** papers.
- 10 read both **Hindi and Telugu** papers.

Question: How many students read **all three** papers?

Find: $|\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C}|=?$
Where A = number of English paper readers
B = number of Hindi paper readers
C = number of Telugu paper readers
Solution
By the Principle of Inclusion and Exclusion,

$$|\boldsymbol{A} \cup \boldsymbol{B} \cup \boldsymbol{C}|=|\boldsymbol{A}|+|\boldsymbol{B}|+|\boldsymbol{C}|-|\boldsymbol{A} \cap \boldsymbol{B}|-|\boldsymbol{B} \cap \boldsymbol{C}|-|\boldsymbol{A} \cap \boldsymbol{C}|+|\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C}|$$

Solution
Given

- A= Students who read the **English** paper.
- B= Students who read the **Hindi** paper.
- C= Students who read the **Telugu** paper.

Also Given,

$$|A|=70$$
$$|B|=60$$
$$|C|=50$$
$$|A \cap B|=20$$
$$|A \cap C|=15$$
$$|B \cap C|=10$$

Using the Principle of Inclusion and Exclusion,

$$|A \cup B \cup C|=|A|+|B|+|C|-|A \cap B|-|A \cap C|-|B \cap C|+|A \cap B \cap C|$$

Since all 100 students are considered,

$$150=70+60+50-20-15-10+|A \cap B \cap C|$$
$$150=135+|A \cap B \cap C|$$
$$|A \cap B \cap C|=15$$

2. ***Find the number of integers from 1 to 1000 (inclusive) that are divisible by 5, 6, or 8. Also find the number of integers that are not divisible by 5, 6, or 8.***

Solution
Given,

$$S=\{1,2,3,\ldots,1000\}$$
$$N=1000$$

Let

- *A= Set of numbers divisible by **5***
- *B= Set of numbers divisible by **6***
- *C= Set of numbers divisible by **8***

Then,

$$| A |= \left\lfloor \frac{1000}{5} \right\rfloor = 200$$
$$| B |= \left\lfloor \frac{1000}{6} \right\rfloor = 166$$
$$| C |= \left\lfloor \frac{1000}{8} \right\rfloor = 125$$

Also,

Pairwise intersections

$$| A \cap B |= \left\lfloor \frac{1000}{LCM\ (5,6)} \right\rfloor = \left\lfloor \frac{1000}{30} \right\rfloor = 33$$
$$| B \cap C |= \left\lfloor \frac{1000}{LCM\ (6,8)} \right\rfloor = \left\lfloor \frac{1000}{24} \right\rfloor = 41$$
$$| C \cap A |= \left\lfloor \frac{1000}{LCM\ (8,5)} \right\rfloor = \left\lfloor \frac{1000}{40} \right\rfloor = 25$$

Triple intersection

$$| A \cap B \cap C |= \left\lfloor \frac{1000}{LCM\ (5,6,8)} \right\rfloor = \left\lfloor \frac{1000}{120} \right\rfloor = 8$$

Using the ***Principle of Inclusion and Exclusion****,*

$$| A \cup B \cup C |=| A | +| B | +| C | -| A \cap B | -| B \cap C | -| C \cap A | +| A \cap B \cap C |$$

Substituting the values,

$$\begin{aligned} &= 200 + 166 + 125 - 33 - 41 - 25 + 8 \\ &= 491 - 99 + 8 \\ &= 499 - 99 \\ &= 400 \end{aligned}$$

Hence,

400 numbers are divisible by 5, 6, or 8.

The number of integers ***not divisible*** *by 5, 6, or 8 is*

$$\begin{aligned} | A \cup \bar{B} \cup C | &= N - | A \cup B \cup C | \\ &= 1000 - 400 \\ &= 600 \end{aligned}$$

Therefore,

Numbers divisible by 5, 6, or 8 = 400

Numbers not divisible by 5, 6, or 8 = 600

Pigeonhole Principle

If m pigeons occupy n pigeon holes and $m > n$, then **at least one pigeonhole contains** $P + 1$**(or more) pigeons**, where

- $P = \frac{m-1}{n}$
- **Generalization**

- If m pigeons occupy $n$pigeonholes ($m > n$), then **at least one pigeonhole contains** $(P+1)$**or more pigeons**, where
- $P = \frac{m-1}{n}$

Problems

1. **Prove that if 30 dictionaries in a library contain a total of 61,327 pages, then at least one dictionary must have 2,045 pages.**

Solution
Given,
Total number of pages, $m = 61{,}327$
Number of dictionaries, $n = 30$
By the **Pigeonhole Principle**,

$$P = \frac{m-1}{n}$$

$$= \frac{61{,}327 - 1}{30}$$

$$= \frac{61{,}326}{30}$$

$$= 2044.2 \approx 2044$$

Therefore, at least one dictionary must contain
$P + 1 = 2044 + 1 = 2045$ pages.
Answer: At least one dictionary has **2,045 pages**.

2. **Seven members of a family have a total of ₹2,886 in their pockets. Show that one of them must have at least ₹413 in his pocket.**

Solution
Given,
Total amount, $m = 2886$
Number of family members, $n = 7$
Using the **Pigeonhole Principle**,

$$P = \frac{m-1}{n}$$
$$= \frac{2886 - 1}{7}$$
$$= \frac{2885}{7}$$
$$= 412$$

Therefore, at least one member must have
$P + 1 = 412 + 1 = 413$ rupees in his/her pocket.
Answer: At least one family member has **₹413** in his/her pocket.

Functions
Let A and B be any two non-empty sets. A **function** is a relation
$$f: A \to B$$
such that for every element $a \in A$, there exists a **unique** element $b \in B$ satisfying
$$f(a) = b.$$
Here,

- b is called the **image** of a.
- a is called the **preimage** of b.

Note

1. In the above mapping, A is called the **domain**.
2. B is called the **codomain**.
3. The subset of B containing all images of elements of A under f is called the **range** of f.

One-to-One (Injective) Function
A function
$$f: A \to B$$
is said to be a **one-to-one (1–1) function** if for every $x_1, x_2 \in A$,
$$f(x_1) = f(x_2) \implies x_1 = x_2.$$
Equivalently,
Different elements of the domain have different images under the function.
Example
A function
$$A = \{1,2,3,4\}, B = \{a, b, c, d, e\}$$
with mappings
$$1 \mapsto a, 2 \mapsto b, 3 \mapsto c, 4 \mapsto d$$
is a **one-to-one function**.
Not a One-to-One Function
If two different elements of the domain have the same image, then the function is **not one-to-one**.
Example:
$$a \mapsto 1, b \mapsto 1, c \mapsto 2$$
Since both a and b map to the same element, the function is **not one-to-one**.
Onto (Surjective) Function
A function
$$f: A \to B$$
is said to be an **onto (surjective) function** if every element of the codomain B has at least one preimage in the domain A.
That is,
$$\forall\, b \in B,\ \exists\, a \in A \text{ such that } f(a) = b.$$
Equivalently,
$$\text{Range}(f) = \text{Codomain}(f).$$

Example (Onto Function)
Let
$$A = \{1,2,3\}, B = \{a,b,c\}$$
with mappings
$$1 \mapsto b, 2 \mapsto a, 3 \mapsto c.$$
Every element of Bhas a preimage, so the function is **onto**.
Example (Not Onto Function)
Let
$$A = \{1,2,3\}, B = \{a,b,c\}$$
with mappings
$$1 \mapsto a, 2 \mapsto b, 3 \mapsto b.$$
Here, the element $c$has no preimage.
Therefore, the function is **not onto**.
Bijective Function
A function
$$f: A \to B$$
is said to be a **bijective function** if it is **both one-to-one (injective) and onto (surjective)**.
In other words, every element of the codomain has exactly one preimage in the domain.
Composite Function
Let A, B, and Cbe any three non-empty sets, and let
$$f: A \to B$$
and
$$g: B \to C.$$
Then the **composition** of the functions $f$and $g$is denoted by
$$g \circ f.$$
The composite function
$$g \circ f: A \to C$$
is defined by
$$(g \circ f)(x) = g(f(x)), \forall x \in A.$$
That is, first apply the function f, and then apply the function $g$to the result.
Illustration
If
$$A \xrightarrow{f} B \xrightarrow{g} C,$$
then
$$A \xrightarrow{g \circ f} C.$$

Inverse Function
Consider a function $f: A \to B$.
If there exists a function $f^{-1}: B \to A$
such that $f^{-1}(f(x)) = x, \forall x \in A$,
and $f(f^{-1}(y)) = y, \forall y \in B$,

then f^{-1} is called the **inverse function** of f.
Note: An inverse function exists **only if** the function f is **bijective** (both one-to-one and onto).

Problem

1. Consider the function $f: \mathbb{R} \to \mathbb{R}$

defined by $f(x) = 4x + 2$.
Prove that f is **one-to-one**, **onto**, and find its inverse f^{-1}.
Solution
Given

$$f: \mathbb{R} \to \mathbb{R}$$

defined by $f(x) = 4x + 2$.
To Prove that f is One-to-One (Injective)
Assume $f(x_1) = f(x_2)$.
Then, $4x_1 + 2 = 4x_2 + 2$.
Subtracting 2 from both sides, $4x_1 = 4x_2$.
Dividing both sides by 4, $x_1 = x_2$.
Hence,

$$\boxed{f \text{ is one-to-one (injective).}}$$

To Prove that f is Onto (Surjective)
Let $y = f(x)$.
Then, $y = 4x + 2$.
Subtracting 2 from both sides, $y - 2 = 4x$.
Therefore,

$$x = \frac{y-2}{4}.$$

Since $x \in \mathbb{R}$ for every $y \in \mathbb{R}$, every element of the codomain has a preimage.
Hence, $\boxed{f \text{ is onto (surjective).}}$
Finding the Inverse Function
Let $y = 4x + 2$.
Solving for x, $x = \frac{y-2}{4}$.
Therefore, $f^{-1}(y) = \frac{y-2}{4}$.
Replacing y by x,

$$\boxed{f^{-1}(x) = \frac{x-2}{4}.}$$

Final Answer

- $f(x) = 4x + 2$ is **one-to-one (injective)**.
- $f(x) = 4x + 2$is **onto (surjective)**.
- Therefore, f is **bijective**.
- The inverse function is

$$\boxed{f^{-1}(x) = \frac{x-2}{4}.}$$

2. Let $f: \mathbb{R} \to \mathbb{R}$

be a function defined by $f(x) = 3 + \frac{1}{x}$
Show that f is **one-to-one**, **onto**, and find f^{-1}.

$$f: \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{3\}.$$

Solution
Given

$$f(x) = 3 + \frac{1}{x}.$$

To Prove that f is One-to-One (Injective)
Assume $f(x_1) = f(x_2)$.
Then, $3 + \frac{1}{x_1} = 3 + \frac{1}{x_2}$.
Subtracting 3 from both sides,

$$\frac{1}{x_1} = \frac{1}{x_2}.$$

Therefore, $x_1 = x_2$.
Hence,

$$\boxed{f \text{ is one-to-one (injective).}}$$

To Prove that f is Onto (Surjective)
Let $f(x) = y$.
Then, $y = 3 + \frac{1}{x}$.
Subtracting 3 from both sides, $y - 3 = \frac{1}{x}$.
Taking the reciprocal, $x = \frac{1}{y-3}$.
Since every $y \neq 3$has a corresponding $x \neq 0$,

$$\boxed{f \text{ is onto.}}$$

Finding the Inverse Function
Let $y = 3 + \frac{1}{x}$.
Then, $y - 3 = \frac{1}{x}$.
Hence, $x = \frac{1}{y-3}$.
Therefore, $f^{-1}(y) = \frac{1}{y-3}$.
Replacing $y$with x,

$$\boxed{f^{-1}(x) = \frac{1}{x-3}.}$$

Final Answer

- $f(x) = 3 + \frac{1}{x}$is **one-to-one (injective)**.
- $f(x) = 3 + \frac{1}{x}$is **onto (surjective)** on the correct domain and codomain.
- Therefore, $f$is **bijective**.
- The inverse function is

$$\boxed{f^{-1}(x) = \frac{1}{x-3}.}$$

3. Find the inverse of the function $f(x) = 4e^{6x+2}$.

Solution

Given $f(x) = 4e^{6x+2}$.
Let $f(x) = y$.
Then, $y = 4e^{6x+2}$.
Dividing both sides by 4, $\frac{y}{4} = e^{6x+2}$.
Taking the natural logarithm (log to base e) on both sides,

$$\ln\left(\frac{y}{4}\right) = 6x + 2.$$

Subtracting 2 from both sides, $\ln\left(\frac{y}{4}\right) - 2 = 6x$.
Dividing both sides by 6, $x = \frac{\ln\left(\frac{y}{4}\right) - 2}{6}$.
This can also be written as $x = \frac{\ln\left(\frac{y}{4}\right)}{6} - \frac{1}{3}$.
Hence, $f^{-1}(y) = \frac{\ln\left(\frac{y}{4}\right)}{6} - \frac{1}{3}$.
Replacing $y$by x,

$$\boxed{f^{-1}(x) = \frac{\ln\left(\frac{x}{4}\right)}{6} - \frac{1}{3}.}$$

Final Answer

The inverse function is

$$\boxed{f^{-1}(x) = \frac{\ln\left(\frac{x}{4}\right)}{6} - \frac{1}{3}.}$$

4. Let $f: \mathbb{R} \to \mathbb{R}andg: \mathbb{R} \to \mathbb{R}$

be defined by $f(x) = 2x + 1$, $g(x) = \frac{x}{3}$.
Show that

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Solution
Given

$$f: \mathbb{R} \to \mathbb{R}, f(x) = 2x + 1,$$
$$g: \mathbb{R} \to \mathbb{R}, g(x) = \frac{x}{3}.$$

To Find $(\boldsymbol{g} \circ \boldsymbol{f})^{-1}$
First, compute the composite function:

$$(g \circ f)(x) = g(f(x)).$$

Substituting $f(x) = 2x + 1$,

$$(g \circ f)(x) = g(2x + 1) = \frac{2x + 1}{3}.$$

Let $(g \circ f)(x) = y$.
Then, $y = \frac{2x+1}{3}$.
Multiply both sides by 3: $3y = 2x + 1$.
Subtract 1: $3y - 1 = 2x$.
Hence,

$$x = \frac{3y - 1}{2}.$$

Therefore,

$$(g \circ f)^{-1}(y) = \frac{3y - 1}{2}.$$

Replacing $y$with x,

$$\boxed{(g \circ f)^{-1}(x) = \frac{3x - 1}{2}.}$$

To Find $\boldsymbol{f}^{-1}$
Let $f(x) = y$.
Then, $y = 2x + 1$.
Subtract 1: $y - 1 = 2x$.
Hence, $x = \frac{y-1}{2}$.
Therefore, $f^{-1}(y) = \frac{y-1}{2}$,
or

$$\boxed{f^{-1}(x) = \frac{x - 1}{2}.}$$

To Find $\boldsymbol{g}^{-1}$
Let $g(x) = y$.
Then, $y = \frac{x}{3}$.
Multiplying both sides by 3, $x = 3y$.
Therefore, $g^{-1}(y) = 3y$,
or

$$\boxed{g^{-1}(x) = 3x.}$$

To Find $\boldsymbol{f}^{-1} \circ \boldsymbol{g}^{-1}$
Using the definition of composition,

$$(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)).$$

Since $g^{-1}(x) = 3x$

we get $f^{-1}(3x) = \frac{3x-1}{2}$.

Thus,

$$\boxed{(f^{-1} \circ g^{-1})(x) = \frac{3x-1}{2}.}$$

Conclusion

Since $(g \circ f)^{-1}(x) = \frac{3x-1}{2}$

And $(f^{-1} \circ g^{-1})(x) = \frac{3x-1}{2}$,

we conclude that

$$\boxed{(g \circ f)^{-1} = f^{-1} \circ g^{-1}.}$$

Hence, the result is proved.

5. Let $f(x) = x^2, g(x) = x^2 + 1$.

Find:

1. $g \circ f$
2. $f \circ g$
3. f^2
4. g^2

Solution

Given

$$f(x) = x^2$$
$$g(x) = x^2 + 1$$

(i) Find $g \circ f$

Using the definition of composition,

$$(g \circ f)(x) = g(f(x)).$$

Substituting $f(x) = x^2$,

$$(g \circ f)(x) = g(x^2).$$

Since $g(x) = x^2 + 1$,

we get $g(x^2) = (x^2)^2 + 1 = x^4 + 1$.

Hence,

$$\boxed{(g \circ f)(x) = x^4 + 1.}$$

(ii) Find $f \circ g$

Using the definition of composition,

$$(f \circ g)(x) = f(g(x)).$$

Substituting $g(x) = x^2 + 1$,

$$(f \circ g)(x) = f(x^2 + 1).$$

Since $f(x) = x^2$,

we get $f(x^2 + 1) = (x^2 + 1)^2$.

Expanding, $(x^2 + 1)^2 = x^4 + 2x^2 + 1$.

Hence,

$$\boxed{(f \circ g)(x) = x^4 + 2x^2 + 1.}$$

(iii) Find f^2

Here, $f^2 = f \circ f$.

Therefore $(f \circ f)(x) = f(f(x))$.

Substituting $f(x) = x^2$,

$$f(x^2) = (x^2)^2 = x^4.$$

Hence,

$$\boxed{f^2(x) = x^4.}$$

(iv) Find g^2

Here, $g^2 = g \circ g$.

Therefore, $(g \circ g)(x) = g(g(x))$.

Substituting $g(x) = x^2 + 1$,

$$g(x^2 + 1) = (x^2 + 1)^2 + 1.$$

Expanding,

$$= (x^4 + 2x^2 + 1) + 1$$
$$= x^4 + 2x^2 + 2.$$

Hence,

$$\boxed{g^2(x) = x^4 + 2x^2 + 2.}$$

Final Answers

$$\boxed{(g \circ f)(x) = x^4 + 1}$$
$$\boxed{(f \circ g)(x) = x^4 + 2x^2 + 1}$$
$$\boxed{f^2(x) = x^4}$$
$$\boxed{g^2(x) = x^4 + 2x^2 + 2}$$

Recursive Function

A **recursive function** is a function that **calls itself** or uses its own previous values to compute subsequent values. It is defined in terms of itself, together with one or more **base cases** that stop the recursion.

In other words, a recursive function repeatedly applies the same rule to calculate the next value until a stopping condition is reached.

Example 1

Let $f(n) = f(n-1) + 3, f(0) = 1$

Find $f(2)$.

Given:

$$f(n) = f(n-1) + 3$$

For $n = 1$,

$$f(1) = f(0) + 3$$
$$= 1 + 3 = 4$$

For $n = 2$,

$$f(2) = f(1) + 3$$
$$= 4 + 3 = 7$$

Answer:

$$\boxed{f(2) = 7}$$

Example 2

Consider a sequence a_n defined recursively by

$$a_0 = 4, a_n = a_{n-1} + n, n \geq 1$$

Find the explicit form of $f(n) = a_n$.

Solution:

Given, $a_0 = 4, a_n = a_{n-1} + n$

Expanding recursively,

$$\begin{aligned} a_n &= (a_{n-2} + n - 1) + n \\ &= (a_{n-3} + (n-2) + (n-1)) + n \\ &= (a_{n-4} + (n-3) + (n-2) + (n-1)) + n \\ &= a_0 + 1 + 2 + 3 + \cdots + n \end{aligned}$$

Since $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$,

we get $a_n = 4 + \frac{n(n+1)}{2}$.

Hence, the explicit form is

$$\boxed{f(n) = a_n = 4 + \frac{n(n+1)}{2}}.$$

Relation

Let A and B be two non-empty sets.

A **binary relation** from A to B is any **subset of** $A \times B$.

Example

Let $A = \{1,2\}, B = \{2,3,4\}$.

Then $A \times B = \{(1,2), (1,3), (1,4), (2,2), (2,3), (2,4)\}$.

Any subset of $A \times B$ is a **relation from** A **to** B.

Properties of Relations

1. Reflexive

A relation R on a set A is **reflexive** if $(a, a) \in R, \forall a \in A$.

2. Antisymmetric

A relation R is **antisymmetric** if $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$

3. Transitive

A relation R is **transitive** if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

Partially Ordered Set (Poset)

A relation Ron a set S is called a **Partially Ordered Set (Poset)** if it satisfies the following properties:

- Reflexive
- Antisymmetric
- Transitive

Lattice

A **lattice** is a partially ordered set $(P, \leq)$such that for every pair of elements $a, b \in P$,

- there exists a **Least Upper Bound (LUB)** (Join), and
- there exists a **Greatest Lower Bound (GLB)** (Meet).

Properties of Lattice

1. Idempotent Law

$$a \wedge a = a$$
$$a \vee a = a$$

2. Associative Law $(a \vee b) \vee c = a \vee (b \vee c)$

$$(a \wedge b) \wedge c = a \wedge (b \wedge c)$$

3. Distributive Law

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$
$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

4. Commutative Law

$$a \vee b = b \vee a$$
$$a \wedge b = b \wedge a$$

5. Absorption Law

$$a \wedge (a \vee b) = a$$
$$a \vee (a \wedge b) = a$$

Problem

1. If $A = \{1,2,3,5,30\}$and $R$is the divisibility relation, prove that $(A, R)$is a lattice but not a distributive lattice.

Given:

$$A = \{1,2,3,5,30\}$$

and $R$is the divisibility relation.

We have to prove that $(A, R)$is a lattice but not a distributive lattice.

Least Upper Bound (LUB) Table:

∨	1	2	3	5	30
1	1	2	3	5	30
2	2	2	30	30	30
3	3	30	3	30	30
5	5	30	30	5	30
30	30	30	30	30	30

Hasse Diagram:

The poset under divisibility can be represented as:

- 1is the least element
- 2,3,5cover 1
- 30is the greatest element, covered by 2,3,5

So the structure is:

$$1 \to \{2,3,5\} \to 30$$

Conclusion:

- For every pair of elements in A, both LUB (join) and GLB (meet) exist. Hence, $(A, R)$is a **lattice**.
- However, the lattice is **not distributive**.

Greatest Lower Bound (GLB)

Consider the set $A = \{1,2,3,5,30\}$.

GLB Table

∧	1	2	3	5	30
1	1	1	1	1	1
2	1	2	1	1	2
3	1	1	3	1	3
5	1	1	1	5	5
30	1	2	3	5	30

Hasse Diagram

A Hasse diagram is drawn for the set Ashowing the divisibility relations among elements.

Observation

By observing the above table, for every pair of elements in A, a greatest lower bound (GLB) exists.

Therefore, for every pair of elements in A, both LUB and GLB exist.
Hence, $(A, R)$is a lattice.

Checking Distributive Property

We know the distributive property is given by:

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

Let $a = 2, b = 3, c = 5$.

$$2 \vee (3 \wedge 5) = (2 \vee 3) \wedge (2 \vee 5)$$

Now, $\quad 3 \wedge 5 = 1$

So,

$$2 \vee (3 \wedge 5) = 2 \vee 1$$
$$2 \vee 1 = 2$$

On the other hand,

$$(2 \vee 3) = 30, (2 \vee 5) = 30$$

So,

$$(2 \vee 3) \wedge (2 \vee 5) = 30 \wedge 30 = 30$$

Conclusion

$$2 \neq 30$$

Hence, LHS ≠ RHS.
Therefore, the distributive property does not hold, and $(A, R)$is **not a distributive lattice**.

Binary Operation

Let Sbe a non-empty set. A function

$$f: S \times S \rightarrow S$$

is called a **binary operation** or **binary composition** on S.
Binary operations are usually denoted by symbols such as $+, \cdot, -, \times$, etc.
We observe:

- Addition ($+$), multiplication ($\times$), and subtraction ($-$)are binary operations in $\mathbb{R}$.
- Division ($\div$)is **not** a binary operation in $\mathbb{R}$, since division by zero is not defined.

Examples

1. $a, b \in S \Rightarrow a \circ b \in S$, then $\circ$is a binary operation.
2. $a, b \in S \Rightarrow a \cdot b \in S$, then $\cdot$is a binary operation.
3. $a, b \in S \Rightarrow a + b \in S$, then $+$is a binary operation.

Algebraic Structure

A non-empty set Gequipped with a binary operation $*$is called an **algebraic structure**.
It is denoted by:

$$(G, *)$$

Common Examples of Algebraic Structures

$$(\mathbb{Z}, +), (\mathbb{R}, +), (\mathbb{R}, \times), (\mathbb{C}, +)$$

Closure Axiom
Let $(G,*)$ be an algebraic structure.
G is said to be **closed** under the operation $*$ if:

$$a * b \in G \forall a, b \in G$$

If closure holds, G is called **closed with respect to** $*$.
Associative Axiom
Let $(G,*)$ be an algebraic structure.
If for all $a, b, c \in G$,

$$(a * b) * c = a * (b * c)$$

then $*$ is said to be **associative** on G.
If closure and associativity both hold, then $(G,*)$ is called a **semigroup**.
Identity Axiom
Let $(G,*)$ be an algebraic structure.
An element $e \in G$ is called an **identity element** if:

$$a * e = e * a = a \forall a \in G$$

Note

- Additive identity is 0
- Multiplicative identity is 1

Examples

- $a + 0 = a = 0 + a$
- $a \times 1 = a = 1 \times a$

Inverse Axiom
Let $(G,*)$ be an algebraic structure and let $e \in G$ be the identity element. If for every $a \in G$, there exists a unique element $b \in G$ such that

$$a * b = b * a = e,$$

then b is called the **inverse** of a.
It is denoted by

$$b = a^{-1} \text{or} a^{-1} = b.$$

If every element of $(G,*)$ has an inverse, then $(G,*)$ is said to satisfy the **Inverse Axiom**.
Note:

- The additive inverse of a is $-a$.
- The multiplicative inverse of a is $\frac{1}{a}$.

Examples:

- $a + (-a) = 0$
- $a \times \frac{1}{a} = 1 (\text{for } a \neq 0)$

Commutative Axiom

Let $(G,*)$ be an algebraic structure. If for all $a, b \in G$,

$$a * b = b * a,$$

then $(G,*)$ is said to satisfy the **Commutative Axiom**.

Group

Let G be a non-empty set and let $*$ be a binary operation on G. Then the algebraic structure $(G,*)$ is called a **Group** if it satisfies the following axioms:

1. Closure Axiom

For all $a, b \in G$,

$$a * b \in G.$$

2. Associative Axiom

For all $a, b, c \in G$,

$$a * (b * c) = (a * b) * c.$$

3. Identity Axiom

There exists an identity element $e \in G$ such that for every $a \in G$,

$$a * e = e * a = a.$$

4. Inverse Axiom

For every $a \in G$, there exists an element $b \in G$ such that

$$a * b = b * a = e.$$

Here, $b = a^{-1}$ is called the inverse of a.

Examples

- $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, and $(\mathbb{Q}, +)$ are all groups.
- However, $(\mathbb{N}, +)$ is **not** a group because additive inverses do not exist in $\mathbb{N}$.

Monoid

Let G be a non-empty set and let $*$ be a binary operation on G. If the algebraic structure $(G,*)$ satisfies:

- Closure Axiom
- Associative Axiom
- Identity Axiom

then $(G,*)$ is called a **Monoid**.

Semigroup

Let G be a non-empty set and let $*$ be a binary operation on G. If the algebraic structure $(G,*)$ satisfies:

- Closure Axiom
- Associative Axiom

then $(G,*)$ is called a **Semigroup**.

Monomorphism

A function $f: G_1 \to G_2$ is said to be a **monomorphism** if f is **one-to-one (1-1)**.

Epimorphism

A function $f: G_1 \to G_2$ is said to be an **epimorphism** if f is **onto**.

Homomorphism
Let $(G_1,*)$and $(G_2,\circ)$be any two groups. A function

$$f: G_1 \to G_2$$

is called a **homomorphism** from group $G_1$to group G_2if

$$f(a * b) = f(a) \circ f(b), \forall\, a, b \in G_1.$$

Isomorphism
A function

$$f: G_1 \to G_2$$

is said to be an **isomorphism** if $f$is **one-to-one, onto, and a homomorphism**.
Automorphism
An **isomorphism** of a group G**onto itself** is called an **automorphism**.
Endomorphism
A **homomorphism** of a group G**into itself** is called an **endomorphism**.

Subgroup
Let $(G,*)$be any group. Let Hbe a non-empty subset of G. If $(H,*)$is also a group under the same binary operation, then $H$is called a **subgroup** of G.
Abelian Group
If $(G,*)$satisfies the following axioms:

- Closure
- Associative
- Identity
- Inverse
- Commutative

then $(G,*)$is called an **Abelian group**.
Finite Group
If the set Gcontains a **finite number of elements**, then the group $(G,*)$is called a **finite group**.
Infinite Group
If the set Gcontains an **infinite number of elements**, then the group $(G,*)$is called an **infinite group**.
Order of a Group
The **number of elements** in a finite group $(G,\cdot)$is called the **order of the group** and is denoted by

$$O(G) \text{or} \mid G \mid.$$

Examples
(i) $G = \{1, -1, i, -i\}$
The order of G is $\mid G \mid = 4$.
(ii) $G = \{1,5,7,10,11\}$
The order of $G$is $\mid G \mid = 5$.

Problem
Prove that the set $\mathbb{Z}$ of all integers forms an abelian group with respect to the operation defined by

$$a * b = a + b + 1.$$

Proof
To prove that $(\mathbb{Z}, *)$is an abelian group, we verify the group axioms.
(i) Closure Axiom
Let $a, b \in \mathbb{Z}$.
Then,

$$a * b = a + b + 1.$$

Since the sum of integers is an integer,

$$a + b + 1 \in \mathbb{Z}.$$

Hence,

$$a * b \in \mathbb{Z}.$$

Therefore, $(\mathbb{Z}, *)$satisfies the ***closure axiom****.*
(ii) Associative Axiom
Let $a, b, c \in \mathbb{Z}$.
Consider the left-hand side:

$$a * (b * c) = a * (b + c + 1) = a + (b + c + 1) + 1 = a + b + c + 2.$$

Now consider the right-hand side:

$$(a * b) * c = (a + b + 1) * c = (a + b + 1) + c + 1 = a + b + c + 2.$$

Since

$$a * (b * c) = (a * b) * c,$$

the operation is ***associative****.*
Therefore, $(\mathbb{Z}, *)$satisfies the ***associative axiom****.*
(iii) Identity Axiom
Let ebe the identity element.
Then, $a * e = a$.
Using the operation, $a + e + 1 = a$.
Hence, $e + 1 = 0$,
which gives

$$e = -1.$$

Also,

$$e * a = -1 + a + 1 = a.$$

Therefore,

$$e = -1$$

is the ***identity element*** *of $(\mathbb{Z}, *)$.*
(iv) Inverse Axiom
Let $a \in \mathbb{Z}$.
Suppose bis the inverse of a. Then,

$$a * b = e = -1.$$

Using the operation,

$$a + b + 1 = -1.$$

Therefore,

$$a + b = -2,$$
which gives
$$b = -2 - a.$$
Also,
$$b * a = (-2 - a) + a + 1 = -1 = e.$$
Hence,
$$a^{-1} = -2 - a.$$
Thus, every element of $\mathbb{Z}$*has an* ***inverse****.*
(v) Commutative Axiom
Let $a, b \in \mathbb{Z}$*.*
Then,
$$a * b = a + b + 1,$$
and
$$b * a = b + a + 1.$$
Since integer addition is commutative,
$$a + b + 1 = b + a + 1.$$
Hence,
$$a * b = b * a.$$
Therefore, the operation is ***commutative****.*
Conclusion
The set $\mathbb{Z}$*with the operation*
$$a * b = a + b + 1$$
satisfies:

1. *Closure*
2. *Associativity*
3. *Identity (identity element* $= -1$*)*
4. *Inverses (inverse of* a*is* $-2 - a$*)*
5. *Commutativity*

Hence,
$$(\mathbb{Z}, *)$$
forms an abelian group.
Problem
Prove that the cube roots of unity form an abelian group with respect to multiplication.
Proof
Let
$$\boldsymbol{G = \{1, \omega, \omega^2\}}$$
be the set of cube roots of unity, where
$$\boldsymbol{\omega^3 = 1, \omega \neq 1}.$$
Also,
$$\boldsymbol{\omega^0 = 1, \omega^1 = \omega, \omega^2 = \omega^2, \omega^3 = 1},$$

and

$$\omega \cdot \omega = \omega^2, \omega \cdot \omega^2 = \omega^3 = 1, \omega^2 \cdot \omega^2 = \omega^4 = \omega.$$

Let "·" denote the binary operation of multiplication on G.
We prove that $(G,\cdot)$is an abelian group.
Composition (Multiplication) Table

.	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

(i) Closure Axiom
From the multiplication table, the product of any two elements of $G$is again an element of G.
That is,

$$\forall\, a, b \in G, a \cdot b \in G.$$

Hence, $(G,\cdot)$satisfies the closure axiom.
(ii) Associative Axiom
Since multiplication of complex numbers is associative,

$$\forall\, a, b, c \in G,$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

For example,

$$1 \cdot (\omega \cdot \omega^2) = (1 \cdot \omega) \cdot \omega^2 = 1.$$

Thus, $(G,\cdot)$satisfies the associative axiom.
(iii) Identity Axiom
From the multiplication table, it is clear that

$$1 \cdot a = a \cdot 1 = a$$

for every $a \in G$.
Hence,

$$1$$

is the identity element of G.
Therefore, $(G,\cdot)$satisfies the identity axiom.
(iv) Inverse Axiom
Every element of Ghas an inverse.

- **The inverse of 1is 1, since**

$$1 \cdot 1 = 1.$$

- **The inverse of $\omega$is ω^2, since**

$$\omega \cdot \omega^2 = \omega^3 = 1.$$

- **The inverse of $\omega^2$is ω, since**

$$\omega^2 \cdot \omega = \omega^3 = 1.$$

Thus, every element of Ghas an inverse.

Hence, $(G,\cdot)$satisfies the inverse axiom.

(v) Commutative Axiom

For all $a, b \in G$,

$$a \cdot b = b \cdot a,$$

since multiplication of complex numbers is commutative.

For example,

$$\omega \cdot \omega^2 = \omega^2 \cdot \omega = 1.$$

Hence, $(G,\cdot)$satisfies the commutative axiom.

Conclusion

The set

$$G = \{1, \omega, \omega^2\}$$

under multiplication satisfies:

1. Closure
2. Associativity
3. Identity (identity element $= 1$)
4. Inverses
5. Commutativity

Therefore,

$$(G,\cdot)$$

forms an abelian group.

Problem

Prove that the fourth roots of unity

$$\{1, -1, i, -i\}, \text{where } i = \sqrt{-1},$$

form an abelian group under multiplication.

Proof

Let

$$G = \{1, -1, i, -i\}$$

be the set of fourth roots of unity.

The binary operation on $G$is multiplication "$\cdot$".

We shall prove that $(G,\cdot)$is an abelian group.

Also,

$$i^2 = -1, i^3 = -i, i^4 = 1, (-i)^2 = -1.$$

Composition (Multiplication) Table

.	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

(i) Closure Axiom
From the multiplication table, it is clear that the product of any two elements of G is again an element of G.
That is,
$$\forall\, a, b \in G, a \cdot b \in G.$$
Hence, $(G,\cdot)$ satisfies the closure axiom.
(ii) Associative Axiom
Since multiplication of complex numbers is associative,
$$\forall\, a, b, c \in G,$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$
For example,
$$1 \cdot ((-1) \cdot i) = (1 \cdot (-1)) \cdot i.$$
Now,
$$(-1) \cdot i = -i,$$
so
$$1 \cdot (-i) = -i,$$
and
$$(-1) \cdot i = -i.$$
Thus,
$$1 \cdot ((-1) \cdot i) = (1 \cdot (-1)) \cdot i.$$
Hence, $(G,\cdot)$ satisfies the associative axiom.

(iii) Identity Axiom
Let e be the identity element.
From the multiplication table, it is clear that
$$1 \cdot a = a \cdot 1 = a$$
for every $a \in G$.
Hence,
$$e = 1.$$
Therefore, $(G,\cdot)$ satisfies the identity axiom.
(iv) Inverse Axiom
Every element of G has an inverse.

- **The inverse of 1 is 1, since**
$$1 \cdot 1 = 1.$$
- **The inverse of -1 is -1, since**
$$(-1) \cdot (-1) = 1.$$
- **The inverse of i is $-i$, since**
$$i \cdot (-i) = 1.$$
- **The inverse of $-i$ is i, since**
$$(-i) \cdot i = 1.$$

Thus, every element of G has an inverse.
Hence, $(G,\cdot)$ satisfies the inverse axiom.
(v) Commutative Axiom

For all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{G}$,

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a},$$

since multiplication of complex numbers is commutative.

For example,

$$\mathbf{1} \cdot (-\mathbf{1}) = (-\mathbf{1}) \cdot \mathbf{1} = -\mathbf{1}.$$

Hence, $(\boldsymbol{G}, \cdot)$satisfies the commutative axiom.

Conclusion

The set

$$\boldsymbol{G} = \{\mathbf{1}, -\mathbf{1}, \boldsymbol{i}, -\boldsymbol{i}\}$$

under multiplication satisfies:

1. **Closure**
2. **Associativity**
3. **Identity (identity element = 1)**
4. **Inverses**
5. **Commutativity**

Therefore,

$$(\boldsymbol{G}, \cdot)$$

forms an abelian group.

Theorem (Subgroup Test)

The necessary and sufficient condition for a non-empty subset $H$of a group $(G,*)$to be a subgroup is that

$$\forall\, a, b \in H, a * b^{-1} \in H,$$

where $b^{-1}$is the inverse of b.

Proof

Let $(G,*)$be a group and let Hbe a non-empty subset of G.

We prove the theorem in two parts.

(i) Necessary Condition

Assume that $H$is a subgroup of G.

Since $H$is itself a group, it satisfies all the group axioms.

Let $a, b \in H$.

Since $H$is a group,

$$b^{-1} \in H$$

(by the inverse axiom).

Again, since $a, b^{-1} \in H$,

$$a * b^{-1} \in H$$

(by the closure axiom).

Hence,

$$\forall\, a, b \in H, a * b^{-1} \in H.$$

Therefore, the condition is **necessary**.

(ii) Sufficient Condition
Assume that
$$\forall\, a, b \in H, a * b^{-1} \in H. \qquad (1)$$
We have to prove that $H$is a subgroup of G.
It is enough to show that Hsatisfies the group axioms.
(a) Closure Axiom
Take any $a, b \in H$.
Since $b \in H$, by applying (1) with $a = b$,
$$b * b^{-1} = e \in H.$$
Now, since $e, a \in H$, apply (1) with $a = a$and $b = b^{-1}$. As
$$(b^{-1})^{-1} = b,$$
we obtain
$$a * (b^{-1})^{-1} = a * b \in H.$$
Hence, Hsatisfies the **closure axiom**.
(b) Associative Axiom
All elements of $H$are elements of G.
Since the operation $*$is associative in G,
$$(a * b) * c = a * (b * c)$$
for all $a, b, c \in H$.
Hence, Halso satisfies the **associative axiom**.
(c) Identity Axiom
Put $a = b$ in (1).
Then,
$$a * a^{-1} \in H.$$
But,
$$a * a^{-1} = e.$$
Therefore,
$$e \in H.$$
Hence, Hcontains the identity element and satisfies the **identity axiom**.
(d) Inverse Axiom
Since $e \in H$, put
$$a = e, b = a$$
in (1).
Then,
$$e * a^{-1} \in H.$$
Since
$$e * a^{-1} = a^{-1},$$
it follows that
$$a^{-1} \in H.$$
Thus, every element of Hhas its inverse in H.
Hence, Hsatisfies the **inverse axiom**.

Conclusion

The subset Hsatisfies:

1. Closure
2. Associativity
3. Identity
4. Inverses

Therefore, H is a subgroup of G.
Hence, a non-empty subset $H$of a group $(G,*)$is a subgroup **if and only if**

$$\boxed{\forall\, a, b \in H,\ a * b^{-1} \in H.}$$

Thus, every element of Hhas its inverse in H.
Hence, Hsatisfies the **inverse axiom**.
Therefore, Hsatisfies all the four group axioms.
Hence,

$$H \text{ is a group.}$$

Since $H \subseteq G$,

$$H \text{ is a subgroup of } G.$$

Therefore,

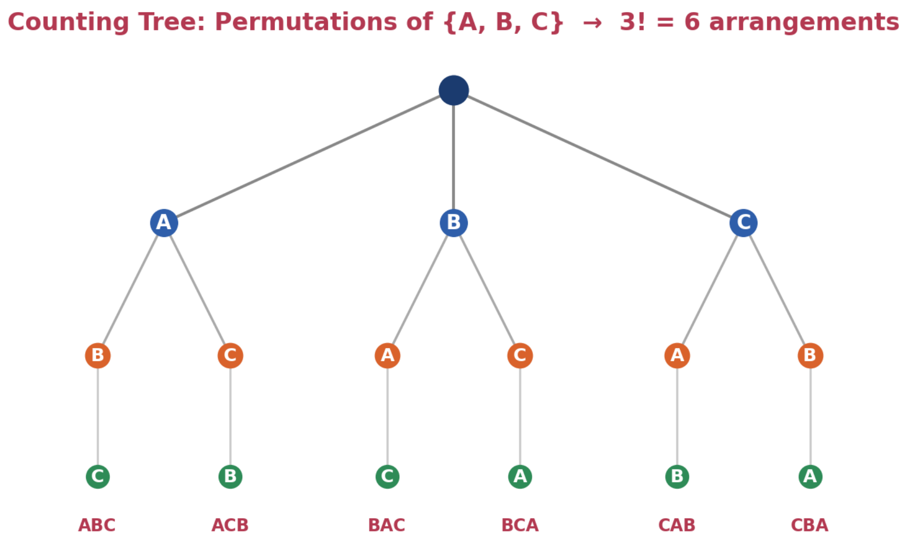
$$\boxed{H \text{ is a subgroup of } G \iff \forall\, a, b \in H,\ a * b^{-1} \in H.}$$

UNIT 3: ELEMENTARY COMBINATORICS

Elementary Combinatorics is the branch of discrete mathematics concerned with **counting** — determining the number of ways objects can be **arranged** (permutations), **selected** (combinations), or distributed, often without listing every possibility individually.

Origin of Combinatorics

Counting problems appear in some of the oldest known mathematical writings, including ancient Indian, Chinese and Greek texts. The formal mathematical study of combinatorics began to flourish in the **17th century**, driven largely by the study of **games of chance**. The correspondence between **Blaise Pascal** and **Pierre de Fermat** in **1654** — triggered by a gambling problem posed by the Chevalier de Méré — is widely credited as the starting point of formal combinatorics and probability theory. Pascal's Triangle, which elegantly encodes combinations $C(n,r)$, was popularized in this period, although similar triangles were known earlier in India, Persia and China.



A counting tree showing all 3! = 6 permutations of the letters {A, B, C}

Uses and Real-Life Examples

- **Passwords & Security —** calculating how many possible passwords exist (and how long brute-force attacks would take) is a direct combinatorics problem.
- **Lottery & Games —** the odds of winning a lottery are computed using combinations, e.g. $C(49,6)$ for a 6-number lottery from 49 numbers.
- **Scheduling & Timetabling —** counting valid arrangements of classes, exams or shifts without conflicts uses permutations and combinations.
- **Genetics —** counting the possible combinations of genes/alleles inherited from parents.
- **Sports Tournaments —** determining the number of possible match pairings or tournament brackets.

- **Computer Algorithms —** Big-O analysis of algorithms that generate permutations/combinations (e.g. the travelling salesman problem).

1. Basics of Counting

There are two fundamental principles of counting:

- Sum Rule (Addition Principle / Disjunctive Counting)
- Product Rule (Multiplication Principle / Sequential Counting)

1.1 Sum Rule

If a set X is the union of k disjoint non-empty subsets $S_1, S_2, \ldots, S_n$, then:

$$|X| = |S_1| + |S_2| + \ldots + |S_n|$$

Equivalently: if $E_1, E_2, \ldots, E_n$ are mutually exclusive events, and E_i can happen in e_i ways, then "E_1 or E_2 or … or E_n" can happen in:

$$e_1 + e_2 + \ldots + e_n \text{ ways}$$

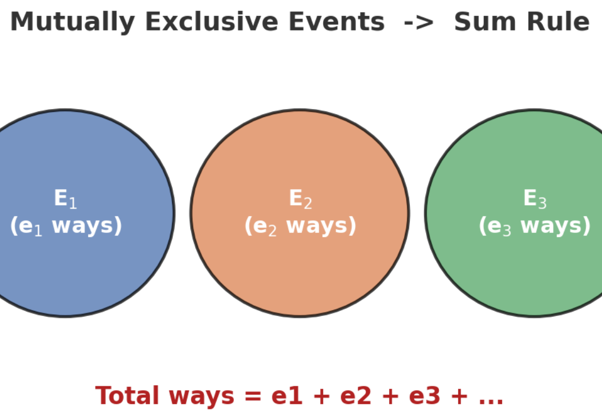


Figure 1: Sum Rule — disjoint/mutually exclusive events are added together.

Solved Examples — Sum Rule

Q1. There are 14 boys and 12 girls in a class. Find the number of ways of selecting one student.

Solution:

Selecting a boy among 14 boys can happen in 14 ways.

Selecting a girl among 12 girls can happen in 12 ways.

✓ By Sum Rule: 14 + 12 = 26 ways.

Q2. In how many ways can we draw a heart or a spade from an ordinary deck of cards? (a) A heart or an ace (b) An ace or a king (c) A numbered card (2–10) or a king?

Solution:

(i) Heart = 13 ways, Spade = 13 ways ⇒ 13 + 13 = 26 ways

(ii) Heart = 13 ways, Ace = 4 − 1 = 3 ways (excluding heart-ace) ⇒ 13 + 3 = 16 ways

(iii) Ace = 4 ways, King = 4 ways ⇒ 4 + 4 = 8 ways

(iv) Numbered cards (2–10) = 9 × 4 = 36 ways, King = 4 ways ⇒ 36 + 4 = 40 ways

✓ Answers: 26, 16, 8, 40 ways respectively.

1.2 Product Rule

If an event can occur in m ways and a second (independent) event can occur in n ways, both events together can occur in m × n ways. In general, if $E_1, E_2, \ldots, E_n$ happen in $n_1, n_2, \ldots, n_n$ ways, the sequence of events can happen in:

$$n_1 \times n_2 \times \ldots \times n_n \text{ ways}$$

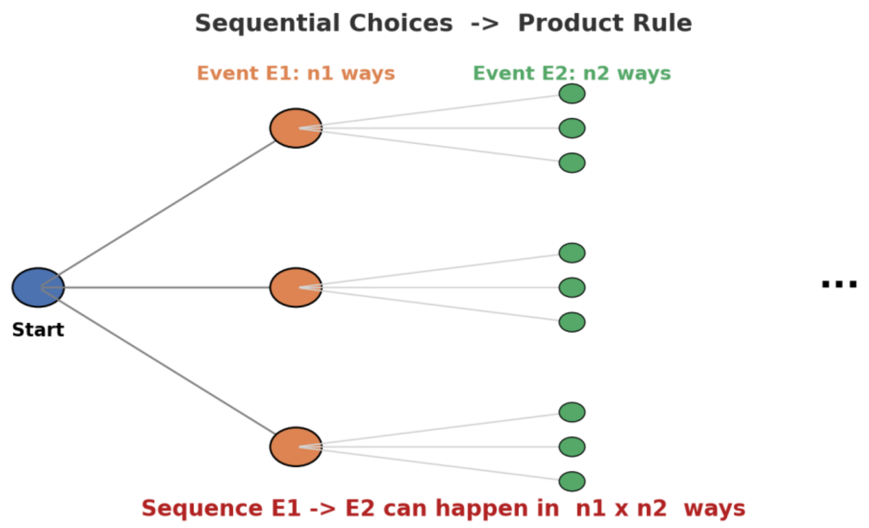


Figure 2: Product Rule — each sequential choice branches independently, so counts multiply.

Solved Examples — Product Rule

Q1. Three persons enter a car with 5 seats. In how many ways can they take up their seats?

Solution:

1st person: 5 choices, 2nd person: 4 choices, 3rd person: 3 choices.

✓ Total ways = 5 × 4 × 3 = 60 ways.

Q2. In how many ways can one select two books of different subjects among six distinct CS books, three distinct maths books, and two distinct chemistry books?

Solution:

One CS & one Maths: 6 × 3 = 18 ways

One CS & one Chemistry: 6 × 2 = 12 ways

One Maths & one Chemistry: 3 × 2 = 6 ways

These sets are pairwise disjoint, so apply the Sum Rule over the three products:

✓ Total = 18 + 12 + 6 = 36 ways.

2. Factorial & Combination

Factorial: The product of the first n natural numbers 1, 2, 3, …, n is denoted by n!

$$n! = n(n-1)(n-2)...3.2.1, \quad 0! = 1! = 1$$

$$n! = n\,.\,(n-1)!$$

2.1 Combination

An unordered selection of objects, taken some or all at a time, is called a Combination. An unordered selection of r objects from a set of n objects is an r-combination of n objects (without repetition), denoted C(n, r) or ${}^nC_r$:

$$C(n, r) = {}^nC_r = n! \,/\, [\, r!\,(n-r)!\,]$$

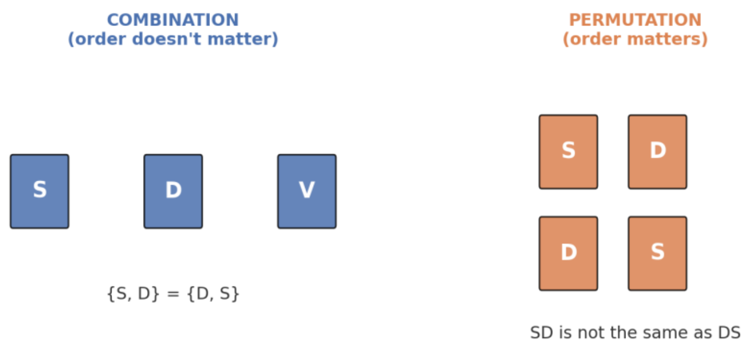


Figure 3: Combination (order irrelevant) vs. Permutation (order relevant).

Example: Selection of 2 batsmen from a set of 3 batsmen {Sachin, Dravid, Virat}. Here n = 3, r = 2:

$${}^3C_2 = 3! \,/\, [2!(3-2)!] = 3 \quad \text{i.e. } \{S,D\}, \{D,V\}, \{V,S\}$$

Key Results — Combinations

#	Result
1	${}^nC_0 = 1$, ${}^nC_1 = n$
2	${}^nC_n = 1$
3	${}^nC_r = {}^nC_{n-r}$
4	${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$
5	Combinations of r objects among n with repetition allowed: $C(n+r-1, n-1) = (n+r-1)! / [r!(n-1)!]$
6	No. of ways to distribute n objects to r people so each gets at least one object: ${}^{n-1}C_{r-1}$
7	No. of non-negative integer solutions of $x_1+x_2+...+x_r = n$ with $x_i > 0$: ${}^{n-1}C_{n-r}$
8	No. of non-negative integer solutions of $x_1+x_2+...+x_r = n$ with $x_i \geq 0$: ${}^{n+r-1}C_n$

3. Permutation

An arrangement of a given set of objects, taken some or all of them at a time, is called a Permutation. Any arrangement of r objects from a set of n objects (without repetition) is an r-permutation of n objects, denoted P(n, r) or ${}^nP_r$:

$$P(n, r) = {}^nP_r = n! \,/\, (n-r)!$$

Example: Number of ways of arranging the letters of “HAT” taken 2 at a time. Here n = 3, r = 2:

$${}^3P_2 = 3!/(3-2)! = 6 \quad \text{i.e. HA, AH, AT, TA, HT, TH}$$

Key Results — Permutations

#	Result
1	${}^{n}P_{0} = 1$
2	${}^{n}P_{n} = n!$
3	${}^{n}P_{1} = n$
4	${}^{n}P_{n-1} = {}^{n}P_{n}$
5	Permutations of n objects with r_1 of type 1, r_2 of type 2, ..., r_m of type m (rest distinct): $n! / (r_1!\, r_2! \ldots r_m!)$
6	Circular permutation of n objects = $(n - 1)!$

4. Solved Problems — Counting, Permutations & Combinations

Q1. In how many ways can five students be selected from 12 students (i) without replacement, (ii) if two students are not to be included?

Solution:

(i) 5 students can be selected from 12 without replacement in ${}^{12}C_{5} = 792$ ways.

(ii) If 2 students are excluded, select 5 students from the remaining 10 students: ${}^{10}C_{5} = 252$ ways.

✓ 792 ways ; 252 ways

Q2. How many four-digit numbers can be formed using digits 0,1,2,3,4,5 (i) if repetition is allowed, (ii) if repetition is not allowed?

Solution:

(ii) No repetition: thousands place ≠ 0 → 5 ways (1,2,3,4,5); remaining 3 places filled from remaining digits: 5 × 4 × 3 = 300... wait: 1st=5, 2nd=5, 3rd=4, 4th=3 (including 0 from 2nd place onward).

By Product Rule: 5 × 5 × 4 × 3 = 300 ways.

(i) Repetition allowed: 1st place excludes 0 → 5 ways; remaining 3 places each have 6 choices: 5 × 6 × 6 × 6 = 1080 ways.

✓ Without repetition: 300 ways ; With repetition: 1080 ways

Q3. How many different license plates are there that involve 1, 2 or 3 letters followed by 4 digits?

Solution:

1 letter + 4 digits: 26×10^4 ways

2 letters + 4 digits: $26^2 \times 10^4$ ways

3 letters + 4 digits: $26^3 \times 10^4$ ways

✓ By Sum Rule, total = $(26 + 26^2 + 26^3) \times 10^4 = 18278 \times 10^4$ ways

Q4. How many ways are there to seat 10 girls around a circular table?

Solution:

Total people = 10. Circular permutations = $(n - 1)! = (10 - 1)! = 9!$

✓ 9! = 362880 ways

Q5. Out of 5 men and 3 women, a committee of 3 is to be formed. In how many ways can it be formed if at least one woman is to be included?

Solution:

2 Men, 1 Woman: ${}^5C_2 \times {}^3C_1 = 10 \times 3 = 30$

1 Man, 2 Women: ${}^5C_1 \times {}^3C_2 = 5 \times 3 = 15$

0 Men, 3 Women: ${}^5C_0 \times {}^3C_3 = 1 \times 1 = 1$

✓ Total = 30 + 15 + 1 = 46 ways

Q6. Find the number of arrangements of the letters in the word ACCOUNTANT.

Solution:

Letters: A-2, C-2, O-1, U-1, N-2, T-2 → Total letters = 10

Arrangements = 10! / (2!2!2!2!)

✓ 10!/(2!2!2!2!) = 226800

Q7. In how many ways can the letters of the word COMPUTER be arranged? How many begin with C and end with R? How many do NOT begin with C but end with R?

Solution:

Total letters = 8, all distinct → 8! = 40320 ways.

Fix C first & R last → remaining 6 letters arranged: ${}^6P_6 = 6! = 720$ ways.

End with R, not starting with C: remaining 6 letters (excluding R fixed at end, C excluded from 1st place) arranged in 6×6×5×4×3×2×1 = 4320 ways.

✓ 40320 ways ; 720 ways (C...R) ; 4320 ways (not C...R)

Q8. A student is to answer 12 out of 15 questions. How many choices if (i) in all, (ii) must answer first two, (iii) first or second but not both, (iv) exactly 3 of first 5, (v) at least 3 of first 5?

Solution:

(i) ${}^{15}C_{12} = 455$

(ii) Must answer first two → choose remaining 10 from 13: ${}^{13}C_{10} = 286$

(iii) One of first two (${}^2C_1$ ways) × 11 from remaining 13 (${}^{13}C_{11}$): 2 × 78 = 156

(iv) Exactly 3 of first 5, 9 of remaining 10: ${}^5C_3 \times {}^{10}C_9 = 10 \times 10 = 100$

(v) At least 3 of first 5: (3,9)+(4,8)+(5,7) = ${}^5C_3{}^{10}C_9 + {}^5C_4{}^{10}C_8 + {}^5C_5{}^{10}C_7$ = 100+225+120 = 445

✓ 455 ; 286 ; 156 ; 100 ; 445

Q9. Find the number of arrangements of the letters of the word MATHEMATICS.

Solution:

Total letters = 11: M-2, A-2, T-2, H-1, E-1, I-1, C-1, S-1

Arrangements = 11! / (2!2!2!)

✓ 11!/(2!2!2!) = 4,989,600

Q10. There are 21 consonants and 5 vowels. Consider 10-letter words with 4 different vowels and 6 consonants. (i) How many such words? (ii) How many contain letter 'a'? (iii) How many begin with 'b' and end with 'c'?

Solution:

(i) Choose 4 vowels (${}^5C_4$), 6 consonants (${}^{21}C_6$), arrange 10 letters (10!): ${}^5C_4 \times {}^{21}C_6 \times 10!$

(ii) Word contains 'a': choose remaining 3 vowels from 4 (${}^4C_3$), 6 consonants from 21 (${}^{21}C_6$), arrange 10!: ${}^4C_3 \times {}^{21}C_6 \times 10!$

(iii) Begins with b, ends with c: remaining 4 vowels from 5 (${}^{5}C_4$), remaining 4 consonants from 19 (${}^{19}C_4$), arrange middle 8 letters (8!): ${}^{19}C_4 \times {}^{5}C_4 \times 8!$

✓ ${}^{5}C_4 \cdot {}^{21}C_6 \cdot 10!$; ${}^{4}C_3 \cdot {}^{21}C_6 \cdot 10!$; ${}^{19}C_4 \cdot {}^{5}C_4 \cdot 8!$

Q11. Consider the word TALLAHASSEE. How many arrangements (i) altogether, (ii) two A's together, (iii) both S's together and both E's together, (iv) if 4 letters are taken?

Solution:

Letters: T-1, A-3, L-2, H-1, S-2, E-2 → Total = 11

(i) 11!/(3!2!2!2!) = 831600

(ii) Treat 2 A's as one unit → 10 letters: 10!/(2!2!2!2!) = 226800

(iii) Treat 2 S's as one & 2 E's as one → 9 letters: 9!/(3!2!) = 30240

(iv) Cases by letter types — see table below.

✓ 831600 ; 226800 ; 30240 ; 896 (for 4-letter selections)

Q12. A man has 15 close friends of whom 6 are women. (i) Ways to invite 3 or more friends? (ii) Ways to invite 3 or more with equal men and women?

Solution:

(i) ${}^{15}C_3+{}^{15}C_4+...+{}^{15}C_{15}$ = 32647

(ii) Men=9, Women=6 available. Equal counts: (2,2),(3,3),(4,4),(5,5),(6,6): ${}^{9}C_2 \cdot {}^{6}C_2 + {}^{9}C_3 \cdot {}^{6}C_3 + {}^{9}C_4 \cdot {}^{6}C_4 + {}^{9}C_5 \cdot {}^{6}C_5 + {}^{9}C_6 \cdot {}^{6}C_6$ = 540+1680+1890+756+84

✓ (i) 32647 ways (ii) 540+1680+1890+756+84 = 4950 ways

Q13. A committee of 5 men and 3 women is formed from 7 men and 6 women. If two particular women are not to be together, how many committees are possible?

Solution:

Case 1 — both excluded: ${}^{7}C_5 \times {}^{4}C_3 = 21 \times 4 = 84$

Case 2 — exactly one included: ${}^{7}C_5 \times {}^{2}C_1 \times {}^{4}C_2 = 21 \times 2 \times 6 = 252$

✓ By Sum Rule: 84 + 252 = 336 committees

Q14. Enumerate the non-negative integral solutions to $x_1+x_2+x_3+x_4+x_5 \le 19$.

Solution:

For each value n = 0,1,...,19 of the sum, non-negative solutions with 5 variables = ${}^{n+4}C_4$ (since r = 5).

Sum over n = 0 to 19 of ${}^{n+4}C_4$ gives the total.

✓ Total = 42504 solutions

Q15. How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5 = 20$ where $x_1 \ge -3, x_2 \ge 0, x_3 \ge 4, x_4 \ge 2, x_5 \ge 2$?

Solution:

Substitute $y_1=x_1+3$, $y_2=x_2$, $y_3=x_3-4$, $y_4=x_4-2$, $y_5=x_5-2$ (all ≥ 0).

Equation becomes $y_1+y_2+y_3+y_4+y_5 = 20 - (-3+0+4+2+2) = 20 - 5 = 15$.

n = 15, r = 5 → solutions = ${}^{19}C_{15}$

✓ ${}^{19}C_{15}$ = 3876 solutions

Q16. Compute P(8, 5) and C(6, 3).

Solution:

$P(8,5) = 8!/(8-5)! = 8\times7\times6\times5\times4 = 6720$

$C(6,3) = 6!/[3!(6-3)!] = 20$

✓ P(8,5) = 6720 ; C(6,3) = 20

Q17. How many integers between 1 and 10,00,000 have digit sum equal to 18?

Solution:

Let $x_1,\ldots,x_6$ be the digits ($0 \le x_i \le 9$). Solve $x_1+\ldots+x_6 = 18$.

Unrestricted non-negative solutions: $^{23}C_{18} = 33649$

Subtract solutions where some $x_i \ge 10$ (set $y_i=x_i-10$): $6 \times {}^{13}C_8 = 6 \times 1287 = 7722$

✓ 33649 – 7722 = 25927 integers

Q18. How many integral solutions are there to $x_1+x_2+x_3+x_4+x_5 = 30$ for each case: (i) $x_i \ge 1$, (ii) $x_1\ge2, x_2\ge2, x_3\ge4, x_4\ge2, x_5\ge0$, (iii) $x_i \ge i+1$?

Solution:

(i) Set $y_i=x_i-1 \ge 0 \rightarrow \Sigma y_i = 30-5 = 25$, n=25, r=5 $\rightarrow {}^{29}C_{25} = 23751$

(ii) Set y_i accordingly $\rightarrow \Sigma y_i = 30-20 = 10$, n=10... (careful: r=5) $\rightarrow {}^{24}C_{20} = 10626$

(iii) Set $y_i=x_i-(i+1) \rightarrow \Sigma y_i = 30-20 = 10$, n=10, r=5 $\rightarrow {}^{14}C_{10} = 1001$

✓ (i) 23751 (ii) 10626 (iii) 1001

Problem 11(iv) — Selecting 4 letters from TALLAHASSEE

#	Result
1	4 different letters: $^6C_4 \times 4! = 360$
2	2 alike + 2 different: $^4C_1 \times {}^5C_2 \times (4!/2!) = 480$
3	2 alike + 2 alike: $^4C_2 \times (4!/2!2!) = 36$
4	3 alike + 1 different: $1 \times {}^5C_1 \times (4!/3!) = 20$

✓ Total arrangements = 360 + 480 + 36 + 20 = 896

5. Binomial Theorem

If $n \in \mathbb{N}$ is any non-negative integer, the Binomial Theorem is given by:

$$(x+y)^n = \Sigma_{r=0}^{n}\, {}^nC_r\, x^r\, y^{n-r} \;=\; \Sigma_{r=0}^{n}\, {}^nC_r\, x^{n-r}\, y^r$$

Notes:

- ${}^nC_r$ is called the binomial coefficient.
- The expansion has $(n + 1)$ terms.

Example: $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

6. Multinomial Theorem

The multinomial theorem generalizes the binomial theorem to more than two terms. For any positive integer n:

$$(x_1+x_2+\ldots+x_t)^n = \Sigma\, [n! / (r_1!r_2!\ldots r_t!)]\, x_1\text{^}r_1\, x_2\text{^}r_2 \ldots x_t\text{^}r_t$$

where $r_1 + r_2 + \ldots + r_t = n$, $r_i \geq 0$. The quantity $n!/(r_1!r_2!\ldots r_t!)$ is the multinomial coefficient, also written as the multinomial symbol $\pm(n; r_1,r_2,\ldots,r_t)$.

7. Solved Problems — Binomial & Multinomial Theorem

Q1. Find the binomial expansion of $(2a - 3b)^4$.

Solution:

Here $x = 2a$, $y = -3b$, $n = 4$.

$(2a-3b)^4 = \Sigma_{r=0}^{4}\, {}^4C_r\, (2a)^{4-r}(-3b)^r$

✓ **$(2a-3b)^4 = 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$**

Q2. Find the coefficient of x^3y^7 in $(x+y)^{10}$.

Solution:

$n=10$. Consider $r=7$ term: ${}^{10}C_7\, x^3y^7 = 120\, x^3y^7$

✓ **Coefficient = 120**

Q3. Find the coefficient of x^2y^4 in $(x - 2y)^6$.

Solution:

$n=6$, $y\to-2y$. Consider $r=4$ term: ${}^6C_4\, x^2(-2y)^4 = 15 \times x^2 \times 16y^4 = 240x^2y^4$

✓ **Coefficient = 240**

Q4. Find the coefficient of x^6y^3 in $(x - 3y)^9$.

Solution:

$n=9$. Consider $r=3$ term: ${}^9C_3\, x^6(-3y)^3 = 84 \times x^6 \times (-27)y^3 = -2268x^6y^3$

✓ **Coefficient = −2268**

Q5. Find the coefficient of x^9y^3 in $(2x - 3y)^{12}$.

Solution:

$n=12$, $x\to2x$, $y\to-3y$. Consider $r=3$ term: ${}^{12}C_3(2x)^9(-3y)^3 = 220\times2^9\times(-27)\, x^9y^3 = 3{,}041{,}280\, x^9y^3$

✓ **Coefficient = 3,041,280**

Q6. Find the coefficient of $x^3y^2z^2$ in $(2x - y + z)^7$.

Solution:

$n=7$. Take $r_1=3$ (x), $r_2=2$ (y), $r_3=2$ (z): term $= 7!/(3!2!2!) \times 2^3x^3(-y)^2z^2 = 1680\, x^3y^2z^2$

✓ Coefficient = 1680

Q7. Find the coefficient of xyz^2 in $[2x - y - z]^4$.

Solution:

$n=4$. Take $r_1=1, r_2=1, r_3=2$: term $= 4!/(1!1!2!) \times (2x)^1(-y)^1(-z)^2 = -24\, xyz^2$

✓ Coefficient = −24

Q8. Find the coefficient of $ab^2c^3d^4$ in $[a + 2b - 3c + 2d + 5]^{13}$.

Solution:

$n=13$, terms: $a, 2b, -3c, 2d, 5$. Take $r_1=1, r_2=2, r_3=3, r_4=4 \Rightarrow r_5 = 13-(1+2+3+4)=3$.

Term $= 13!/(1!2!3!4!3!) \times a^1(2b)^2(-3c)^3(2d)^4(5)^3$

✓ Coefficient of $ab^2c^3d^4 = -13!/(1!2!3!4!3!) \times 125$ (a large negative constant)

Q9. Find the coefficient of x^2yz in $[2x - y + z + 1]^7$.

Solution:

$n=7$, terms: $2x, -y, z, 1$. Take $r_1=2, r_2=1, r_3=1 \Rightarrow r_4 = 7-4 = 3$.

Term $= 7!/(2!1!1!3!) \times (2x)^2(-y)^1(z)^1(1)^3 = -1680\, x^2yz$

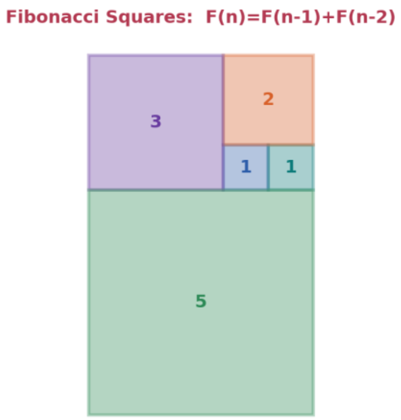
✓ Coefficient = −1680

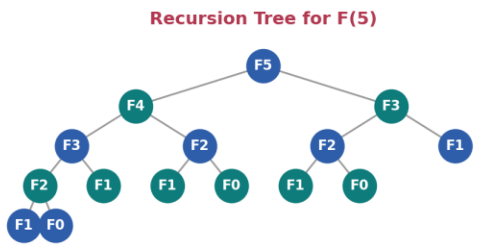
UNIT 4: RECURRENCE RELATIONS

Recurrence Relation: is an equation that defines each term of a sequence as a **function of its preceding terms**, together with one or more **initial conditions**. A classic example is the **Fibonacci relation**: $F(n) = F(n-1) + F(n-2)$, with $F(0) = 0, F(1) = 1$.

Origin of Recurrence Relations

The most famous early recurrence relation comes from **Leonardo of Pisa (Fibonacci)**, who introduced the Fibonacci sequence in his **1202** book *Liber Abaci*, originally posed as a puzzle about the growth of a rabbit population. Long before Fibonacci, similar recursive counting patterns had already appeared in the work of Indian mathematicians **Pingala**, **Virahanka** and **Hemachandra** (6th–12th centuries) while analyzing rhythmic patterns in Sanskrit poetry. Recurrence relations were later formalized as a general mathematical tool in the **18th century** through the work of **Abraham de Moivre** and **Daniel Bernoulli**, who developed methods to solve them in closed form, and today they are central to the analysis of algorithms in computer science.





Fibonacci tiling (each square's side = sum of the two before it) and the recursion tree for F(5)

Uses and Real-Life Examples

- **Algorithm Analysis —** the running time of recursive algorithms (like Merge Sort, $T(n) = 2T(n/2) + n$) is expressed and solved as a recurrence relation.
- **Finance —** compound interest follows the recurrence $A(n) = A(n-1) \times (1 + r)$, where each year's balance depends on the previous years.
- **Population Growth Models —** biological population sizes over generations are often modelled using recurrence relations.
- **Computer Programming —** recursive functions (a function calling itself) directly implement recurrence relations in code.
- **Fibonacci in Nature —** the arrangement of leaves, sunflower seeds, and pinecones frequently follows Fibonacci-like patterns, linked to efficient packing.
- **Dynamic Programming —** problems like the Knapsack problem and shortest paths are solved by breaking them into recurrence relations over smaller subproblems.

1. Generating Functions of Sequences

Let $a_0, a_1, a_2, \ldots, a_n, \ldots$ be a sequence of real numbers. The function

$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots = \Sigma_{n=0}\text{^}\infty\, a_nx^n$$

is called the generating function. Here $\{a_0, a_1, a_2, \ldots, a_n, \ldots\}$ is called the numeric function of a_n.

Example: The generating function for the numeric function $\{3^0, 3^1, 3^2, \ldots, 3^n, \ldots\}$ is $f(x) = \Sigma\, 3^nx^n$.

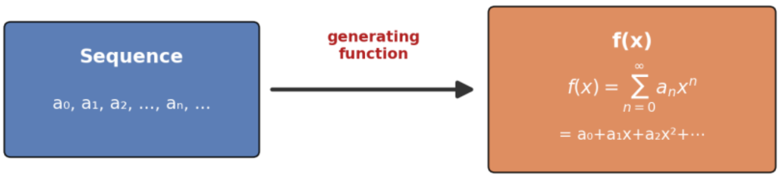


Figure 1: A sequence and its generating function represent the same information.

Solved Examples — Finding the Generating Function

Q1. Find the generating function for the sequence 1, 1, 0, 1, 1, 1, …

Solution:

Let $a_0=1, a_1=1, a_2=0, a_3=1, a_4=1, a_5=1, \ldots$

$f(x) = a_0+a_1x+a_2x^2+\ldots = 1+x+0.x^2+x^3+\ldots = (1+x+x^2+x^3+\ldots) - x^2$

✓ $f(x) = (1 - x)^{-1} - x^2$

Q2. Find the generating function for the sequence 1, 1, 1, 3, 1, 1, …

Solution:

Let $a_0=1, a_1=1, a_2=1, a_3=3, a_4=1, a_5=1, \ldots$

$f(x) = (1+x+x^2+x^3+x^4+\ldots) + 2x^3$

✓ $f(x) = (1 - x)^{-1} + 2x^3$

Standard Generating Function Expansions (Notes)

#	Result
1	$1/(1-x) = (1-x)^{-1} = 1+x+x^2+x^3+\ldots = \Sigma\, x^n$
2	$1/(1+x) = (1+x)^{-1} = 1-x+x^2-x^3+\ldots = \Sigma\, (-x)^n$
3	$1/(1-x)^2 = (1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots = \Sigma\, (n+1)x^n$
4	$1/(1+x)^2 = (1+x)^{-2} = 1-2x+3x^2-4x^3+\ldots = \Sigma\, (n+1)(-x)^n$
5	$e^x = 1+x+x^2/2!+x^3/3!+\ldots$

Solved Examples — Determine the Sequence from a Given Generating Function

Q1. $f(x) = 2e^x + 3x^2$.

Solution:

$2e^x = 2[1+x+x^2/2!+x^3/3!+x^4/4!+\ldots] = 2+2x+2x^2/2!+2x^3/3!+2x^4/4!+\ldots$

$f(x) = 2+2x+4x^2+2x^3/3!+2x^4/4!+\ldots + 3x^2$

✓ Sequence: {2, 2, 4, 2/3!, 2/4!, 2/5!, …}

Q2. **$f(x) = 7e^{8x} - 4e^{3x}$.**

Solution:

$7e^{8x} = 7[1+8x+(8x)^2/2!+(8x)^3/3!+...] = 7+56x+224x^2+...$

$4e^{3x} = 4[1+3x+(3x)^2/2!+(3x)^3/3!+...] = 4+12x+18x^2+...$

$f(x) = (7-4)+(56-12)x+(224-18)x^2+... = 3+44x+206x^2+...$

✓ Sequence: {3, 44, 206, ...}

Q3. **$f(x) = (2x - 3)^3$.**

Solution:

By Binomial Theorem: $(2x-3)^3 = {}^3C_0(2x)^3(-3)^0+{}^3C_1(2x)^2(-3)^1+{}^3C_2(2x)^1(-3)^2+{}^3C_3(2x)^0(-3)^3$

$= 8x^3 - 36x^2 + 54x - 27$

✓ Sequence: {−27, 54, −36, 8}

Q4. **$f(x) = x^4/(1 - x)$.**

Solution:

$x^4/(1-x) = x^4(1-x)^{-1} = x^4(1+x+x^2+x^3+...) = x^4+x^5+x^6+x^7+...$

✓ Sequence: {0, 0, 0, 0, 1, 1, 1, 1, ...}

Q5. **$f(x) = 1/(1 - ax)$.**

Solution:

$1/(1-ax) = (1-ax)^{-1} = 1+ax+(ax)^2+(ax)^3+... = 1+ax+a^2x^2+a^3x^3+a^4x^4+...$

✓ Sequence: {1, a, a^2, a^3, a^4, ...}

2. Calculating Coefficients of Generating Functions

#	Result
1	$C(n,r) = (n \text{ choose } r) = {}^{n}C_r = n! / [r!(n-r)!]$
2	$1 + x + x^2 + \ldots + x^n = (1 - x^{n+1}) / (1 - x)$
3	Coefficient of x^r in $(1+x+x^2+\ldots)^n = {}^{n+r-1}C_r$, since $(1+x+x^2+\ldots)^n = (1-x)^{-n} = \Sigma_r\, {}^{n+r-1}C_r\, x^r$

Solved Examples — Coefficient Extraction

Q1. Find the coefficient of x^{20} in $(x^3+x^4+x^5+\ldots)^5$.

Solution:

$(x^3+x^4+x^5+\ldots)^5 = [x^3(1+x+x^2+\ldots)]^5 = x^{15}(1-x)^{-5} = x^{15}\, \Sigma_r\, {}^{4+r}C_r\, x^r$

For x^{20}, we need $r = 5$: coefficient $= {}^{4+5}C_5 = {}^{9}C_5 = 126$

✓ Coefficient of x^{20} = 126

Q2. Find the coefficient of x^5 in $(1 - 2x)^{-7}$.

Solution:

$(1-2x)^{-7} = \Sigma_r\, {}^{6+r}C_r\, (2x)^r$

For $r = 5$: ${}^{11}C_5\, (2x)^5 = {}^{11}C_5 \times 2^5 \times x^5 = 462 \times 32 \times x^5 = 14784\, x^5$

✓ Coefficient of x^5 = 14784

Q3. Find the coefficient of x^{20} in $(x^2+x^3+x^4+x^5+x^6)^5$.

Solution:

$(x^2+x^3+x^4+x^5+x^6)^5 = [x^2(1+x+x^2+x^3+x^4)]^5 = x^{10}\,(1-x^5)^5\,(1-x)^{-5}$

$(1-x^5)^5 = {}^5C_0 - {}^5C_1x^5 + {}^5C_2x^{10} - {}^5C_3x^{15} + {}^5C_4x^{20} - {}^5C_5x^{25}$

$(1-x)^{-5} = \Sigma_r\, {}^{4+r}C_r\, x^r$. Need coefficient of x^{10} from the product (since x^{10} already factored out, target $x^{20} \Rightarrow$ remaining x^{10}):

Coefficient of $x^{20} = {}^5C_0 \cdot {}^{14}C_{10} - {}^5C_1 \cdot {}^9C_5 + {}^5C_2 \cdot {}^4C_0$

✓ Coefficient of x^{20} = 3817

3. Recurrence Relations

A recurrence relation is a mathematical equation that defines a sequence by relating each term to one or more previous terms in the same sequence.

One can carry out step-by-step computation to determine a_n from $a_{n-1}, a_{n-2}, \ldots$, provided that the values of the function at one or more points are given. These given values are called initial conditions or boundary conditions of the recurrence relation.

Example: The numeric function {5, 8, 11, 14, ...} is defined by the recurrence relation $a_n = a_{n-1} + 3, n \geq 1$, with initial condition $a_0 = 5$.

3.1 Order of a Recurrence Relation

The order of the recurrence relation is the difference between the largest and smallest subscript appearing in the relation.

- Ex 1: $a_n = -3a_{n-1}$ is an RR with order $n - (n-1) = 1$
- Ex 2: $a_{n+2} = a_{n+1} - 2a_n$ is an RR with order $(n+2) - n = 2$

3.2 Linear Recurrence Relation with Constant Coefficients

$$c_0 a_n + c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k} = f(n)$$

where c_i's are constant, is called a linear recurrence relation of k^{th} order, provided both c_0 and c_k are non-zero.

- If $f(n) = 0$, the relation is a Linear Homogeneous Equation.
- If $f(n) \neq 0$, the relation is a Linear Non-Homogeneous Equation.

3.3 Solution Methods for Recurrence Relations

An explicit formula that satisfies the recurrence relation together with its initial condition is called a solution of the RR. Three key methods are used:

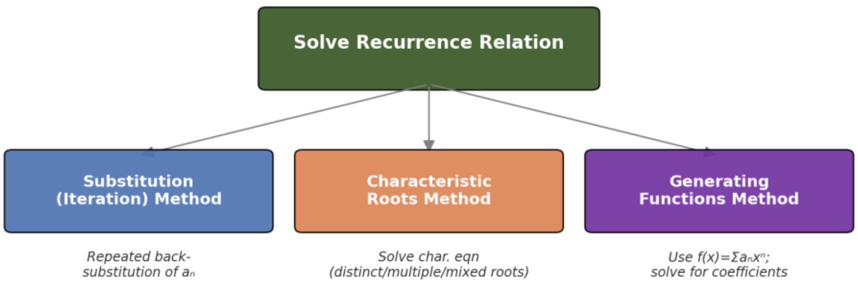


Figure 2: Three principal techniques used to solve recurrence relations.

4. Solving Recurrence Relations by Substitution Method

In this method the RR for a_n is used repeatedly to solve for a general expression for a_n in terms of n.

Q1. Solve $a_n = a_{n-1} + 2$, $n \geq 2$, subject to $a_1 = 3$.

Solution:

n=2: $a_2 = a_1+2 = 3+2 = a_1+2(1)$

n=3: $a_3 = a_2+2 = a_1+2(2)$

n=4: $a_4 = a_3+2 = a_1+2(3)$

In general: $a_n = a_1 + (n-1)(2) = 3 + 2n - 2$

✓ $\mathbf{a_n = 2n + 1}$

Q2. $a_n = a_{n-1} + 3^n$, $n \geq 1$, $a_0 = 1$.

Solution:

n=1: $a_1 = a_0+3$

n=2: $a_2 = a_1+3^2 = a_0+3+3^2$

n=3: $a_3 = a_2+3^3 = a_0+3+3^2+3^3$

Similarly: $a_n = a_0+3+3^2+\dots+3^n = 1+3+3^2+\dots+3^n = (1-3^{n+1})/(1-3)$

✓ $\mathbf{a_n = (3^{n+1} - 1)/2}$

Q3. $a_n + 3n \cdot a_{n-1} = 0$, $a_0 = 1$.

Solution:

n=1: $a_1 = -3(1)a_0$

n=2: $a_2 = -3(2)a_1 = -3(2)\cdot[-3(1)a_0] = 3^2 \cdot 2! \cdot a_0$

n=3: $a_3 = -3(3)a_2 = -3^3 \cdot 3! \cdot a_0$

✓ $\mathbf{a_n = (-3)^n \cdot n!}$ **(since $a_0 = 1$)**

5. Solving Recurrence Relations by Generating Functions Method

Let $a_n + A\,a_{n-1} + B\,a_{n-2} = 0$ for $n \ge 2$, be a 2nd order recurrence relation. Multiply by x^n and take summation over $n \ge 2$:

$$G(x)[1 + Ax + Bx^2] = a_0 + a_1x + Ax \cdot a_0$$

$$G(x) = [a_0 + x(a_1 + Aa_0)] / (1 + Ax + Bx^2)$$

Substitute a_0, a_1 values, then resolve into partial fractions. The coefficient of x^n in the resulting expansion of $G(x)$ determines a_n.

Q1. $\mathbf{a_n - 2a_{n-1} - 3a_{n-2} = 0,\ n \ge 2,\ \text{with}\ a_0 = 3,\ a_1 = 1.}$

Solution:

Let $G(x) = \Sigma a_n x^n$. Multiplying and summing gives: $G(x)[1 - 2x - 3x^2] = a_0 + (a_1 - 2a_0)x = 3 - 5x$

$G(x) = (3 - 5x)/[(1-3x)(1+x)] = A/(1-3x) + B/(1+x)$

Solving: $A = 1$, $B = 2$

$G(x) = 1/(1-3x) + 2/(1+x) = \Sigma 3^n x^n + 2\Sigma(-1)^n x^n$

✓ $\mathbf{a_n = 3^n + 2(-1)^n}$

Q2. $\mathbf{a_n - 5a_{n-1} + 6a_{n-2} = 2^n,\ n \ge 2,\ \text{with}\ a_0 = 1,\ a_1 = 1.}$

Solution:

$G(x)[1 - 5x + 6x^2] = 1/(1-2x) - 1 - 2x + 5x \cdot a_0 \Rightarrow G(x)(1-2x)(1-3x) = (1-6x+12x^2)/(1-2x)$

$G(x) = (12x^2-6x+1) / [(1-2x)^2(1-3x)]$

Partial fractions: $A/(1-2x) + B/(1-2x)^2 + C/(1-3x)$, giving $A=0$, $B=-2$, $C=3$

$G(x) = -2(1-2x)^{-2} + 3(1-3x)^{-1} = -2\Sigma(n+1)2^n x^n + 3\Sigma 3^n x^n$

✓ $\mathbf{a_n = -(n+1)2^{n+1} + 3^{n+1}}$

Q3. $\mathbf{a_n = 8a_{n-1} + 10^{n-1},\ \text{with}\ a_0 = 1,\ a_1 = 9.}$

Solution:

$a_n - 8a_{n-1} = 10^{n-1}$. Multiply by x^n and sum: $G(x) - a_0 - 8xG(x) = x/(1-10x)$

$G(x)(1-8x) = x/(1-10x) + 1 = (x+1-10x)/(1-10x) = (1-9x)/(1-10x)$

$G(x) = (1-9x) / [(1-10x)(1-8x)]$

Partial fractions: $A/(1-10x) + B/(1-8x)$, giving $A = 1/2$, $B = 1/2$

$G(x) = \tfrac{1}{2}\Sigma 10^n x^n + \tfrac{1}{2}\Sigma 8^n x^n$

✓ $\mathbf{a_n = \tfrac{1}{2}(8^n + 10^n)}$

Q4. $\mathbf{a_n - 9a_{n-1} + 20a_{n-2} = 0,\ n \ge 2,\ \text{with}\ a_0 = -3,\ a_1 = -10.}$

Solution:

$G(x)[1 - 9x + 20x^2] = a_0 + (a_1 - 9a_0)x = -3 + 17x$

$G(x) = (-3+17x) / [(1-5x)(1-4x)] = A/(1-5x) + B/(1-4x)$

Solving: $A = 2$, $B = -5$

$G(x) = 2\Sigma 5^n x^n - 5\Sigma 4^n x^n$

✓ $\mathbf{a_n = 2(5^n) - 5(4^n)}$

Q5. A person deposits ₹10,000 in a savings account yielding 12% per year, compounded annually. How much will be in the account after 30 years?

Solution:

Let P_n = amount after n years, $P_0 = 10{,}000$. Then $P_n = P_{n-1} + 0.12 \cdot P_{n-1} = 1.12 \cdot P_{n-1}$

Solving this first-order RR by substitution: $P_n = (1.12)^n P_0$

General solution: $P_n = 10{,}000\ (1.12)^n$

$P_{30} = 10{,}000\ (1.12)^{30}$

✓ P_{30} = ₹2,99,599.22 (approx.)

6. Method of Characteristic Roots

In this method, the solution of a recurrence relation is obtained as a sum of two parts:

- The homogeneous solution $a_n^{(h)}$, which satisfies the RR when f(n) = 0.
- The particular solution $a_n^{(p)}$, which satisfies the relation with f(n) on the RHS.

$$a_n = a_n^{(h)} + a_n^{(p)}$$

Note: If the RHS of the given recurrence relation is zero, the general solution is simply $a_n = a_n^{(h)}$.

6.1 Solving Linear Homogeneous Equations

Let $a_n = r^n$ be a solution of $c_0a_n+c_1a_{n-1}+\dots+c_ka_{n-k} = 0$. Dividing by r^{n-k} gives the characteristic equation:

$$c_0r^k + c_1r^{k-1} + \dots + c_k = 0$$

This equation has k roots, called characteristic roots.

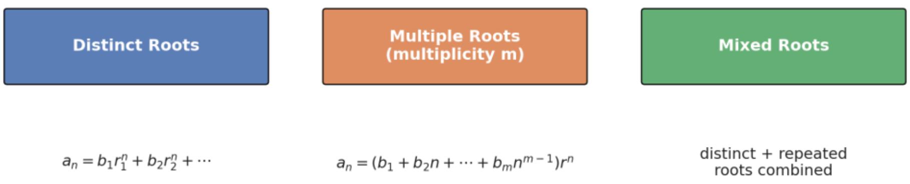


Figure 3: The three characteristic-root cases and their general solution forms.

#	Result
1	Distinct roots $r_1,r_2,\dots,r_k$: general solution $a_n = b_1{r_1}^n + b_2{r_2}^n + \dots + b_k{r_k}^n$
2	Repeated root r with multiplicity m: general solution $a_n = (b_1+b_2n+b_3n^2+\dots+b_mn^{m-1})r^n$
3	Mixed roots: combine both distinct-root terms and repeated-root terms in the same general solution

Solved Examples — Characteristic Roots Method

Q1. $\mathbf{a_n - a_{n-1} - 2a_{n-2} = 0.}$

Solution:

Let $a_n = r^n$. Characteristic equation: $r^2 - r - 2 = 0$

$r(r-2) + 1(r-2) = 0 \Rightarrow (r-2)\,(r+1) = 0 \Rightarrow r = 2, -1$

✓ $\mathbf{a_n = b_1\,(2)^n + b_2\,(-1)^n}$

Q2. $\mathbf{a_n - 7a_{n-1} + 12a_{n-2} = 0,\ n \geq 2,\ with\ a_0 = 2,\ a_1 = 5.}$

Solution:

Characteristic equation: $r^2 - 7r + 12 = 0 \Rightarrow (r-3)\,(r-4) = 0 \Rightarrow r = 3, 4$

General solution: $a_n = b_1(3)^n + b_2(4)^n$

$a_0=2 \Rightarrow b_1+b_2=2$; $a_1=5 \Rightarrow 3b_1+4b_2=5$

Solving: $b_2 = -1$, $b_1 = 3$

✓ $\mathbf{a_n = 3^{n+1} - 4^n}$

Q3. $\mathbf{a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}, n \geq 3}$**, with** $\mathbf{a_0=3, a_1=1, a_2=0}$**.**

Solution:

Characteristic equation: $r^3 - 2r^2 - r + 2 = 0$. Testing $r=1$: root found.

Factor: $(r-1)(r^2-r-2) = 0 = (r-1)(r-2)(r+1) \Rightarrow r = 1, 2, -1$

General solution: $a_n = b_1(1)^n + b_2(2)^n + b_3(-1)^n$

Using $a_0=3$, $a_1=1$, $a_2=0$ → solving the 3 equations gives $b_1=7/2$, $b_2=-1$, $b_3=1/2$

✓ $\mathbf{a_n = \frac{1}{2}[7 - 2^{n+1} + (-1)^n]}$

Q4. **Find the solution for** $\mathbf{a_{n+2} - 6a_{n+1} + 9a_n = 0}$**.**

Solution:

Characteristic equation: $r^2 - 6r + 9 = 0 \Rightarrow (r-3)^2 = 0 \Rightarrow r = 3, 3$ (repeated root)

✓ $\mathbf{a_n = (b_1 + b_2 n)3^n}$

7. Solution of Inhomogeneous Recurrence Relations

The general solution of a linear non-homogeneous recurrence relation $c_0a_n+c_1a_{n-1}+\ldots+c_ka_{n-k} = f(n)$ is $a_n = a_n^{(h)} + a_n^{(p)}$. There is no general method for finding the particular solution for every f(n); instead, we use the method of inspection with the trial solutions below:

f(n)	Trial particular solution
f(n) = constant	$a_n^{(p)} = A_0$
f(n) = polynomial of degree m	$a_n^{(p)} = A_0+A_1n+A_2n^2+\ldots+A_mn^m$
$f(n) = b^n$ (b is not a root)	$a_n^{(p)} = A_0b^n$
$f(n) = b^n$ (b is a root of multiplicity m)	$a_n^{(p)} = A_0n^mb^n$
$f(n) = b^n$ × polynomial (b a root of multiplicity m)	$a_n^{(p)} = b^nn^m(A_0+A_1n+\ldots)$

Solved Examples — Inhomogeneous Recurrence Relations

Q1. Find the particular solution of $a_n = 3a_{n-1} + 7$.

Solution:

$a_n - 3a_{n-1} = 7$. RHS is constant, so let $a_n^{(p)} = A_0$

$A_0 - 3A_0 = 7 \Rightarrow -2A_0 = 7 \Rightarrow A_0 = -7/2$

✓ $a_n^{(p)} = -7/2$

Q2. Solve $a_{n+2} + 4a_{n+1} + 4a_n = 7$, $n \geq 0$, with $a_0 = 1$, $a_1 = 2$.

Solution:

Homogeneous part: $r^2+4r+4=0 \Rightarrow (r+2)^2=0 \Rightarrow r=-2,-2 \Rightarrow a_n^{(h)}=(b_1+b_2n)(-2)^n$

Particular part: f(n)=constant, let $a_n^{(p)}=A_0 \Rightarrow A_0+4A_0+4A_0=7 \Rightarrow 9A_0=7 \Rightarrow A_0=7/9$

General: $a_n=(b_1+b_2n)(-2)^n+7/9$. Using $a_0=1$: $b_1=2/9$. Using $a_1=2$: $b_2=-5/6$

✓ $a_n = (2/9 - 5n/6)(-2)^n + 7/9$

Q3. Solve $y_{n+2} - y_{n+1} - 2y_n = n^2$, $n \geq 0$.

Solution:

Homogeneous part: $r^2-r-2=0 \Rightarrow (r-2)(r+1)=0 \Rightarrow r=2,-1 \Rightarrow y_n^{(h)}=b_12^n+b_2(-1)^n$

Particular part: $f(n)=n^2$ (polynomial degree 2), let $y_n^{(p)}=A_0+A_1n+A_2n^2$

Substituting & equating coefficients: $A_2=-1/2$, $A_1=-1/2$, $A_0=-1$

✓ $y_n = b_1(2)^n + b_2(-1)^n - 1 - n/2 - n^2/2$

8. Solving Recurrence Relations by Iterative Method

In this method, we use the recurrence relation iteratively (i.e. repeated back-substitution) to solve for a closed-form expression.

Q1. $a_n = a_{n-1} + 2, a_0 = 3.$

Solution:

Replace n by n−1: $a_{n-1} = a_{n-2} + 2$. Substituting: $a_n = a_{n-2} + 2(2)$

Replace n by n−2: $a_{n-2} = a_{n-3} + 2$. Substituting: $a_n = a_{n-3} + 3(2)$

Similarly, $a_n = a_{n-k} + k(2)$. Put k = n: $a_n = a_0 + 2n$

✓ $a_n = 3 + 2n$

Q2. $a_n = a_{n-1} + 2n + 3, a_0 = 1.$

Solution:

Replace n by n−1: $a_{n-1} = a_{n-2} + 2(n-1) + 3$. Substituting into original relation repeatedly:

$a_n = a_{n-k} + [2(n-(k-1))+3] + [2(n-(k-2))+3] + \dots + [2n+3]$

Put k = n: $a_n = a_0 + \Sigma_{r=1}^{n} (2r+3) = a_0 + 2 \cdot n(n+1)/2 + 3n = 1 + n^2 + n + 3n$

✓ $a_n = n^2 + 4n + 1$

UNIT 5: GRAPH THEORY

Introduction

Graph theory is the branch of mathematics that studies graphs — structures made up of points, called vertices or nodes, connected by lines, called edges or links. Despite its simple building blocks, graph theory provides a powerful language for describing relationships and connections between objects, whether those objects are cities on a map, people in a social network, web pages on the internet, or molecules in a chemical compound.

A graph is formally denoted as $G = (V, E)$, where V is a set of vertices and E is a set of edges connecting pairs of vertices. Graphs can be directed (edges have a direction, such as one-way streets) or undirected (edges are bidirectional, such as friendships). They can also be weighted, where each edge carries a numerical value representing cost, distance, or capacity.

Origin and History

Graph theory traces its origin to 1736, when the Swiss mathematician Leonhard Euler solved the now-famous "Seven Bridges of Königsberg" problem. The city of Königsberg (now Kaliningrad, Russia) was divided by a river into four land areas connected by seven bridges. The puzzle asked whether it was possible to walk through the city crossing each bridge exactly once and returning to the starting point.

Euler proved that no such route existed by representing the land masses as vertices and the bridges as edges, reducing the geographic puzzle to a question about the structure of a graph. This abstraction — replacing real-world objects with points and connections — is considered the birth of graph theory and, more broadly, an early milestone in the field of topology.

Over the following two centuries, mathematicians such as Gustav Kirchhoff (electrical circuits), Arthur Cayley (chemical structures and trees), and later Dénes Kőnig, who wrote the first textbook on graph theory in 1936, expanded the field. The 20th century saw explosive growth as computer science, operations research, and network science adopted graph theory as a foundational tool.

Uses and Applications

Graph theory underlies many of the technologies and systems used every day. Because it models relationships so naturally, it appears across computer science, engineering, biology, social science, and logistics.

Computer Science and Networking

- Modeling computer networks, the internet, and data routing between servers.
- Powering search engine algorithms, such as Google's PageRank, which treats web pages as vertices and hyperlinks as edges.
- Representing data structures like trees and linked lists, and solving problems such as shortest-path search (Dijkstra's algorithm) and network flow.

Transportation and Logistics

- Designing road, rail, and flight networks, and finding optimal routes (used by GPS and navigation apps).
- Solving the traveling salesman and vehicle routing problems for delivery and supply-chain optimization.

Social Sciences

- Analyzing social networks, where people are vertices and relationships (friendships, follows) are edges.
- Identifying influential individuals, communities, and the spread of information or trends.

Biology and Chemistry

- Modeling molecular structures, where atoms are vertices and chemical bonds are edges.
- Mapping neural networks in the brain and protein-interaction networks in genetics.

Other Fields

- Scheduling and resource allocation (exam timetabling, register allocation in compilers).
- Electrical circuit design, using Kirchhoff's original graph-based analysis.
- Epidemiology, modeling how diseases spread through contact networks.

Real-Life Examples

Example 1: Google Maps and GPS Navigation

When a navigation app finds the fastest route between two locations, it models the road network as a weighted graph — intersections are vertices, roads are edges, and travel time or distance is the weight. Shortest-path algorithms then compute the optimal route in seconds.

Example 2: Social Media Networks

Platforms such as Facebook and LinkedIn represent users as vertices and connections (friendships or follows) as edges. Graph algorithms suggest new friends, detect communities, and measure how information or content spreads through the network.

Example 3: The Internet and Web Search

The World Wide Web is itself a massive directed graph, with web pages as vertices and hyperlinks as directed edges. Search engines analyze this graph structure to rank pages by importance and relevance.

Example 4: Airline Flight Routes

Airlines model their route networks as graphs, with airports as vertices and flights as edges. This helps optimize scheduling, minimize layovers, and identify the most efficient hub-and-spoke structures.

Example 5: Epidemiology and Disease Spread

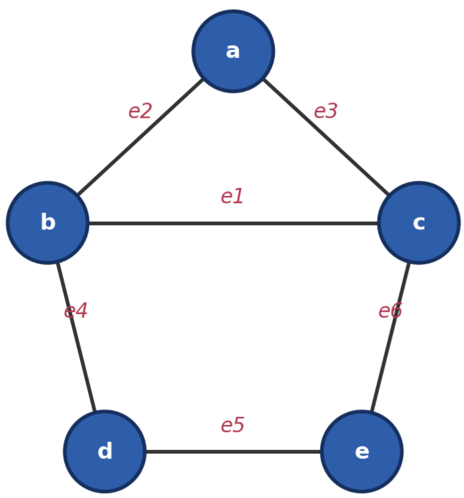
Public health researchers model populations as graphs, with individuals as vertices and contact events as edges, to simulate and predict how infectious diseases spread and to evaluate the impact of interventions such as vaccination or quarantine.

1. Introduction to Graphs

Graph: A graph G is a pair of sets (V, E), where V is the set of points called **vertices**, and E is a set of lines called **edges**. Both V and E are assumed to be finite sets.

Key Notes

- **Vertex / Node / Junction —** A point in a graph is also called a node or a junction.
- **Edge / Line / Branch / Arc —** A line joining two vertices in G is called an edge, line, branch or arc.
- **Order —** The number of vertices in G is denoted by $|V|$ or $|V(G)|$.
- **Size —** The number of edges in G is denoted by $|E|$ or $|E(G)|$.



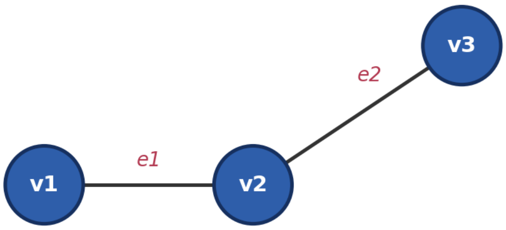
Example graph — $|V| = 5$, $|E| = 6$

Here the graph has five vertices *a, b, c, d, e* and six edges *e1, e2, e3, e4, e5, e6*. So **$|V| = 5$** and **$|E| = 6$**.

2. Adjacent Vertices and Adjacent Edges

Adjacent Vertices: Two vertices u and v are said to be adjacent if there exists an edge connecting u and v.

Adjacent Edges: If two edges have a common vertex, then they are called adjacent edges.

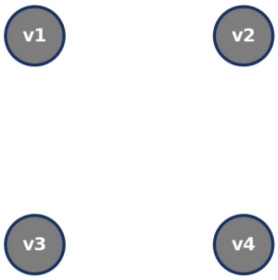


v1 and v2 are adjacent vertices; v1 and v3 are NOT adjacent

3. Special Types of Graphs

Finite, Infinite, Trivial and Null Graphs

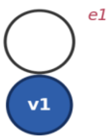
- **Finite graph:** A graph with a finite number of vertices and finite number of edges.
- **Infinite graph:** A graph with an infinite number of vertices.
- **Trivial graph:** A graph with only 0 or 1 vertex.
- **Null graph:** A graph without any edges (contains only isolated vertices).



Null graph — |V| = 4, |E| = 0

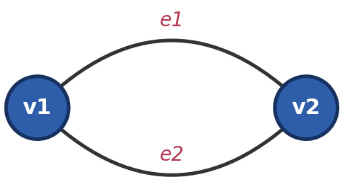
Loop, Parallel Edges and Multiple Edges

- **Loop:** An edge is a loop if it connects a vertex to itself (also called a self-loop).



e1 is a loop at v1

- **Parallel edges:** Two edges are parallel if they have the same pair of end vertices.

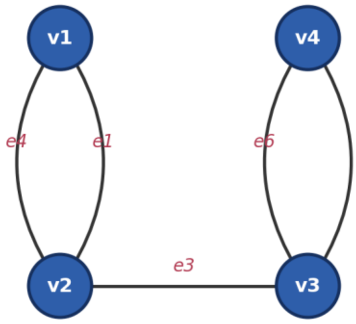


e1 and e2 are parallel edges

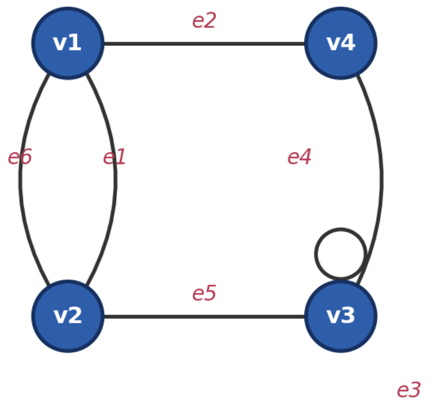
- **Multiple edges:** If two or more edges have the same vertices, they are called multiple edges.

Simple Graph, Multigraph and Pseudograph

- **Simple graph:** A graph with neither a loop nor a parallel edge.
- **Multigraph:** A graph containing multiple/parallel edges (no loops).
- **Pseudograph:** A graph in which both loops and multiple edges are allowed.



Multigraph — parallel edges present



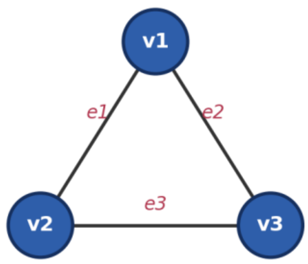
Pseudograph — loops and parallel edges both present

Note:

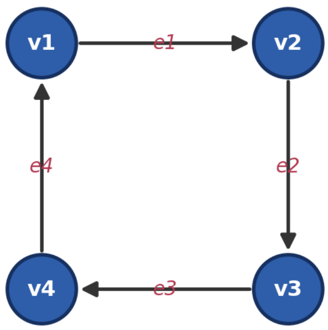
- In some networks and programs, multigraphs are allowed to contain loops; in others they are not.

4. Directed, Undirected and Mixed Graphs

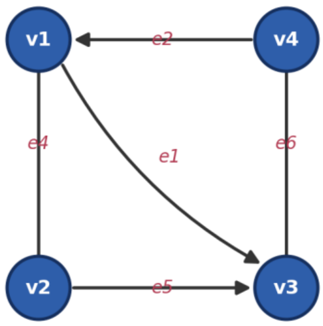
- **Undirected graph:** A graph in which every edge is undirected.
- **Directed graph (Digraph):** A graph in which every edge is directed.
- **Mixed graph:** A graph containing both directed and undirected edges.



Undirected graph



Directed graph (digraph)

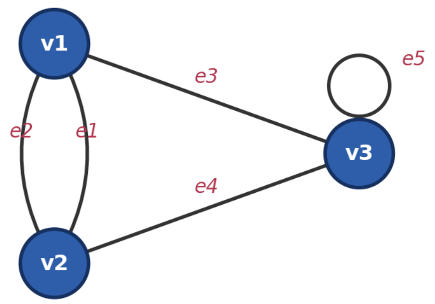


Mixed graph

Note: Suppose e1 = (v1, v2) is a directed edge of a digraph, then (i) v1 is the **initial vertex** of e1, (ii) v2 is the **terminal vertex** of e1, (iii) e1 is said to be incident from v1 to v2, and (iv) v1, v2 are called adjacent nodes of e1.

5. Degree of a Vertex

Degree of a vertex: The degree of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes **twice** to the degree of that vertex. It is denoted by $deg(v)$ or $d(v)$.



Graph G with a loop and parallel edges

Here $deg(v1) = 3$, $deg(v2) = 3$, $deg(v3) = 4$ (the loop e5 adds 2 to the degree of v3).

Minimum and Maximum Degree

The minimum and maximum degrees of the vertices of a graph G are denoted by $\delta(G)$ and $\Delta(G)$ respectively.

Isolated and Pendant Vertices

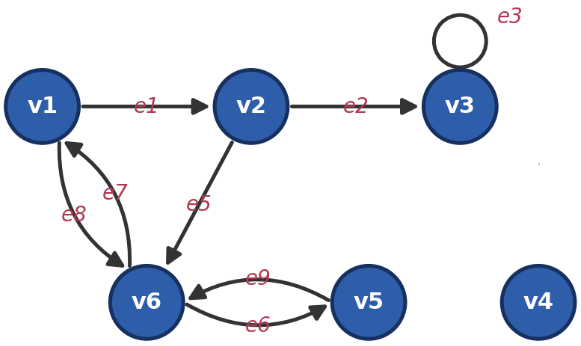
- **Isolated vertex:** A vertex with no edges connecting to it, i.e. degree zero.
- **Pendant vertex:** A vertex whose degree is exactly one.

Note:

- A graph containing only isolated vertices is called a null graph.

Indegree and Outdegree (for directed graphs)

- **Indegree:** The indegree of a vertex in a digraph is the number of inward-directed edges to that vertex.
- **Outdegree:** The outdegree of a vertex in a digraph is the number of outward-directed edges from that vertex.



Directed graph — indegree / outdegree example

Vertex	Indegree	Outdegree
v1	1	2
v2	1	2
v3	1	1
v5	1	2
v6	3	1

6. Handshaking Theorem (Euler's First Theorem)

The sum of the degrees of all vertices in a graph G is **twice** the number of edges in that graph:

$$\Sigma \deg(v_i) = 2e$$

where e is the number of edges and n is the number of vertices.

Solved Problems

Q. How many vertices will the following graphs have, if the conditions are:

- (i) 16 edges and all the vertices of degree 2.
- (ii) 21 edges, three vertices of degree 4 and other vertices of degree 3.
- (iii) 24 edges and all the vertices of the same degree.

Solution (i):

Given e = 16, all vertices have degree 2. By handshaking theorem, $\Sigma \deg(v_i) = 2e$

$2 + 2 + \ldots + 2\ (n\ \text{times}) = 2(16) \implies 2n = 32 \implies n = 16$

∴ Number of vertices = 16

Solution (ii):

Given e = 21, three vertices of degree 4 and (n − 3) vertices of degree 3.

$4+4+4+3+3+...+3(n-3 \text{ times}) = 2(21) \implies 12 + 3(n-3) = 42$

$3(n-3) = 30 \implies n - 3 = 10 \implies n = 13$

∴ Total number of vertices = 13

Solution (iii):

Given e = 24, all vertices have the same degree, say d.

$n \times d = 48 \quad (\text{using } \Sigma deg = 2e = 48)$

For a simple graph, $0 \le d \le n - 1$. Testing factor pairs of 48 satisfying $d \le n - 1$:

$(n,d) = (8,6), (12,4), (16,3), (24,2), (48,1)$

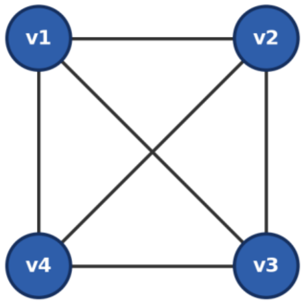
∴ Possible vertices = 8, 12, 16, 24 or 48, with degree 6, 4, 3, 2, 1 respectively.

7. Matrix Representation of Graphs

Although graphs are geometric figures, their representation in computer memory is through matrices — mainly the **Adjacency Matrix** and the **Incidence Matrix**.

Adjacency Matrix of a Simple Undirected Graph

For a graph G with n vertices, the n×n adjacency matrix A = $[a_{ij}]$ is defined as: $a_{ij} = 1$ if v_i is adjacent to v_j, and $a_{ij} = 0$ otherwise.



Graph G (K4)

	v1	v2	v3	v4
v1	0	1	1	1
v2	1	0	1	1
v3	1	1	0	1
v4	1	1	1	0

Adjacency Matrix for a Multigraph

a_{ij} = m, where m = number of edges between v_i and v_j (m ≥ 1); otherwise a_{ij} = 0.

Adjacency Matrix for a Directed Graph

a_{ij} = m, where m = number of edges beginning at v_i and ending at v_j; otherwise a_{ij} = 0.

Example: Draw the Graph Represented by a Given Adjacency Matrix

Given the adjacency matrix:

	v1	v2	v3	v4
v1	1	2	0	1
v2	2	0	3	0
v3	0	3	1	1
v4	1	0	1	0

Corresponding graph has a loop at v1, parallel edges (v1–v2), (v2–v3) and a loop at v3

Incidence Matrix

For a graph G with n vertices and m edges, the n×m incidence matrix X = [x_{ij}] is defined as: $x_{ij} = 1$ if v_i is incident to edge e_j, $x_{ij} = 2$ if e_j is a loop at v_i, and $x_{ij} = 0$ otherwise.

Incidence Matrix of a Directed Graph

$x_{ij} = 1$ if v_i is the initial vertex of e_j; $x_{ij} = -1$ if v_i is the terminal vertex of e_j; $x_{ij} = 0$ if v_i is not incident on e_j.

8. Walks, Paths, Trails, Circuits and Cycles

Walk: A sequence of vertices and edges v1, e1, v2, e2, ..., e_{n-1}, v_n in which each edge's endpoints are the preceding and succeeding vertices. Repetition of vertices and edges is allowed. The length of a walk is the number of edges.

- **Closed walk:** A walk that begins and ends at the same vertex.
- **Open walk:** A walk whose terminal vertices are different.

Path: A walk with no repeated vertices (hence no repeated edges).

Trail: A walk with no repeated edges. Repetition of vertices is allowed.

Circuit: A closed walk with no repeated edges, i.e. start and end vertex are the same. Repetition of vertices is allowed. A circuit is a closed trail.

Cycle: A closed walk with no repeated edges and no repeated vertices other than the starting and ending vertex. A cycle is a closed path with no repetition of edges and vertices.

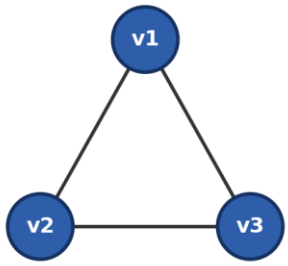
Note:

- Every cycle is a circuit, but not every circuit is a cycle.

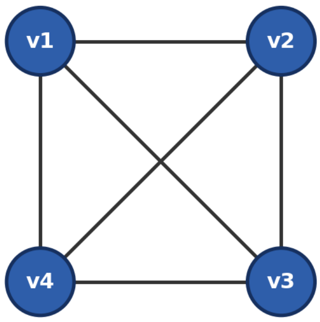
Structure	Repeated Vertices	Repeated Edges	Closed
Walk	✓	✓	✓ / ✗
Path	✗	✗	✓ / ✗
Trail	✓	✗	✓ / ✗
Circuit	✓	✗	✓
Cycle	✗	✗	✓

9. Regular Graph

Regular graph: A graph in which all vertices are of equal degree is called a regular graph. If every vertex has degree k, it is called a k-regular graph.



2-regular graph — $deg(v_i) = 2$



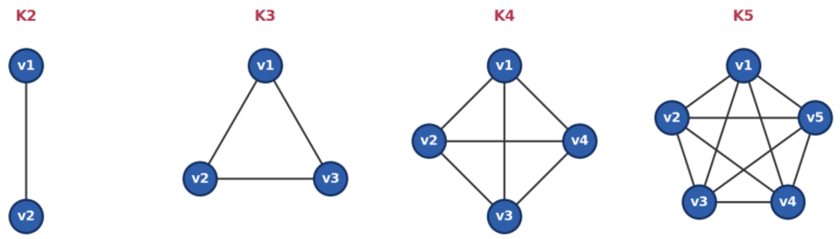
3-regular graph — $deg(v_i) = 3$

Note:

- Every null graph is a regular graph of degree 0 (0-regular graph).
- A complete graph K_n is a regular graph of degree n − 1 (i.e. (n−1)-regular graph).

10. Complete Graph

Complete graph: A simple graph G in which every pair of distinct vertices is adjacent is called a complete graph. A complete graph of n vertices is denoted by K_n and has exactly $n(n-1)/2$ edges.

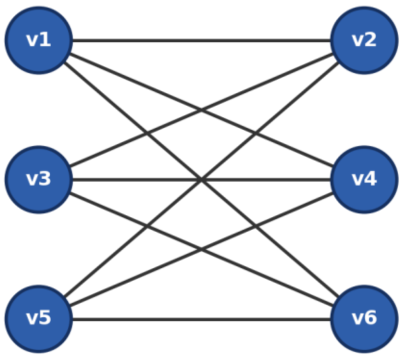


Complete graphs K2, K3, K4 and K5

11. Bipartite Graph and Complete Bipartite Graph

Bipartite graph: A loop-free graph G in which the vertex set V is partitioned into two disjoint sets X and Y such that every edge of G has one end in X and the other end in Y. (X, Y) is called the bi-partition of G.

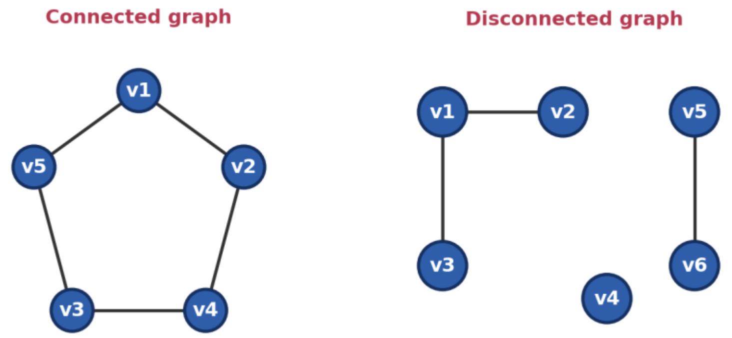
Complete bipartite graph: For a simple bipartite graph with bipartition (X, Y), if there is an edge between every pair of vertices from X and Y, then the graph is a complete bipartite graph, denoted $K_{m,n}$.



Complete bipartite graph K3,3 with bi-partition X = {v1, v3, v5}, Y = {v2, v4, v6}

12. Connected Graph

Connected graph: A graph G = (V, E) is said to be connected if, for every pair of vertices u, v ∈ V, there exists a path between u and v. Otherwise, G is disconnected.



Connected graph vs. Disconnected graph

13. Isomorphism of Graphs

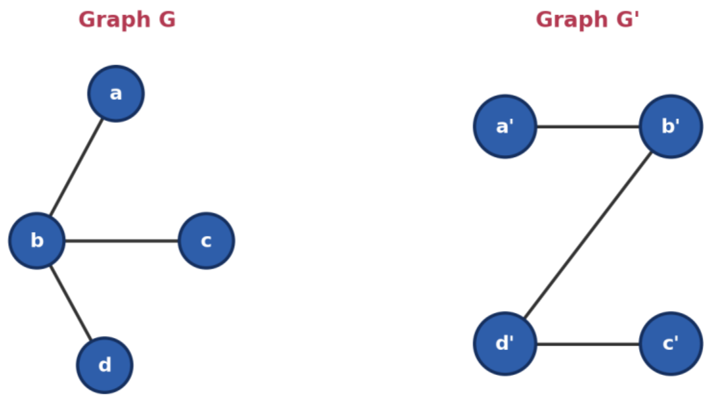
Two graphs G and G' are isomorphic if there is a function f : V(G) → V(G') such that: (i) f is one-one, (ii) f is onto, (iii) for each pair of vertices u, v of G, $(u,v) \in E(G) \Leftrightarrow (f(u), f(v)) \in E(G')$.

Consequences of an Isomorphism

- |V(G)| = |V(G')| — same number of vertices.
- |E(G)| = |E(G')| — same number of edges.
- The degree sequences of G and G' are identical.
- Loops and cycles map to loops and cycles of the same length.

Solved Problem

Q. Verify whether the following graphs are isomorphic or not.



Graph G and Graph G'

Solution:

Order and size: $|V(G)| = |V(G')| = 4$, $|E(G)| = |E(G')| = 3$

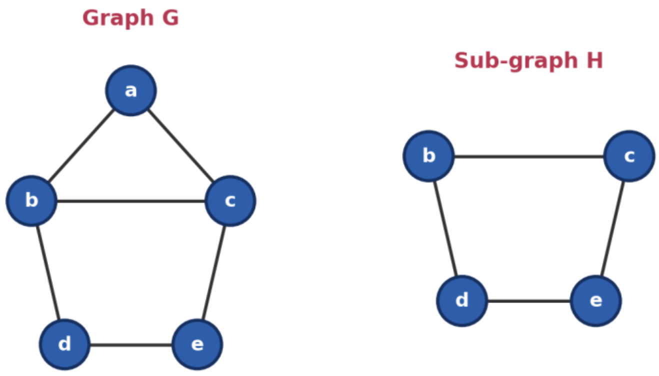
Degree sequence of G = {1, 1, 2, 2}, Degree sequence of G' = {1, 1, 2, 2} — same.

Define f(a) = a', f(b) = b', f(c) = d', f(d) = c'. f is one-one and onto, and it preserves adjacency: {a,b} ∈ E(G) ⟺ {a',b'} ∈ E(G'), and similarly for the remaining edges.

∴ The adjacency matrices match, hence the graphs are isomorphic.

14. Subgraphs

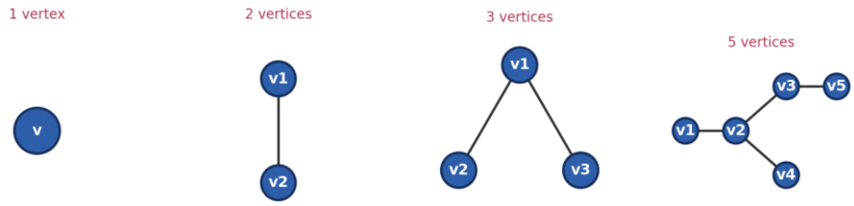
Subgraph: Let G and H be two graphs with vertex sets V(G), V(H) and edge sets E(G), E(H). If V(H) ⊆ V(G) and E(H) ⊆ E(G), then H is called a subgraph of G.



Graph G and one of its sub-graphs H

15. Trees and Spanning Trees

Tree: A tree is a connected, acyclic graph. Its edges are called branches. A tree with a single vertex is a trivial tree; otherwise, it is a non-trivial tree.

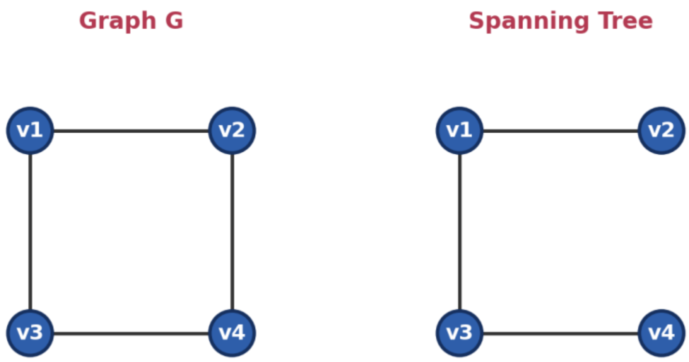


Trees with 1, 2, 3 and 5 vertices

Note:

- If in a graph G there is exactly one path between every pair of vertices, then G is a tree.
- A tree with n vertices has exactly n − 1 edges.

Spanning tree: A subgraph H of a graph G is a spanning tree of G if (i) H is a tree, and (ii) H contains all the vertices of G.

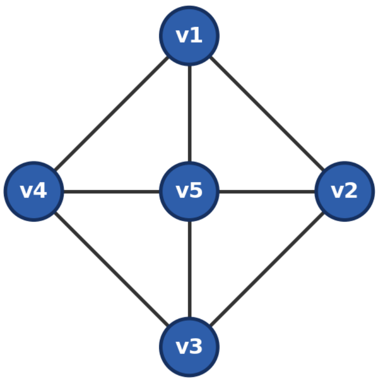


Graph G and one of its spanning trees

16. Planar Graphs and Euler's Formula

Planar graph: A graph is said to be planar if it can be drawn in a plane such that its edges do not intersect each other except at vertices.

Euler's Formula: Let G be a connected planar graph, then $r = e - v + 2$, where r = number of regions, e = number of edges, v = number of vertices.



Planar graph — v = 5, e = 8, r = 5

$r = e - v + 2 = 8 - 5 + 2 = 5$ ✓ **Verified**

17. Euler Path, Euler Circuit and Euler Graph

Euler path: A path that passes through each edge of the graph exactly once, but vertices may be repeated.

Note:

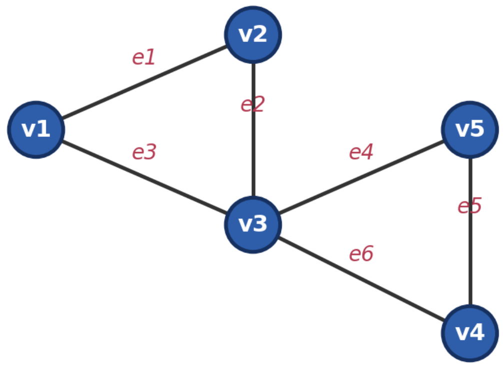
- At most 2 vertices in the graph may have odd degree for an Euler path to exist.

Euler circuit: A graph G is said to have an Euler circuit if there is a circuit in G that traverses every edge exactly once, i.e. the start and end vertices are the same. In an Euler circuit, every vertex must have even degree.

Euler graph: A graph G is called an Euler graph if it has an Euler circuit.

Solved Problem

Q. Determine whether the following graph has an Euler path or Euler circuit.



Two triangles joined at a common vertex v3

Solution:

All vertices have even degree, so we can trace an Euler circuit:

v1, e1, v2, e2, v3, e4, v5, e5, v4, e6, v3, e3, v1

∴ The graph is an Euler circuit, and hence also has an Euler path.

18. Hamiltonian Path and Hamiltonian Circuit

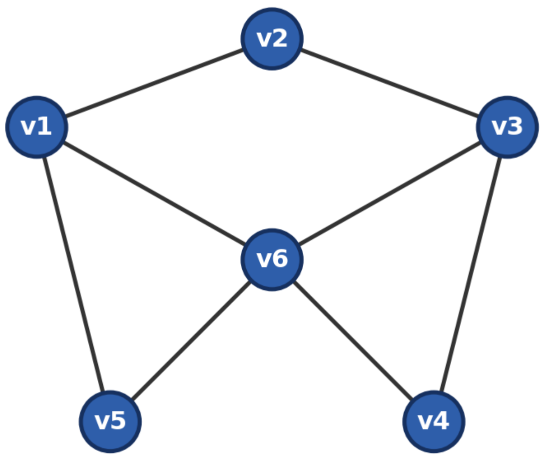
Hamiltonian path: A path in graph G that uses each vertex of the graph exactly once.

Hamiltonian circuit: A circuit that contains each vertex in G exactly once, except the start and end vertices, which are the same.

Hamiltonian graph: A graph containing a Hamiltonian cycle (circuit) is called a Hamiltonian graph.

Solved Problem

Q. Determine whether the following graph is Hamiltonian (path or circuit).



Example graph

Solution:

Tracing the vertices: v1, v2, v3, v4, v6, v5 visits every vertex exactly once $\Longrightarrow$ Hamiltonian path.

Extending back to v1: v1, v2, v3, v4, v6, v5, v1 revisits only the start vertex $\Longrightarrow$ Hamiltonian circuit.

∴ The graph is both a Hamiltonian path and a Hamiltonian circuit, hence a Hamiltonian graph.

Key Distinction:

- A Eulerian graph has a circuit that uses every EDGE exactly once (vertices may repeat).
- A Hamiltonian graph has a circuit that uses every VERTEX exactly once, except the first/last (edges may be skipped).