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Department of Electronics and Communication Engineering

23ECE361T – Digital Signal Processing

UNIT IV – FIR FILTERS

Introduction, Characteristics of FIR filters with linear phase, Frequency response of linear phase FIR filters, Design of FIR filters using Fourier series and windowing methods (Rectangular, Triangular, Raised Cosine, Hanging, Hamming, Blackman), Comparison of IIR & FIR filters, Basic structures of FIR Filters – Direct form, Cascade form, Linear phase realizations.

1. Introduction to FIR Digital Filters

Definition

A Finite Impulse Response (FIR) filter is a digital filter whose impulse response becomes zero after a finite number of samples. A standard FIR filter is non-recursive and uses present and past input samples without output feedback.

The general difference equation is

$$y[n] = \sum_{k=0}^{M} b_k x[n-k]$$

where M is the filter order, $M+1$ is the number of coefficients, b_k are filter coefficients, and $x[n]$ is the input.

Its transfer function is

$$H(z) = \sum_{k=0}^{M} b_k z^{-k}$$

Characteristics

- Finite-duration impulse response.
- Usually implemented without feedback.
- Always BIBO stable if all coefficients are finite.
- Can be designed for exact linear phase.
- May require a higher order than an IIR filter for comparable magnitude specifications.

Example 1: Determine the output sequence

Given

$$y[n] = 0.25x[n] + 0.5x[n-1] + 0.25x[n-2]$$

Input: $x[n] = \{2, 4, 6\}$, with zero initial conditions.

$$y[0] = 0.25(2) = 0.5$$

$$y[1] = 0.25(4) + 0.5(2) = 2$$

$$y[2] = 0.25(6) + 0.5(4) + 0.25(2) = 4$$

$$\boxed{y[n] = \{0.5, 2, 4\}}$$

The filter computes a weighted average of three input samples.

Example 2: Find the transfer function and order

Given

$$y[n] = x[n] + 2x[n-1] + x[n-2]$$

Taking the z-transform,

$$Y(z) = X(z) + 2z^{-1}X(z) + z^{-2}X(z)$$

Therefore,

$$\boxed{H(z) = 1 + 2z^{-1} + z^{-2}}$$

There are three coefficients, so the filter order is $M = 2$.

2. Characteristics of FIR Filters with Linear Phase

Definition

An FIR filter has linear phase if its phase response is a linear function of frequency over the frequency range where the response is nonzero. Linear phase means that the filter delays the frequency components by the same amount of time.

For real FIR coefficients, a sufficient symmetry condition is

$$b[n] = b[M - n]$$

or an antisymmetry condition:

$$b[n] = -b[M - n]$$

These are called symmetric and antisymmetric impulse responses.

Four types of linear-phase FIR filters

Type	Order M	Coefficient property	Typical use
I	Even	Symmetric	General linear-phase filters
II	Odd	Symmetric	Useful low-pass filters; response zero at Nyquist
III	Even	Antisymmetric	Differentiators and Hilbert transformers
IV	Odd	Antisymmetric	Hilbert transformers and fractional-delay applications

Here, “even order” means M is even, and “odd order” means M is odd.

Example 1: Identify symmetry

Given

$$h[n] = \{1, 2, 3, 2, 1\}$$

There are five coefficients, so

$$M = 5 - 1 = 4$$

Check symmetry:

$$h[0] = h[4] = 1, \quad h[1] = h[3] = 2$$

The coefficients are symmetric and the order is even.

Answer: Type I linear-phase FIR filter.

Example 2: Identify antisymmetry

Given

$$h[n] = \{1, 2, 0, -2, -1\}$$

Check the coefficient pairs:

$$h[0] = -h[4], \qquad h[1] = -h[3]$$

The centre coefficient is zero, and the sequence is antisymmetric. There are five coefficients, so $M = 4$.

Answer: Type III linear-phase FIR filter.

Advantages of linear phase

- Preserves the relative timing of frequency components.
- Reduces phase distortion in audio and communication systems.
- Useful for waveform-preserving applications.
- Can be obtained by enforcing symmetry or antisymmetry in the FIR coefficients.

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3. Frequency Response of Linear-Phase FIR Filters

Theory

The frequency response is obtained by evaluating the transfer function on the unit circle:

$$H(e^{j\omega}) = \sum_{n=0}^{M} h[n] e^{-j\omega n}$$

For a symmetric impulse response, the frequency response can be factored into a linear-phase term and a real-valued amplitude function:

$$H(e^{j\omega}) = e^{-j\omega M/2} A(\omega)$$

where $A(\omega)$ is real-valued.

The phase term is

$$\phi(\omega) = -\frac{M\omega}{2}$$

when $A(\omega)$ is positive. Where $A(\omega)$ changes sign, phase may include jumps of π.

The group delay is

$$\tau_g = -\frac{d\phi(\omega)}{d\omega} = \frac{M}{2}$$

Example 1: Find the frequency response

Given

$$h[n] = \{1, 2, 1\}$$

The transfer function is

$$H(z) = 1 + 2z^{-1} + z^{-2}$$

Substitute $z = e^{j\omega}$:

$$H(e^{j\omega}) = 1 + 2e^{-j\omega} + e^{-j2\omega}$$

Factor out $e^{-j\omega}$:

$$H(e^{j\omega}) = e^{-j\omega}(e^{j\omega} + 2 + e^{-j\omega})$$

Using $e^{j\omega} + e^{-j\omega} = 2\cos\omega$,

$$\boxed{H(e^{j\omega}) = e^{-j\omega}(2 + 2\cos\omega)}$$

The phase is linear, and the group delay is

$$\boxed{\tau_g = M/2 = 1 \text{ sample}}$$

Example 2: Find group delay

An FIR filter has order $M = 8$ and symmetric coefficients.

For a linear-phase FIR filter,

$$\tau_g = \frac{M}{2} = \frac{8}{2}$$

$$\boxed{\tau_g = 4 \text{ samples}}$$

If the sampling frequency is $f_s = 8000$ Hz, the time delay is

$$t_d = \frac{4}{8000} = 0.0005 \text{ s}$$

$$\boxed{t_d = 0.5 \text{ ms}}$$

4. FIR Filter Design Using the Fourier Series Method

Theory

The Fourier series method starts with the desired frequency response and calculates the corresponding impulse response by inverse Fourier transformation. The infinite impulse response is truncated to a finite length to obtain an FIR filter.

For an ideal low-pass filter with cutoff frequency ω_c,

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| \le \pi \end{cases}$$

The ideal impulse response is

$$h_d[n] = \begin{cases} \frac{\omega_c}{\pi}, & n = 0 \\ \frac{\sin(\omega_c n)}{\pi n}, & n \neq 0 \end{cases}$$

This ideal response is infinitely long. A finite FIR filter is obtained by truncating and centering the sequence.

Design steps

1. Specify the desired frequency response.
2. Choose the cutoff frequency.
3. Calculate the ideal impulse response.
4. Truncate it to the desired number of coefficients.
5. Apply a window if required.
6. Verify the frequency response.

Example 1: Calculate the ideal impulse response

Given a low-pass cutoff frequency

$$\omega_c = \frac{\pi}{2}$$

At $n = 0$,

$$h_d[0] = \frac{\omega_c}{\pi} = \frac{\pi/2}{\pi} = 0.5$$

At $n = 1$,

$$h_d[1] = \frac{\sin(\pi/2)}{\pi} = \frac{1}{\pi} \approx 0.3183$$

At $n = 2$,

$$h_d[2] = \frac{\sin(\pi)}{2\pi} = 0$$

By symmetry,

$$\boxed{h_d[-1] = 0.3183, \quad h_d[0] = 0.5, \quad h_d[1] = 0.3183}$$

These are ideal low-pass impulse-response samples. Truncating the infinite sequence creates a practical FIR approximation.

Example 2: Construct a three-tap FIR approximation

For the same cutoff $\omega_c = \pi/2$, truncate the ideal impulse response to $n = -1, 0, 1$:

$$h_d[-1] = \frac{1}{\pi}, \quad h_d[0] = 0.5, \quad h_d[1] = \frac{1}{\pi}$$

Shift the indices to $0, 1, 2$:

$$h[0] = \frac{1}{\pi}, \quad h[1] = 0.5, \quad h[2] = \frac{1}{\pi}$$

Therefore,

$$\boxed{H(z) = 0.3183 + 0.5z^{-1} + 0.3183z^{-2}}$$

This is a finite, symmetric FIR approximation to the ideal low-pass filter.

5. FIR Filter Design Using the Windowing Method

Theory

Windowing truncates the ideal impulse response while controlling the ripple and transition behaviour of the resulting FIR filter.

$$h[n] = h_d[n]w[n]$$

where $h_d[n]$ is the ideal impulse response and $w[n]$ is the selected window.

A window trades transition width against stopband attenuation. A narrow main lobe generally gives a sharper transition, while lower sidelobes generally reduce stopband leakage.

Example 1: Apply a rectangular window

Suppose the ideal impulse-response samples are

$$h_d[n] = \{0.2, 0.4, 0.6, 0.4, 0.2\}$$

A length-5 rectangular window is

$$w_R[n] = \{1, 1, 1, 1, 1\}$$

Multiply sample by sample:

$$h[n] = h_d[n]w_R[n]$$

Therefore,

$$\boxed{h[n] = \{0.2, 0.4, 0.6, 0.4, 0.2\}}$$

The coefficients remain unchanged because the rectangular window has a value of 1 across its full length.

Example 2: Apply a triangular window

Use the same ideal response and a length-5 triangular window:

$$w_T[n] = \{0, 0.5, 1, 0.5, 0\}$$

Multiply sample by sample:

$$h[0] = 0.2(0) = 0$$

$$h[1] = 0.4(0.5) = 0.2$$

$$h[2] = 0.6(1) = 0.6$$

$$h[3] = 0.4(0.5) = 0.2$$

$$h[4] = 0.2(0) = 0$$

Therefore,

$$\boxed{h[n] = \{0, 0.2, 0.6, 0.2, 0\}}$$

This demonstrates how the window changes the finite impulse-response coefficients.

6. Rectangular Window

Theory

The rectangular window keeps all selected samples and removes samples outside the window.

For a window of length N,

$$w_R[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Characteristics: Narrow main lobe, but relatively high sidelobes and significant spectral leakage.

Example 1: Calculate window coefficients

For $N = 5$,

$$\boxed{w_R[n] = \{1, 1, 1, 1, 1\}}$$

Example 2: Window an impulse response

Given

$$h_d[n] = \{0.1, 0.3, 0.5, 0.3, 0.1\}$$

Using a length-5 rectangular window,

$$h[n] = h_d[n] w_R[n]$$

Hence,

$$\boxed{h[n] = \{0.1, 0.3, 0.5, 0.3, 0.1\}}$$

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7. Triangular Window

Theory

A triangular window gives a linear taper towards the endpoints. A common symmetric triangular window is the Bartlett window:

$$w_T[n] = 1 - \left| \frac{n - (N-1)/2}{(N-1)/2} \right|$$

for $0 \le n \le N-1$, with $N > 1$.

Characteristics: Reduced sidelobes compared with the rectangular window, but a wider main lobe.

Example 1: Calculate a length-5 triangular window

For $N = 5$,

$$w_T[n] = 1 - \left| \frac{n-2}{2} \right|$$

Evaluating for $n = 0, 1, 2, 3, 4$,

$$\boxed{w_T[n] = \{0, 0.5, 1, 0.5, 0\}}$$

Example 2: Apply the window

Given

$$h_d[n] = \{1, 2, 3, 2, 1\}$$

Multiply by the triangular window:

$$h[n] = \{1(0), 2(0.5), 3(1), 2(0.5), 1(0)\}$$

$$\boxed{h[n] = \{0, 1, 3, 1, 0\}}$$

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8. Raised-Cosine Window

Theory

The term *raised-cosine window* can refer to different cosine-based windows. A commonly used symmetric raised-cosine form is the Hann window, which is also known as the Hanning window in many DSP syllabi:

$$w_H[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right]$$

for $0 \le n \le N-1$.

Example 1: Calculate a length-5 Hann window

For $N = 5$,

$$w_H[n] = \frac{1}{2}\left[1 - \cos\left(\frac{\pi n}{2}\right)\right]$$

Calculate the values:

- $n = 0: \ w_H[0] = 0$
- $n = 1: \ w_H[1] = 0.5$
- $n = 2: \ w_H[2] = 1$
- $n = 3: \ w_H[3] = 0.5$
- $n = 4: \ w_H[4] = 0$

Therefore,

$$\boxed{w_H[n] = \{0, 0.5, 1, 0.5, 0\}}$$

For this length, it matches the symmetric triangular window numerically.

Example 2: Window the coefficients

Given

$$h_d[n] = \{0.2, 0.4, 0.6, 0.4, 0.2\}$$

Apply the Hann window:

$$h[n] = h_d[n] w_H[n]$$

$$\boxed{h[n] = \{0, 0.2, 0.6, 0.2, 0\}}$$

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9. Hanning (Hann) Window

The Hann window is often called the Hanning window in older textbooks. Its formula is the same raised-cosine expression used above.

$$w_{\text{Hann}}[n] = \frac{1}{2}\left[1 - \cos\left(\frac{2\pi n}{N-1}\right)\right]$$

Example 1: Find the length-7 Hann window

For $N = 7$,

$$w[n] = \frac{1}{2}\left[1 - \cos\left(\frac{\pi n}{3}\right)\right]$$

Evaluating gives

$$\boxed{w[n] = \{0, 0.25, 0.75, 1, 0.75, 0.25, 0\}}$$

Example 2: Calculate one windowed coefficient

For $N = 7$, suppose the ideal coefficient at $n = 2$ is $h_d[2] = 0.4$.

Since $w[2] = 0.75$,

$$h[2] = h_d[2] w[2]$$

$$h[2] = 0.4(0.75)$$

$$\boxed{h[2] = 0.3}$$

10. Hamming Window

Theory

The Hamming window is a cosine window designed to reduce the first sidelobe compared with the Hann window.

$$w_{\text{Ham}}[n] = 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right)$$

for $0 \le n \le N-1$.

Characteristics: Its endpoints are nonzero (approximately 0.08), and it provides good sidelobe suppression for many FIR design applications.

Example 1: Calculate a length-5 Hamming window

For $N = 5$,

$$w[n] = 0.54 - 0.46 \cos\left(\frac{\pi n}{2}\right)$$

Evaluate:

- $n = 0 :\ 0.54 - 0.46 = 0.08$
- $n = 1 :\ 0.54$
- $n = 2 :\ 0.54 + 0.46 = 1$
- $n = 3 :\ 0.54$
- $n = 4 :\ 0.08$

Therefore,

$$\boxed{w[n] = \{0.08, 0.54, 1, 0.54, 0.08\}}$$

Example 2: Apply the Hamming window

Given

$$h_d[n] = \{1, 2, 3, 2, 1\}$$

Multiply the coefficients by the length-5 Hamming window:

$$h[n] = \{0.08, 1.08, 3, 1.08, 0.08\}$$

$$\boxed{h[n] = \{0.08, 1.08, 3, 1.08, 0.08\}}$$

11. Blackman Window

Theory

The Blackman window uses two cosine terms to reduce sidelobes further, usually at the cost of a wider main lobe.

$$w_B[n] = 0.42 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right)$$

for $0 \leq n \leq N-1$.

Example 1: Calculate a length-5 Blackman window

For $N = 5$, evaluate the formula at $n = 0, 1, 2, 3, 4$.

At $n = 0$,

$$w_B[0] = 0.42 - 0.5 + 0.08 = 0$$

At $n = 1$,

$$w_B[1] = 0.42 - 0.5 \cos(\pi/2) + 0.08 \cos(\pi)$$

$$w_B[1] = 0.42 - 0 - 0.08 = 0.34$$

At $n = 2$,

$$w_B[2] = 0.42 - 0.5\cos(\pi) + 0.08\cos(2\pi)$$

$$w_B[2] = 0.42 + 0.5 + 0.08 = 1$$

By symmetry,

$$\boxed{w_B[n] = \{0, 0.34, 1, 0.34, 0\}}$$

Example 2: Apply the Blackman window

Given

$$h_d[n] = \{1, 2, 3, 2, 1\}$$

Multiply by the window coefficients:

$$h[n] = \{0, 0.68, 3, 0.68, 0\}$$

$$\boxed{h[n] = \{0, 0.68, 3, 0.68, 0\}}$$

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12. Comparison of FIR Window Functions

The values below are approximate characteristics of common windows; exact results depend on the convention and window length.

Window	Main-lobe width (relative)	Peak sidelobe (approx.)	General use
Rectangular	Narrow	−13 dB	Sharp transition, more leakage
Triangular / Bartlett	Wider	−25 dB	Simple tapered window
Hann / Hanning	Wider	−31 dB	General spectral analysis and FIR design
Hamming	Wider	−43 dB	Good first-sidelobe suppression
Blackman	Widest among these common windows	−58 dB	Stronger sidelobe reduction

A practical design balances transition width and stopband attenuation. For stronger stopband rejection, Hamming or Blackman is often preferred over rectangular.

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13. Comparison of IIR and FIR Filters

Theory

FIR filters use a finite number of input samples. IIR filters use feedback and typically have an impulse response that continues indefinitely.

Feature	FIR	IIR
Impulse response	Finite	Infinite in general
Feedback	Not required in standard form	Usually present
Stability	BIBO stable for finite coefficients	Must check pole locations

Linear phase	Can be exact	Generally not exact with conventional designs
Order for similar specifications	Often higher	Often lower
Computational cost	Can be higher	Often lower
Quantization sensitivity	Often more robust	May be more sensitive
Common design methods	Windowing, frequency sampling, optimal methods	Butterworth, Chebyshev, elliptic, analog transformations

Example 1: Identify and compare filter types

Given:

$$H_1(z) = 1 + 0.5z^{-1} + 0.25z^{-2}$$

$$H_2(z) = \frac{1}{1 - 0.8z^{-1}}$$

- $H_1(z)$ is FIR, order 2, and has a finite impulse response.
- $H_2(z)$ is IIR, order 1, and has an impulse response proportional to $(0.8)^n u[n]$.

The IIR filter has a lower order in this example but requires feedback.

Example 2: Calculate the FIR group delay and IIR pole

For a symmetric FIR filter of order $M = 6$,

$$\tau_g = \frac{M}{2} = 3 \text{ samples}$$

For the IIR filter

$$H(z) = \frac{1}{1 - 0.7z^{-1}}$$

the pole is

$$z = 0.7$$

Since $|0.7| < 1$, the causal IIR filter is stable.

Conclusion: FIR offers straightforward linear-phase design, while IIR often achieves a target magnitude response with fewer coefficients.

14. Basic Structure of FIR Filters – Direct Form

Theory

The direct form implements the FIR difference equation using a delay line, coefficient multipliers and an adder.

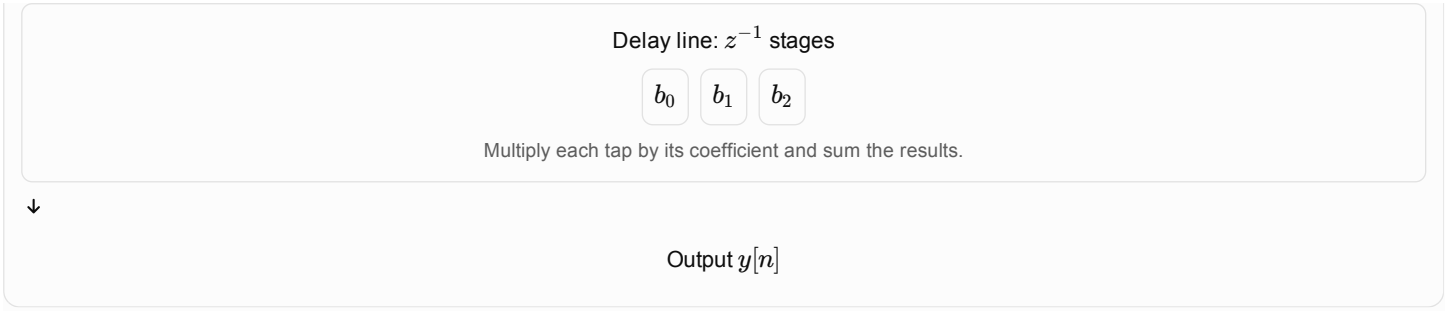
$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M]$$

It is also called the tapped-delay-line structure.

Direct-form FIR structure

Input $x[n]$

↓



Example 1: Obtain the difference equation

Given

$$H(z) = 2 + 3z^{-1} + z^{-2}$$

Therefore,

$$\boxed{y[n] = 2x[n] + 3x[n-1] + x[n-2]}$$

The direct form requires two delay elements.

Example 2: Calculate output

Given the same filter and input $x[n] = \{1, 2, 3\}$, assume zero initial conditions.

$$y[0] = 2(1) = 2$$

$$y[1] = 2(2) + 3(1) = 7$$

$$y[2] = 2(3) + 3(2) + 1(1) = 13$$

Thus,

$$\boxed{y[n] = \{2, 7, 13\}}$$

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15. Cascade Form of FIR Filters

Theory

In cascade form, the FIR transfer function is factored into lower-order sections connected in series.

$$H(z) = H_1(z)H_2(z)\cdots H_L(z)$$

For real coefficients, a higher-order FIR filter may be factored into first-order and second-order sections. Each section can be implemented using a small delay line and multipliers.

Example 1: Factor the transfer function

Given

$$H(z) = 1 + 2z^{-1} + z^{-2}$$

Factor:

$$H(z) = (1 + z^{-1})^2$$

Therefore,

$$\boxed{H_1(z) = 1 + z^{-1}, \qquad H_2(z) = 1 + z^{-1}}$$

The cascade consists of two identical first-order FIR sections.

Example 2: Calculate the cascade output

Each section satisfies

$$v[n] = x[n] + x[n-1]$$

and the second section satisfies

$$y[n] = v[n] + v[n-1]$$

For $x[n] = \{1, 2, 1\}$, with zero initial conditions, the first section produces

$$v[n] = \{1, 3, 3, 1\}$$

The second section produces

$$y[n] = \{1, 4, 6, 4, 1\}$$

Therefore,

$$\boxed{y[n] = \{1, 4, 6, 4, 1\}}$$

This matches the convolution of the two sections' impulse responses.

16. Linear-Phase Realizations of FIR Filters

Theory

Linear-phase FIR filters use symmetric or antisymmetric coefficients. Symmetry allows paired input samples to be added before multiplication, reducing the number of multipliers.

For an even-length symmetric filter with $M + 1$ coefficients,

$$b[n] = b[M - n]$$

The output can be grouped as

$$y[n] = \sum_{k=0}^{\lfloor (M-1)/2 \rfloor} b[k]\{x[n-k] + x[n-(M-k)]\}$$

with a separate centre term when the number of coefficients is odd.

For antisymmetric coefficients, the paired samples are subtracted instead of added.

Example 1: Symmetric coefficients

Given

$$h[n] = \{1, 2, 3, 2, 1\}$$

The difference equation is

$$\begin{aligned} y[n] = x[n] + 2x[n-1] + 3x[n-2] \\ + 2x[n-3] + x[n-4] \end{aligned}$$

Group the symmetric terms:

$$\boxed{y[n] = \{x[n] + x[n-4]\} + 2\{x[n-1] + x[n-3]\} + 3x[n-2]}$$

Instead of five separate coefficient multiplications, the paired form requires two pair additions, two multiplications for the paired results, and one centre multiplication.

Example 2: Antisymmetric coefficients

Given

$$h[n] = \{1, 2, 0, -2, -1\}$$

The coefficient pairs satisfy

$$h[0] = -h[4], \qquad h[1] = -h[3]$$

The output is

$$y[n] = x[n] + 2x[n-1] - 2x[n-3] - x[n-4]$$

Grouping terms gives

$$\boxed{y[n] = \{x[n] - x[n-4]\} + 2\{x[n-1] - x[n-3]\}}$$

The antisymmetric realization uses paired differences and has a zero centre coefficient.

Important Questions:

1. Define a digital filter. Explain the characteristics and advantages of FIR filters.
2. What is a linear-phase FIR filter? Derive the conditions on the impulse response for a linear-phase FIR filter.
3. Derive the frequency response of linear-phase FIR filters and explain the four types of linear-phase FIR filters (Types I, II, III, and IV).
4. Explain the conditions for obtaining linear phase in FIR filters and discuss the significance of symmetric and antisymmetric impulse responses.

5. Explain the Fourier series method of FIR filter design with the necessary equations and design steps.
6. Explain the windowing method of FIR filter design and describe its general procedure.
7. Explain the Rectangular window and Triangular window methods for FIR filter design, including their characteristics.
8. Explain the Raised Cosine, Hanning (Hann), Hamming, and Blackman windows and compare their characteristics.
9. Design an FIR low-pass filter using the windowing method for the given specifications.
10. Explain the effect of window selection on the transition width, main-lobe width, and side-lobe attenuation of the designed FIR filter.
11. Compare the different window functions used in FIR filter design with respect to transition width and stopband attenuation.

12. Compare IIR and FIR digital filters in terms of impulse response, stability, phase response, complexity, and implementation.
13. Explain why FIR filters are preferred in applications requiring exact linear phase.

14. Explain the Direct Form structure of an FIR filter with a neat block diagram.
15. Explain the Cascade Form realization of an FIR filter with a suitable block diagram.
16. Explain linear-phase FIR filter realizations and discuss how symmetry reduces the number of multiplications.
17. Compare the Direct Form and Cascade Form realizations of FIR filters.