Sreenivasa Institute of Technology and Management Studies [SITAMS]
[Autonomous]
Chittoor, Andhra Pradesh

Department of Electronics and Communication Engineering

23ECE361T – Digital Signal Processing

Unit I - Discrete Time Signals And Systems & Z- Transform

Introduction to digital signal processing, Review of discrete-time signals and systems, Analysis of discrete-time linear time invariant systems, frequency domain representation of discrete time signals and systems Definition, ROC, Properties, Poles and Zeros in Z-plane, the inverse Z Transform, System analysis, Transfer function.

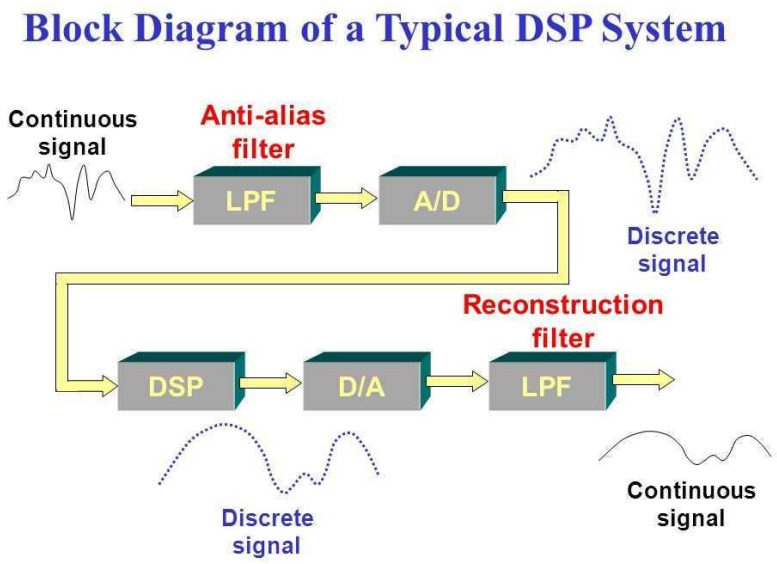
1.Introduction to digital signal processing:

Digital Signal Processing (DSP) is the technique of processing signals in digital form using mathematical operations and digital systems such as microprocessors, DSP processors, and computers.

A signal carries information about a physical quantity, such as sound, temperature, pressure, or an electrical voltage.

Examples:

- Speech and audio signals
- ECG and EEG biomedical signals
- Radar and communication signals
- Image and video signals
- Sensor data in IoT applications



This diagram illustrates the Block Diagram of a Typical Digital Signal Processing (DSP) System. It outlines the sequence of stages required to take a real-world continuous analog signal, process it digitally, and convert it back into an analog output signal.

Here is a step-by-step breakdown of each block and signal transformation shown in the diagram:

1. Continuous Signal (Input): A continuous analog signal (e.g., sound waves from a microphone, temperature variations, or voltage signals) varying continuously over time.

2. Anti-Alias Filter (Low-Pass Filter - LPF): Function: Removes high-frequency components that exceed half the sampling rate.

Purpose: Prevents aliasing, a phenomenon where high frequencies fold over and distort lower frequencies during sampling.

3. A/D Converter (Analog-to-Digital Converter): Function: Converts the bandwidth-limited analog signal into a Discrete Signal.

Process: Involves sampling (taking discrete-time snapshots of the waveform) and quantization (assigning discrete voltage level values).

4. Digital Signal Processor (DSP): Function: The core processing unit (e.g., a DSP microprocessor, FPGA, or microcontroller).

Purpose: Performs algorithmic calculations and mathematical manipulations on the discrete numerical data (e.g., filtering, noise reduction, compression, equalization, FFT).

Output: Produces a modified Discrete Signal.

5. D/A Converter (Digital-to-Analog Converter): Function: Transforms the processed digital numbers back into a continuous-time staircase-like analog voltage signal.

6. Reconstruction Filter (Low-Pass Filter - LPF): Function: Filters out high-frequency staircase artifacts and quantization noise produced by the D/A conversion.

Purpose: Smooths the signal, resulting in a clean Continuous Signal output ready for real-world playback or action (e.g., output to speakers, motors, or displays).

Advantages of DSP

1. High accuracy and repeatability.
2. Easy implementation of complex algorithms.
3. Better noise reduction and filtering.
4. Flexible modification through software.
5. Reliable storage and transmission of digital data

2.Review of Discrete-Time Signals:

A discrete-time signal is a signal defined only at discrete time instants. It is represented using an integer index ‘n’.

$$x[n],\ n=0,1,2,\ldots..$$

Here, x[n] represents the signal value at the n^{th} sample.

For example, a discrete-time signal may be

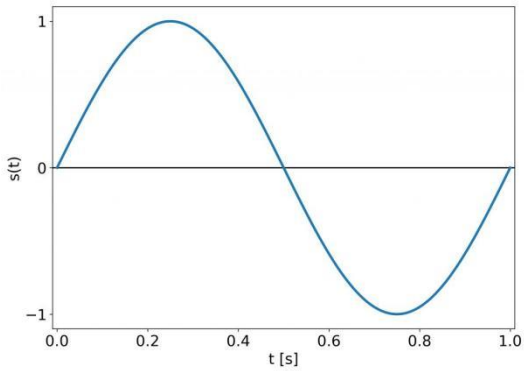
x[n]={1,2,3,4,5}

where the samples are taken at successive discrete time instants.

Types of signals

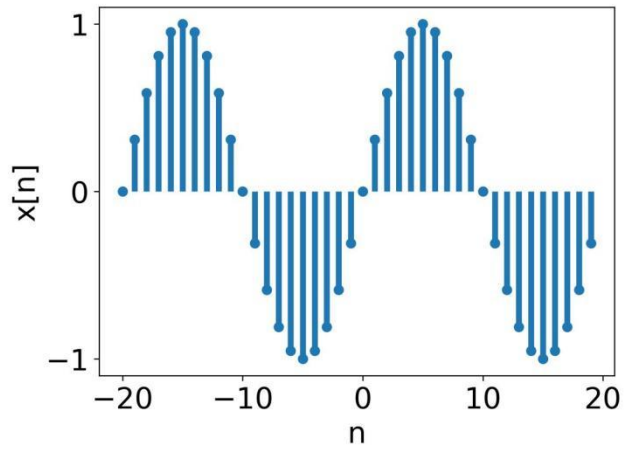
1. Continuous-time signal: Defined for every instant of time.

Example: $x(t)=\sin(2*pi *f_t)$



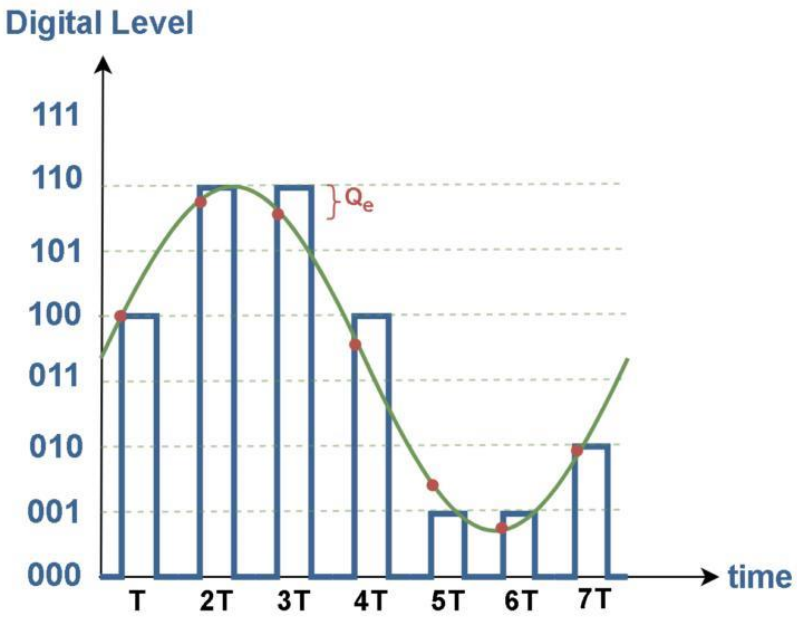
2. Discrete-time signal: Defined only at discrete sample indices.

Example: x[n]=sin(w n)



3. Digital signal

Has discrete time indices and quantized amplitude values, usually encoded in binary.



Elementary discrete-time signals

A. Unit impulse signal

It has a value of 1 at n=0 and zero everywhere else.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

B. Unit step signal

It changes from zero to one at n=0.

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

C. Unit ramp signal

$$r[n] = nu[n]$$

Its values are 0,1,2,3,… for n=0,1,2,3,…

D. Exponential signal

$$x[n] = a^n$$

The signal grows when |a|>1 for positive n, and decays when 0<|a|<1.

E. Sinusoidal signal

$$x[n] = A\cos(\omega_0 n + \phi)$$

where:

- A = amplitude
- W_0 = angular frequency in radians/sample
- Ø = phase angle

3.Analysis of Discrete time Linear Time Invariant System:

A Linear Time-Invariant (LTI) system is a discrete-time system that satisfies two important properties:

1. Linearity — the system obeys the principles of superposition.
2. Time invariance — the system's behaviour does not change with time.

LTI systems are important in DSP because their output can be calculated using the input signal and the system's impulse response.

Properties of LTI systems

1 Linearity

A system is linear if it satisfies both additivity and homogeneity.

Mathematically,

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\}$$

where:

- $x_1[n]$, $x_2[n]$ are input signals.
- a, b are constants.
- $T\{\cdot\}$ represents the system operation.

Example: y[n]=3x[n]

This is a linear system because multiplying the input by a constant multiplies the output by the same constant.

However, $y[n]=x^2[n]$

is nonlinear because squaring does not satisfy the superposition property in general.

2 Time invariance

A system is time-invariant if a shift in the input produces the same shift in the output.

If, x[n] -> y[n]

then the shifted input must produce, $x[n-n_0]$ -> $y[n-n_0]$

for any integer shift n_0.

Example 1:y[n]=x[n-2]

This system is time-invariant because the delay remains fixed.

Example 2: y[n]=nx[n]

This system is time-varying because its output depends explicitly on the time index \(n\).

Impulse response of an LTI system

The **impulse response**, denoted by $h[n]$, is the output of a system when the input is a unit impulse $\delta[n]$.

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

For an LTI system,

$$\boxed{h[n] = T\{\delta[n]\}}$$

The impulse response completely characterizes a discrete-time LTI system.

Convolution sum

The output of a discrete-time LTI system is obtained by convolving the input x[n] with the impulse response h[n].

$$y[n] = x[n] * h[n]$$

The mathematical expression is

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Here:

- $x[k]$ is the input sequence.
- $h[n-k]$ is the shifted and reversed impulse response.
- $y[n]$ is the output sequence.

Example: Calculate the output using convolution

Given:

x[n]={1,2,1} ; h[n]=\{1,1\}

Assume both sequences start at n=0.

Using convolution,

$$y[n] = \sum_{k} x[k]h[n-k]$$

Calculate each output sample:

y[0]=1(1)=1
y[1]=1(1)+2(1)=3
y[2]=2(1)+1(1)=3
y[3]=1(1)=1

Therefore,

y[n]={1,3,3,1}

Causality of LTI systems

An LTI system is causal if the output depends only on present and past input values.

The necessary and sufficient condition is

$$h[n]=0 \text{ for } n<0$$

That means the impulse response must be zero before time \(n=0\).

Example: $h[n]=(1/2)^n u[n]$

This system is causal because $u[n]=0$ for $n<0$.

Stability of LTI systems

An LTI system is BIBO stable if every bounded input produces a bounded output.

The necessary and sufficient condition is

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

In other words, the absolute sum of the impulse response must be finite.

Example 1: Stable system

$$h[n]=(1/2)^n u[n]$$

Its absolute sum is

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2$$

Since the sum is finite, the system is stable.

Example 2: Unstable system

$$h[n]=u[n]$$

Its absolute sum is infinite, so the system is not BIBO stable.

Difference equation representation

An LTI system can be represented by a linear constant-coefficient difference equation.

$y[n]+a_1y[n-1]+a_2y[n-2] =b_0x[n]+b_1x[n-1]$

where:

- x[n] is the input.

- y[n] is the output.
- a_1,a_2,b_0,b_1 are constant coefficients.

Example: y[n]=0.5y[n-1]+x[n]

This is a linear time-invariant system when the coefficients are constant and the system is considered with a fixed initial-rest condition.

Frequency response of an LTI system

The frequency response describes how an LTI system changes the amplitude and phase of sinusoidal input signals at different frequencies.

It is obtained by taking the discrete-time Fourier transform (DTFT) of the impulse response:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

The magnitude $|H(e^{j\omega})|$ represents the amplitude response, and the angle $\angle H(e^{j\omega})$ represents the phase response.

Frequency response is used to analyse low-pass, high-pass, band-pass and band-stop digital filters.

4. Frequency Domain Representation of Discrete-Time Signals and Systems

Definition

Frequency domain representation is a method of representing a discrete-time signal or system in terms of its frequency components rather than describing it only by its amplitude variation with time.

It helps us understand the different frequencies present in a signal and how a system modifies those frequencies.

- Time domain: Represents the signal amplitude as a function of discrete-time index n.
- Frequency domain: Represents the signal in terms of angular frequency ‘w’, measured in radians/sample.

Frequency domain representation of discrete-time signals

A discrete-time signal x[n] can be represented in the frequency domain using the Discrete-Time Fourier Transform (DTFT).

The DTFT is defined as

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

where:

- $x[n]$ = discrete-time signal.
- $X(e^{j\omega})$ = frequency domain representation.
- ω = angular frequency in radians/sample.
- $j = \sqrt{-1}$.

The inverse DTFT is

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

It converts the frequency domain representation back into the time domain.

Frequency domain representation of discrete-time systems

For a discrete-time Linear Time-Invariant (LTI) system, the frequency response is the DTFT of its impulse response h[n]

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

The output in the frequency domain is

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

where:

- $X(e^{j\omega})$ = input spectrum.
- $H(e^{j\omega})$ = system frequency response.
- $Y(e^{j\omega})$ = output spectrum.

Thus, convolution in the time domain becomes multiplication in the frequency domain.

$$y[n] = x[n] * h[n] \quad \Longleftrightarrow \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Important properties

1. The DTFT is periodic with period 2*pi.
2. The magnitude response $|H(e^{jw}|$ indicates how much each frequency is amplified or attenuated.
3. The phase response angle of $H(e^{jw})$ indicates the phase shift introduced by the system.
4. Frequency domain analysis simplifies the analysis and design of digital filters.

Applications

- Digital low-pass and high-pass filter design.
- Noise reduction in speech and audio signals.
- Spectrum analysis in communication systems.
- Biomedical signal processing, such as ECG analysis.
- Frequency analysis of wireless sensor network signals.

5. Z-Transform: ROC, Properties, Poles and Zeros, Inverse Z-Transform, System Analysis and Transfer Function

Z-Transform — Definition

The Z-transform is a mathematical tool used to analyse discrete-time signals and systems in the complex frequency domain. For a discrete-time signal x[n], the bilateral Z-transform is

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

where x[n] is the signal, z is a complex variable (z = r e^(jω)), and X(z) is its Z-transform. It is used to solve difference equations and study causality, stability, and digital filters.

Region of Convergence (ROC)

The Region of Convergence (ROC) is the set of values of z for which the Z-transform summation converges to a finite value.

$$\sum_{n=-\infty}^{\infty} |x[n] z^{-n}| < \infty$$

Types of ROC:

- Right-sided sequence: ROC lies outside the outermost pole. Example: x[n] = a^n u[n], ROC |z| > |a|.
- Left-sided sequence: ROC lies inside the innermost pole. Example: x[n] = −a^n u[−n−1], ROC |z| < |a|.
- Two-sided sequence: ROC is an annular region between poles, for example |a| < |z| < |b|.

Important properties: the ROC does not contain poles. A right-sided sequence has an ROC extending outward to infinity; a left-sided sequence has an ROC extending inward toward the origin. For a stable LTI system, the ROC includes the unit circle.

Properties of the Z-Transform

Property	Time domain	Z-domain
Linearity	a x[n] + b y[n]	aX(z) + bY(z)
Time shifting	x[n − n_0]	z^(−n_0) X(z)
Time reversal	x[−n]	X(z^(−1))
Multiplication by a^n	a^n x[n]	X(z/a)
Convolution	x[n] * h[n]	X(z) H(z)
First difference	x[n] − x[n−1]	(1 − z^(−1)) X(z)

The time-shifting and difference properties are stated for bilateral Z-transforms. The ROC must also be considered.

Poles and Zeros in the Z-Plane

A rational transfer function can be written as $H(z) = B(z)/A(z)$.

- Zeros are values of z that make the numerator zero.
- Poles are values of z that make the denominator zero.

Example: $H(z) = (z − 0.5)/(z − 0.8)$

Zero: $z − 0.5 = 0$, so $z = 0.5$.

Pole: $z − 0.8 = 0$, so $z = 0.8$.

In a standard pole-zero plot, zeros are marked by circles and poles by crosses. Poles influence natural and transient response; zeros influence attenuation and cancellation at particular values of z.

For a causal rational LTI system, BIBO stability requires all poles of the reduced transfer function to lie strictly inside the unit circle, assuming no hidden unstable internal modes.

Inverse Z-Transform

The inverse Z-transform converts X(z) back into the discrete-time sequence x[n]:

$$x[n] = Z^{-1}\{X(z)\}.$$

Common methods:

- Inspection using standard transform pairs
- Partial fraction expansion
- Power-series expansion

Example 1: By inspection

$X(z) = 1 / (1 − a\,z^{-1})$

For ROC $|z| > |a|$, the corresponding sequence is $x[n] = a^n u[n]$.

The ROC is essential because the same algebraic expression can represent different sequences with different ROCs.

Example 2: Partial fraction expansion

$X(z) = 1 / [(1 − 0.5z^{-1})(1 − 0.25z^{-1})]$

Partial fractions: $X(z) = 2/(1 − 0.5z^{-1}) − 1/(1 − 0.25z^{-1})$.

For the right-sided ROC $|z| > 0.5$, $x[n] = [2(0.5)^n − (0.25)^n]u[n]$.

System Analysis Using the Z-Transform

Consider the difference equation: $y[n] - 0.5y[n-1] = x[n]$.

Under the initial-rest assumption, taking the Z-transform gives:
$Y(z) - 0.5z^{-1}Y(z) = X(z)$
$Y(z)(1 - 0.5z^{-1}) = X(z)$
Therefore, $H(z) = Y(z)/X(z) = 1/(1 - 0.5z^{-1})$.
For the causal system, the ROC is $|z| > 0.5$. Its pole is at $z = 0.5$, so it is stable.

Transfer Function

The transfer function of a discrete-time LTI system is the ratio of the Z-transform of the output to the Z-transform of the input, under zero initial conditions:

$$H(z) = Y(z)/X(z).$$

General rational form:

$$H(z) = [b_0 + b_1z^{-1} + \dots + b_M\, z^{-M}] / [1 + a_1z^{-1} + \dots + a_N\, z^{-N}].$$

Corresponding difference equation:

$$y[n] + a_1y[n-1] + \dots + a_Ny[n-N] = b_0x[n] + b_1x[n-1] + \dots + b_Mx[n-M].$$

Applications include digital filter design, stability analysis, pole-zero analysis, and frequency-response calculation.

Important Question:

1. Explain the basic elements of a Digital Signal Processing (DSP) system with a neat block diagram. Discuss the advantages and applications of DSP.

2. Define discrete-time signals and classify them with suitable examples. Explain unit impulse, unit step, ramp, and exponential signals.

3. Explain the properties of discrete-time systems: linearity, time invariance, causality, stability, and memoryless property, with suitable examples.

4. Explain the convolution-sum representation and derive the expression for the output of an LTI system.

5. Given $x[n]=\{1,2,1\}$ and $h[n]=\{1,1,2\}$, calculate the linear convolution $y[n]=x[n]*h[n]$.

6. Explain the Discrete-Time Fourier Transform (DTFT) and its important properties. Obtain the frequency-domain representation of a discrete-time signal.

7. Define the Z-transform and Region of Convergence (ROC). Explain the significance of ROC and determine the Z-transform and ROC of common discrete-time signals.

8. Explain the important properties of the Z-transform, including linearity, time shifting, time reversal, convolution, and multiplication by an exponential sequence, with equations.

9. Explain the methods of finding the inverse Z-transform. Find x[n] for the given transform: $X(z)=1/(1-0.5z^{-1}),\ |z|>0.5$

10. Explain poles and zeros in the Z-plane and their relationship to system stability and causality. Define the system function or transfer function H(z), and determine the poles, zeros, and stability of a given system.