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23ECE361T – Digital Signal Processing

UNIT III – IIR FILTERS

Introduction to digital filters, Analog filter approximations – Butterworth and Chebyshev, Design of IIR Digital filters from analog filters by Impulse invariant and bilinear transformation methods, Frequency transformations, Basic structures of IIR Filters - Direct form I, Direct form-II, Cascade form and Parallel form realizations.

1. Introduction to Digital Filters

1.1 Definition

A digital filter processes a discrete-time signal $x[n]$ to produce an output $y[n]$. It can remove noise, suppress unwanted frequencies, or select a particular frequency range.

The general IIR difference equation is

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]$$

Its transfer function is

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Here, b_k are feedforward coefficients and a_k are feedback coefficients.

Example 1: Calculate the output of a digital filter

Given the difference equation

$$y[n] = 0.5x[n] + 0.5x[n-1]$$

and input $x[n] = \{2, 4, 6\}$, assume zero initial conditions.

Solution:

$$y[0] = 0.5(2) + 0.5(0) = 1$$

$$y[1] = 0.5(4) + 0.5(2) = 3$$

$$y[2] = 0.5(6) + 0.5(4) = 5$$

Therefore,

$$\boxed{y[n] = \{1, 3, 5\}}$$

This is a simple averaging filter that smooths changes in the input.

Example 2: Identify the transfer function

Given

$$y[n] = x[n] + 0.4x[n-1] - 0.5y[n-1]$$

Taking the z-transform under zero initial conditions,

$$Y(z) = X(z) + 0.4z^{-1}X(z) - 0.5z^{-1}Y(z)$$

Collect the output terms:

$$Y(z)(1+0.5z^{-1}) = X(z)(1+0.4z^{-1})$$

Hence,

$$\boxed{H(z) = \frac{1+0.4z^{-1}}{1+0.5z^{-1}}}$$

The pole is at $z = -0.5$, which is inside the unit circle. This causal first-order filter is stable.

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2. Types of Digital Filters: FIR and IIR

FIR filter

A finite impulse response filter has no feedback in its usual direct implementation. Its impulse response ends after a finite number of samples.

$$H(z) = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}$$

IIR filter

An infinite impulse response filter normally uses feedback. Its impulse response may continue indefinitely.

$$H(z) = \frac{B(z)}{A(z)}$$

Example 1: Identify the filter type

$$H_1(z) = 1 + 2z^{-1} + z^{-2}$$

There is no denominator other than 1 and no feedback term.

Answer: FIR filter of order 2.

Example 2: Identify the filter type

$$H_2(z) = \frac{1}{1-0.8z^{-1}}$$

The denominator contains a feedback term. Its impulse response is

$$h[n] = (0.8)^n u[n]$$

This response theoretically continues indefinitely.

Answer: IIR filter of order 1. Its pole is at $z = 0.8$, so the causal filter is stable.

3. Analog Filter Approximations

An analog filter approximation is a mathematical method for obtaining a practical transfer function that satisfies specified passband and stopband requirements.

Two common approximations are:

- Butterworth: maximally flat passband.

- Chebyshev: permits ripple in one frequency region to obtain a sharper transition.

The analog transfer function is expressed in the Laplace variable s.

Example 1: Calculate the Butterworth magnitude at cutoff

For an N-th order low-pass Butterworth filter,

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

Let $N = 2$ and $\Omega = \Omega_c$.

$$|H(j\Omega_c)|^2 = \frac{1}{1 + 1^4} = \frac{1}{2}$$

Therefore,

$$|H(j\Omega_c)| = \frac{1}{\sqrt{2}} = 0.7071$$

In decibels,

$$20 \log_{10}(0.7071) \approx -3.01 \text{ dB}$$

Answer: The cutoff magnitude is 0.7071, or approximately -3.01 dB.

Example 2: Compare attenuation for two Butterworth orders

Let $\Omega/\Omega_c = 2$.

For $N = 1$,

$$|H| = \frac{1}{\sqrt{1 + 2^2}} = \frac{1}{\sqrt{5}} = 0.4472$$

For $N = 2$,

$$|H| = \frac{1}{\sqrt{1 + 2^4}} = \frac{1}{\sqrt{17}} = 0.2425$$

The second-order filter attenuates this frequency more strongly.

Conclusion: Increasing Butterworth filter order sharpens the transition between passband and stopband.

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4. Butterworth Analog Filter Design

4.1 Theory

The Butterworth filter has a maximally flat magnitude response in the passband and a monotonic response in the stopband.

Its magnitude-squared response is

$$|H(j\Omega)|^2 = \frac{1}{1 + (\Omega/\Omega_c)^{2N}}$$

The minimum order for a low-pass Butterworth filter is

$$N \geq \frac{\log_{10}\left[\frac{10^{A_s/10}-1}{10^{A_p/10}-1}\right]}{2\log_{10}(\Omega_s/\Omega_p)}$$

where A_p is the maximum passband attenuation, A_s is the minimum stopband attenuation, and Ω_p, Ω_s are the analog passband and stopband edge frequencies.

Example 1: Find the Butterworth filter order

Given:

- Passband edge: $\Omega_p = 1000$ rad/s
- Stopband edge: $\Omega_s = 2000$ rad/s
- Passband attenuation: $A_p = 3$ dB
- Stopband attenuation: $A_s = 20$ dB

Substitute:

$$N \geq \frac{\log_{10}\left[\frac{10^{20/10}-1}{10^{3/10}-1}\right]}{2\log_{10}(2000/1000)}$$

Using $10^2 - 1 = 99$ and $10^{0.3} - 1 \approx 0.9953$,

$$N \geq \frac{\log_{10}(99/0.9953)}{2\log_{10} 2}$$

$$N \geq \frac{1.998}{0.6021} \approx 3.32$$

Round upward:

$$\boxed{N = 4}$$

Answer: A fourth-order Butterworth filter is required.

Example 2: Calculate the cutoff frequency

For a Butterworth filter of order $N = 2$, suppose the attenuation at $\Omega = 2000$ rad/s must be 12 dB.

The magnitude response in decibels is

$$A(\Omega) = 10\log_{10}\left[1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}\right]$$

Given $A = 12$ dB,

$$10^{12/10} = 1 + \left(\frac{2000}{\Omega_c}\right)^4$$

$$15.849 - 1 = \left(\frac{2000}{\Omega_c}\right)^4$$

$$\Omega_c = \frac{2000}{(14.849)^{1/4}}$$

$$\boxed{\Omega_c \approx 1019 \text{ rad/s}}$$

Answer: The cutoff frequency is approximately 1019 rad/s.

5. Chebyshev Analog Filter Design

5.1 Theory

A Chebyshev Type I filter permits equal ripple in the passband to achieve a sharper transition than a Butterworth filter of the same order.

Its magnitude-squared response is

$$|H(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_N^2(\Omega/\Omega_p)}$$

The ripple factor is

$$\boxed{\epsilon = \sqrt{10^{A_p/10} - 1}}$$

The minimum Type I order is

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{A_s/10}-1}{10^{A_p/10}-1}}}{\cosh^{-1}(\Omega_s/\Omega_p)}$$

for a low-pass filter where $\Omega_s > \Omega_p$.

Example 1: Calculate the ripple factor

Given passband ripple $A_p = 1$ dB,

$$\epsilon = \sqrt{10^{1/10} - 1}$$

$$\epsilon = \sqrt{1.2589 - 1}$$

$$\boxed{\epsilon \approx 0.509}$$

Answer: The Chebyshev ripple factor is approximately 0.509.

Example 2: Find the Chebyshev filter order

Given:

- $\Omega_p = 1000$ rad/s
- $\Omega_s = 2000$ rad/s
- $A_p = 1$ dB
- $A_s = 20$ dB

First,

$$\epsilon^2 = 10^{0.1} - 1 = 0.2589$$

Then,

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^2-1}{0.2589}}}{\cosh^{-1}(2)}$$

$$N \geq \frac{\cosh^{-1}(19.56)}{\cosh^{-1}(2)}$$

$$N \geq \frac{3.666}{1.317} \approx 2.78$$

Round upward:

$$\boxed{N = 3}$$

Answer: A third-order Chebyshev Type I filter satisfies these specifications. For comparison, the earlier Butterworth example required fourth order.

6. IIR Design by the Impulse Invariance Method

6.1 Theory

The impulse invariance method forms a digital filter by sampling the impulse response of an analog filter.

Using the convention $h[n] = T h_a(nT)$,

$$H_a(s) = \frac{A}{s-p} \quad \Longrightarrow \quad H(z) = \frac{TA}{1 - e^{pT} z^{-1}}$$

Thus, the analog pole p maps to

$$\boxed{z = e^{pT}}$$

The method can introduce aliasing because the analog spectrum is replicated periodically in the digital frequency domain.

Example 1: Convert a first-order analog filter

Given

$$H_a(s) = \frac{1}{s+2}, \qquad T = 0.1 \text{ s}$$

The analog pole is $p = -2$, and the residue is $A = 1$.

Using impulse invariance,

$$H(z) = \frac{T}{1 - e^{-2T} z^{-1}}$$

Substitute $T = 0.1$:

$$H(z) = \frac{0.1}{1 - e^{-0.2} z^{-1}}$$

Since $e^{-0.2} \approx 0.8187$,

$$\boxed{H(z) = \frac{0.1}{1 - 0.8187 z^{-1}}}$$

The digital pole is $z = 0.8187$, inside the unit circle.

Example 2: Convert a two-pole analog filter

Given

$$H_a(s) = \frac{1}{(s+1)(s+3)}$$

First perform partial fractions:

$$H_a(s) = \frac{1/2}{s+1} - \frac{1/2}{s+3}$$

Let $T = 0.2$ s. Apply impulse invariance to each term:

$$H(z) = \frac{0.2(1/2)}{1 - e^{-0.2}z^{-1}} - \frac{0.2(1/2)}{1 - e^{-0.6}z^{-1}}$$

Therefore,

$$\boxed{H(z) = \frac{0.1}{1 - 0.8187z^{-1}} - \frac{0.1}{1 - 0.5488z^{-1}}}$$

Answer: The digital filter is obtained by mapping each analog pole and preserving the corresponding sampled impulse-response contribution.

Important: Impulse invariance has different scaling conventions in some textbooks. Always follow the convention stated in the question; the examples here use $h[n] = Th_a(nT)$.

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7. IIR Design by the Bilinear Transformation

7.1 Theory

The bilinear transformation converts an analog filter into a digital filter by replacing s with

$$\boxed{s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}}$$

It does not create transformation-induced frequency aliasing, but it warps the frequency axis. Prewarping is used when exact digital passband and stopband edge locations are required.

Example 1: Convert an analog first-order filter

Given

$$H_a(s) = \frac{1}{s+1}, \qquad T = 0.1 \text{ s}$$

Since $2/T = 20$,

$$H(z) = \frac{1}{20\frac{1-z^{-1}}{1+z^{-1}} + 1}$$

Multiply the numerator and denominator by $1 + z^{-1}$:

$$H(z) = \frac{1 + z^{-1}}{20(1 - z^{-1}) + (1 + z^{-1})}$$

Simplify:

$$H(z) = \frac{1 + z^{-1}}{21 - 19z^{-1}}$$

Normalize the denominator:

$$\boxed{H(z) = \frac{0.047619 + 0.047619z^{-1}}{1 - 0.904762z^{-1}}}$$

The pole is $z = 19/21 \approx 0.9048$, so the filter is stable.

Example 2: Design using prewarping

Suppose the desired digital cutoff is $f_c = 100$ Hz and the sampling frequency is $f_s = 1000$ Hz.

First, calculate the digital angular cutoff:

$$\omega_c = 2\pi \frac{f_c}{f_s} = 2\pi \frac{100}{1000} = 0.2\pi$$

The analog prewarped cutoff is

$$\Omega_c = \frac{2}{T} \tan\left(\frac{\omega_c}{2}\right)$$

Since $T = 1/1000 = 0.001$ s,

$$\Omega_c = 2000 \tan(0.1\pi)$$

$$\Omega_c \approx 2000(0.32492)$$

$$\boxed{\Omega_c \approx 649.84 \text{ rad/s}}$$

Answer: Design the analog prototype with a cutoff of approximately 649.84 rad/s before applying the bilinear transformation. This preserves the specified digital cutoff of 100 Hz.

8. Analog Frequency Transformations

Frequency transformations convert a normalized low-pass analog prototype into a low-pass, high-pass, band-pass, or band-stop filter.

The main substitutions are:

Transformation	Substitution
Low-pass to low-pass	$s \to s/\Omega_c$
Low-pass to high-pass	$s \to \Omega_c/s$
Low-pass to band-pass	$s \to (s^2 + \Omega_0^2)/(Bs)$
Low-pass to band-stop	$s \to Bs/(s^2 + \Omega_0^2)$

For band-pass and band-stop filters,

$$B = \Omega_2 - \Omega_1, \qquad \Omega_0 = \sqrt{\Omega_1 \Omega_2}$$

where Ω_1, Ω_2 are the two band-edge frequencies.

Example 1: Low-pass to high-pass transformation

Given a normalized first-order low-pass filter,

$$H_{LP}(s) = \frac{1}{s+1}$$

Convert it into a high-pass filter with cutoff $\Omega_c = 10$ rad/s.

Use

$$s \to \frac{\Omega_c}{s} = \frac{10}{s}$$

Therefore,

$$H_{HP}(s) = \frac{1}{10/s + 1}$$

Multiply the numerator and denominator by s:

$$\boxed{H_{HP}(s) = \frac{s}{s+10}}$$

This filter attenuates low frequencies and passes high frequencies.

Example 2: Low-pass to band-pass transformation

Given the normalized low-pass prototype

$$H_{LP}(s) = \frac{1}{s+1}$$

Design a band-pass analog filter with edges

$$\Omega_1 = 100, \qquad \Omega_2 = 400 \text{ rad/s}$$

Calculate the bandwidth:

$$B = 400 - 100 = 300 \text{ rad/s}$$

Calculate the center frequency:

$$\Omega_0 = \sqrt{100(400)} = 200 \text{ rad/s}$$

Apply the low-pass to band-pass substitution:

$$s \to \frac{s^2 + 200^2}{300s}$$

Thus,

$$H_{BP}(s) = \frac{1}{\frac{s^2+40000}{300s} + 1}$$

Multiply the numerator and denominator by $300s$:

$$\boxed{H_{BP}(s) = \frac{300s}{s^2 + 300s + 40000}}$$

Answer: The transformed band-pass filter has a center frequency of 200 rad/s and a bandwidth of 300 rad/s.

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9. Direct Form I Realization of IIR Filters

9.1 Theory

Direct Form I implements the numerator and denominator parts of the transfer function with separate delay lines.

For

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

the difference equation is

$$y[n] = b_0 x[n] + b_1 x[n-1] - a_1 y[n-1]$$

It uses separate storage for past input samples and past output samples.

Example 1: Obtain the difference equation

Given

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.2z^{-1}}$$

Multiply both sides by the denominator:

$$Y(z)(1 - 0.2z^{-1}) = X(z)(1 + 0.5z^{-1})$$

Expand:

$$Y(z) - 0.2z^{-1}Y(z) = X(z) + 0.5z^{-1}X(z)$$

Convert to the time domain:

$$\boxed{y[n] = x[n] + 0.5x[n-1] + 0.2y[n-1]}$$

This equation directly determines the Direct Form I implementation.

Example 2: Calculate output samples

Use the same filter, with input

$$x[n] = \{1, 2, 0\}$$

Assume zero initial conditions.

At $n = 0$:

$$y[0] = 1 + 0.5(0) + 0.2(0) = 1$$

At $n = 1$:

$$y[1] = 2 + 0.5(1) + 0.2(1) = 2.7$$

At $n = 2$:

$$y[2] = 0 + 0.5(2) + 0.2(2.7) = 1.54$$

Therefore,

$$\boxed{y[n] = \{1, 2.7, 1.54\}}$$

10. Direct Form II Realization of IIR Filters

10.1 Theory

Direct Form II shares the delay elements of the feedforward and feedback sections, reducing memory requirements.

For the transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

define the internal state $w[n]$:

$$w[n] = x[n] - a_1 w[n-1]$$

Then calculate

$$y[n] = b_0 w[n] + b_1 w[n-1]$$

Direct Form II uses one delay element for this first-order example.

Example 1: Derive Direct Form II equations

Given

$$H(z) = \frac{1 + 0.5z^{-1}}{1 - 0.2z^{-1}}$$

Compare its denominator with $1 + a_1 z^{-1}$:

$$a_1 = -0.2$$

Therefore,

$$w[n] = x[n] - (-0.2) w[n-1]$$

$$\boxed{w[n] = x[n] + 0.2 w[n-1]}$$

The output equation is

$$\boxed{y[n] = w[n] + 0.5w[n-1]}$$

These two equations form the Direct Form II realization.

Example 2: Calculate output samples using Direct Form II

Given $x[n] = \{1, 2, 0\}$, with $w[-1] = 0$:

At $n = 0$:

$$w[0] = 1 + 0.2(0) = 1$$

$$y[0] = 1 + 0.5(0) = 1$$

At $n = 1$:

$$w[1] = 2 + 0.2(1) = 2.2$$

$$y[1] = 2.2 + 0.5(1) = 2.7$$

At $n = 2$:

$$w[2] = 0 + 0.2(2.2) = 0.44$$

$$y[2] = 0.44 + 0.5(2.2) = 1.54$$

Hence,

$$\boxed{y[n] = \{1, 2.7, 1.54\}}$$

Conclusion: Direct Form I and Direct Form II produce the same output for the same transfer function and initial conditions, although their internal structures differ.

11. Cascade Form Realization

11.1 Theory

In cascade form, the transfer function is factored into lower-order sections connected in series.

$$H(z) = H_1(z)H_2(z)\cdots H_L(z)$$

The output of one section becomes the input to the next section. Higher-order filters are often implemented as a cascade of second-order sections.

Example 1: Factor a transfer function into cascade sections

Given

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 + 0.25z^{-1})}$$

Separate it into two sections:

$$H_1(z) = \frac{1}{1 - 0.5z^{-1}}$$

$$H_2(z) = \frac{1}{1 + 0.25z^{-1}}$$

Therefore,

$$\boxed{H(z) = H_1(z)H_2(z)}$$

The first section has a pole at 0.5; the second has a pole at -0.25. Both are inside the unit circle.

Example 2: Calculate the cascade output

Let the input be $x[n] = \{1, 0\}$, with zero initial conditions.

First section:

$$y_1[n] = x[n] + 0.5y_1[n-1]$$

Thus,

$$y_1[0] = 1, \qquad y_1[1] = 0.5$$

Second section:

$$y[n] = y_1[n] - 0.25y[n-1]$$

Therefore,

$$y[0] = 1$$

$$y[1] = 0.5 - 0.25(1) = 0.25$$

$$y[2] = 0.25 - 0.25(0.25) = 0.1875$$

Hence, the beginning of the output sequence is

$$\boxed{y[n] = \{1, 0.25, 0.1875, \ldots\}}$$

The response continues because both sections are recursive.

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12. Parallel Form Realization

12.1 Theory

In parallel form, the transfer function is expanded into partial fractions. Each partial-fraction component is implemented as an independent filter, and all branch outputs are added.

$$H(z) = C + H_1(z) + H_2(z) + \cdots + H_L(z)$$

For example,

$$H(z) = \frac{A}{1 - p_1 z^{-1}} + \frac{B}{1 - p_2 z^{-1}}$$

The branch outputs are added to produce the final output.

Example 1: Convert to parallel form

Given

$$H(z) = \frac{1}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$

Let $q = z^{-1}$. Use partial fractions:

$$\frac{1}{(1 - 0.5q)(1 - 0.25q)} = \frac{A}{1 - 0.5q} + \frac{B}{1 - 0.25q}$$

Multiplying through by the denominator,

$$1 = A(1 - 0.25q) + B(1 - 0.5q)$$

Compare constant terms and coefficients of q:

$$A + B = 1$$

$$-0.25A - 0.5B = 0$$

Solving gives

$$A = 2, \qquad B = -1$$

Therefore,

$$\boxed{H(z) = \frac{2}{1 - 0.5z^{-1}} - \frac{1}{1 - 0.25z^{-1}}}$$

This is the required parallel realization.

Example 2: Calculate the parallel output

Use the parallel transfer function from Example 1 and input $x[n] = \{1, 0\}$.

Branch 1:

$$H_1(z) = \frac{2}{1 - 0.5z^{-1}}$$

Its difference equation is

$$y_1[n] = 0.5y_1[n - 1] + 2x[n]$$

Thus,

$$y_1[0] = 2, \qquad y_1[1] = 1$$

Branch 2:

$$H_2(z) = \frac{-1}{1 - 0.25z^{-1}}$$

Its difference equation is

$$y_2[n] = 0.25y_2[n-1] - x[n]$$

Thus,

$$y_2[0] = -1, \qquad y_2[1] = -0.25$$

Add the two branch outputs:

$$y[0] = y_1[0] + y_2[0] = 2 - 1 = 1$$

$$y[1] = y_1[1] + y_2[1] = 1 - 0.25 = 0.75$$

Therefore,

$$\boxed{y[n] = \{1, 0.75, \ldots\}}$$

The output matches the original transfer function.

Important Questions:

1. Define a digital filter. Explain the classification of digital filters and distinguish between FIR and IIR filters.
2. Explain the characteristics and design specifications of digital filters.
3. Explain the Butterworth filter approximation and derive its magnitude-squared frequency response.
4. Explain the Chebyshev filter approximation and derive its magnitude response.
5. Compare Butterworth and Chebyshev filters with respect to passband characteristics, stopband attenuation, and transition width.

6. Explain the impulse invariance method for designing IIR digital filters from analog filters. Derive the transformation equations and discuss its limitations.
7. Explain the bilinear transformation method for designing IIR digital filters from analog filters. Derive the transformation equation and explain frequency warping.
8. Design an IIR digital filter using the impulse invariance method for a given analog transfer function.
9. Design an IIR digital filter using the bilinear transformation method for a given analog transfer function.
10. Compare the impulse invariance and bilinear transformation methods of IIR filter design.
11. Explain frequency transformations used in IIR filter design. Discuss the transformations required to convert a low-pass prototype into high-pass, band-pass, and band-stop filters.

12. Explain the Direct Form-I realization of an IIR digital filter with a block diagram.
13. Explain the Direct Form-II realization of an IIR digital filter with a block diagram. Discuss its advantages over Direct Form-I.
14. Explain the cascade form realization of an IIR filter and describe how the transfer function is expressed as a product of second-order sections.
15. Explain the parallel form realization of an IIR filter using partial fraction expansion.
16. Compare Direct Form-I, Direct Form-II, cascade form, and parallel form realizations of IIR digital filters.