SREENIVASA INSTITUTE OF TECHNOLOGY AND MANAGEMENT STUDIES (SITAMS)

(AUTONOMOUS)

(Approved by AICTE, New Delhi & Affiliated to JNTUA, Ananthapuramu)

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DEPARTMENT OF SCIENCE AND HUMANITIES

NUMERICAL METHODS & TRANSFORM TECHNIQUES-(23BSC234)

Lecturer Notes prepared by
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II- B. TECH / I- SEMESTER REGULATION: R23

Branch: Mechanical

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

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YEAR & SEM: II-I

Branch: Mechanical

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

Department of Science & Humanities
IIB Tech- I Semester

23BSC234	NUMRICAL METHODS AND TRANSFORM TECHNIQUES	L	T	P	C
	(MECH)	2	1	-	3

PRE-REQUISITES: Algebra, Calculus, Differential equations, Set theory, Series

COURSE EDUCATIONAL OBJECTIVES:

1. To develop skill to analyze appropriate method to find the root of the Algebraic and Transcendental Equations and apply the concept of interpolation and Curve fitting for the Prediction of required values
2. To learn the method of evaluation of numerical derivative, numerical integration
3. To learn the method of solve ordinary differential equation using numerical methods.
4. To learn the concept of Laplace Transform and Inverse Laplace Transform
5. To develop skill to design sine and cosine waves with the help of Fourier Series and to learn the concept of Fourier Transform and Inverse Fourier Transform.

UNIT I: Solution of Algebraic & Transcendental Equations and Interpolation 09

Introduction-Bisection Method-Iterative method, Regula-falsi method and Newton Raphson method Finite differences-Newton's forward and backward interpolation formulae – Lagrange's formulae.

UNIT II: Numerical Differentiation, Numerical Integration and Curve Fitting 09

Numerical differentiation: Newton's forward and backward formulae
Numerical integration: Trapezoidal rule - Simpson's 1/3 Rule - Simpson's 3/8 Rule
Curve fitting: Fitting of straight line, second-degree and Exponential curve by method of least squares

UNIT III: Solution of Initial value problems to Ordinary differential equations 09

Numerical solution of Ordinary Differential equations: Solution by Taylor's series-Picard's Method of successive Approximations-Euler's and modified Euler's methods-Runge-Kutta methods (second and fourth order).

UNIT IV: Laplace Transforms 09

Definition-Laplace transform of standard functions-existence of Laplace Transform – Inverse transform – First shifting Theorem, transforms of derivatives and integrals - Convolution theorem –Application of Laplace transforms to ordinary differential equations of first and second order.

UNIT V: Fourier series & Fourier transforms 09

Determination of Fourier coefficients (Euler's) – Dirichlet conditions for the existence of Fourier series -Fourier series of Even and odd functions – Fourier series in an arbitrary interval – Half-range Fourier sine and cosine expansions
Fourier integral theorem (without proof) – Fourier sine and cosine integrals-complex form of Fourier integral. Fourier transform – Fourier sine and cosine transforms – Properties – Inverse transforms – convolution theorem.

COURSE OUTCOMES:

On successful completion of the course, students will be able to		Po's related to CO's
CO1	Apply numerical methods to solve algebraic and transcendental equations and to Derive interpolating polynomials using interpolation formulae.	**PO1, PO2, PO3**
CO2	Demonstrate knowledge in finding the numerical values to derivatives, integrals through different mathematical methods and constructing a curve, or mathematical function, that has the best fit to a series of data points	**PO1, PO2, PO3**
CO3	Demonstrate knowledge in solving ordinary differential equations numerically through various methods	**PO1, PO2, PO3**
CO4	Understand the use of Laplace transform in system modeling, digital signal processing, process control, solving Boundary Value Problems.	**PO1, PO2, PO3**
CO5	Apply Fourier series and Fourier transform in communication theory and signal analysis, image processing and filters, data processing and analysis, solving partial differential equations for problems on gravity.	**PO1, PO2, PO3**

TEXT BOOKS:

1. B.S. Grewal, Higher Engineering Mathematics, KhannaPublishers,2017, 44th Edition
2. Erwin Kreyszig, Advanced Engineering Mathematics, Wiley India

REFERENCE BOOKS:

1. R.K. Jain and S.R.K. Iyengar, Advanced Engineering Mathematics, Alpha Science International Ltd.,2021 5th Edition (9th reprint).
2. B.V. Ramana, Higher Engineering Mathematics, Mc Graw Hill publishers
3. Alan Jeffrey, Advanced Engineering Mathematics, Elsevier

ONLINE LEARNING RESOURCES:

1. https://onlinecourses.nptel.ac.in/noc20_ma50/preview
2. *https://archive.nptel.ac.in/courses/111/106/111106111/*

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CO\PO	PO1	PO2	PO3	PO4	PO5	PO6	PO7	PO8	PO9	PO10	PO11	PO12
CO.1	3	3	3	-	-	-	-	-	-	-	-	-
CO.2	3	3	3	-	-	-	-	-	-	-	-	-
CO.3	3	3	3	-	-	-	-	-	-	-	-	-
CO.4	3	3	3	-	-	-	-	-	-	-	-	-
CO.5	3	3	3	-	-	-	-	-	-	-	-	-
CO*	**3**	**3**	**3**	**-**	**-**	**-**	**-**	**-**	**-**	**-**	**-**	-

🎓 Chapter 0: Introduction to Numerical Methods and Transform Techniques

💡 Thought for the Day

"Engineering is not about solving easy problems; it is about finding smart solutions to difficult problems."

🚀 What is Numerical Methods?

Definition

Numerical Methods are mathematical techniques used to obtain approximate solutions to engineering and scientific problems when exact analytical solutions are impossible or difficult to obtain.

📌 Simple Definition

Numerical Methods = Approximate Solutions + Computer Algorithms

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🤔 Why Do We Need Numerical Methods?

Many engineering problems involve:

- **Complex equations**
- **Large amounts of data**
- **Nonlinear systems**
- **Experimental observations**

These cannot always be solved using ordinary mathematics.

Hence,

➡️ We use Numerical Methods.

🌍 Real-Life Examples

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Problem	Numerical Method Used
Google Maps shortest route	Optimization Algorithms
Weather Forecasting	Numerical Differential Equations
ISRO Rocket Launch	Newton-Raphson Method
Medical CT Scan	Fourier Transform
Mobile Signal Processing	Fourier Transform
Bridge Design	Numerical Integration

📖 What Will We Learn?

UNIT–I

Solution of Algebraic & Transcendental Equations

📌 Finding roots of equations

Methods:

- Bisection Method
- Regula-Falsi Method
- Newton-Raphson Method

Example:

$$x^3 - x - 1 = 0$$

UNIT–II

Numerical Differentiation & Integration

Instead of solving by formulas,
we estimate

✓ Derivatives

✓ Areas

using numerical techniques.

Methods:

- Trapezoidal Rule
- Simpson's 1/3 Rule

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- **Simpson's 3/8 Rule**

UNIT–III

Numerical Solution of Differential Equations

Engineering systems are described using differential equations.

Example:

- **Cooling of coffee** ☕
- **Motion of vehicles** 🚗
- **Population growth** 👪

Methods:

- **Euler Method**
- **Modified Euler Method**
- **Runge-Kutta Method**

UNIT–IV

Laplace Transform

Transforms difficult differential equations into simple algebraic equations.

Applications:

- **Electrical circuits**
- **Mechanical vibrations**
- **Control systems**

UNIT–V

Fourier Series & Fourier Transform

Converts complicated signals into simple sine and cosine waves.

Applications:

🎵 **Audio Compression**
📷 **Image Processing**
📡 **Wireless Communication**
🩺 **ECG Signal Analysis**

📌 Engineering Applications

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Mechanical Engineering

✓ Heat Transfer
✓ Fluid Mechanics
✓ Machine Design
✓ Vibration Analysis
✓ CFD

Civil Engineering

✓ Bridge Design
✓ Structural Analysis
✓ Earthquake Engineering

Electrical Engineering

✓ Power Systems
✓ Control Systems
✓ Signal Processing

Computer Science

✓ Artificial Intelligence
✓ Machine Learning
✓ Image Processing
✓ Data Analytics

🎯 Course Outcomes

After completing this course, students can

☑ Solve engineering equations
☑ Analyze engineering data
☑ Solve differential equations
☑ Apply Laplace Transform
☑ Use Fourier Transform in signal processing

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🧠 Prerequisites

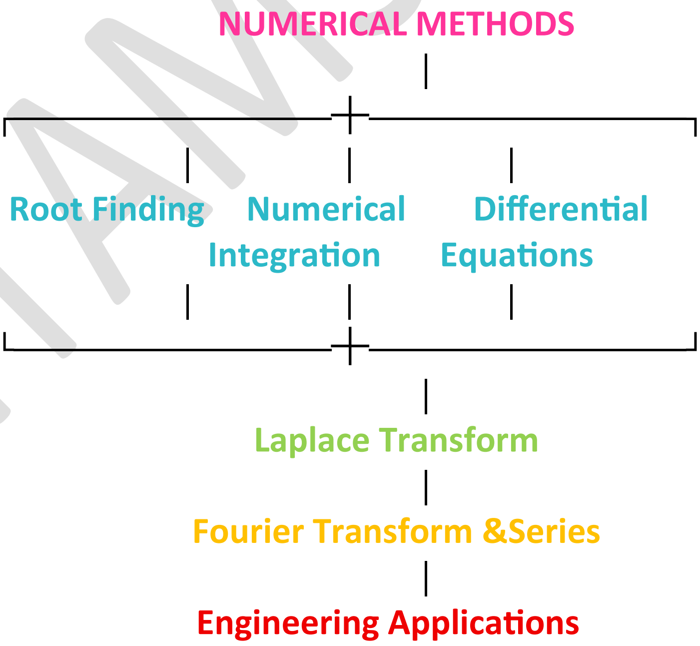
Students should know

- 📌 **Algebra**
- 📌 **Calculus**
- 📌 **Differential Equations**
- 📌 **Trigonometry**
- 📌 **Matrices**

💻 Software Used

- **MATLAB**
- **Python**
- **Scilab**
- **Excel**

🌈 Mind Map



NUMERICAL METHODS

Root Finding | **Numerical Integration** | **Differential Equations**

Laplace Transform

Fourier Transform &Series

Engineering Applications

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🌟 Fun Facts

- ◆ ISRO uses Numerical Methods for satellite trajectory calculations.
- ◆ Google Maps uses optimization algorithms to suggest the fastest route.
- ◆ Netflix and YouTube use Fourier Transform for video compression.
- ◆ Weather forecasting uses millions of numerical calculations every second.
- ◆ Artificial Intelligence is built on numerical optimization techniques.

💬 Classroom Discussion

Think & Answer

1. Can every mathematical equation be solved exactly?
2. Why do engineers prefer numerical methods over analytical methods?
3. Name two real-life applications of Fourier Transform.
4. Why are computers essential for Numerical Methods?
5. Which engineering field uses Laplace Transform extensively?

✨ Key Takeaways

- Numerical Methods provide approximate solutions to complex engineering problems.
- They are essential for solving real-world problems that lack exact analytical solutions.
- Laplace Transform simplifies differential equations into algebraic equations.
- Fourier Transform converts signals into frequency components for easier analysis.
- These techniques are widely used in mechanical, civil, electrical, electronics, computer science, aerospace, biomedical, and data science applications.

🌟 Inspirational Quote

"Engineering begins where exact mathematics ends. Numerical Methods bridge the gap between theory and real-world engineering solutions."

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UNIT-1

SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL

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UNIT-1
SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

Introduction:

The determination of the roots (zeros) of an equation of the form

$$f(x) = 0$$

is an important problem in science, engineering, economics, and mathematics. Many equations cannot be solved exactly, so numerical methods are used to obtain approximate solutions.

🎯 Learning Outcomes

After this class, students will be able to:

- **Differentiate between algebraic and transcendental equations.**
- **Understand the concept of the root of an equation.**
- **Recognize the need for numerical methods.**
- **Identify real-world engineering applications.**

🤔 Warm-Up Activity (Think–Pair–Share)

Question 1

Solve:

$$2x + 5 = 9$$

Answer:

$$x = 2$$

Easy! ✓

Question 2

Now solve:

$$x = \cos x$$

Can you solve it using ordinary algebra?
No!
This is where Numerical Methods come into the picture.

📌 What is an Equation?

An equation is a mathematical statement containing one or more unknown variables.

Example:

$$2x + 3 = 7$$

📌 What is the Root of an Equation?

A root (or zero) of an equation is the value of x that satisfies the equation.

Example:

$$x^2 - 9 = 0$$

Roots are:

$$x = 3, x = -3$$

Importance of Finding Roots:

Finding roots is essential in many fields such as:

- **Engineering design**
- **Electrical circuit analysis**
- **Structural analysis**
- **Fluid mechanics**
- **Economics**
- **Computer science**
- **Physics**

Many practical problems lead to equations that cannot be solved analytically. Numerical methods provide approximate solutions with the desired accuracy.

📘 Types of Equations

1️⃣ Algebraic Equation

An equation containing only algebraic operations such as addition, subtraction, multiplication, division, powers, and roots of the variable.

Examples

$$x^2 - 5x + 6 = 0$$
$$x^3 - 4x + 1 = 0$$
$$x^4 + 2x^2 - 7 = 0$$

OR

A polynomial equation of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0$$

where $a_0, a_1, \dots, a_n$are constants and $a_0 \neq 0$, is called an algebraic equation.
Examples : $x^4 - 5x^2 + 6 = 0$,$x^2 + x + 1 = 0$

Characteristics

✓ **Polynomial equations**
✓ **Finite number of terms**
✓ **Can often be solved exactly (for lower degrees)**

2️⃣ Transcendental Equation

An equation involving one or more transcendental functions such as:

- **Trigonometric functions**
- **Exponential functions**
- **Logarithmic functions**
- **Hyperbolic functions**

is called a transcendental equation.

Examples

$$\tan x - e^{x} = 0$$
$$xe^{x} = \cos x$$
$$\sin x = xe^{2x}$$

These equations usually cannot be solved exactly, so numerical methods are required

Characteristics

- ✓ **Cannot usually be solved exactly.**
- ✓ **Numerical methods are required.**

📊 Comparison

Algebraic Equation	Transcendental Equation
Polynomial equation	Contains exponential, logarithmic, or trigonometric functions
Exact solution may exist	Exact solution is generally not possible
Easier to solve	Requires numerical methods

📌 Important Terms

Equation: **A mathematical statement with an unknown variable.**

Root (Zero): **The value that satisfies the equation.**

Approximate Solution: **A value close to the exact solution.**

Iteration: **Repeating calculations until the required accuracy is achieved.**

🧠 Quick Quiz

1. What is the root of an equation?

2. Which equation is transcendental?

a) $x^2 - 5 = 0$

b) $e^x - x = 0$

3. Which engineering field uses numerical methods?

a) Mechanical

b) Civil

c) Electrical

d) All of the above

Answer: d) All of the above

📚 Key Points to Remember

- **A root is the value that satisfies an equation.**
- **Equations are classified as algebraic and transcendental.**
- **Transcendental equations generally require numerical methods.**
- **Numerical methods provide accurate approximate solutions.**

🌟 Activity

Find whether the following equations are Algebraic or Transcendental:

1. $x^3 - 2x + 1 = 0$
2. $e^x - x = 0$
3. $\sin x = x/2$
4. $x^4 + 5x - 3 = 0$
5. $\log x + x = 2$

✨ Motivational Quote

"Every complex engineering problem begins with an equation, and every successful engineer knows how to solve it."

Graphical Method and Bracketing of Roots

🎯 Case Study:

Designing a Water Storage Tank

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🌍 Real-Life Scenario

A civil engineer is designing a water storage tank. During the design process, the engineer needs to determine the height of water (h) that satisfies the equation:

$$f(h) = h^3 - 6h - 4 = 0$$

Since this equation cannot be solved easily using ordinary algebra, the engineer first decides to identify an interval where the solution exists before applying a numerical method.

🔍 Step 1: Check the Interval

Choose the interval:

$$h = 2\text{and}h = 3$$

Calculate the function values:

$$f(2) = 2^3 - 6(2) - 4 = 8 - 12 - 4 = -8$$
$$f(3) = 3^3 - 6(3) - 4 = 27 - 18 - 4 = 5$$

📌 Observation

Since

$$f(2) \times f(3) = (-8)(5) = -40 < 0$$

the function changes its sign between 2 and 3.

Therefore,

✅ At least one root lies in the interval (2, 3).

This interval is called the Bracketing Interval.

📈 Graphical Interpretation

If the graph of $f(h) = h^3 - 6h - 4$is plotted,

- **At h = 2, the graph is below the x-axis.**
- **At h = 3, the graph is above the x-axis.**

Hence, the graph crosses the x-axis between 2 and 3, indicating the presence

of a root.

💡 Engineer's Decision

The engineer successfully identifies the interval (2, 3) containing the root. The next step is to apply a numerical method such as the Bisection Method or Regula-Falsi Method to determine the exact height of the water.

🌍 Engineering Significance

Bracketing of roots is widely used in:

- 🏗 **Structural Engineering – Load and stress analysis.**
- 🚗 **Mechanical Engineering – Equilibrium and vibration analysis.**
- ⚡ **Electrical Engineering – Circuit parameter calculations.**
- 🌊 **Civil Engineering – Water resource and hydraulic system design.**
- 🚀 **Aerospace Engineering – Flight trajectory calculations.**

💬 Classroom Discussion

1. **Why is it important to identify a bracketing interval before using the Bisection Method?**
2. **What condition guarantees the existence of a root in an interval?**
3. **Can the Graphical Method provide the exact value of a root? Why or why not?**
4. **Which numerical methods require a bracketing interval?**

🎯 Learning Outcome

By the end of this case study, students will understand that Graphical Method and Bracketing of Roots are essential first steps in solving nonlinear engineering equations. These techniques help engineers locate the interval containing the solution before applying numerical methods to obtain an accurate root.
Takeaway: *"Before finding the solution, an engineer must first know where the solution exists."*

🌟 Final Inspirational Message

"Numerical Methods are the bridge between mathematical theory and real-world

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engineering solutions. Every modern engineer uses them to solve problems that shape the future."

Numerical Methods Used to Find Roots

The common methods are:

1. **Bisection Method**
2. **Regula-Falsi (False Position) Method**
3. **Newton-Raphson Method**
4. **Secant Method**
5. **Fixed Point Iteration Method**

Geometrical Interpretation

(a) Simple Root

A number α is called a **simple root** if $f(\alpha) = 0$
And $f'(\alpha) \neq 0$.
The function can be written as $f(x) = (x - \alpha)g(x)$,
where

$$g(\alpha) \neq 0.$$

Geometrical Meaning:

- The graph **crosses the x-axis** at the root.
- The slope at the root is **not zero**.

Example:

$$f(x) = x - 3.$$

The graph cuts the x-axis at

$$x = 3.$$

(b) Multiple Root:

A number α is called a **multiple root of multiplicity** m if

$$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$$

but

$$f^{(m)}(\alpha) \neq 0.$$

The function can be written as

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$$f(x) = (x-\alpha)^m g(x),$$

where

$$g(\alpha) \neq 0.$$

Geometrical Meaning:

- If m is **even** (2, 4, 6, ...), the graph **touches the x-axis and turns back**.
- If m is **odd** (3, 5, ...), the graph **crosses the x-axis**, but more flatly than a simple root.

Example:

$$f(x) = (x-2)^2.$$

Here,

- Root = 2
- Multiplicity = 2

The graph touches the x-axis at $x = 2$and turns back.

Difference Between Simple and Multiple Root:

Simple Root	Multiple Root
$f(\alpha) = 0$	$f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$
$f'(\alpha) \neq 0$	First non-zero derivative is the m-th derivative
Graph crosses x-axis	Even multiplicity: touches x-axis; Odd multiplicity: crosses x-axis
Multiplicity = 1	Multiplicity > 1

Number and Methods of Finding Roots:

(a) Number of Roots:

Polynomial Equations:

A polynomial of degree $n$has exactly n**roots** (counting repeated and complex roots).

Example

$$x^3 - 6x^2 + 11x - 6 = 0$$

has 3 roots:

$$1,\ 2,\ 3.$$

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Transcendental Equations:

A transcendental equation may have:

- One root
- No root
- Two or more roots
- Infinitely many roots

depending on the function.

Example

$$\sin x = 0$$

has infinitely many roots:

$$x = n\pi, n = 0, \pm 1, \pm 2, \ldots$$

Methods of Finding Roots:

1. Direct Method

- Gives the **exact value** of the root.
- Solves the equation in a finite number of steps.
- Suitable for simple algebraic equations.

Example:

$$x^2 - 9 = 0$$

Roots: $x = \pm 3$.

---

2. Numerical Method:

Numerical methods are used when the equation cannot be solved exactly.
Starting with an initial approximation

$$x_0,$$

successive approximations

$$x_1,\ x_2,\ x_3, \ldots$$

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are obtained until they converge to the actual root

$$\alpha.$$

Mathematically,

$$x_k \to \alpha as k \to \infty.$$

Examples of numerical methods:

- Bisection Method
- Regula-Falsi Method
- Newton-Raphson Method
- Secant Method
- Fixed Point Iteration Method

- **Root:** A value α such that $f(\alpha) = 0$.
- **Geometrical Interpretation:** Root is the point where the graph meets the x-axis.
- **Simple Root:** $f(\alpha) = 0,\ f'(\alpha) \neq 0$; graph crosses the x-axis.
- **Multiple Root:** $f(\alpha) = f'(\alpha) = \cdots = f^{(m-1)}(\alpha) = 0$; even multiplicity touches the x-axis, odd multiplicity crosses it.
- **Polynomial of degree n:** Has exactly $n$roots.
- **Numerical methods:** Used to find approximate roots when exact solutions are not possible.

1. Bisection Method (Bolzano Method):

Definition

The **Bisection Method** is used to find the approximate root of an equation

$$f(x) = 0$$

by repeatedly dividing the interval into two equal parts.
This method works only if the function is **continuous** in the interval $[a, b]$and

$$f(a) \times f(b) < 0.$$

This means:

- $f(a)$is negative and $f(b)$is positive, or
- $f(a)$is positive and $f(b)$is negative.

Since the signs are opposite, there is at least one root between $a$and b.

Algorithm (Procedure)

Step 1

Choose two values $a$and $b$such that

$$f(a) \times f(b) < 0.$$

Step 2

Find the midpoint

$$x_1 = \frac{a+b}{2}.$$

This is the **first approximation**.

Step 3

Calculate

$$f(x_1).$$

- If $f(x_1) = 0,$ then x_1 is the exact root. Otherwise continue.

Step 4 If

$$f(a) \times f(x_1) < 0,$$

then the root lies between

$$a \text{ and } x_1.$$

Take the new interval as

$$[a, x_1].$$

Otherwise,
if

$$f(x_1) \times f(b) < 0,$$

the root lies between

$$x_1 \text{ and } b.$$

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Take the new interval as

$$[x_1, b].$$

Step 5

Again find the midpoint of the new interval.
For example,
if the new interval is

$$[a, x_1],$$

then

$$x_2 = \frac{a + x_1}{2}.$$

Step 6

Repeat the same process until the required accuracy is obtained.

Formulae:

First Approximation

$$x_1 = \frac{a + b}{2}$$

Second Approximation

If the root lies in $[a, x_1]$,

$$x_2 = \frac{a + x_1}{2}$$

If the root lies in $[x_1, b]$,

$$x_2 = \frac{x_1 + b}{2}$$

Similarly,

$$x_3,\ x_4, \ldots$$

are obtained by repeatedly bisecting the interval.

Flow of the Method

Choose a and b

|
▼
Check $f(a)\times f(b)<0$?

|
▼
Find midpoint

$x_1=(a+b)/2$

|
▼
Calculate $f(x_1)$

|
▼
Choose the interval
where sign changes

|
▼
Repeat until
required accuracy

Geometrical Interpretation

```
↑y
|
|          • (b,f(b))
|         /
|        /
|       /
|      /
|-------●-------------→ x-axis
|    Root
|
|  • (a,f(a))
|+-------------------------------→ x
```

$f(a)<0$ $f(b)>0$

The graph crosses the x-axis between **a** and **b**.

The midpoint

$$x_1 = \frac{a+b}{2}$$

is calculated.
If $f(x_1)$is positive, the new interval becomes

$$[a, x_1].$$

If $f(x_1)$is negative, the new interval becomes

$$[x_1, b].$$

📖 Case Study

Engine Safe Operating Temperature
The temperature satisfies

$$\boldsymbol{T^3 - 4T - 9 = 0}$$

Checking

$$\boldsymbol{f(2) = -9}$$
$$\boldsymbol{f(3) = 6}$$

Since

$$\boldsymbol{(-9)(6) < 0}$$

the engineer applies the Bisection Method.
After several iterations, the safe operating temperature is obtained accurately

Advantages:

- Very simple and easy to understand.
- Always converges if $f(a)$and $f(b)$have opposite signs.
- Gives accurate results with repeated iterations.
- Suitable for continuous functions.

Disadvantages:

- Converges slowly compared to other numerical methods.
- Applicable only when the initial interval has opposite signs.
- Cannot be used if $f(a) \times f(b) > 0$.

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Important Conditions:

The Bisection Method can be applied only when:

1. The function is continuous.
2. The interval $[a, b]$is known.
3. $f(a) \times f(b) < 0$.

Exam Points (2 Marks)

- **Bisection Method:** A numerical method that finds the root by repeatedly dividing the interval into two equal parts.
- **Also called: Bolzano Method**.
- **First approximation:**

$$x_1 = \frac{a+b}{2}$$

- **Condition for applicability:**

$$f(a) \times f(b) < 0$$

- **Main idea:** Select the subinterval in which the sign of the function changes and continue bisecting until the desired accuracy is reached.

PROBLEMS

Problem 1: Find the Real Root of $x^3 - x - 1 = 0$by Bisection Method

Given

$$f(x) = x^3 - x - 1$$

Step 1: Find the Interval

Calculate the values of the function.

$$f(0) = 0 - 0 - 1 = -1 < 0$$
$$f(1) = 1 - 1 - 1 = -1 < 0$$
$$f(2) = 8 - 2 - 1 = 5 > 0$$

Since

$$f(1) < 0, f(2) > 0,$$

the root lies between **1 and 2**.

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So,

$$a = 1, b = 2.$$

Step 2: Apply Bisection Formula
The formula is

$$x_n = \frac{a+b}{2}.$$

At each step:

- Find the midpoint.
- Calculate $f(x_n)$.
- Choose the interval where the sign changes.
- Repeat until the required accuracy is obtained.

Iteration Table

Iteration (n)	A	B	$x_n = \frac{a+b}{2}$	$f(x_n)$	Sign
1	1	2	1.500	0.875	+ve
2	1	1.500	1.250	-0.297	-ve
3	1.250	1.500	1.375	0.225	+ve
4	1.250	1.375	1.3125	-0.052	−ve
5	1.3125	1.375	1.34375	0.082	+ve
6	1.3125	1.34375	1.328125	0.014	+ve
7	1.3125	1.328125	1.3203125	-0.019	−ve
8	1.3203125	1.328125	1.32421875	0.001	+ve
9	1.3203125	1.32421875	1.322265625	-0.009	−ve
10	1.322265625	1.32421875	1.3232421875	-0.004 (approx.)	−ve
11	1.3232421875	1.32421875	1.32373 (approx.)	≈0	

Result
Therefore, the required real root is

$$\boxed{x \approx 1.322}$$

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Problem 2: Find the Real Root of $\mathbf{x}\boldsymbol{e^x} - \mathbf{1} = \mathbf{0}$ by Bisection Method
Given

$$f(x) = \mathbf{x}\boldsymbol{e^x} - \mathbf{1}$$

Solution:
Step 1: Find the Interval
Calculate the values of the function.

$$f(0) = \mathbf{0}\boldsymbol{e^0} - \mathbf{1} = -1 < 0 \quad \text{(-ve)}$$
$$f(1) = \mathbf{1}\boldsymbol{e^1} - \mathbf{1} = 1.7183 > 0 \quad \text{(+ve)}$$

Since

$$f(0) < 0, f(1) > 0,$$

the root lies between **0 and 1**.
So,

$$a = 0, b = 1.$$

Table representing Bisection Method iterations:

N	a (-ve)	b (+ve)	xn = (a+b)/2	f(xn) = xn*e^xn - 1	Sign of f(xn)
1	0	1	0.5	-0.1756	-ve
2	0.5	1	0.75	0.5878	(+ve)
3	0.5	0.75	0.625	0.1677	(+ve)
4	0.5	0.625	0.5625	-0.0128	(+ve)
5	0.5625	0.625	0.5938	0.0751	+ve
6	0.5625	0.5938	0.5782	0.0307	(+ve)
7	0.5625	0.5782	0.5704	0.0089	(+ve)
8	0.5685	0.5704	0.5685	-0.0019	(-ve)
9	0.5665	0.5704	0.5685	0.0036	(+ve)
10	0.5665	0.5685	0.5675	0.0010	(+ve)

11	0.5665	0.5675	0.567	-0.0004	(-ve)
12	0.567	0.5675	0.5673	0.0003	(+ve)
13	0.567	0.5673	0.5672	0.0002	(+ve)
14	0.567	0.5672	0.5671	-0.0001	(-ve)
15	0.567	0.5672	0.5672	0.00002	(+ve)
16	0.5671	0.5672	0.5672	0.00002	(+ve)

The root is 0.5672

3. Find a real root of the equation f(x) = x^3 - 4x - 9, using Bisection Method

Let f(x) = x^3 - 4x - 9
put x=0 => f(0) = -9 (-ve)
put x=1 => f(1) = -12 (-ve)
put x=2 => f(2) = -9 (-ve)
put x=3 => f(3) = 6 (+ve)
.. f(2) & f(3) have opposite signs
.. root lies b/w 2 & 3
Let xn = (a+b)/2

n	a (-ve)	b (+ve)	xn = (a+b)/2	f(xn) = x^3-4x-9	Sign
0	2	3	2.5	-3.3150	-ve
1	2.5	3	2.75	0.7969	+ve
2	2.5	2.75	2.6250	-1.4121	-ve
3	2.625	2.75	2.6875	-0.3371	-ve
4	2.6875	2.75	2.7188	0.2209	+ve
5	2.6875	2.7188	2.7032	-0.0606	-ve
6	2.7032	2.7188	2.7110	0.0806	+ve
7	2.7032	2.7110	2.7071	0.0103	+ve
8	2.7032	2.7071	2.7052	-0.0248	-ve

9	2.7052	2.7071	2.7062	-0.0068	-ve
10	2.7062	2.7071	2.7067	0.0022	+ve
11	2.7062	2.7067	2.7065	-0.0014	-ve
12	2.7065	2.7067	2.7066	0.0013	+ve
13	2.7065	2.7066	2.7066	0.004	+ve

.. x12 = x13
.. The root is 2.7066

2. Regula Falsi Method (or) False Position Method:

The given equation is f(x)=0. Suppose we want to find the roots of equation f(x)=0 first we select the interval [a,b] such that f(a) & f(b) have the approximation signs then root of f(x)=0 lies b/w a & b
Then find first approximation x1 = (a*f(b) - b*f(a)) / (f(b) - f(a))
If f(x1) = 0, then x1 is root
If f(x1) is +ve and f(a) is -ve then root lies b/w a & x1
Now 2nd approximation is x2 = (a*f(x1) - x1*f(a)) / (f(x1) - f(a))
Suppose f(x1) & f(b) are opposite signs then the roots lies b/w x1 & b.
Now x3 = (x1*f(b) - b*f(x1)) / (f(b) - f(x1))

Advantages of Regula Falsi (False Position) Method

1. It is **simple and easy to understand**.
2. It **always converges** if the function is continuous and the initial interval brackets the root (i.e., $f(a)$and $f(b)$have opposite signs).
3. It does **not require the derivative** of the function.
4. It generally converges **faster than the Bisection Method**.
5. The root always remains **within the selected interval**, making it a reliable method.

Disadvantages of Regula Falsi (False Position) Method

1. The **rate of convergence may be slow**, especially when one endpoint remains fixed for many iterations.
2. It may require **many iterations** to obtain the desired accuracy.
3. It is applicable **only when the root is bracketed** (i.e., the function changes sign over the interval).
4. It may converge **more slowly than methods like the** Newton–Raphson Method **or** Secant Method.
5. The method is **not suitable for functions that do not change sign** over the chosen interval, even if a root exists.

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Problems:

1. Find the root of the equation x^3 - 2x - 5 = 0

let f(x) = x^3 - 2x - 5
put x=0 => f(0) = -5 (-ve)
put x=1 => f(1) = 1 - 2 - 5 = -6 (-ve)
put x=2 => f(2) = 2^3 - 4 - 5 = -1 (-ve)
put x=3 => f(3) = 3^3 - 6 - 5 = 16 (+ve)
.. f(2) & f(3) have opposite signs
.. Root lies b/w 2 & 3

In Regula falsi method $x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$

n	A	B	f(a)	f(b)	$x_n = \frac{af(b) - bf(a)}{f(b) - f(a)}$	$f(x_n)$ = x^3-2x-5
1	2	3	-1	16	2.0588	-0.390
2	2.0588	3	-0.3908	16	2.0813	-0.147
3	2.0813	3	-0.1470	16	2.0896	-0.054
4	2.0896	3	-0.0546	16	2.0927	-0.020
5	2.0927	3	-0.0202	16	2.0939	-0.007
6	2.0939	3	-0.0075	16	2.0943	-0.002
7	2.0943	3	-0.0028	16	2.0945	-0.0010
8	2.0945	3	-0.0010	16	2.0946	-0.00037
9	2.0946	3	-0.00037	16	2.0946	-0.00014

.. x8 = x9
.. The root is 2.0946

---

2. Solve x*logx - 1.2 = 0 using Regula falsi method

let f(x) = x*logx - 1.2
put x=1 => f(1) = -1.2 < 0 (-ve)
put x=2 => f(2) = -0.5979 < 0 (-ve)
put x=3 => f(3) = 0.2313 > 0 (+ve)
.. f(2) & f(3) have opposite signs

.. Root lies b/w 2 & 3

$x_n = \frac{\mathrm{af(b) - bf(a)}}{\mathrm{f(b) - f(a)}}$

N	a (-ve)	b (+ve)	f(a)	f(b)	$x_n = \frac{\mathrm{af(b) - bf(a)}}{\mathrm{f(b) - f(a)}}$	f(xn)
1	2	3	-0.5979	0.2313	2.7210	-0.0171
2	2.7210	3	-0.0171	0.2313	2.7402	-0.0003
3	2.7402	3	-0.0003	0.2313	2.7405	-0.0001
4	2.7405	3	-0.0001	0.2313	2.7406	-0.0004
5	2.7406	3	-0.0004	0.2313	2.7406	

.. x4 = x5

.. The root is 2.7406

---

3. Find root of equation x*e^x = 2

let f(x) = x*e^x - 2

put x=0 => f(0) = -2 (-ve)

put x=1 => f(1) = 1e - 2 = 0.7183 (+ve)

.. f(0) & f(1) have opposite signs

.. Root lies b/w 0 & 1

$x_n = \frac{\mathrm{af(b) - bf(a)}}{\mathrm{f(b) - f(a)}}$

N	a (-ve)	b (+ve)	f(a)	f(b)	$x_n = \frac{\mathrm{af(b) - bf(a)}}{\mathrm{f(b) - f(a)}}$	f(xn)
1	0	1	-2	0.7183	0.7358	-0.4643
2	0.7358	1	-0.4643	0.7183	0.8395	-0.0564
3	0.8395	1	-0.0564	0.7183	0.8512	-0.0061
4	0.8512	1	-0.0061	0.7183	0.8526	-0.0009
5	0.8526	1	-0.0009	0.7183	0.8526	-0.0002

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For upto 3 decimals x4 = x5
.. The root is 0.852

The Iteration Method (or) The method of successive approximation method:

Suppose we want to find the root of the equation f(x)=0 reduce the equation as x=ø(x).
Suppose the initial approximation is x0 then the first approximation is x1=ø(x0)
The remaining successive approximation are x2=ø(x1), x3=ø(x2), ..., xn=ø(xn-1)
At one stage of the two successive approximation are equal say xn = xn-1 then the root is xn.
But the iteration method is applied when ø(x) is continuous i.e |ø'(x)| < 1

Iteration (Fixed Point) Method

The **Iteration Method** (also called the **Fixed Point Iteration Method**) is a numerical method used to find the approximate root of an equation by repeatedly substituting an initial approximation into a suitable function.

Advantages of Iteration (Fixed-Point Iteration) Method

1. It is **simple and easy to implement**.
2. It **does not require the derivative** of the function.
3. It requires **less computational effort** per iteration.
4. It is suitable for solving **nonlinear equations** when an appropriate iteration function is chosen.
5. The method can converge **rapidly** if the iteration function satisfies the convergence condition (i.e., $|g'(x)| < 1$near the root).

Disadvantages of Iteration (Fixed-Point Iteration) Method

1. **Convergence is not guaranteed** for every iteration function.
2. Choosing a suitable iteration function $g(x)$can be **difficult**.
3. The method may **converge slowly** compared to methods like the Newton–Raphson Method.
4. If the convergence condition is not satisfied, the method may **diverge or oscillate**.
5. The number of iterations required depends **strongly on the initial guess** and the choice of the iteration function.

Procedure (Algorithm):

Given the equation

$$f(x) = 0,$$

rewrite it in the form

$$x = \phi(x).$$

Choose an initial approximation x_0.

Calculate

$$x_1 = \phi(x_0),$$
$$x_2 = \phi(x_1),$$
$$x_3 = \phi(x_2),$$

and so on until two successive values are equal (or nearly equal).
The common value is the required root.

Condition for Convergence

The iteration method converges only if $|\ \phi'(x)\ | < 1$
in the interval containing the root.
If

$$|\ \phi'(x)\ | > 1,$$

the method diverges.

Problems

1. Find the root of x^3-x-11=0 using iteration method

let f(x) = x^3-x-11
put x=0 => f(0) = -11 < 0
put x=1 => f(1) = -11 < 0
put x=2 => f(2) = -5 < 0
put x=3 => f(3) = 13 > 0
.. f(2) & f(3) have opposite signs
.. Root lies b/w 2 & 3
x0 = (a+b)/2 = (2+3)/2 = 2.5
From given x^3-x-11=0
x^3 = x+11
x = (x+11)^(1/3)
let ø(x) = (x+11)^(1/3)
ø'(x) = 1/3*(x+11)^(-2/3) = 1/3*(x+11)^(-2/3)
|ø'(x)| < 1/3*(1)^(-2/3) < 1
.. |ø'(x)| < 1

$$x = \phi(x) = (x + 11)^{1/3}$$

Take $x_0 = 2.5$

Iterations

$$x_1 = (2.5 + 11)^{1/3} = 2.3781$$
$$x_2 = (2.3781 + 11)^{1/3} = 2.3740$$

$$x_3 = (2.3740 + 11)^{1/3} = 2.3736$$
$$x_4 = (2.3736 + 11)^{1/3} = 2.3736$$

Since

$$x_3 = x_4,$$

the required root is

$$\boxed{x = 2.3736}$$

Problem 2

Solve $3x = \cos x + 1$ using the Iteration Method.

Solution:

Step 1: Write the equation $3x = \cos x + 1$

or

$$f(x) = \cos x - 3x + 1.$$

Step 2: Locate the root

Calculate

$$f(0) = \cos 0 - 3(0) + 1 = 2 > 0$$
$$f\left(\frac{\pi}{4}\right) = 0.7071 - 2.3562 + 1 = -0.6491 < 0$$
$$f\left(\frac{\pi}{2}\right) = 0 - 4.7124 + 1 = -3.7124 < 0$$

Since

$$f(0) > 0, f\left(\frac{\pi}{4}\right) < 0,$$

the root lies between

$$\boxed{\left(0, \frac{\pi}{4}\right)}$$

(or more generally within $\left(0, \frac{\pi}{2}\right)$).

Step 3: Convert into Iteration Form

$$3x = \cos x + 1$$

$$x = \frac{\cos x + 1}{3}$$

Therefore,

$$\boxed{\phi(x) = \frac{\cos x + 1}{3}}$$

Step 4: Check the Convergence Condition
Differentiate

$$\phi(x) = \frac{\cos x + 1}{3}$$
$$\phi'(x) = \frac{-\sin x}{3}$$

At

$$x = 0,$$
$$\mid \phi'(0) \mid = 0 < 1$$

At

$$x = \frac{\pi}{2},$$
$$\mid \phi'\left(\frac{\pi}{2}\right) \mid = \frac{1}{3} = 0.333 < 1$$

Hence,

$$\mid \phi'(x) \mid < 1.$$

Therefore, the Iteration Method is applicable.

Step 5: Iteration Table
Take

$$x_0 = 0$$

and use

$$x_{n+1} = \frac{\cos x_n + 1}{3}.$$

Iteration	Formula	Value
x_0	Initial value	0.00000

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Iteration	Formula	Value
x_1	$\frac{\cos 0 + 1}{3}$	0.66667
x_2	$\frac{\cos 0.66667 + 1}{3}$	0.59530
x_3	$\frac{\cos 0.59530 + 1}{3}$	0.60933
x_4	$\frac{\cos 0.60933 + 1}{3}$	0.60668
x_5	$\frac{\cos 0.60668 + 1}{3}$	0.60718
x_6	$\frac{\cos 0.60718 + 1}{3}$	0.60709
x_7	$\frac{\cos 0.60709 + 1}{3}$	0.60710
x_8	$\frac{\cos 0.60710 + 1}{3}$	0.60710

Since

$$x_7 = x_8,$$

the iterations have converged.

Final Answer

$$\boxed{x = 0.60710}$$

Exam Steps (5 Marks)

1. Write the equation $f(x) = 0$.
2. Rewrite it as $x = \phi(x)$.
3. Check the convergence condition | $\phi'(x)$ |< 1.
4. Choose an initial approximation x_0.
5. Compute $x_1, x_2, x_3, \ldots$.
6. Stop when two successive values are equal.
7. Write the final root.

---

2 MARKS

1. What is the Iteration Method?

It is a numerical method in which the equation is written as $x = \phi(x)$, and successive approximations are used to obtain the root.

2. What is the convergence condition?

$$\boxed{|\ \phi'(x)\ |< 1}$$

3. Formula used in the Iteration Method

$$\boxed{x_{n+1} = \phi(x_n)}$$

Newton–Raphson Method

The Newton–Raphson Method is one of the fastest numerical methods used to find the approximate root of an equation.

It starts with an initial approximation and improves it by repeated iterations.

Formula

If

$$\boldsymbol{f(x) = 0},$$

then the Newton–Raphson iteration formula is

$$\boxed{\boldsymbol{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}}$$

where

- **x_n= current approximation**
- **x_{n+1}= next approximation**
- **$f'(x)$= derivative of $f(x)$**

The iterations are continued until two successive values become equal (or nearly equal).

Algorithm (Procedure)

Step 1

Write the equation in the form

$$\boldsymbol{f(x) = 0}.$$

Step 2

Find the derivative

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$f'(x)$.

Step 3
Choose an initial approximation x_0.
Step 4
Use the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Step 5
Repeat the process until the desired accuracy is obtained.

Advantages

- **Very fast convergence.**
- **Requires fewer iterations.**
- **Highly accurate.**

Disadvantages

- **Requires the derivative $f'(x)$.**
- **A poor initial guess may lead to divergence.**
- **Cannot be applied if $f'(x) = 0$.**

Solved Problem 1

Find the root of

$$x^3 - x - 1 = 0$$

using the Newton–Raphson Method.

Solution:

Step 1: Write the function

$$f(x) = x^3 - x - 1.$$

Step 2: Locate the root
Calculate

$$f(1) = 1 - 1 - 1 = -1 < 0$$
$$f(2) = 8 - 2 - 1 = 5 > 0$$

**Since the signs are opposite,
the root lies between**

1 and 2.

Step 3: Initial Approximation
Take

$$x_0 = \frac{1+2}{2} = 1.5.$$

Step 4: Differentiate

$$f'(x) = 3x^2 - 1.$$

Step 5: Apply Newton–Raphson Formula

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}.$$

Iteration Table

Iteration (n)	x_n	x_{n+1}
0	1.5000	1.3478
1	1.3478	1.3252
2	1.3252	1.3247
3	1.3247	1.3247

Since

$$x_3 = x_4,$$

the iterations have converged.

Final Answer

$$\boxed{x = 1.3247}$$

Solved Problem 2 (Partial)

Find the root of

$$e^x \sin x = 1$$

using the Newton–Raphson Method.

Solution:

Step 1: Write the function

$$f(x) = e^x \sin x - 1.$$

Step 2: Test the function

At

$$x = 0,$$
$$f(0) = e^0 \sin 0 - 1 = 1 \times 0 - 1 = -1.$$

Since

$$f(0) < 0,$$

we test another value (such as $x = 1$) to locate the interval containing the root.

Step 3: Differentiate

Using the product rule,

$$\frac{d}{dx}(e^x \sin x) = e^x \sin x + e^x \cos x.$$

Therefore,

$$\boxed{f'(x) = e^x(\sin x + \cos x)}$$

Step 4: Newton–Raphson Formula

The iteration formula becomes

$$\boxed{x_{n+1} = x_n - \frac{e^{x_n}\sin x_n - 1}{e^{x_n}(\sin x_n + \cos x_n)}}$$

After choosing a suitable initial approximation, this formula is used repeatedly until two successive values are equal.

Formula Summary

Newton–Raphson Formula

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

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For $x^3 - x - 1 = 0$

$$\boxed{x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}}$$

For $e^x \sin x = 1$

$$\boxed{x_{n+1} = x_n - \frac{e^{x_n}\sin x_n - 1}{e^{x_n}(\sin x_n + \cos x_n)}}$$

Exam Points

1. Newton–Raphson Method:

A numerical method used to find the root of an equation by successive approximations.

2. Main Formula:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

3. Advantage:

Converges very rapidly (faster than the Bisection Method).

4. Disadvantage:

Requires the derivative of the function and may fail if $f'(x) = 0$.

5. Root of $x^3 - x - 1 = 0$:

$$\boxed{x \approx 1.3247}$$

Note: Compared with the Bisection Method (which gave approximately 1.322 after several iterations), the Newton–Raphson Method reaches the more accurate value 1.3247 in only a few iterations because it converges much faster.

---

3. Using Newton–Raphson Method find the root of

$$x^4 - x - 9 = 0$$

Solution:

Let

$$f(x) = x^4 - x - 9$$

Then, $f'(x) = 4x^3 - 1$

Put $x = 0$, $\quad f(0) = 0 - 0 - 9 = -9$ **(-ve)**

Put $x = 1,$ $f(1) = 1 - 1 - 9 = -9$ (-ve)
Put $x = 2,$ $f(2) = 16 - 2 - 9 = 5$ (+ve)
Since $f(1)$and $f(2)$have opposite signs,
$\therefore$ Root lies between 1 and 2.

Take the initial approximation,

$$x_0 = \frac{1+2}{2} = 1.5$$

By Newton–Raphson Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Iteration Table

n	x_n	x_{n+1}
0	1.5000	1.9230
1	1.9230	1.8245
2	1.8245	1.8137
3	1.8137	1.8136
4	1.8136	1.8136

Since

$$x_3 = x_4,$$

$$\boxed{\therefore \text{The root is } 1.8136}$$

Interpolation

Let $y = f(x)$be a function. Let

$$y_0, y_1, y_2, \dots, y_n$$

be the values of $y$corresponding to the values of x,

$$x_0, x_1, x_2, \dots, x_n.$$

These values are called entries.

The process of computing the value of the function $y = f(x)$within the given range of data is called Interpolation.

The process of computing the value of the function outside the given range is called Extrapolation.

To determine the value of the function $f(x)$at some intermediate value of x, the following difference operators are used:

1. **Forward Difference Operator (Δ)**
2. **Backward Difference Operator (∇)**
3. **Central Difference Operator (δ)**

---

1. Forward Difference

Consider the function

$$y = f(x),$$

where

$$y_0, y_1, \dots, y_n$$

are the values corresponding to

$$x_0, x_1, \dots, x_n.$$

First Order Forward Difference

$$\begin{aligned}
\Delta y_0 &= y_1 - y_0 \\
\Delta y_1 &= y_2 - y_1 \\
\Delta y_2 &= y_3 - y_2 \\
\Delta y_3 &= y_4 - y_3 \\
&\dots \\
\Delta y_{n-1} &= y_n - y_{n-1}
\end{aligned}$$

Second Order Forward Difference

$$\begin{aligned}
\Delta^2 y_0 &= \Delta y_1 - \Delta y_0 \\
\Delta^2 y_1 &= \Delta y_2 - \Delta y_1 \\
\Delta^2 y_2 &= \Delta y_3 - \Delta y_2 \\
&\dots \\
\Delta^2 y_{n-2} &= \Delta y_{n-1} - \Delta y_{n-2}
\end{aligned}$$

Third Order Forward Difference

$$\begin{aligned}
\Delta^3 y_0 &= \Delta^2 y_1 - \Delta^2 y_0 \\
\Delta^3 y_1 &= \Delta^2 y_2 - \Delta^2 y_1 \\
&\dots \\
\Delta^3 y_{n-3} &= \Delta^2 y_{n-2} - \Delta^2 y_{n-3}
\end{aligned}$$

General r^{th} Order Forward Difference

$$\boxed{\Delta^r y_i = \Delta^{r-1} y_{i+1} - \Delta^{r-1} y_i}$$

where

$$r = 1, 2, \dots, n.$$

Forward Difference Table

x	y	Δy	Δ^2y	Δ^3y	Δ^4y
x_0	y_0	$\Delta y_0 = y_1 - y_0$ $\Delta y_1 = y_2 - y_1$ $\Delta y_2 = y_3 - y_2$ $\Delta y_3 = y_4 - y_3$	$\Delta^2 y_0 = \Delta y_1 - \Delta y_0$ $\Delta^2 y_1 = \Delta y_2 - \Delta y_1$ $\Delta^2 y_2 = \Delta y_3 - \Delta y_2$	$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$ $\Delta^3 y_1 = \Delta^2 y_2 - \Delta^2 y_1$	$\Delta^4 y_0 = \Delta^3 y_1 - \Delta^3 y_0$
x_1	y_1				
x_2	y_2				
x_3	y_3				
x_4	y_4				

Example

Q1. Write down the Forward Difference Table for the data:

X	4	6	8	10
Y	1	3	8	16

Solution:

Forward Difference Table
For the data:

X	y	Δy	Δ^2y	Δ^3y
4	1	$\Delta y_0 = 2$ $\Delta y_1 = 5$ $\Delta y_2 = 8$	$\Delta^2 y_0 = 3$ $\Delta^2 y_1 = 3$	$\Delta^3 y_0 = 0$
6	3			
8	8			
10	16			

Newton's Forward Difference Formula (Newton's Forward Interpolation Formula)

Let

$$y = f(x)$$

where

$$x_0, x_1, x_2, \dots, x_n$$

are equally spaced values of x, and

$$y_0, y_1, y_2, \dots, y_n$$

are the corresponding values of y.
Suppose it is required to find the value of the function at some point x, where x lies between x_0 and x_n, and the value of x is near the beginning of the interval. Then we use Newton's Forward Interpolation Formula.
Formula

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots + \frac{P(P-1)(P-2)\cdots(P-n+1)}{n!}\Delta^n y_0$$

where

$$\boxed{P = \frac{x - x_0}{h}}$$

Here,

- **x= value at which interpolation is required**
- **x_0= initial value of x**
- **h= common interval (equal spacing)**

This formula is also called Newton's Gregory Forward Interpolation Formula.

Example

Find $f(1.6)$using Newton's Forward Interpolation Formula for the following data:

x	1	1.4	1.8	2.2
y	3.49	4.82	5.96	6.50

Forward Difference Table

x	Y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	3.49	$\Delta y_0 = 1.33$		
1.4	4.82		$\Delta^2 y_0 = -0.19$	
1.8	5.96	$\Delta y_1 = 1.14$	$\Delta^2 y_1 = -0.60$	$\Delta^3 y_0 = -0.41$
2.2	6.50	$\Delta y_2 = 0.54$		

From the given data, the difference between successive values of $x$is equal.

$$h = 0.4$$

We have to find

$$f(1.6)$$

using Newton's Forward Interpolation Formula.

Using

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots$$

where

$$P = \frac{x - x_0}{h}$$

Here,

- $x = 1.6$
- $x_0 = 1$

- $h = 0.4$

Therefore,

$$P = \frac{1.6 - 1}{0.4} = \frac{0.6}{0.4} = 1.5$$

Using Newton's Forward Interpolation Formula,

$$\begin{aligned} y &= 3.49 + 1.5(1.33) + \frac{1.5(1.5-1)}{2!}(-0.19) \\ &\quad + \frac{1.5(0.5)(-0.5)}{3!}(-0.41) \\ &= 3.49 + 1.995 - 0.07125 + 0.025625 \\ &\boxed{y = 5.439375} \end{aligned}$$

Hence,

$$\boxed{f(1.6) = 5.439375}$$

Example

Find $f(2.5)$using Newton's Forward Interpolation Formula for the following data.

Given Data

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**
y	0	1	16	81	256	625	1296

From the given data,

$$h = 1$$

We have to find

$$f(2.5)$$

using Newton's Forward Interpolation Formula.

Forward Difference Table

x	**Y**	**Δy**	**$\Delta^2 y$**	**$\Delta^3 y$**	**$\Delta^4 y$**	**$\Delta^5 y$**	**$\Delta^6 y$**
0	0	1	14	36	24	0	
1	1	15	50	60	24	0	0
2	16	65	110	84	24		

x	Y	Δy	Δ^2y	Δ^3y	Δ^4y	Δ^5y	Δ^6y
3	81	175	194	108			
4	256	369	302				
5	625	671					
6	1296						

Using Newton's Forward Formula,

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0 + \frac{P(P-1)(P-2)(P-3)(P-4)}{5!}\Delta^5 y_0 + \frac{P(P-1)(P-2)(P-3)(P-4)(P-5)}{6!}\Delta^6 y_0$$

where

$$P = \frac{2.5-0}{1} = 2.5$$

Substituting,

$$y = 0 + 0.5 + 26.25 + 11.25 + 0.9375$$

$$\boxed{y = 39.0625}$$

Hence,

$$\boxed{f(2.5) = 39.0625}$$

Example

For $x = 0,1,2,3,4$ and $f(x) = 1,14,15,5,6,$

find the interpolation polynomial using Newton's Forward Interpolation Formula.

Solution:

From the given data,

$$h = 1, x_0 = 0$$

$$P = \frac{x - x_0}{h} = x$$

Difference Table

x	Y	Δy	Δ^2y	Δ^3y	Δ^4y
0	1	13 1 -10 1	-12 -11 11	1 22	21
1	14				
2	15				
3	5				
4	6				

Using Newton's Forward Formula,

$$y = f(x) \quad = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

Substituting,

$$= 1 + x(13) + \frac{x(x-1)}{2}(-12) + \frac{x(x-1)(x-2)}{6}(1)$$

Expanding,

$$= 1 + 13x - 6x(x-1) + \frac{x(x-1)(x-2)}{6}$$

$$= \frac{x^3 - 39x^2 + 116x + 6}{6}$$

Therefore,

$$\boxed{f(x) = \frac{1}{6}x^3 - \frac{13}{2}x^2 + \frac{58}{3}x + 1}$$

Example

If $u_0 = 1,\ u_1 = 0,\ u_2 = 5,\ u_3 = 22,\ u_4 = 57,$ find $u(0.5)$.

Solution:

Difference Table

u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	-1 5 17 35	6 12 18	6 6	0
1	0				
2	5				
3	22				
4	57				

Given,

$$x = 0.5, x_0 = 0, h = 1$$

Hence,

$$P = \frac{0.5 - 0}{1} = 0.5$$

Using Newton's Forward Formula,

$$y = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

Substituting,

$$= 1 + 0.5(-1) + \frac{0.5(-0.5)}{2}(6) + \frac{0.5(-0.5)(-1.5)}{6}(6)$$

$$= 1 - 0.5 - 0.75 + 0.375$$

$$\boxed{y = 0.125}$$

Therefore,

$$\boxed{u(0.5) = 0.125}$$

Backward Differences

Let

$$y = f(x)$$

be the function.

Let

$$y_0, y_1, \dots, y_n$$

be the functional values corresponding to

$$x_0, x_1, \dots, x_n.$$

Then the backward differences are

$$\nabla y_1 = y_1 - y_0,$$
$$\nabla y_2 = y_2 - y_1,$$
$$\nabla y_3 = y_3 - y_2,$$
$$\dots$$
$$\nabla y_n = y_n - y_{n-1}.$$

These are called **first-order backward differences.**
The **second-order backward differences** are

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1,$$
$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2,$$
$$\dots$$
$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}.$$

In general,

$$\boxed{\nabla^r y_n = \nabla^{r-1} y_n - \nabla^{r-1} y_{n-1}}$$

where

$$n = 1,2,3,\dots$$

These are called the r^{th}**-order backward differences.**

Backward Difference Table

x	**y**	**∇y**	**∇²y**	**∇³y**	**∇⁴y**
x_0	y_0				
x_1	y_1	$\nabla y_1 = y_1 - y_0$			
x_2	y_2	$\nabla y_2 = y_2 - y_1$	$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$		
x_3	y_3	$\nabla y_3 = y_3 - y_2$	$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$	$\nabla^3 y_3 = \nabla^2 y_3 - \nabla^2 y_2$	
x_4	y_4	$\nabla y_4 = y_4 - y_3$	$\nabla^2 y_4 = \nabla y_4 - \nabla y_3$	$\nabla^3 y_4 = \nabla^2 y_4 - \nabla^2 y_3$	$\nabla^4 y_4 = \nabla^3 y_4 - \nabla^3 y_3$

Newton's Backward Difference Formula (Newton's Backward Interpolation Formula)

Let the function

$$y = f(x)$$

satisfy the values

$$y_0, y_1, \dots, y_n$$

corresponding to

$$x_0, x_1, \dots, x_n$$

of the independent variable x.
Let the interval between successive values of $x$be equal.
Suppose it is required to find the value of $y$at some point $x$lying inside the interval (x_0, x_n), and the required value is **near the end of the table**.
Then Newton's Backward Difference Formula is used.

Formula

$$\boxed{y = f(x) = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n + \cdots}$$

where

$$\boxed{P = \frac{x - x_n}{h}}$$

Here,

- x_n= last value of x
- x= value where interpolation is required
- h= common interval

Notes

1. Newton's Backward Difference Formula is applicable only for **equal intervals**.
2. It is useful when the required value lies **near the end of the table**.

Example 1

Find the value of y at $x = 9$ using Newton's Backward Difference Formula.

Given Data

x	2	5	8	11
y	94.8	87.9	81.3	75.1

Solution:

Backward Difference Table

x	Y	∇y	∇²y	∇³y
2	94.8	-6.9	0.3	0.1

x	Y	∇y	∇²y	∇³y
5	87.9	-6.6	0.4	
8	81.3	-6.2		
11	75.1			

Here,

$$x = 9, x_n = 11, h = 3$$

Therefore,

$$P = \frac{9-11}{3} = -\frac{2}{3} \approx -0.67$$

Using Newton's Backward Formula,

$$y = y_n + P\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

Substituting,

$$\begin{aligned} y \quad &= 75.1 + (-0.67)(-6.2) \\ &+ \frac{(-0.67)(0.33)}{2}(0.4) \\ &+ \frac{(-0.67)(0.33)(1.33)}{6}(0.1) \end{aligned}$$

$$= 75.1 + 4.154 - 0.04422 - 0.0049105$$

$$\boxed{y = 79.204}$$

Example 2

Use Newton's Backward Difference Formula to find $f(7.5)$.

Given Data

x	1	2	3	4	5	6	7	8
y	1	8	27	64	125	216	343	512

Solution:

Difference Table

x	y	∇y	∇²y	∇³y	∇⁴y	∇⁵y
1	1	7	12	6	0	0
2	8	19	18	6	0	0

x	y	∇y	∇²y	∇³y	∇⁴y	∇⁵y
3	27	37	24	6	0	0
4	64	61	30	6	0	
5	125	91	36	6		
6	216	127	42			
7	343	169				
8	512					

Here,

$$x = 7.5, x_n = 8, h = 1$$

$$P = \frac{7.5 - 8}{1} = -0.5$$

Using Newton's Backward Formula,

$$\begin{aligned} y &= 512 + (-0.5)(169) \\ &\quad + \frac{(-0.5)(0.5)}{2}(42) \\ &\quad + \frac{(-0.5)(0.5)(1.5)}{6}(6) \\ &= 512 - 84.5 - 5.25 - 0.375 \\ &\boxed{f(7.5) = 421.875} \end{aligned}$$

Example 3

Using Backward Difference Formula, find the polynomial passing through

$$(3,6),\ (4,24),\ (5,60),\ (6,120)$$

Solution:

Difference Table

x	y	∇y	∇²y	∇³y
3	6			
4	24	18 36 60	18 24	6
5	60			

x	y	∇y	∇²y	∇³y
6	120			

Here,

$$x_n = 6, h = 1, P = x - 6$$

Using Newton's Backward Formula,

$$\begin{aligned} y &= 120 + P(60) \\ &+ \frac{P(P+1)}{2}(24) \\ &+ \frac{P(P+1)(P+2)}{6}(6) \end{aligned}$$

Substituting $P = x - 6$,

$$\begin{aligned} y &= 120 + 60(x-6) \\ &+ 12(x-6)(x-5) \\ &+ (x-6)(x-5)(x-4) \end{aligned}$$

Expanding,

$$\boxed{y = x^3 - 3x^2 + 2x}$$

Interpolation with Unequal Intervals (Lagrange's Interpolation)

Let

$$y = f(x)$$

be the function, and let

$$f(x_0),\ f(x_1),\ f(x_2),\ \dots,\ f(x_n)$$

be the values of $y$corresponding to

$$x_0,\ x_1,\ x_2,\ \dots,\ x_n.$$

If the values of $x$are **not equally spaced**, then Newton's interpolation formulas cannot be used.
In such cases, **Lagrange's Interpolation Formula** is used.
Lagrange's Formula

$$\boxed{\begin{aligned} f(x) = {} & \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}f(x_0) \\ & + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}f(x_1) \\ & + \cdots \end{aligned}}$$

Lagrange Interpolation Formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0) + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2) + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}f(x_n)$$

Example 1

Using the Lagrange Interpolation Formula find the value of $y(10)$ from the following table.

x	5	6	9	11
y	12	13	14	16

Solution:

From the given table, the difference between the x-values is **unequal**, so we use the **Lagrange Interpolation Formula (LIF)**.

Given

$$x_0 = 5, x_1 = 6, x_2 = 9, x_3 = 11$$
$$f(x_0) = 12, f(x_1) = 13, f(x_2) = 14, f(x_3) = 16$$

Using Lagrange interpolation,

$$\begin{aligned} y = f(10) = {} & \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) \\ & + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13) \\ & + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) \\ & + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16) \end{aligned}$$

Substituting,

$$= 2 + (-4.33) + 11.66 + 5.33$$

$$\boxed{y(10) \approx 14.66 = \frac{44}{3}}$$

Example 2

Calculate $f(10)$ given the following values:

x	**1**	**7**	**15**
$f(x)$	168	192	336

Solution:

Given

$$x_0 = 1, x_1 = 7, x_2 = 15$$
$$f(x_0) = 168, f(x_1) = 192, f(x_2) = 336$$

Since the x-values are **unequally spaced**, use the **Lagrange Interpolation Formula**.

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)$$

For $x = 10$,

$$y = f(10) = \frac{(10-7)(10-15)}{(1-7)(1-15)}(168) + \frac{(10-1)(10-15)}{(7-1)(7-15)}(192) + \frac{(10-1)(10-7)}{(15-1)(15-7)}(336)$$

Simplifying,

$$= \frac{3(-5)(168)}{(-6)(-14)} + \frac{9(-5)(192)}{6(-8)} + \frac{9(3)(336)}{14(8)}$$

$$= -30 + 180 + 126$$
$$\boxed{f(10) = 276}$$

PRACTICE QUESTIONS:

1.Find the root of $x^3 + x - 1 = 0$ using Newton Raphson method
2.Find the root of $x^3 - 9 = 0$ using Newton Raphson method
3.Solve $x^3 - x - 2 = 0$ using the Iteration method.
4. Solve $x^3 - 4x - 9 = 0$by the Bisection method.
5. Find the root of $x^2 - 5 = 0$using the Bisection method.
6. Solve $e^x - 3x = 0$using the Bisection method.
7. Solve $x^3 - 2x - 5 = 0$using the Regula-Falsi method
8. Find the root of $x\log_{10} x - 1.2 = 0$using the Regula-Falsi method.
9. Solve $e^{-x} - x = 0$using the Newton-Raphson method.
10. Find the positive root of $x - \sin x - 0.5 = 0$using the Newton-Raphson method.

Unit–2

Numerical Differentiation, Numerical Integration and Curve Fitting

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

Unit–2

Numerical Differentiation, Numerical Integration and Curve Fitting

Numerical Integration

Numerical integration is the approximate computation of a definite integral

$$\int_a^b f(x)\,dx$$

using numerical techniques. Numerical integration is also known as **Numerical Quadrature**. The most straightforward numerical integration technique uses the **Newton–Cotes formula**.

Newton–Cotes Formula

Let $y_0, y_1, y_2, \dots, y_n$be the values of

$$y = f(x)$$

corresponding to

$$a = x_0,\ x_1,\ x_2,\ \ \dots,\ x_n = b$$

which are equally spaced with step size h.
Then,

$$x_r = x_0 + rh, x_n = x_0 + nh$$

By Newton's Forward Interpolation Formula,

$$y = y_0 + \frac{P}{1!}\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots \qquad (1)$$

where

$$P = \frac{x - x_0}{h}$$

Now,

$$\int_{x_0}^{x_n} y\,dx = \int_{x_0}^{x_n} \left[y_0 + \frac{P}{1!}\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \cdots \right] dx$$

where

- h= interval size (step size)
- n= number of sub-intervals
- $a$and b= limits of integration

Since

$$x = x_0 + Ph,$$

therefore,

$$dx = h\,dP.$$

Hence,

$$\int_{x_0}^{x_n} y\,dx = h\int_0^n \left[y_0 + P\Delta y_0 + \frac{P^2 - P}{2}\Delta^2 y_0 + \frac{P^3 - 3P^2 + 2P}{6}\Delta^3 y_0 + \cdots\right] dP$$

Integrating,

$$\int_{x_0}^{x_n} y\,dx = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \cdots\right]$$

This is called the **Newton–Cotes Quadrature Formula**.

Trapezoidal Rule

Put $n = 1$in the above formula.
Then,

$$\int_{x_0}^{x_1} y\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right)$$

Since higher-order differences do not exist for $n = 1$,

$$\Delta y_0 = y_1 - y_0.$$

Therefore,

$$\int_{x_0}^{x_1} y\,dx = h\left[y_0 + \frac{1}{2}(y_1 - y_0)\right]$$

or,

$$\boxed{\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1)}$$

This is known as the **Trapezoidal Rule**.
Putting $n = 2$,

$$\int_{x_0}^{x_2} y\,dx = h\left[y_0 + \Delta y_0 + \frac{1}{3}\Delta^2 y_0\right]$$

which simplifies to

$$\int_{x_0}^{x_2} y\,dx = \frac{h}{2}(y_0 + y_2)$$

Similarly,

$$\int_{x_0+(n-1)h}^{x_0+nh} y\,dx = \frac{h}{2}(y_{n-1} + y_n)$$

Since $y = f(x)$,

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} f(x)\,dx$$

Therefore,

$$\begin{aligned}\int_{x_0}^{x_n} f(x)\,dx &= \frac{h}{2}(y_0 + y_1) + \frac{h}{2}(y_1 + y_2) + \cdots + \frac{h}{2}(y_{n-1} + y_n)\\ &= \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \cdots + y_{n-1})]\end{aligned}$$

Hence,

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}[(\text{Sum of first and last ordinates}) + 2 \times (\text{Sum of remaining ordinates})]}$$

Simpson's $\frac{1}{3}$Rule:

This is another popular numerical integration method.
By putting $n = 2$in the Newton–Cotes Quadrature Formula,

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]}$$

where

$$h = \frac{x_n - x_0}{n} = \frac{b-a}{n}$$

or

$$\boxed{\frac{h}{3}[(\text{Sum of first and last ordinates}) + 4 \times (\text{Sum of odd ordinates}) + 2 \times (\text{Sum of even ordinates})]}$$

This is known as **Simpson's $\frac{1}{3}$Rule**.

Simpson's $\frac{3}{8}$Rule:

$$\boxed{\int_{x_0}^{x_n} f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + y_8 + \cdots) + 2(y_3 + y_6 + y_9 + \cdots)]}$$

Example 1

Evaluate

$$\int_0^1 x^3\ dx$$

using the **Trapezoidal Rule** with **5 subintervals** and compare with the exact value.

Given

$$a = 0, b = 1, n = 5$$
$$h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$
$$f(x) = x^3$$

Table

x	**0**	**0.2**	**0.4**	**0.6**	**0.8**	**1**
$y = f(x) = x^3$	0	0.008	0.064	0.216	0.512	1

Using the Trapezoidal Rule,

$$\int_0^1 x^3\ dx = \frac{h}{2}[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)]$$
$$= \frac{0.2}{2}[(0 + 1) + 2(0.008 + 0.064 + 0.216 + 0.512)]$$
$$= 0.1[1 + 2(0.800)]$$
$$= 0.1(2.600) = 0.2600$$

Exact Value

$$\int_0^1 x^3\ dx = \left[\frac{x^4}{4}\right]_0^1 = \frac{1}{4} = 0.2500$$

Comparison

- Trapezoidal Rule = **0.2600**
- Exact Value = **0.2500**

Example 2

Use the Trapezoidal Rule with $n = 4$to estimate

$$\int_0^1 \frac{1}{1 + x^2}\ dx$$

Solution:

Given

$$a = 0, b = 1, n = 4$$
$$h = \frac{1 - 0}{4} = 0.25$$
$$f(x) = \frac{1}{1 + x^2}$$

Table

x	**0**	**0.25**	**0.50**	**0.75**	**1**
$y = f(x)$	1	0.9412	0.8000	0.6400	0.5000

Using the Trapezoidal Rule,

$$\int_0^1 \frac{1}{1 + x^2}\ dx = \frac{0.25}{2}[(1 + 0.5) + 2(0.9412 + 0.8 + 0.64)]$$
$$= 0.125[1.5 + 2(2.3812)]$$
$$= 0.125(6.2624)$$
$$\boxed{= 0.7828 \text{ (approximately)}}$$

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Exercise

Evaluate

$$\int_0^{1.2} e^x \, dx$$

taking **6 intervals** using the **Trapezoidal Rule**.

Simpson's $\frac{1}{3}$Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}[(y_0 + y_n) + 4(y_1 + y_3 + \cdots + y_{n-1}) + 2(y_2 + y_4 + \cdots + y_{n-2})]$$

where

$$h = \frac{b-a}{n}$$

Simpson's $\frac{3}{8}$Rule

$$\int_a^b f(x)\,dx = \frac{3h}{8}[(y_0 + y_n) + 2(y_3 + y_6 + \cdots) + 3(y_1 + y_2 + \cdots + y_{n-1})]$$

This method is applicable only when $n$is a multiple of **3**. It is not as accurate as Simpson's $\frac{1}{3}$rule.

Example

Given that

x	**4**	**4.2**	**4.4**	**4.6**	**4.8**	**5.0**	**5.2**
$y = \log_e x$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

Evaluate

$$\int_4^{5.2} \log_e x \, dx$$

using:

1. Trapezoidal Rule
2. Simpson's $\frac{1}{3}$Rule

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3. Simpson's $\frac{3}{8}$Rule

Given

$$h = 0.2$$

(i) Trapezoidal Rule

$$\int_4^{5.2} \log_e x \, dx = \frac{0.2}{2}[(1.3863 + 1.6487) + 2(1.4351 + 1.4816 + 1.5261 + 1.5686 + 1.6094)]$$

$$= 0.1\,[3.0350 + 2(7.6208)]$$

$$= 0.1(18.2766)$$

$$\boxed{= 1.82766}$$

(ii) Simpson's $\frac{1}{3}$Rule

$$\int_4^{5.2} \log_e x \, dx = \frac{0.2}{3}[(1.3863 + 1.6487) + 4(1.4351 + 1.5261 + 1.6094) + 2(1.4816 + 1.5686)]$$

$$= 0.0667\,(3.0350 + 18.2824 + 6.1004)$$

$$\boxed{= 1.8288}$$

(iii) Simpson's $\frac{3}{8}$Rule

$$\int_4^{5.2} \log_e x \, dx = \frac{3(0.2)}{8}[(1.3863 + 1.6487) + 3(1.4351 + 1.4816 + 1.5686 + 1.6094) + 2(1.5261)]$$

$$= 0.0750\,(3.0350 + 18.2841 + 3.0522)$$

$$\boxed{= 1.8278}$$

Example

Divide the range into **10 equal parts**. Find the approximate value of

$$\int_0^{\pi} \sin x \, dx$$

using:

- Trapezoidal Rule

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- Simpson's $\frac{1}{3}$Rule
- Simpson's $\frac{3}{8}$Rule

Given

$$a = 0, b = \pi, n = 10$$
$$h = \frac{b-a}{n} = \frac{\pi}{10}$$

Table of values

x	0	$\frac{\pi}{10}$	$\frac{2\pi}{10}$	$\frac{3\pi}{10}$	$\frac{4\pi}{10}$	$\frac{5\pi}{10}$	$\frac{6\pi}{10}$	$\frac{7\pi}{10}$	$\frac{8\pi}{10}$	$\frac{9\pi}{10}$	π
$y = \sin x$	0	0.3090	0.5878	0.8090	0.9511	1.0000	0.9511	0.8090	0.5878	0.3090	0

(i) Trapezoidal Rule

$$\int_0^{\pi} \sin x\, dx = \frac{\pi}{10(2)}[(0+0) + 2(0.3090 + 0.5878 + 0.8090 + 0.9511 + 1 + 0.9511 + 0.8090 + 0.5878 + 0.3090)]$$
$$= 0.1571(12.6276)$$
$$\boxed{= 1.9838}$$

(ii) Simpson's $\frac{1}{3}$Rule

$$\int_0^{\pi} \sin x\, dx = \frac{\pi}{30}[(0+0) + 4(0.3090 + 0.8090 + 1 + 0.8090 + 0.3090) + 2(0.5878 + 0.9511 + 0.9511 + 0.5878)]$$
$$= 0.1047(19.099)$$
$$\boxed{= 1.9997}$$

(iii) Simpson's $\frac{3}{8}$Rule

$$\int_0^{\pi} \sin x\, dx = \frac{3\pi}{80}[0 + 3(0.3090 + 0.5878 + 0.9511 + 1 + 0.8090 + 0.5878) + 2(0.8090 + 0.9511 + 0.3090)]$$
$$= 0.1178\,[0 + 3(4.2447) + 2(2.0691)]$$
$$\boxed{= 1.9876}$$

Numerical Differentiation

Newton's Forward Difference Formula for Differentiation

(i) First Derivative

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 - \cdots\right]$$

(ii) Second Derivative

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

Example 1

From the following table of values of $x$and y, obtain

$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}$$

at $x = 1.2$.

Given Table

x	y
1.0	2.7183
1.2	3.3201
1.4	4.0552
1.6	4.9530
1.8	6.0496
2.0	7.3891
2.2	9.0250

Solution:

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	2.7183	0.6018	0.1333	0.0294	0.0067	0.0013	0.0001
1.2	3.3201	0.7351	0.1627	0.0361	0.0080	0.0014	
1.4	4.0552	0.8978	0.1988	0.0441	0.0094		

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.6	4.9530	1.0966	0.2429	0.0535			
1.8	6.0496	1.3395	0.2964				
2.0	7.3891	1.6359					
2.2	9.0250						

First Derivative

Using Newton’s forward difference formula,

$$h = 1.4 - 1.2 = 0.2$$

$$\left(\frac{dy}{dx}\right)_{x=1.2} = \frac{1}{0.2}\left[0.7351 - \frac{1}{2}(0.1627) + \frac{1}{3}(0.0361) - \frac{1}{4}(0.0080) + \frac{1}{5}(0.0014)\right]$$

$$= 5\,[0.7351 - 0.0814 + 0.0120 - 0.0020 + 0.0003]$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=1.2} = 3.3205}$$

Second Derivative

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.2} = \frac{1}{0.04}\left[0.1627 - 0.0361 + \frac{11}{12}(0.0080) - \frac{5}{6}(0.0014)\right]$$

$$= 25[0.1627 - 0.0361 + 0.0073 - 0.0011]$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=1.2} = 3.32}$$

Example 2

Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ at $x = 0$using the following table.

Given

x	**0**	**2**	**4**	**6**	**8**	**10**
$f(x)$	0	12	248	1284	4080	9980

Solution:

Forward Difference Table

x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0	12	224	576	384	0

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x	$y = f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
2	12	236	800	960	384	
4	248	1036	1760	1344		
6	1284	2796	3104			
8	4080	5900				
10	9980					

First Derivative

Using Newton's forward difference formula,

$$h = 2$$

$$\left(\frac{dy}{dx}\right)_{x=0} = \frac{1}{2}\left[12 - \frac{1}{2}(224) + \frac{1}{3}(576) - \frac{1}{4}(384) + 0\right]$$

$$= 0.5[12 - 112 + 192 - 96]$$

$$= 0.5(-4)$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=0} = -2}$$

Second Derivative

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = \frac{1}{2^2}\left[224 - 576 + \frac{11}{12}(384)\right]$$

$$= \frac{1}{4}[224 - 576 + 352]$$

$$= 0.25(0)$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=0} = 0}$$

Example 3

Find

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}$$

at $x = 1.5$using the following table.

Given

x	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
$f(x)$	3.375	7.0	13.625	24.0	38.875	59.0

Solution:

Forward Difference Table

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.5	3.375	3.625 6.625 10.375 14.875 20.125	3.000 3.750 4.500 5.250	0.750 0.750 0.750	0 0	0
2.0	7.000					
2.5	13.625					
3.0	24.000					
3.5	38.875					
4.0	59.000					

Using Newton's forward difference formula,

$$h = 2.0 - 1.5 = 0.5$$

$$\left(\frac{dy}{dx}\right)_{x=1.5} = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75)\right]$$

(The remaining part of the solution is cut off in the provided image.)

First Derivative

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \cdots\right]$$

$$= \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3) + \frac{1}{3}(0.75)\right]$$

$$= 2(3.625 - 1.5 + 0.25)$$

$$= 2(2.375)$$

$$= 4.75$$

Second Derivative

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \cdots\right]$$

$$= \frac{1}{(0.5)^2}(3 - 0.75)$$

$$= \frac{2.25}{0.25} = 9$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=1.5} = 9}$$

Newton's Backward Formula for Differentiation

(i) First Derivative

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \cdots\right]$$

(ii) Second Derivative

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \cdots\right]$$

Example 4

Find

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}$$

for $x = 2.2$.

Given Table

x	**1.0**	**1.2**	**1.4**	**1.6**	**1.8**	**2.0**	**2.2**
y	2.7183	3.3201	4.0552	4.9530	6.0496	7.3891	9.0250

Solution:

Backward Difference Table:

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
1.0	2.7183						
1.2	3.3201	0.6018					
1.4	4.0552	0.7351	0.1333				
1.6	4.9530	0.8978	0.1627	0.0294			
1.8	6.0496	1.0966	0.1988	0.0361	0.0067		
2.0	7.3891	1.3395	0.2429	0.0441	0.0080	0.0013	
2.2	9.0250	1.6359	0.2964	0.0535	0.0094	0.0014	0.0001

Given,

$$h = 0.2$$

(i) First Derivative

$$\left(\frac{dy}{dx}\right)_{x=2.2} = \frac{1}{0.2}\left[1.6359 + \frac{1}{2}(0.2964) + \frac{1}{3}(0.0535) + \frac{1}{4}(0.0094) + \frac{1}{5}(0.0014) + \frac{1}{6}(0.0001)\right]$$

$$= 5(1.6359 + 0.1482 + 0.0178 + 0.00235 + 0.00028 + 0.00001)$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=2.2} = 9.022}$$

(ii) Second Derivative

$$\left(\frac{d^2y}{dx^2}\right)_{x=2.2} = \frac{1}{0.04}\left[0.2964 + 0.0535 + \frac{11}{12}(0.0094) + \frac{5}{6}(0.0014) + \frac{137}{180}(0.0001)\right]$$

$$= 25(0.2964 + 0.0535 + 0.0086 + 0.0011 + 0)$$

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=2.2} = 8.99}$$

Example 5

Find

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}$$

when $x = 6$.

Given Table

x	**0**	**1**	**2**	**3**	**4**	**5**	**6**
y	6.9897	7.4036	7.7815	8.1291	8.4510	8.7506	9.0309

Solution:

Backward Difference Table:

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
0	6.9897	0.4139	-0.0360	0.0057	-0.0014	0.0005	0
1	7.4036	0.3779	-0.0303	0.0043	-0.0009	0.0005	
2	7.7815	0.3476	-0.0257	0.0034	-0.0004		

x	y	∇	∇^2	∇^3	∇^4	∇^5	∇^6
3	8.1291	0.3219	-0.0223	0.0030			
4	8.4510	0.2996	-0.0193				
5	8.7506	0.2803					
6	9.0309						

Given,

$$h = 6 - 5 = 1$$

(i) First Derivative

$$\left(\frac{dy}{dx}\right)_{x=6} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \frac{1}{5}\nabla^5 y_n\right]$$

$$= \frac{1}{1}\left[0.2803 + \frac{1}{2}(-0.0193) + \frac{1}{3}(0.0030) + \frac{1}{4}(-0.0004) + \frac{1}{5}(0.0005)\right]$$

$$= 0.2803 - 0.00965 + 0.0010 - 0.0001 + 0.0001$$

$$\boxed{\left(\frac{dy}{dx}\right)_{x=6} = 0.27165}$$

Numerical Differentiation

6. Estimate the Rate of Growth of the Population in the Year 1981 from the Following Table

Given:

Year (x)	**1951**	**1961**	**1971**	**1981**	**1991**
Population (y)	19.96	39.65	58.81	77.21	94.61

SOLUTION:

Difference Table

x	**Y**	**∇**	**∇²**	**∇³**	**∇⁴**
1951	19.96				
1961	39.65	19.69	-0.53		
1971	58.81	19.16	-0.76	-0.23	-0.01
1981	77.21	18.40	-1.00	-0.24	
1991	94.61	17.40			

Here,

$$h = 1991 - 1981 = 10$$

Using Newton's Backward Formula,

$$\left(\frac{dy}{dx}\right)_{x=x_n}=\frac{1}{h}\left[\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+\frac{1}{4}\nabla^4 y_n+\cdots\right]$$
$$=\frac{1}{10}\left[18.4+\frac{1}{2}(-0.76)+\frac{1}{3}(-0.23)\right]$$
$$=0.1(18.4-0.38-0.076)$$
$$=0.1(17.9434)$$
$$=1.79434$$

Therefore,

$$\boxed{\left(\frac{dy}{dx}\right)_{x=1981}=1.79434}$$

Using Newton's Backward Formula for Second Derivative,

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n}=\frac{1}{h^2}\left[\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n+\frac{5}{6}\nabla^5 y_n+\cdots\right]$$

Here,

$$h=10, h^2=100$$
$$=\frac{1}{100}\left[-1+(-0.24)+\frac{11}{12}(-0.23)\right]$$
$$=-0.01429$$

Hence,

$$\boxed{\left(\frac{d^2y}{dx^2}\right)_{x=1981}=-0.01429}$$

Least Squares Curve Fitting Procedure

The method of **Least Squares** is probably the most systematic procedure to fit a unique curve through the given data points and is widely used in practical computations. It can also be easily implemented on a digital computer.

Let the set of data points be

$$(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$$

and let the curve

$$y=f(x)$$

be fitted to these data.
If e_i is the error of approximation at $x = x_i$, then

$$\begin{aligned} e_1 &= y_1 - f(x_1) \\ e_2 &= y_2 - f(x_2) \\ &\vdots \\ e_n &= y_n - f(x_n) \end{aligned}$$

Hence,

$$S = [y_1 - f(x_1)]^2 + [y_2 - f(x_2)]^2 + \cdots + [y_n - f(x_n)]^2$$

or

$$S = e_1^2 + e_2^2 + \cdots + e_n^2$$

The method of least squares consists of minimizing the quantity S.

Derivation of Normal Equations for Least Square Approximation

Let the straight line to be fitted be

$$y = a_0 + a_1 x$$

Then,

$$S = [y_1 - (a_0 + a_1 x_1)]^2 + [y_2 - (a_0 + a_1 x_2)]^2 + \cdots + [y_m - (a_0 + a_1 x_m)]^2$$

For minimum error,

$$\frac{dS}{da_0} = 0$$

and

$$\frac{dS}{da_1} = 0$$

Differentiating,

$$\frac{dS}{da_0} = -2[y_1 - (a_0 + a_1 x_1)] - 2[y_2 - (a_0 + a_1 x_2)] - \cdots - 2[y_m - (a_0 + a_1 x_m)] = 0$$

Similarly,

$$\frac{dS}{da_1} = -2x_1[y_1 - (a_0 + a_1 x_1)] - 2x_2[y_2 - (a_0 + a_1 x_2)] - \cdots - 2x_m[y_m - (a_0 + a_1 x_m)] = 0$$

These simplify to

First Normal Equation

$$m a_0 + a_1 \sum_{i=1}^{m} x_i = \sum_{i=1}^{m} y_i$$

Second Normal Equation

$$a_0 \sum_{i=1}^{m} x_i + a_1 \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i \, y_i$$

Equations (1) and (2) are called the **Normal Equations** for the least squares straight-line fit.

Least Squares Method

Example 1

Fit a Straight Line to the Following Data and Find $y$when $x = 6$

Given Data

x	**0**	**5**	**10**	**15**	**20**	**25**
y	12	15	17	22	24	30

Solution

Let the straight line be

$$y = a_0 + a_1 x$$

Construct the table:

x_i	y_i	x_i^2	$x_i y_i$
0	12	0	0
5	15	25	75
10	17	100	170
15	22	225	330
20	24	400	480
25	30	625	750

$$\sum x_i = 75, \sum y_i = 120$$
$$\sum x_i^2 = 1375, \sum x_i y_i = 1805$$

Here,

$$m = 6$$

First Normal Equation

$$ma_0 + a_1\sum x_i = \sum y_i$$
$$6a_0 + 75a_1 = 120 \qquad (1)$$

Second Normal Equation

$$a_0\sum x_i + a_1\sum x_i^2 = \sum x_i y_i$$
$$75a_0 + 1375a_1 = 1805 \qquad (2)$$

Multiply (1) by 75:

$$450a_0 + 5625a_1 = 9000$$

Multiply (2) by 6:

$$450a_0 + 8250a_1 = 10830$$

Subtracting,

$$2625a_1 = 1830$$
$$a_1 = \frac{1830}{2625} = 0.697$$

Substitute into (1):

$$6a_0 + 75(0.697) = 120$$
$$6a_0 = 67.725$$
$$a_0 = \frac{67.725}{6} = 11.28$$

Hence the required straight line is

$$\boxed{y = 11.28 + 0.697x}$$

For $x = 6$,

$$y = 11.28 + 0.697(6)$$
$$= 11.28 + 4.182$$
$$\boxed{y = 15.462}$$

Example 2

The Following Table Gives the Temperature T(°C) and the Length l(mm) of a Heated Rod. If

$$l = a_0 + a_1T,$$

find the best values of $a_0$and a_1.

Given Data

T(°C)	**20**	**30**	**40**	**50**	**60**	**70**
l(mm)	800.3	800.4	800.6	800.7	800.9	801.0

Solution

Let

$$l = a_0 + a_1 T$$

Prepare the table.

T_i	l_i	T_i^2	$T_i l_i$
20	800.3	400	16006
30	800.4	900	24012
40	800.6	1600	32024
50	800.7	2500	40035
60	800.9	3600	48054
70	801.0	4900	56070

$$\sum T_i = 270$$
$$\sum l_i = 4803.9$$
$$\sum T_i^2 = 13900$$
$$\sum T_i l_i = 216201$$

First Normal Equation

$$6a_0 + 270a_1 = 4803.9 \qquad (1)$$

Second Normal Equation

$$270a_0 + 13900a_1 = 216201 \qquad (2)$$

Multiply (1) by 270:

$$1620a_0 + 72900a_1 = 1297053$$

Multiply (2) by 6:

$$1620a_0 + 83400a_1 = 1297206$$

Subtracting,

$$10500a_1 = 153$$
$$a_1 = \frac{153}{10500} = 0.01457 \approx 0.0146$$

Substitute into (1):

$$6a_0 + 270(0.0146) = 4803.9$$
$$6a_0 + 3.942 = 4803.9$$
$$6a_0 = 4799.958$$
$$a_0 \approx 799.99 \approx 800$$

Hence,

$$\boxed{a_0 \approx 800, a_1 \approx 0.0146}$$

Example 3

Certain Experimental Values of $x$and $y$are Given Below.

If

$$y = a_0 + a_1 x,$$

find the approximate values of $a_0$and a_1.

Given Data

x	**0**	**2**	**5**	**7**
y	-1	5	12	20

Solution

Let

$$y = a_0 + a_1 x$$

Construct the table.

x_i	y_i	x_i^2	$x_i y_i$
0	-1	0	0
2	5	4	10
5	12	25	60
7	20	49	140

$$\sum x_i = 14$$
$$\sum y_i = 36$$
$$\sum x_i^2 = 78$$
$$\sum x_i y_i = 210$$

Here,

$$m = 4$$

First Normal Equation

$$4a_0 + 14a_1 = 36 \qquad (1)$$

Second Normal Equation

$$14a_0 + 78a_1 = 210 \qquad (2)$$

Multiply (1) by 14:

$$56a_0 + 196a_1 = 504$$

Multiply (2) by 4:

$$56a_0 + 312a_1 = 840$$

Subtracting,

$$116a_1 = 336$$
$$a_1 = \frac{336}{116} = 2.89$$

Substitute into (1):

$$4a_0 + 14(2.89) = 36$$
$$4a_0 = 36 - 40.46 = -4.46$$
$$a_0 = -1.115$$

Hence,

$$\boxed{a_0 = -1.115, a_1 = 2.89}$$

Example 4

By the Method of Least Squares, Find the Straight Line that Best Fits the Following Data.

Given Data

x	**1**	**2**	**3**	**4**	**5**
y	14	27	40	55	68

Solution

Let the straight line be

$$y = a_0 + a_1 x$$

Construct the table.

x_i	y_i	x_i^2	$x_i y_i$
1	14	1	14
2	27	4	54

x_i	y_i	x_i^2	$x_i y_i$
3	40	9	120
4	55	16	220
5	68	25	340

$$\sum x_i = 15$$
$$\sum y_i = 204$$
$$\sum x_i^2 = 55$$
$$\sum x_i y_i = 748$$

From (1) and (2):

$$75 \times (1) \Rightarrow 4500a_0 + 5625a_1 = 9000$$
$$6 \times (2) \Rightarrow 4500a_0 + 8250a_1 = 10830$$

Subtracting,

$$2625a_1 = 1830$$
$$a_1 = \frac{1830}{2625} = 0.697$$

Substituting in (1),

$$6a_0 + 75(0.697) = 120$$
$$6a_0 = 120 - 52.275 = 67.725$$
$$a_0 = \frac{67.725}{6} = 11.28$$

Hence, the straight line passing through the given points is

$$\boxed{y = 11.28 + 0.697x}$$

For $x = 6$,

$$y = 11.28 + 0.697(6) = 15.462 \approx 15.468$$

---

2. The table below gives the temperature T(in °C) and length l(in mm) of a heated rod. If

$$l = a_0 + a_1 T,$$

find the best values of $a_0$and a_1.

Given Data

T**(°C)**	l**(mm)**
20	800.3
30	800.4

T(°C)	l(mm)
40	800.6
50	800.7
60	800.9
70	801.0

Solution

Let the straight line be

$$l = a_0 + a_1T$$

T_i	l_i	T_i^2	T_il_i
20	800.3	400	16006
30	800.4	900	24012
40	800.6	1600	32024
50	800.7	2500	40035
60	800.9	3600	48054
70	801.0	4900	56070

$$\sum T_i = 270, \sum l_i = 4803.9,$$
$$\sum T_i^2 = 13900, \sum T_il_i = 216201$$

First Normal Equation

$$6a_0 + 270a_1 = 4803.9 \qquad (1)$$

Second Normal Equation

$$270a_0 + 13900a_1 = 216201 \qquad (2)$$

From (1) and (2),

$$(1) \times 270:$$
$$1620a_0 + 72900a_1 = 1297053$$
$$(2) \times 6:$$
$$1620a_0 + 83400a_1 = 1297206$$

Subtracting,

$$10500a_1 = 153$$
$$a_1 = \frac{153}{10500} = 0.01457 \approx 0.0146$$

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Substituting in (1),

$$6a_0 + 270(0.0146) = 4803.9$$
$$6a_0 = 4803.9 - 3.942 = 4799.958$$
$$a_0 = \frac{4799.958}{6} \approx 799.99 \approx 800$$

Hence,

$$\boxed{a_0 \approx 800, a_1 \approx 0.0146}$$

Y=800+0.0146X

---

3. Certain experimental values of $x$and $y$are given below. If

$$y = a_0 + a_1 x,$$

find the approximate values of $a_0$and a_1.

Given Data

x	**0**	**2**	**5**	**7**
y	-1	5	12	20

Solution

Let the straight line be

$$y = a_0 + a_1 x$$

x_i	y_i	x_i^2	$x_i y_i$
0	-1	0	0
2	5	4	10
5	12	25	60
7	20	49	140

$$\sum x_i = 14, \sum y_i = 36,$$
$$\sum x_i^2 = 78, \sum x_i y_i = 210$$

Here,

$$m = 4$$

First Normal Equation

$$4a_0 + 14a_1 = 36 \qquad (1)$$

Second Normal Equation

$$14a_0 + 78a_1 = 210 \qquad (2)$$

From (1) and (2),

$$(1) \times 14:$$
$$56a_0 + 196a_1 = 504$$
$$(2) \times 4:$$
$$56a_0 + 312a_1 = 840$$

Subtracting,

$$116a_1 = 336$$
$$a_1 = \frac{336}{116} \approx 2.89$$

Substituting in (1),

$$4a_0 + 14(2.89) = 36$$
$$4a_0 = 36 - 40.46 = -4.46$$
$$a_0 = -1.115$$

Hence,

$$\boxed{a_0 \approx -1.115, a_1 \approx 2.89}$$
Y=$-1.115 + 2.89X$

---

4. By the method of Least Squares, find the straight line that best fits the following data.

Given Data

x	**1**	**2**	**3**	**4**	**5**
y	14	27	40	55	68

Solution

Let the straight line be

$$y = a_0 + a_1 x$$

x_i	y_i	x_i^2	$x_i y_i$
1	14	1	14
2	27	4	54
3	40	9	120
4	55	16	220
5	68	25	340

$$\sum x_i = 15, \sum y_i = 204,$$
$$\sum x_i^2 = 55, \sum x_i y_i = 748$$

Here,

$$m = 5$$

First Normal Equation

$$5a_0 + 15a_1 = 204 \qquad (1)$$

Second Normal Equation

$$15a_0 + 55a_1 = 748 \qquad (2)$$

From (1) and (2),

$$(1) \times 15:$$
$$75a_0 + 225a_1 = 3060$$
$$(2) \times 5:$$
$$75a_0 + 275a_1 = 3740$$

Subtracting,

$$50a_1 = 680$$
$$a_1 = 13.6$$

Substituting in (1),

$$5a_0 + 15(13.6) = 204$$
$$5a_0 = 0$$
$$a_0 = 0$$

Hence, the required straight line is

$$\boxed{y = 13.6x}$$

Fit a Polynomial of Second Degree:

Let

$$\boxed{y = a_0 + a_1x + a_2x^2}$$

(parabola)

Normal Equations

1. First Normal Equation

$$ma_0 + \sum x_i a_1 + \sum x_i^2 a_2 = \sum y_i$$

2. Second Normal Equation

$$\sum x_i a_0 + \sum x_i^2 a_1 + \sum x_i^3 a_2 = \sum x_i y_i$$

3. Third Normal Equation

$$\sum x_i^2 a_0 + \sum x_i^3 a_1 + \sum x_i^4 a_2 = \sum x_i^2 y_i$$

Problem 1

Find the best values of a_0, a_1, a_2 so that the parabola

$$y = a_0 + a_1x + a_2x^2$$

fits the data:

x	**1.0**	**1.5**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
y	1.1	1.2	1.5	2.6	2.8	3.3	4.1

SOLUTION

Computation Table

x_i	y_i	x_i^2	x_i^3	x_i^4	x_iy_i	$x_i^2y_i$
1.0	1.1	1.000	1.000	1.000	1.10	1.10
1.5	1.2	2.250	3.375	5.0625	1.80	2.70
2.0	1.5	4.000	8.000	16.000	3.00	6.00
2.5	2.6	6.250	15.625	39.0625	6.50	16.25
3.0	2.8	9.000	27.000	81.000	8.40	25.20
3.5	3.3	12.250	42.875	150.0625	11.55	40.425
4.0	4.1	16.000	64.000	256.000	16.40	65.60
Σ	**16.6**	**50.75**	**161.875**	**548.125**	**48.75**	**157.275**

Normal Equations

Using the above totals,

First Normal Equation

$$7a_0 + 17.5a_1 + 50.75a_2 = 16.6$$

Second Normal Equation

$$17.5a_0 + 50.75a_1 + 161.875a_2 = 48.75$$

Third Normal Equation

$$50.75a_0 + 161.875a_1 + 548.125a_2 = 157.275$$

Solving the Equations

After elimination,

$$49a_1 + 245a_2 = 50.750$$
$$245a_1 + 1261.75a_2 = 285.44$$

Multiplying the first equation by 5,

$$245a_1 + 1225a_2 = 253.75$$

Subtracting,

$$36.75a_2 = 4.69$$
$$\boxed{a_2 = 0.1276}$$

Now,

$$49a_1 + 245(0.1276) = 50.750$$
$$49a_1 = 19.488$$
$$\boxed{a_1 = 0.3929}$$

Using the first normal equation,

$$7a_0 = 16.6 - 17.5(0.3929) - 50.75(0.1276)$$
$$7a_0 = 3.1976$$
$$\boxed{a_0 = 0.4568}$$

Best-Fit Polynomial

$$\boxed{y = 0.4568 + 0.3929x + 0.1276x^2}$$

This is the required second-degree polynomial (parabola) that best fits the given data.

2. Fit a Polynomial of the Second Degree

Fit a polynomial of the second degree to the data given in the following table.

x	**0.0**	**1.0**	**2.0**
y	1.0	6.0	17.0

Assume

$$y = a_0 + a_1x + a_2x^2$$

Computation Table

x_i	y_i	x_i^2	x_i^3	x_i^4	x_iy_i	$x_i^2y_i$
0	1	0	0	0	0	0

x_i	y_i	x_i^2	x_i^3	x_i^4	x_iy_i	$x_i^2y_i$
1	6	1	1	1	6	6
2	17	4	8	16	34	68
Σ	**24**	**5**	**9**	**17**	**40**	**74**

Normal Equations

First Normal Equation

$$3a_0 + 3a_1 + 5a_2 = 24(1)$$

Second Normal Equation

$$3a_0 + 5a_1 + 9a_2 = 40(2)$$

Third Normal Equation

$$5a_0 + 9a_1 + 17a_2 = 74(3)$$

Solution

From (2) – (1),

$$2a_1 + 4a_2 = 16$$
$$a_1 + 2a_2 = 8(4)$$

Multiply (2) by 5:

$$15a_0 + 25a_1 + 45a_2 = 200$$

Multiply (3) by 3:

$$15a_0 + 27a_1 + 51a_2 = 222$$

Subtracting,

$$2a_1 + 6a_2 = 22$$
$$a_1 + 3a_2 = 11(5)$$

Subtract (4) from (5):

$$a_2 = 3$$

Substitute into (4):

$$a_1 + 2(3) = 8$$
$$a_1 = 2$$

Substitute $a_1 = 2$ and $a_2 = 3$ into (1):

$$3a_0 + 3(2) + 5(3) = 24$$
$$3a_0 = 3$$
$$a_0 = 1$$

Required Polynomial

$$\boxed{y = a_0 + a_1x + a_2x^2}$$
$$\boxed{y = 1 + 2x + 3x^2}$$

Problem 3

Find the best values of a, b, and c so that the parabola

$$y = a + bx + cx^2$$

fits the following data.

x	**0**	**1**	**2**	**3**	**4**
y	1	0	3	10	21

Computation Table

x_i	y_i	x_i^2	x_i^3	x_i^4	x_iy_i	$x_i^2y_i$
0	1	0	0	0	0	0
1	0	1	1	1	0	0
2	3	4	8	16	6	12
3	10	9	27	81	30	90
4	21	16	64	256	84	336
Σ	**35**	**30**	**100**	**354**	**120**	**438**

Hence, the required polynomial is assumed as

$$\boxed{y = a + bx + cx^2}$$

Normal Equations

First Normal Equation

$$5a + 10b + 30c = 35(1)$$

Second Normal Equation

$$10a + 30b + 100c = 120(2)$$

Third Normal Equation

$$30a + 100b + 354c = 438(3)$$

Solution

From equations (1) and (2):

Multiply (1) by 10:

$$50a + 100b + 300c = 350$$

Multiply (2) by 5:

$$50a + 150b + 500c = 600$$

Subtracting,

$$-50b - 200c = -250$$
$$b + 4c = 5(4)$$

From equations (2) and (3):

Multiply (2) by 10:

$$100a + 300b + 1000c = 1200$$

Multiply (3) by 3:

$$90a + 300b + 1062c = 1314$$

Subtracting,

$$10a - 62c = -114$$

Using the elimination shown in the notes,

$$100b + 540c = 780$$
$$5b + 27c = 39(5)$$

From equations (4) and (5):

Multiply (4) by 5:

$$5b + 20c = 25$$

Subtract (5):

$$(5b + 27c) - (5b + 20c) = 39 - 25$$
$$7c = 14$$
$$\boxed{c = 2}$$

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Substitute into equation (4):

$$b + 4(2) = 5$$
$$b = -3$$

Substitute $b = -3$and $c = 2$ into equation (1):

$$5a + 10(-3) + 30(2) = 35$$
$$5a - 30 + 60 = 35$$
$$5a = 5$$
$$\boxed{a = 1}$$

Required Polynomial

$$\boxed{y = a + bx + cx^2}$$
$$\boxed{y = 1 - 3x + 2x^2}$$

PRACTICE QUESTIONS

Question 1

Fit a second-degree polynomial to the following data:

x	0	1	2	3	4
y	1	2	5	10	17

Question 2

Fit a parabola

y=a+bx+cx
2

to the following observations.

x	1	2	3	4	5
y	2	5	10	17	26

Question 3

Fit a second-degree polynomial for the data.

x 0 2 4 6 8

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y	3	7	15	27	43

Question 4

Fit a parabola using the least squares method.

x	1	2	3	4
y	4	7	12	19

PROBLEM 5.

Using the second-order Runge–Kutta method, find $y(0.2)$if

$$\frac{dy}{dx} = x + y, y(0) = 1,$$

taking $h = 0.2$.

PROBLEM 6

Using the Trapezoidal Rule, evaluate

$$\int_1^3 (x^3 + 2x)\, dx$$

taking $n = 4$.

PROBLEM 7. Using Simpson's $\frac{1}{3}$Rule, evaluate

$$\int_0^2 (x^3 + 1)\, dx$$

taking four equal intervals.

PROBLEM 8. Using Simpson's $\frac{3}{8}$Rule, evaluate

$$\int_0^3 (x^2 + 2x)\, dx.$$

PROBLEM 9. Using Newton's Forward Interpolation Formula, find $y(23)$.

x	**20**	**22**	**24**	**26**	**28**
y	142	183	228	277	330

PROBLEM 10. Using Newton's Forward Interpolation, find $y(18)$.

x	**15**	**17**	**19**	**21**	**23**
y	225	289	361	441	529

PROBLEM 11. Using Newton's Backward Interpolation Formula, find $y(37)$.

x	**25**	**30**	**35**	**40**	**45**
y	112	148	190	238	292

UNIT–3

Solution of Initial Value Problem to Ordinary Differential Equations

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UNIT–3

Solution of Initial Value Problem to Ordinary Differential Equations

Numerical Solution of Ordinary Differential Equations

Many problems in Science and Engineering can be formulated into ordinary differential equations. The analytical methods of solving differential equations are applicable only to a selected class of differential equations. Quite often, equations occurring in physical problems do not belong to any of these simple types. Hence, numerical methods are used for solving such differential equations.

The following numerical methods are discussed:

1. Taylor's Series Method
2. Picard's Method
3. Euler's Method
4. Runge–Kutta Method

---

1. Taylor's Series Method

To find the numerical solution of the differential equation

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition

$$y(x_0) = y_0$$

Method

Differentiate the given equation repeatedly to obtain

$$y', y'', y''', y^{(4)}, \dots$$

Substitute the values at

$$x = x_0, y = y_0$$

into Taylor's expansion:

$$\boxed{y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + \cdots}$$

Thus, Taylor's series gives the approximate value of $y$for nearby values of x.

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Example 1

Solve

$$\frac{dy}{dx} = xy + 1, y(0) = 1$$

using Taylor's method and compute $y(0.1)$.

Solution

Given

$$y' = xy + 1$$

Differentiate successively:

$$\begin{aligned} y'' &= y + xy' \\ y''' &= 2y' + xy'' \\ y^{(4)} &= 3y'' + xy''' \end{aligned}$$

At

$$x = 0, y = 1$$

we obtain

$$\begin{aligned} y_0' &= 1 \\ y_0'' &= 1 \\ y_0''' &= 2 \\ y_0^{(4)} &= 3 \end{aligned}$$

Hence,

$$y(x) \quad = 1 + x + \frac{x^2}{2!} + \frac{2x^3}{3!} + \frac{3x^4}{4!} + \cdots$$

At $x = 0.1$,

$$\begin{aligned} y(0.1) \quad &= 1 + 0.1 + \frac{0.01}{2} + \frac{2(0.001)}{6} + \frac{3(0.0001)}{24} \\ &= 1 + 0.1 + 0.005 + 0.000333 + 0.0000125 \\ &\approx \boxed{1.1053} \end{aligned}$$

2. Picard's Method of Successive Approximation

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

Method

Integrate both sides:

$$dy = f(x,y)\,dx$$

Applying the limits,

$$\boxed{y(x) = y_0 + \int_{x_0}^{x} f(x,y)\,dx}$$

Since $y$appears inside the integral, successive approximations are used.
The Picard iteration formula is

$$\boxed{y_{n+1} = y_0 + \int_{x_0}^{x} f(t,y_n)\,dt}$$

where

First approximation

$$y_1 = y_0 + \int_{x_0}^{x} f(t,y_0)\,dt$$

Second approximation

$$y_2 = y_0 + \int_{x_0}^{x} f(t,y_1)\,dt$$

Similarly,

$$y_3,\ y_4, \dots$$

are obtained until the desired accuracy is reached.

Example 1

Using Picard's method, find the solution of

$$\frac{dy}{dx} = 1 + xy, y(0) = 0$$

up to the third approximation.

Solution

Picard iteration formula:

$$y_{n+1} = 0 + \int_{0}^{x} (1 + t\ y_n)\,dt$$

First approximation

Take

$$y_0 = 0$$

Then

$$\begin{aligned} y_1 &= \int_0^x 1\,dt \\ &= x \end{aligned}$$

Second approximation

$$\begin{aligned} y_2 &= \int_0^x (1 + t\ y_1)\,dt \\ &= \int_0^x (1 + t^2)\,dt \\ &= x + \frac{x^3}{3} \end{aligned}$$

Third approximation

$$\begin{aligned} y_3 &= \int_0^x \left(1 + t\left(t+\frac{t^3}{3}\right)\right) dt \\ &= \int_0^x \left(1 + t^2 + \frac{t^4}{3}\right) dt \\ &= x + \frac{x^3}{3} + \frac{x^5}{15} \end{aligned}$$

Hence,

$$\boxed{y_3 = x + \frac{x^3}{3} + \frac{x^5}{15}}$$

EXAMPLE 2

Using Picard's method, find the value of y at $x = 0.4$ for

$$\frac{dy}{dx} = x^2 + y^2, y(0) = 0.$$

Solution:

Given

$$\frac{dy}{dx} = x^2 + y^2, y(0) = 0, x_0 = 0$$
$$[\therefore \ y(x_0) = y_0]$$

We know that Picard's iteration formula is

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx (i)$$

Let

1st approximation

Put $n = 1$in Eq. (i), we get

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

Here,

$$f(x, y) = x^2 + y^2$$
$$f(x, y_0) = x^2$$

Hence,

$$y_1 = 0 + \int_0^x x^2 \ dx$$
$$= \left[\frac{x^3}{3}\right]_0^x$$
$$= \frac{x^3}{3}$$

Also,

$$f(x, y_1) = x^2 + \left(\frac{x^3}{3}\right)^2 = x^2 + \frac{x^6}{9}$$

2nd approximation

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$
$$= 0 + \int_0^x \left(x^2 + \frac{x^6}{9}\right) dx$$
$$= \frac{x^3}{3} + \frac{x^7}{63}$$

Calculation of $y_3$is difficult and hence the approximate value is taken as y_2.
For $x = 0.4$,

$$y_2 = \frac{(0.4)^3}{3} + \frac{(0.4)^7}{63}$$
$$= 0.02136 + 0.00003$$
$$\boxed{\approx 0.02139}$$

Example 3

Find the value of y for $x = 0.25,\ 0.5,\ 1$ by Picard's method

Given that

$$\frac{dy}{dx} = \frac{x^2}{y^2 + 1}, y(0) = 0$$

Solution:

Given

$$\frac{dy}{dx} = \frac{x^2}{y^2 + 1}, y(0) = 0, x_0 = 0$$

We know Picard's iteration formula

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$$

1st approximation

$$y_1 = y_0 + \int_0^x f(x, y_0)\, dx$$

Since

$$f(x, y_0) = \frac{x^2}{0^2 + 1} = x^2$$

Therefore,

$$y_1 = 0 + \int_0^x x^2\ dx$$
$$= \left[\frac{x^3}{3}\right]_0^x$$
$$= \frac{x^3}{3}$$

Now,

$$f(x, y_1) = \frac{x^2}{\left(\frac{x^3}{3}\right)^2 + 1} = \frac{x^2}{\frac{x^6}{9} + 1} = \frac{9x^2}{x^6 + 9}$$

2nd approximation

$$y_2 = y_0 + \int_0^x f(x, y_1)\, dx$$
$$= 0 + \int_0^x \frac{9x^2}{x^6 + 9}\, dx$$

Let

$$x^3 = t$$

Then

$$3x^2\, dx = dt$$

When $x = 0,\ t = 0$
When $x = x,\ t = x^3$
Hence,

$$= 3\int_0^{x^3} \frac{dt}{t^2 + 3^2}$$

Using

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

we get

$$= 3\left[\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right)\right]_0^{x^3}$$
$$= \tan^{-1}\left(\frac{x^3}{3}\right)$$

Therefore,

$$\boxed{y_2 = \tan^{-1}\left(\frac{x^3}{3}\right)}$$

Hence, we use $y_2$as the value of y.

Values

$$y(0.25) = \tan^{-1}\left(\frac{(0.25)^3}{3}\right) \approx 0.0052$$
$$y(0.5) = \tan^{-1}\left(\frac{(0.5)^3}{3}\right) \approx 0.0416$$
$$y(1) = \tan^{-1}\left(\frac{1}{3}\right) \approx 0.3218$$

Euler's Method

We have so far discussed the methods which yield the solution of the differential equation in the form of a function.
We will now describe the method which gives the solution in the form of a set of tabulated values. Suppose we wish to solve the equation

$$\frac{dy}{dx} = f(x,y)$$

given

$$y(x_0) = y_0.$$

In this method, starting from x_0, we take equally spaced values of x, say

$$x_0,\ x_1,\ x_2,\ \ldots,\ x_n$$

with step length h.
Therefore,

$$x_n = x_{n-1} + h, n = 1,2,\ldots$$

The successive approximation values of $y$are given by

$$y(x_1) = y_1 = y_0 + h\,f(x_0, y_0)$$

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$$y(x_2) = y_2 = y_1 + h\,f(x_1, y_1)$$
$$y(x_3) = y_3 = y_2 + h\,f(x_2, y_2)$$
$$\vdots$$
$$\boxed{y_{n+1} = y_n + h\,f(x_n, y_n), n = 0,1,2, \dots}$$

This is known as **Euler's algorithm** and can be used to evaluate

$$y_1,\ y_2,\ \dots,\ y_n,$$

i.e.,

$$y(x_1),\ y(x_2),\ \dots,\ y(x_n),$$

starting from the initial condition.

Example 1
Using Euler's method solve for $y$at $x = 2$from

$$\frac{dy}{dx} = 3x^2 + 1, y(1) = 2,$$

taking step size

1. $h = 0.5$
2. $h = 0.25$

Solution
Given

$$\frac{dy}{dx} = 3x^2 + 1$$
$$f(x, y) = 3x^2 + 1, y(1) = 2$$

Here,

$$x_0 = 1, y_0 = 2.$$

We know Euler's algorithm is

$$y_{n+1} = y_n + h\,f(x_n, y_n) \dots \dots \dots \dots.\ (1)$$

Clearly,

$$x_n = x_0 + nh, x_0 = 1.$$

Taking $n = 0$in Eq. (1), we have

$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 2 + 0.5\,[\,3(1)^2 + 1\,] \\ &= 2 + 0.5(4) \\ &= 4. \end{aligned}$$

Also,

$$f(1,2) = 3(1)^2 + 1 = 4.$$

Taking $n = 1$in Eq. (1), we have

$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$= 4 + 0.5\,[\,3(1.5)^2 + 1\,] = 7.8750$$

Taking $n = 2$in Eq. (1), we have

$$\begin{aligned} y_3 &= y_2 + h\,f(x_2, y_2) \\ &= 7.8750 + 0.5(3(2)^2 + 1) \\ &= 7.8750 + 0.5(13) \\ &= 14.375 \end{aligned}$$

Given $h = 0.25$

$$x_0 = 1, x_1 = 1.25, x_2 = 1.5, x_3 = 1.75$$

Taking $n = 0$in Eq. (1), we have

$$\begin{aligned} y_1 &= y_0 + h\,f(x_0, y_0) \\ &= 2 + 0.25(4) \\ y_1 &= 3 \end{aligned}$$

Taking $n = 1$in Eq. (1), we have

$$\begin{aligned} y_2 &= y_1 + h\,f(x_1, y_1) \\ &= 3 + 0.25(3(1.25)^2 + 1) \\ &= 3 + 0.25(5.6875) \\ y_2 &= 4.4219 \end{aligned}$$

Taking $n = 2$in Eq. (1), we have

$$\begin{aligned} y_3 &= y_2 + h\,f(x_2, y_2) \\ &= 4.4219 + 0.25(7.75) \\ &= 6.3594 \end{aligned}$$

Example 2

Using Euler's method to find $y(0.1)$& $y(0.2)$given

$$y' = (x^3 + xy)e^{-x}$$

Solution

Given

$$\frac{dy}{dx} = (x^3 + xy)e^{-x}$$

Here,

$$x_0 = 0, y_0 = 1, h = 0.1$$
$$x_0 = 0, x_1 = 0.1, x_2 = 0.2, x_3 = 0.3$$

We know Euler's algorithm

$$y_{n+1} = y_n + h\, f(x_n, y_n)(1)$$

Taking $n = 0$in Eq. (1), we have

$$\begin{aligned} y_1 &= y_0 + h\, f(x_0, y_0) \\ &= 1 + 0.1(0) \\ &= 1 \end{aligned}$$

Also,

$$f(x_0, y_0) = (0^3 + 0(1))e^{-0} = 0$$

Taking $n = 1$in Eq. (1), we have

$$\begin{aligned} y_2 &= y_1 + h\, f(x_1, y_1) \\ &= 1 + 0.1(0.0914) \\ &= 1.0091 \end{aligned}$$

where

$$\begin{aligned} f(x_1, y_1) &= ((0.1)^3 + (0.1)(1))e^{-0.1} \\ &= (0.101)e^{-0.1} \\ &= 0.0914 \end{aligned}$$

Taking $n = 2$in Eq. (1), we have

$$\begin{aligned} y_3 &= y_2 + h\, f(x_2, y_2) \\ &= 1.0091 + 0.1(0.1733) \\ &\approx 1.0264 \end{aligned}$$

where

$$f(x_2, y_2) = ((0.2)^3 + (0.2)(1.0091))e^{-0.2}$$
$$\approx 0.1733$$

Hence,

$$\boxed{y(0.1) \approx 1.0000, y(0.2) \approx 1.0091}$$

(The next iteration gives $y(0.3) \approx 1.0264$.)

Modified Euler's Theorem:

Suppose we have to solve

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$$

to find

$$y(x_1) = y_1, \text{at } x = x_1 = x_0 + h.$$

Compute

$$y_1^{(0)} = y_0 + h\,f(x_0, y_0)$$
$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$y_1^{(n+1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(n)})\right], n = 0,1,2, \dots$$

where $y_1^{(n)}$ is the n^{th} approximation to y_1.

If two successive values $y_1^{(n)}$ and $y_1^{(n+1)}$ are almost equal, we stop there and take

$$y_1 = y_1^{(n)}.$$

Now we have

$$\frac{dy}{dx} = f(x, y)$$

with $y = y_1 \quad x = x_1$.

To get

$$y_2 = y(x_2),$$

we use the above procedure again.

---

1. **Solve**

$$y' = y + e^x, y(0) = 0,$$

for

$$x = 0.2,\ 0.4.$$

Slution:
Given

$$f(x,y) = y + e^x, x_0 = 0, y_0 = 0,$$
$$h = 0.2.$$

To find $\boldsymbol{y_1 = y(x_1) = y(0.2)}$
Predictor:

$$y_1^{(0)} = y_0 + h\, f(x_0, y_0)$$

Since

$$f(0,0) = 1,$$
$$y_1^{(0)} = 0 + 0.2(1) = 0.2.$$

Corrector:

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$
$$= 0 + \frac{0.2}{2}[\,1 + 1.4214\,]$$
$$= \frac{0.2}{2}(2.4214) = 0.2421.$$

Here,

$$f(0.2,0.2) = 0.2 + e^{0.2} = 1.4214.$$

Next correction:

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= 0 + \frac{0.2}{2}[\,1 + 1.4635\,]$$
$$= \frac{0.2}{2}(2.4635) = 0.2464.$$

where

$$f(0.2{,}0.2421) = 0.2421 + e^{0.2} = 1.4635.$$

Next,

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

$$= 0 + \frac{0.2}{2}[1{+}0.2464{+}e^{0.2}].$$

=20.2(2.4678)=0.2468.

Since successive values are almost equal,

$$\boxed{y(0.2) \approx 0.2468.}$$

y1(4)=y0+2h[f(x0,y0)+f(x1,y1(3))]

$$= 0 + \frac{0.2}{2}[1{+}0.2468{+}e^{0.2}]$$

$$= 0.2468$$

Since we find $y_1^{(3)}$and $y_1^{(4)}$are equal,
we take

$$y_1 = y(x_1) = y(0.2) = 0.2468.$$

Now we find y_2.
We take

$$y_1 = y(x_1) = y(0.2) = 0.2468,$$
$$x_1 = 0.2, y_1 = 0.2468, x_2 = 0.4, h = 0.2.$$

To find y_2

$$y_2^{(0)} = y(x_2) = y(0.4)$$
$$= y_1 + h\, f(x_1, y_1)$$
$$= 0.2468 + 0.2(0.2468 + 1.2214)$$
$$y_2^{(0)} = 0.5404$$

where

$$f(x_1, y_1) = 0.2468 + e^{0.2} = 1.4682.$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(0)}\right)\right]$$

$$= 0.2468 + \frac{0.2}{2}[1.4682{+}0.5404{+}1.4918]$$

$$= 0.5968$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$
$$= 0.2468 + \frac{0.2}{2}[1.4682+0.5968+1.4918]$$
$$= 0.6035$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$
$$= 0.2468 + 0.1[1.4682+0.6035+1.4918]$$
$$= 0.6031$$

$$y_2^{(4)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(3)})\right]$$
$$= 0.2468 + \frac{0.2}{2}[1.4682+0.6031+1.4918]$$
$$= 0.6031$$

Since

$$y_2^{(3)} = y_2^{(4)} = 0.6031,$$

we have

$$\boxed{y_2 = y(0.4) = 0.6031.}$$

Using **Modified Euler's Method**, compute

$$y(0.02) \text{ and } y(0.04)$$

given

$$\frac{dy}{dx} = x^2 + y, y(0) = 1.$$

Given

$$f(x, y) = x^2 + y$$
$$x_0 = 0, y_0 = 1, h = 0.02$$

Step 1: Compute $\boldsymbol{y(0.02)}$
At

$$x_0 = 0, y_0 = 1$$

First,

$$f(x_0, y_0) = 0^2 + 1 = 1$$

Predictor

$$y_1^{(0)} = 1 + 0.02(1)$$
$$= 1.02$$

First Corrector

At

$$x_1 = 0.02$$
$$f(x_1, y_1^{(0)}) = (0.02)^2 + 1.02$$
$$= 0.0004 + 1.02$$
$$= 1.0204$$

Hence,

$$y_1^{(1)} = 1 + \frac{0.02}{2}(1 + 1.0204)$$
$$= 1 + 0.01(2.0204)$$
$$= 1.020204$$

Second Corrector

$$f(0.02, 1.020204) = 0.0004 + 1.020204$$
$$= 1.020604$$

Therefore,

$$y_1^{(2)} = 1 + 0.01(1 + 1.020604)$$
$$= 1.02020604$$

Third Corrector

$$f(0.02, 1.02020604) = 1.02060604$$
$$y_1^{(3)} = 1 + 0.01(1 + 1.02060604)$$
$$= 1.02020606$$

Since the values agree up to six decimal places,

$$\boxed{y(0.02) = 1.020206}$$

Step 2: Compute $\boldsymbol{y(0.04)}$

Now,

$$x_1 = 0.02, y_1 = 1.020206$$

First,

$$f(x_1, y_1) = (0.02)^2 + 1.020206$$
$$= 0.0004 + 1.020206$$
$$= 1.020606$$

Predictor

$$y_2^{(0)} = 1.020206 + 0.02(1.020606)$$
$$= 1.020206 + 0.02041212$$
$$= 1.04061812$$

First Corrector

At

$$x_2 = 0.04$$
$$f(0.04, 1.04061812) = (0.04)^2 + 1.04061812$$
$$= 0.0016 + 1.04061812$$
$$= 1.04221812$$

Hence,

$$y_2^{(1)} = 1.020206 + \frac{0.02}{2}(1.020606 + 1.04221812)$$
$$= 1.020206 + 0.01(2.06282412)$$
$$= 1.04083424$$

Second Corrector

$$f(0.04, 1.04083424) = 1.04243424$$
$$y_2^{(2)} = 1.020206 + 0.01(1.020606 + 1.04243424)$$
$$= 1.020206 + 0.02063040$$
$$= 1.04083640$$

Third Corrector

$$f(0.04, 1.04083640) = 1.04243640$$
$$y_2^{(3)} = 1.020206 + 0.01(1.020606 + 1.04243640)$$
$$= 1.04083642$$

The values have converged.
Therefore,

$$\boxed{y(0.04) = 1.040836}$$

Final Answers

$$\boxed{y(0.02) = 1.020206}$$
$$\boxed{y(0.04) = 1.040836}$$

Runge–Kutta Method:

First Order Runge–Kutta Method:

We know that by Euler's method

$$y_1 = y_0 + h\,f(x_0, y_0)$$

Then

$$y_1 = y_0 + h\,y_0'$$

Since

$$\frac{dy}{dx} = y' = f(x, y)$$
$$y_0' = f(x_0, y_0)$$

Expanding the left-hand side by Taylor's series, we get

$$y_1 = y(x_1) = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \cdots + \frac{h^n}{n!}y_0^{(n)}$$

It follows that Euler's method agrees with Taylor's series solution up to the term in h. Hence Euler's method is the ***Runge–Kutta method of first order****.*

Second Order RK Method

We know that by the modified RK method

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$
$$= y_0 + \frac{h}{2}[f_0 + f(x_0 + h,\ y_0 + hf_0)](1)$$

where

$$hf_0 = f(x_0, y_0)$$
$$k_1 = h\, f_0$$
$$k_2 = h\, f(x_0 + h,\ y_0 + k_1)$$

Then Eq. (1) becomes

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

*which is the **second order RK formula**.*

Hence,

2nd Order RK Formula

$$\boxed{y_1 = y_0 + \frac{1}{2}(k_1 + k_2)}$$

where

$$k_1 = h\, f(x_0, y_0)$$
$$k_2 = h\, f(x_0 + h,\ y_0 + k_1)$$

Third Order RK Method

The 3rd order RK method is defined as

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)}$$

where

$$k_1 = h\, f(x_0, y_0)$$
$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{3}k_1\right)$$
$$k_3 = h\, f(x_0 + h,\ y_0 + 2k_2 - k_1)$$

Fourth Order RK Method

The 4th order RK method is given by

$$\boxed{y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$

where

$$\begin{aligned} k_1 &= h\,f(x_0, y_0) \\ k_2 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_1\right) \\ k_3 &= h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{1}{2}k_2\right) \\ k_4 &= h\,f(x_0 + h,\ y_0 + k_3) \end{aligned}$$

PROBLEMS

1. ***Apply RK Method of 2nd Order, find the approximate value of ywhen $x = 1.1$, given***

$$\frac{dy}{dx} = 3x + y^2, y(1) = 1.2$$

Solution

Given

$$\begin{aligned} &\frac{dy}{dx} = f(x,y) = 3x + y^2 \\ &y(1) = 1.2 \\ &x_0 = 1, y_0 = 1.2 \end{aligned}$$

Here,

$$h = 0.1, x_1 = 1.1$$

1st Order RK Method

$$\begin{aligned} &y_1 = y_0 + h\,f(x_0, y_0) \\ &= 1.2 + 0.1\,f(1,1.2) \\ &= 1.2 + 0.1[3 + (1.2)^2] \\ &= 1.2 + 0.1(4.44) \\ &y_1' = 1.644 \end{aligned}$$

2nd Order RK Method

$$\begin{aligned} y_1 &= y_0 + \frac{1}{2}(k_1 + k_2) \\ k_1 &= h\,f(x_0, y_0) \\ k_2 &= h\,f(x_0 + h,\ y_0 + k_1) \end{aligned}$$

$$\begin{aligned}
&k_1 = 0.1(4.44) = 0.444\\
&k_2 = 0.1\,f(1+0.1,\ 1.2+0.444)\\
&= 0.1\,f(1.1,\ 1.644)\\
&= 0.1[3(1.1)+(1.644)^2]\\
&= 0.1(6.0027)\\
&= 0.6003
\end{aligned}$$

Hence,

$$\begin{aligned}
&y_1 = 1.2 + \frac{1}{2}(0.444+0.6003)\\
&= 1.2 + 0.5(1.0443)\\
&\boxed{y_1 = 1.7222}
\end{aligned}$$

---

2 . Using RK Method of Second Order, Compute $\boldsymbol{y(2.5)}$ ***from***

$$\frac{dy}{dx} = \frac{x+y}{x}, y(2) = 2$$

taking

$$h = 0.25$$

Given

$$\begin{aligned}
&\frac{dy}{dx} = f(x,y) = \frac{x+y}{x}\\
&y(2) = 2, x_0 = 2, y_0 = 2, h = 0.25, x_1 = 2.5
\end{aligned}$$

1st Order RK Method

$$\begin{aligned}
&y_1 = y_0 + h\,f(x_0,y_0)\\
&= 2 + 0.25\left[\frac{2+2}{2}\right]\\
&= 2 + 0.25(2)\\
&y_1 = 2.5
\end{aligned}$$

2nd Order RK Method

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where

$$k_1 = h\,f(x_0, y_0)$$
$$k_2 = h\,f(x_0 + h,\ y_0 + k_1)$$

Now,

$$\begin{aligned} k_1 &= 0.25(2) = 0.5 \\ k_2 &= 0.25\,f(2.25,\ 2.5) \\ &= 0.25\left[\frac{2.25 + 2.5}{2.25}\right] \\ &= 0.25(2.1111) \\ k_2 &= 0.5278 \end{aligned}$$

Hence,

$$\begin{aligned} y_1 &= 2 + \frac{1}{2}(0.5 + 0.5278) \\ &= 2 + 0.5(1.0278) \\ &\boxed{y_1 = 2.5139} \end{aligned}$$

Runge–Kutta Method (Fourth Order) – Solved Problem

1. **Problem**

Using the **Runge–Kutta Fourth Order (RK4) Method**, find the value of $y$at $x = 0.2$given that

$$\frac{dy}{dx} = x + y, y(0) = 1,$$

taking step size

$$h = 0.2.$$

Formula (RK4)

$$k_1 = h\,f(x_n, y_n)$$
$$k_2 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$
$$k_3 = h\,f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$
$$k_4 = h\,f(x_n + h, y_n + k_3)$$

Then,

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Given $f(x,y) = x + y$

$$x_0 = 0, y_0 = 1, h = 0.2$$

Step 1: Calculate k_1

$$k_1 = 0.2(0 + 1)$$
$$k_1 = 0.2$$

Step 2: Calculate k_2

$$k_2 = 0.2\left[0.1 + \left(1 + \frac{0.2}{2}\right)\right]$$
$$= 0.2(0.1 + 1.1)$$
$$= 0.2(1.2)$$
$$k_2 = 0.24$$

Step 3: Calculate k_3

$$k_3 = 0.2\left[0.1 + \left(1 + \frac{0.24}{2}\right)\right]$$
$$= 0.2(0.1 + 1.12)$$
$$= 0.2(1.22)$$
$$k_3 = 0.244$$

Step 4: Calculate k_4

$$k_4 = 0.2[0.2 + (1 + 0.244)]$$
$$= 0.2(1.444)$$
$$k_4 = 0.2888$$

Step 5: Calculate y_1

$$y_1 = 1 + \frac{1}{6}(0.2 + 2(0.24) + 2(0.244) + 0.2888)$$
$$= 1 + \frac{1}{6}(1.4568)$$
$$= 1 + 0.2428$$
$$\boxed{y(0.2) \approx 1.2428}$$

Final Answer

$$\boxed{y(0.2) = 1.2428}$$

PRACTICE QUESTIONS:

1. Solve $\frac{dy}{dx} = x + y,\ y(0) = 1$and find $y(0.1)$using Taylor's series method.
2. Solve $\frac{dy}{dx} = x^2 + y,\ y(0) = 1$and compute $y(0.2)$
3. Solve $\frac{dy}{dx} = x + y,\ y(0) = 1$and find $y(0.2)$using Euler's method with $h = 0.1$.
4. Solve $\frac{dy}{dx} = x - y,\ y(0) = 1$and find $y(0.3)$using Euler's method with $h = 0.1$.
5. Solve $\frac{dy}{dx} = y - x,\ y(0) = 1$and compute $y(0.2)$using the Modified Euler's method.
6. Solve $\frac{dy}{dx} = x + y^2,\ y(0) = 1$and find $y(0.2)$
7. Solve $\frac{dy}{dx} = x + y,\ y(0) = 1$and determine $y(0.2)$using RK2 with $h = 0.1$.
8. Solve $\frac{dy}{dx} = y - x^2,\ y(0) = 1$and find $y(0.2)$using RK4 with $h = 0.1$.
9. Solve $\frac{dy}{dx} = x + y,\ y(1) = 2$and compute $y(1.2)$using RK4 with $h = 0.1$.
10. Solve $\frac{dy}{dx} = x + y,\ y(0) = 1$using Picard's method and obtain the first three approximations

UNIT-4

LAPLACE TRANSFORMS

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

UNIT-4

LAPLACE TRANSFORMS

Introduction

The **Laplace Transform** is a powerful mathematical tool used to convert a function of time (time domain) into a function of a complex variable (frequency or p-**domain**).

It simplifies the process of solving **linear differential equations** by converting them into **algebraic equations**, making it very useful in engineering, mathematics, and physics.

Definition

Suppose $f(t)$ is a real-valued function defined on $(-\infty, \infty)$ such that

$$f(t) = 0, \qquad t < 0.$$

Then the **Laplace Transform** of $f(t)$ is defined as

$$\boxed{L\{f(t)\} = \bar{f}(p) = \int_0^{\infty} e^{-pt} f(t)\, dt}$$

where

- L denotes the Laplace Transform operator.
- p is a real or complex parameter.

Applications of Laplace Transforms

Laplace Transforms have many practical applications in science and engineering.

1. Control Systems

- Stability analysis of automatic control systems.

- Design of feedback systems.
- Applications in cruise control and robotics.

2. Electrical Circuit Analysis

- Solving RLC circuit equations.
- Analysis of current and voltage.
- Circuit transient response.

3. Signal Processing

- Design and analysis of communication systems.
- Digital and analog filter design.
- Signal transformation.

4. Mechanical Engineering

- Vibration analysis.
- Vehicle suspension systems.
- Heat conduction and dynamic system modeling.

5. Nuclear Physics

- Radioactive decay analysis.
- Nuclear reactor calculations.

6. Aerospace Engineering

- Aircraft and spacecraft dynamic analysis.
- Flight control systems.

7. Biomedical Engineering

- ECG and EEG signal analysis.
- Medical imaging systems.
- Physiological system modeling.

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

Standard Laplace Transforms

Function $f(t)$	Laplace Transform $L\{f(t)\}$
1	$\frac{1}{p}$
t^n	$\frac{n!}{p^{n+1}}$
e^{at}	$\frac{1}{p-a}$
e^{-at}	$\frac{1}{p+a}$
$\sin at$	$\frac{a}{p^2+a^2}$
$\cos at$	$\frac{p}{p^2+a^2}$
$\sinh at$	$\frac{a}{p^2-a^2}$
$\cosh at$	$\frac{p}{p^2-a^2}$

Standard Inverse Laplace Transforms

$L^{-1}\left\{\frac{1}{p}\right\} = 1$ $L^{-1}\left\{\frac{1}{p^{n+1}}\right\} = \frac{t^n}{n!}$ $L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at}$ $L^{-1}\left\{\frac{p}{p^2+a^2}\right\} = \cos at$ $L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{1}{a}\sin at$

Properties of Laplace Transforms

Property 1: Constant Multiplication

If c is a constant, then

$$\boxed{L\{cf(t)\} = cL\{f(t)\} = cF(s)}$$

Property 2: Addition and Subtraction

$$\boxed{L\{f_1(t) \pm f_2(t)\} = L\{f_1(t)\} \pm L\{f_2(t)\}}$$

Property 3: Linearity

If c_1 and c_2 are constants,

$$\boxed{L\{c_1 f_1(t) \pm c_2 f_2(t)\} = c_1 L\{f_1(t)\} \pm c_2 L\{f_2(t)\}}$$

Laplace Transform of Standard Functions

Problem 1

Show that

$$L\{1\} = \frac{1}{s}.$$

Solution

Given, $f\ (t) = 1.$

By the definition of Laplace Transform,

$$L\{f(t)\} = \int_0^\infty e^{-st}\ f(t)\, dt.$$

Therefore,

$$L\{1\} = \int_0^\infty e^{-st}\ dt.$$

Integrating,

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty.$$

Since

$$e^{-\infty} = 0, e^0 = 1,$$

we get

$$= -\frac{1}{s}(0 - 1) = \frac{1}{s}.$$

Hence,

$$\boxed{L\{1\} = \frac{1}{s}}$$

Problem 2

Show that $L\{t^n\} = \frac{n!}{s^{n+1}} = \frac{\Gamma(n+1)}{s^{n+1}}.$

Solution

Given,

$$f(t) = t^n.$$

From the definition,

$$L\{t^n\} = \int_0^\infty e^{-st}\, t^n\, dt.$$

Using integration by parts,

$$\int u\, dv = uv - \int v\, du$$

L

$$u = t^n, dv = e^{-st} dt.$$

Then,

$$du = nt^{n-1} dt,$$

and

$$v = -\frac{e^{-st}}{s}.$$

Hence,

$$L\{t^n\} = \left[\ -\frac{t^n e^{-st}}{s}\right]_0^\infty + \frac{n}{s}\int_0^\infty e^{-st}\, t^{n-1} dt.$$

Since the boundary term is zero,

$$L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}.$$

Repeating the process,

$$L\{t^n\} = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdots\frac{1}{s}L\{1\}.$$

Therefore,

$$= \frac{n!}{s^n}\left(\frac{1}{s}\right).$$

Hence,

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

or,

$$\boxed{L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}}$$

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where

$$\Gamma(n+1) = n!,$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Problem 3
Find

$$L\{e^{at}\}.$$

Solution
Given,

$$f(t) = e^{at}.$$

By the definition of Laplace Transform,

$$L\{e^{at}\} = \int_0^\infty e^{-st}\, e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\, dt.$$

Integrating,

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty .$$

For $s > a$,

$$e^{-(s-a)\infty} = 0.$$

Therefore,

$$= -\frac{1}{s-a}(0-1).$$

Hence,

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}, s > a.}$$

Problem 3
Find

$$L\{e^{at}\}.$$

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Solution

Given,

$$f(t) = e^{at}.$$

From the definition of Laplace Transform,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Therefore,

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt.$$
$$= \int_0^\infty e^{-(s-a)t}\, dt.$$

Integrating,

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty.$$
$$= \frac{1}{-(s-a)}[e^{-\infty} - e^0].$$

Since

$$e^{-\infty} = 0, e^0 = 1,$$

we obtain

$$= \frac{1}{-(s-a)}(0-1).$$

Hence,

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$

Problem 4

Find

$$L\{e^{-at}\}.$$

Solution

Given,

$$f(t) = e^{-at}.$$

From the definition of Laplace Transform,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

Therefore,

$$L\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt.$$

$$= \int_0^\infty e^{-(s+a)t}\; dt.$$

Integrating,

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty .$$

$$= \frac{1}{-(s+a)}[e^{-\infty} - e^0].$$

Since

$$e^{-\infty} = 0, e^0 = 1,$$

we get

$$= \frac{-1}{s+a}(0-1).$$

Hence,

$$\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$$

Note

$$\boxed{L\{e^{iat}\} = \frac{1}{s-ia}}$$

$$\boxed{L\{e^{-iat}\} = \frac{1}{s+ia}}$$

Problem 5

Find

$$L\{\sin at\}\text{and}L\{\cos at\}.$$

Solution
We know that

$$L\{e^{iat}\} = \frac{1}{s-ia}.$$

Multiplying the numerator and denominator by the conjugate $(s+ia)$,

$$L\{e^{iat}\} = \frac{1}{s-ia} \times \frac{s+ia}{s+ia}.$$

Therefore,

$$= \frac{s+ia}{(s-ia)(s+ia)}.$$

Since

$$(ia)^2 = -a^2,$$

we have

$$= \frac{s+ia}{s^2+a^2}.$$

Using **Euler's Formula**,

$$e^{iat} = \cos at + i\sin at,$$

therefore,

$$L\{\cos at + i\sin at\} = \frac{s+ia}{s^2+a^2}.$$

By the linearity property,

$$L\{\cos at\} + iL\{\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}.$$

Comparing the real and imaginary parts,

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

and

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

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Final Results

$$\boxed{L\{e^{at}\} = \frac{1}{s-a}}$$
$$\boxed{L\{e^{-at}\} = \frac{1}{s+a}}$$
$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$
$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

Laplace Transforms of Hyperbolic Functions and Solved Examples

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Derivation of Laplace Transforms of Trigonometric Functions

Using Euler's formula,

$$e^{iat} = \cos at + i\sin at$$

Taking Laplace Transform on both sides,

$$L\{\cos at + i\sin at\} = \frac{s+ia}{s^2+a^2}$$

Hence,

$$L\{\cos at\} + iL\{\sin at\} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$$

Comparing the real and imaginary parts,

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$
$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

Problem 6: Find $L\{\sinh at\}$and $L\{\cosh at\}$
Solution

We know that

$$\sinh at = \frac{e^{at} - e^{-at}}{2}$$

and

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

(i) Laplace Transform of sinh *at*

$$L\{\sinh at\} = \frac{1}{2}[L\{e^{at}\} - L\{e^{-at}\}]$$

Substituting the known transforms,

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

Taking LCM,

$$= \frac{1}{2}\left[\frac{(s+a)-(s-a)}{(s-a)(s+a)}\right]$$
$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right]$$

Therefore,

$$\boxed{L\{\sinh at\} = \frac{a}{s^2-a^2}}$$

(ii) Laplace Transform of cosh *at*

$$L\{\cosh at\} = \frac{1}{2}[L\{e^{at}\} + L\{e^{-at}\}]$$

Substituting,

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

Taking LCM,

$$= \frac{1}{2}\left[\frac{(s+a)+(s-a)}{s^2-a^2}\right]$$
$$= \frac{1}{2}\left[\frac{2s}{s^2-a^2}\right]$$

Hence,

$$\boxed{L\{\cosh at\} = \frac{s}{s^2 - a^2}}$$

Solved Problems

Problem 1

Find

(a) $L\{\cos 4t\}$

(b) $L\{\sinh 2t\}$

Solution

(a)

Using

$$L\{\cos at\} = \frac{p}{p^2 + a^2},$$

we get

$$L\{\cos 4t\} = \frac{p}{p^2 + 4^2} = \boxed{\frac{p}{p^2 + 16}}.$$

(b)

Using

$$L\{\sinh at\} = \frac{a}{p^2 - a^2},$$

we obtain

$$L\{\sinh 2t\} = \frac{2}{p^2 - 2^2} = \boxed{\frac{2}{p^2 - 4}}.$$

Problem 2

Find

$$L\{\sin^2 2t\}.$$

Solution

Using the trigonometric identity,

$$\sin^2 x = \frac{1 - \cos 2x}{2},$$

we have

$$\sin^2 2t = \frac{1 - \cos 4t}{2}.$$

Applying Laplace Transform,

$L\{\sin^2 2t\} = L\left\{\frac{1}{2} - \frac{1}{2}\cos 4t\right\} = \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 4t\} = \frac{1}{2p} - \frac{1}{2}\left(\frac{p}{p^2+16}\right)$.

Therefore,

$$\boxed{L\{\sin^2 2t\} = \frac{1}{2p} - \frac{p}{2(p^2+16)}}$$

Problem 3

Evaluate

$L\{\sin 3t \mathrm{Sin}\, 2t\}$.

Solution

$$=1/2L\{2\sin 3t \sin 2t\}.$$

W.K.T 2SinASinB=cos(A-B)-cos(A+B)

$$=1/2L\{\cos(3\mathrm{t}-2\mathrm{t}) - \cos(3\mathrm{t}+2\mathrm{t})\}$$

$$=1/2L\{\cos(\mathrm{t}) - \cos(5\mathrm{t})\}$$

$$= \frac{1}{2}\left[\frac{p}{p^2+1} - \frac{p}{p^2+25}\right]$$

$$= \boxed{\frac{p}{2}\left(\frac{1}{p^2+1} - \frac{1}{p^2+25}\right)}$$

Problem 4

Evaluate

(i)

$$L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 5t\}$$

(ii)

$$L\{\cos^2 2t\}$$

Solution

(i)

Using linearity of Laplace Transform,

$$L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 5t\}$$
$$= L\{e^{2t}\} + 4L\{t^3\} - 2L\{\sin 3t\} + 3L\{\cos 5t\}.$$

Using the standard formulas,

$$L\{e^{at}\} = \frac{1}{p-a},$$
$$L\{t^n\} = \frac{n!}{p^{n+1}},$$
$$L\{\sin at\} = \frac{a}{p^2+a^2},$$
$$L\{\cos at\} = \frac{p}{p^2+a^2},$$

we obtain

$$= \frac{1}{p-2} + 4\left(\frac{3!}{p^4}\right) - 2\left(\frac{3}{p^2+9}\right) + 3\left(\frac{p}{p^2+25}\right).$$

Since

$$3! = 6,$$

therefore,

$$\boxed{L\{e^{2t} + 4t^3 - 2\sin 3t + 3\cos 5t\} = \frac{1}{p-2} + \frac{24}{p^4} - \frac{6}{p^2+9} + \frac{3p}{p^2+25}}$$

(ii)

Find

$$L\{\cos^2 2t\}.$$

Using

$$\cos^2 x = \frac{1+\cos 2x}{2},$$

we have

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$$\cos^2 2t = \frac{1+\cos 4t}{2}.$$

Applying Laplace Transform,

$$L\{\cos^2 2t\} = L\left\{\frac{1+\cos 4t}{2}\right\}$$
$$= \frac{1}{2}L\{1\} + \frac{1}{2}L\{\cos 4t\}$$
$$= \frac{1}{2}\left(\frac{1}{p}\right) + \frac{1}{2}\left(\frac{p}{p^2+16}\right).$$

Hence,

$$\boxed{L\{\cos^2 2t\} = \frac{1}{2}\left(\frac{1}{p} + \frac{p}{p^2+16}\right)}$$

Problem 5

Find

$$L\{\sin 2t \cos 2t\}.$$

Solution

Using the identity,

$$\sin A \cos A = \frac{1}{2}\sin 2A,$$

we get

$$\sin 2t \cos 2t = \frac{1}{2}\sin 4t.$$

Therefore,

$$L\{\sin 2t\cos 2t\} = \frac{1}{2}L\{\sin 4t\}$$
$$= \frac{1}{2}\left(\frac{4}{p^2+16}\right).$$

Hence,

$$\boxed{L\{\sin 2t\cos 2t\} = \frac{2}{p^2+16}}$$

Problem 6

Find

$$L\{\sin^2 t\}$$

Solution

Using the identity,

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

Therefore,

$$L\{\sin^2 t\} = \frac{1}{2}[L\{1\} - L\{\cos 2t\}]$$
$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Taking LCM,

$$= \frac{1}{2}\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$
$$= \frac{1}{2}\left[\frac{4}{s(s^2+4)}\right]$$

Answer

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

$$\boxed{L\{\sin^2 t\} = \frac{2}{s(s^2+4)}}$$

Problem 7

Find

$$L\{5e^{2t} + 3\cos 4t\}$$

Solution

$$= 5L\{e^{2t}\} + 3L\{\cos 4t\}$$
$$= 5\left(\frac{1}{s-2}\right) + 3\left(\frac{s}{s^2+16}\right)$$

Answer

$$\boxed{\frac{5}{s-2} + \frac{3s}{s^2+16}}$$

Problem 8

Find

$$L\{e^{3t} + 3e^{-2t}\}$$

Solution

Using linearity,

$$L\{e^{3t} + 3e^{-2t}\} = L\{e^{3t}\} + 3L\{e^{-2t}\}$$
$$= \frac{1}{s-3} + \frac{3}{s+2}$$

Answer

$$\boxed{\frac{1}{s-3} + \frac{3}{s+2}}$$

Problem 9

Find

$$\mathcal{L}\{\sin^2 t\cos 2t\}$$

Solution

Using the identity,

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

Therefore,

$$\mathcal{L}\{\sin^2 t\cos 2t\} = \mathcal{L}\left\{\left(\frac{1-\cos 2t}{2}\right)\cos 2t\right\}$$
$$= \frac{1}{2}\mathcal{L}\{\cos 2t - \cos^2 2t\}$$

Using

$$\cos^2 2t = \frac{1+\cos 4t}{2}$$

we get

$$= \frac{1}{2}\mathcal{L}\left\{\cos 2t - \frac{1+\cos 4t}{2}\right\}$$
$$= \frac{1}{2}\left[\mathcal{L}\{\cos 2t\} - \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos 4t\}\right]$$

Substituting the known transforms,

$$= \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{1}{2s} - \frac{s}{2(s^2+16)}\right]$$

Hence,

$$\boxed{\mathcal{L}\{\sin^2 t\cos 2t\} = \frac{1}{4}\left[\frac{2s}{s^2+4} - \frac{1}{s} - \frac{s}{s^2+16}\right]}$$

Problem 10

Find

$$\mathcal{L}\{(\sin t-\cos t)^2\}$$

Solution

Expanding,

$$(\sin t-\cos t)^2=\sin^2 t+\cos^2 t-2\sin t\cos t$$

Using the identities,

$$\sin^2 t+\cos^2 t=1$$

and

$$2\sin t\cos t=\sin 2t$$

Therefore,

$$=1-\sin 2t$$

Taking Laplace Transform,

$$=\mathcal{L}\{1\}-\mathcal{L}\{\sin 2t\}$$
$$=\frac{1}{s}-\frac{2}{s^2+4}$$

Answer

$$\boxed{\mathcal{L}\{(\sin t-\cos t)^2\}=\frac{1}{s}-\frac{2}{s^2+4}}$$

Problem 11

Find:

1. $\mathcal{L}\{\sin^3 t\}$

2. $\mathcal{L}\{\sin(\omega t + \alpha)\}$
3. $\mathcal{L}\{8t^3 + \cos 4t + e^{-2t}\}$
4. $\mathcal{L}\{t^2 + at + b\}$

Solutions

Problem 1

Find

$$\mathcal{L}\{\sin^3 t\}$$

Solution

Using the identity,

$$\sin 3t = 3\sin t - 4\sin^3 t$$

Therefore,

$$\sin^3 t = \frac{3\sin t - \sin 3t}{4}$$

Taking Laplace Transform,

$$= \frac{3}{4}\mathcal{L}\{\sin t\} - \frac{1}{4}\mathcal{L}\{\sin 3t\}$$

Substituting,

$$\begin{aligned} &= \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right) \\ &= \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] \\ &= \frac{3}{4}\left[\frac{8}{(s^2+1)(s^2+9)}\right] \end{aligned}$$

Answer

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

$$\boxed{\mathcal{L}\{\sin^3 t\} = \frac{6}{(s^2+1)(s^2+9)}}$$

Problem 2

Find

$$\mathcal{L}\{\sin(\omega t+\alpha)\}$$

Solution

Using

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

we have

$$\sin(\omega t+\alpha) = \sin\omega t\cos\alpha + \cos\omega t\sin\alpha$$

Taking Laplace Transform,

$$= \cos\alpha\,\mathcal{L}\{\sin\omega t\} + \sin\alpha\,\mathcal{L}\{\cos\omega t\}$$

Substituting,

$$= \cos\alpha\left(\frac{\omega}{s^2+\omega^2}\right) + \sin\alpha\left(\frac{s}{s^2+\omega^2}\right)$$

Answer

$$\boxed{\mathcal{L}\{\sin(\omega t+\alpha)\} = \frac{\omega\cos\alpha + s\sin\alpha}{s^2+\omega^2}}$$

Problem 3

Find

$$\mathcal{L}\{8t^3 + \cos 4t + e^{-2t}\}$$

Solution

Using linearity,

$$= 8\mathcal{L}\{t^3\} + \mathcal{L}\{\cos 4t\} + \mathcal{L}\{e^{-2t}\}$$

Since,

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$$

Therefore,

$$= 8\left(\frac{6}{s^4}\right) + \frac{s}{s^2+16} + \frac{1}{s+2}$$

Answer

$$\boxed{\mathcal{L}\{8t^3 + \cos 4t + e^{-2t}\} = \frac{48}{s^4} + \frac{s}{s^2+16} + \frac{1}{s+2}}$$

Problem 4

Find

$$\mathcal{L}\{t^2 + at + b\}$$

Solution

Using linearity,

$$\mathcal{L}\{t^2 + at + b\} = \mathcal{L}\{t^2\} + a\mathcal{L}\{t\} + b\mathcal{L}\{1\}$$

Substituting the standard transforms,

$$\mathcal{L}\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}, \mathcal{L}\{t\} = \frac{1}{s^2}, \mathcal{L}\{1\} = \frac{1}{s}$$

Hence,

$$\boxed{\mathcal{L}\{t^2 + at + b\} = \frac{2}{s^3} + \frac{a}{s^2} + \frac{b}{s}}$$

PROBLEM 12

Find

$$L\{(\sin t - \cos t)^2\}$$

Solution

Expanding,

$$(\sin t - \cos t)^2 = \sin^2 t + \cos^2 t - 2\sin t\cos t$$

Using the identities,

$$\sin^2 t + \cos^2 t = 1, 2\sin t\cos t = \sin 2t$$

Therefore,

$$L\{(\sin t - \cos t)^2\} = L\{1 - \sin 2t\}$$
$$= L\{1\} - L\{\sin 2t\}$$
$$= \frac{1}{s} - \frac{2}{s^2 + 4}$$

Answer

$$\boxed{L\{(\sin t - \cos t)^2\} = \frac{1}{s} - \frac{2}{s^2 + 4}}$$

Problem 13

Find

(i) $L\{(t^2+1)^2\}$

Solution

Expand, $(t^2+1)^2 = t^4 + 2t^2 + 1$

Using linearity,

$$L\{(t^2+1)^2\} = L\{t^4\} + 2L\{t^2\} + L\{1\}$$

Substituting,

$$L\{t^4\} = \frac{4!}{s^5} = \frac{24}{s^5}$$
$$L\{t^2\} = \frac{2!}{s^3} = \frac{2}{s^3}$$

Hence,

$$= \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}$$

Answer

$$\boxed{L\{(t^2+1)^2\} = \frac{24}{s^5} + \frac{4}{s^3} + \frac{1}{s}}$$

(ii)

Find

$$L\left\{\frac{e^{-at}-1}{a}\right\}$$

Solution

Using linearity,

$$= \frac{1}{a}[L\{e^{-at}\} - L\{1\}]$$

Substituting,

$$= \frac{1}{a}\left[\frac{1}{s+a} - \frac{1}{s}\right]$$

Simplifying,

$$= \frac{1}{a}\left[\frac{s-(s+a)}{s(s+a)}\right]$$

$$= -\frac{1}{s(s+a)}$$

Answer

$$\boxed{L\left\{\frac{e^{-at}-1}{a}\right\} = -\frac{1}{s(s+a)}}$$

(iii)

Find

$$L\{\cos t\cos 2t\cos 3t\}$$

Solution

Using $\cos A\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

First, $\cos 2t\cos 3t = \frac{1}{2}(\cos 5t + \cos t)$

Therefore,

$$=\frac{1}{2}L\{\cos t\cos 5t+\cos^2 t\}$$

Again,

$$\cos t\cos 5t=\frac{1}{2}(\cos 6t+\cos 4t)$$

and

$$\cos^2 t=\frac{1+\cos 2t}{2}$$

Hence,

$$=\frac{1}{4}[L\{\cos 6t\}+L\{\cos 4t\}+L\{1\}+L\{\cos 2t\}]$$

Substituting,

$$=\frac{1}{4}\left[\frac{s}{s^2+36}+\frac{s}{s^2+16}+\frac{1}{s}+\frac{s}{s^2+4}\right]$$

Answer

$$\boxed{L\{\cos t\cos 2t\cos 3t\}=\frac{1}{4}\left[\frac{s}{s^2+36}+\frac{s}{s^2+16}+\frac{1}{s}+\frac{s}{s^2+4}\right]}$$

Laplace Transform of Piecewise Functions

Problem 1

Find the Laplace Transform of

$$f(t)=\begin{cases}1, & 0<t<2,\\ 2, & 2<t<4,\\ 3, & 4<t<6,\\ 0, & t>6.\end{cases}$$

Solution

From the definition of Laplace Transform,

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$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

Splitting the integral over the given intervals,

$$L\{f(t)\} = \int_0^2 e^{-st}\ dt + 2\int_2^4 e^{-st}\ dt + 3\int_4^6 e^{-st}\ dt + \int_6^\infty 0\, dt$$

Evaluating each integral,

$$= \left[\frac{-e^{-st}}{s}\right]_0^2 + 2\left[\frac{-e^{-st}}{s}\right]_2^4 + 3\left[\frac{-e^{-st}}{s}\right]_4^6$$

$$= -\frac{1}{s}(e^{-2s} - 1) - \frac{2}{s}(e^{-4s} - e^{-2s}) - \frac{3}{s}(e^{-6s} - e^{-4s})$$

Expanding,

$$= \frac{1}{s}[1 - e^{-2s} - 2e^{-4s} + 2e^{-2s} - 3e^{-6s} + 3e^{-4s}]$$

Combining like terms,

$$= \frac{1}{s}[1 + e^{-2s} + e^{-4s} - 3e^{-6s}]$$

Answer

$$\boxed{L\{f(t)\} = \frac{1}{s}(1 + e^{-2s} + e^{-4s} - 3e^{-6s})}$$

Problem 2

Find the Laplace Transform of

$$f(t) = \begin{cases} \sin t, & 0 < t < \pi, \\ 0, & t > \pi. \end{cases}$$

Solution

By definition,

$$L\{f(t)\} = \int_0^{\pi} e^{-st} \sin t \, dt$$

Using the standard integral,

$$\int e^{-st} \sin t \, dt = \frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)$$

Therefore,

$$L\{f(t)\} = \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^{\pi}$$

Substituting the limits,

$$= \frac{1}{s^2+1}[e^{-s\pi}(-s\sin\pi - \cos\pi) - (-s\sin 0 - \cos 0)]$$

Using

$$\sin\pi = 0, \cos\pi = -1,$$
$$\sin 0 = 0, \cos 0 = 1,$$

we get

$$= \frac{1}{s^2+1}[e^{-s\pi}(1) - (-1)]$$

Hence,

$$= \frac{1 + e^{-s\pi}}{s^2+1}$$

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Answer

$$\boxed{L\{f(t)\} = \frac{1+e^{-s\pi}}{s^2+1}}$$

EXISTENCE OF LAPLACE TRANSFORM

The existence of the Laplace transform is determined by the convergence of the improper integral:
F(s)= $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
For the integral to exist (converge) for some value of s , the function f(t) must generally satisfy two primary conditions.

1. Piecewise Continuity

The function f(t) must be piecewise continuous on every finite interval [0, A] where A > 0. This means that within any finite time range, the function may have a finite number of jump discontinuities, but it cannot have any infinite discontinuities (singularities) that are not integrable.

2. Exponential Order

To ensure the integral converges as , $t \to \infty$ the function must not grow faster than an exponential function. We say f(t) is of **exponential order** if there exist constants M > 0, c, and T such that:

$|F(t)| \leq Me^{ct}$ for all t>T

Conditions for Existence of Laplace Transform

A function $f(t)$possesses a Laplace Transform if it satisfies the following conditions:

1. **Piecewise Continuity**

The function $f(t)$must be piecewise continuous on every finite interval

$$a \leq t \leq b.$$

2. **Exponential Order**

There exist constants $M$and $a$such that

$$|\, f(t)\, | \leq Me^{at},$$

or equivalently,

$$\lim_{t\to\infty} e^{-at} f(t)$$

is finite.

First Shifting Theorem

Theorem

If

$$L\{f(t)\} = \bar{f}(p),$$

then

$$\boxed{L\{e^{at} f(t)\} = \bar{f}(p-a)}$$

provided

$$p - a > 0.$$

Example 1

Find

$$L\{e^{2t} \cos 3t\}.$$

Solution

Since

$$L\{\cos 3t\} = \frac{p}{p^2+9},$$

by the First Shifting Theorem,

$$L\{e^{2t} \cos 3t\} = \bar{f}(p-2).$$

Hence,

$$\boxed{L\{e^{2t}\cos 3t\} = \frac{p-2}{(p-2)^2+9}}$$

Example 2

Find

$$L\{e^{-5t}\sinh 2t\}.$$

Solution

Since

$$L\{\sinh 2t\} = \frac{2}{p^2-4},$$

by the First Shifting Theorem,

$$L\{e^{-5t}\sinh 2t\} = \bar{f}(p+5).$$

Therefore,

$$\boxed{L\{e^{-5t}\sinh 2t\} = \frac{2}{(p+5)^2-4}}$$

Example 3

Find

$$L\{t^2e^{-3t}\}.$$

Solution

Since

$$L\{t^2\} = \frac{2!}{p^3} = \frac{2}{p^3},$$

using the First Shifting Theorem,

$$L\{e^{-3t}t^2\} = \bar{f}(p+3).$$

Hence,

$$\boxed{L\{t^2e^{-3t}\} = \frac{2}{(p+3)^3}}$$

Laplace Transform of Derivatives

If $f(t)$is continuous and of exponential order, then:

$$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$$
$$\boxed{\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)}$$
$$\boxed{\mathcal{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)}$$

where

$$F(s) = \mathcal{L}\{f(t)\}.$$

General Formula

$$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)}$$

Example 1: Find $\mathcal{L}\{\sin at\}$

Let

$$f(t) = \sin at.$$

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Then

$$f(0) = 0,$$
$$f'(t) = a\cos at, f'(0) = a,$$
$$f''(t) = -a^2\sin at.$$

Using

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0),$$

we get

$$\mathcal{L}\{-a^2\sin at\} = s^2F(s) - a.$$

Since

$$\mathcal{L}\{-a^2\sin at\} = -a^2F(s),$$

therefore

$$-a^2F(s) = s^2F(s) - a.$$

Hence

$$a = (s^2 + a^2)F(s),$$

so

$$\boxed{\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2}}.$$

Example 2: Find

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\}$$

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Given

$$\mathcal{L}\{\sin\sqrt{t}\} = \frac{\sqrt{\pi}\, e^{-1/(4s)}}{2s^{3/2}}.$$

Let

$$f(t) = \sin\sqrt{t}.$$

Then

$$f'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}.$$

Using

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0),$$

and since

$$f(0) = \sin 0 = 0,$$

we have

$$\frac{1}{2}\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = s\left(\frac{\sqrt{\pi}e^{-1/(4s)}}{2s^{3/2}}\right).$$

Multiplying both sides by 2,

$$\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \frac{\sqrt{\pi}\, e^{-1/(4s)}}{\sqrt{s}}.$$

Therefore,

$$\boxed{\mathcal{L}\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-1/(4s)}}.$$

Laplace Transform of Integrals

If

$$\mathcal{L}\{f(t)\} = F(s),$$

then

$$\boxed{\mathcal{L}\left\{\int_0^t f(x)\,dx\right\} = \frac{F(s)}{s}}$$

since

$$\frac{d}{dt}\left(\int_0^t f(x)\,dx\right) = f(t)$$

and the integral is zero at $t = 0$.

Similarly,

$$\boxed{\mathcal{L}\left\{\int_0^t \int_0^u f(\tau)\,d\tau\,du\right\} = \frac{F(s)}{s^2}}$$

Useful Property (Division by t)

If

$$\mathcal{L}\{f(t)\} = F(s),$$

then

$$\boxed{\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du}$$

where $u$is a dummy variable of integration.

Problem 1

Find

$$\mathcal{L}\left\{\int_0^t \frac{1-e^{-t}}{t}\,dt\right\}.$$

Step 1

Using the integral property,

$$\mathcal{L}\left\{\int_0^t \frac{1-e^{-t}}{t}\,dt\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\}.$$

Step 2

Using the division-by-t property,

$$\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\} = \int_s^\infty \left(\frac{1}{u} - \frac{1}{u+1}\right) du.$$

Step 3

Evaluate the integral:

$$= [\ln u - \ln(u+1)]_s^\infty.$$

As $u \to \infty$,

$$\ln\frac{u}{u+1} \to 0.$$

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Therefore,

$$= 0 - \ln\left(\frac{s}{s+1}\right) = \ln\left(\frac{s+1}{s}\right).$$

Final Answer

$$\boxed{\mathcal{L}\left\{\int_0^t \frac{1-e^{-t}}{t}\,dt\right\} = \frac{1}{s}\ln\left(\frac{s+1}{s}\right)}$$

Problem 2

Find

$$\mathcal{L}\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\}.$$

Step 1

Using the integral property,

$$\mathcal{L}\left\{\int_0^t \frac{e^t \sin t}{t}\,dt\right\} = \frac{1}{s}\mathcal{L}\left\{\frac{e^t \sin t}{t}\right\}.$$

Step 2

Using the division-by-t property,

$$\mathcal{L}\left\{\frac{e^t \sin t}{t}\right\} = \int_s^\infty \mathcal{L}\{e^t \sin t\}\,du.$$

Since

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1},$$

by the first shifting theorem,

$$\mathcal{L}\{e^t \sin t\} = \frac{1}{(s-1)^2+1}.$$

Hence,

$$= \int_s^\infty \frac{du}{(u-1)^2+1}.$$

Step 3

Evaluate:

$$= [\tan^{-1}(u-1)]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s-1).$$

Using the identity

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x,$$

we get

$$= \cot^{-1}(s-1).$$

Final Answer

$$\boxed{\mathcal{L}\left\{\int_0^t \frac{e^t \sin t}{t}\, dt\right\} = \frac{1}{s}\cot^{-1}(s-1)}$$

1. Evaluate

$$\mathcal{L}\left\{\int_0^t e^{-u} \cos u\, du\right\}$$

Note: The integration variable should be u(or τ), not t.

Solution

Using the integral property,

$$\mathcal{L}\left\{\int_0^t f(u)\,du\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}.$$

Here,

$$f(t) = e^{-t}\cos t.$$

Therefore,

$$= \frac{1}{s}\mathcal{L}\{e^{-t}\cos t\}.$$

Using the First Shifting Theorem,

$$\mathcal{L}\{e^{-t}\cos t\} = \frac{s+1}{(s+1)^2+1}.$$

Hence,

$$\boxed{\mathcal{L}\left\{\int_0^t e^{-u}\cos u\,du\right\} = \frac{s+1}{s[(s+1)^2+1]}}$$

---

2. Evaluation of Integrals Using Laplace Transforms

Problem 1

Evaluate

$$\int_0^\infty e^{-2t}\sin 3t\,dt.$$

Solution

Using

$$\mathcal{L}\{\sin at\} = \frac{a}{s^2 + a^2},$$

with $a = 3$,

$$= \left[\frac{3}{s^2 + 9}\right]_{s=2} = \frac{3}{4+9} = \boxed{\frac{3}{13}}.$$

Problem 2

Evaluate

$$\int_0^\infty t\,e^{-3t}\cos t\,dt.$$

Solution

Using

$$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s),$$

where

$$F(s) = \frac{s}{s^2 + 1}.$$

Differentiate:

$$\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1) - 2s^2}{(s^2+1)^2} = \frac{1 - s^2}{(s^2+1)^2}.$$

Therefore,

$$\mathcal{L}\{t\cos t\} = -\frac{1-s^2}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}.$$

At $s = 3$,

$$= \frac{9-1}{(9+1)^2} = \frac{8}{100} = \boxed{\frac{2}{25}}.$$

---

3. Evaluate

$$\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\, dt.$$

Solution

Using

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\, du,$$

where

$$F(u) = \frac{1}{u+1} - \frac{1}{u+2}.$$

Thus,

$$\int_s^\infty \left(\frac{1}{u+1} - \frac{1}{u+2}\right) du.$$

Integrating,

$$= [\ln(u+1) - \ln(u+2)]_s^\infty.$$

As $u \to \infty$,

$$\ln\left(\frac{u+1}{u+2}\right) \to 0.$$

At $s = 0$,

$$0 - \ln\left(\frac{1}{2}\right) = \ln 2.$$

Hence,

$$\boxed{\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\, dt = \ln 2}$$

Evaluate

$$\boxed{\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\, dt}$$

Note: The denominator $t$is essential. Without it, the answer would not be $\ln 2$.

Solution

Using the property

$$\boxed{\int_0^\infty \frac{f(t)}{t}\, dt = \int_0^\infty F(s)\, ds},$$

where

$$F(s) = \mathcal{L}\{f(t)\}.$$

Let

$$f(t) = e^{-t} - e^{-2t}.$$

Then

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$$\mathcal{L}\{f(t)\} = \frac{1}{s+1} - \frac{1}{s+2}.$$

Hence,

$$\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\, dt = \int_0^\infty \left(\frac{1}{s+1} - \frac{1}{s+2}\right) ds.$$

Integrating,

$$= [\ln(s+1) - \ln(s+2)]_0^\infty.$$

As $s \to \infty$,

$$\ln\left(\frac{s+1}{s+2}\right) \to 0.$$

Therefore,

$$= 0 - \ln\left(\frac{1}{2}\right) = \ln 2.$$

Final Answer

$$\boxed{\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\, dt = \ln 2}$$

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2. Inverse Laplace Transform (I.L.T.)

Definition

If

$$\mathcal{L}\{f(t)\} = F(s),$$

then

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = f(t)}.$$

Standard Inverse Laplace Transforms

S.No	$F(s)$	$\mathcal{L}^{-1}\{F(s)\}$
1	$\frac{1}{s}$	1
2	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$
3	$\frac{1}{s-a}$	e^{at}
4	$\frac{1}{s+a}$	e^{-at}
5	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin(at)$
6	$\frac{s}{s^2+a^2}$	$\cos(at)$
7	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh(at)$
8	$\frac{s}{s^2-a^2}$	$\cosh(at)$

---

3. First Shifting Theorem (Inverse Laplace Transform)

If

$$\mathcal{L}^{-1}\{F(s)\} = f(t),$$

then

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at} f(t)}$$

or equivalently,

$$\boxed{\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\,\mathcal{L}^{-1}\{F(s)\}}$$

Important Correction

The first problem in your transcription should be

$$\boxed{\int_0^\infty \frac{e^{-t} - e^{-2t}}{t}\,dt}$$

not

$$\int_0^\infty (e^{-t} - e^{-2t})\,dt.$$

This is because:

- $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt = \boxed{\ln 2}$
- whereas $\int_0^\infty (e^{-t} - e^{-2t})\,dt = 1 - \frac{1}{2} = \boxed{\frac{1}{2}}$

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Problem A

Find

$$\mathcal{L}^{-1}\left\{\frac{s^2-3s+4}{s^3}\right\}.$$

Solution

$$=\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{3}{s^2}+\frac{4}{s^3}\right\}$$

Using

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}=1, \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}=t, \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\}=\frac{t^2}{2},$$

we obtain

$$=1-3t+4\left(\frac{t^2}{2}\right)$$
$$\boxed{=1-3t+2t^2.}$$

Problem B

Find

$$\mathcal{L}^{-1}\left\{\frac{3(s^2-2)^2}{2s^5}\right\}.$$

Solution

Expand:

$$(s^2-2)^2=s^4-4s^2+4.$$

Therefore,

$$= \frac{3}{2}\mathcal{L}^{-1}\left\{\frac{s^4 - 4s^2 + 4}{s^5}\right\}$$
$$= \frac{3}{2}\left[\frac{1}{s} - \frac{4}{s^3} + \frac{4}{s^5}\right].$$

Now,

$$\mathcal{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{t^4}{4!}.$$

Hence

$$= \frac{3}{2}\left[1 + \frac{t^4}{6} - 2t^2\right].$$

You may also write

$$\boxed{\frac{3}{2} - 3t^2 + \frac{t^4}{4}.}$$

Problem C

Find

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2 - a^2}\right\}.$$

Solution

Factor:

$$s^2 - a^2 = (s-a)(s+a).$$

Partial fractions:

$$\frac{s}{(s-a)(s+a)} = \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right).$$

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Taking inverse transforms,

$$=\frac{1}{2}(e^{at}+e^{-at})$$

Using

$$\cosh(at)=\frac{e^{at}+e^{-at}}{2},$$

we get

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s}{s^2-a^2}\right\}=\cosh(at).}$$

Problem D

Find

$$\mathcal{L}^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\}.$$

Solution

Partial fractions:

$$\frac{4}{(s+1)(s+2)}=4\left(\frac{1}{s+1}-\frac{1}{s+2}\right).$$

Hence,

$$=4(e^{-t}-e^{-2t}).$$

Therefore,

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

$$\boxed{\mathcal{L}^{-1}\left\{\frac{4}{(s+1)(s+2)}\right\} = 4(e^{-t} - e^{-2t}).}$$

Problem E

Find

$$\mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\}.$$

Factorization

$$s^4 + 4a^4 = (s^2 + 2as + 2a^2)(s^2 - 2as + 2a^2).$$

After resolving into partial fractions and applying the First Shifting Theorem,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{s^2}{s^4+4a^4}\right\} = \frac{1}{2a}[\cosh(at)\sin(at) + \sinh(at)\cos(at)].}$$

Problem 1

Simplify

$$\frac{1}{4a}[e^{at}(\cos at + \sin at) - e^{-at}(\cos at - \sin at)].$$

Solution

Expand:

$$= \frac{1}{4a}[e^{at}\cos at + e^{at}\sin at - e^{-at}\cos at + e^{-at}\sin at].$$

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

Group like terms:

$$= \frac{1}{4a}[(e^{at} - e^{-at})\cos at + (e^{at} + e^{-at})\sin at].$$

Using

$$\sinh(at) = \frac{e^{at} - e^{-at}}{2},$$

and

$$\cosh(at) = \frac{e^{at} + e^{-at}}{2},$$

we obtain

$$\boxed{\frac{1}{2a}[\cos(at)\sinh(at) + \sin(at)\cosh(at)].}$$

Problem 2

Find

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^3}\right\}.$$

Solution

Using the First Shifting Theorem,

$$\mathcal{L}^{-1}\{F(s-a)\} = e^{at}\mathcal{L}^{-1}\{F(s)\}.$$

Since

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \frac{t^2}{2!},$$

we get

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{(s-a)^3}\right\} = \frac{t^2 e^{at}}{2}.}$$

Inverse Laplace Transform by Partial Fractions

Problem 1

Find

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\}.$$

Solution

Write

$$\frac{1}{s(s+1)(s+2)} = \frac{1}{s}\left(\frac{1}{s+1} - \frac{1}{s+2}\right).$$

Therefore,

$$= \frac{1}{s(s+1)} - \frac{1}{s(s+2)}.$$

Now,

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1},$$

and

$$\frac{1}{s(s+2)} = \frac{1}{2}\left(\frac{1}{s} - \frac{1}{s+2}\right).$$

Hence,

$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\cdot\frac{1}{s+2}.$$

Taking inverse transforms,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)(s+2)}\right\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.}$$

Problem 2

Find

$$\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\}.$$

Solution

Factor the denominator:

$$s^2 - 2s - 3 = (s+1)(s-3).$$

Assume

$$\frac{3s+7}{(s+1)(s-3)} = \frac{A}{s+1} + \frac{B}{s-3}.$$

Then

$$3s + 7 = A(s-3) + B(s+1).$$

Substitute $s = -1$:

$$4 = -4A$$
$$A = -1.$$

Substitute $s = 3$:

$$16 = 4B$$
$$B = 4.$$

Therefore,

$$\frac{3s+7}{(s+1)(s-3)} = -\frac{1}{s+1} + \frac{4}{s-3}.$$

Taking inverse Laplace transforms,

$$\boxed{\mathcal{L}^{-1}\left\{\frac{3s+7}{s^2-2s-3}\right\} = -e^{-t} + 4e^{3t}.}$$

Problem 3

Find

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

Solution

Using partial fractions,

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{A}{s^2+a^2} + \frac{B}{s^2+b^2}$$

Multiplying throughout,

$$s^2 = A(s^2+b^2) + B(s^2+a^2)$$

Comparing coefficients,

$A + B = 1 \qquad \cdots (1)\ Ab^2 + Ba^2 = 0 \qquad \cdots (2)$

From (2),

$$A = -\frac{Ba^2}{b^2}$$

Substituting into (1),

$-\frac{Ba^2}{b^2}+B=1\; B\left(\frac{b^2-a^2}{b^2}\right)=1\; B=\frac{b^2}{b^2-a^2}$

Similarly,

$$A=-\frac{a^2}{b^2-a^2}$$

Hence,

$$\frac{s^2}{(s^2+a^2)(s^2+b^2)}=\frac{-a^2}{(b^2-a^2)(s^2+a^2)}+\frac{b^2}{(b^2-a^2)(s^2+b^2)}$$

Taking inverse Laplace transform,

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{1}{b^2-a^2}\left[-a^2L^{-1}\left\{\frac{1}{s^2+a^2}\right\}+b^2L^{-1}\left\{\frac{1}{s^2+b^2}\right\}\right]$$

Since

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\}=\frac{1}{a}\sin(at)$$

and

$$L^{-1}\left\{\frac{1}{s^2+b^2}\right\}=\frac{1}{b}\sin(bt),$$

therefore,

$$\boxed{L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{b\sin(bt)-a\sin(at)}{b^2-a^2}}$$

Convolution Theorem

If $f(t)$ and $g(t)$ are two functions defined for $t\geq 0$, then their convolution is defined as

$$(f*g)(t)=\int_0^t f(u)\,g(t-u)\,du \qquad \cdots(1)$$

Theorem

If

$$L\{f(t)\} = \bar{f}(s),$$

and

$$L\{g(t)\} = \bar{g}(s),$$

then

$$L\{f * g\} = L\{f(t)\}\,L\{g(t)\} = \bar{f}(s)\bar{g}(s).$$

Hence,

$$\boxed{L^{-1}\{\bar{f}(s)\bar{g}(s)\} = f * g} \qquad \cdots (2)$$

Example Problems Using Convolution Theorem

Problem 1

Find

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\}$$

Solution

Given,

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right\}$$

Let

$$\bar{f}(s) = \frac{1}{s}, \qquad \bar{g}(s) = \frac{1}{s^2+a^2}$$

Then,

$$f(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$$

and

$$g(t) = L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{1}{a}\sin(at)$$

Using the Convolution Theorem,

$$L^{-1}\{\bar{f}(s)\bar{g}(s)\} = \int_0^t f\,(u)\,g(t-u)\,du = \int_0^t 1\cdot\frac{1}{a}\sin\big(a(t-u)\big)\,du = \frac{1}{a}\int_0^t \sin\,\big(a(t-u)\big)\,du$$
$$= \frac{1}{a^2}\big[\cos\big(a(t-u)\big)\big]_0^t = \frac{1}{a^2}[\cos 0 - \cos(at)]$$

Therefore,

$$\boxed{L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1-\cos(at)}{a^2}}$$

Problem 2

Find

$$L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$$

Solution

Let

$$F(s) = \frac{s}{s^2+4}, \qquad G(s) = \frac{1}{s^2+4}.$$

Then,

$$f(t) = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

Hence,

$$f(u) = \cos 2u.$$

Also,

$$g(t) = L^{-1}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin2t$$

Hence,

$$g(t-u) = \frac{1}{2}\sin2(t-u).$$

By the Convolution Theorem,

$L^{-1}\{F(s)G(s)\} = \int_0^t f\,(u)g(t-u)\,du = \int_0^t \cos 2u\cdot\frac{1}{2}\sin(2t-2u)\,du$

Using

$$2\sin A\cos B = \sin(A+B)+\sin(A-B),$$

we get

$= \frac{1}{4}\int_0^t[\sin2t+\sin(2t-4u)]\ du = \frac{1}{4}\left[u\sin2t+\frac{\cos(2t-4u)}{4}\right]_0^t = \frac{1}{4}\left[t\sin2t+\frac{\cos(-2t)}{4}-\frac{\cos2t}{4}\right]$

Since

$$\cos(-2t) = \cos2t,$$

the cosine terms cancel.

Therefore,

$$\boxed{L^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{t}{4}\sin2t}$$

Problem 3

Using Convolution Theorem, find

$$L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}$$

Solution

Let

$$F(s)=\frac{1}{s^2+1},\qquad G(s)=\frac{1}{s-2}.$$

Then,

$$f(t)=L^{-1}\left\{\frac{1}{s^2+1}\right\}=\sin t$$

Hence,

$$f(u)=\sin u.$$

Also,

$$g(t)=L^{-1}\left\{\frac{1}{s-2}\right\}=e^{2t}$$

Hence,

$$g(t-u)=e^{2(t-u)}.$$

Applying the Convolution Theorem,

$L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}=\int_0^t \sin u\, e^{2(t-u)}\,du=e^{2t}\int_0^t e^{-2u}\sin u\,du.$

Using

$$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx),$$

with

$$a=-2,\qquad b=1,$$

we obtain

$=e^{2t}\left[\frac{e^{-2u}}{5}(-2\sin u-\cos u)\right]_0^t=\frac{e^{2t}}{5}\left[e^{-2t}(-2\sin t-\cos t)-(-1)\right]$

Therefore,

$$\boxed{L^{-1}\left\{\frac{1}{(s-2)(s^2+1)}\right\}=\frac{1}{5}(e^{2t}-2\sin t-\cos t)}$$

Problem 4

Using Convolution Theorem, find

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}$$

Solution

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = \int_0^t \cos au \cdot \cos b(t-u)\,du$$

Using the identity,

$$\cos A \cos B = \frac{1}{2}[\cos(A+B)+\cos(A-B)],$$

we get

$= \frac{1}{2}\int_0^t[\cos(au+bt-bu)+\cos(au-bt+bu)]\,du = \frac{1}{2}\left[\int_0^t \cos\big((a-b)u+bt\big)\,du + \int_0^t \cos\big((a+b)u-bt\big)\,du\right]$

Using

$\cos(x+y) = \cos x\cos y - \sin x \sin y, = \frac{1}{2}\left[\int_0^t\{\cos(a-b)u\cos bt - \sin(a-b)u\sin bt\}\,du + \int_0^t\{\cos(a+b)u\cos bt + \sin(a+b)u\sin bt\}\,du\right]$

Integrating,

$$= \frac{1}{2}\left[\frac{\sin(a-b)u}{a-b}\cos bt + \frac{\cos(a-b)u}{a-b}\sin bt + \frac{\sin(a+b)u}{a+b}\cos bt - \frac{\cos(a+b)u}{a+b}\sin bt\right]_0^t$$

Substituting the limits,

$= \frac{1}{2}\left[\left\{\frac{\cos bt\sin(a-b)t}{a-b} + \frac{\cos(a-b)t\sin bt}{a-b} - \frac{\sin bt}{a-b}\right\} + \left\{\frac{\sin(a+b)t\cos bt}{a+b} - \frac{\cos(a+b)t\sin bt}{a+b} + \frac{\sin bt}{a+b}\right\}\right]$

Using the identities,

$\sin(A+B) = \sin A\cos B + \cos A \sin B, \sin(A-B) = \sin A\cos B - \cos A\sin B,$

we obtain

$$= \frac{1}{2}\left[\frac{\text{sin}at-\text{sin}bt}{a-b}+\frac{\text{sin}at+\text{sin}bt}{a+b}\right]=\frac{1}{2}\left[\frac{(a+b)\text{sin}at-(a+b)\text{sin}bt+(a-b)\text{sin}at+(a-b)\text{sin}bt}{a^2-b^2}\right]=\frac{1}{2}\left[\frac{2a\text{sin}at-2b\text{sin}bt}{a^2-b^2}\right]$$

Hence,

$$\boxed{L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}=\frac{a\text{sin}at-b\text{sin}bt}{a^2-b^2}}$$

Application of Laplace Transforms

Solving Differential Equations with Constant Coefficients

Procedure

Step 1: Write the given differential equation and apply the Laplace Transform on both sides.

Step 2: Use the given initial conditions and the following formulas:

$$L\{y'(t)\}=sL\{y(t)\}-y(0)$$

$$L\{y''(t)\}=s^2L\{y(t)\}-sy(0)-y'(0)$$

$$L\{y'''(t)\}=s^3L\{y(t)\}-s^2y(0)-sy'(0)-y''(0)$$

Step 3: Rearrange the equation and take the inverse Laplace Transform to obtain the required solution satisfying the initial conditions.

Example

Using Laplace Transform, solve

$$(D^2+4)y=6\text{cos}4t,$$

if

$$y(0)=3,\qquad y'(0)=1.$$

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Solution

Given,

$$(D^2+4)y=6\cos 4t$$

or

$$y''+4y=6\cos 4t.$$

Taking Laplace Transform on both sides,

$L\{y''+4y\}=L\{6\cos 4t\}$ $L\{y''\}+4L\{y\}=6L\{\cos 4t\}$ $[s^2L\{y\}-sy(0)-y'(0)]+4L\{y\}=\frac{6s}{s^2+16}$.

Using

$$y(0)=3,\qquad y'(0)=1,$$

we obtain

$$[s^2L\{y\}-3s-1]+4L\{y\}=\frac{6s}{s^2+16}.$$

Therefore,

$$(s^2+4)L\{y\}=\frac{6s}{s^2+16}+3s+1.$$

Hence,

$$L\{y\}=\frac{6s}{(s^2+16)(s^2+4)}+\frac{3s+1}{s^2+4}.$$

Taking inverse Laplace Transform,

$$y=6L^{-1}\left\{\frac{s}{(s^2+16)(s^2+4)}\right\}+3L^{-1}\left\{\frac{s}{s^2+4}\right\}+L^{-1}\left\{\frac{1}{s^2+4}\right\}.$$

Using partial fractions,

$$\frac{s}{(s^2+16)(s^2+4)}=\frac{1}{16-4}\left(\frac{s}{s^2+4}-\frac{s}{s^2+16}\right),$$

we get

$$y = \frac{6}{12}(\cos 2t - \cos 4t) + 3\cos 2t + \frac{1}{2}\sin 2t.$$

Therefore,

$$\boxed{y = \frac{7}{2}\cos 2t - \frac{1}{2}\cos 4t + \frac{1}{2}\sin 2t}$$

Application of Laplace Transforms

Problem 1

Using Laplace Transform, solve

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t,$$

given that

$$y(0) = 0, \qquad y'(0) = 1.$$

Solution

Given,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$$

or

$$y'' + 2y' + 5y = e^{-t}\sin t.$$

Taking Laplace Transform on both sides,

$$L\{y'' + 2y' + 5y\} = L\{e^{-t}\sin t\}.$$

Therefore,

$$L\{y''\} + 2L\{y'\} + 5L\{y\} = L\{e^{-t}\sin t\}.$$

Using the Laplace Transform formulas,

$$[s^2L\{y\} - sy(0) - y'(0)] + 2[sL\{y\} - y(0)] + 5L\{y\} = L\{\sin t\}_{s\to s+1}.$$

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Applying the initial conditions,

$$y(0) = 0, \qquad y'(0) = 1,$$

we get

$$[s^2L\{y\} - 1] + 2sL\{y\} + 5L\{y\} = \left(\frac{1}{s^2+1}\right)_{s\to s+1}.$$

Hence,

$$(s^2 + 2s + 5)L\{y\} - 1 = \frac{1}{(s+1)^2 + 1}.$$

Therefore,

$(s^2 + 2s + 5)L\{y\} = \frac{1}{s^2+2s+2} + 1. (s^2 + 2s + 5)L\{y\} = \frac{s^2+2s+3}{s^2+2s+2}.$

Thus,

$$L\{y\} = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}.$$

Taking inverse Laplace Transform,

$$y = L^{-1}\left[\frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}\right].$$

Let

$$x = s^2 + 2s + 2.$$

Then,

$$\frac{x+1}{x(x+3)} = \frac{A}{x} + \frac{B}{x+3}.$$

Hence,

$$x + 1 = A(x + 3) + Bx.$$

Putting

$x = 0, 1 = 3A\ A = \frac{1}{3}.$

Putting

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$x=-3, -2=-3B\ B=\frac{2}{3}.$

Therefore,

$y=L^{-1}\left[\frac{1/3}{s^2+2s+2}+\frac{2/3}{s^2+2s+5}\right].=\frac{1}{3}\left[L^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}+2L^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}\right].$

Using the First Shifting Theorem,

$$=\frac{1}{3}\left[e^{-t}L^{-1}\left\{\frac{1}{s^2+1}\right\}+2e^{-t}L^{-1}\left\{\frac{1}{s^2+4}\right\}\right].$$

Hence,

$$=\frac{1}{3}\left[e^{-t}\text{sin}t+\frac{2e^{-t}}{2}\text{sin}2t\right].$$

Therefore,

$$\boxed{y=\frac{e^{-t}}{3}(\text{sin}t+\text{sin}2t)}$$

Problem 2

Solve

$$(D^2+n^2)x=a\text{sin}(nt+\alpha),$$

given that

$$x(0)=0, \qquad x'(0)=0.$$

Solution

Given,

$$(D^2+n^2)x=a\text{sin}(nt+\alpha).$$

or

$$x''+n^2x=a[\text{sin}nt\text{cos}\alpha+\text{cos}nt\text{sin}\alpha].$$

Taking Laplace Transform,

$$L\{x''\}+n^2L\{x\}=acos\alpha\, L\{\text{sin}nt\}+asin\alpha\, L\{\text{cos}nt\}.$$

Therefore,

$$[s^2L\{x\}-sx(0)-x'(0)]+n^2L\{x\}=acos\alpha\left(\frac{n}{s^2+n^2}\right)+asin\alpha\left(\frac{s}{s^2+n^2}\right).$$

Since

$$x(0)=0,\qquad x'(0)=0,$$

we obtain

$$(s^2+n^2)L\{x\}=ancos\alpha\left(\frac{1}{s^2+n^2}\right)+asin\alpha\left(\frac{s}{s^2+n^2}\right).$$

Hence,

$$L\{x\}=ancos\alpha\left(\frac{1}{(s^2+n^2)^2}\right)+asin\alpha\left(\frac{s}{(s^2+n^2)^2}\right).$$

Taking inverse Laplace Transform,

$$x=ancos\alpha L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\}+asin\alpha L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\}.$$

Using the standard results,

$$L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\}=\frac{1}{2n^3}(\text{sin}nt-nt\text{cos}nt),$$

and

$$L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\}=\frac{t}{2n}\text{sin}nt.$$

Therefore,

$$x=ancos\alpha\cdot\frac{1}{2n^3}(\text{sin}nt-nt\text{cos}nt)+asin\alpha\cdot\frac{t}{2n}\text{sin}nt.$$

Hence,

$$\boxed{x=\frac{acos\alpha}{2n^2}(\text{sin}nt-nt\text{cos}nt)+\frac{atsin\alpha}{2n}\text{sin}nt}$$

Problem 3

Solve

$$y''' + 2y'' - y' - 2y = 0,$$

given that

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 6.$$

Solution

Given,

$$y''' + 2y'' - y' - 2y = 0$$

with

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 6.$$

Taking Laplace Transform on both sides,

$$L\{y'''\} + 2L\{y''\} - L\{y'\} - 2L\{y\} = L\{0\}.$$

Using the Laplace Transform formulas,

$$[p^3L\{y\} - p^2y(0) - py'(0) - y''(0)] + 2[p^2L\{y\} - py(0) - y'(0)] - (pL\{y\} - y(0)) - 2L\{y\} = 0.$$

Applying the initial conditions,

$$y(0) = 0, \qquad y'(0) = 0, \qquad y''(0) = 6,$$

we get

$$p^3L\{y\} - 6 + 2p^2L\{y\} - pL\{y\} - 2L\{y\} = 0.$$

Hence,

$$L\{y\}(p^3 + 2p^2 - p - 2) = 6.$$

Therefore,

$$L\{y\} = \frac{6}{p^3 + 2p^2 - p - 2}.$$

Factorizing,

$$p^3 + 2p^2 - p - 2 = (p-1)(p+1)(p+2).$$

Thus,

$$L\{y\} = \frac{6}{(p-1)(p+1)(p+2)}.$$

Using partial fractions,

$$\frac{1}{(p-1)(p+1)(p+2)} = \frac{A}{p-1} + \frac{B}{p+1} + \frac{C}{p+2}.$$

Multiplying throughout,

$$1 = A(p+1)(p+2) + B(p-1)(p+2) + C(p-1)(p+1).$$

Putting

$p = -1, 1 = -2B\ B = -\frac{1}{2}.$

Putting

$p = -2, 1 = 3C\ C = \frac{1}{3}.$

Putting

$p = 1, 1 = 6A\ A = \frac{1}{6}.$

Hence,

$$\frac{1}{(p-1)(p+1)(p+2)} = \frac{1}{6(p-1)} - \frac{1}{2(p+1)} + \frac{1}{3(p+2)}.$$

Taking inverse Laplace Transform,

$$y = 6\left[\frac{1}{6}L^{-1}\left\{\frac{1}{p-1}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{p+1}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{p+2}\right\}\right].$$

Using

$$L^{-1}\left\{\frac{1}{p-a}\right\} = e^{at},$$

we obtain

$$y = 6\left[\frac{1}{6}e^{t} - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}\right].$$

Therefore,

$$\boxed{y = e^{t} - 3e^{-t} + 2e^{-2t}}$$

Problem 4

Solve

$$(D^2 - 2D + 2)y = 0,$$

given that

$$y(0) = 1, \qquad y'(0) = 1.$$

Solution

Given,

$$(D^2 - 2D + 2)y = 0$$

or

$$y'' - 2y' + 2y = 0.$$

Taking Laplace Transform on both sides,

$$L\{y''\} - 2L\{y'\} + 2L\{y\} = 0.$$

Using the Laplace Transform formulas,

$$p^2L\{y\} - py(0) - y'(0) - 2[pL\{y\} - y(0)] + 2L\{y\} = 0.$$

Applying the initial conditions,

$$y(0) = 1, \qquad y'(0) = 1,$$

we get

$$p^2L\{y\} - p - 1 - 2pL\{y\} + 2 + 2L\{y\} = 0.$$

Hence,

$$(p^2 - 2p + 2)L\{y\} - p + 1 = 0.$$

Therefore,

$$L\{y\} = \frac{p-1}{p^2 - 2p + 2}.$$

Completing the square,

$$L\{y\} = \frac{p-1}{(p-1)^2 + 1}.$$

Taking inverse Laplace Transform,

$$y = L^{-1}\left\{\frac{p-1}{(p-1)^2 + 1}\right\}.$$

Using the **First Shifting Theorem**,

$$L^{-1}\left\{\frac{p-1}{(p-1)^2 + 1}\right\} = e^t L^{-1}\left\{\frac{p}{p^2 + 1}\right\}.$$

Since

$$L^{-1}\left\{\frac{p}{p^2 + 1}\right\} = \cos t,$$

the required solution is $\boxed{y = e^t \cos t}$

PRACTICE QUESTIONS

1 Find the Laplace Transform of:

1. e^{5t}
2. t^4
3. $\sin 6t$
4. $\cos 4t$
5. $\sinh 3t$
6. $\cosh 2t$
7. $t\sin 2t$

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8. $t^2 e^{-3t}$
9. $e^{2t}\cos 5t$
10. $e^{-4t}\sin 3t$

2.If $L\{\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$,find $L\left\{\frac{1}{\sqrt{t}}\right\}$.

3.If $L\{\cos\sqrt{t}\} = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-1/(4s)}$, find $L\left\{\frac{\sin\sqrt{t}}{\sqrt{t}}\right\}$.

4.Find $L\{y'' + 4y' + 3y\}$
given $y(0) = 2, y'(0) = 1.$

5.Evaluate using Laplace Transforms: $\int_0^{\infty} e^{-5t}\, sin\, 2t\, dt$

6.Find $L^{-1}\left\{\frac{5}{(s+2)(s+4)}\right\}$

7.Find $L^{-1}\left\{\frac{1}{s(s+a)(s+b)}\right\}$.

UNIT – V
Fourier Series & Fourier Transforms

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UNIT – V

Fourier Series & Fourier Transforms

Introduction :

Fourier Series and Fourier Transforms are important mathematical tools used to represent complex functions or signals in terms of simple sine and cosine functions. These techniques were developed by the French mathematician Joseph Fourier while studying heat conduction problems. They play a vital role in engineering, physics, mathematics, and computer science.
A Fourier Series is used to express a periodic function as an infinite sum of sine and cosine terms. It helps analyze periodic signals by separating them into different frequency components. Fourier Series is widely used in solving heat conduction, wave equations, vibration analysis, electrical circuits, and signal processing.
A Fourier Transform is an extension of the Fourier Series used for non-periodic functions. It converts a signal from the time domain to the frequency domain, making it easier to analyze its frequency components. Fourier Transforms are extensively used in digital signal processing, image processing, communication systems, control systems, medical imaging, radar, sonar, and many other engineering applications.
Both Fourier Series and Fourier Transforms simplify the analysis of complex engineering problems and provide efficient methods for solving differential equations, filtering signals, and designing engineering systems.

Specific Uses of Fourier Series

- Represents **periodic functions** as a sum of sine and cosine terms.
- Solves heat conduction and wave equations.
- Studies mechanical vibrations.
- Performs harmonic analysis of electrical signals.
- Analyzes periodic signals in communication systems.
- Used in structural vibration analysis.

Specific Uses of Fourier Transforms

- Converts a **time-domain signal** into the **frequency domain**.
- Performs frequency spectrum analysis.
- Enables signal compression and reconstruction.
- Used in image processing and enhancement.
- Designs digital filters.
- Analyzes radar and sonar signals.
- Processes biomedical signals such as ECG and EEG.
- Used in speech and audio processing.

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- Supports wireless communication systems.
- Solves partial differential equations (PDEs).

Periodic Function

A function $f(x)$ is said to be **periodic** if it satisfies the relation

$$f(x+p)=f(x), \forall x$$

where p is some fixed positive real number. The real number p is called the **period** of the function. The smallest positive number p for which this relation holds is called the **period** (or **fundamental period**) of $f(x)$.

Fourier Series

A periodic function $f(x)$, defined in an interval of length 2π, say

$$[c,\ c+2\pi], [0,\ 2\pi], [-\pi,\ \pi]$$

can be expressed in the form

$$\begin{aligned} f(x) = {} & \frac{a_0}{2}+a_1\cos x+a_2\cos 2x+\cdots+a_n\cos nx+\cdots \\ & +b_1\sin x+b_2\sin 2x+\cdots+b_n\sin nx+\cdots \end{aligned}$$

or

$$\begin{aligned} f(x) = {} & \frac{a_0}{2}+(a_1\cos x+b_1\sin x)+(a_2\cos 2x+b_2\sin 2x) \\ & +\cdots+(a_n\cos nx+b_n\sin nx)+\cdots \end{aligned}$$

Hence,

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)}$$

where

$$a_0,\ a_n,\ b_n(n=1,2,\dots)$$

are constants. Such a series is known as the **Fourier Series**. The constants a_0, a_n, and b_n are called the **Fourier coefficients** of $f(x)$.

Fourier Series is an infinite series expansion of a periodic function in terms of sines and cosines of an angle and its multiples.

Fourier Series is useful to solve ordinary and partial differential equations, particularly with periodic functions appearing as non-homogeneous terms.

Euler's Formulae

The Fourier series for the function $f(x)$ in the interval

$$c \le x \le c + 2\pi$$

is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos(nx)\,dx$$
$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin(nx)\,dx$$

The values of $a_0,\ a_n,\ b_n$ are known as **Euler's Formulae**.

Dirichlet Conditions:

A function $f(x)$ has a valid Fourier series expansion if

1. $f(x)$ is well-defined, periodic and single-valued.
2. $f(x)$ has only a finite number of discontinuities in any one period.
3. $f(x)$ has at most a finite number of maxima and minima in the interval of definition.

Formulae:

1.

$$\sin\int (nx)\ dx = -\frac{\cos(nx)}{n} + C$$

2.

$$\cos\int (nx)\ dx = \frac{\sin(nx)}{n} + C$$

3.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

(Integration by Parts)

4.

$$e^{ax}\sin\int(bx)\ dx=\frac{e^{ax}}{a^2+b^2}(a\sin bx-b\cos bx)+C$$

5.

$$e^{ax}\cos\int(bx)\ dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)+C$$

6.

$$\sin 0=0,\sin(n\pi)=0,\cos 0=1,\cos(n\pi)=(-1)^n$$

Some Special Cases

(i) If $c=0$, then

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos(nx)\,dx$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin(nx)\,dx$$

(ii) If $c=-\pi$, then

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx \qquad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx$$

Even and Odd Functions

Even Function

A function $f(x)$is said to be **even** if

$$f(-x)=f(x)$$

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Examples: $x^2, \cos x, \tan^2 x$

Odd Function

A function $f(x)$is said to be **odd** if

$$f(-x) = -f(x)$$

Examples: $\sin x, x^3$

Problems

Problem 1

Obtain the Fourier series of $f(x) = x^2$in the interval $(0, 2\pi)$.

Solution:

The Fourier series of $f(x) = x^2$in $0 < x < 2\pi$
is given by

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) \qquad (1)$$

Using Euler's formulae, determine $a_0,\ a_n,\ b_n$.

Constant Term $a_0 = \frac{1}{\pi}\int_0^{2\pi} x^2\ dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{(2\pi)^3}{3} - 0\right] = \frac{8\pi^2}{3}$

Cosine coefficient

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\ dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos(nx)\ dx$$

$$= \frac{1}{\pi}\left[\frac{x^2\sin nx}{n}\right]_0^{2\pi} + \frac{2}{\pi n^2}[x\cos nx]_0^{2\pi} - \frac{2}{\pi n^3}[\sin nx]_0^{2\pi}$$

$$= \frac{1}{n\pi}[4\pi^2\sin 2n\pi - 0] + \frac{2}{\pi n^2}[2\pi\cos 2n\pi - 0] - \frac{2}{\pi n^3}[\sin 2n\pi - 0]$$

$$= \frac{1}{n\pi}[0 - 0] + \frac{2}{\pi n^2}[2\pi(1) - 0] - \frac{2}{\pi n^3}[0 - 0] = \frac{4}{n^2}$$

Sine coefficient

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\ dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\ dx$$

$$= \frac{1}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= -\frac{1}{n\pi}[x^2\cos nx]_0^{2\pi} + \frac{2}{\pi n^2}[x\sin nx]_0^{2\pi} + \frac{2}{\pi n^3}[\cos nx]_0^{2\pi}$$

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$$= -\frac{1}{n\pi}[4\pi^2\cos 2n\pi - 0] + \frac{2}{\pi n^2}[2\pi\sin 2n\pi - 0] + \frac{2}{\pi n^3}[\cos 2n\pi - \cos 0]$$

$$= -\frac{1}{n\pi}[4\pi^2 - 0] + \frac{2}{\pi n^2}[0 - 0] + \frac{2}{\pi n^3}[1 - 1] = -\frac{4\pi}{n}$$

Substituting the values of a_0, $a_n$and $b_n$in ①,

$$f(x) = x^2 = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \left(\frac{-4\pi}{n}\right)\sin nx$$

$$= \frac{4}{3}\pi^2 + 4\left[\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-4\pi\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots\right].$$

Problem 2

Find the Fourier series representing

$$f(x) = x, 0 < x < 2\pi.$$

Solution:

The Fourier series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$.

Using Euler's formulae,

Constant term

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi} = 2\pi.$$

Hence,

$$\frac{a_0}{2} = \pi.$$

Cosine coefficient

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x\sin(nx)}{n} + \frac{\cos(nx)}{n^2}\right]_0^{2\pi}$$

$$= 0.$$

Sine coefficient

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[-\frac{x\cos(nx)}{n} + \frac{\sin(nx)}{n^2}\right]_0^{2\pi}$$

$$= -\frac{2}{n}.$$

Hence,

$$\boxed{f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}}$$

or

$$\boxed{x = \pi - 2\left(\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right).}$$

Problem 3

Find the Fourier series of $f(x) = x + x^2$ in the interval $[-\pi, \pi]$ Hence deduce $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \frac{1}{12}$.

Solution:

Given

$$f(x) = x + x^2 \text{in the interval } [-\pi, \pi]$$

$$f(x) = x + x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \rightarrow (1)$$

Using Euler's formulae, we determine a_0, a_n, b_n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{2\pi}[\pi^2 - (-\pi)^2] + \frac{1}{3\pi}[\pi^3 - (-\pi)^3]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} - (1+2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{n\pi}[(x+x^2)\sin nx]_{-\pi}^{\pi} + \frac{1}{n^2\pi}[\cos nx]_{-\pi}^{\pi} + \frac{2}{n^2\pi}(x\cos nx)_{-\pi}^{\pi} - \frac{2}{n^3\pi}(\sin nx)_{-\pi}^{\pi}$$

$$= \frac{1}{n\pi}[(\pi+\pi^2)\sin n\pi - (-\pi+\pi^2)\sin(-n\pi)] + \frac{1}{n^2\pi}[\cos n\pi - \cos(-n\pi)]$$

$$+ \frac{2}{n^2\pi}[\pi\cos n\pi + \pi\cos(-n\pi)] - \frac{2}{n^3\pi}[\sin n\pi - \sin(-n\pi)]$$

$$= \frac{1}{n\pi}(0-0) + \frac{1}{n^2\pi}[\cos n\pi - \cos n\pi] + \frac{2}{n^2\pi}\,2\pi(-1)^n - \frac{2}{n^3\pi}(0)$$

$$= (-1)^n\frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{n\pi}[(x+x^2)\cos nx]_{-\pi}^{\pi} - \frac{1}{n^2\pi}(\sin nx)_{-\pi}^{\pi} - \frac{2}{n^2\pi}(x\sin nx)_{-\pi}^{\pi} + \frac{2}{n^3\pi}(\cos nx)_{-\pi}^{\pi}$$

$$= -\frac{1}{n\pi}[(\pi+\pi^2)\cos n\pi - (-\pi+\pi^2)\cos(-n\pi)] - \frac{1}{n^2\pi}[\sin n\pi - \sin(-n\pi)]$$

$$- \frac{2}{n^2\pi}[\pi\sin n\pi + \pi\sin(-n\pi)] + \frac{2}{n^3\pi}[\cos n\pi - \cos(-n\pi)]$$

Substituting the values of a_0, a_n& $b_n$in eqn (1)

$$f(x) = x + x^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty}(-1)^n\frac{4}{n^2}\cos nx + (-1)^{n+1}\frac{2}{n}\sin nx$$

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

put $x = 0$, we get

$$0 + 0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos 0^\circ + \frac{1}{2^2}\cos 0^\circ - \frac{1}{3^2}\cos 0^\circ + \frac{1}{4^2}\cos 0^\circ + \cdots\right]$$

$$+2\left[\frac{1}{1}\sin 0^\circ - \frac{1}{2}\sin 0^\circ + \frac{1}{3}\sin 0^\circ - \cdots\right]$$

$$0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right]$$

$$\Rightarrow -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right] + \frac{\pi^2}{3} = 0$$

$$\Rightarrow 4\left[\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\right]=\frac{\pi^2}{3}$$

$$\therefore \frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$$

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4. Obtain the Fourier Series of $f(x)=(\pi-x)^2$ in the interval $(0, 2\pi)$

Hence deduce that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}$$

Sol:

The Fourier Series of

$f(x)=(\pi-x)^2$ in $(0,2\pi)$ is given by

$$f(x)=(\pi-x)^2=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx \rightarrow (1)$$

Using Euler's formulae, we determine the values of

$a_0,\ a_n\ \&\ b_n$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\,dx=\frac{1}{\pi}\left[\frac{(\pi-x)^3}{3(-1)}\right]_0^{2\pi}$$

$$=-\frac{1}{3\pi}\left[(\pi-2\pi)^3-(\pi-0)^3\right]$$

$$=-\frac{1}{3\pi}\left[\ \ -\pi^3-\pi^3\right]=\frac{2\pi^3}{3\pi}=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\cos nx\,dx$$

$$=\frac{1}{\pi}\left[(\pi-x)^2\left(\frac{\sin nx}{n}\right)-2(\pi-x)(-1)\left(\frac{-\cos nx}{n^2}\right)+(-2)(-1)\left(\frac{\sin nx}{n^3}\right)\right]$$

$$=\frac{1}{n\pi}[0-0]-\frac{2}{\pi n^2}[-\pi-\pi]-\frac{2}{\pi n^3}(0-0)=\frac{4}{n^2}$$

$$=\frac{1}{\pi}\left[(\pi-x)^2\left(\frac{-\cos nx}{n}\right)-2(\pi-x)(-1)\left(\frac{-\sin nx}{n^2}\right)+(-2)(-1)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$=-\frac{1}{n\pi}\left[(\pi-x)^2\cos nx\right]_0^{2\pi}-\frac{2}{\pi n^2}\left[(\pi-x)\sin nx\right]_0^{2\pi}+\frac{2}{\pi n^3}(\cos nx)_0^{2\pi}$$

$$=-\frac{1}{n\pi}\left[(\pi-2\pi)^2\cos 2n\pi-(\pi-0)^2\cos 0^\circ\right]-\frac{2}{\pi n^2}\left[(\pi-2\pi)\sin 2n\pi-(\pi-0)\sin 0^\circ\right]$$

$$+\frac{2}{\pi n^3}\left[\cos 2n\pi-\cos 0^\circ\right]$$

$$= -\frac{1}{n\pi}[\pi^2 - \pi^2] - \frac{2}{\pi n^2}[0-0] + \frac{2}{\pi n^3}[1-1] = 0$$

Substituting these values $a_0,\ a_n,\ \&\ b_n$

$$(\pi - x)^2 = \frac{2\pi^2}{2(3)} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx + 0$$

$$= \frac{\pi^2}{3} + 4\left[\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

put $x = 0$, we get

$$(\pi - 0)^2 = \frac{\pi^2}{3} + 4\left[\frac{1}{1^2}\cos 0^\circ + \frac{1}{2^2}\cos 0^\circ + \frac{1}{3^2}\cos 0^\circ + \cdots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{1}{4}\left(\frac{2\pi^2}{3}\right) = \frac{\pi^2}{6}$$

---

5. Determine the Fourier series expansion of the function

$f(x) = e^x$ in $(0, 2\pi)$

Sol:

The Fourier series of the function $f(x) = e^x$ in the interval $(0, 2\pi)$ is given by

$$f(x) = e^x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Using Euler's formulae, we determine $a_0,\ a_n,\ \&\ b_n$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x\,dx = \frac{1}{\pi}[e^x]_0^{2\pi}$$

$$= \frac{1}{\pi}(e^{2\pi} - e^0) = \frac{1}{\pi}(e^{2\pi} - 1)$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_0^{2\pi}$$

$$\left(\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right)$$

$$= \frac{1}{\pi(n^2+1)}[e^{2\pi}(\cos 2n\pi + n\sin 2n\pi) - e^0(\cos 0 + n\sin 0)]$$

$$= \frac{1}{\pi(n^2+1)}[e^{2\pi}(1+0) - 1(1+0)] = \frac{1}{\pi(n^2+1)}(e^{2\pi}-1) = \frac{e^{2\pi}-1}{\pi(n^2+1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\sin nx - n\cos nx)\right]_0^{2\pi}$$

$$\left(\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right)$$

$$= \frac{1}{\pi(n^2+1)}[e^{2\pi}(\sin 2n\pi - n\cos 2n\pi) - e^0(\sin 0^\circ - n\cos 0^\circ)]$$

$$= \frac{1}{\pi(n^2+1)}[e^{2\pi}(0-n) - 1(0-n)] = \frac{1}{\pi(n^2+1)}(n - ne^{2\pi})$$

Substitute the values of a_0, a_n& $b_n$in (1)

$$f(x) = e^x = \frac{e^{2\pi}-1}{2\pi} + \sum_{n=1}^{\infty}\frac{e^{2\pi}-1}{\pi(n^2+1)}\cos nx + \sum_{n=1}^{\infty}\frac{n(1-e^{2\pi})}{\pi(n^2+1)}\sin nx$$

$$= \frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2+1}\cos nx - \sum_{n=1}^{\infty}\frac{n}{n^2+1}\sin nx\right]$$

6. Find the Fourier Series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right]$$

$$= \frac{1}{\pi}\int_0^{\pi} 1\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0\,dx = \frac{1}{\pi}[x]_0^{\pi} + \frac{1}{\pi}(\pi - 0) = 1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 1 \cdot \cos nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0 \cdot \cos nx\, dx$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} = \frac{1}{n\pi}(0-0) = 0$$

Similarly,

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{\pi} f(x)\sin nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} 1 \cdot \sin nx\, dx + \frac{1}{\pi}\int_{\pi}^{2\pi} 0 \cdot \sin nx\, dx = \frac{1}{n\pi}[1-(-1)^n]$$

Substitute these values in $f(x)$, we get

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}[1-(-1)^n]\sin nx$$

$$= \frac{1}{2} + \frac{1}{\pi}\left[2\sin x + \frac{2}{3}\sin 3x + \frac{2}{5}\sin 5x + \cdots\right]$$

$$\therefore\ f(x) = \frac{1}{2} + \frac{2}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right]$$

Fourier Series of odd and even functions:

A function $f(x)$, defined in $(-\pi,\pi)$can be represented by Fourier series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

Case (i): let $f(x)$be an even function in $(-\pi,\pi)$

i.e.

$$f(-x) = f(x)$$

Then

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

∴ Fourier series of an even function contains only cosine terms and is known as **Fourier cosine series**.

i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

&

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

Case (2):

let $f(x)$be an odd function in $(-\pi, \pi)$

i.e.

$$f(-x) = -f(x)$$
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0, a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

Thus the Fourier series of an odd function contains only sine terms and is known as **Fourier sine series**.

is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Here,

$$b_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

Problems

1. Find the Fourier Series to represent the function $f(x) = x^2$ in the interval $(-\pi, \pi)$

$$f(-x) = (-x)^2 = x^2 = f(x)$$

Hence $f(x)$is an even function.

Hence, $b_n = 0$and the Fourier series reduces to Fourier cosine series given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{3\pi}[\pi^3 - 0] = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^2 \sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{4}{n^2}(-1)^n$$

Substitute these values in $f(x)$, we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$x^2 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right]$$

2. Expand the function $f(x) = x^3$**as a Fourier series in the interval** $(-\pi, \pi)$

Sol: We have

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

$\therefore f(x)$is odd function. Hence $a_0 = 0$& $a_n = 0$

Hence Fourier expansion reduces to Fourier sine series.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\, dx$$

$$= \frac{2}{\pi}\left[-\frac{x^3\cos nx}{n} + \frac{3x^2\sin nx}{n^2} + \frac{6x\cos nx}{n^3} - \frac{6\sin nx}{n^4}\right]_0^{\pi}$$

$$= -\frac{2}{n\pi}[\pi^3(-1)^n - 0] + \frac{6}{n^2\pi}(0-0) + \frac{12}{n^3\pi}[\pi(-1)^n - 0] - \frac{12}{n^4\pi}(0-0)$$

$$= \frac{2\pi^2}{n}(-1)^{n+1} + \frac{12(-1)^n}{n^3}$$

Substitute these value in $f(x)$, we get

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = \sum_{n=1}^{\infty}\left[\frac{2\pi^2}{n}(-1)^{n+1} + \frac{12(-1)^n}{n^3}\right]\sin nx$$

$$= 2\pi^2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right] - 12\left[\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \cdots\right]$$

Fourier series in an arbitrary interval (Change of interval)

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, c < x < c + 2l$$

Case 1: If we put $c = 0$, interval $0 < x < 2l$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\, dx, a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\, dx, b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\, dx$$

Case 2: If $c = -l, \ -l < x < l$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx, a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\, dx, b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\, dx$$

f $f(x)$is an odd function,

$$a_0 = a_n = 0, b_n = \frac{2}{l}\int_0^{l} f(x)\sin\frac{n\pi x}{l}\, dx$$

If $f(x)$is an even function, we have

$$a_0 = \frac{2}{l}\int_0^{l} f(x)\, dx, a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\, dx, b_n = 0$$

Problems:

1. Find a Fourier series to represent $f(x) = x^2$ in $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

If $f(x)$ is even i.e.

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$$\therefore b_n = 0$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\int_{-l}^{l} x^2\,dx = \frac{2}{l}\int_{0}^{l} x^2\,dx = \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{2l^2}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_{0}^{l} x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[\frac{x^2\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} + \frac{2x\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} - \frac{2\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^l$$

$$= \frac{2}{n\pi}\left[x^2\sin\frac{n\pi x}{l}\right]_0^l + \frac{4l}{n^2\pi^2}\left[x\cos\frac{n\pi x}{l}\right]_0^l - \frac{4l^2}{n^3\pi^3}\left[\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4l^2}{n^2\pi^2}(-1)^n$$

Substitute these values in $f(x)$, we get

$$f(x) = \frac{l^2}{3} + \sum\frac{4l^2}{n^2\pi^2}(-1)^n\cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} - \cdots\right]$$

2. **Find the Fourier series expansion for $f(x)$ if**

$$f(x) = \begin{cases} 0, \text{if } -l < x < 0 \\ 1, \text{if } 0 < x < l \end{cases}$$

Sol:

Let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx = \frac{1}{l}\left[\int_{-l}^{0} f(x)\,dx + \int_{0}^{l} f(x)\,dx\right]$$

$$= \frac{1}{l}\int_{-l}^{0} 0\,dx + \frac{1}{l}\int_{0}^{l} 1\,dx = \frac{1}{l}(l-0) = 1$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{0} f(x)\cos\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_{-l}^{0} 0\cdot\cos\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l} 1\cdot\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{n\pi}[\sin n\pi - 0] = 0$$

$$= \frac{1}{l}\int_{-l}^{0} 0\cdot\sin\frac{n\pi x}{l}\,dx + \frac{1}{l}\int_{0}^{l} 1\cdot\sin\frac{n\pi x}{l}\,dx$$

$$= -\frac{1}{n\pi}[\cos n\pi - 1] = \frac{1}{n\pi}[1-(-1)^n]$$

If $n$is odd,

$$b_n = \frac{1}{n\pi}[1+1] = \frac{2}{n\pi}$$

If $n$is even,

$$b_n = \frac{1}{n\pi}[1-1] = 0$$

Substituting these values in $f(x)$, we get

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{n\pi}[1-(-1)^n]\sin\frac{n\pi x}{l} = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,3,5}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \frac{1}{5}\sin\frac{5\pi x}{l} + \cdots\right]$$

Half Range Series

Fourier cosine and sine series:

$$-l < x < l$$

Then

The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx, a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

The half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\,dx,$$

where

$$b_n = \frac{2}{\pi}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

If the range is $0 < x < \pi$, then

(i) The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

(ii) The half-range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

1. Find half range cosine series of $f(x) = 1,\ 0 < x < 2$

Sol:

The half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 1\,dx = [x]_0^2 = 2, \frac{a_0}{2} = \frac{2}{2} = 1$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx = \int_0^2 \cos\frac{n\pi x}{2}\,dx = \left[\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right]_0^2$$
$$= \frac{2}{n\pi}\sin(n\pi) = 0$$
$$\therefore f(x) = 1 + 0 = 1$$

Problem 2

Find the half-range cosine series of

$$f(x) = e^x, 0 < x < 1.$$

Solution

The half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l},$$

where $l = 1$.

Hence,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x.$$

Constant term

$$a_0 = \frac{2}{1}\int_0^1 e^x\,dx = 2[e^x]_0^1 = 2(e-1).$$

Therefore,

$$\frac{a_0}{2} = e - 1.$$

Cosine coefficients

$$a_n = \frac{2}{1}\int_0^1 e^x\cos(n\pi x)\,dx.$$

Using

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx),$$

we obtain

$$a_n = \frac{2}{1+n^2\pi^2}[e(\cos n\pi + n\pi\sin n\pi) - 1].$$

Since

$$\sin n\pi = 0, \cos n\pi = (-1)^n,$$
$$a_n = \frac{2\,[\,e(-1)^n - 1\,]}{1 + n^2\pi^2}.$$

Hence,

$$\boxed{e^x = (e-1) + \sum_{n=1}^{\infty} \frac{2\,[\,e(-1)^n - 1\,]}{1 + n^2\pi^2}\cos(n\pi x)}$$

for

$$0 < x < 1.$$

Problem 3
Find the half-range cosine series for $f(x) = x, 0 < x < \pi$.
Solution:

Since $f(-x) = -x = -f(x)$, the function is **odd**.
Hence,

$$a_0 = 0, a_n = 0,$$

The half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

Constant term

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi.$$

Hence,

$$\frac{a_0}{2} = \frac{\pi}{2}.$$

Cosine coefficients

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx$$

Integrating by parts,

$$= \frac{2}{\pi}\left[\frac{x\sin(nx)}{n}+\frac{\cos(nx)}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{(\quad -1)^n-1}{n^2}\right].$$

Therefore,

$$\boxed{x=\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(\quad -1)^n-1}{n^2}\cos(nx)}$$

or equivalently,

$$x=\frac{\pi}{2}-\frac{4}{\pi}\left(\cos x+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right).$$

Find half-range sine series for $f(x)=x(\pi-x),\ 0<x<\pi$ and hence deduce that

$$1-\frac{1}{3^3}+\frac{1}{5^3}-\cdots=\frac{\pi^3}{32}$$

Sol:

$$f(x)=x(\pi-x), 0<x<\pi$$

$$f(x)=\sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n=\frac{2}{\pi}\int_0^{\pi} x(\pi-x)\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}(\pi x-x^2)\sin nx\,dx$$

$$=\frac{2}{\pi}\left[\pi\int_0^{\pi} x\sin nx\,dx-\int_0^{\pi} x^2\sin nx\,dx\right]$$

$$=\frac{2}{\pi}\pi\left[x\left(\frac{-\cos nx}{n}\right)-\left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}-\frac{2}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right)-2x\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$=2\left[\left(\frac{\pi(-(-1)^n)}{n}-0\right)-(0+0)\right]$$

$$-\frac{2}{\pi}\left[\left(\frac{\pi^2(-(-1)^n)}{n}-2\pi(0)+\frac{2(-1)^n}{n^3}\right)-\left(0-0+\frac{2}{n^3}\right)\right]$$

$$=\frac{4}{n^3\pi}[1-(-1)^n]$$

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$$x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)$$

Deduction
Put

$$x = \frac{\pi}{2}$$

$$\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left(\frac{\sin\frac{\pi}{2}}{1^3} + \frac{\sin\frac{3\pi}{2}}{3^3} + \frac{\sin\frac{5\pi}{2}}{5^3} + \cdots\right)$$

$$\frac{\pi^2}{4} \times \frac{\pi}{8} = 1 - \frac{1}{3^3} + \frac{1}{5^3} + \cdots$$

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32}$$

Fourier Integral (without Proof)

The Fourier integral of $f(x)$is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\lambda(t - x)\,dt\,d\lambda$$

The Fourier cosine integral of $f(x)$is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty \int_0^\infty f(t)\cos\lambda t\cos\lambda x\,dt\,d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \cos\lambda x\left(\int_0^\infty f\,(t)\cos\lambda t\,dt\right)d\lambda$$

The Fourier sine integral of $f(x)$is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\lambda x\left(\int_0^\infty f\,(t)\sin\lambda t\,dt\right)d\lambda$$

Problem

Using Fourier Integral show that

$$e^{-ax} - e^{-bx} = \frac{e(b^2 - a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2 + a^2)(\lambda^2 + b^2)}\,d\lambda, a, b > 0$$

Sol:

RHS contain sin-terms so use Fourier sine integral

$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty f(t)\sin\lambda t\,dt\right]d\lambda$$

Let

$$f(x)=e^{-ax}-e^{-bx}$$
$$f(t)=e^{-at}-e^{-bt}$$
$$f(x)=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty (e^{-at}-e^{-bt})\sin\lambda t\,dt\right]d\lambda$$
$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\int_0^\infty e^{-at}\sin\lambda t\,dt-\int_0^\infty e^{-bt}\sin\lambda t\,dt\right]d\lambda$$
$$=\frac{2}{\pi}\int_0^\infty \sin\lambda x\left[\frac{\lambda}{a^2+\lambda^2}-\frac{\lambda}{b^2+\lambda^2}\right]d\lambda$$
$$=\frac{2}{\pi}\int_0^\infty \lambda\sin\lambda x\left[\frac{b^2-a^2}{(a^2+\lambda^2)(b^2+\lambda^2)}\right]d\lambda$$
$$e^{-ax}-e^{-bx}=\frac{2(b^2-a^2)}{\pi}\int_0^\infty \frac{\lambda\sin\lambda x}{(\lambda^2+a^2)(\lambda^2+b^2)}\,d\lambda$$

Fourier Transform of $f(x)$

The Fourier transform of a function $f(x)$is given by

$$\mathcal{F}\{f(x)\}=F(p)=\int_{-\infty}^{\infty} f(x)\,e^{ipx}\,dx$$

The inverse Fourier transform of $F(p)$is given by

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(p)\,e^{-ipx}\,dp$$

Fourier Sine Transform

Fourier sine transform of $f(x)$is

$$F_s(p)=F_s\{f(x)\}=\int_0^\infty f(x)\sin px\,dx$$

Inverse Fourier sine transform of $F_s(p)$is given by

$$f(x)=\frac{2}{\pi}\int_0^\infty F_s(p)\sin px\,dp$$

Fourier Cosine Transform

Fourier cosine transform of $f(x)$is

$$F_c(p) = F_c\{f(x)\} = \int_0^\infty f(x)\cos px\,dx$$

Inverse Fourier Cosine Transform of $F_c(p)$is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_c\,(p)\cos px\,dp$$

Dirichlet Conditions:

A function $f(x)$is said to satisfy Dirichlet's conditions in the interval (a, b)if
(i) $f(x)$defined and is single valued function except possibly at finite no. of points in the interval (a, b).
(ii) $f(x)$& $f'(x)$are piecewise continuous in (a, b).

Fourier Transforms

Definition: Let $f(x)$be a function indefinite or finite on $(-\infty, \infty)$and be piecewise continuous in each finite partial interval and absolutely integrable in $(-\infty, \infty)$, then

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}\,f(x)\,dx$$

is called the **Fourier transform** of $f(x)$and it is denoted by

$$F\{f(x)\} = F(p)$$

The function $f(x)$is called the **inverse Fourier transform** of $F(p)$.

$$F\{f(x)\} = F(p)$$

or

$$f(x) = F^{-1}\{F(p)\}$$

Inverse Fourier transform: The function $f(x)$is called the inverse Fourier transform of $F(p)$, i.e.,

$$f(x) = F^{-1}\{F(p)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\,F(p)\,dp$$

Some authors also define Fourier transform in the following forms:

$$F\{f(x)\} = F(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\,f(x)\,dx$$

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$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} F(p)\, dp$$

or

$$F\{f(x)\} = F(p) = \int_{-\infty}^{\infty} e^{-ipx} f(x)\, dx$$
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} F(p)\, dp$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} F(p)\, dp$$

Fourier integral formula:
The function

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(\lambda) \left[\int_{-\infty}^{\infty} e^{i\omega(x-\lambda)}\, d\omega \right] d\lambda$$

Dirichlet's conditions:

The function $f(x)$is satisfy Dirichlet's conditions in the conditions (a, b)if

- $f(x)$is defined and single valued, finite no. of points in the interval $(a, b)$and
- $f(x)$and $f'(x)$are piecewise condition in the interval (a, b)

Fourier Sine Transform:

The infinite Fourier sine transform of $f(x)$in $(0, \infty)$is defined by

$F_s\{f(x)\}$ or $\bar{F}_s(p)$,

such that

$$F_s\{f(x)\} = \bar{F}_s(p) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin px\, dx$$

The function $f(x)$is called the inverse Fourier sine transform of $\bar{F}_s(p)$, i.e.,

$$f(x) = F_s^{-1}\{\bar{F}_s(p)\}$$

Inverse formula of Fourier Sine Transform:
If

$$f(x) = F_s^{-1}\{\bar{F}_s(p)\}$$

and the function $f(x)$satisfy Dirichlet's condition in every finite interval $(0, \Omega)$and is such that

$$\int_0^\infty |\, f(x) \,|\ dx$$

exist such that

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{F_s}\,(p) \sin px \, dp$$

at every point.
of continuity of $f(x)$.

Fourier Cosine Transform:

The infinite Fourier cosine transform of $f(x)$in $(0, \infty)$is defined by
$F_c\{f(x)\}$ or $\bar{F_c}(p)$,

such that

$$F_c\{f(x)\} = \bar{F_c}(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx$$

The function $f(x)$is called the inverse Fourier cosine transform of $\bar{F_c}(p)$, i.e.,
$f(x) = F_c^{-1}\{\bar{F_c}(p)\}$

Inverse formula for Fourier Cosine Transform:

If

$$f(x) = F_c^{-1}\{\bar{F_c}(p)\}$$

and the function $f(x)$satisfies Dirichlet's condition in every finite interval $(0, l)$and is such that

$$\int_0^\infty |\, f(x) \,|\ dx$$

exist such that

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{F_c}\,(p) \cos px \, dp$$

at every point of continuity of $f(x)$.

Theorem:

Show that linearity Property of Fourier Transform

Statement: If

$$F\{f(x)\} = \bar{F}(p)$$

and

$$F\{g(x)\} = \bar{G}(p)$$

then

$$F\{af(x) + bg(x)\} = a\bar{F}(p) + b\bar{G}(p),$$

where $a, b$are constants.

Proof:

We have

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$
$$F\{g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} g(x)\, dx$$

Hence the proof.

Problems:

1) Find the Fourier transform of $f(x)$defined by

$$f(x) = \begin{cases} 1, & \text{if } |x| < a, \\ 0, & \text{if } |x| > a. \end{cases}$$

and hence evaluate

$$(a)\ \int_{-\infty}^{\infty} \frac{\sin pa\cos px}{p}\, dp, (b)\ \int_{0}^{\infty} \frac{\sin p}{p}\, dp.$$

Given that

$$f(x) = \begin{cases} 1, & \text{if } |x| \leq a, \\ 0, & \text{if } |x| > a. \end{cases}$$

i.e.,

$$f(x) = \begin{cases} 1, & \text{if } -a \leq x \leq a, \\ 0, & \text{otherwise.} \end{cases}$$

By the def of Fourier transform

$$F\{f(x)\} = F(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$
$$F(p) = \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{-a} + + \int_{-a}^{a} + + \int_{a}^{\infty}\right) e^{ipx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-a} e^{ipx}\, f(x)\, dx + \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ipx}\, f(x)\, dx + \frac{1}{\sqrt{2\pi}}\int_{a}^{\infty} e^{ipx}\, f(x)\, dx$$

$$= 0 + \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ipx}\,(1)\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} e^{ipx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{ipx}}{ip}\right]_{-a}^{a}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{ip}[2i\sin pa]$$

$$= \frac{1}{p} \cdot \frac{1}{\sqrt{2\pi}}[2\sin pa]$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}}\left(\frac{\sin pa}{p}\right)$$

$$= \sqrt{\frac{2}{\pi}}\,\frac{\sin pa}{p}$$

$$F(p) = \sqrt{\frac{2}{\pi}}\,\frac{\sin pa}{p}$$

$$F\{f(x)\} = F(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ipx}\, f(x)\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\, F(p)\, dp$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ipx}\, \sqrt{\frac{2}{\pi}}\frac{\sin pa}{p}\, dp$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} \frac{e^{-ipx}\sin pa}{p}\, dp$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}}\int_{-\infty}^{\infty} (\cos px - i\sin px)\,\frac{\sin pa}{p}\, dp$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} (\cos px - i\sin px)\,\frac{\sin pa}{p}\, dp$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos px \sin pa}{p}\, dp - \int_{-\infty}^{\infty} \frac{i\sin px \sin pa}{p}\, dp$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin p a \cos px}{p}\, dp = \begin{cases} \pi, & \text{if } \mid x \mid < a, \\ 0, & \text{if } \mid x \mid > a. \end{cases}$$

Now comparing the real & imaginary part
of the above equation on both sides,

$$\int_{-\infty}^{\infty} \frac{\sin p a \cos px}{p}\, dp = \begin{cases} \pi, & \mid x \mid < a, \\ 0, & \mid x \mid > a. \end{cases}$$

put $x = 0,\ a = 1$

$$= \int_{-\infty}^{\infty} \frac{\sin p \cos(p \cdot 0)}{p}\, dp - \int_{-\infty}^{\infty} \frac{i \sin p \sin p}{p}\, dp$$

$$2 \int_{0}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \frac{\sin p}{p}\, dp$$

$$\pi = 2 \int_{-\infty}^{\infty} \frac{\sin p}{p}\, dp$$

$$\pi = 2 \int_{0}^{\infty} \frac{\sin p}{p}\, dp$$

$$\therefore \int_{0}^{\infty} \frac{\sin p}{p}\, dp = \frac{\pi}{2}$$

---

(2) Find the Fourier transform of

$$f(x) = \begin{cases} e^{-x}, & a < x < b, \\ 0, & x < a,\ x > b. \end{cases}$$

we know that

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left(\int_{-\infty}^{a} + + \int_{a}^{b} + + \int_{b}^{\infty} \right) e^{ipx} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[0 + \int_{a}^{b} e^{ipx} f(x)\, dx + 0 \right]$$

$$= \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{ipx} f(x)\, dx$$

2) Find the Fourier sine & cosine transform of

$F(x)$ is

$$F(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 0, & x > 1. \end{cases}$$

we have

$$F_s\{f(x)\} = \bar{F}_s(p)$$

$$\bar{F}_s(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty F(x) \sin px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^1 + + \int_1^\infty \right) F(x) \sin px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin px \, (1) \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \sin px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\quad -\frac{\cos px}{p} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \left(\quad -\frac{1}{p} \right) [\cos px]_0^1$$

Cosine

$$\bar{F}_c(p) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos px \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left(\int_0^1 f\,(x) \cos px \, dx + \int_1^\infty f\,(x) \cos px \, dx \right)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 1 \cdot \cos px \, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin px}{p} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{p} [\sin p(1) - \sin p(0)]$$

$$\bar{F}_c(p) = \sqrt{\frac{2}{\pi}} \frac{\sin p}{p}$$

---

(3) Find the Fourier sine transform of $f(x)$is

$$F(x)=\begin{cases}0, & 0<x<a,\\ x, & a\le x\le b,\\ 0, & x>b.\end{cases}$$

$$F_s\{f(x)\}=\sqrt{\frac{2}{\pi}}\int_0^\infty F(x)\sin px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^a++\int_a^b++\int_b^\infty\right]f(x)\sin px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[0+\int_a^b x\sin px\,dx+0\right]$$

$$=\sqrt{\frac{2}{\pi}}\int_a^b x\sin px\,dx$$

$$F_s\{f(x)\}=\sqrt{\frac{2}{\pi}}\int_a^b x\sin px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[-\frac{x\cos px}{p}+\frac{\sin px}{p^2}\right]_a^b$$

$$=\sqrt{\frac{2}{\pi}}\left[\left(-\frac{b\cos pb}{p}+\frac{\sin pb}{p^2}\right)-\left(-\frac{a\cos pa}{p}+\frac{\sin pa}{p^2}\right)\right]$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{a\cos pa-b\cos pb}{p}+\frac{\sin pb-\sin pa}{p^2}\right]$$

Hence,

$$\boxed{F_s(p)=\sqrt{\frac{2}{\pi}}\left[\frac{a\cos pa-b\cos pb}{p}+\frac{\sin pb-\sin pa}{p^2}\right]}$$

Hence the proof.

(6) Find the cosine transform of the function $f(x)$is

$$f(x)=\begin{cases}\cos ax, & 0\le x\le a,\\ 0, & x>a.\end{cases}$$

Given

$$F_c\{f(x)\}=\bar{F}_c(p)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\textstyle\int_0^a ++\int_a^\infty\right]f(x)\cos px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\int_0^a \cos ax\cdot\cos px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\int_0^a 2\cos ax\cdot\cos px\,dx$$

$$[2\cos A\cos B=\cos(A+B)+\cos(A-B)]$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{2}\left[\int_0^a \cos\,(a+p)x\,dx+\int_0^a \cos\,(a-p)x\,dx\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{\sin a(a+p)}{a+p}+\frac{\sin a(a-p)}{a-p}\right]$$

(7) Find the Fourier transform of $f(x)$is

$$f(x)=\begin{cases}x, & 0<x<1,\\ 2-x, & 1<x<2,\\ 0, & x>2.\end{cases}$$

Given

$$f(x)=\begin{cases}x, & 0<x<1,\\ 2-x, & 1<x<2,\\ 0, & x>2.\end{cases}$$

$$\bar{F}_s(p)=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left(\textstyle\int_0^1 ++\int_1^2 ++\int_2^\infty\right)f(x)\sin px\,dx$$

$$=\sqrt{\frac{2}{\pi}}\left[\int_0^1 x\sin px\,dx+\int_1^2 (2-x)\sin px\,dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[x\left(\frac{-\cos px}{p}\right)_0^1 - \int_0^1 \frac{-\cos px}{p}\,dx + (2-x)\left(\frac{-\cos px}{p}\right)_1^2\right.$$

$$\left.-\int_1^2 (-1)\left(\frac{-\cos px}{p}\right)dx\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos p}{p} + 0 + \frac{1}{p}\left(\frac{\sin px}{p}\right)_0^1 + (2-x)\left(\frac{-\cos px}{p}\right)_1^2 - \left(\frac{\sin px}{p}\right)_1^2\right].$$

(11) Find the Fourier transform of sine $F(0,\infty)$

$$f(x) = e^{-x}$$

Sine :-
Given

$$f(x) = e^{-x}$$

$$\bar{F}_s(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin pn\,dn$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-n}\sin pn\,dn$$

$$\left[\int e^{-an}\sin bn\,dn = \frac{e^{-an}}{a^2+b^2}(-a\sin bn - b\cos bn)\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{e^0}{1+p^2}\{-\sin p(0) - p\cos p(0)\}\right]$$

$$\bar{F}_s(p) = \sqrt{\frac{2}{\pi}}\left[\frac{p}{1+p^2}\right]$$

Cosine :-
Given

$$f(n) = e^{-n}$$

$$\bar{F}_c(p) = \sqrt{\frac{2}{\pi}}\int_0^\infty F(n)\cos pn\,dn$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-n}\cos pn\,dn.$$

$$\left[\int e^{-an}\cos bn\,dn = \frac{e^{-an}}{a^2+b^2}[-a\cos bn + b\sin bn]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-an}}{a^2+p^2}\{-(-1)\cos pn + p\sin pn\}\right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}}\left[0 - \frac{1}{1^2+p^2}\{-\cos(0) + p\sin p(0)\}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{1}{1^2+p^2}[-1]\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{1}{1+p^2}\right]$$

Q. Find the Fourier cosine transform of

$$f(x) = \frac{1}{1+x^2}$$

and hence find the Fourier sine transform of

$$\frac{x}{1+x^2}.$$

Solution:

The Fourier cosine transform is defined by

$$\boldsymbol{F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx.}$$

Given

$$\boldsymbol{f(x) = \frac{1}{1+x^2},}$$

we have

$$\boldsymbol{F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos sx}{1+x^2}\,dx.}$$

Now, using the standard result,

$$\boldsymbol{\int_0^\infty \frac{\cos sx}{1+x^2}\,dx = \frac{\pi}{2}e^{-s}, s > 0.}$$

Therefore,

$$F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{2}{\pi}}\cdot\frac{\pi}{2}e^{-s}.$$

Hence,

$$\boxed{F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\,e^{-s}}$$

Hence find the Fourier sine transform of

$$\frac{x}{1+x^2}.$$

We use the property

$$F_s\{xf(x)\} = -\frac{d}{ds}[F_c\{f(x)\}].$$

Since

$$F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\,e^{-s},$$

differentiate with respect to s,

$$\frac{d}{ds}\left(\sqrt{\frac{\pi}{2}}\,e^{-s}\right) = -\sqrt{\frac{\pi}{2}}\,e^{-s}.$$

Hence,

$$F_s\left\{\frac{x}{1+x^2}\right\} = -\left(\ -\sqrt{\frac{\pi}{2}}\,e^{-s}\right).$$

Therefore,

$$\boxed{F_s\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\,e^{-s}}$$

Final Answers

1. **Fourier Cosine Transform**

$$\boxed{\boldsymbol{F_c\left\{\frac{1}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\, e^{-s}}}$$

2. Fourier Sine Transform

$$\boxed{\boldsymbol{F_s\left\{\frac{x}{1+x^2}\right\} = \sqrt{\frac{\pi}{2}}\, e^{-s}}}$$

Hence proved.

Multiple Fourier Transforms

Let $f(x,y)$be a function of two variables x& y. It's Fourier transform,

$$\bar{F}(p,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x,y)\, e^{ipx}\, dx.$$

Now regarding $\bar{F}(p,y)$as a function of y, it's Fourier transform is

$$\bar{F}(p,q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{F}\,(p,y)e^{iqy}\, dy.$$

$$\bar{F}(p,q) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(px+qy)}\, dx\, dy.$$

which is Fourier transform of $f(x,y)$.

Inversion Formula :-

Using inversion formula for Fourier transforms we have

$$f(x,y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \bar{F}\,(p,q)e^{-iqy}\, dy.$$

$$f(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \bar{F}\,(p,q)e^{-i(px+qy)}\, dx\, dy.$$

which is the inversion formula for the Fourier transform of $f(x,y)$.

Convolution :-

Definition :- The function $H(x) = f * g$

$$H(x) = f * g$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\, g(x-u)\, du.$$

is called the **convolution (or) folding** of two integral functions f& $g$over the interval $(-\infty,\infty)$.

Theorem (1)

The convolution (or) folding theorem for Fourier transform's.

Statement : If $F\{f(x)\}$, & $F\{g(x)\}$are the Fourier transform of the function $f(x)$& $g(x)$respectively, then the Fourier transform of the convolution of $f(x)$& $g(x)$is the product of their Fourier transforms.

$$F\{\, f(x) * g(x)\,\} = F\{f(x)\}\, F\{g(x)\}$$

Proof :-

We have

$$f(x) * g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du.$$

$$F\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx.$$

$$F\{f(x) * g(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} f\,(u)\; g(x-u)\, du \right] e^{isx}\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u) \left[\int_{-\infty}^{\infty} g\,(x-u)\, du \right] e^{isx}\, dx.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{ipu}\, du \int_{-\infty}^{\infty} g(x-u)\, e^{ip(x-u)}\, dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{ipu}\, du \int_{-\infty}^{\infty} g(x-u)\, e^{ip(x-u)}\, dx$$

Put $x - u = y$

$$dx = dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} f(u)\, e^{ipu}\, du \int_{-\infty}^{\infty} g(y)\, e^{ipy}\, dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(u)\, e^{ipu}\, du\; \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(y)\, e^{ipy}\, dy$$

$$= F\{f(u)\} \cdot F\{g(y)\}$$

$$= F\{f(x)\} \times F\{g(x)\}$$

Hence proved.

Problem:

Show that $\int_0^\infty e^{-x} \cos \lambda x \, dx = \frac{1}{1+\lambda^2}$

solution:

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

$$\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}$$

$$I_c\{f(x)\} = F_c(\lambda) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}$$

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty F_c\,(\lambda)\cos\lambda x\,d\lambda$$

$$e^{-x} = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}\cos\lambda x\,d\lambda$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2}\cos\lambda x\,d\lambda$$

$$\int_0^\infty \frac{1}{1+\lambda^2}\cos\lambda x\,d\lambda = \frac{\pi}{2}e^{-x}.$$

PRACTICE QUESTIONS:

1. Find the half-range cosine series for

$$f(x) = x, 0 < x < \pi.$$

2.Find the half-range sine series for

$$f(x) = x, 0 < x < \pi.$$

3.Find the half-range sine series of

$$f(x) = x(\pi - x), 0 < x < \pi.$$

Hence prove that

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \cdots = \frac{\pi^3}{32}.$$

4. Expand

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

into a Fourier series.

5. Find the half-range cosine series for

$$f(x) = x^2, 0 < x < \pi.$$

6. Derive the Fourier Sine Integral.
7. Expand f(x)=|x| in a Fourier series.
8. Find the half-range cosine series for f(x)=x,0<x<π.
9. Find the half-range sine series for f(x)=x,0<x<π.
10. Obtain the Fourier series of $x$in $(-\pi, \pi)$

"Engineering is not just about finding answers—it is about finding the smartest path to the answer through Numerical Methods and Transform Techniques."

11. Find the Inverse Fourier Transform of

$$F(s) = \frac{2a}{a^2 + s^2}.$$

⭐ Final Inspirational Quote:

"Master Numerical Methods to solve the unknown, master Transform Techniques to simplify the complex, and master mathematics to shape the future of engineering."

THE END