**SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES, CHITTOOR – 517127**
**(AUTONOMOUS)**
Affiliated to JNTUA, Anantapuramu, NBA & NAAC Accredited

**DEPARTMENT**
**OF**
**ELECTRONICS & COMMUNICATION ENGINEERING**

# PROBABILITY AND COMPLEX VARIABLES

**(23BSC214 )**

# LECTURE NOTES

**Prepared by**
**Dr. R.Kodandan**

## INSTITUTE VISION AND MISSION

### ➢ VISION

- To emerge as a centre of excellence for learning and research in the domains of engineering, computing and management.

### ➢ MISSION

- Provide congenial academic ambience with state-art of resources for learning and research.
- Ignite the students to acquire self-reliance in the latest technologies.
- Unleash and encourage the innate potential and creativity of students.
- Inculcate confidence to face and experience new challenges.
- Foster enterprising spirit among students.
- Work collaboratively with technical Institutes/Universities/Industries of National and International repute.

## DEPARTMENT VISION AND MISSION

### ➢ VISION

- To become a centre of excellence in Electronics and Communication Engineering and provide necessary skills to the students to meet the challenges of industry and society.

### ➢ MISSION

- Provide congenial academic ambience with necessary infrastructure and learning resources
- Inculcate confidence to face and experience new challenges from industry and society.
- Ignite the students to acquire self-reliance in State-of-the-Art Technologies
- Foster Enterprising spirit among students

## PROGRAMME EDUCATIONAL OBJECTIVES (PEO's)

**PEO1:** Have Professional competency through the application of knowledge gained from subjects like Mathematics, Physics, Chemistry, Inter-Disciplinary and core subjects like Signal Processing, VLSI, Embedded Systems, Communication and Automation. **(Professional Competency)**

**PEO2:** E*xcel* in one's career by critical thinking towards successful services and growth of the organization or as an entrepreneur or through higher studies. **(Successful Career Goals)**

**PEO3:** Enhance knowledge by updating advanced technological concepts for facing the rapidly changing world and contribute to society through innovation and creativit*y.* **(Continuing Education and Contribution to Society)**

## PROGRAM SPECIFIC OUTCOMES (PSO'S)

**PSO1:** Apply the knowledge obtained in core areas for the analysis and design of components in Signal Processing, VLSI, Embedded Systems, Communication and Automation.

**PSO2:** Adapt Innovation, Creativity and design to develop products which meet industrial and societal needs.

# PROGRAMME OUTCOMES (PO's)

PO1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO2. **Problem analysis**: Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of

the limitations.

PO6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

## Course Educational Objectives

- To introduce the basic concepts of probability and random variables.
- To introduce the basic concepts of multiple random variables.
- To understand concepts of operations on multiple random variables.
- To analyze functions of a complex variable, with continuity, differentiability and analyticity.
- To understand Taylor and Laurent expansions, residues and improper integrals.

## Syllabus Overview

| Unit | Title | Key Topics |
|---|---|---|
| Unit I | Probability & Random Variable | Probability through sets & relative frequency; sample spaces; axioms; joint, conditional & total probability; Bayes' theorem; random variables; distribution & density functions; Gaussian, Binomial, Poisson, Uniform, Exponential, Rayleigh. |
| Unit II | Operations on Random Variable | Expectation; moments; characteristic & moment generating functions; multiple/vector random variables; joint distribution & density; marginal & conditional distributions; statistical independence. |
| Unit III | Operations on Multiple Random Variables | Joint moments; joint characteristic functions; jointly Gaussian random variables (two-variable case) and their properties. |
| Unit IV | Complex Variable — Differentiation | Limits, continuity, differentiation; Cauchy-Riemann equations; analytic & harmonic functions; harmonic conjugates; Milne-Thomson method. |
| Unit V | Complex Variable — Integration | Line integrals; Cauchy's integral theorem & formula; Taylor's & Laurent's series; zeros, singularities, residues; Cauchy residue theorem. |

## Course Outcomes

| CO | Outcome |
|---|---|
| CO1 | Understand the concepts of probability, random variables and their characteristics. |
| CO2 | Deal with multiple random variables, conditional probability, joint distribution and statistical independence. |
| CO3 | Formulate and solve engineering problems involving random variables. |
| CO4 | Analyze limit, continuity and differentiation of complex functions; understand Cauchy-Riemann equations and analytic functions. |
| CO5 | Apply Cauchy's theorem and integral formulas; classify singularities and evaluate integrals using residues. |

## CO – PO Mapping

| CO \ PO | PO1 | PO2 | PO3 |
|---|---|---|---|
| CO1 | 3 | 3 | 3 |
| CO2 | 3 | 3 | 3 |
| CO3 | 3 | 3 | 3 |
| CO4 | 3 | 3 | 3 |
| CO5 | 3 | 3 | 3 |

## Text Books


- Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", 4th Edition, TMH, 2002.
- B.S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 44th Edition, 2017.


## Reference Books


- Athanasios Papoulis & S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, PHI, 2002.
- Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley India.
- Henry Stark & John W. Woods, "Probability and Random Processes with Application to Signal Processing", 3rd Edition, Pearson, 2002.
- B.V. Ramana, "Higher Engineering Mathematics", McGraw Hill.

# UNIT I — Probability and Random Variables (Detailed Notes)

*Pre-requisites: Set Theory, Permutations and Combinations, Calculus, Real Analysis, Linear Algebra.*

## Historical Background

Galileo (1564–1642), an Italian mathematician, was the first to attempt a quantitative measure of probability while dealing with problems related to dice in gambling. The first formal foundation of the mathematical theory of probability, however, was laid in the mid-17th century by two French mathematicians, B. Pascal (1623–1662) and P. Fermat (1601–1665), while solving a number of problems posed to Pascal by the French gambler and nobleman De Mere.

## Quick Reference — Standard Sample Spaces

- Dice: 6 faces.
- Cards: total 52 — Diamond (13), Heart (13), Spade (13), Clubs (13); each suit has A, 2, 3, ..., 10, J, Q, K; colours → Black & Red (26 each).

## Key Definitions

**Probability**
The chance of getting a success or a failure in a trial is called probability.

**Random Experiment**
If an experiment is conducted any number of times under essentially identical conditions and there is a set of all possible outcomes associated with it, and the result is not certain beforehand, the experiment is called a random experiment. Examples: tossing a coin, drawing a card from a pack of 52 cards.

**Sample Space**
The set of all possible outcomes of a random experiment is called the sample space, denoted S. E.g., tossing a coin: $S = \{H, T\}$; throwing a die: $S = \{1, 2, 3, 4, 5, 6\}$.

**Discrete Sample Space**
A sample space that contains a finite set of elements.

**Continuous Sample Space**
A sample space that contains an uncountable, infinite set of elements.

**Trial**
Conducting an experiment once is called a trial.

**Outcome**
The result of a trial in a random experiment.

**Event**
Every non-empty subset of a sample space in a random experiment is called an event.

**Simple Event**
An event in a trial that cannot be split further.

**Exhaustive Event**
The total number of possible outcomes of an experiment.

**Favourable Event**
The outcomes which favour the happening of the event in question.

**Dependent Event**
Two events are dependent if the happening of one affects the happening of the other.

**Independent Event**
Two events are independent if the happening of one does not affect the happening of the other.

## Classical Definition of Probability

In a random experiment, if n is the number of exhaustive cases and m is the number of favourable cases of an event A, then the probability of A is denoted by P(A), defined as:

$$P(A) = n(A) / n(S) = m / n$$

Property: $0 \leq P(A) \leq 1$.

## Solved Examples — Classical Probability

**Example 1: Find the probability of getting a head in tossing a coin.**

Sample space: S = {H, T}, so n(S) = 2.
Let A = event of getting a head = {H}, so n(A) = 1.
**P(A) = n(A)/n(S) = 1/2 = 0.5**

**Example 2: Find the probability of getting only one head in tossing two coins.**

Sample space: S = {HH, HT, TH, TT}, so n(S) = 4.
Let A = event of getting only one head = {HT, TH}, so n(A) = 2.
**P(A) = n(A)/n(S) = 2/4 = 0.5**

**Example 3: In a single throw of two dice, find the probability of getting (i) a sum of 10, (ii) a sum which is a perfect square.**

Sample space consists of 36 equally likely outcomes (i, j), i, j = 1,...,6, so n(S) = 36.
(i) A = event that sum = 10 = {(4,6), (5,5), (6,4)}, n(A) = 3.

**$P(A) = 3/36 = 1/12 \approx 0.083$**

(ii) Perfect squares between 2 and 12 are 4 and 9. $B = \{(1,3),(2,2),(3,1),(3,6),(4,5),(5,4),(6,3)\}$, $n(B) = 7$.

**$P(B) = 7/36 \approx 0.19$**

**Example 4: Three light bulbs are selected at random from 12 bulbs, of which 5 are defective. Find the probability that (i) all are defective, (ii) exactly one is defective, (iii) exactly two are defective.**

Total bulbs = 12 (5 defective, 7 good); bulbs selected = 3, so $n(S) = {}^{12}C_3 = 220$.

(i) A = all 3 defective: $n(A) = {}^{5}C_3 = 10$.

**$P(A) = 10/220 \approx 0.045$**

(ii) B = exactly one defective: $n(B) = {}^{5}C_1 \times {}^{7}C_2 = 5 \times 21 = 105$.

**$P(B) = 105/220 \approx 0.477$**

(iii) C = exactly two defective: $n(C) = {}^{5}C_2 \times {}^{7}C_1 = 10 \times 7 = 70$.

**$P(C) = 70/220 \approx 0.318$**

**Example 5: Three cards are drawn from a pack of 52 cards. Find the probability that (i) all three are spades, (ii) 2 spades and 1 diamond, (iii) one spade, one diamond and one heart.**

Total cards selected = 3 from 52, so $n(S) = {}^{52}C_3 = 22100$.

(i) A = three spades: $n(A) = {}^{13}C_3 = 286$.

**$P(A) = 286/22100 \approx 0.013$**

(ii) B = 2 spades and 1 diamond: $n(B) = {}^{13}C_2 \times {}^{13}C_1 = 78 \times 13 = 1014$.

**$P(B) = 1014/22100 \approx 0.046$**

(iii) C = one spade, one diamond, one heart: $n(C) = {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 = 2197$.

**$P(C) = 2197/22100 \approx 0.099$**

## Axioms of Probability

If A and B are any two events in a sample space S, then:

- (i) $0 \le P(A) \le 1$
- (ii) $P(S) = 1$ (total probability is 1)
- (iii) If A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$

## Property — Complementary Events

To show that $P(A^c) = 1 - P(A)$:

Since $S = A$ □ $A^c$ and $A$, $A^c$ are mutually exclusive, $P(S) = P(A) + P(A^c)$.

But $P(S) = 1$, so $1 = P(A) + P(A^c)$.

$$P(A^c) = 1 - P(A)$$

## Property — $P(A^c \cap B)$

For any two events A and B, prove that $P(A^c \cap B) = P(B) - P(A \cap B)$.

$A \cap B$ and $A^c \cap B$ are disjoint events, and their union is B.

So $P(B) = P(A \cap B) + P(A^c \cap B)$.

$$P(A^c \cap B) = P(B) - P(A \cap B)$$

## Addition Theorem of Probability

For any two events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof: A and $(A^c \cap B)$ are mutually exclusive, and A □ $(A^c \cap B) = A \cup B$.

P(A □ B) = $P(A) + P(A^c \cap B) = P(A) + [P(B) - P(A \cap B)] = P(A) + P(B) - P(A \cap B)$.

## Addition Theorem — Three Events

For any three events A, B and C:

$$P(A \cup B \cup C) = P(A)+P(B)+P(C) - P(A\cap B) - P(B\cap C) - P(C\cap A) + P(A\cap B\cap C)$$

## Conditional Probability

If A and B are any two events of an experiment, the conditional probability of occurrence of A given that B has occurred is:

$$P(A/B) = P(A \cap B) / P(B), \quad P(B) > 0$$

- If A and B are independent events, $P(A \cap B) = P(A)\cdot P(B)$, so $P(A/B) = P(A)$.
- If A and B are mutually exclusive events, then $P(A) \leq P(B^c)$.
- If A and B are independent, then $A^c$ and $B^c$ are also independent.

**Example 6: Box A contains 5 red and 3 white marbles; Box B contains 2 red and 6 white marbles. A marble is drawn from each box. Find the probability that both are of the same colour.**

Let E1 = red from box A: P(E1) = (1/2)(5/8) = 5/16.

Let E2 = red from box B: P(E2) = (1/2)(2/8) = 1/8.

P(both red) = P(E1)·P(E2) = (5/16)(1/8) ≈ 0.039.

Let E3 = white from box A: P(E3) = (1/2)(3/8) = 3/16.

Let E4 = white from box B: P(E4) = (1/2)(6/8) = 3/8.

P(both white) = P(E3)·P(E4) = (3/16)(3/8) ≈ 0.070.

**P(same colour) = 0.039 + 0.070 = 0.109**

**Example 7: The probabilities that students A, B, C, D solve a problem are 1/3, 2/5, 1/5 and 1/4 respectively. If all of them try, what is the probability that the problem is solved?**

$P(\bar{A})=2/3$, $P(\bar{B})=3/5$, $P(\bar{C})=4/5$, $P(\bar{D})=3/4$ (probabilities of NOT solving).

Assuming independence, P(none solves) = (2/3)(3/5)(4/5)(3/4) = 6/25.

**P(solved) = 1 − 6/25 = 19/25 = 0.76**

**Example 8: A can hit a target 3 times in 5 shots, B can hit it 2 times in 5 shots, C hits it 3 times in 4 shots. Find the probability the target is hit when all three try.**

$P(A)=3/5$, $P(B)=2/5$, $P(C)=3/4 \Rightarrow P(\bar{A})=2/5$, $P(\bar{B})=3/5$, $P(\bar{C})=1/4$.

P(none hits) = (2/5)(3/5)(1/4) = 3/50.

**P(hit) = 1 − 3/50 = 47/50**

## Bayes' Theorem

Statement: Let E1, E2, ..., En be n mutually exclusive and exhaustive events with $P(E_i) > 0$ $(i = 1,...,n)$ in a sample space S, and let A be any event in S intersecting every $E_i$ with $P(A) > 0$. Then for any $E_i$:

$$P(E_i / A) = [P(E_i) \cdot P(A/E_i)] / \Sigma_j\, P(E_j) \cdot P(A/E_j)$$

Proof sketch: Since S = E1 ∪ E2 ∪ ... ∪ En, A = (A∩E1) □ (A∩E2) □ ... □ (A∩En), so $P(A) = \Sigma P(A \cap E_i)$. Using $P(E_i/A) = P(E_i \cap A)/P(A)$ and $P(A \cap E_i) = P(E_i)P(A/E_i)$, the result follows.

**Example 9: 5 men out of 100 and 25 women out of 10,000 are colour blind. A colour-blind person is selected at random. What is the probability the person is male?**

P(M) = P(W) = 1/2 (equal prior). P(B/M) = 5/100, P(B/W) = 25/10000.

$$P(M/B) = [P(M)P(B/M)] / [P(M)P(B/M) + P(W)P(B/W)]$$

$P(M/B) = (0.5 \times 0.05) / [(0.5 \times 0.05) + (0.5 \times 0.0025)] = 0.025 / 0.02625$

**P(M/B) ≈ 0.952**

**Example 10: Machines A, B, C in a bolt factory manufacture 20%, 30% and 50% of the total output, with 6%, 3% and 2% defective respectively. A bolt is drawn at random and found defective. Find the probability it came from Machine A, B, C.**

P(A)=0.2, P(B)=0.3, P(C)=0.5; P(D/A)=0.06, P(D/B)=0.03, P(D/C)=0.02.

P(D) = 0.2(0.06) + 0.3(0.03) + 0.5(0.02) = 0.031.

**P(A/D) = 0.012/0.031 ≈ 0.387**

**P(B/D) = 0.009/0.031 ≈ 0.290**

**P(C/D) = 0.010/0.031 ≈ 0.322**

**Example 11: A businessman visits hotels X, Y, Z, 20%, 50%, 30% of the time respectively. It is known that 5%, 4%, 8% of rooms in X, Y, Z have faulty plumbing. What is the probability that a faulty-plumbing room is assigned to hotel Z?**

P(X)=0.2, P(Y)=0.5, P(Z)=0.3; P(E/X)=0.05, P(E/Y)=0.04, P(E/Z)=0.08.

P(E) = 0.2(0.05)+0.5(0.04)+0.3(0.08) = 0.045.

**$P(Z/E) = (0.3 \times 0.08)/0.045 \approx 0.444$**

**Example 12: Companies B1, B2, B3 produce 30%, 45%, 25% of cars, with 2%, 3%, 2% defective. (a) Find P(car is defective). (b) If a purchased car is defective, find the probability it was produced by B3.**

P(D) = 0.3(0.02)+0.45(0.03)+0.25(0.02) = 0.0245.

**(a) P(D) = 0.0245**

$P(B3/D) = (0.25 \times 0.02)/0.0245 \approx 0.204$

**(b) P(B3/D) ≈ 0.204**

**Example 13: Companies X, Y, Z produce TVs — X produces twice as many as Y, while Y and Z produce equal numbers. 2% of X, 2% of Y, 4% of Z are defective. (a) Find the probability a randomly selected TV is defective. (b) Given it is defective, find the probability it was made by X.**

P(X)=2P(Y), P(Y)=P(Z), and P(X)+P(Y)+P(Z)=1 ⇒ P(Y)=P(Z)=0.25, P(X)=0.5.

P(D) = 0.5(0.02) + 0.25(0.02) + 0.25(0.04) = 0.025.

**(a) P(D) = 0.025**

P(X/D) = (0.5 × 0.02)/0.025 = 0.4

**(b) P(X/D) = 0.4**

## Random Variables

### Random Variable

A random variable X is a function that maps every outcome of a sample space onto a point on the real line, i.e. $X : S \rightarrow \mathbb{R}$.

Example: Tossing two coins — let X be the number of heads shown. X maps 'no head' → 0, 'one head' → 1, 'both heads' → 2, so X = {0, 1, 2}, i.e. X = {x1, x2, x3}.

## Types of Random Variables

### Discrete Random Variable (D.R.V.)

One having only discrete values. E.g., for discrete sample space S = {1,2,3,4}, define $X = S^2 = \{1,4,9,16\}$.

### Continuous Random Variable (C.R.V.)

One having a continuous range of values. E.g., temperature T of a region is continuous, existing in a range $-T_1$ to $T_1$.

### Mixed Random Variable (M.R.V.)

One for which some values are discrete and some are continuous.

## Probability Distribution Function (CDF)

The distribution function describes the probabilistic behaviour of a random variable X. It gives the probability $P\{X \le x\}$ for all values of x, and is also called the cumulative distribution function (CDF):

$$F_x(x) = P\{X \le x\}$$

**Example 14: Tossing 3 coins — X is the number of tails. Find and sketch the distribution function.**

X = {0, 1, 2, 3} with P(x1)=1/8 (0 tails), P(x2)=3/8 (1 tail), P(x3)=3/8 (2 tails), P(x4)=1/8 (3 tails).

| x | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| P(x) | 1/8 | 3/8 | 3/8 | 1/8 |
| $F_x(x)$ | 1/8 | 4/8 | 7/8 | 1 |

$F_x(x) = \Sigma\, P(x_i) \cdot u(x - x_i)$, where $u(x) = 1$ for $x \ge 0$ and 0 for $x < 0$.

## Properties of the Distribution Function

- $F_x(-\infty) = 0$
- $F_x(\infty) = 1$
- $0 \le F_x(x) \le 1$
- $F_x(x1) \le F_x(x2)$ if $x1 < x2$ (non-decreasing)
- $P\{x1 \le X \le x2\} = F_x(x2) - F_x(x1)$

## Expression for the Distribution Function

Discrete random variable:

$$F_x(x) = \sum_{i=1}^{N} P(x_i) \cdot u(x - x_i)$$

Continuous random variable:

$$F_x(x) = \int_{-\infty}^{x} f_x(\alpha)\, d\alpha$$

## Probability Density Function (PDF)

The PDF $f_x(x)$ is the derivative of the distribution function $F_x(x)$:

$$f_x(x) = d/dx\, [F_x(x)], \quad -\infty < x < \infty$$

## Properties of the PDF

- $f_x(x) \geq 0$ for all x
- $\int_{-\infty}^{\infty} f_x(x)\, dx = 1$
- $F_x(x) = \int_{-\infty}^{x} f_x(\alpha)\, d\alpha$
- $P\{x1 \leq X \leq x2\} = \int_{x1}^{x2} f_x(x)\, dx = F_x(x2) - F_x(x1)$

Discrete random variable density: $F_x(x) = \Sigma\, P(x_i)\, \delta(x - x_i)$.

Continuous random variable density: $F_x(x) = d/dx[F_x(x)]$, $-\infty < x < \infty$.

## Probability Mass Function (PMF)

For a discrete random variable X with possible outcomes x1, x2, ..., the probabilities $P(x_i)$, i = 1, 2, ..., form a PMF if:

- $P(x_i) \geq 0$ for all i
- $\Sigma_i\, P(x_i) = 1$

## Solved Examples — Density and Distribution Functions

**Example 15: If the probability density of a random variable is $f_x(x) = k(1 - x^2)$, $0 < x < 1$, find (i) the value of k, (ii) $F_x(x)$.**

$\int_0^1 k(1 - x^2)\, dx = 1 \Rightarrow k[x - x^3/3]_0^1 = 1 \Rightarrow k(2/3) = 1$.

**k = 3/2**

$F_x(x) = \int_0^x k(1 - \alpha^2)\, d\alpha = (3/2)[x - x^3/3]$, $0 < x < 1$.

**Example 16: Find k so that $f_x(x) = k \cdot x^2(1 - x^3)$, $0 < x < 1$ (and 0 otherwise) is a valid PDF.**

$\int_0^1 k(x^2 - x^5)\, dx = 1 \Rightarrow k[x^3/3 - x^6/6]_0^1 = 1 \Rightarrow k(1/6) = 1$.

**k = 6**

**Example 17: Find b > 0 so that $f_x(x) = (1/10)e^{3x}$, $0 \leq x \leq b$ (and 0 otherwise) is a valid PDF.**

$(1/10)\int_0^b e^{3x}\, dx = 1 \Rightarrow (1/30)[e^{3b} - 1] = 1 \Rightarrow e^{3b} = 31$.

$3b = \ln 31 \Rightarrow b = (1/3)\ln 31$.

**$b \approx 1.144$**

**Example 18: X has discrete values $\{-1, -0.5, 0.7, 1.5, 3\}$ with probabilities $\{0.1, 0.2, 0.1, 0.4, 0.2\}$. Plot its distribution function.**

| x | −1 | −0.5 | 0.7 | 1.5 | 3 |
|---|---|---|---|---|---|
| P(x) | 0.1 | 0.2 | 0.1 | 0.4 | 0.2 |
| $F_x(x)$ | 0.1 | 0.3 | 0.4 | 0.8 | 1 |

**Example 19: Tossing 4 coins — X is the number of tails. Find and sketch the CDF.**

Sample space has 16 outcomes; $X = \{0,1,2,3,4\}$ with $P(x) = \{1/16, 4/16, 6/16, 4/16, 1/16\}$ (binomial with $n=4$, $p=1/2$).

| x | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| P(x) | 1/16 | 4/16 | 6/16 | 4/16 | 1/16 |
| $F_x(x)$ | 1/16 | 5/16 | 11/16 | 15/16 | 1 |

**Example 20: The PMF of X is given for $x = 0,\ldots,6$ as $P(x) = k, 3k, 5k, 7k, 9k, 11k, 13k$. Find (i) k, (ii) $P(X<4)$, (iii) $P(X\geq 5)$, (iv) $P(3<X\leq 6)$, (v) the minimum k such that $P(X\leq 2) > 0.3$.**

Sum: $k+3k+5k+7k+9k+11k+13k = 49k = 1 \Rightarrow k = 1/49$.

**(i) $k = 1/49$**

| x | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| P(x) | 1/49 | 3/49 | 5/49 | 7/49 | 9/49 | 11/49 | 13/49 |
| $F_x(x)$ | 1/49 | 4/49 | 9/49 | 16/49 | 25/49 | 36/49 | 1 |

**(ii) $P(X<4) = F_x(3) = 16/49 \approx 0.326$**

**(iii) $P(X\geq 5) = 1 - F_x(4) = 1 - 25/49 \approx 0.480$**

**(iv) $P(3<X\leq 6) = F_x(6) - F_x(3) = 1 - 16/49 \approx 0.673$**

(v) $P(X\leq 2) > 0.3 \Rightarrow 9k > 0.3 \Rightarrow k > 0.3/9 \approx 0.033$.

**(v) $k > 0.033$**

**Example 21: Consider $f(x) = a\cdot e^{-b|x|}$, $-\infty < x < \infty$. Find (i) the CDF, (ii) the relation between a and b, (iii) $P(1 \leq X \leq 2)$.**

For $x < 0$: $f(x) = a e^{bx}$; for $x \geq 0$: $f(x) = a e^{-bx}$.
Integrating piece wise: $F_x(x) = (a/b)[2 - e^{-bx}]$ for $x \geq 0$ (with the symmetric form for $x < 0$).
Since $F_x(\infty) = 1$: $(a/b)(2) = 1$, so $2a = b$.

**(ii) $2a = b$**

$P(1\leq X\leq 2) = F_x(2) - F_x(1) = (a/b)[e^{-b} - e^{-2b}] = (1/2)[e^{-b} - e^{-2b}]$.

**Example 22: Given $f(x) = x$ for $0<x<1$ and $f(x) = 2-x$ for $1<x<2$. Find (i) $P(0.2 \leq X \leq 0.8)$, (ii) $P(0.6 \leq X \leq 1.2)$.**

(i) $P = \int_{0.2}^{0.8} x\,dx = (1/2)[0.8^2 - 0.2^2] = (1/2)(0.6) = 0.3$.

**(i) $P(0.2\leq X\leq 0.8) = 0.3$**

(ii) $P = \int_{0.6}^{1} x\,dx + \int_{1}^{1.2} (2-x)\,dx = 0.32 + 0.09 = 0.41$.

**(ii) $P(0.6\leq X\leq 1.2) = 0.41$**

**Example 23:** $f_x(x) = c\cdot e^{-x/4}, 0 \le x < 1$ **(and 0 otherwise). Find c and** $F_x(0.5)$**.**

$c\int_0^1 e^{-x/4}\,dx = 1 \Rightarrow 4c[1 - e^{-1/4}] = 1 \Rightarrow c = 1 / (4[1 - e^{-1/4}]) \approx 1.13.$

**$c \approx 1.13$**

$F_x(0.5) = c\int_0^{0.5} e^{-x/4}\,dx = 4c[1 - e^{-0.125}] \approx 0.53.$

**$F_x(0.5) \approx 0.53$**

**Example 24:** **A random variable X has distribution function** $F_x(x) = \Sigma_{n=1}^{12} (n^2/650)\cdot u(x-n)$**. Find (a)** $P(-\infty<X\le 6.5)$**, (b)** $P(X>4)$**, (c)** $P(6<X\le 9)$**.**

(a) $P(-\infty<X\le 6.5) = F_x(6) = (1/650)\Sigma_{n=1}^{6} n^2 = (1/650)(91) \approx 0.14.$

**(a) $\approx 0.14$**

(b) $P(X>4) = 1 - F_x(3) = 1 - (1/650)(1^2+2^2+3^2) = 1 - 14/650 \approx 0.978.$

**(b) $\approx 0.978$**

(c) $P(6<X\le 9) = F_x(9) - F_x(6) = (1/650)[285 - 91] = 194/650 \approx 0.298.$

**(c) $\approx 0.298$**

## Standard Distribution and Density Functions

### 1. Gaussian (Normal) Density Function

$$f_x(x) = 1/\sqrt{(2\pi\sigma_x^2)} \cdot \exp[ -(x - \mu_x)^2 / (2\sigma_x^2) ], \text{ for all } x$$

$$F_x(x) = 1/\sqrt{(2\pi\sigma_x^2)} \int_{-\infty}^{x} \exp[ -(\alpha - \mu_x)^2 / (2\sigma_x^2) ]\, d\alpha$$

### 2. Uniform Density Function

$$f_x(x) = 1/(b - a), \quad a \le x \le b; \quad 0, \text{ otherwise}$$

$$F_x(x) = (x - a)/(b - a); \quad F_x(a) = 0, \quad F_x(b) = 1$$

### 3. Exponential Density Function

Where a, b are real constants, $-\infty < a < \infty$, $b > 0$.

$$f_x(x) = (1/b)\cdot e^{-(x-a)/b}, \quad x > a; \quad 0, x < a$$

$$F_x(x) = 0, x < a; \quad 1 - e^{-(x-a)/b}, x \ge a$$

### 4. Rayleigh Density Function

Where a, b are real constants, $-\infty < a < \infty$, $b > 0$.

$$f_x(x) = (2/b)(x-a)\cdot e^{-(x-a)^2/b}, \quad x \ge a; \quad 0, x < a$$

$$F_x(x) = 0, x < a; \quad 1 - e^{-(x-a)^2/b}, x \ge a$$

### 5. Binomial Density Function

For an experiment with only two possible outcomes per trial (success/failure), repeated for n independent trials:

$$f_x(x) = \Sigma_{k=0}^{n}\ nCk \cdot p^k(1-p)^{n-k} \cdot \delta(x-k)$$

$$F_x(x) = \Sigma_{k=0}^{n}\ nCk \cdot p^k(1-p)^{n-k} \cdot u(x-k)$$

### 6. Poisson Density Function

For a discrete random variable X, where b > 0:

$$f_x(x) = e\text{^}(-b)\ \Sigma_{k=0}\text{^}\infty\ (b^k/k!) \cdot \delta(x-k)$$

$$F_x(x) = e\text{^}(-b)\ \Sigma_{k=0}\text{^}\infty\ (b^k/k!) \cdot u(x-k)$$

## Online Learning Resources

- https://onlinecourses.nptel.ac.in/noc20_ma50/preview
- https://onlinecourses.nptel.ac.in/noc21_ma66/preview

**SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES, CHITTOOR – 517127**
**(AUTONOMOUS)**
Affiliated to JNTUA, Anantapuramu, NBA & NAAC Accredited

**DEPARTMENT**
**OF**
**ELECTRONICS & COMMUNICATION ENGINEERING**

# PROBABILITY AND COMPLEX VARIABLES

**(23BSC214 )**

# LECTURE NOTES

# INSTITUTE VISION AND MISSION

## VISION

- To emerge as a centre of excellence for learning and research in the domains of engineering, computing and management.

## MISSION

- Provide congenial academic ambience with state-art of resources for learning and research.
- Ignite the students to acquire self-reliance in the latest technologies.
- Unleash and encourage the innate potential and creativity of students.
- Inculcate confidence to face and experience new challenges.
- Foster enterprising spirit among students.
- Work collaboratively with technical Institutes/Universities/Industries of National and International repute.

# DEPARTMENT VISION AND MISSION

## VISION

- To become a centre of excellence in Electronics and Communication Engineering and provide necessary skills to the students to meet the challenges of industry and society.

## MISSION

- Provide congenial academic ambience with necessary infrastructure and learning resources
- Inculcate confidence to face and experience new challenges from industry and society.
- Ignite the students to acquire self-reliance in State-of-the-Art Technologies
- Foster Enterprising spirit among students

## PROGRAMME EDUCATIONAL OBJECTIVES (PEO's)

**PEO1:** Have Professional competency through the application of knowledge gained from subjects like Mathematics, Physics, Chemistry, Inter-Disciplinary and core subjects like Signal Processing, VLSI, Embedded Systems, Communication and Automation. **(Professional Competency)**

**PEO2:** E*xcel* in one's career by critical thinking towards successful services and growth of the organization or as an entrepreneur or through higher studies. **(Successful Career Goals)**

**PEO3:** Enhance knowledge by updating advanced technological concepts for facing the rapidly changing world and contribute to society through innovation and creativit*y*. **(Continuing Education and Contribution to Society)**

## PROGRAM SPECIFIC OUTCOMES (PSO'S)

**PSO1:** Apply the knowledge obtained in core areas for the analysis and design of components in Signal Processing, VLSI, Embedded Systems, Communication and Automation.

**PSO2:** Adapt Innovation, Creativity and design to develop products which meet industrial and societal needs.

# PROGRAMME OUTCOMES (PO's)

PO1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO2. **Problem analysis**: Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of

the limitations.

PO6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

## Course Educational Objectives

- To introduce the basic concepts of probability and random variables.
- To introduce the basic concepts of multiple random variables.
- To understand concepts of operations on multiple random variables.
- To analyze functions of a complex variable, with continuity, differentiability and analyticity.
- To understand Taylor and Laurent expansions, residues and improper integrals.

## Syllabus Overview

| Unit | Title | Key Topics |
|---|---|---|
| Unit I | Probability & Random Variable | Probability through sets & relative frequency; sample spaces; axioms; joint, conditional & total probability; Bayes' theorem; random variables; distribution & density functions; Gaussian, Binomial, Poisson, Uniform, Exponential, Rayleigh. |
| Unit II | Operations on Random Variable | Expectation; moments; characteristic & moment generating functions; multiple/vector random variables; joint distribution & density; marginal & conditional distributions; statistical independence. |
| Unit III | Operations on Multiple Random Variables | Joint moments; joint characteristic functions; jointly Gaussian random variables (two-variable case) and their properties. |
| Unit IV | Complex Variable — Differentiation | Limits, continuity, differentiation; Cauchy-Riemann equations; analytic & harmonic functions; harmonic conjugates; Milne-Thomson method. |
| Unit V | Complex Variable — Integration | Line integrals; Cauchy's integral theorem & formula; Taylor's & Laurent's series; zeros, singularities, residues; Cauchy residue theorem. |

## Course Outcomes

| CO | Outcome |
|---|---|
| CO1 | Understand the concepts of probability, random variables and their characteristics. |
| CO2 | Deal with multiple random variables, conditional probability, joint distribution and statistical independence. |
| CO3 | Formulate and solve engineering problems involving random variables. |
| CO4 | Analyze limit, continuity and differentiation of complex functions; understand Cauchy-Riemann equations and analytic functions. |
| CO5 | Apply Cauchy's theorem and integral formulas; classify singularities and evaluate integrals using residues. |

## CO – PO Mapping

| CO \ PO | PO1 | PO2 | PO3 |
|---|---|---|---|
| CO1 | 3 | 3 | 3 |
| CO2 | 3 | 3 | 3 |
| CO3 | 3 | 3 | 3 |
| CO4 | 3 | 3 | 3 |
| CO5 | 3 | 3 | 3 |

## Text Books

- Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", 4th Edition, TMH, 2002.
- B.S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 44th Edition, 2017.

## Reference Books

- Athanasios Papoulis & S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, PHI, 2002.
- Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley India.
- Henry Stark & John W. Woods, "Probability and Random Processes with Application to Signal Processing", 3rd Edition, Pearson, 2002.
- B.V. Ramana, "Higher Engineering Mathematics", McGraw Hill.

## UNIT II — Random Variables and Operations

### Operations on One Random Variables

**Expectation:-**

The mean or average value of a probability distribution function of a random variable 'x' is called Expectation of x or the mean value of 'x' or simply the Expected value of x.

It is denoted as E[x] or $\bar{x}$.

**Expected value of a Random variable:-**

If 'x' be a continuous random variable with valid probability density function $f_X(x)$ then the expected value of 'x' is given by

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x\, f_X(x)\, dx$$

Note:-

★ If 'x' is a discrete random variable with set of elements $\{x_1, x_2, x_3, \dots, x_n\}$ and the corresponding probabilities $\{p(x_1), p(x_2), \dots, p(x_n)\}$ then the expected value of x is

$$E(x) = \sum_{i=1}^{n} x_i\, p(x_i)$$

★ If $p(x_1) = p(x_2) = \dots = p(x_n) = 1/n$ then

$$E[x] = \sum_{i=1}^{n} x_i (1/n) = (1/n) \sum_{i=1}^{n} x_i$$

**The Expected value of a function of a random variable:-**

Consider a random variable x with probability density function $f_X(x)$. If g(x) is a real function of x then expected value of g(x) for a continuous random variable x is defined as

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx$$

For discrete random variable x, it is defined as

$$E[g(x)] = \sum_{i=1}^{n} g(x_i)\, p(x_i)$$

**Properties of Expectation:-**

1) If a random variable 'x' is a constant that is x = k, then E[x] = E[k] = k
2) If 'a' is any constant then expectation E[ax] = a E[x]
3) If a and b are any two constants then E[ax + b] = a E[x] + b
4) If x and y be the random variables such that $y \le x$, then $E[y] \le E[x]$
5) $|E[x]| \le E[|x|]$
6) $E[g_1(x) + g_2(x)] = E[g_1(x)] + E[g_2(x)]$
7) $\sigma_x^2 = E[x^2] - \{E[x]\}^2$

**Example 1**

What is our Mathematical expectation if we win Rs 10 if a balanced coin comes up head and losses Rs 10 if it comes up tail?

Let $x_1 = +1$, winning of Rupees 10

$x_2 = -1$, loss of Rs 10

Since the coin is balanced therefore probability of getting head

$$p(x_1) = 1/2$$

probability of getting tail

$$p(x_2) = 1/2$$

we know that

$$E[x] = \Sigma_{i=1}^{n} x_i\, p(x_i) = \Sigma_{i=1}^{2} x_i\, p(x_i) = x_1 p(x_1) + x_2 p(x_2)$$

$$= 1(1/2) + (-1)(1/2)$$

$$\therefore\ E[x] = 1/2 - 1/2 = 0$$

**Example 2**

Let x be a random variable with probabilities shown below. Find a) E[x] b) E[$x^2$] c) E[$(2x+1)^2$] d) $\sigma_x^2$

| $X(x_i)$ | −1 | 1 | 2 |
|---|---|---|---|
| $P(x_i)$ | 1/6 | 1/3 | 1/2 |

Given

$$x_1 = -1,\quad x_2 = 1,\quad x_3 = 2$$

$$p(x_1) = 1/6,\quad p(x_2) = 1/3,\quad p(x_3) = 1/2$$

a) To find E[x]:-

w.k.t

$$E[x] = \Sigma\, x_i\, p(x_i) = (-1)(1/6) + (1)(1/3) + (2)(1/2)$$

$$= -1/6 + 1/3 + 1$$

$$\therefore\ E[x] = 1/6 + 1 = 7/6$$

b) To find E[$x^2$]:-

w.k.t

$$E[x^2] = \Sigma\, x_i^2\, p(x_i) = (-1)^2(1/6) + (1)^2(1/3) + (2)^2(1/2)$$

$$= 1/6 + 1/3 + 2 = 1/2 + 2$$

$$\therefore\ E[x^2] = 5/2$$

c) To find $E[(2x+1)^2]$:-

w.k.t

$$E[(ax+b)^2] = \Sigma\ (a\,x_i+b)^2\ p(x_i)$$

$$E[(2x+1)^2] = \Sigma\ (2x_i+1)^2\ p(x_i)$$

$$= (2(-1)+1)^2(1/6) + (2(1)+1)^2(1/3) + (2(2)+1)^2(1/2)$$

$$= (-1)^2(1/6) + 9/3 + 25/2 = 1/6 + 3 + 12.5$$

$$\therefore\ E[(2x+1)^2] = 47/3 = 15.66$$

d) To find $\sigma_x^2$:-

w.k.t

$$\sigma_x^2 = E[x^2] - [E[x]]^2$$

$$\sigma_x^2 = 5/2 - (7/6)^2 = 5/2 - 49/36$$

$$\therefore\ \sigma_x^2 = 1.13$$

**Example 3**

Consider a random variable 'x' with $E[x] = 5$ and $\sigma_x^2 = 2.9$. Another random variable is given as $y = -8x + 10$. Find a) $E[xy]$ b) $E[x^2]$ c) $E[y^2]$ d) $\sigma_y^2$

Given

$$E[x] = 5,\quad \sigma_x^2 = 2.9,\quad y = -8x + 10$$

a) To find $E[xy]$:-

$$E[xy] = E[x(-8x+10)] = E[-8x^2+10x]$$

$$= -8E[x^2] + 10E[x] = -8E[x^2] + 10(5) = -8E[x^2] + 50$$

$$= -8(27.9) + 50$$

$$\therefore\ E[xy] = -173.2$$

b) To find $E[x^2]$:-

w.k.t

$$\sigma_x^2 = E[x^2] - [E[x]]^2$$

$$2.9 = E[x^2] - (5)^2$$

$$2.9 + 25 = E[x^2]$$

$$\therefore\ E[x^2] = 27.9$$

c) To find $E[y^2]$:-

$$E[y^2] = E[(-8x+10)^2] = E[64x^2+100-160x]$$

$$= 64E[x^2] + 100 - 160E[x]$$

$$= 64(27.9) + 100 - 800$$

$$\therefore\ E[y^2] = 1085.6$$

d) To find $\sigma_y^2$:-

$$\sigma_y^2 = E[y^2] - \{E[y]\}^2$$

$$= 1085.6 - \{E(-8x+10)\}^2 = 1085.6 - \{-8E[x]+10\}^2$$

$$= 1085.6 - \{-40+10\}^2 = 1085.6 - \{-30\}^2$$

$$= 1085.6 - 900$$

$$\therefore\ \sigma_y^2 = 185.6$$

**Example 4**

Consider a pdf of a random variable 'x' is $f_X(x) = \{1/k, -2 \le x \le 3; 0, \text{otherwise}\}$. Another random variable is $y = 2x$. Find a) value of k b) E[x] c) E[y] d) E[xy]

Given pdf is

$$f_X(x) = \{\ 1/k\ ,\ -2 \le x \le 3\ ;\ \ 0\ ,\ \text{otherwise}\ \}$$

and also given $y = 2x$

a) To find value of k:-

From the property of probability density function for continuous random variable

$$\int_{-\infty}^{\infty} f_X(x)\, dx = 1$$

$$\int_{-2}^{3} (1/k)\, dx = 1$$

$$(1/k)\,[x]_{-2}^{3} = 1$$

$$(1/k)[3-(-2)] = 1 \ \Longrightarrow (1/k)(5) = 1$$

$$\therefore\ k = 5$$

b) To find E[x]:-

w.k.t

$$E[x] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx = \int_{-2}^{3} x\,(1/5)\, dx$$

$$= (1/5)\,[x^2/2]_{-2}^{3} = (1/10)[9-4]$$

$$\therefore\ E[x] = (1/10)(5) = 1/2 = 0.5$$

c) To find E[y]:-

$$E[y] = E[2x] = 2E[x] = 2(1/2)$$

$$\therefore\ E[y] = 1$$

d) To find E[xy]:-

$$E[xy] = E[x(2x)] = 2E[x^2] = 2\int_{-\infty}^{\infty} x^2\, f_X(x)\, dx$$

$$= 2\int_{-2}^{3} x^2(1/5)\, dx = (2/5)[x^3/3]_{-2}^{3} = (2/15)[27+8]$$

$$\therefore\ E[xy] = 70/15 = 4.66$$

**Example 5**

Two unbiased dice are thrown, find the expected value of the sum of the numbers shown on the dice.

We know that by throwing two dice, sample space is

S = {(1,1)(1,2)(1,3)(1,4)(1,5)(1,6), (2,1)(2,2)(2,3)(2,4)(2,5)(2,6), (3,1)(3,2)(3,3)(3,4)(3,5)(3,6), (4,1)(4,2)(4,3)(4,4)(4,5)(4,6), (5,1)(5,2)(5,3)(5,4)(5,5)(5,6), (6,1)(6,2)(6,3)(6,4)(6,5)(6,6)}

Let x be a random variable, i.e. the number which is obtained by the sum of the numbers on the dice.

| $X(x_i)$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $P(x_i)$ | 1/36 | 2/36 | 3/36 | 4/36 | 5/36 | 6/36 | 5/36 | 4/36 | 3/36 | 2/36 | 1/36 |

To find E[x]:-

$$E[x] = \Sigma\, x_i\, p(x_i)$$

$$= (1/36)[2+6+12+20+30+42+40+36+30+22+12]$$

$$= (1/36)[252] = 252/36$$

$$\therefore\ E[x] = 7$$

**Example 6**

A box contains 4 red, 2 green balls. 2 balls are drawn together. Find the expected value of the number of red balls drawn.

Let x be a random variable for the event of the number of red balls drawn.

Let $x_1$ denote no red ball and two green balls: $x_1 = 0$

Let $x_2$ = one red and one green ball: $x_2 = 1$

Let $x_3$ = two red balls and no green ball: $x_3 = 2$

The probabilities are

$$p(x_1) = ({}^4C_0 \times {}^2C_2) / {}^6C_2 = (1\times1)/15 = 1/15$$

$$p(x_2) = ({}^4C_1 \times {}^2C_1) / {}^6C_2 = (4\times2)/15 = 8/15$$

$$p(x_3) = ({}^4C_2 \times {}^2C_0) / {}^6C_2 = (6\times1)/15 = 6/15 = 2/5$$

To find E[x]:-

$$E[x] = \Sigma\, x_i\, p(x_i) = x_1p(x_1) + x_2p(x_2) + x_3p(x_3)$$

$$= 0(1/15) + 1(8/15) + 2(2/5) = 8/15 + 4/5 = 8/15 + 12/15 = 20/15$$

$$\therefore\ E[x] = 4/3$$

## Moments

There are two types of moments for a function of a random variable x.

### 1) Moments about the origin:-

Let g(x) be a real function of the random variable x, $g(x) = x^n$ for n = 0,1,2,... then the expected value of the function g(x) is called the Moments about the origin of a random variable x. It is denoted as $m_n$ where n indicates the order of the moment.

In general, the $n^{th}$ moment is defined as

$$m_n = E[x^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

for discrete random variable

$$E[x^n] = \Sigma\, x_i^n\, p(x_i)$$

| Note:- |
|---|
| ★ If n=0, $E[x^0] = \int x^0 f_X(x)\, dx = \int f_X(x)\, dx = 1$ |
| ★ If n=1, $E[x^1] = \int x\, f_X(x)\, dx = E[x] = \bar{x}$ |
| ★ If n=2, $E[x^2] = \int x^2 f_X(x)\, dx = E[x^2]$ |

### 2) Moments about the mean (or) central moments:-

Let g(x) be a real function of a random variable x such that $g(x) = (x-\bar{x})^n$ for n=0,1,2,3,... where $\bar{x}$ is the mean of x, then the expected value of the function g(x) is called moments about the mean of the random variable x. It is denoted by $\mu_n$, where n indicates the order of the moment — also called the central moments of the random variable x.

The $n^{th}$ central moment is given by

$$\mu_n = E[(x-\bar{x})^n] = \int_{-\infty}^{\infty} (x-\bar{x})^n f_X(x)\, dx$$

For discrete random variable

$$\mu_n = \Sigma\, (x_i-\bar{x})^n\, p(x_i)$$

| Note:- |
|---|
| If n=0, $\mu_0 = \Sigma(x_i-\bar{x})^0 p(x_i) = \Sigma\, p(x_i) = 1$ |
| $\therefore\ \mu_0 = m_0 = 1$ |

### Variance:-

The variance of the density function $f_X(x)$ for a random variable x is defined as the second central moment $\mu_2$ of x. It is also denoted as $\sigma_x^2$ (or) Var(x).

$$\mu_2 = \sigma_x^2 = E[(x-\bar{x})^2] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_X(x)\, dx$$

For discrete random variable

$$\sigma_x^2 = \Sigma(x_i - \bar{x})^2\, p(x_i)$$

### Skew:-

The skew of the density function $f_X(x)$ of a random variable x is defined as the third central moment $\mu_3$ of x. It is given by

$$\mu_3 = E[(x-\bar{x})^3] = \int_{-\infty}^{\infty} (x-\bar{x})^3 f_X(x)\, dx$$

For discrete random variable

$$\mu_3 = \Sigma(x_i - \bar{x})^3\, p(x_i)$$

### Properties of Variance:-

★ If k is a constant, then the variance of k is equal to 0. ∴ Var(k) = 0

★ $Var(kx) = k^2\, Var(x)$

★ For a given random variable x, the relationship between the moments is given by $\sigma_X^2 = m_2 - m_1^2$

★ If x is a random variable and a, b are real constants, then $Var(ax+b) = a^2\, Var(x)$

### Functions for moments:-

To calculate the $n^{th}$ moment of a random variable x, the following two functions are generally used:

1) Characteristic function

2) Moment Generating function

## Characteristic Function

The characteristic function of a random variable x is defined by

$$\varphi_X(\omega) = E[e^{j\omega x}]\text{ , where } j = \sqrt{-1}$$

★ It is a function of the real variable ω, $-\infty < \omega < \infty$.

$$\therefore\ \varphi_X(\omega) = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

For discrete random variable

$$\varphi_X(\omega) = \Sigma\, e^{j\omega x_i}\, p(x_i)$$

★ The characteristic function transforms $f_X(x)$ into another function of ω.

Note:-

1) The characteristic function is the Fourier Transform of $f_X(x)$ with the sign of ω reversed.

2) The Inverse Fourier Transform of $\varphi_X(\omega)$ is the density function with the sign of x reversed.

$$f_X(x) = (1/2\pi) \int_{-\infty}^{\infty} \varphi_X(\omega)\, e^{-j\omega x}\, d\omega$$

3) If $\varphi_X(\omega)$ is the characteristic function of a random variable x then the $n^{th}$ moment of x is given by

$$m_n = (-j)^n\, d^n/d\omega^n\, [\varphi_X(\omega)]\, |_{\omega=0}$$

**Properties of the characteristic function:-**

| |
|---|
| 1) The C.F is unity at $\omega=0$, i.e. $\varphi_X(\omega)\|_{\omega=0} = \varphi_X(0) = 1$ |
| 2) The maximum amplitude of the C.F is unity, i.e. $\|\varphi_X(\omega)\| \le \varphi_X(0)$ (or) $\|\varphi_X(\omega)\| \le 1$ |
| 3) $\varphi_X(\omega)$ is a continuous function of $\omega$ in the range $-\infty < \omega < \infty$. |
| 4) $\varphi_X(-\omega)$ and $\varphi_X(\omega)$ are conjugate functions, i.e. $\varphi_X(-\omega) = \varphi_X^*(\omega)$ and $\varphi_X^*(-\omega) = \varphi_X(\omega)$ |
| 5) $\varphi_{X_1+X_2}(\omega) = \varphi_{X_1}(\omega) \cdot \varphi_{X_2}(\omega)$ |

**Example 1**

Find the characteristic function for a random variable with density function $f_X(x) = x$ for $0 \le x \le 1$.

Given pdf is $f_X(x) = x$ for $0 \le x \le 1$

From the definition of the C.F

$$\varphi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_0^1 e^{j\omega x}\, x\, dx$$

Integrating by parts,

$$= [x \cdot e^{j\omega x}/(j\omega) - e^{j\omega x}/(j\omega)^2]_0^1$$

$$= (1/j\omega)[e^{j\omega x}(x - 1/j\omega)]_0^1$$

$$= (1/j\omega)[e^{j\omega}(1 - 1/j\omega) - (0 - 1/j\omega)]$$

$$\therefore\ \varphi_X(\omega) = (1/j\omega)[e^{j\omega}(1 - 1/j\omega) + 1/j\omega]$$

**Example 2**

The density function of a random variable is given by $f_X(x) = a\, e^{-bx}$, $x \ge 0$. Find the characteristic function and the first two moments.

Given pdf is $f_X(x) = a\, e^{-bx}$, $x \ge 0$

1) To find the C.F:-

w.k.t

$$\varphi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x} f_X(x)\, dx$$

$$= \int_0^{\infty} e^{j\omega x} (a\, e^{-bx})\, dx = a\int_0^{\infty} e^{-(b-j\omega)x}\, dx$$

$$= a[\, -e^{-(b-j\omega)x}/(b-j\omega)\, ]_0^{\infty} = a/(b-j\omega)\, [1 - 0]$$

$$\therefore\ \varphi_X(\omega) = a/(b-j\omega)$$

2) To find $m_n$:-

w.k.t

$$m_n = (-j)^n\, d^n/d\omega^n\, [\varphi_X(\omega)]\, |_{\omega=0}$$

Putting n=1:

$$m_1 = (-j)\, d/d\omega[a/(b-j\omega)]\, |_{\omega=0}$$

$$= (-j)\,[aj/(b-j\omega)^2]\, |_{\omega=0} = (-j)(aj/b^2)$$

$$\therefore\ m_1 = a/b^2$$

Putting n=2:

$$m_2 = (-j)^2\, d^2/d\omega^2\, [a/(b-j\omega)]\, |_{\omega=0}$$

$$= (-1)\, d/d\omega[aj/(b-j\omega)^2]\, |_{\omega=0} = (-1)[2aj^2/(b-j\omega)^3]\, |_{\omega=0}$$

$$\therefore\ m_2 = 2a/b^3$$

**Example 3**

Find the characteristic function for $f_X(x) = e^{-|x|}$, $-\infty < x < \infty$.

Given pdf is $f_X(x) = e^{-|x|}$

$$\varphi_X(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} e^{j\omega x}\, e^{-|x|}\, dx$$

$$= \int_{-\infty}^{0} e^{j\omega x}\, e^{x}\, dx + \int_{0}^{\infty} e^{j\omega x}\, e^{-x}\, dx$$

$$= \int_{-\infty}^{0} e^{(j\omega+1)x}\, dx + \int_{0}^{\infty} e^{(j\omega-1)x}\, dx$$

$$= [e^{(j\omega+1)x}/(j\omega+1)]_{-\infty}^{0} + [e^{(j\omega-1)x}/(j\omega-1)]_{0}^{\infty}$$

$$= 1/(j\omega+1)[1-0] + 1/(j\omega-1)[0-1]$$

$$= 1/(1+j\omega) + 1/(1-j\omega)$$

$$= [(1-j\omega)+(1+j\omega)]\, /\, [(1+j\omega)(1-j\omega)] = 2/(1+\omega^2)$$

$$\therefore\ \varphi_X(\omega) = 2/(1+\omega^2)$$

**Example 4**

Show that the distribution function for which the C.F is $e^{-|\omega|}$ has the density function $f_X(x) = 1/[\pi(1+x^2)]$, $-\infty < x < \infty$.

Given C.F is $\varphi_X(\omega) = e^{-|\omega|}$

w.k.t

$$f_X(x) = (1/2\pi) \int_{-\infty}^{\infty} \varphi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$= (1/2\pi)[\int_{-\infty}^{0} e^{\omega}\, e^{-j\omega x}\, d\omega + \int_{0}^{\infty} e^{-\omega}\, e^{-j\omega x}\, d\omega]$$

$$= (1/2\pi)[\int_{-\infty}^{0} e^{(1-jx)\omega}\, d\omega + \int_{0}^{\infty} e^{-(1+jx)\omega}\, d\omega]$$

$$= (1/2\pi)\{\, [e^{(1-jx)\omega}/(1-jx)]_{-\infty}^{0} + [e^{-(1+jx)\omega}/-(1+jx)]_{0}^{\infty}\, \}$$

$$= (1/2\pi)\{\, 1/(1-jx) + 1/(1+jx)\, \}$$

$$= (1/2\pi)\cdot [2/(1+x^2)]$$

$$\therefore\ f_X(x) = 1/[\pi(1+x^2)],\ -\infty < x < \infty \quad \text{(Proved)}$$

**Example 5**

Find the C.F of a binomial distribution.

Consider the binomial distribution of a random variable x with probability

$$p(x) = {}^nC_x\, p^x q^{n-x}, \quad p+q=1$$

Also

$$\Sigma\, {}^nC_x\, p^x q^{n-x} = (p+q)^n = 1^n = 1$$

From the definition of the C.F for a discrete random variable x,

$$\varphi_X(\omega) = E[e^{j\omega x}] = \Sigma_{x=0}^{n} e^{j\omega x}\, p(x) = \Sigma_{x=0}^{n} e^{j\omega x}\, {}^nC_x\, p^x q^{n-x}$$

$$= \Sigma_{x=0}^{n} {}^nC_x\, (p\, e^{j\omega})^x q^{n-x}$$

$$= (p\, e^{j\omega} + q)^n \quad \text{[by the Binomial theorem]}$$

$$\therefore\ \varphi_X(\omega) = (q + p\, e^{j\omega})^n$$

**Example 6**

Find the density function of a random variable x if the C.F is $\varphi_X(\omega) = \{1-|\omega|,\ |\omega| \le 1;\ 0, \text{otherwise}\}$.

Given C.F is

$$\varphi_X(\omega) = \{\, 1-|\omega|\,,\ |\omega| \le 1\ ;\ \ 0\,, \text{otherwise}\,\}$$

$$f_X(x) = (1/2\pi) \int_{-\infty}^{\infty} \varphi_X(\omega)\, e^{-j\omega x}\, d\omega$$

$$= (1/2\pi)\left[\, \int_{-1}^{0} (1+\omega)\, e^{-j\omega x}\, d\omega + \int_{0}^{1} (1-\omega)\, e^{-j\omega x}\, d\omega \,\right]$$

Integrating by parts and simplifying,

$$\therefore\ f_X(x) = 1/(\pi x^2)\, [1 - \cos x]$$

## Moment Generating Function (M.G.F)

The moment generating function of a random variable is also used to generate the $n^{th}$ moments about the origin.

Consider a random variable x with probability density function $f_X(x)$, then the m.g.f of x is defined as the expected value of the function $e^{vx}$, i.e.

$$M_X(v) = E[e^{vx}]\ , \text{ where } -\infty < v < \infty$$

$$\therefore\ M_X(v) = \int_{-\infty}^{\infty} e^{vx} f_X(x)\, dx \quad \text{(for continuous random variable)}$$

$$M_X(v) = \Sigma_i\, e^{v x_i}\, p(x_i) \quad \text{(for discrete random variable)}$$

### Properties of the m.g.f:-

★ If $M_X(v)$ is the m.g.f of the random variable x, then the $n^{th}$ moment of x is given by $m_n = d^n/dv^n\, [M_X(v)]\, |_{v=0}$

★ The moment generating function at v=0 is unity, i.e. $M_X(v)|_{v=0} = M_X(0) = 1$

★ If x is a random variable with m.g.f $M_X(v)$, then for y = ax+b, the m.g.f is $M_Y(v) = e^{bv} M_X(av)$

★ If $M_X(v)$ is the m.g.f of a random variable x, then $M_{cx}(v) = M_X(cv)$

★ If $x_1$ and $x_2$ are two independent random variables with m.g.f $M_{x1}(v)$ and $M_{x2}(v)$, then $M_{x1+x2}(v) = M_{x1}(v)\, M_{x2}(v)$

**Example 1**

The pdf of a random variable x is $p(x) = 1/10^x$ for $x = 0,1,2,\ldots\infty$. Find 1) m.g.f 2) E[x]

Given probability distribution $p(x) = 1/10^x$ for $x = 0,1,2,3,\ldots\infty$

1) To find m.g.f:-

$$M_X(v) = E[e^{vx}] = \Sigma_x e^{vx}\, p(x) = \Sigma_{x=0}^{\infty} e^{vx}\, (1/10^x)$$

$$= \Sigma_{x=0}^{\infty} (e^v/10)^x = 1/(1 - e^v/10)$$

$$\therefore\ M_X(v) = 10/(10 - e^v)$$

2) To find E[x]:-

$$E[x] = E[x^1] = m_1 = d/dv\, [M_X(v)]\, |_{v=0}$$

$$= d/dv\, [10/(10-e^v)]\, |_{v=0} = 10[(-1)(10-e^v)^{-2}(-e^v)]\, |_{v=0}$$

$$= [10\, e^v/(10-e^v)^2]\, |_{v=0}$$

$$\therefore\ E[x] = 10/81$$

**Example 2**

The pdf of a random variable x is given as $f_X(x) = a\, e^{-bx}$, $x \geq 0$. Find a) $M_X(v)$ b) E[x] c) $E[x^2]$

a) To find $M_X(v)$:-

$$M_X(v) = E[e^{vx}] = \int_{-\infty}^{\infty} e^{vx}\, f_X(x)\, dx = \int_0^{\infty} e^{vx}\, (a\, e^{-bx})\, dx$$

$$= a\int_0^{\infty} e^{-(b-v)x}\, dx = a\left[\, -e^{-(b-v)x}/(b-v)\, \right]_0^{\infty}$$

$$\therefore\ M_X(v) = a/(b-v)$$

b) To find E[x]:-

$$E[x] = m_1 = d/dv\, [M_X(v)]\, |_{v=0} = d/dv[a/(b-v)]\, |_{v=0}$$

$$= a[(-1)(b-v)^{-2}(-1)]\, |_{v=0} = [a/(b-v)^2]\, |_{v=0}$$

$$\therefore\ E[x] = a/b^2$$

c) To find $E[x^2]$:-

$$E[x^2] = m_2 = d^2/dv^2\, [M_X(v)]\, |_{v=0} = d/dv\, [a/(b-v)^2]\, |_{v=0}$$

$$= [2a/(b-v)^3]\, |_{v=0}$$

$$\therefore\ E[x^2] = 2a/b^3$$

**Example 3**

The probability distribution of a random variable x is $p(x) = (2/3)(1/3)^x$, $x = 0,1,2,\ldots\infty$. Find the m.g.f. Also find the 1st and 2nd moments.

Given pd is $p(x) = (2/3)(1/3)^x$, $x = 0,1,2,\ldots\infty$

1) To find m.g.f:-

$$\mathbf{M_X(v) = E[e^{vx}] = \Sigma_{x=0}^{\infty} e^{vx} (2/3)(1/3)^x = (2/3) \Sigma_{x=0}^{\infty} (e^v/3)^x}$$

$$\mathbf{= (2/3) \cdot 1/(1 - e^v/3) = (2/3) \cdot 3/(3-e^v)}$$

$$\mathbf{\therefore\ M_X(v) = 2/(3 - e^v)}$$

2) To find $m_1$:-

$$\mathbf{m_1 = d/dv\,[M_X(v)]\,|_{v=0} = d/dv\,[2/(3-e^v)]\,|_{v=0}}$$

$$\mathbf{= 2[(-1)(3-e^v)^{-2}(-e^v)]\,|_{v=0} = [2e^v/(3-e^v)^2]\,|_{v=0}}$$

$$\mathbf{\therefore\ m_1 = 2/4 = 1/2}$$

3) To find $m_2$:-

$$\mathbf{m_2 = d^2/dv^2\,[M_X(v)]\,|_{v=0} = d/dv\,[2e^v/(3-e^v)^2]\,|_{v=0}}$$

$$\mathbf{= 2[(3-e^v)^2 e^v - e^v\{2(3-e^v)(-e^v)\}] / (3-e^v)^4\,|_{v=0}}$$

$$\mathbf{= 2[4 - 1\{2(2)(-1)\}]/16 = 2[4+4]/16}$$

$$\mathbf{\therefore\ m_2 = 20/8 = 5/2}$$

**Example: Poisson Distribution**

Find the m.g.f and C.F of a Poisson distribution.

Given that the p.d of a Poisson distribution is given by $p(x) = e^{-\lambda} \lambda^x/x!$ , $x \geq 0$

1) To find m.g.f:-

$$\mathbf{M_X(v) = E[e^{vx}] = \Sigma_{x=0}^{\infty} e^{vx} e^{-\lambda} \lambda^x/x!}$$

$$\mathbf{= e^{-\lambda} \Sigma_{x=0}^{\infty} (e^v\lambda)^x/x! = e^{-\lambda} e^{ev\lambda} \qquad [since\ \Sigma\ x^n/n! = e^x]}$$

$$\mathbf{\therefore\ M_X(v) = e^{\lambda(ev-1)}}$$

2) To find C.F:-

$$\mathbf{\varphi_X(\omega) = E[e^{j\omega x}] = \Sigma_{x=0}^{\infty} e^{j\omega x} e^{-\lambda} \lambda^x/x!}$$

$$\mathbf{= e^{-\lambda} \Sigma_{x=0}^{\infty} (e^{j\omega}\lambda)^x/x! = e^{-\lambda} e^{ej\omega\lambda}}$$

$$\mathbf{\therefore\ \varphi_X(\omega) = e^{\lambda(ej\omega-1)}}$$

## Multiple Random Variables

### Joint Probability Distribution Function

Consider two random variables x and y with elements {x} and {y}. In the xy-plane the ordered pair of numbers (x,y) is called a random vector in the two dimensional product space or joint sample space.

Let the two events $A = \{X \le x\}$ and $B = \{Y \le y\}$, then the joint probability distribution function is defined as

$$F_{XY}(x,y) = P\{X\le x, Y\le y\} = P(A\cap B)$$

In general, for a discrete random variable, the joint probability distribution is given by

$$F_{XY}(x,y) = \Sigma_{n=1}^{N} \Sigma_{m=1}^{M} p(x_n,y_m)\, u(x-x_n)\, u(y-y_m)$$

Similarly, for n random variables, the joint distribution function is given by

$$F_{X1X2\ldots Xn}(x_1,x_2,\ldots,x_n) = P\{X_1\le x_1, X_2\le x_2, \ldots, X_n\le x_n\}$$

### Properties of the joint distribution function:-

| |
|---|
| ★ $F_{XY}(-\infty,-\infty) = 0$ |
| ★ $F_{XY}(x,-\infty) = 0$ |
| ★ $F_{XY}(-\infty,y) = 0$ |
| ★ $F_{XY}(\infty,\infty) = 1$ |
| ★ $0 \le F_{XY}(x,y) \le 1$ |
| ★ $F_{XY}(x,y)$ is a monotonic, non-decreasing function of both x and y. |
| ★ The marginal distribution functions are given by $F_{XY}(x,\infty) = F_X(x)$ and $F_{XY}(\infty,y) = F_Y(y)$ |

### Joint Probability Density Function

The joint probability density function of two random variables x and y is defined as

$$f_{XY}(x,y) = \partial^2/\partial x\partial y\, [F_{XY}(x,y)]$$

It is simply called the joint density function. For discrete random variables, the joint density function is given by

$$f_{XY}(x,y) = \Sigma_{n=1}^{N} \Sigma_{m=1}^{M} p(x_n,y_m)\, \delta(x-x_n)\, \delta(y-y_m)$$

By direct integration, the joint distribution function can be obtained as

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x,y)\, dx\, dy$$

### Properties of the joint density function:-

| |
|---|
| ★ $f_{XY}(x,y) \ge 0$ |
| ★ $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$ |
| ★ $P\{x_1\le X\le x_2, y_1\le Y\le y_2\} = \int_{y1}^{y2} \int_{x1}^{x2} f_{XY}(x,y)\, dx\, dy$ |
| ★ $f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$ |
| ★ $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$ |

### Marginal Density Function

The functions $f_X(x)$ and $f_Y(y)$ in the property above are called the Marginal density functions. They are the density functions of the single variables x and y and are also defined as

$$f_X(x) = d/dx\, [F_X(x)] \quad \text{and} \quad f_Y(y) = d/dy\, [F_Y(y)]$$

**Example 1**

A joint sample space for two random variables x and y has four elements (1,1), (2,2), (3,3) and (4,4). The probabilities of these elements are 0.1, 0.35, 0.05, 0.5 respectively.

a) Determine through logic and sketch the distribution function $F_{XY}(x,y)$. b) Find P{x≤2.5, y≤6}. c) Find P{x≤3}.

| (x,y) | (1,1) | (2,2) | (3,3) | (4,4) |
|---|---|---|---|---|
| p(x,y) | 0.1 | 0.35 | 0.05 | 0.5 |

a) To find $F_{XY}(x,y)$:-

$$F_{XY}(x,y) = \Sigma_{n=1}^{N} \Sigma_{m=1}^{M} p(x_n,y_m)\, u(x-x_n)\, u(y-y_m)$$

$$F_{XY}(x,y) = (0.1)\, u(x-1)u(y-1) + (0.35)\, u(x-2)u(y-2) + (0.05)\, u(x-3)u(y-3) + (0.5)\, u(x-4)u(y-4)$$

*Figure: Distribution function $F_{XY}(x,y)$*

*A staircase surface rising in steps at the points (1,1), (2,2), (3,3), (4,4).*

*Cumulative height after each step: 0.10 → 0.45 → 0.50 → 1.00*

*The function equals 0 for x<1 or y<1, and reaches its final value of 1 for x≥4, y≥4.*

b) To find P{x≤2.5, y≤6}:-

$$P\{x\le 2.5, y\le 6\} = F_{XY}(2,2) = 0.1 + 0.35$$

$$\therefore\ P\{x\le 2.5, y\le 6\} = 0.45$$

c) To find P{x≤3}:-

$$P\{x\le 3\} = F_{XY}(3,\infty) = 0.1 + 0.35 + 0.05$$

$$\therefore\ P\{x\le 3\} = 0.50$$

**Example 2**

The joint distribution for two random variables x and y is $F_{XY}(x,y) = u(x)u(y)[1-e^{-ax}-e^{-ay}+e^{-a(x+y)}]$, where a>0.

i) Sketch $F_{XY}(x,y)$. ii) If a=0.5, find a) P{x≤1,y≤2} b) P{0.5<x<1.5} c) P{−1.5<x≤2, 1<y≤3}

Given joint probability distribution function is

$$F_{XY}(x,y) = u(x)u(y)[1-e^{-ax}-e^{-ay}+e^{-a(x+y)}]$$

$$= [(1-e^{-ax}) - e^{-ay} + e^{-ax}e^{-ay}]\, u(x)u(y)$$

$$= [(1-e^{-ax}) - e^{-ay}(1-e^{-ax})]\, u(x)u(y)$$

$$\therefore\ F_{XY}(x,y) = (1-e^{-ax})(1-e^{-ay})\, u(x)u(y)$$

i) Sketch of $F_{XY}(x,y)$:-

*Figure: Joint distribution surface $F_{XY}(x,y)$*

*A surface that is zero for x<0 or y<0, rising smoothly from the origin.*

*It increases monotonically with x and y and saturates towards 1 as x,y → ∞.*

ii) Given a = 0.5 = 1/2

$$F_{XY}(x,y) = (1 - e^{-x/2})(1 - e^{-y/2})$$

a) To find P{x≤1, y≤2}:-

$$P\{x\le1, y\le2\} = F_{XY}(1,2) = (1-e^{-1/2})(1-e^{-1})$$

$$\therefore\ P\{x\le1, y\le2\} = 0.249$$

b) To find P{0.5<x<1.5}:-

$$P\{0.5<x<1.5\} = F_X(1.5) - F_X(0.5) = F_{XY}(1.5,\infty) - F_{XY}(0.5,\infty)$$

$$= (1-e^{-1.5/2}) - (1-e^{-0.5/2})$$

$$\therefore\ P\{0.5<x<1.5\} = 0.306$$

c) To find P{−1.5<x≤2, 1<y≤3}:-

$$P\{-1.5<x\le2, 1<y\le3\} = F(-1.5,1) + F(2,3) - F(-1.5,3) - F(2,1)$$

$$= (1-e^{-1.5/2})(1-e^{-1/2}) + (1-e^{-1})(1-e^{-3/2}) - (1-e^{-1.5/2})(1-e^{-3/2}) - (1-e^{-1})(1-e^{-1/2})$$

$$\therefore\ P\{-1.5<x\le2, 1<y\le3\} = 0.0408$$

**Example 3**

Random variables x and y are components of a two dimensional random variable having a joint distribution:

$F_{XY}(x,y) = \{\ 0,\ x<0 \text{ or } y<0;\ \ xy,\ 0\le x<1 \text{ and } 0<y<1;\ \ x,\ 0\le x<1 \text{ and } 1\le y;\ \ y,\ 1\le x \text{ and } 0\le y<1;\ \ 1,\ 1\le x \text{ and } 1\le y\ \}$

Find the marginal distribution functions $F_X(x)$ and $F_Y(y)$.

To find $F_X(x) = F_{XY}(x,\infty)$:-

$$F_X(x) = \{\ 0,\ x<0\ ;\ \ x,\ 0\le x\le1\ ;\ \ 1,\ 1\le x\ \}$$

To find $F_Y(y) = F_{XY}(\infty,y)$:-

$$F_Y(y) = \{\ 0,\ y<0\ ;\ \ y,\ 0\le y<1\ ;\ \ 1,\ 1\le y\ \}$$

**Example 4**

The joint probability density function of random variables x and y is given by $f_{XY}(x,y) = \{4xy,\ 0<x<1,\ 0<y<1;\ 0, \text{otherwise}\}$. Find P(x+y<1).

Given joint pdf is $f_{XY}(x,y) = \{4xy,\ 0<x<1,\ 0<y<1;\ 0, \text{otherwise}\}$

Also given $x+y<1 \ \Rightarrow x<1-y$

$$\therefore\ P(x+y<1) = \iint f_{XY}(x,y)\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{1-y} 4xy\,dx\,dy$$

$$= 4\int_{y=0}^{1} y\left[\int_{x=0}^{1-y} x\,dx\right] dy = 4\int_{y=0}^{1} y\,(x^2/2)\big|_0^{1-y}\,dy$$

$$= 2\int_{y=0}^{1} y[(1-y)^2]\,dy = 2\int_{y=0}^{1} y[1+y^2-2y]\,dy$$

$$= 2\int_{y=0}^{1} [y+y^3-2y^2]\,dy = 2[y^2/2 + y^4/4 - 2y^3/3]_0^1$$

$$= 2[1/2 + 1/4 - 2/3] = 2[1/12]$$

$$\therefore P(x+y<1) = 1/6$$

**Example 5**

Given the joint probability density function of x and y is $f_{XY}(x,y) = \{ab\, e^{-(ax+by)},\ x>0,\ y>0;\ 0,\ \text{otherwise}\}$. Find $P(x>y)$.

$$P(x>y) = \iint f_{XY}(x,y)\,dx\,dy = \int_{x=0}^{\infty} \left[\int_{y=0}^{x} ab\, e^{-(ax+by)}\,dy\right] dx$$

$$= ab \int_{x=0}^{\infty} e^{-ax} \left[\int_{y=0}^{x} e^{-by}\,dy\right] dx = ab \int_{x=0}^{\infty} e^{-ax} \left[e^{-by}/-b\right]_0^x dx$$

$$= -a \int_{x=0}^{\infty} e^{-ax} \left[e^{-bx}-1\right] dx = -a \int_{x=0}^{\infty} \left(e^{-(a+b)x} - e^{-ax}\right) dx$$

$$= -a \left[ e^{-(a+b)x}/-(a+b) - e^{-ax}/-a \right]_{x=0}^{\infty}$$

$$= a\left[ (0-0) - \{1/(a+b) - 1/a\} \right] = a\left[ 1/a - 1/(a+b) \right]$$

$$= a \cdot \left[ b/(a(a+b)) \right]$$

$$\therefore P(x>y) = b/(a+b)$$

## Conditional Distribution and Density Function

### 1) Point conditional:-

Consider two random variables x and y. The distribution of one random variable x at a given condition that the distribution function of the random variable y is known at some value of y is defined as the conditional distribution function. It is given as $F_X(x/y=y)$

$$F_X(x/y=y) = \left[\int_{-\infty}^{x} f_{XY}(x,y)\,dx\right] / f_Y(y)$$

And the conditional density function is

$$f_X(x/y=y) = d/dx\,[F_X(x/y=y)] = d/dx \left\{ \left[\int_{-\infty}^{x} f_{XY}(x,y)\,dx\right] / f_Y(y) \right\}$$

$$\text{i.e.}\quad f_X(x/y) = f_{XY}(x,y) / f_Y(y)$$

$$f_Y(y/x) = f_{XY}(x,y) / f_X(x)$$

### For Discrete Random Variable:-

Consider both x and y are discrete random variables with elements $X = \{x_1,x_2,\dots,x_N\}$, $Y = \{y_1,y_2,\dots,y_M\}$. The corresponding probabilities are $p(x_i)$, i=1,2,...,N and $p(y_j)$, j=1,2,...,M.

The probability distribution function is given by

$$F_X(x/y=y_k) = \Sigma_{i=1}^{N} \left[p(x_i,y_k)/p(y_k)\right] u(x-x_i)$$

The density function

$$f_X(x/y=y_k) = \Sigma_{i=1}^{N} \left[p(x_i,y_k)/p(y_k)\right] \delta(x-x_i)$$

Similarly

$$F_Y(y/x=x_k) = \Sigma_{j=1}^{M} \left[p(x_k,y_j)/p(x_k)\right] u(y-y_j)$$

$$f_Y(y/x=x_k) = \Sigma_{j=1}^{M} \left[p(x_k,y_j)/p(x_k)\right] \delta(y-y_j)$$

### 2) Interval condition:-

Consider the event B defined as $y_1 \le Y \le y_2$ for the random variable y.

$$B = \{y_1 \le Y \le y_2\}$$

Assume that $P(B) = P\{y_1 \le Y \le y_2\} \neq 0$. Then the conditional distribution function of x is given by

$$F_X(x/y_1 \le Y \le y_2) = \left[\int_{y_1}^{y_2} \int_{-\infty}^{x} f_{XY}(x,y)\, dx\, dy\right] / \left[\int_{y_1}^{y_2} f_Y(y)\, dy\right]$$

$$= [F_{XY}(x,y_2) - F_{XY}(x,y_1)] / [F_Y(y_2) - F_Y(y_1)]$$

On differentiating, we get the conditional density function, i.e.

$$f_X(x/y_1 \le Y \le y_2) = \left[\int_{y_1}^{y_2} f_{XY}(x,y)\, dy\right] / \left[\int_{y_1}^{y_2} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy\right]$$

## Statistical Independence of Random Variables

Consider two random variables x and y with events $A = \{X \le x\}$, $B = \{Y \le y\}$ for two real numbers x and y. The two random variables are said to be statistically independent if and only if

$$P\{X \le x, Y \le y\} = P\{X \le x\} \cdot P\{Y \le y\}$$

The distribution function is

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y)$$

The density function is

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y)$$

Note:-

The conditional distribution function for independent random variables:

$F_X(x/y) = F_X(x)$  $F_Y(y/x) = F_Y(y)$

Similarly the conditional density functions are:

$f_X(x/y) = f_X(x)$  $f_Y(y/x) = f_Y(y)$

if they are statistically independent.

### Example 1

The joint probability density function of random variables x, y is given by $f_{XY}(x,y) = \{K e^{-(ax+by)}, x>0, y>0; 0, \text{otherwise}\}$. i) Find the value of k. ii) Are x and y independent?

Given JPdf is $f_{XY}(x,y) = \{K e^{-(ax+by)}, x>0, y>0; 0, \text{otherwise}\}$

i) To find 'k' value:-

From the properties of the joint density function,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy = 1$$

$$\int_0^{\infty} \int_0^{\infty} K e^{-(ax+by)}\, dx\, dy = 1$$

$$K \int_0^{\infty} e^{-ax}\, dx \int_0^{\infty} e^{-by}\, dy = 1$$

$$K [e^{-ax}/-a]_0^{\infty} [e^{-by}/-b]_0^{\infty} = 1$$

$$(K/ab)[0-1][0-1] = 1$$

$$\therefore K = ab$$

ii) To show that x and y are independent:-

w.k.t the Marginal density function for the random variable is given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{\infty} ab\, e^{-(ax+by)}\,dy$$

$$= ab\, e^{-ax} \int_0^{\infty} e^{-by}\,dy = ab\, e^{-ax}\,[e^{-by}/-b]_0^{\infty}$$

$$= (ab/b)\, e^{-ax}\,[0-1]$$

$$\therefore\ f_X(x) = a\, e^{-ax}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^{\infty} ab\, e^{-(ax+by)}\,dx$$

$$= ab\, e^{-by} \int_0^{\infty} e^{-ax}\,dx = ab\, e^{-by}\,[e^{-ax}/-a]_0^{\infty}$$

$$= -(ab/a)\, e^{-by}\,[0-1]$$

$$\therefore\ f_Y(y) = b\, e^{-by}$$

$$\therefore\ f_X(x) \cdot f_Y(y) = (a\, e^{-ax})(b\, e^{-by}) = ab\, e^{-(ax+by)}$$

$$\therefore\ f_X(x) \cdot f_Y(y) = f_{XY}(x,y)$$

∴ x and y are independent events.

**Example 2**

The JPDF for a random variable x and y is given by $f_{XY}(x,y) = \{K(x+y), 0<x<2, 0<y<1; 0, \text{otherwise}\}$.

a) Find the value of k. b) Find the marginal probability density functions of x and y. c) Are x and y independent?

a) To find 'k' value:-

From the properties of the JPdf,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\int_{x=0}^{2}\int_{y=0}^{1} K(x+y)\,dy\,dx = 1$$

$$K \int_{x=0}^{2} [xy + y^2/2]_0^1\,dx = 1$$

$$K \int_{x=0}^{2} (x + 1/2)\,dx = 1$$

$$K\,[x^2/2 + x/2]_0^2 = 1$$

$$K[\{4/2+2/2\} - 0] = 1 \implies K(3) = 1$$

$$\therefore\ K = 1/3$$

b) To find the Marginal density functions $f_X(x)$ and $f_Y(y)$:-

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dy = \int_0^{1} (1/3)(x+y)\,dy$$

$$= (1/3)[xy+y^2/2]_0^1 = (1/3)(x + 1/2)$$

$$\therefore\ f_X(x) = \{ (1/3)(x+1/2),\ 0<x<2\ ;\ 0,\ \text{otherwise} \}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_0^2 (1/3)(x+y)\,dx$$

$$= (1/3)[x^2/2+yx]_0^2 = (1/3)(2y+2)$$

$$\therefore\ f_Y(y) = \{ (2/3)(y+1),\ 0<y<1\ ;\ 0,\ \text{otherwise} \}$$

c) To verify x and y are independent:-

$$f_X(x) \cdot f_Y(y) = (1/3)(x+1/2) \cdot (2/3)(1+y)$$

$$= (2/9)(x+1/2)(1+y) = (2/9)[x+xy+1/2+y/2]$$

$$\neq (1/3)(x+y)$$

$$\therefore\ f_{XY}(x,y) \neq f_X(x) \cdot f_Y(y)$$

∴ x and y are not independent random variables.

**SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES, CHITTOOR – 517127**

**(AUTONOMOUS)**

Affiliated to JNTUA, Anantapuramu, NBA & NAAC Accredited

**DEPARTMENT**

**OF**

**ELECTRONICS & COMMUNICATION ENGINEERING**

# PROBABILITY AND COMPLEX VARIABLES

**(23BSC214 )**

## LECTURE NOTES

## INSTITUTE VISION AND MISSION

### ➢ VISION

- To emerge as a centre of excellence for learning and research in the domains of engineering, computing and management.

### ➢ MISSION

- Provide congenial academic ambience with state-art of resources for learning and research.
- Ignite the students to acquire self-reliance in the latest technologies.
- Unleash and encourage the innate potential and creativity of students.
- Inculcate confidence to face and experience new challenges.
- Foster enterprising spirit among students.
- Work collaboratively with technical Institutes/Universities/Industries of National and International repute.

## DEPARTMENT VISION AND MISSION

### ➢ VISION

- To become a centre of excellence in Electronics and Communication Engineering and provide necessary skills to the students to meet the challenges of industry and society.

### ➢ MISSION

- Provide congenial academic ambience with necessary infrastructure and learning resources
- Inculcate confidence to face and experience new challenges from industry and society.
- Ignite the students to acquire self-reliance in State-of-the-Art Technologies
- Foster Enterprising spirit among students

## PROGRAMME EDUCATIONAL OBJECTIVES (PEO's)

**PEO1:** Have Professional competency through the application of knowledge gained from subjects like Mathematics, Physics, Chemistry, Inter-Disciplinary and core subjects like Signal Processing, VLSI, Embedded Systems, Communication and Automation. **(Professional Competency)**

**PEO2:** E*xcel* in one's career by critical thinking towards successful services and growth of the organization or as an entrepreneur or through higher studies. **(Successful Career Goals)**

**PEO3:** Enhance knowledge by updating advanced technological concepts for facing the rapidly changing world and contribute to society through innovation and creativit*y*. **(Continuing Education and Contribution to Society)**

## PROGRAM SPECIFIC OUTCOMES (PSO'S)

**PSO1:** Apply the knowledge obtained in core areas for the analysis and design of components in Signal Processing, VLSI, Embedded Systems, Communication and Automation.

**PSO2:** Adapt Innovation, Creativity and design to develop products which meet industrial and societal needs.

# PROGRAMME OUTCOMES (PO's)

PO1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO2. **Problem analysis**: Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of

the limitations.

PO6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

## Course Educational Objectives

- To introduce the basic concepts of probability and random variables.
- To introduce the basic concepts of multiple random variables.
- To understand concepts of operations on multiple random variables.
- To analyze functions of a complex variable, with continuity, differentiability and analyticity.
- To understand Taylor and Laurent expansions, residues and improper integrals.

## Syllabus Overview

| Unit | Title | Key Topics |
|---|---|---|
| Unit I | Probability & Random Variable | Probability through sets & relative frequency; sample spaces; axioms; joint, conditional & total probability; Bayes' theorem; random variables; distribution & density functions; Gaussian, Binomial, Poisson, Uniform, Exponential, Rayleigh. |
| Unit II | Operations on Random Variable | Expectation; moments; characteristic & moment generating functions; multiple/vector random variables; joint distribution & density; marginal & conditional distributions; statistical independence. |
| Unit III | Operations on Multiple Random Variables | Joint moments; joint characteristic functions; jointly Gaussian random variables (two-variable case) and their properties. |
| Unit IV | Complex Variable — Differentiation | Limits, continuity, differentiation; Cauchy-Riemann equations; analytic & harmonic functions; harmonic conjugates; Milne-Thomson method. |
| Unit V | Complex Variable — Integration | Line integrals; Cauchy's integral theorem & formula; Taylor's & Laurent's series; zeros, singularities, residues; Cauchy residue theorem. |

## Course Outcomes

| CO | Outcome |
|---|---|
| CO1 | Understand the concepts of probability, random variables and their characteristics. |
| CO2 | Deal with multiple random variables, conditional probability, joint distribution and statistical independence. |
| CO3 | Formulate and solve engineering problems involving random variables. |
| CO4 | Analyze limit, continuity and differentiation of complex functions; understand Cauchy-Riemann equations and analytic functions. |
| CO5 | Apply Cauchy's theorem and integral formulas; classify singularities and evaluate integrals using residues. |

## CO – PO Mapping

| CO \ PO | PO1 | PO2 | PO3 |
|---|---|---|---|
| CO1 | 3 | 3 | 3 |
| CO2 | 3 | 3 | 3 |
| CO3 | 3 | 3 | 3 |
| CO4 | 3 | 3 | 3 |
| CO5 | 3 | 3 | 3 |

## Text Books


- Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", 4th Edition, TMH, 2002.
- B.S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 44th Edition, 2017.


## Reference Books


- Athanasios Papoulis & S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, PHI, 2002.
- Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley India.
- Henry Stark & John W. Woods, "Probability and Random Processes with Application to Signal Processing", 3rd Edition, Pearson, 2002.
- B.V. Ramana, "Higher Engineering Mathematics", McGraw Hill.

# UNIT – III

## Operations on Multiple Random Variables

### 1. Function of Joint Random Variables

If g(x, y) is a function of two random variables X and Y with joint density function $f_{XY}(x, y)$, then the expected value of the function g(x, y) is given by:

$$\bar{g} = E[g(X, Y)] = \iint_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

Similarly, for n random variables $X_1, X_2, ..., X_n$ with joint density function $f_{x_1 x_2 \ldots x_n}(x_1, x_2, ..., x_n)$, the expected value of $g(X_1, X_2, ..., X_n)$ is given by:

$$\bar{g} = E[g(X_1,\ldots,X_n)] = \int\ldots\int g(x_1,\ldots,x_n)\, f_{x_1\ldots x_n}(x_1,\ldots,x_n)\, dx_1\ldots dx_n$$

### 2. Joint Moments About the Origin

The joint moments about the origin for two random variables X, Y is the expected value of the function g(X,Y) = $X^n Y^k$. It is denoted by $m_{nk}$.

$$m_{nk} = E[X^n Y^k] = \iint_{-\infty}^{\infty} x^n y^k f_{XY}(x,y)\, dx\, dy$$

- The sum n + k is called the order of the moment.
- If k = 0, then $m_{n0} = E[X^n]$ — the moments of X.
- If n = 0, then $m_{0k} = E[Y^k]$ — the moments of Y.

**First order moments**

$$m_{10} = E[X] = \iint x\, f_{XY}(x,y)\, dx\, dy$$

$$m_{01} = E[Y] = \iint y\, f_{XY}(x,y)\, dx\, dy$$

**Second order moments**

$$m_{20} = E[X^2] \qquad m_{02} = E[Y^2] \qquad m_{11} = E[XY]$$

### 3. Correlation

Consider two random variables X and Y. The second order joint moment $m_{11}$ is called the correlation of X and Y, denoted as $R_{XY}$.

$$R_{XY} = m_{11} = E[XY] = \iint_{-\infty}^{\infty} xy\, f_{XY}(x,y)\, dx\, dy \quad \text{(continuous)}$$

$$R_{XY} = \Sigma_m \Sigma_n\, x_m y_n\, P_{XY}(x_m, y_n) \quad \text{(discrete)}$$

### Properties of Correlation

- If X and Y are statistically independent, then X and Y are said to be uncorrelated, i.e. $R_{XY} = E[XY] = E[X] \cdot E[Y]$.
- If the random variables X and Y are orthogonal, their correlation is zero, i.e. $R_{XY} = 0$.

## 4. Joint Central Moments

Consider two random variables X and Y. The expected values of the function $g(x,y) = (x - \bar{x})^n (y - \bar{y})^k$ are called joint central moments, denoted by $\mu_{nk}$.

$$\mu_{nk} = E[(X - \bar{x})^n (Y - \bar{y})^k] = \iint_{-\infty}^{\infty} (x-\bar{x})^n (y-\bar{y})^k f_{XY}(x,y)\, dx\, dy$$

The order of the central moment is n + k.

### Key Results

- 0th order central moment: $\mu_{00} = E[1] = 1$
- First order central moments: $\mu_{10} = E[x - \bar{x}] = 0$, $\mu_{01} = E[y - \bar{y}] = 0$
- Second order central moments: $\mu_{20} = E[(x - \bar{x})^2] = \sigma_X^2$, $\mu_{02} = E[(y - \bar{y})^2] = \sigma_Y^2$, $\mu_{11} = E[(x - \bar{x})(y - \bar{y})] = \sigma_{XY}^2$

## 5. Covariance

Consider the random variables X and Y. The second order joint central moment $\mu_{11}$ is called the covariance of X and Y, expressed as $C_{XY}$.

$$C_{XY} = \mu_{11} = E[(X - \bar{x})(Y - \bar{y})] = \iint_{-\infty}^{\infty} (x-\bar{x})(y-\bar{y})\, f_{XY}(x,y)\, dx\, dy \quad \text{(continuous)}$$
$$C_{XY} = \Sigma_n \Sigma_m (x_n - \bar{x}_n)(y_m - \bar{y}_m)\, P(x_n, y_m) \quad \text{(discrete)}$$

### Correlation Coefficient

The normalized second order central moment is called the correlation coefficient, denoted $\rho$:

$$\rho = \mu_{11} / \sqrt{(\mu_{20}\, \mu_{02})} = C_{XY} / \sqrt{(\sigma_X^2 \sigma_Y^2)} = C_{XY} / (\sigma_X \sigma_Y)$$

**Note:** *The range of correlation coefficient is $-1 \le \rho \le 1$. If X and Y are independent, $\rho = 0$. If correlation is perfect, $\rho = \pm 1$. If X = Y, then $\rho = 1$.*

### Properties of Covariance

- $C_{XY} = R_{XY} - \bar{x} \cdot \bar{y}$
- If X and Y are independent random variables, the covariance is zero: $C_{XY} = 0$. But the converse is not true.
- $Var(X + Y) = Var(X) + Var(Y) + 2C_{XY}$
- $Var(X - Y) = Var(X) + Var(Y) - 2C_{XY}$
- $|\sigma_{XY}| \le \sigma_X\, \sigma_Y$
- $Cov(X + a, Y + b) = Cov(X, Y) = C_{XY}$
- $Cov(aX, bY) = ab \cdot Cov(X, Y) = ab \cdot C_{XY}$

## Worked Examples

### Example 1

Random variables X and Y have the joint density $f_{XY}(x,y) = 1/24$, $0<x<6$, $0<y<4$. Find $E[g(X,Y)]$ for $g(X,Y) = (XY)^2$.

**Solution:**

$E[g(X,Y)] = \int_0^6\int_0^4 (xy)^2 (1/24)\, dy\, dx = (1/24)\int_0^6 x^2dx \int_0^4 y^2dy$

$= (1/24)[6^3/3][4^3/3] = (1/24)(72)(21.33) = 64$

**$E[g(X,Y)] = 64$**

### Example 2

Show that $X_1$ and $X_2$ with joint pdf $f(x_1,x_2) = 1/16$ for $|x_1|\le 4$, $2<x_2<4$ (0 otherwise) are independent, and show they are orthogonal.

**Solution:**

$f\_X_1(x_1) = \int_2^4 (1/16)\, dx_2 = 2/16 = 1/8$, for $-4 \le x_1 \le 4$

$f\_X_2(x_2) = \int_{-4}^4 (1/16)\, dx_1 = 8/16 = 1/2$, for $2 < x_2 < 4$

$f\_X_1(x_1)\cdot f\_X_2(x_2) = (1/8)(1/2) = 1/16 = f\_X_1X_2(x_1,x_2) \Rightarrow X_1, X_2$ are independent.

$E[X_1X_2] = \int_{-4}^4\int_2^4 x_1x_2 (1/16)\, dx_2dx_1 = (1/16)[x_1^2/2]_{-4}^4 \cdot [x_2^2/2]_2^4 = (1/64)(0)(6) = 0$

**$E[X_1X_2] = 0 \Rightarrow X_1$ and $X_2$ are orthogonal.**

### Example 3 & 4

A joint density function is $f_{XY}(x,y) = x(y+1.5)$, $0<x<1$, $0<y<1$ (0 otherwise). Find the first & second order moments, and in general find $m_{nk}$ for $n,k = 0,1,2,\dots$

**Solution:**

$m_{nk} = E[X^nY^k] = \int_0^1 x^{n+1}dx \cdot \int_0^1 y^k(y + 3/2)dy = [1/(n+2)] \cdot [1/(k+2) + 3/(2(k+1))]$

**Simplifying: $m_{nk} = (5k + 8) / [2(k+1)(k+2)(n+2)]$**

Special cases: $m_{10} = E[X] = 2/3$, $m_{01} = E[Y] = 13/24$, $m_{20} = E[X^2] = 1/2$, $m_{02} = E[Y^2] = 3/8$, $m_{11} = E[XY] = 13/36$

### Example 5

Independent random variables X and Y have moments $m_{10} = 2$, $m_{20} = 14$, $m_{02} = 12$, $m_{11} = -6$. Find $\mu_{22}$.

**Solution:**

$\mu_{22} = E[(X-\bar{x})^2(Y-\bar{y})^2] = [E(X^2) - \{E(X)\}^2]\cdot[E(Y^2) - \{E(Y)\}^2] = [\mu_{20} - m_{10}^2][\hat{\mu}_{02} - m_{01}^2]$

Since $m_{11} = m_{10} \times m_{01} \Rightarrow m_{01} = m_{11}/m_{10} = -6/2 = -3$

**$\mu_{22} = [14 - (2)^2][12 - (-3)^2] = [14-4][12-9] = (10)(3) = 30$**

### Example 6

Three statistically independent random variables $X_1, X_2, X_3$ have means $\bar{x}_1=3$, $\bar{x}_2=6$, $\bar{x}_3=-2$. Find the mean values of:

(a) $g = X_1 + 3X_2 + 4X_3$ (b) $g = X_1X_2X_3$ (c) $g = -2X_1X_2 - 3X_1X_3 + 4X_2X_3$ (d) $g = X_1 + X_2 + X_3$

**Solution:**

(a) $E[g] = E[X_1] + 3E[X_2] + 4E[X_3] = 3 + 3(6) + 4(-2) = 3 + 18 - 8 = 13$

(b) $E[g] = E[X_1]E[X_2]E[X_3] = (3)(6)(-2) = -36$

(c) $E[g] = -2(3)(6) - 3(3)(-2) + 4(6)(-2) = -36 + 18 - 48 = -66$

(d) $E[g] = 3 + 6 - 2 = 7$

### Example 7

If X and Y are independent, with $E[X]=\lambda_1$, $Var(X)=\sigma_1^2$, $E[Y]=\lambda_2$, $Var(Y)=\sigma_2^2$, prove $V[XY] = \sigma_1^2\sigma_2^2 + \lambda_1^2\sigma_2^2 + \lambda_2^2\sigma_1^2$.

**Solution:**

$V[XY] = E[(XY)^2] - \{E[XY]\}^2 = E[X^2]E[Y^2] - \{E[X]\}^2\{E[Y]\}^2$ (since X, Y independent)

Since $\sigma_1^2 = E[X^2] - \lambda_1^2 \Rightarrow E[X^2] = \sigma_1^2 + \lambda_1^2$, similarly $E[Y^2] = \sigma_2^2 + \lambda_2^2$

$V[XY] = (\sigma_1^2+\lambda_1^2)(\sigma_2^2+\lambda_2^2) - \lambda_1^2\lambda_2^2 = \sigma_1^2\sigma_2^2 + \sigma_1^2\lambda_2^2 + \sigma_2^2\lambda_1^2 + \lambda_1^2\lambda_2^2 - \lambda_1^2\lambda_2^2$

**$\therefore$ $V[XY] = \sigma_1^2\sigma_2^2 + \lambda_1^2\sigma_2^2 + \lambda_2^2\sigma_1^2$ (Hence proved)**

### Example 8

The joint density function is $f_{xy}(x,y) = (x+y)^2/40$, $-1<x<1$, $-3<y<3$ (0 otherwise). Find (a) $\sigma_x^2$, $\sigma_y^2$ (b) correlation coefficient $\rho_{xy}$.

**Solution:**

$m_{10} = E[X] = 0$, $E[X^2] = 9/25 = 0.36 \Rightarrow \sigma_x^2 = 0.36 - 0 = 0.36$

$m_{01} = E[Y] = 0$, $E[Y^2] = 5.16$ □ $\sigma_y^2 = 5.16 - 0 = 5.16$

$E[XY] = \int_{-1}^{1}\int_{-3}^{3} xy(x+y)^2/40 \, dy \, dx = 0.6$

$C_{xy} = E[XY] - E[X]E[Y] = 0.6$

**$\rho_{xy} = C_{xy} / (\sigma_x\sigma_y) = 0.6 / (\sqrt{0.36}\sqrt{5.16}) = 0.440$**

### Example 9

For X and Y with joint density $f_{xy}(x,y) = 0.15\delta(x+1)\delta(y) + 0.1\delta(x)\delta(y) + 0.1\delta(x)\delta(y-2) + 0.4\delta(x-1)\delta(y+2) + 0.2\delta(x-1)\delta(y-1) + 0.05\delta(x-1)\delta(y-3)$. Find (a) correlation (b) correlation coefficient (c) whether X, Y are uncorrelated or orthogonal.

**Probability mass table:**

| (x,y) | (-1,0) | (0,0) | (0,2) | (1,-2) | (1,1) | (1,3) |
|---|---|---|---|---|---|---|
| P(x,y) | 0.15 | 0.1 | 0.1 | 0.4 | 0.2 | 0.05 |

### Solution — Example 9

$R_{xy} = E[XY] = \Sigma x_iP(x_i,y_i) = (-1)(0)(0.15) + (0)(0)(0.1) + (0)(2)(0.1) + (1)(-2)(0.4) + (1)(1)(0.2) + (1)(3)(0.05) = -0.45$

$E[X] = 0.5$, $E[X^2] = 0.8$, $E[Y] = -0.25$, $E[Y^2] = 2.65$

$C_{xy} = R_{xy} - E[X]E[Y] = -0.45 - (0.5)(-0.25) = -0.325$

$\sigma_x^2 = 0.8 - (0.5)^2 = 0.55$, $\sigma_y^2 = 2.65 - (-0.25)^2 = 2.58$

**$\rho_{xy} = -0.325/\sqrt{(0.55)(2.58)} = -0.272$**

**Since $C_{xy} \neq 0$, X and Y are NOT uncorrelated; since $R_{xy} \neq 0$, X and Y are NOT orthogonal.**

**Example 10**

If the joint density function is $f_{xy}(x,y) = x+y$, $0 \le x \le 1$, $0 \le y \le 1$ (0 otherwise), find the correlation coefficient.

**Solution:**

$f_X(x) = x + 1/2$, $f_Y(y) = 1/2 + y$ (marginal densities)

$E[X] = E[Y] = 7/12$, $E[X^2] = E[Y^2] = 10/24$

$E[XY] = \int_0^1\int_0^1 xy(x+y)\, dy\, dx = 1/3$

$C_{xy} = 1/3 - (7/12)(7/12) = -0.006$

$\sigma_x^2 = \sigma_y^2 = 10/24 - (7/12)^2 = 0.076$

**$\rho_{xy} = -0.006/\sqrt{(0.076)(0.076)} = -0.078$**

**Example 11**

For X and Y with $f_{xy}(x,y) = 0.5\delta(x+1)\delta(y) + 0.1\delta(x)\delta(y-2) + 0.1\delta(x)\delta(y+2) + 0.4\delta(x-1)\delta(y+2) + 0.2\delta(x-1)\delta(y-1) + 0.5\delta(x-1)\delta(y-3)$. Find (a) correlation (b) covariance (c) correlation coefficient (d) are X, Y uncorrelated or orthogonal?

| (x,y) | (-1,0) | (0,0) | (0,2) | (1,-2) | (1,1) | (1,3) |
|---|---|---|---|---|---|---|
| P(x,y) | 0.5 | 0.1 | 0.1 | 0.4 | 0.2 | 0.5 |

**Solution — Example 11**

$R_{xy} = E[XY] = (1)(-2)(0.4) + (1)(1)(0.2) + (1)(3)(0.5) = 0.9$

$E[X] = 0.6$, $E[X^2] = 1.6$, $E[Y] = 1.1$, $E[Y^2] = 6.7$

$C_{xy} = 0.9 - (0.6)(1.1) = 0.24$

$\sigma_x^2 = 1.6 - (0.6)^2 = 1.24$, $\sigma_y^2 = 6.7 - (1.1)^2 = 5.49$

**$\rho_{xy} = 0.24/\sqrt{(1.24)(5.49)} = 0.091$**

$Cov(X,Y) = C_{xy} = 0.24$

**Since $C_{xy} \neq 0$, X and Y are NOT uncorrelated; since $R_{xy} \neq 0$, X and Y are NOT orthogonal.**

## 6. Joint Characteristic Function

The joint characteristic function (JCF) of two random variables X and Y is defined as the expected value of the joint function $g(x,y) = e^{(j\omega_1 x + j\omega_2 y)}$. It is given as:

$$\varphi_{xy}(\omega_1,\omega_2) = E[e^{(j(\omega_1 X+\omega_2 Y))}] = \iint_{-\infty}^{\infty} e^{(j\omega_1 x+j\omega_2 y)} f_{xy}(x,y)\, dx\, dy$$

where $\omega_1$ and $\omega_2$ are real variables.

### Notes

- This expression is the two-dimensional Fourier transform of the joint density function.
- The inverse Fourier transform of the JCF gives the joint density: $f_{xy}(x,y) = \iint \varphi_{xy}(\omega_1,\omega_2)\, e^{(-j(\omega_1 x+\omega_2 y))}\, d\omega_1\, d\omega_2$
- If X, Y are discrete random variables with pmf P(x,y): $\varphi_{xy}(\omega_1,\omega_2) = \Sigma_m\Sigma_n e^{(j\omega_1 x_m+j\omega_2 y_n)} P(x_m,y_n)$

### Properties of the Joint Characteristic Function

- Marginal characteristic functions: $\varphi_x(\omega_1) = \varphi_{xy}(\omega_1, 0)$, $\varphi_y(\omega_2) = \varphi_{xy}(0, \omega_2)$
- If X and Y are independent, the JCF equals the product of the individual characteristic functions: $\varphi_{xy}(\omega_1,\omega_2) = \varphi_x(\omega_1)\cdot\varphi_y(\omega_2)$
- If X and Y are independent, $\varphi_{x+y}(\omega) = \varphi_x(\omega)\cdot\varphi_y(\omega)$
- Joint moments can be derived from the JCF as:

$$m_{n\,k} = (-j)^{n+k}\cdot \partial^{n+k}\varphi_{xy}(\omega_1,\omega_2) / \partial\omega_1{}^n \partial\omega_2{}^k \mid \omega_1=\omega_2=0$$

## 7. Joint Moment Generating Function

The joint moment generating function (JMGF) of two random variables X and Y is defined as the expected value of the joint function $g(x,y) = e^{(v_1 x+v_2 y)}$. It is expressed as:

$$M_{xy}(v_1,v_2) = E[e^{(v_1 X+v_2 Y)}] = \iint_{-\infty}^{\infty} e^{(v_1 x+v_2 y)} f_{xy}(x,y)\, dx\, dy \quad \text{(continuous)}$$

$$M_{xy}(v_1,v_2) = \Sigma_m\Sigma_n e^{(v_1 x_m+v_2 y_n)} P_{xy}(x_m,y_n) \quad \text{(discrete)}$$

### Properties of the Joint MGF

- Marginal MGFs: $M_x(v_1) = M_{xy}(v_1, 0)$, $M_y(v_2) = M_{xy}(0, v_2)$
- If X and Y are independent: $M_{xy}(v_1,v_2) = M_x(v_1)\cdot M_y(v_2)$, and $M_{x+y}(v) = M_x(v)\cdot M_y(v)$
- Joint moments can be derived from the JMGF:

$$m_{n\,k} = \partial^{n+k} M_{xy}(v_1,v_2) / \partial v_1{}^n \partial v_2{}^k \mid v_1=v_2=0$$

### Worked Example — Joint Characteristic Function

**Example 12**

Two random variables X and Y have joint characteristic function $\varphi_{xy}(\omega_1,\omega_2) = \exp(-2\omega_1^2 - 8\omega_2^2)$. (i) Show X and Y are zero-mean random variables. (ii) Are X and Y correlated?

**Solution:**

$m_{nk} = (-j)^{n+k}\, \partial^{n+k}\varphi_{xy}(\omega_1,\omega_2)/\partial\omega_1{}^n\partial\omega_2{}^k$ at $\omega_1=\omega_2=0$

(i) $m_{10} = (-j)\cdot\partial\varphi/\partial\omega_1|_0 = (-j)[e^{(-2\omega_1^2-8\omega_2^2)}(-4\omega_1)]|_0 = 0 \Rightarrow E[X] = 0$

$m_{01} = (-j)\cdot\partial\varphi/\partial\omega_2|_0 = (-j)[e^{(-2\omega_1^2-8\omega_2^2)}(-16\omega_2)]|_0 = 0 \Rightarrow E[Y] = 0$

**Hence X and Y are zero mean random variables.**

(ii) $C_{xy} = R_{xy} - E[X]E[Y] = R_{xy} = E[XY] = m_{11}$

$m_{11} = (-j)^2 \cdot \partial^2\varphi/\partial\omega_1\partial\omega_2|_0 = \partial(e^{(-2\omega_1^2)})/\partial\omega_1 \cdot \partial(e^{(-8\omega_2^2)})/\partial\omega_2 |_0 = 0$

**∴ E[XY] = 0 ⇒ X and Y are uncorrelated.**

# 8. Gaussian Random Variables

## (i) Two Random Variables

If two random variables X and Y are said to be jointly Gaussian, the joint density function is given as (bivariate Gaussian density function):

$$f_{xy}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \times \exp\left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(x-\bar{x})^2}{\sigma_x^2} - \frac{2\rho(x-\bar{x})(y-\bar{y})}{\sigma_x\sigma_y} + \frac{(y-\bar{y})^2}{\sigma_y^2} \right] \right\}$$

where:

- $\bar{x} = E[X]$, $\bar{y} = E[Y]$
- $\sigma_x^2 = E[X^2] - \{E[X]\}^2$, $\sigma_y^2 = E[Y^2] - \{E[Y]\}^2$
- $\rho = C_{xy} / (\sigma_x\sigma_y)$

$$\text{Maximum of } f_{xy}(x,y) = 1 / (2\pi\sigma_x\sigma_y\sqrt{1-\rho^2})$$

## (ii) N Random Variables

Consider N random variables $X_n$, n = 1,2,3,...,N. They are said to be jointly Gaussian if their joint density function is given by:

$$f_{x_1x_2\ldots x_n}(x_1,x_2,\ldots,x_n) = \frac{1}{(2\pi)^{N/2}|C_x|^{1/2}} \times \exp\left\{ -[x-\bar{x}]^T[C_x]^{-1}[x-\bar{x}] / 2 \right\}$$

where $[C_x]$ is the N×N covariance matrix with elements $C_{ij}$, and $[x - \bar{x}]$ is the column vector of deviations $(x_n - \bar{x}_n)$.

## Properties of Gaussian Random Variables

- Gaussian random variables are completely defined by their means, variances, and covariances.
- If Gaussian random variables are uncorrelated, then they are statistically independent.
- All marginal density functions derived from an N-Gaussian density function are also Gaussian.
- All conditional density functions are also Gaussian.

The linear transformation of Gaussian random variables produces Gaussian random variable

**SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES, CHITTOOR – 517127**

**(AUTONOMOUS)**

Affiliated to JNTUA, Anantapuramu, NBA & NAAC Accredited

**DEPARTMENT**

**OF**

**ELECTRONICS & COMMUNICATION ENGINEERING**

# PROBABILITY AND COMPLEX VARIABLES

**(23BSC214 )**

# LECTURE NOTES

## INSTITUTE VISION AND MISSION

### VISION

- To emerge as a centre of excellence for learning and research in the domains of engineering, computing and management.

### MISSION

- Provide congenial academic ambience with state-art of resources for learning and research.
- Ignite the students to acquire self-reliance in the latest technologies.
- Unleash and encourage the innate potential and creativity of students.
- Inculcate confidence to face and experience new challenges.
- Foster enterprising spirit among students.
- Work collaboratively with technical Institutes/Universities/Industries of National and International repute.

## DEPARTMENT VISION AND MISSION

### VISION

- To become a centre of excellence in Electronics and Communication Engineering and provide necessary skills to the students to meet the challenges of industry and society.

### MISSION

- Provide congenial academic ambience with necessary infrastructure and learning resources
- Inculcate confidence to face and experience new challenges from industry and society.
- Ignite the students to acquire self-reliance in State-of-the-Art Technologies
- Foster Enterprising spirit among students

## PROGRAMME EDUCATIONAL OBJECTIVES (PEO's)

**PEO1:** Have Professional competency through the application of knowledge gained from subjects like Mathematics, Physics, Chemistry, Inter-Disciplinary and core subjects like Signal Processing, VLSI, Embedded Systems, Communication and Automation. **(Professional Competency)**

**PEO2:** E*xcel* in one's career by critical thinking towards successful services and growth of the organization or as an entrepreneur or through higher studies. **(Successful Career Goals)**

**PEO3:** Enhance knowledge by updating advanced technological concepts for facing the rapidly changing world and contribute to society through innovation and creativit*y*. **(Continuing Education and Contribution to Society)**

## PROGRAM SPECIFIC OUTCOMES (PSO'S)

**PSO1:** Apply the knowledge obtained in core areas for the analysis and design of components in Signal Processing, VLSI, Embedded Systems, Communication and Automation.

**PSO2:** Adapt Innovation, Creativity and design to develop products which meet industrial and societal needs.

# PROGRAMME OUTCOMES (PO's)

PO1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO2. **Problem analysis**: Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of

the limitations.

PO6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

## Course Educational Objectives

- To introduce the basic concepts of probability and random variables.
- To introduce the basic concepts of multiple random variables.
- To understand concepts of operations on multiple random variables.
- To analyze functions of a complex variable, with continuity, differentiability and analyticity.
- To understand Taylor and Laurent expansions, residues and improper integrals.

## Syllabus Overview

| Unit | Title | Key Topics |
|---|---|---|
| Unit I | Probability & Random Variable | Probability through sets & relative frequency; sample spaces; axioms; joint, conditional & total probability; Bayes' theorem; random variables; distribution & density functions; Gaussian, Binomial, Poisson, Uniform, Exponential, Rayleigh. |
| Unit II | Operations on Random Variable | Expectation; moments; characteristic & moment generating functions; multiple/vector random variables; joint distribution & density; marginal & conditional distributions; statistical independence. |
| Unit III | Operations on Multiple Random Variables | Joint moments; joint characteristic functions; jointly Gaussian random variables (two-variable case) and their properties. |
| Unit IV | Complex Variable — Differentiation | Limits, continuity, differentiation; Cauchy-Riemann equations; analytic & harmonic functions; harmonic conjugates; Milne-Thomson method. |
| Unit V | Complex Variable — Integration | Line integrals; Cauchy's integral theorem & formula; Taylor's & Laurent's series; zeros, singularities, residues; Cauchy residue theorem. |

## Course Outcomes

| CO | Outcome |
|---|---|
| CO1 | Understand the concepts of probability, random variables and their characteristics. |
| CO2 | Deal with multiple random variables, conditional probability, joint distribution and statistical independence. |
| CO3 | Formulate and solve engineering problems involving random variables. |
| CO4 | Analyze limit, continuity and differentiation of complex functions; understand Cauchy-Riemann equations and analytic functions. |
| CO5 | Apply Cauchy's theorem and integral formulas; classify singularities and evaluate integrals using residues. |

## CO – PO Mapping

| CO \ PO | PO1 | PO2 | PO3 |
|---|---|---|---|
| CO1 | 3 | 3 | 3 |
| CO2 | 3 | 3 | 3 |
| CO3 | 3 | 3 | 3 |
| CO4 | 3 | 3 | 3 |
| CO5 | 3 | 3 | 3 |

## Text Books


- Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", 4th Edition, TMH, 2002.
- B.S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 44th Edition, 2017.


## Reference Books


- Athanasios Papoulis & S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, PHI, 2002.
- Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley India.
- Henry Stark & John W. Woods, "Probability and Random Processes with Application to Signal Processing", 3rd Edition, Pearson, 2002.
- B.V. Ramana, "Higher Engineering Mathematics", McGraw Hill.


# UNIT – IV

# Complex Variables – Differentiation

## 1. Introduction to Complex Numbers

### Complex Number

An expression of the form $x + iy$, where x and y are real numbers and $i = \sqrt{-1}$, is said to be a complex number. It is denoted by 'z'. Here x is called the real part of z, i.e. Re(z), and y is called the imaginary part of z, i.e. Im(z).

### Conjugate of a Complex Number

Consider a complex number $z = x + i(-y)$. This is said to be the conjugate of z, denoted by $\bar{z}$. Thus $\bar{z} = x + i(-y)$.

**Note:** *$x = (z + \bar{z})/2$ and $y = (z - \bar{z})/2i$*

### Geometrical Representation of a Complex Number

Consider a complex number $z = x + iy$. Geometrically, a complex number is denoted by a point in the two-dimensional plane whose first coordinate is the real part and second coordinate is the imaginary part, i.e. (x, y).

- The horizontal axis is called the Real axis.
- The vertical axis is called the Imaginary axis.
- The plane in which a complex number is represented geometrically is called the Complex plane (or) Argand plane.

### Polar Form of a Complex Number

If $z = x + iy$ is any complex number, then it is represented in polar form, i.e. $(r, \theta)$, given by:

$$z = r e^{i\theta}, \quad \text{where } r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}(y/x)$$

### Complex Variable

A quantity which is varying with respect to complex numbers is said to be a complex variable, and it is denoted by 'z'. Thus $z = x + iy$ where x and y are real variables.

### Complex Function

Consider two complex variables 'z' and 'w'. If for each value of z, there is a value for w, then w is said to be a function of z, and it is denoted by $w = f(z)$.

**Note:** *Let $w = u + iv$, where u and v are real variables (functions of x and y).*

## 2. Types of Complex Functions

### (i) Single Valued Function

Consider a complex function $w = f(z)$. If for each value of z, there is only a single value for w, then $w = f(z)$ is said to be a single valued function.

**Note:** *Example: $w = z^2$*

### (ii) Multiple Valued Function

Consider a complex function $w = f(z)$. If for any one value of z, there are multiple values for w, then $w = f(z)$ is said to be a multiple valued function.

**Note:** *Example:* $w = \sqrt{z}$

## 3. Limit of a Function Nearer to a Point

Consider a function f(z) and a point 'a'. If f(z) values are nearer to a finite value for nearer values of the point 'a', then the finite value is said to be the limit of the point function f(z) for the nearer values of point 'a', and it is denoted by:

$$\lim_{z\to a} f(z) = [\text{finite value}]$$

### Method for Evaluation of $\lim_{z\to a} f(z)$

Evaluate lim f(z) along all 5 paths from z to a:

- Rectangular path (P, L, Q)
- Rectangular path (P, M, Q)
- Straight line (P, Q)
- Parabola path (P, B, Q)
- Parabola path (P, C, Q)

### Process

- Convert f(z) into two real variables x and y by substituting $z = x + iy$, and then convert the two variables into a single variable x or y depending on the mathematical form of the path.
- If $\lim_{z\to a} f(z)$ is unique (or the same) along all 5 paths, then $\lim_{z\to a} f(z)$ exists and the unique value is the limit value.

## 4. Continuity of a Function

### Continuity at a Point

Consider a point function f(z) and point 'A'. f is continuous at $z = a$ if:

- $\lim_{z\to a} f(z)$ exists
- $\lim_{z\to a} f(z) = f(a)$

### Continuity over a Region

Consider a function f(z) and a region R. If f(z) is continuous at every point over R, then f(z) is continuous over R.

### Worked Examples — Continuity

**Example 1**

Discuss the continuity of $f(z) = (x^3 - y^3)/(x^3 + y^3)$, $z\neq 0$; $f(z) = 0$, $z = 0$, at $z = 0$.

**Solution:**

Along PLO ($y = k$, $x\to 0$): $\lim f(z) = \lim (x^3-k^3)/(x^3+k^3) = -k^3/k^3 = -1$

Along PMO ($x = k$, $y\to 0$): $\lim f(z) = \lim (k^3-y^3)/(k^3+y^3) = k^3/k^3 = 1$

Along PO (straight line, $y=x$): $\lim f(z) = \lim x^3/x^3 = 1$

**Since the limit along PLO (-1) differs from the limit along PMO and PO (1), $\lim_{z \to 0} f(z)$ does NOT exist.**

**∴ f(z) is NOT continuous at z = 0.**

### Example 2

Discuss the continuity of $f(z) = [x^3(1+i) - y^3(1-i)]/(x^2+y^2), z \neq 0$; $f(z) = 0, z = 0$.

**Solution:**

Along PLO ($x=0, y \to 0$): $Lt\ f(z) = Lt\ [-y^3(1+i)]/y^2 = 0$

Along PMO ($y=0, x \to 0$): $Lt\ f(z) = Lt\ [x^3(1+i)]/x^2 = 0$

Along PAO ($y = mx$): $Lt\ f(z) = Lt\ [x^3(1+i) - (mx)^3(1-i)]/[x^2+(mx)^2] = 0$

Along PBO ($y^2 = mx$): $Lt\ f(z) = 0$

Along PCO ($x^2 = my$): $Lt\ f(z) = 0$

**Since lim f(z) is the same (= 0) along all 5 paths, $\lim_{z \to 0} f(z)$ exists and equals 0 = f(0).**

**∴ f is continuous at z = 0.**

# 5. Derivative of a Function

## Derivative at a Point

Consider a function f(z) and a point $z_0$. If:

$$Lt_{h \to 0}\ [f(z_0+h) - f(z_0)] / h \quad \text{exists,}$$

then the derivative of f(z) at $z_0$ exists, and the limit value is the derivative of f at $z_0$, denoted by $df/dz|_{z_0}$ (or) $f'(z_0)$.

## Derivative over a Region

Consider a function f(z) and region (R). If the derivative of f(z) at each point exists over the region (R), then the derivative of 'f' exists over the region R.

# 6. Analyticity of a Function

## Analyticity at a Point

Consider a function f(z) and a point 'a'. If:

- f(z) is derivable at 'a', and
- f(z) is derivable in the neighbourhood of 'a' (a circular region with centre 'a' and radius ε, a very small quantity),

then f(z) is analytic at the point a.

## Analyticity over a Region

Consider a function f(z) and a region 'R'. If f(z) is analytic at every point over the region 'R', then f(z) is analytic over the region 'R'.

## Result 1 — Necessary Condition for Analyticity

If $f(z) = w = u + iv$ is analytic over a region R, then:

$$\partial u/\partial x = \partial v/\partial y \quad \text{and} \quad \partial u/\partial y = -\partial v/\partial x$$
$$f'(z) = \partial u/\partial x + i\, \partial v/\partial x$$

## Result 2 — Sufficient Condition for Analyticity

If:

- $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ (Cauchy–Riemann equations, or C-R equations), and
- $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$ are continuous,

then $f(z) = u + iv$ is analytic everywhere.

**Note:** *The sufficient condition is useful for easily proving f(z) is analytic over a region.*

## Process to Test Analyticity

- Write the real part (u) and imaginary part (v) of f(z) by substituting $z = x + iy$.
- Evaluate $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$.
- Check the C-R equations: $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$.
- Check the continuity of $\partial u/\partial x$, $\partial u/\partial y$, $\partial v/\partial x$, $\partial v/\partial y$.

## Notes on Continuity

- An nth degree polynomial in x and y is continuous everywhere.
- An exponential function in x and y is continuous everywhere.
- Sine and cosine functions in x and y are continuous everywhere.
- If f and g are continuous, then $f+g$, $f-g$, $f \cdot g$, $f/g$ $(g \neq 0)$ are continuous.

# Worked Examples — Testing Analyticity

### Example 1

Show that $f(z) = z^2$ is analytic everywhere and hence find its derivative.

**Solution:**

$f(z) = z^2 = (x+iy)^2 = (x^2-y^2) + i(2xy) = u + iv$

$u = x^2 - y^2, \quad v = 2xy$

$\partial u/\partial x = 2x, \ \partial v/\partial x = 2y, \ \partial u/\partial y = -2y, \ \partial v/\partial y = 2x$

$\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$; all four partials are continuous.

**∴ f(z) is analytic everywhere.**

**$f'(z) = \partial u/\partial x + i\partial v/\partial x = 2x + i(2y) = 2(x+iy) = 2z$**

### Example 2

Show that $f(z) = z^3$ is analytic everywhere and hence find its derivative.

**Solution:**

$f(z) = z^3 = (x+iy)^3 = (x^3-3xy^2) + i(3x^2y-y^3) = u + iv$

$u = x^3-3xy^2, \quad v = 3x^2y-y^3$

$\partial u/\partial x = 3x^2-3y^2, \ \partial v/\partial x = 6xy, \quad \partial u/\partial y = -6xy, \ \partial v/\partial y = 3x^2-3y^2$

**C-R equations satisfied and all partials continuous ⇒ f(z) is analytic everywhere.**

**$f'(z) = (3x^2-3y^2) + i(6xy) = 3(x^2-y^2+2ixy) = 3(x+iy)^2 = 3z^2$**

## Example 3

Show that $f(z) = z + 2\bar{z}$ is NOT analytic everywhere.

**Solution:**

$f(z) = z + 2\bar{z} = (x+iy) + 2(x-iy) = 3x - iy = u + iv$

$u = 3x, \quad v = -y$

$\partial u/\partial x = 3, \ \partial v/\partial x = 0, \quad \partial u/\partial y = 0, \ \partial v/\partial y = -1$

$\partial u/\partial x \neq \partial v/\partial y$ and $\partial u/\partial y \neq -\partial v/\partial x$

**∴ f(z) is NOT analytic everywhere.**

## Example 4

Show that f(z) = e^z is analytic everywhere and find its derivative.

**Solution:**

$f(z) = e^z = e^{(x+iy)} = e^x(\cos y + i \sin y) = u + iv$

$u = e^x \cos y, \quad v = e^x \sin y$

$\partial u/\partial x = e^x \cos y = \partial v/\partial y, \quad \partial u/\partial y = -e^x \sin y = -\partial v/\partial x$

**All four partials continuous ⇒ f(z) is analytic everywhere.**

**$f'(z) = e^x \cos y + i\, e^x \sin y = e^x(\cos y + i \sin y) = e^z$**

## Example 5

Show that f(z) = sin z is analytic everywhere and find its derivative.

**Solution:**

$f(z) = \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y = u + iv$

$u = \sin x \cosh y, \quad v = \cos x \sinh y$

$\partial u/\partial x = \cos x \cosh y = \partial v/\partial y, \quad \partial u/\partial y = \sin x \sinh y = -\partial v/\partial x$

**All partials continuous ⇒ f(z) is analytic everywhere.**

**$f'(z) = \cos x \cosh y - i \sin x \sinh y = \cos(x+iy) = \cos z$**

## Example 6

Show that f(z) = log z is analytic everywhere except at the origin, and find its derivative.

**Solution:**

$f(z) = \log(x+iy) = (1/2)\log(x^2+y^2) + i \tan^{-1}(y/x) = u + iv$

$u = (1/2)\log(x^2+y^2), \quad v = \tan^{-1}(y/x)$

$\partial u/\partial x = x/(x^2+y^2) = \partial v/\partial y$, $\partial u/\partial y = y/(x^2+y^2) = -\partial v/\partial x$

**All partials continuous except at origin (x=y=0) ⇒ f(z) is analytic everywhere except at the origin.**

**$f'(z) = x/(x^2+y^2) - iy/(x^2+y^2) = 1/(x+iy) = 1/z$**

## Cauchy–Riemann Equations in Polar Coordinates

*$\partial u/\partial r = (1/r)\, \partial v/\partial \theta$ and $\partial u/\partial \theta = -r\, \partial v/\partial r$*

### Example 7

Show that $f(z) = z^n$, $n \in Z^+$, is analytic everywhere and hence find its derivative.

**Solution:**

$f(z) = z^n = (r\, e^{i\theta})^n = r^n(\cos n\theta + i \sin n\theta) = u + iv$

$u = r^n \cos n\theta$, $v = r^n \sin n\theta$

$\partial u/\partial r = n\, r^{n-1} \cos n\theta$, $\partial v/\partial \theta = n\, r^n \cos n\theta \Rightarrow \partial u/\partial r = (1/r)\partial v/\partial \theta$

$\partial u/\partial \theta = -n\, r^n \sin n\theta$, $\partial v/\partial r = n\, r^{n-1} \sin n\theta \Rightarrow \partial u/\partial \theta = -r\partial v/\partial r$

**All partials continuous ⇒ $f(z) = z^n$ is analytic everywhere.**

**$f'(z) = n\, z^{n-1}$**

*Additional practice problems (not analytic — verify using C-R equations):*

- Show that $f(z) = \sin x \sin y - i \cos x \cos y$ is not analytic everywhere.
- Show that $f(z) = z + 3\bar{z}$ is not analytic.
- Show that $f(z) = \cos z$ is analytic.
- Show that $f(z) = |z|^2$ is not analytic.

### Solution — sin x sin y - i cos x cos y

$f(z) = \sin x \sin y - i \cos x \cos y = u + iv$, $u = \sin x \sin y$, $v = -\cos x \cos y$

$\partial u/\partial x = \cos x \sin y$, $\partial v/\partial x = \sin x \cos y$, $\partial u/\partial y = \sin x \cos y$, $\partial v/\partial y = \cos x \sin y$

$\partial u/\partial x = \partial v/\partial y$ but $\partial u/\partial y \neq -\partial v/\partial x$

**∴ f(z) is NOT analytic everywhere.**

### Solution — $z + 3\bar{z}$

$f(z) = z + 3\bar{z} = x+iy + 3(x-iy) = 4x - i2y = u + iv$, $u = 4x$, $v = -2y$

$\partial u/\partial x = 4$, $\partial v/\partial x = 0$, $\partial u/\partial y = 0$, $\partial v/\partial y = -2$

$\partial u/\partial x \neq \partial v/\partial y$ and $\partial u/\partial y \neq -\partial v/\partial x$

**∴ f(z) is NOT analytic everywhere.**

### Solution — cos z

$f(z) = \cos(x+iy) = \cos x \cosh y - i \sin x \sinh y = u+iv$, $u = \cos x \cosh y$, $v = -\sin x \sinh y$

$\partial u/\partial x = -\sin x \cosh y = \partial v/\partial y$, $\partial u/\partial y = \cos x \sinh y = -\partial v/\partial x$

**All partials continuous ⇒ f(z) is analytic everywhere.**

**$f'(z) = -\sin x \cosh y - i \cos x \sinh y = -\sin(x+iy) = -\sin z$**

# 7. Laplace Equation and Harmonic Functions

## Laplace Equation

An equation of the form $\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = 0$, where f is a function in two variables x, y, is said to be a Laplace equation.

## Harmonic Function

A function that satisfies the Laplace equation is said to be a harmonic function.

## Properties of Analytic Functions

- If $f(z) = w = u + iv$ is an analytic function, then u and v are harmonic functions. (Corollary: the real and imaginary parts of an analytic function are harmonic functions.)
- If $f(z) = w = u + iv$ is an analytic function, then $u(x,y) = c_1$ and $v(x,y) = c_2$ form orthogonal curves (Corollary: the real and imaginary parts of an analytic function form orthogonal curves).
- If $f(z) = w = u + iv$ is an analytic function and $u = c$, then f(z) is a constant function. (Corollary: an analytic function whose real part is constant is a constant function.)
- If an analytic function's imaginary part is constant, it is a constant function.
- If $f(z) = u + iv$ is analytic and $|f(z)| = c$, then f(z) is a constant function.

# Worked Examples — Complex Identities & Analyticity Proofs

### Example 1

If $\cosh(u+iv) = x+iy$, prove that (i) $x^2/\cosh^2 u + y^2/\sinh^2 u = 1$ (ii) $x^2/\cos^2 v - y^2/\sin^2 v = 1$.

**Solution:**

$\cosh(u+iv) = \cosh u \cos v + i(\sinh u \sin v) = x + iy$

$\Rightarrow x = \cosh u \cos v, \quad y = \sinh u \sin v$

**(i) $x^2/\cosh^2 u + y^2/\sinh^2 u = \cos^2 v + \sin^2 v = 1$**

**(ii) $x^2/\cos^2 v - y^2/\sin^2 v = \cosh^2 u - \sinh^2 u = 1$**

### Example 2

Solve $\sinh z = i$.

**Solution:**

$\sinh z = i \Rightarrow (e^z - e^{-z})/2 = i \Rightarrow e^z - e^{-z} = 2i$

$(e^z)^2 - 2i(e^z) - 1 = 0 \Rightarrow (e^z - i)^2 = 0 \Rightarrow e^z = i$

**$z = \ln(i) = (1/2)\ln(0^2+1^2) + i \tan^{-1}(1/0) + 2n\pi i = i(\pi/2) + 2n\pi i, \quad n \in Z$**

**Example 3**

Test whether or not f(z) = sin x sin y - i cos x cos y is analytic.

**Solution:**

$u = \sin x \sin y, \quad v = -\cos x \cos y$

$\partial u/\partial x = \cos x \sin y, \ \partial v/\partial x = \sin x \cos y, \quad \partial u/\partial y = \sin x \cos y, \ \partial v/\partial y = \cos x \sin y$

$\partial u/\partial x = \partial v/\partial y$ but $\partial u/\partial y \neq -\partial v/\partial x$

**Hence f(z) is NOT analytic.**

**Example 4**

Separate the real and imaginary parts of tanh z.

**Solution:**

$\tanh z = \sinh z/\cosh z = (\sinh x \cos y + i \cosh x \sin y)/(\cosh x \cos y + i \sinh x \sin y)$

Multiplying numerator and denominator by the conjugate and simplifying using double-angle identities:

$\tanh z = \sinh 2x/(\cosh 2x + \cos 2y) + i \sin 2y/(\cosh 2x + \cos 2y)$

## Results Involving |f(z)| and f'(z) for Analytic f

**Example 5**

Show that $[\partial^2/\partial x^2 + \partial^2/\partial y^2]\, |f(z)|^2 = 4|f'(z)|^2$, where f(z) is analytic.

**Solution:**

Using $\partial^2/\partial x^2 + \partial^2/\partial y^2 = 4\partial^2/\partial z\partial\bar{z}$ and $|z|^2 = z\cdot\bar{z}$:

$[\partial^2/\partial x^2+\partial^2/\partial y^2]|f(z)|^2 = 4\partial/\partial z[\partial/\partial\bar{z}(f(z)\cdot f(\bar{z}))] = 4\partial/\partial z[f(z)\cdot f'(\bar{z})] = 4f'(\bar{z})\cdot f'(z) = 4|f'(z)|^2$

**Example 6**

Show that $[\partial^2/\partial x^2 + \partial^2/\partial y^2]\, |f(z)|^p = p^2|f(z)|^{(p-2)}\, |f'(z)|^2$, where f(z) is analytic.

**Solution:**

$[\partial^2/\partial x^2+\partial^2/\partial y^2]|f(z)|^p = 4\partial/\partial z[\partial/\partial\bar{z}\, (f(z)\cdot f(\bar{z}))^{(p/2)}]$

$= 4\partial/\partial z[(f(z))^{(p/2)} \cdot (p/2)(f(\bar{z}))^{(p/2-1)}\, f'(\bar{z})]$

$= p^2\, [f(z)\cdot f(\bar{z})]^{((p-2)/2)} \times [f'(z)\cdot f'(\bar{z})] = p^2|f(z)|^{(p-2)}|f'(z)|^2$

**Example 7**

Show that $[\partial^2/\partial x^2 + \partial^2/\partial y^2] \log|f'(z)| = 0$, where f(z) is analytic.

**Solution:**

$[\partial^2/\partial x^2+\partial^2/\partial y^2] \log|f'(z)| = 4\partial/\partial z[\partial/\partial\bar{z} \log(f'(z)f'(\bar{z}))^{(1/2)}]$

$= 4\partial/\partial z[(1/2)(f''(\bar{z})/f'(\bar{z}))] = 4 \cdot 0 = 0$

**Example 8**

Show that $[\partial^2/\partial x^2 + \partial^2/\partial y^2]\,|\text{Re } f(z)|^2 = 2|f'(z)|^2$, where f(z) is analytic.

**Solution:**

Since $|\text{Re } f(z)|^2 = (1/4)[f(z)+f(\bar{z})]^2$:

$[\partial^2/\partial x^2+\partial^2/\partial y^2]|\text{Re } f(z)|^2 = 4\partial/\partial z[\partial/\partial \bar{z}((1/4)(f(z)+f(\bar{z}))^2)]$

$= 4\partial/\partial z[(1/4)\times 2(f(z)+f(\bar{z}))\times f'(\bar{z})] = 4\times(2/4)f'(\bar{z})[f'(z)+0] = 2|f'(z)|^2$

## 8. Construction of an Analytic Function When One Part is Known

### Method 1

Consider an analytic function f(z) = u + iv.

- Case (i): Let u be known.

$v = \int[-\partial u/\partial y\, dx + \partial u/\partial x\, dy]$, *evaluated using* $\int M\, dx + \int N\, dy + c$ *(y constant; terms of N not containing x)*

- Case (ii): Let v be known.

$u = \int[\partial v/\partial y\, dx - \partial v/\partial x\, dy]$, *evaluated using* $\int M\, dx + \int N\, dy + c$ *(y constant; terms of N not containing x)*

### Method 2 — Milne–Thomson Method

- Case (i): Let u be known.

$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]\, dz + c$, where $\phi_1(x,y) = \partial u/\partial x$ and $\phi_2(x,y) = \partial u/\partial y$

- Case (ii): Let v be known.

$f(z) = \int[\psi_1(z,0) + i\psi_2(z,0)]\, dz + c$, where $\psi_1(x,y) = \partial v/\partial y$ and $\psi_2(x,y) = \partial v/\partial x$

### Worked Examples — Constructing f(z) from Real or Imaginary Part

**Example 1**

Find the analytic function whose real part is $x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$.

**Solution:**

$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$

By Milne–Thomson method: $\phi_1(x,y) = \partial u/\partial x = 3x^2-3y^2+6x$, $\phi_2(x,y) = \partial u/\partial y = -6xy-6y$

$f(z) = \int[(3z^2+6z) - i(0)]\, dz + c = z^3 + 3z^2 + c$

**Example 2**

Find the analytic function whose imaginary part is $3x^2y - y^3$.

**Solution:**

$v = 3x^2y - y^3$

By Milne–Thomson method: $\psi_1(x,y) = \partial v/\partial y = 3x^2-3y^2$, $\psi_2(x,y) = \partial v/\partial x = 6xy$

$f(z) = \int[3z^2 + i(0)]\, dz + c = z^3 + c$

### Example 3

Find the analytic function whose imaginary part is $e^{-x}(x \cos y + y \sin y)$.

**Solution:**

$v = e^{-x}(x \cos y + y \sin y)$

$\psi_1(x,y) = \partial v/\partial y = e^{-x}[-x \sin y + \sin y + y \cos y]$, $\psi_2(x,y) = \partial v/\partial x = -e^{-x}[x \cos y + y \sin y] + e^{-x} \cos y$

Setting y = 0 (Milne–Thomson): $\psi_1(z,0) = 0$, $\psi_2(z,0) = -e^{-z}\cdot z + e^{-z} = e^{-z}(1-z)$

**$f(z) = \int[0 + i(e^{-z}(1-z))]\, dz + c = i\int e^{-z}(1-z)\, dz + c = i[z e^{-z}] + c$**

### Example 4

If $u = (1/2)\log(x^2+y^2)$, show that u is harmonic and hence evaluate v and then f(z).

**Solution:**

$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = (y^2-x^2)/(x^2+y^2)^2 + (x^2-y^2)/(x^2+y^2)^2 = 0$

**∴ u satisfies Laplace's equation and u is harmonic.**

$v = \int[-\partial u/\partial y\, dx + \partial u/\partial x\, dy] = \int[-y/(x^2+y^2)]dx + \int 0\, dy + c = -\tan^{-1}(x/y) + c$

**$f(z) = u + iv = (1/2)\log(x^2+y^2) + i(-\tan^{-1}(x/y)) + c$**

### Example 5

If v = cos x sinh y, show that v is harmonic and hence evaluate u and then f(z).

**Solution:**

$\partial^2v/\partial x^2 + \partial^2v/\partial y^2 = -\sinh y \cos x + \cos x \sinh y = 0$

**∴ v is Harmonic.**

$u = \int[\partial v/\partial y\, dx - \partial v/\partial x\, dy] = \int(\cos x \cosh y)dx + \int 0\, dy + c = \cosh y \sin x + c$

**$f(z) = u + iv = (\cosh y \sin x + c) + i(\cos x \sinh y) = \cosh y \sin x + i \cos x \sinh y + c$**

**Note:** *If u+v (or) u-v is given, then construct (1+i)f(z) or similar combinations, since f(z) = u+iv and if(z) = iu-v, giving (1+i)f(z) = (u-v) + i(u+v) = U + iV.*

### Example 6

Find f(z), where $u + v = x/(x^2+y^2)$.

**Solution:**

Let (1+i)f(z) = U + iV, where U + iV = (u-v) + i(u+v), so $V = u+v = x/(x^2+y^2)$.

By Milne–Thomson method:

$\psi_1(x,y) = \partial V/\partial y = -2xy/(x^2+y^2)^2$, $\psi_2(x,y) = \partial V/\partial x = (x^2+y^2-2x^2)/(x^2+y^2)^2$

Setting y=0: $\psi_1(z,0)=0$, $\psi_2(z,0) = -1/z^2$

$(1+i)f(z) = \int[0 + i(-1/z^2)]\, dz = -i\int z^{-2}\, dz = -i(z^{-1}/-1) + c = iz^{-1} + c$

**$f(z) = [i/(1+i)]\, z^{-1} + c/(1+i)$**

**Example 7**

Find f(z), where $u - v = e^x(\cos y - \sin y)$.

**Solution:**

Let $(1+i)f(z) = U + iV$, where $U + iV = (u-v) + i(u+v)$, so $U = u-v = e^x(\cos y - \sin y)$.

$\phi_1(x,y) = \partial U/\partial x = (\cos y - \sin y)e^x$, $\phi_2(x,y) = \partial U/\partial y = e^x[-\sin y - \cos y]$

By Milne–Thomson: $(1+i)f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]\, dz + c = \int[e^z - i(-e^z)]\, dz + c$

$= \int(1+i)e^z\, dz + c = (1+i)e^z + c$

**$f(z) = e^z + c/(1+i)$**

## Additional Worked Examples — Harmonic Conjugates

**Example ($u = e^x \cos y$)**

Show that $u(x,y) = e^x \cos y$ is harmonic. Determine its harmonic conjugate v(x,y) and the analytic function f(z) = u + iv.

**Solution:**

$\partial u/\partial x = e^x \cos y$, $\partial u/\partial y = -e^x \sin y$, $\partial^2u/\partial x^2 = e^x \cos y$, $\partial^2u/\partial y^2 = -e^x \cos y$

$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0 \Rightarrow$ u is harmonic.

By C-R equations: $\partial v/\partial x = -\partial u/\partial y = e^x \sin y$. Integrating: $v = e^x \sin y + f(y)$

$\partial v/\partial y = e^x \cos y + f'(y)$. Also $\partial v/\partial y = \partial u/\partial x = e^x \cos y \Rightarrow f'(y) = 0 \Rightarrow f(y) = c$

$v = e^x \sin y + c$

**$f(z) = u+iv = e^x \cos y + ie^x \sin y + ic = e^x e^{(iy)} + ic = e^z + ic$**

**Example ($u = x^3 - 3xy^2$)**

Show that $u(x,y) = x^3 - 3xy^2$ is harmonic and find its harmonic conjugate and corresponding f(z) in terms of z.

**Solution:**

$\partial u/\partial x = 3x^2-3y^2$, $\partial u/\partial y = -6xy$, $\partial^2u/\partial x^2 = 6x$, $\partial^2u/\partial y^2 = -6x$

$\partial^2u/\partial x^2 + \partial^2u/\partial y^2 = 0 \Rightarrow$ u is harmonic.

$dv = \partial v/\partial x\, dx + \partial v/\partial y\, dy = -\partial u/\partial y\, dx + \partial u/\partial x\, dy = 6xy\, dx + (3x^2-3y^2)\, dy$ (exact differential)

Integrating: $v = 6xy(x^2/2)$... simplified $\rightarrow v = 3x^2y - y^3 + c$

**$f(z) = u+iv = x^3-3xy^2 + i(3x^2y-y^3) + ic = (x+iy)^3 + ic = z^3 + k$, where $k = ic$**

**Example ($f = x^3 + 3kxy^2$)**

Find k such that $f(x,y) = x^3 + 3kxy^2$ may be harmonic, and find its conjugate.

**Solution:**

$\partial f/\partial x = 3x^2+3ky^2$, $\partial f/\partial y = 6kxy$, $\partial^2f/\partial x^2 = 6x$, $\partial^2f/\partial y^2 = 6kx$

Harmonic $\Rightarrow 6x + 6kx = 0 \Rightarrow x(1+k) = 0 \Rightarrow k = -1$ (since $x \neq 0$)

**Hence $f(x,y) = x^3 - 3xy^2$.**

Let g be the conjugate: $dg = -\partial f/\partial y\, dx + \partial f/\partial x\, dy = 6xy\, dx + (3x^2-3y^2)\, dy$ (using k=-1, exact)

**Integrating: $g = 3x^2y - y^3 + c$**

### Example ($u = x^2 - y^2$)

Find the conjugate harmonic function of the harmonic function $u = x^2 - y^2$.

**Solution:**

$\partial u/\partial x = 2x$, $\partial^2u/\partial x^2 = 2$; $\partial u/\partial y = -2y$, $\partial^2u/\partial y^2 = -2$ $\Rightarrow$ sum = 0 $\Rightarrow$ u is harmonic.

$f'(z) = \partial u/\partial x + i\partial v/\partial x = \partial u/\partial x - i\partial u/\partial y = 2x + i2y$

Using Milne–Thomson: $f'(z) = 2z$. Integrating: $f(z) = z^2 + c$

**$u+iv = (x+iy)^2 + ik = x^2-y^2 + i2xy + ik$. Equating imaginary parts: $v = 2xy + k$**

### Example ($u = e^{2x}(x \cos 2y - y \sin 2y)$)

Show that $u(x,y) = e^{2x}(x \cos 2y - y \sin 2y)$ is harmonic and find its harmonic conjugate. (Find f(z) whose real part is u.)

**Solution:**

$\partial u/\partial x = e^{2x}[(2x+1)\cos 2y - 2y \sin 2y]$

$\partial^2u/\partial x^2 = e^{2x}[(4x+4)\cos 2y - 4y \sin 2y]$

$\partial u/\partial y = e^{2x}[-2x \sin 2y - 2y \cos 2y - \sin 2y]$

**$\partial^2u/\partial y^2 = -e^{2x}[(4x+4)\cos 2y - 4y \sin 2y] = -\partial^2u/\partial x^2$ $\Rightarrow$ sum = 0 $\Rightarrow$ u harmonic.**

$f'(z) = \partial u/\partial x - i\partial u/\partial y$. By Milne–Thomson ($x\to z$, $y\to 0$):

$f'(z) = e^{2z}[(2z+1)\cos 0 - 0] - i\, e^{2z}(0) = e^{2z}(2z+1) = 2z\cdot e^{2z} + e^{2z}$

Integrating: $f(z) = z\cdot e^{2z} + c$

$u+iv = (x+iy)e^{2x+2iy} + c = e^{2x}(x \cos 2y - y \sin 2y) + i[e^{2x}(x \sin 2y + y \cos 2y) + k]$

**$\therefore$ $v = e^{2x}(2 \sin 2y + y \cos 2y) + k$**

### Example ($u = e^{x^2-y^2} \cos 2xy$)

Find the conjugate harmonic of $u = e^{x^2-y^2} \cos 2xy$. Hence find f(z) in terms of z.

**Solution:**

$\partial u/\partial x = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy)$

$\partial u/\partial y = -2e^{x^2-y^2}(x \sin 2xy + y \cos 2xy)$

$f'(z) = \partial u/\partial x - i\partial u/\partial y = 2e^{x^2-y^2}(x \cos 2xy - y \sin 2xy) + i2e^{x^2-y^2}(x \sin 2xy + y \cos 2xy)$

By Milne–Thomson ($x\to z$, $y\to 0$): $f'(z) = 2z\cdot e^{z^2}$

Integrating (put $z^2=t$): $f(z) = \int 2z\, e^{z^2}\, dz + c = e^{z^2} + c$

$u+iv = e^{(x+iy)^2} + c = e^{x^2-y^2}(\cos 2xy + i \sin 2xy) + c$

**$\therefore$ $v = e^{x^2-y^2} \sin 2xy + k$**

**SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES, CHITTOOR – 517127**

**(AUTONOMOUS)**

Affiliated to JNTUA, Anantapuramu, NBA & NAAC Accredited

**DEPARTMENT**

**OF**

**ELECTRONICS & COMMUNICATION ENGINEERING**

# PROBABILITY AND COMPLEX VARIABLES

**(23BSC214 )**

# LECTURE NOTES

## INSTITUTE VISION AND MISSION

### VISION

- To emerge as a centre of excellence for learning and research in the domains of engineering, computing and management.

### MISSION

- Provide congenial academic ambience with state-art of resources for learning and research.
- Ignite the students to acquire self-reliance in the latest technologies.
- Unleash and encourage the innate potential and creativity of students.
- Inculcate confidence to face and experience new challenges.
- Foster enterprising spirit among students.
- Work collaboratively with technical Institutes/Universities/Industries of National and International repute.

## DEPARTMENT VISION AND MISSION

### VISION

- To become a centre of excellence in Electronics and Communication Engineering and provide necessary skills to the students to meet the challenges of industry and society.

### MISSION

- Provide congenial academic ambience with necessary infrastructure and learning resources
- Inculcate confidence to face and experience new challenges from industry and society.
- Ignite the students to acquire self-reliance in State-of-the-Art Technologies
- Foster Enterprising spirit among students

## PROGRAMME EDUCATIONAL OBJECTIVES (PEO's)

**PEO1:** Have Professional competency through the application of knowledge gained from subjects like Mathematics, Physics, Chemistry, Inter-Disciplinary and core subjects like Signal Processing, VLSI, Embedded Systems, Communication and Automation. **(Professional Competency)**

**PEO2:** E*xcel* in one's career by critical thinking towards successful services and growth of the organization or as an entrepreneur or through higher studies. **(Successful Career Goals)**

**PEO3:** Enhance knowledge by updating advanced technological concepts for facing the rapidly changing world and contribute to society through innovation and creativit*y*. **(Continuing Education and Contribution to Society)**

## PROGRAM SPECIFIC OUTCOMES (PSO'S)

**PSO1:** Apply the knowledge obtained in core areas for the analysis and design of components in Signal Processing, VLSI, Embedded Systems, Communication and Automation.

**PSO2:** Adapt Innovation, Creativity and design to develop products which meet industrial and societal needs.

# PROGRAMME OUTCOMES (PO's)

PO1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO2. **Problem analysis**: Identify, formulate, research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.

PO4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of

the limitations.

PO6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.

PO8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.

PO11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO12. **Life-long learning:** Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

## Course Educational Objectives

- To introduce the basic concepts of probability and random variables.
- To introduce the basic concepts of multiple random variables.
- To understand concepts of operations on multiple random variables.
- To analyze functions of a complex variable, with continuity, differentiability and analyticity.
- To understand Taylor and Laurent expansions, residues and improper integrals.

## Syllabus Overview

| Unit | Title | Key Topics |
|---|---|---|
| Unit I | Probability & Random Variable | Probability through sets & relative frequency; sample spaces; axioms; joint, conditional & total probability; Bayes' theorem; random variables; distribution & density functions; Gaussian, Binomial, Poisson, Uniform, Exponential, Rayleigh. |
| Unit II | Operations on Random Variable | Expectation; moments; characteristic & moment generating functions; multiple/vector random variables; joint distribution & density; marginal & conditional distributions; statistical independence. |
| Unit III | Operations on Multiple Random Variables | Joint moments; joint characteristic functions; jointly Gaussian random variables (two-variable case) and their properties. |
| Unit IV | Complex Variable — Differentiation | Limits, continuity, differentiation; Cauchy-Riemann equations; analytic & harmonic functions; harmonic conjugates; Milne-Thomson method. |
| Unit V | Complex Variable — Integration | Line integrals; Cauchy's integral theorem & formula; Taylor's & Laurent's series; zeros, singularities, residues; Cauchy residue theorem. |

## Course Outcomes

| CO | Outcome |
|---|---|
| CO1 | Understand the concepts of probability, random variables and their characteristics. |
| CO2 | Deal with multiple random variables, conditional probability, joint distribution and statistical independence. |
| CO3 | Formulate and solve engineering problems involving random variables. |
| CO4 | Analyze limit, continuity and differentiation of complex functions; understand Cauchy-Riemann equations and analytic functions. |
| CO5 | Apply Cauchy's theorem and integral formulas; classify singularities and evaluate integrals using residues. |

## CO – PO Mapping

| CO \ PO | PO1 | PO2 | PO3 |
|---|---|---|---|
| CO1 | 3 | 3 | 3 |
| CO2 | 3 | 3 | 3 |
| CO3 | 3 | 3 | 3 |
| CO4 | 3 | 3 | 3 |
| CO5 | 3 | 3 | 3 |

## Text Books


- Peyton Z. Peebles, "Probability, Random Variables & Random Signal Principles", 4th Edition, TMH, 2002.
- B.S. Grewal, "Higher Engineering Mathematics", Khanna Publishers, 44th Edition, 2017.


## Reference Books


- Athanasios Papoulis & S. Unnikrishna Pillai, "Probability, Random Variables and Stochastic Processes", 4th Edition, PHI, 2002.
- Erwin Kreyszig, "Advanced Engineering Mathematics", Wiley India.
- Henry Stark & John W. Woods, "Probability and Random Processes with Application to Signal Processing", 3rd Edition, Pearson, 2002.
- B.V. Ramana, "Higher Engineering Mathematics", McGraw Hill.

# UNIT – V

# COMPLEX INTEGRATION

## 1. Continuous Curve (or Arc)

If x(t) and y(t) are two continuous real functions of a variable t in the interval $a \le t \le b$, the parametric equations

$$x = x(t), \quad y = y(t) \quad \ldots\ldots(1)$$

define a continuous curve (or arc).

### Closed Curve

A curve is said to be closed if the initial point and final point coincide,

$$z(a) = z(b)$$

Otherwise it is called an arc.

### Multiple Point of the Arc

If equation (1) is satisfied by more than one value of t in the given range, then the point z or (x, y) is called a multiple point of the arc.

### Jordan Arc

A continuous arc without multiple points is called a Jordan arc.

### Jordan Curve

A Jordan curve consists of a chain of a finite number of continuous arcs; equivalently, a simple closed curve is defined as a Jordan curve.

### Regular Arc

If z(t) has a continuous derivative in $a \le t \le b$, the curve is said to be a regular arc. It is also known as a smooth curve — a smooth curve has no corners.

### Contour

A continuous Jordan curve consisting of a finite number of regular arcs is called a contour.

---

## 2. Line Integral

Let f(z) be a function which is continuous at all points on a curve C with end points a and b. Then the line integral of f(z) along the curve C is given by

$$\int_C f(z)\,dz \quad \text{or} \quad \int_a^b f(z)\,dz$$

## Properties of the Line Integral

$$1.\ \int_C f(z)\,dz = \int_{C1} f(z)\,dz + \int_{C2} f(z)\,dz$$

$$2.\ \int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$$

$$3.\ \text{If } f(z) = 1, \text{ then } \int_a^b f(z)\,dz = b - a$$

$$4.\ \int_C [f(z) \pm g(z)]\,dz = \int_C f(z)\,dz \pm \int_C g(z)\,dz$$

$$5.\ \int_C [\lambda f(z)]\,dz = \lambda \int_C f(z)\,dz$$

**Problem 1: Evaluate $\int_C f(z)\,dz$, where $f(z) = y - x - 3x^2 i$ and C is the straight-line segment from $z = 0$ to $z = 1 + i$.**

***Solution:***

Given $f(z) = y - x - 3x^2 i$. Since $z = x + iy$, $dz = dx + i\,dy$.

C is the straight-line segment joining $z = 0$ to $z = 1 + i$, i.e. $(x, y)$ goes from $(0, 0)$ to $(1, 1)$.

Equation of the line joining (0,0) and (1,1):

$$y - 0 = [(1-0)/(1-0)]\,(x-0) \Rightarrow y = x \Rightarrow dy = dx$$

Substituting $y = x$ and $dy = dx$:

$$\int_C f(z)dz = \int_{x=0}^{1} (x - x - 3x^2 i)(dx + i\,dx) = \int_0^1 (-3i\,x^2)(1+i)\,dx$$

$$= (-3i)(1+i)\int_0^1 x^2\,dx = (-3i - 3i^2)(x^3/3)|_0^1 = 3(1-i)(1/3)$$

$$\therefore \int_C f(z)\,dz = 1 - i$$

**Problem 2: Evaluate $\int_0^{2+i} (2x + iy + 1)\,dz$ along the straight line $2y = x$.**

***Solution:***

$$\int_0^{2+i} (2x+iy+1)\,dz = \int_{(0,0)}^{(2,1)} (2x+iy+1)(dx + i\,dy) \quad \ldots\ldots(1)$$

Given equation of the curve C: $2y = x \Rightarrow x = 2y \Rightarrow dx = 2\,dy$.

Substituting into (1):

$$= \int_{y=0}^{1} [2(2y) + iy + 1][2\,dy + i\,dy] = \int_0^1 (4y + iy + 1)(2+i)\,dy$$

$$= (2+i)\,[\,(4+i)(y^2/2) + y\,]_0^1 = (2+i)\,[\,(4+i)(1/2) + 1\,]$$

$$= (2+i)\,[\,(6+i)/2\,] = ½\,(12 + 2i + 6i - 1)$$

$$\therefore \int_0^{2+i} (2x+iy+1)\,dz = ½\,(11 + 8i)$$

**Problem 3: Evaluate $\int_0^{1+i} (x^2 - iy)\,dz$ along the path (a) $y = x$ (b) $y = x^2$.**

***Solution:***

***(a) Along $y = x$ ( $\Rightarrow dy = dx$ )***

$$\int_0^{1+i}(x^2-iy)\,dz = \int_{x=0}^{1} (x^2 - ix)(1+i)\,dx = (1+i)\int_0^1 (x^2 - ix)\,dx$$

$$= (1+i)\,[\,x^3/3 - i\,x^2/2\,]_0^1 = (1+i)\,(1/3 - i/2)$$

$$\therefore \text{ Result (a)} = (1 - i)/2 \quad \text{[after simplification, as in worked notes]}$$

***(b) Along $y = x^2$ ( $\Rightarrow dy = 2x\,dx$ )***

$$\int_0^{1+i}(x^2-iy)\,dz = \int_{x=0}^{1} [x^2 - i\,x^2](1 + 2ix)\,dx = (1-i)\int_0^1 2x^3\,dx$$

$$= 2(1-i)\,[x^4/4]_0^1 = 2(1-i)(1/4)$$

$$\therefore \text{ Result (b)} = (1 - i)/2$$

*Note: The two different paths from 0 to 1+i give the same value here; in general the value of a contour integral of a function that is not analytic can depend on the path chosen.*

## 3. Cauchy's Integral Theorem

Statement: If f(z) is an analytic function and f'(z) is continuous within and on a simple closed curve C, then

$$\oint_C f(z)\, dz = 0$$

*Note: 1. If f(z) is analytic in a region R, and P and Q are any two points in R, then $\int_P^Q f(z)\, dz$ is independent of the path joining P and Q lying inside R.*

*Note: 2. If f(z) is analytic in the region bounded by two simple closed curves C1 and C2, then $\oint_{C1} f(z)\, dz = \oint_{C2} f(z)\, dz$.*

**Problem: Verify Cauchy's theorem for the integral of $z^3$ taken over the boundary of the rectangle with vertices −1, 1, 1+i, −1+i.**

***Solution:***

Given $f(z) = z^3$. The closed curve C is the rectangle with vertices A(−1,0), B(1,0), C(1,1), D(−1,1), traversed A → B → C → D → A.

$$\int_C z^3\, dz = \left[\int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}\right] z^3\, dz \quad ......(1)$$

***Along AB : y = 0 ⇒dy = 0***

$$\int_{AB} z^3\, dz = \int_{x=-1}^{1} x^3\, dx = [x^4/4]_{-1}^{1} = \tfrac{1}{4}(1 - 1) = 0$$

***Along BC : x = 1 ⇒dx = 0***

$$\int_{BC} z^3\, dz = \int_{y=0}^{1} (1+iy)^3\, (i\, dy) = i \int_0^1 [1 - iy^3 + 3iy - 3y^2]\, dy$$
$$= i\,[\,y - iy^4/4 + 3iy^2/2 - y^3\,]_0^1 = i\,(3i/2 - \tfrac{1}{4}) = -3/2 - i/4 = -5/4$$

***Along CD : y = 1 ⇒dy = 0***

$$\int_{CD} z^3\, dz = \int_{x=1}^{-1} (x+i)^3\, dx = \int_1^{-1} (x^3 - i + 3ix^2 - 3x)\, dx$$
$$= [x^4/4 - ix + ix^3 - 3x^2/2]_1^{-1} = (\tfrac{1}{4} + i - i - 3/2) - (\tfrac{1}{4} - i + i - 3/2) = 0$$

***Along DA : x = −1 ⇒dx = 0***

$$\int_{DA} z^3\, dz = \int_{y=1}^{0} (-1+iy)^3\, (i\, dy) = i \int_1^0 [-1 - iy^3 + 3iy + 3y^2]\, dy$$
$$= i\,[\,-y - iy^4/4 + 3iy^2/2 + y^3\,]_1^0 = i\,[\tfrac{1}{4} - 3i/2] = 5/4$$

Adding all four contributions (using equation 1):

$$\int_C z^3\, dz = 0 - 5/4 + 0 + 5/4 = 0$$

**Hence Cauchy's theorem is verified.**

## 4. Cauchy's Integral Formula

Statement: If f(z) is analytic within and on a closed curve C, and 'a' is any point within C, then

$$\oint_C f(z)/(z-a)\, dz = 2\pi i\, f(a)$$

### Generalized Cauchy's Integral Formula

If a function f(z) is analytic at each point of a region R and on a closed curve C, and 'a' is any point within C, then

$$\oint_C f(z)/(z-a)^{n+1}\, dz = (2\pi i/n!)\, f^{(n)}(a)$$

**Problem 1: Evaluate $\oint_C \sin z /(z - \pi/6)^2\, dz$, where C is the circle $|z| = 1$.**

***Solution:***

Comparing with $f(z)/(z-a)^{n+1}$, we have $f(z) = \sin z$, $a = \pi/6$, $n = 1$.

Given closed curve C : $|z| = 1$ (i.e. $x^2 + y^2 = 1$). Let $\phi(z) = |z| - 1$; at $z = a = \pi/6$, $\phi(a) = \pi/6 - 1 \approx -0.476 < 0$, so $z = \pi/6$ lies inside C.

By the generalized Cauchy's integral formula:

$$\oint_C \sin z/(z-\pi/6)^2\, dz = (2\pi i/1!)\, f'(\pi/6) = 2\pi i\, [\, d/dz(\sin z)\, ]_{z=\pi/6} = 2\pi i \cos(\pi/6)$$
$$= 2\pi i\, (\sqrt{3}/2) = \pi i\sqrt{3}$$

**Problem 2: Evaluate $\oint_C e^z /[(z-1)(z-2)]\, dz$, where C is the circle $|z| = 3$.**

***Solution:***

Here $f(z) = e^z$, and the poles $a = 1$, $b = 2$ both lie inside $|z| = 3$.

Resolving into partial fractions: $1/[(z-1)(z-2)] = 1/(z-2) - 1/(z-1)$.

$$\oint_C e^z/[(z-1)(z-2)]\, dz = \oint_C e^z/(z-2)\, dz - \oint_C e^z/(z-1)\, dz$$
$$= 2\pi i\, f(2) - 2\pi i\, f(1) = 2\pi i\, [e^2 - e^1]$$

**Problem 3: Use Cauchy's integral formula to evaluate $\oint_C (\sin \pi z^2 + \cos \pi z^2)/[(z-1)(z-2)]\, dz$, where C is the circle $|z| = 3$.**

***Solution:***

Here $f(z) = \sin \pi z^2 + \cos \pi z^2$, $a = 1$, $b = 2$. Since $|z| = 3$ ($x^2+y^2=3$) contains both $z = 1$ and $z = 2$:

$$\oint_C f(z)/[(z-1)(z-2)]\, dz = \oint_C f(z)/(z-2)\, dz - \oint_C f(z)/(z-1)\, dz = 2\pi i[f(2) - f(1)]$$
$$= 2\pi i\, [\, (\sin 4\pi + \cos 4\pi) - (\sin\pi + \cos\pi)\, ] = 2\pi i\, [\, 0 + 1 - 0 - (-1)\, ] = 2\pi i(1+1) = 4\pi i$$

---

## 5. Complex Power Series

### Sequence

A function defined on the set of natural numbers into the set of complex numbers is called a sequence of complex numbers, denoted by $\{u_n\}$, where n = 1, 2, 3, …, i.e. u1, u2, u3, …, un, …. Here un is called the nth term of the sequence.

A sequence $\{u_n\}$ is said to be convergent if $\lim_{n\to\infty} u_n = a$, a finite quantity (any constant); otherwise it is called divergent.

### Infinite Series

Let u1(z), u2(z), …, un(z), … be a sequence of functions defined on a set of complex numbers. Then we can form a new sequence Sn(z), defined by

$$S_n(z) = \Sigma_{n=1..\infty}\, u_n(z) = u_1(z) + u_2(z) + \ldots + u_n(z) + \ldots$$

This is called an infinite series, also written as Σ un. The series is said to be convergent on the set of complex numbers if $\lim_{(n\to\infty)} S_n(z) = S(z)$.

## Power Series

A power series is defined as a series of the form

$$\Sigma_{n=0..\infty}\, a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_n(z-a)^n + \ldots$$

where 'a' is any fixed point; the point 'a' in the complex plane is known as the centre of the power series, and a0, a1, a2, …, an are called the coefficients of the power series.

## Taylor's Series

An analytic function can be represented by a power series called a Taylor's series of the function, as given in the following theorem.

### *Taylor's Theorem*

Statement: If a function f(z) is analytic inside a circle C with centre 'a', then for all z inside C:

$$f(z) = f(a) + (z-a)/1!\, f'(a) + (z-a)^2/2!\, f''(a) + (z-a)^3/3!\, f'''(a) + \ldots + (z-a)^n/n!\, f^{(n)}(a) + \ldots \quad \ldots\ldots(1)$$

*Note: From equation (1), it is clear that a complex analytic function can always be represented by a power series.*

*Note: Putting a = 0 in equation (1), we get $f(z) = f(0) + z/1!\, f'(0) + z^2/2!\, f''(0) + \ldots$, which is called Maclaurin's series.*

## Laurent's Series

If the complex function f(z) has a singularity at 'a', we cannot apply Taylor's theorem to get an expansion of f(z) in non-negative powers of (z–a). We shall show that certain functions can be expanded as an infinite series, even though the function is not analytic at a point of the given region, by Laurent's theorem, which consists of positive and negative powers of (z–a).

### *Laurent's Theorem*

Statement: If f(z) is analytic inside and on the boundary of the ring-shaped region R, bounded by two concentric circles C1 and C2 of radii r1 and r2 (r1 > r2) respectively, having centre at 'a', then for all z in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$$

$$\text{where } a_n = 1/(2\pi i) \oint_{C1} f(w)/(w-a)^{n+1}\, dw,\ n = 0,1,2,\ldots$$

$$\text{and } a_{-n} = 1/(2\pi i) \oint_{C2} f(w)/(w-a)^{-n+1}\, dw,\ n = 1,2,3,\ldots$$

**Problem 1: Obtain the Taylor's series expansion of f(z) = 1/z about the point z = 1.**

***Solution:***

Taylor's series for f(z) about z = a is:

$$f(z) = f(a) + (z-a)/1!\, f'(a) + (z-a)^2/2!\, f''(a) + (z-a)^3/3!\, f'''(a) + \ldots \quad \ldots\ldots(1)$$

If $f(z) = 1/z$, then $f(a) = f(1) = 1$. $f'(z) = -1/z^2 \Rightarrow f'(1) = -1$. $f''(z) = 2/z^3 \Rightarrow f''(1) = 2$. $f'''(z) = -6/z^4 \Rightarrow f'''(1) = -6$.

Substituting into (1):

$$1/z = 1 + (z-1)/1!\,(-1) + (z-1)^2/2!\,(2) + (z-1)^3/3!\,(-6) + \dots$$
$$1/z = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots$$

Alternative method: put $z - 1 = w \Rightarrow z = 1 + w$.

$$f(z) = 1/z = 1/(1+w) = (1+w)^{-1} = 1 - w + w^2 - w^3 + \dots = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots$$

**Problem 2: Expand $f(z) = 1/z^2$ (a) in powers of $(z+1)$, (b) in powers of $(z-2)$.**

***Solution:***

### *(a) Powers of (z+1)*

Let $z + 1 = w \Rightarrow z = w - 1$.

$$f(z) = 1/z^2 = 1/(w-1)^2 = (1-w)^{-2} = 1 + 2w + 3w^2 + 4w^3 + \dots \quad (|w| < 1)$$
$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \dots$$

### *(b) Powers of (z–2)*

Let $z - 2 = w \Rightarrow z = 2 + w$.

$$f(z) = 1/z^2 = 1/(2+w)^2 = (1/4)\,[1 + w/2]^{-2} = \tfrac{1}{4}\,[1 - 2(w/2) + 3(w/2)^2 - 4(w/2)^3 + \dots]$$
$$= \tfrac{1}{4}\,[\,1 - (z-2) + (3/4)(z-2)^2 - (1/2)(z-2)^3 + \dots\,]$$

*Note: Standard binomial expansions used above: $(1-x)^{-1} = 1+x+x^2+x^3+\dots$; $(1+x)^{-1} = 1-x+x^2-x^3+\dots$; $(1-x)^{-2} = 1+2x+3x^2+4x^3+\dots$; $(1+x)^{-2} = 1-2x+3x^2-4x^3+\dots$, all for $|x| < 1$.*

**Problem 3: Represent $f(z) = z/[(z-1)(z-3)]$ by a series of positive and negative powers of $(z-1)$, which converges to $f(z)$ when $0 < |z-1| < 2$.**

***Solution:***

We expand the function in terms of $(z - 1)$. Let $z - 1 = w \Rightarrow z = 1 + w$, so $z - 3 = w - 2$.

$$f(z) = z/[(z-1)(z-3)] = (1+w)/[w(w-2)] \quad \dots\dots(1)$$

Resolving $(1+w)/[w(w-2)]$ into partial fractions:

$$(1+w)/[w(w-2)] = A/w + B/(w-2) \;\Rightarrow\; 1+w = A(w-2) + Bw \quad \dots\dots(2)$$

Putting $w = 0$ in (2): $1 = -2A \Rightarrow A = -1/2$. Putting $w = 2$ in (2): $3 = 2B \Rightarrow B = 3/2$.

$$\therefore f(z) = -(1/2)(1/w) + (3/2)(1/(w-2)) = -1/(2w) - (3/4)(1/(1-w/2))$$

Since $0 < |z-1| < 2 \Rightarrow |w| < 2 \Rightarrow |w/2| < 1$, expand $(1-w/2)^{-1}$ as a binomial series:

$$= -1/(2w) - (3/4)[1 + w/2 + (w/2)^2 + (w/2)^3 + \dots]$$
$$= -1/(2w) - (3/4)[\,1 + \tfrac{1}{2}w + \tfrac{1}{4}w^2 + (1/8)w^3 + \dots\,]$$

Putting $w = z - 1$:

$$f(z) = -1/[2(z-1)] - (3/4)[\,1 + \tfrac{1}{2}(z-1) + \tfrac{1}{4}(z-1)^2 + (1/8)(z-1)^3 + \dots\,]$$

---

## 6. Zero of an Analytic Function

A zero of an analytic function f(z) is a value of z such that f(z) = 0. In particular, a point 'a' is called a zero of an analytic function f(z) if f(a) = 0.

### Zero of Order m

If an analytic function f(z) can be expressed in the form $f(z) = (z-a)^m \phi(z)$, then z = a is a zero of order m for f(z).

- If $f(z) = (z-1)^3$, then z = 1 is a zero of order 3 of f(z).
- If f(z) = 1/(1–z), then z = ∞ is a simple zero of f(z).
- If f(z) = sin z, then z = 0, ±π, ±2π, ... are simple zeros of f(z).
- If f(z) = e^{tan z}, it has no zeros, since $e^z \neq 0$ for any z.

---

## 7. Singular Points

A singular point of a function f(z) is a point at which the function f(z) ceases to be analytic.

### Different Types of Singularities

#### *(i) Isolated Singularity*

A point z = a is called an isolated singularity of an analytic function f(z) if:

- f(z) is not analytic at the point z = a.
- f(z) is analytic in a neighbourhood of z = a which contains no other singularity.

Example: (i) $f(z) = e^z/(z-1)$ has an isolated singular point at z = 1. (ii) $f(z) = e^z/(z^2+1)$ has two isolated singular points, z = ±i.

#### *(ii) Poles of an Analytic Function*

If z = a is an isolated singular point of an analytic function f(z), then f(z) can be expanded in a Laurent's series about the point z = a:

$$f(z) = \Sigma_{n=0..\infty}\, an(z-a)^n + \Sigma_{n=1..\infty}\, bn/(z-a)^n \quad ......(1)$$

The series of negative integral powers of (z–a), namely $\Sigma\, bn/(z-a)^n$, is known as the principal part of the Laurent's series of f(z). If the principal part contains a finite number of terms, say m, then the singular point z = a is called a pole of order m of f(z).

#### *Simple Pole*

A simple pole is a pole of order one.

#### *(iii) Essential Singularity*

If the principal part of f(z) contains an infinite number of terms, i.e. the series $\Sigma\, bn/(z-a)^n$ contains an infinite number of terms, then the point z = a is called an essential singularity of f(z).

#### *(iv) Removable Singularity*

If the principal part of f(z) contains no term, i.e. bn = 0 for all n, then the singularity z = a is called a removable singularity of f(z). In this case,

$$f(z) = \Sigma_{n=0..\infty}\, an(z-a)^n$$

Example: $f(z) = (1 - \cos z)/z$; then $z = 0$ is a removable singularity.

## 8. Residues

The coefficient of $1/(z-a)$ in the Laurent's expansion of $f(z)$ about the isolated singularity $z = a$ is called the residue of $f(z)$ at the point $z = a$.

$$\text{Residue of } f(z) \text{ at } z = a = a_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz$$
$$\text{i.e. } \oint_C f(z)\,dz = 2\pi i \times [\text{ Residue of } f(z) \text{ at } z=a\ ]$$

where C is a closed curve containing the point $z = a$.

### Cauchy's Residue Theorem

Statement: If $f(z)$ is analytic within and on a closed curve C, except at a finite number of poles $a_1, a_2, \ldots, a_n$ within C, and $R_1, R_2, \ldots, R_n$ are the residues of $f(z)$ at these poles, then

$$\oint_C f(z)\,dz = 2\pi i\,[\,R_1 + R_2 + \ldots + R_n\,]$$

i.e. $\oint_C f(z)\,dz = 2\pi i \times$ [ sum of the residues at the poles within C ].

### Notes — Formulas for Computing Residues

- If $z = z_0$ is a simple pole, the Laurent expansion is $f(z) = \Sigma\, a_n(z-z_0)^n + a_{-1}(z-z_0)^{-1}$.
- If $\lim_{z\to z_0} (z-z_0) f(z) = a$, then $[\text{Res } f(z)]_{z=z_0} = \lim_{z\to z_0} (z-z_0) f(z)$.
- If $f(z) = \phi(z)/\psi(z)$, where $\psi(z_0) = 0$ but $\phi(z_0) \neq 0$, then $[\text{Res } f(z)]_{z=z_0} = \phi(z_0)/\psi'(z_0) = \lim_{z\to z_0} [\ \phi(z)/\psi'(z)\ ]$.
- Residue at a pole of order m: if $f(z)$ is analytic within a curve C and has a pole of order m at $z = z_0$, the residue at $z_0$ is:

$$[\text{Res } f(z)]_{z=z_0} = \frac{1}{(m-1)!} \times \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}\left[\,(z-z_0)^m f(z)\,\right]$$

### Problems Related to Poles and Residues

**Problem 1: Determine the poles of the functions (i) $z/\cos z$ (ii) $\cot z$.**

***Solution:***

(i) The poles of $f(z) = z/\cos z$ are given by $\cos z = 0$, i.e. $z = (2n+1)\pi/2$, $n = 0, \pm1, \pm2, \ldots$. Hence these are simple poles of $f(z)$.

(ii) The poles of $f(z) = \cot z = \cos z/\sin z$ are given by $\sin z = 0$, i.e. $z = n\pi$, $n = 0, \pm1, \pm2, \ldots$, which are simple poles of $f(z)$.

**Problem 2: Find the zeros and poles of $[\,(z+1)/(z^2+1)\,]^n$.**

***Solution:***

Let $g(z) = [\,(z+1)/(z^2+1)\,]^n = (z+1)^n/(z^2+1)^n$.

#### *(a) Zeros*

The zeros of $g(z)$ are given by $(z+1)^n = 0$, i.e. $z = -1$. So $z = -1$ is a zero of order n.

#### *(b) Poles*

The poles of g(z) are obtained by putting the denominator equal to zero:

$$(z^2+1)^n = 0 \Rightarrow (z-i)^n(z+i)^n = 0 \Rightarrow z = i, -i$$

∴ z = i and z = –i are both poles of order n.

**Problem 3: Find the poles of the function f(z) = 1/[(z+1)(z+3)] and the residues at these poles.**

***Solution:***

f(z) has two simple poles at z = –1 and z = –3.

$$Res(f)z{=}{-1} = \lim z{\to}{-1}\ (z+1)\ f(z) = \lim z{\to}{-1}\ 1/(z+3) = 1/2$$

$$Res(f)z{=}{-3} = \lim z{\to}{-3}\ (z+3)\ f(z) = \lim z{\to}{-3}\ 1/(z+1) = -1/2$$

**Problem 4: Determine the poles of the function f(z) = $z^n$ /[$(z–1)^2$(z+2)] and the residue at each pole.**

***Solution:***

z = 1 and z = –2 are the zeros of the denominator, of orders 2 and 1 respectively. So z = 1 is a pole of order 2, and z = –2 is a pole of order 1 for f(z).

$$[Res\ f(z)]z{=}{-2} = \lim z{\to}{-2}\ (z+2)\ f(z) = \lim z{\to}{-2}\ z^n/(z-1)^2 = 4/9 \quad (for\ n = 2)$$

Using the formula for a pole of order m (here m = 2):

$$[Res\ f(z)]z{=}1 = \lim z{\to}1\ (1/1!)\ d/dz\ [\ (z-1)^2 f(z)\ ] = \lim z{\to}1\ d/dz\ [\ z^n/(z+2)\ ]$$

$$= \lim z{\to}1\ [\ (z+2)(nz^{n-1}) - z^n\ ]\ /\ (z+2)^2 = 5/9 \quad (for\ n = 2)$$

**Problem 5: Find the residue of z $e^z$ /$(z–1)^3$ at its pole.**

***Solution:***

Let f(z) = z $e^z/(z–1)^3$. Here z = 1 is a pole of f(z) of order 3 (a = 1, m = 3). Using

$$[Res\ f(z)]z{=}a = 1/(m-1)! \times \lim z{\to}a\ \ d\text{^}(m-1)/dz\text{^}(m-1)\ [\ (z-a)^m f(z)\ ]$$

with a = 1, m = 3:

$$[Res\ f(z)]z{=}1 = (1/2!)\ \lim z{\to}1\ \ d^2/dz^2\ (\ z\,e^z\ )$$

$$= (1/2)\ \lim z{\to}1\ (\ z\,e^z + 2e^z\ ) = (1/2)\ \lim z{\to}1\ (\ z\,e^z + e^z + e^z\ )$$

$$= (1/2)(3e) = 3e/2$$

**Problem 6: Evaluate ∮C (2z–1)/[z(z+2)(2z+1)] dz, where C is the circle |z| = 1.**

***Solution:***

Here f(z) = (2z–1)/[z(z+2)(2z+1)] has 3 simple poles: z = 0, z = –2 and z = –1/2. Only the poles z = 0 and z = –1/2 lie inside the circle |z| = 1.

Residue of f(z) at z = 0:

$$Res(f)z{=}0 = \lim z{\to}0\ z\ f(z) = \lim z{\to}0\ (2z-1)/[(z+2)(2z+1)] = -1/2$$

Residue of f(z) at z = –1/2:

$$Res(f)z{=}{-1/2} = \lim z{\to}{-1/2}\ (2z+1)\ f(z) = \lim z{\to}{-1/2}\ (2z-1)/[z(z+2)] = (-2)/(-3/4) = 8/3$$

By the Residue theorem:

$$\oint C\ (2z-1)/[z(z+2)(2z+1)]\ dz = 2\pi i\ [\ -1/2 + 8/3\ ] = ((13/3)\ \pi i$$